Differential Equations

Abraham Murciano
Elad Harizy
Ezra Dweck
Leib Golovaty
Nati Yudkowsky
Jonah Lawrence

June 19, 2020

1 Chapters 1 and 2

1st-Order Differential Equations, Bernoulli's Equation, Integration Factors

A. Given a differential equation and a function $y(x)$, we are to check that y is a solution to the differential equation.

1. For the following differential equation and function $y(x)$,

$$\frac{dy}{dx} = 3y, \qquad y = 4e^{3x}$$

Since we have y, it can be differentiated to find $\frac{dy}{dx}$, then verify that it in fact is equal to $3y$.

$$\frac{dy}{dx} = 12e^{3x} = 3 \cdot 4e^{3x} = 3y$$

3. For the following differential equation and function $y(x)$,

$$\frac{d^2y}{dx^2} + 16y = 0, \qquad y = \sin(4x)$$

This time we must differentiate y twice to find $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = 4\cos(4x) \Rightarrow \frac{d^2y}{dx^2} = -16\sin(4x)$$

Now we can substitute this back into the original equation, and it quite obviously satisfies it.

$$-16\sin(4x) + 16\sin(4x) = 0$$

5. For the following differential equation and function $y(x)$,

$$\frac{dy}{dx} + 2xy = 1, \qquad y = e^{-x^2}\int_0^x e^{t^2}\,dt + ce^{-x^2}$$

we must find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= e^{-x^2}e^{x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt - 2cxe^{-x^2}\\ &= e^0 - 2x\left(e^{-x^2}\int_0^x e^{t^2}\,dt + ce^{-x^2}\right)\\ &= 1 - 2xy\end{aligned}$$

Now we can substitute this into the original differential equation, and we see that is in fact satisfies it.

$$\frac{dy}{dx} + 2xy = 1 - 2xy + 2xy = 1$$

B. **1.** We are given the following differential equation.

$$\frac{dy}{dx} = 4xe^{2x}$$

To find a solution, both sides can be integrated.

$$y = 4\int xe^{2x}dx$$

Now integration by parts can be applied, with $f(x) = x$ and $g'(x) = e^{2x}$. This implies that $f'(x) = 1$ and $g(x) = \frac{1}{2}e^{2x}$. Therefore, by the rule of integration by parts,

$$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

The next step is to calculate the integral of e^{2x}. To do so, we will substitute $u = 2x$.

$$\int e^{2x}dx = \frac{1}{2}\int e^{u}du = \frac{1}{2}\cdot\frac{e^{u}}{\ln(e)} = \frac{1}{2}e^{2x} + c$$

Therefore we have

$$y = 4\left(\frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right)\right) + c = 2xe^{2x} - e^{2x} + c$$
$$= e^{2x}(2x-1) + c$$

The graphs for y when $c = -5, 0, 1, 4,$ and 9 can be seen in Figure 1

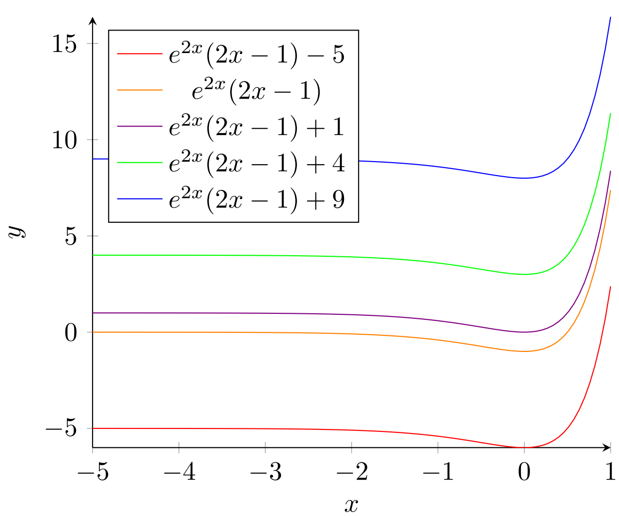

Figure 1: $y = e^{2x}(2x-1) + c$ when $c = -5, 0, 1, 4,$ or 9

2. Given the following differential equation and values for the constant of integration

$$y'(x) = x + 3, \qquad c = 0, 1, -6.$$

We can integrate it to find $y(x)$.

$$y'(x) = x + 3$$
$$\int y'(x)dx = \int (x+3)dx$$
$$y(x) = \frac{x^2}{2} + 3x + c$$

The graphs for y with the given values for c can be seen in Figure 2

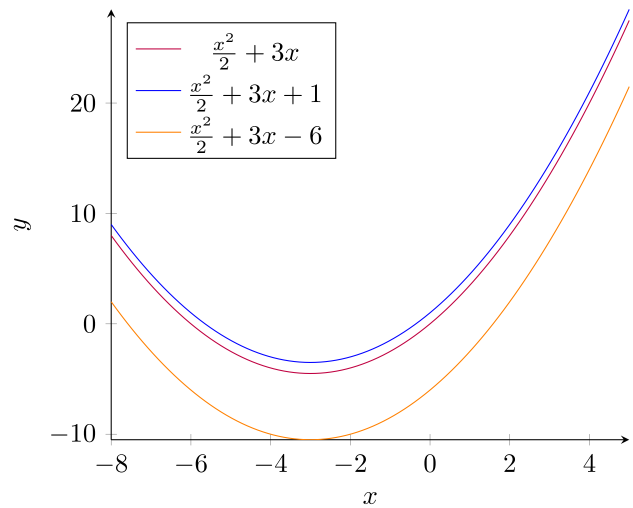

Figure 2: $y = e^{2x}(2x-1) + c$ when $c = 0, 1$, or -6

4. We are given the following differential equation and certain pairs of values for x and y.

$$y' = \frac{2}{x} + 3, \qquad y(1) = 0, y(1) = 1, y(-2) = -6$$

We now integrate to find y, and substitute the given values of x and y to find values for c.

$$y' = \frac{2}{x} + 3$$
$$\int y'(x)dx = \int \left(\frac{2}{x} + 3\right) dx$$
$$y(x) = 2\ln x + 3x + c$$
$$y(1) = 2\ln(1) + 3(1) + c = 0 \;\Rightarrow\; c = -3$$
$$y(1) = 2\ln(1) + 3(1) + c = 1 \;\Rightarrow\; c = -2$$
$$y(-2) = 2\ln(-2) + 3(-2) + c = -6 \;\Rightarrow\; c \text{ is undefined}$$

Once we have these, we can plot the graphs in Figure 3.

C. We are to solve the following differential equations, finding a general solution and a particular solution with the given initial condition. (Note that the right-hand side depends on x only.)

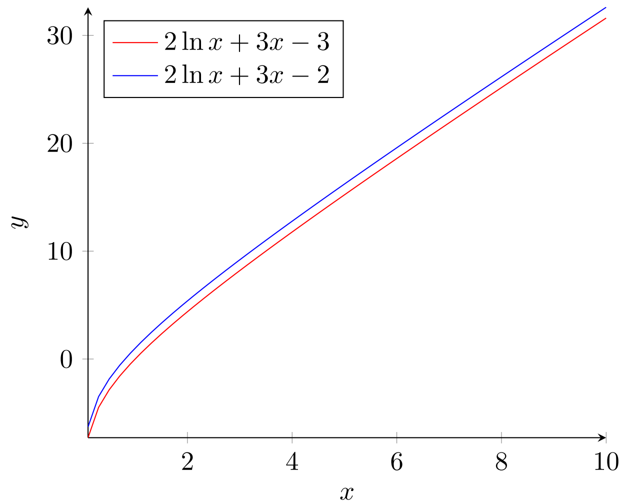

Figure 3: $y = 2\ln x + 3x + c$ when $y(1) = 0, y(1) = 1, y(-2) = -6$

1. We are given this equation and initial condition.

$$\frac{dy}{dx} = 3x^2 + 5, \qquad y(1) = 1$$

Now we can integrate $\frac{dy}{dx}$ to obtain y.

$$\int \frac{dy}{dx} dx = \int (3x^2 + 5) dx$$

The general solution is

$$y(x) = x^3 + 5x + c$$

Now we can substitute our initial condition

$$y(1) = 1^3 + 5(1) + c = 1 \;\Rightarrow\; c = -5$$

The particular solution is

$$y(x) = x^3 + 5x - 5$$

4. We are given this equation and initial condition.

$$\frac{dy}{dx} = \ln|x - 1|, \quad y(0) = 1$$

Now we can integrate $\frac{dy}{dx}$ to obtain y.

$$\int \frac{dy}{dx} dx = \int (\ln|x - 1|) dx$$

The general solution is

$$y(x) = (x - 1)\ln|x - 1| - x + c$$

Now we can substitute our initial condition

$$y(0) = (-1)\ln|-1| + c = 1 \;\Rightarrow\; c = 1$$

The particular solution is

$$y(x) = (x-1)\ln|x-1| - x + 1$$

D. We are to solve the following differential equations by separation of variables.

3. We are given this differential equation.

$$(y-3)\frac{dy}{dx} = \frac{4y}{x}$$

After rearranging and integrating we obtain the following.

$$\frac{(y-3)\frac{dy}{dx}}{y} = \frac{4}{x}$$

$$\frac{dy}{dx} - \frac{3\frac{dy}{dx}}{y} = \frac{4}{x}$$

$$\int \frac{dy}{dx} - \frac{3\frac{dy}{dx}}{y}dx = \int \frac{4}{x}dx$$

From integration by substitution,

$$\int 1 - \frac{3}{y}dy = \int \frac{4}{x}dx$$

$$y - 3\ln y = 4\ln x + c$$

$$\frac{y}{4} - \ln y^{\frac{3}{4}} = \ln x + c$$

$$\ln x + \ln y^{\frac{3}{4}} = \frac{y}{4} - c$$

$$\ln xy^{\frac{3}{4}} = \frac{y}{4} - c$$

$$xy^{\frac{3}{4}} = e^{\frac{y}{4}-c}$$

Let $\tilde{c} \equiv \frac{1}{e^c}$.

$$xy^{\frac{3}{4}} = \tilde{c}e^{\frac{y}{4}}$$

6. We are given this differential equation.

$$\cos^2(x)y'(x) = y + 3$$

Now we rearrange and integrate to obtain the following.

$$\frac{y'(x)}{y+3} = \frac{1}{\cos^2 x}$$

$$\int \frac{y'(x)}{y+3}dx = \int \frac{1}{\cos^2 x}dx$$

$$\int \frac{1}{y+3}dy = \tan x + c$$

$$\ln|y+3| = \tan x + c$$

$$y + 3 = e^{\tan x + c}$$

Let $\tilde{c} \equiv e^c$. Finally we have

$$y = \tilde{c}e^{\tan x} - 3$$

9. We are given this differential equation.

$$y'(x) = 3x^2(y+2)$$

After rearranging and integrating we obtain the following.

$$y' = 3x^2(y+2) \tag{1}$$
$$\frac{y'}{y+2} = 3x^2 \tag{2}$$
$$\int \frac{y'}{y+2}\ dx = \int 3x^2\ dx \tag{3}$$
$$\int \frac{1}{y+2}\ dy = x^3 + c \tag{4}$$
$$\ln|y+2| = x^3 + c \tag{5}$$
$$y + 2 = e^{x^3+c} \tag{6}$$

Let $\tilde{c} \equiv e^c$. Now we have

$$y = \tilde{c}e^{x^3} - 2$$

E. We must solve the following differential equations, using the method of integration-factor.

3. $y'(x) + \left[\frac{1}{x} + 1\right] y = e^x$.

$$y'(x) + \left[\frac{1}{x} + 1\right] y = e^x$$
$$\alpha(x) = \exp\left[\int \frac{1}{x} + 1dx\right] = \exp\left[\ln x + x\right] = xe^x$$

Multiply both sides by the integrating factor, $\alpha(x)$.

$$(xe^x)y' + (xe^x)y\left[\frac{1}{x} + 1\right] = (xe^x)e^x$$

Integrate both sides with respect to x.

$$(xe^x)y = \int (xe^{2x})dx$$

Integrate the right hand side using integration by parts. Let $u = x, v' = e^{2x} \ \Rightarrow\ u' = 1, v = \frac{e^{2x}}{2}$.

$$(xe^x)y = \frac{xe^{2x}}{2} - \int 1\frac{e^{2x}}{2}dx$$
$$(xe^x)y = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$
$$y = \frac{e^x}{2} - \frac{e^x}{4x} + \frac{c}{xe^x}$$

5.

7.

9.

F. 1.

2.

5.

7.

G. 2.

4.

7.

H. 2.

4.

I. 2.

J. 3.

2 Chapters 5 and 7

The law of Existence and Uniqueness, Qualitative Analysis of 1st-Order Differential Equations (Slope Fields)

2.

4.

6.

7.

8.

10.

14. **a.**

b.

e.

j.

k.

3 Chapter 4

Linear 2nd-Order ODEs, Linear nth-Order ODEs, and Linear 2nd-Order ODEs Whose Order can be Reduced

1. **a.**

c.

2. **c.**

f.

5. **a.**

c.

e.

i.

l.

6. **c.**

e.

g.

i.

4 Chapter 3

Systems of Linear 1st-Order ODEs

1. **a.**

d.

2. **c.**

3. **b.**

d.

4. **b.**

5. **b.**

f.

5 Chapter 8

Laplace Transforms

1. **a.**

   b.

   f.

6 Chapter 9

Sturn-Liouville Theory

1. **a.**

   b.

   c.