

Derivation Document

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Let f_w be the rate of wind energy production, f_s be the rate of solar energy production. Let $x = [x_1, x_2]$ be the state vector where x_1 is the amount of energy in the grid and x_2 is the amount of energy in the battery. Let u_1 be the battery output, u_2 be the battery input, u_g be the gas energy production rate, u_w be the wind energy production rate, and u_s be the solar energy production rate. Let d be the demand rate.

We let $\mathbf{u} = [u_1, u_2, u_g, u_w, u_s]^T$ be the control vector. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be tunable parameters.

The state equations are given by:

$$\begin{aligned}\dot{x}_1 &= u_w f_w + u_s f_s + u_g - u_1 + u_2 - d \\ \dot{x}_2 &= u_1 - u_2 - \epsilon x_2\end{aligned}$$

We seek to minimize the cost functional:

$$J[u, x] = \int_{t_0}^{t_f} (\alpha C_M + \beta C_E) u^2 + \gamma (u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta))^2$$

Subject to the constraints on battery capacity and the control input:

$$\begin{aligned}0 &\leq x_2 \leq x_2^{\max} \\ \mathbf{u} &\geq 0\end{aligned}$$

We can use the Pontryagin's Minimum Principle to derive the necessary conditions for optimality. The Lagrangian is given by:

$$\begin{aligned}\mathcal{L}(x, u, \lambda, \mu) &= p_1(u_w f_w + u_s f_s + u_g - u_1 + u_2 - d) + p_2(u_1 - u_2 - \epsilon x_2) \\ &\quad - (\alpha C_M + \beta C_E) \mathbf{u}^2 + \gamma (u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta))^2 \\ &\quad + \mu_1 x_2 + \mu_2 (x_2^{\max} - x_2) - \lambda^T \mathbf{u}\end{aligned}$$

Subject to the conditions:

$$\begin{aligned}\mu_1, \mu_2 &\geq 0 \\ \mu_1 x_2 &= 0 \\ \mu_2 (x_2^{\max} - x_2) &= 0 \\ \lambda &\geq 0 \\ \lambda^T \mathbf{u} &= 0\end{aligned}$$

The costate equations are given by:

$$\begin{aligned}\dot{p}_1 &= -\frac{\partial \mathcal{L}}{\partial x_1} = 0 \\ \dot{p}_2 &= -\frac{\partial \mathcal{L}}{\partial x_2} = -\epsilon p_2 + \mu_1 - \mu_2\end{aligned}$$

Solving for the costate variables and applying the endpoint conditions $p(t_f) = 0$ gives:

$$\begin{aligned}p_1(t) &= 0 \\ p_2(t) &= e^{-\epsilon t} \int_{t_f}^t e^{\epsilon \tau} (\mu_1(\tau) + \mu_2(\tau)) d\tau\end{aligned}$$

The optimality conditions are given by:

$$\begin{aligned}u_1^* &= \frac{\partial \mathcal{L}}{\partial u_1} = -p_1 + p_2 - 2(\alpha C_{M_1} + \beta C_{E_1})u_1 - 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta)) - \lambda_1 = 0 \\ u_2^* &= \frac{\partial \mathcal{L}}{\partial u_2} = -p_1 - p_2 - 2(\alpha C_{M_2} + \beta C_{E_2})u_2 + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta)) - \lambda_2 = 0 \\ u_g^* &= \frac{\partial \mathcal{L}}{\partial u_g} = p_1 - 2(\alpha C_{M_g} + \beta C_{E_g})u_g + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta)) - \lambda_3 = 0 \\ u_w^* &= \frac{\partial \mathcal{L}}{\partial u_w} = p_1 f_w - 2(\alpha C_{M_w} + \beta C_{E_w})u_w + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta))f_w - \lambda_4 = 0 \\ u_s^* &= \frac{\partial \mathcal{L}}{\partial u_s} = p_1 f_s - 2(\alpha C_{M_s} + \beta C_{E_s})u_s + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta))f_s - \lambda_5 = 0\end{aligned}$$

We solve for optimal control inputs

(Solving for u_1):

$$\begin{aligned}p_2 - 2(\alpha C_{M_1} + \beta C_{E_1})u_1 - 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta)) - \lambda_1 &= 0 \\ p_2 - 2(\alpha C_{M_1} + \beta C_{E_1})u_1 + 2\gamma u_1 - 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - (d + \delta)) - \lambda_1 &= 0 \\ -2(\alpha C_{M_1} + \beta C_{E_1})u_1 + 2\gamma u_1 &= \lambda_1 - p_2 + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - (d + \delta)) \\ u_1 &= \frac{\lambda_1 - p_2 + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - (d + \delta))}{2\gamma - 2(\alpha C_{M_1} + \beta C_{E_1})}\end{aligned}$$

(Solving for u_2):

$$\begin{aligned}-p_2 - 2(\alpha C_{M_2} + \beta C_{E_2})u_2 + 2\gamma(u_w f_w + u_s f_s + u_g + u_2 - u_1 - (d + \delta)) - \lambda_2 &= 0 \\ -p_2 + 2\gamma u_2 - 2(\alpha C_{M_2} + \beta C_{E_2})u_2 + 2\gamma(u_w f_w + u_s f_s + u_g - u_1 - (d + \delta)) &= \lambda_2 \\ u_2 &= \frac{\lambda_2 + p_2 - 2\gamma(u_w f_w + u_s f_s + u_g - u_1 - (d + \delta))}{2\gamma - 2(\alpha C_{M_2} + \beta C_{E_2})}\end{aligned}$$