Derivation Document

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Let f_w be the rate of wind energy production, f_s be the rate of solar energy production. Let $x = [x_1, x_2]$ be the state vector where x_1 is the amount of energy in the grid and x_2 is the amount of energy in the battery. Let u_1 be the batery output, u_2 be the battery input, u_g be the gas energy production rate, u_w be the wind energy production rate, and u_s be the solar energy production rate. Let d be the demand rate.

We let $\mathbf{u} = [u_1, u_2, u_g, u_w, u_s]^T$ be the control vector. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be tunable parameters.

The state equations are given by:

$$\begin{aligned}
\dot{x_1} &= u_w f_w + u_s f_s + u_g - u_1 + u_2 - d \\
\dot{x_2} &= u_1 - u_2 - \epsilon x_2
\end{aligned}$$

We seek to minimize the cost functional:

$$J[u, x] = \int_{t_0}^{t_f} (\alpha C_M + \beta C_E) u^2 + \gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta))^2$$

Subject to the constraints on battery capacity and the control input:

$$\begin{aligned}
&0 \leq x_2 \leq x_2^{\max} \\
&\mathbf{u} \geq 0
\end{aligned}$$

We can use the Pontryagin's Minimum Principle to derive the necessary conditions for optimality. The Lagrangian is given by:

$$\begin{aligned}
\mathcal{L}(x, u, \lambda, \mu) = {} & p_1(u_w f_w + u_s f_s + u_g - u_1 + u_2 - d) + p_2(u_1 - u_2 - \epsilon x_2) \\
& - (\alpha C_M + \beta C_E)\mathbf{u}^2 + \gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta))^2 \\
& + \mu_1 x_2 + \mu_2(x_2^{\max} - x_2) - \lambda^T \mathbf{u}
\end{aligned}$$

Subject to the conditions:

$$\begin{aligned}
&\mu_1, \mu_2 \geq 0 \\
&\mu_1 x_2 = 0 \\
&\mu_2(x_2^{\max} - x_2) = 0 \\
&\lambda \geq 0 \\
&\lambda^T \mathbf{u} = 0
\end{aligned}$$

The costate equations are given by:

$$\dot{p_1} = -\frac{\partial \mathcal{L}}{\partial x_1} = 0$$
$$\dot{p_2} = -\frac{\partial \mathcal{L}}{\partial x_2} = -\epsilon p_2 + \mu_1 - \mu_2$$

Solving for the costate variables and applying the endpoint conditions $p(t_f) = 0$ gives:

$$p_1(t) = 0$$
$$p_2(t) = \frac{\mu_1 - \mu_2}{\epsilon} e^{-at}(e^{at} - e^{at_f})$$

The optimality conditions are given by:

$$u_1^* = \frac{\partial \mathcal{L}}{\partial u_1} = -p_1 + p_2 - 2(\alpha C_{M_1} + \beta C_{E_1})u_1 - 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d+\delta)) - \lambda_1 = 0$$
$$u_2^* = \frac{\partial \mathcal{L}}{\partial u_2} = -p_1 - p_2 - 2(\alpha C_{M_2} + \beta C_{E_2})u_2 + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d+\delta)) - \lambda_2 = 0$$
$$u_g^* = \frac{\partial \mathcal{L}}{\partial u_g} = p_1 - 2(\alpha C_{M_g} + \beta C_{E_g})u_g + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d+\delta)) - \lambda_3 = 0$$
$$u_w^* = \frac{\partial \mathcal{L}}{\partial u_w} = p_1 f_w - 2(\alpha C_{M_w} + \beta C_{E_w})u_w + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d+\delta))f_w - \lambda_4 = 0$$
$$u_s^* = \frac{\partial \mathcal{L}}{\partial u_s} = p_1 f_s - 2(\alpha C_{M_s} + \beta C_{E_s})u_s + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d+\delta))f_s - \lambda_5 = 0$$