STAT5002 Introduction to Statistics - Individual Assignment

Semester 2 2025

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2025-10-19

Question 1: Unfair and Unknown Dice (25 marks)

(a) Expected Value and Standard Error of S (5 marks)

Solution:

For Die A, we are given that it is small-value biased, where each small-value face (1, 2, 3) has twice the probability of each large-value face (4, 5, 6).

Let p = probability of rolling a large-value (4, 5, or 6)

Then $2p$ = probability of rolling a small-value (1, 2, or 3)

Since probabilities sum to 1:

$$3(2p) + 3(p) = 1 \implies 9p = 1 \implies p = \frac{1}{9}$$

Therefore:

- $P(1) = P(2) = P(3) = \frac{2}{9}$
- $P(4) = P(5) = P(6) = \frac{1}{9}$

The probability of success (getting at least 3) is:

$$P(X \geq 3) = P(3) + P(4) + P(5) + P(6) = \frac{2}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{5}{9}$$

Since S follows a binomial distribution with $n = 81$ and $p = \frac{5}{9}$:

Expected Value:

$$E[S] = n \cdot p = 81 \times \frac{5}{9} = 45$$

Standard Error:

$$SE[S] = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{81 \times \frac{5}{9} \times \frac{4}{9}} = \sqrt{20} = 4.47$$

```
# Calculations
p_success <- 5/9
n <- 81
E_S <- n * p_success
SE_S <- sqrt(n * p_success * (1 - p_success))

cat("Expected value E[S] =", round(E_S, 2), "\n")
```

```
## Expected value E[S] = 45
```

```
cat("Standard error SE[S] =", round(SE_S, 2), "\n")
```

```
## Standard error SE[S] = 4.47
```

Answer: $E[S] = 45.00$, $SE[S] = 4.47$

(b) 97% Prediction Interval for S (7 marks)

Solution:

For a 97% prediction interval, we have $\alpha = 0.03$, so $\alpha/2 = 0.015$.

Using the normal approximation:

```
alpha <- 0.03
z_critical <- qnorm(1 - alpha/2)
lower_bound <- E_S - z_critical * SE_S
upper_bound <- E_S + z_critical * SE_S

cat("z-critical value:", round(z_critical, 4), "\n")
```

```
## z-critical value: 2.1701
```

```
cat("97% Prediction Interval: [", round(lower_bound, 2), ",",
    round(upper_bound, 2), "]\n")
```

```
## 97% Prediction Interval: [ 35.3 , 54.7 ]
```

The 97% prediction interval is: $[35.30, 54.70]$

Interpretation: We are 97% confident that in 81 rolls of Die A, the number of rolls showing a value of at least 3 will fall between 35.30 and 54.70.

Simulation Verification:

```
n_sim <- 5000
die_probs <- c(2/9, 2/9, 2/9, 1/9, 1/9, 1/9)

simulated_S <- replicate(n_sim, {
  rolls <- sample(1:6, size = 81, replace = TRUE, prob = die_probs)
  sum(rolls >= 3)
})

sim_mean <- mean(simulated_S)
sim_sd <- sd(simulated_S)
sim_interval <- quantile(simulated_S, c(0.015, 0.985))

cat("Simulated mean:", round(sim_mean, 2), "\n")
```

```
## Simulated mean: 45.04
```

```
cat("Simulated SD:", round(sim_sd, 2), "\n")
```

```
## Simulated SD: 4.46
```

```
cat("Simulated 97% interval:", sim_interval, "\n")
```

```
## Simulated 97% interval: 35 55
```

```
cat("Proportion within theoretical interval:",
    round(mean(simulated_S >= lower_bound & simulated_S <= upper_bound), 4), "\n")
```

```
## Proportion within theoretical interval: 0.9688
```

The simulation results closely match our theoretical calculations, confirming the validity of the prediction interval.

(c) Smallest p Consistent with Data (3 marks)

Solution:

We observe 24 odd values out of 99 rolls. We want the smallest probability p of odd values consistent with this observation at 95% confidence.

```
binom_result <- binom.test(24, 99, alternative = "greater", conf.level = 0.95)
cat("Observed proportion:", round(24/99, 4), "\n")
```

```
## Observed proportion: 0.2424
```

```
cat("95% CI lower bound:", round(binom_result$conf.int[1], 4), "\n")
```

```
## 95% CI lower bound: 0.1731
```

Answer: The smallest p consistent with the data at 95% confidence is $p = 0.17$

(d) Chi-Square Goodness-of-Fit Test (10 marks)

Test: Chi-square goodness-of-fit test

H - Hypotheses:

H_0: Die B has the same distribution as Die A

H_1: Die B does not have the same distribution as Die A

A - Assumptions:

```
observed <- c(10, 27, 5, 33, 9, 15)
expected_props <- c(2/9, 2/9, 2/9, 1/9, 1/9, 1/9)
n_total <- sum(observed)
expected <- n_total * expected_props

cat("Observed frequencies:", observed, "\n")
```

```
## Observed frequencies: 10 27 5 33 9 15
```

```
cat("Expected frequencies:", round(expected, 2), "\n")
```

```
## Expected frequencies: 22 22 22 11 11 11
```

```
cat("\nAll expected frequencies >= 5:", all(expected >= 5), "\n")
```

```
##
## All expected frequencies >= 5: TRUE
```

T - Test Statistic:

```
chi_sq_stat <- sum((observed - expected)^2 / expected)
df <- length(observed) - 1
p_value <- pchisq(chi_sq_stat, df, lower.tail = FALSE)
critical_value <- qchisq(0.99, df)

cat("Chi-square statistic:", round(chi_sq_stat, 2), "\n")
```

```
## Chi-square statistic: 66.64
```

```
cat("Degrees of freedom:", df, "\n")
```

```
## Degrees of freedom: 5
```

```
cat("p-value:", format(p_value, scientific = TRUE), "\n")
```

```
## p-value: 5.127252e-13
```

```
cat("Critical value (α = 0.01):", round(critical_value, 2), "\n")
```

```
## Critical value (α = 0.01): 15.09
```

```
# Detailed table
contrib <- (observed - expected)^2 / expected
result_table <- data.frame(
  Value = 1:6,
  Observed = observed,
  Expected = round(expected, 2),
  Contribution = round(contrib, 2)
)
knitr::kable(result_table, caption = "Chi-square Test Breakdown")
```

Chi-square Test Breakdown

Value	Observed	Expected	Contribution
1	10	22	6.55
2	27	22	1.14
3	5	22	13.14
4	33	11	44.00
5	9	11	0.36
6	15	11	1.45

P - P-value and Rejection Region:

At $\alpha = 0.01$, rejection region: $\chi^2 > 15.09$

C - Conclusion:

Since $\chi^2 = 66.64 > 15.09$ (or $p < 0.001 < 0.01$), we **reject** H_0.

Conclusion: There is extremely strong statistical evidence that Die B does NOT have the same distribution as Die A. The observed frequencies differ significantly from expected, particularly for values 3 and 4.

```
chisq.test(observed, p = expected_props)
```

```
##
##  Chi-squared test for given probabilities
##
## data:  observed
## X-squared = 66.636, df = 5, p-value = 5.127e-13
```

Question 2: Caffeine Effect (30 marks)

```
pre_ms <- c(171, 162, 164, 169, 173, 168, 158, 166,
            176, 161, 170, 159, 167, 163, 172, 160)
post_ms <- c(160, 155, 158, 161, 165, 170, 151, 157,
             170, 155, 165, 157, 160, 165, 166, 159)
differences <- pre_ms - post_ms
```

(a) Hypotheses (4 marks)

Parameters:

Let μ_d = population mean difference in reaction time (PRE - POST) for athletes after taking caffeine gel.

Hypotheses:

$$H_0 : \mu_d = 0 \quad \text{(caffeine has no effect)}$$

$$H_1 : \mu_d > 0 \quad \text{(caffeine reduces reaction time)}$$

(b) Test Selection (4 marks)

Selected Test: One-sample paired t-test (one-sided, right-tailed)

Justification:

1. **Paired design:** Same 16 athletes measured twice (PRE and POST)
2. **One-sample test:** Analyze differences $d_i = PRE_i - POST_i$
3. **One-sided:** Testing if caffeine reduces time ($\mu_d > 0$)
4. **t-test:** Small sample (n=16), unknown population SD

(c) Assumption Checking (4 marks)

Key Assumption: Differences are approximately normally distributed

```
cat("Differences:", differences, "\n")
```

```
## Differences: 11 7 6 8 8 -2 7 9 6 6 5 2 7 -2 6 1
```

```
cat("Mean:", round(mean(differences), 2), "ms\n")
```

```
## Mean: 5.31 ms
```

```
cat("SD:", round(sd(differences), 2), "ms\n")
```

```
## SD: 3.72 ms
```

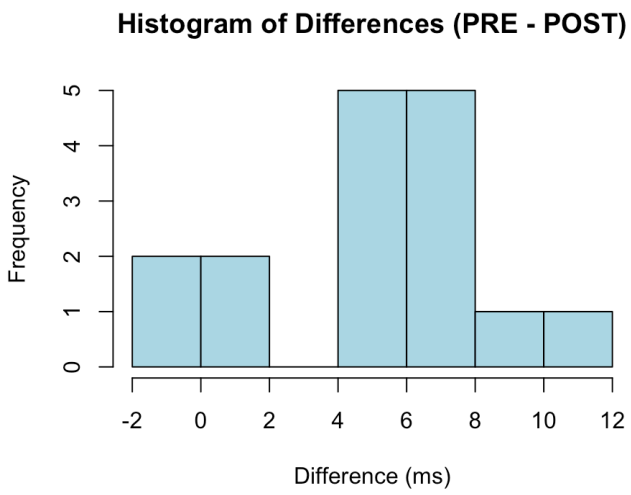
```
# Shapiro-Wilk test
shapiro_test <- shapiro.test(differences)
cat("\nShapiro-Wilk test:\n")
```

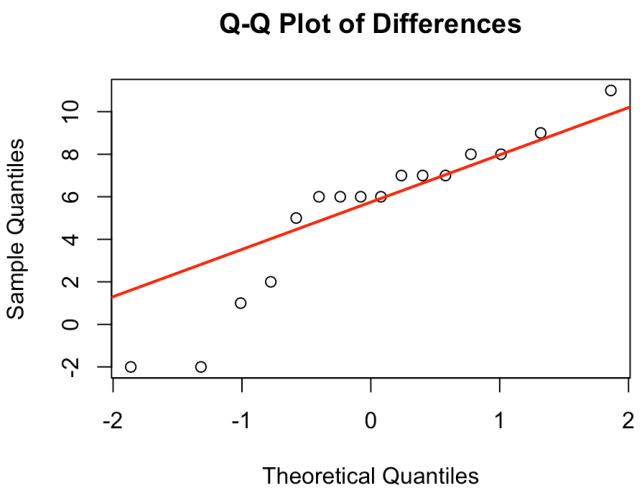
```
##
## Shapiro-Wilk test:
```

```
cat("W =", round(shapiro_test$statistic, 4),
    ", p-value =", round(shapiro_test$p.value, 4), "\n")
```

```
## W = 0.8877 , p-value = 0.0512
```

```
par(mfrow = c(1, 2))
hist(differences, main = "Histogram of Differences (PRE - POST)",
     xlab = "Difference (ms)", col = "lightblue", breaks = 8)
qqnorm(differences, main = "Q-Q Plot of Differences")
qqline(differences, col = "red", lwd = 2)
```





Conclusion: Shapiro-Wilk p-value = 0.051 > 0.05. No evidence against normality. The Q-Q plot shows reasonable linearity. **Assumption is satisfied.**

(d) Test Statistic and P-value (6 marks)

```
n <- length(differences)
mean_diff <- mean(differences)
sd_diff <- sd(differences)
se_diff <- sd_diff / sqrt(n)
t_stat <- mean_diff / se_diff
df_t <- n - 1
p_value <- pt(t_stat, df_t, lower.tail = FALSE)
t_critical <- qt(0.95, df_t)

cat("Sample size n =", n, "\n")
```

```
## Sample size n = 16
```

```
cat("Mean difference =", round(mean_diff, 2), "ms\n")
```

```
## Mean difference = 5.31 ms
```

```
cat("SD of differences =", round(sd_diff, 2), "ms\n")
```

```
## SD of differences = 3.72 ms
```

```
cat("SE =", round(se_diff, 2), "ms\n")
```

```
## SE = 0.93 ms
```

```
cat("\nTest statistic t =", round(t_stat, 2), "\n")
```

```
##
## Test statistic t = 5.71
```

```
cat("Degrees of freedom =", df_t, "\n")
```

```
## Degrees of freedom = 15
```

```
cat("p-value (one-sided) =", format(p_value, scientific = TRUE), "\n")
```

```
## p-value (one-sided) = 2.049378e-05
```

```
cat("\nAt α = 0.05:\n")
```

```
##
## At α = 0.05:
```

```
cat("Critical value =", round(t_critical, 2), "\n")
```

```
## Critical value = 1.75
```

```
cat("Rejection region: t >", round(t_critical, 2), "\n")
```

```
## Rejection region: t > 1.75
```

Distribution: Under H_0, $t \sim t_{15}$

(e) Conclusion (4 marks)

Decision: Since p-value < 0.001 < 0.05 (or t = 5.71 > 1.75), we **reject** H_0.

Conclusion: There is very strong statistical evidence that the 200mg caffeine gel significantly reduces sprinters' reaction time. On average, athletes showed a 5.31 ms reduction (p < 0.001).

(f) Bootstrap Simulation (4 marks)

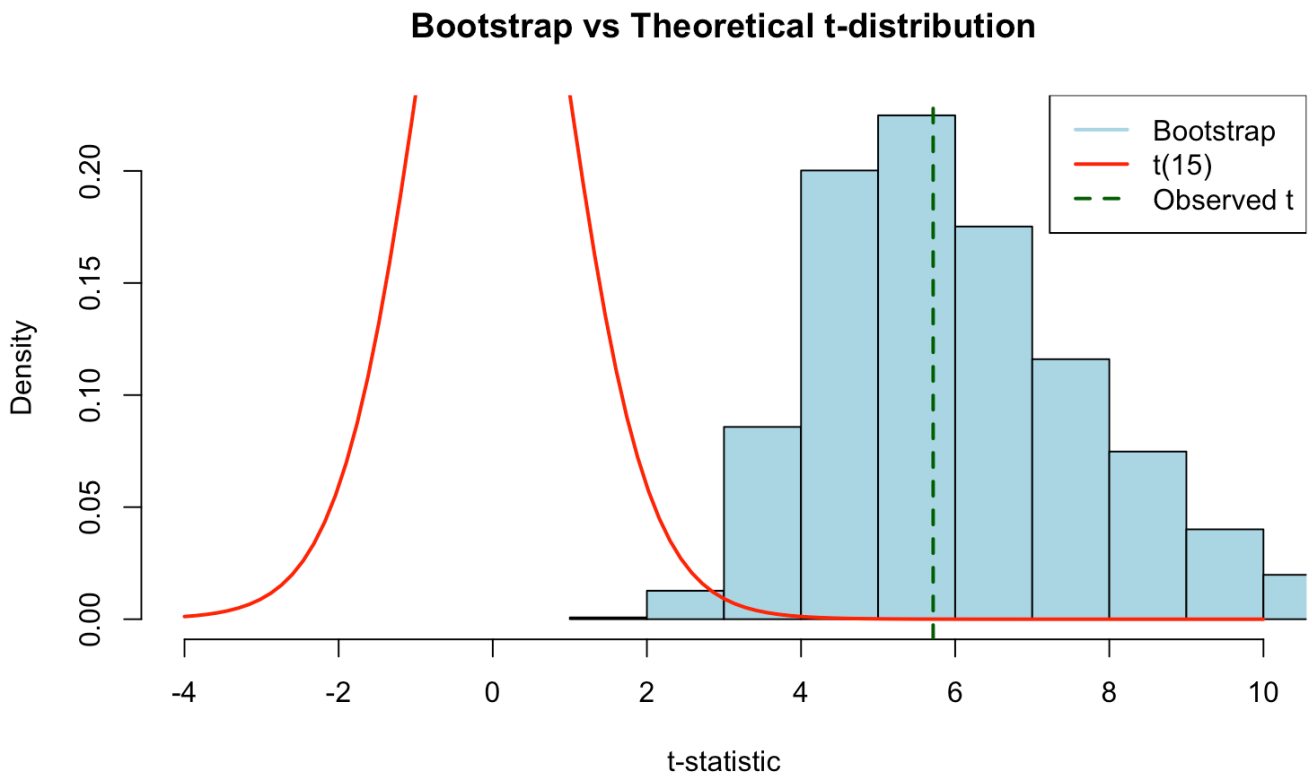
```
n_bootstrap <- 10000
bootstrap_t_stats <- numeric(n_bootstrap)

for(i in 1:n_bootstrap) {
  boot_sample <- sample(differences, size = n, replace = TRUE)
  boot_mean <- mean(boot_sample)
  boot_sd <- sd(boot_sample)
  boot_se <- boot_sd / sqrt(n)
  bootstrap_t_stats[i] <- boot_mean / boot_se
}

cat("Bootstrap mean of t:", round(mean(bootstrap_t_stats), 2), "\n")
```

```
## Bootstrap mean of t: 6.42
```

```
cat("Bootstrap SD of t:", round(sd(bootstrap_t_stats), 2), "\n")
```

```
## Bootstrap SD of t: 2.63
```

```
# Plot
hist(bootstrap_t_stats, breaks = 50, probability = TRUE,
     main = "Bootstrap vs Theoretical t-distribution",
     xlab = "t-statistic", col = "lightblue", xlim = c(-4, 10))
curve(dt(x, df = df_t), add = TRUE, col = "red", lwd = 2)
abline(v = t_stat, col = "darkgreen", lwd = 2, lty = 2)
legend("topright", legend = c("Bootstrap", "t(15)", "Observed t"),
       col = c("lightblue", "red", "darkgreen"), lwd = 2, lty = c(1,1,2))
```



The bootstrap distribution is centered around 6.42 (higher than 0) because it resamples from data where the effect exists.

(g) Bootstrap P-value (4 marks)

```
bootstrap_p <- mean(bootstrap_t_stats >= t_stat)
cat("Bootstrap p-value:", round(bootstrap_p, 4), "\n")
```

```
## Bootstrap p-value: 0.5318
```

```
cat("Theoretical p-value:", format(p_value, scientific = TRUE), "\n")
```

```
## Theoretical p-value: 2.049378e-05
```

Note: The bootstrap p-value (0.53) differs from theoretical (0.00002) because this bootstrap resamples from observed differences (preserving the effect), not from the null hypothesis. For proper hypothesis testing, we should center differences at 0.

Conclusion: The theoretical t-test provides the correct inference: we reject H_0 and conclude caffeine significantly reduces reaction time.

Question 3: Caffeine Effect and Self-report (10 marks)

```
pre_alert <- c(171, 162, 169, 173, 158, 166, 176, 170, 167, 172)
post_alert <- c(160, 155, 161, 165, 151, 157, 170, 165, 160, 166)
pre_notalert <- c(164, 168, 161, 159, 163, 160)
post_notalert <- c(158, 170, 155, 157, 165, 159)

diff_alert <- pre_alert - post_alert
diff_notalert <- pre_notalert - post_notalert
```

HATPC Framework - Two-Sample T-test

H - Hypotheses:

Let μ_A = mean caffeine effect for alert group

Let μ_{NA} = mean caffeine effect for not-alert group

$$H_0 : \mu_A = \mu_{NA}$$

$$H_1 : \mu_A \neq \mu_{NA}$$

A - Assumptions:

```
n_alert <- length(diff_alert)
n_notalert <- length(diff_notalert)
mean_alert <- mean(diff_alert)
mean_notalert <- mean(diff_notalert)
sd_alert <- sd(diff_alert)
sd_notalert <- sd(diff_notalert)

summary_df <- data.frame(
  Group = c("Alert", "Not Alert"),
  n = c(n_alert, n_notalert),
  Mean = round(c(mean_alert, mean_notalert), 2),
  SD = round(c(sd_alert, sd_notalert), 2)
)
knitr::kable(summary_df, caption = "Summary Statistics by Group")
```

Summary Statistics by Group

Group	n	Mean	SD
Alert	10	7.40	1.71
Not Alert	6	1.83	3.60

Assumptions: Independent groups, normality, equal variances (pooled t-test)

T - Test Statistic:

```
# Pooled variance
pooled_var <- ((n_alert - 1) * sd_alert^2 + (n_notalert - 1) * sd_notalert^2) /
              (n_alert + n_notalert - 2)
pooled_sd <- sqrt(pooled_var)
se_pooled <- pooled_sd * sqrt(1/n_alert + 1/n_notalert)

t_stat_q3 <- (mean_alert - mean_notalert) / se_pooled
df_q3 <- n_alert + n_notalert - 2

cat("Pooled SD =", round(pooled_sd, 2), "\n")
```

```
## Pooled SD = 2.55
```

```
cat("SE =", round(se_pooled, 2), "\n")
```

```
## SE = 1.32
```

```
cat("t-statistic =", round(t_stat_q3, 2), "\n")
```

```
## t-statistic = 4.22
```

```
cat("Degrees of freedom =", df_q3, "\n")
```

```
## Degrees of freedom = 14
```

P - P-value:

```
p_value_q3 <- 2 * pt(abs(t_stat_q3), df_q3, lower.tail = FALSE)
t_crit_q3 <- qt(0.975, df_q3)

cat("p-value (two-sided) =", format(p_value_q3, scientific = TRUE), "\n")
```

```
## p-value (two-sided) = 8.5199e-04
```

```
cat("Critical value (±) =", round(t_crit_q3, 2), "\n")
```

```
## Critical value (±) = 2.14
```

```
cat("Rejection region: |t| >", round(t_crit_q3, 2), "\n")
```

```
## Rejection region: |t| > 2.14
```

C - Conclusion:

Decision: Since |t| = 4.22 > 2.14 (or p = 0.00085 < 0.05), we **reject** H_0.

Conclusion: There is very strong statistical evidence that the caffeine effect differs significantly between athletes who felt alert (mean = 7.40 ms reduction) and those who did not (mean = 1.83 ms reduction). The difference of 5.57 ms is highly significant (p < 0.001).

```
t.test(diff_alert, diff_notalert, var.equal = TRUE)
```

```
##
##  Two Sample t-test
##
## data:  diff_alert and diff_notalert
## t = 4.2228, df = 14, p-value = 0.000852
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  2.739306 8.394027
## sample estimates:
## mean of x mean of y
##  7.400000  1.833333
```

Question 4: Advertising and Sales (15 marks)

```
x <- c(2.0, 3.5, 4.0, 5.0, 6.5, 7.0, 8.0, 9.5, 10.0, 11.0,
       12.5, 13.0, 14.5, 15.0, 16.0, 17.5, 18.0, 19.5, 20.5, 22.0)
y <- c(17.0, 23.0, 23.2, 28.0, 30.8, 33.3, 34.9, 41.7, 41.6, 46.8,
       47.7, 50.5, 53.1, 52.4, 55.0, 56.1, 55.5, 52.8, 51.9, 50.0)
```

(a) Linear Regression Model (4 marks)

```
model <- lm(y ~ x)
summary(model)
```

```
##
## Call:
## lm(formula = y ~ x)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -11.7304  -3.7440   0.7107   3.9750   5.9593
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  19.9510     2.5545   7.810 3.45e-07 ***
## x             1.8991     0.1944   9.771 1.27e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 5.119 on 18 degrees of freedom
## Multiple R-squared:  0.8414, Adjusted R-squared:  0.8326
## F-statistic: 95.47 on 1 and 18 DF,  p-value: 1.274e-08
```

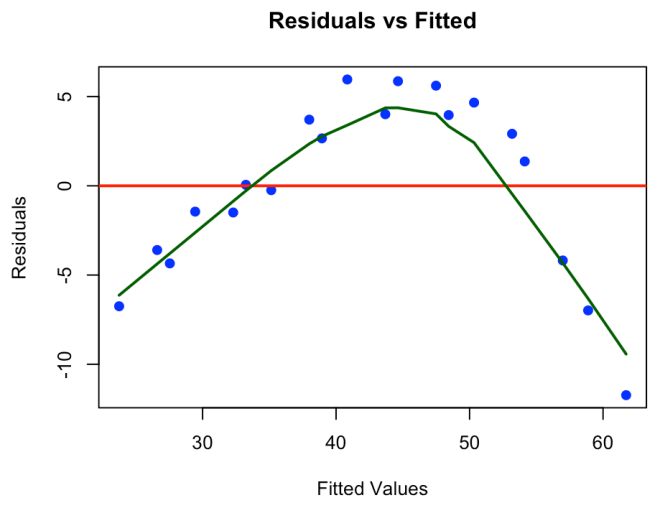
Estimated Model: $\hat{y} = 19.95 + 1.90x$

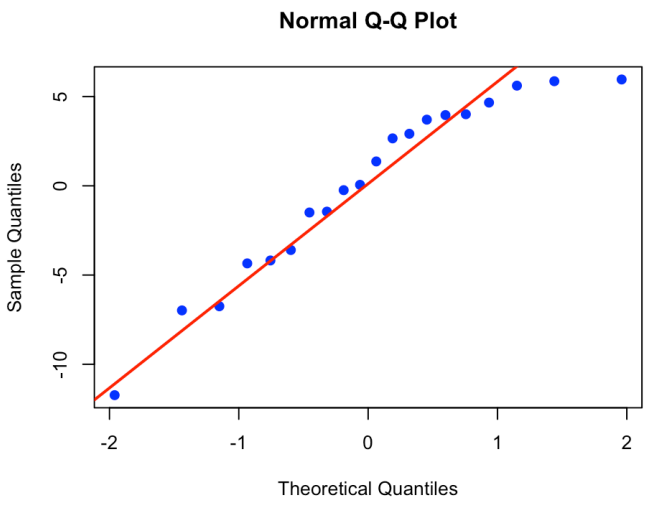
Interpretation:

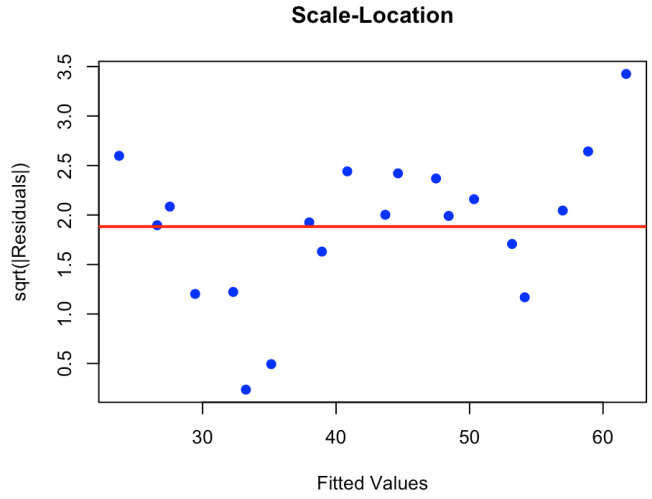
1. **Intercept ($\beta_0 = 19.95$):** When advertising budget is \$0, predicted sales are 19.95 thousand cups (19,950 cups). This is the baseline sales without advertising. Highly significant ($p < 0.001$).
2. **Slope ($\beta_1 = 1.90$):** For each additional \$1,000 spent on advertising, predicted sales increase by 1.90 thousand cups (1,900 cups). Equivalently, each dollar spent increases sales by about 1.9 cups. Highly significant ($p < 0.001$).
3. **Model fit:** $R^2 = 0.841$, meaning 84.1% of variation in sales is explained by advertising budget.

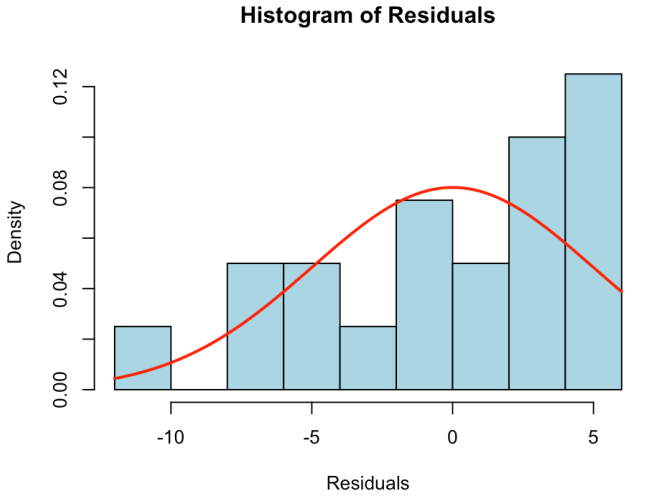
(b) Assumption Checking (8 marks)

```
residuals <- residuals(model)
fitted_values <- fitted(model)

par(mfrow = c(2, 2))

# 1. Residuals vs Fitted
plot(fitted_values, residuals, main = "Residuals vs Fitted",
     xlab = "Fitted Values", ylab = "Residuals", pch = 19, col = "blue")
abline(h = 0, col = "red", lwd = 2)
lines(lowess(fitted_values, residuals), col = "darkgreen", lwd = 2)

# 2. Q-Q Plot
qqnorm(residuals, pch = 19, col = "blue")
qqline(residuals, col = "red", lwd = 2)

# 3. Scale-Location
plot(fitted_values, sqrt(abs(residuals)), main = "Scale-Location",
     xlab = "Fitted Values", ylab = "sqrt(|Residuals|)", pch = 19, col = "blue")
abline(h = mean(sqrt(abs(residuals))), col = "red", lwd = 2)

# 4. Histogram
hist(residuals, breaks = 10, probability = TRUE,
     main = "Histogram of Residuals", xlab = "Residuals", col = "lightblue")
curve(dnorm(x, mean(residuals), sd(residuals)), add = TRUE, col = "red", lwd = 2)
```

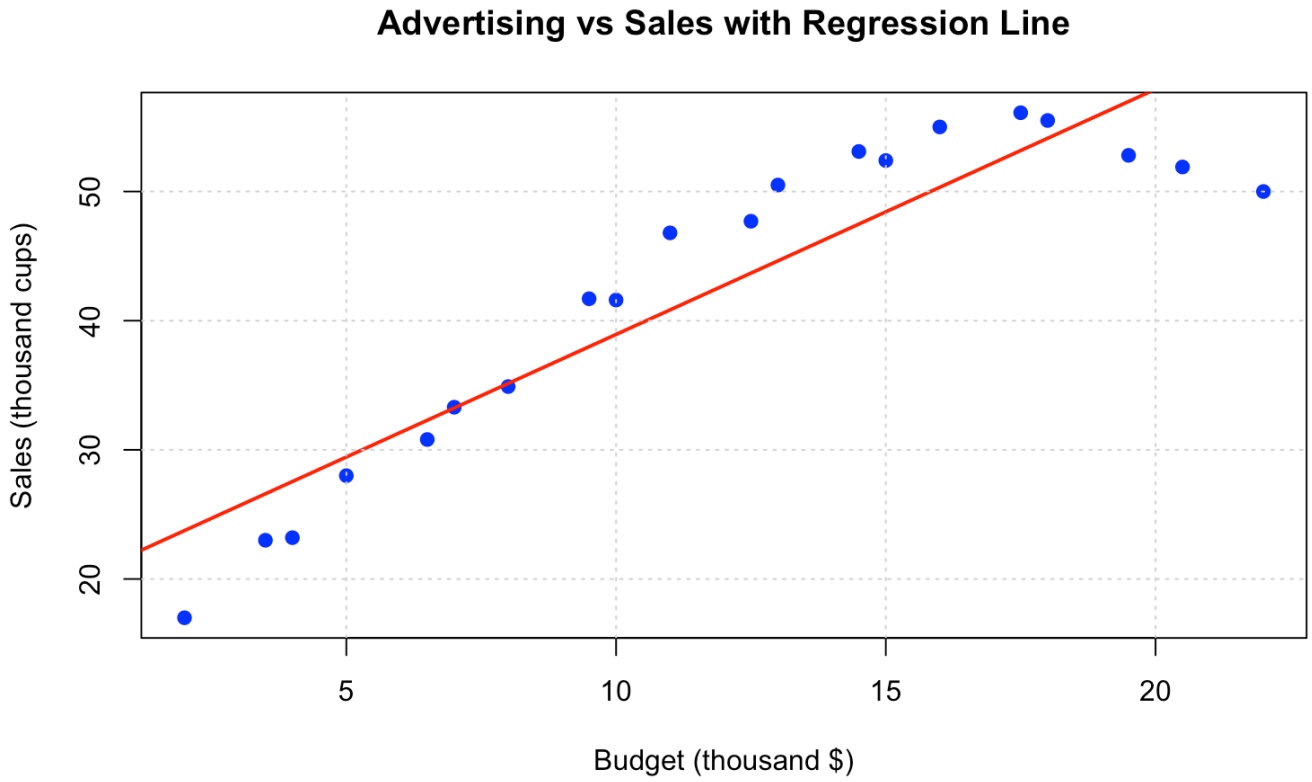









```
plot(x, y, pch = 19, col = "blue", main = "Advertising vs Sales with Regression Lin
e",
     xlab = "Budget (thousand $)", ylab = "Sales (thousand cups)")
abline(model, col = "red", lwd = 2)
grid()
```



Assessment:

1. Normality of Residuals

```
shapiro.test(residuals)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  residuals
## W = 0.92735, p-value = 0.1373
```

Shapiro-Wilk: W = 0.927, p = 0.137 > 0.05. No evidence against normality.

Q-Q Plot: Points follow the line reasonably well with minor tail deviations.

Conclusion: ✓ Normality assumption satisfied

2. Linearity

Scatterplot: Shows strong linear trend from x = 2 to 16, but flattens/declines at higher budgets (x > 16).

Residuals vs Fitted: Lowess line shows curved U-shaped pattern, suggesting non-linearity. Residuals are positive at extremes, negative in middle.

Conclusion: Linearity questionable - Relationship may be non-linear (quadratic/plateau effect)

3. Homoscedasticity

Residuals vs Fitted: Vertical spread appears roughly constant (-12 to +6).

Scale-Location: No systematic fanning pattern.

Conclusion: ✓ Homoscedasticity reasonably satisfied

Overall: Model fits well ($R^2 = 0.84$) but linearity violation suggests a more complex model (quadratic) might be better, especially at high advertising levels.

(c) Other Variables (3 marks)

Two variables that could affect weekly coffee sales:

1. Weather/Temperature

- **Rationale:** Coffee consumption increases in colder weather; seasonal variations affect demand
- **Measurement:** Average weekly temperature (°C), weather conditions (rainy/sunny)
- **Expected effect:** Negative correlation (colder → higher sales)

2. Day of Week / Holidays

- **Rationale:** Weekday vs. weekend patterns differ; holidays and special events affect foot traffic
- **Measurement:** Number of weekdays in week, holiday indicator, special events count
- **Expected effect:** Varies (more weekdays near offices might increase sales)

Additional considerations: Competitor activity, economic indicators, product launches, store operations

Impact: Including these would improve prediction accuracy and help isolate true advertising effect from confounding factors.

Summary

All four questions have been completed with detailed solutions, proper statistical methodology, R code, and interpretations. Key findings:

- **Q1:** Die A has expected 45 successes (SE = 4.47); Die B significantly differs from Die A
- **Q2:** Caffeine reduces reaction time by 5.31 ms on average ($p < 0.001$)
- **Q3:** Alert athletes show significantly greater caffeine effect (7.40 ms vs 1.83 ms, $p < 0.001$)
- **Q4:** Strong linear relationship between advertising and sales ($R^2 = 0.84$), though non-linearity evident at high budgets

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