Exercises for the course: Differential equations

May 21, 2020

1 Chapters 1–2

Exercises to be submitted:

A. 1.,3.,5.	**C.** 1.,4.	**E.** 3.,5.,7.,9.	**G.** 2.,4.,7.	**I.** 2.
B. 2.,4.	**D.** 3.,6.,9.	**F.** 1.,2.,5.,7.	**H.** 2.,4.	**J.** 3.

Sections **A.–J.** are exercises in first-order differential equations, separabe differential equations, Bernoulli's equation, integration factors, differential equations with a homogeneous source and generic problems related to first-order differential equations.

A. Given a differential equation and a function $y(x)$, check that $y(x)$ is a solution to the differential equation.

1. $y'(x) = 3y,$ $\quad y(x) = 4e^{3x}.$

2. $y'(x) = 3xy,$ $\quad y(x) = -3e^{3x^2/2}.$

3. $y''(x) + 16y = 0.,$ $\quad y(x) = \sin 4x.$

4. $x^2y''(x) - 3xy' + 4y = 0,$ $\quad y(x) = x^2 \ln x.$

5. $y'(x) + 2xy = 1,$ $\quad y(x) = e^{-x^2} \int_0^x dt\, e^{t^2} + ce^{-x^2}.$

B. Solve the following differential equations, sketch the graph of the solution for different values of the unknown constant c, or different initial conditions given in each question. Explain for each given c those values of x for which the solution exists (note that the right-hand side depends on x only).

1. $y'(x) = 4xe^{2x}, \quad c = 0, 1, 4, -5, 9.$

2. $y'(x) = x + 3, \quad c = 0, 1, -6.$

3. $y'(x) = 1, \quad c = 0, -2, 8.$

4. $y' = \dfrac{2}{x} + 3, \quad y(1) = 0,\ y(1) = 1,\ y(-2) = -6.$

5. $y'(x) \sin x, \quad c = 0.$

C. Solve the following differential equations. Find a general solution and a particular solution with the given initial condition. (Note that the right-hand side depends on x only.)

1. $y'(x) = 3x^2 + 5, \quad y(1) = 1.$

2. $y'(x) = e^{2x}, \quad y(0) = 2.5.$

3. $y'(x) = (x^2 + x)\, e^x, \quad y(0) = 0.$

4. $y' = \ln|x - 1|, \quad y(0) = 1.$

5. $f'(x) = \dfrac{x}{x^4 + 4}, \quad f(0) = 2.$

D. Solve the following differential equations by separation of variables.

1. $y'(x) = \frac{xy}{1+y}$.
2. $y'(x) = \frac{y^2 + xy^2}{x^2 y - x^2}$.
3. $(y-3)y'(x) = \frac{4y}{x}$.
4. $y'(x) = xy - y$.
5. $y'(x) = \frac{2y}{x\,(y-1)}$.
6. $\cos^2 x y'(x) = y + 3$.
7. $\tan x y'(x) = 1 + y$.
8. $\cos x\,(e^{2y} - y)\,y'(x) = e^y \sin 2x$.
9. $y'(x) = 3x^2\,(y+2)$.

E. Solve the following differential equations, using the *integration-factor*method.

1. $y'(x) + 5y = e^{2x}$.
2. $y'(x) = y + x$.
3. $y'(x) + \left[\frac{1}{x} + 1\right] y = e^x$.
4. $y'(x) + \frac{y}{x} = x$.
5. $\sin x y'(x) + y \cos x = \sin x \cos x$.
6. $xy'(x) + y = x \sin x$.
7. $(1 - x^2)\,y'(x) = xy + 2$.
8. $xy'(x) + y = x \cos x, \qquad y\,(\pi) = 0$.
9. $(x-1)\,y'(x) - y = (x-1)^4, \qquad y(4) = 9$.

F. Solve the following differential equations, (note that the right-hand side is a homogeneous function).

1. $y'(x) = 1 + \dfrac{y}{x}$.
2. $y'(x) = 1 - \dfrac{y}{x} + \dfrac{y^2}{x^2}$.
3. $xy'(x) = y - e^{y/x}$.
4. $2x^2y'(x) = x^2 + y^2$.
5. $x\,(x+y)\,y'(x) - y^2 = xy$.
6. $xy'(x) = y + \sqrt{x^2 - y^2}$.
7. $y'(x) = \dfrac{x^2 + xy + y^2}{x^2}$.

G. Solve the following different examples of Bernoulli's equation.

1. $y'(x) + \dfrac{y}{x} = xy^2$.
2. $y'(x) + xy = \dfrac{2x}{y}$.
3. $y'(x) + \dfrac{y}{x} = 2x^2y^3$.
4. $3y - 2y'(x) = y^4e^{3x}$.
5. $y'(x) + 2y = y^3\,(x-1)$.
6. $x^2y - \dfrac{x^3}{2}y' = y^3\cos x$.
7. $\dfrac{y'(x)}{\sqrt{y}} = e^{x^3} - \dfrac{4\sqrt{y}}{x}$.

H. Each example in the list below is a differential equation together with an intial condition, or in other words each example is an initial value problem. In each example there is a list of different regions provided. Check if the solution exists in each region.

	Differential Equationn.	Initial Condition	Regions
1.	$xy'(x) + y = 0,$	$y(1) = 1,$	$\left(\frac{1}{2}, 12\right)$, $(-3,-1)$,$(0,\infty)$,$(-\infty,0)$.
2.	$y'(x) = -\frac{x}{y},$	$y(3) = 4,$	$(-5,0)$, $(-5,5)$,$(0,10)$,$(0,5)$.
3.	$y'(x) = 25 + y^2,$	$y(0) = 0,$	$(-5\pi, 5\pi)$, $\left(-\frac{\pi}{10}, \frac{\pi}{10}\right)$, $(-\pi,\pi)$,$(-5\pi,0)$.
4.	$2y'(x) = y^3 \cos x,$	$y(0) = 1,$	$(-\infty,\infty)$, $(0,\infty)$,$(-\infty,0)$,$(-10,0)$.
5.	$y'(x) = 2xy,$	$y(0) = 0,$	$(0,\infty)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(-1,1)$.

I. In the following questions a differential equation is given together with a bounded function $y(x)$. Check if the given function is solution to the differential equation and prove your claim.

1.

$$xy' - 2y = 0.$$

$$y\,(x) = \begin{cases} -x^2 & x < 0, \\ x^2 & x \geq 0, \end{cases}$$

where the function y is defined in the region $\mathbb{R}$.

2.

$$y' = -\frac{x}{y}.$$

$$y\,(x) = \begin{cases} \sqrt{25 - x^2} & -5 < x < 0, \\ -\sqrt{25 - x^2} & 0 \leq x < 5, \end{cases}$$

where the function y is defined in the region $(-5, 5)$.

J. Various generic examples of differential equations.

1. Given the differential equation

$$y' = e^{-x^2},$$

a. Explain why the solution is an increasing function in the entire open region $\mathbb{R}$,

b. Calculate the limits

$$\lim_{x\to+\infty} y', \qquad \lim_{x\to-\infty} y'.$$

c. What can be concluded from part **b.** about the function as $x \to \pm\infty$.

2. Given that $y = \sin x$ is a solution to the differential equation

$$y' = \sqrt{1-y^2},$$

find the largest region in which a solution is defined. [Hint: It is not the region $(-\infty, \infty)$.]

3. Prove that the family of functions

$$x^3 + y^3 = 3cxy,$$

which are called the *folia of Descartes*, is a family of solutions to the differential equation

$$y' = \frac{cy - x^2}{y^2 - cx}.$$

4. Given the differential equation

$$y' = y\,(a - by)\,,$$

a. Separate different cases for the values a, b, and find the largest region in which the solution is increasing, and the largest region in which the solution is decreasing.

b. For which values of a, b do we obtain the trivial solution $y(x) = 0$.

c. Solve the differential equation itself.

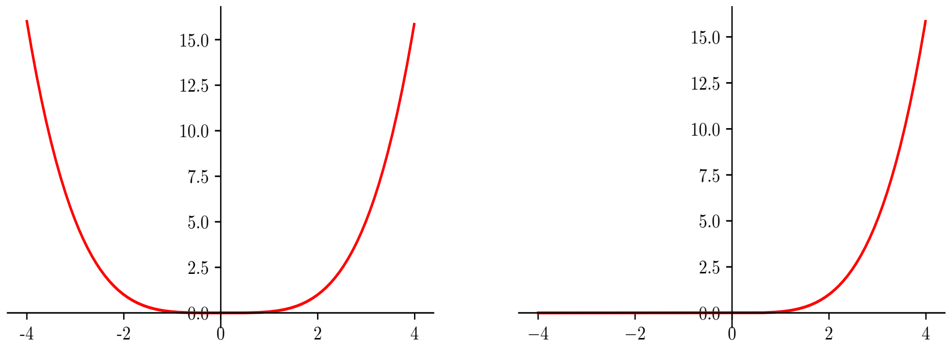
5. The following functions

$$y(x) = \frac{x^4}{16}$$

and

$$y(x) = \begin{cases} 0, & x < 0, \\ \frac{x^4}{16}, x \geq 0, & \end{cases}$$

defined in the region $(-\infty, \infty)$ are given. These functions have the same **range** but different **domains** (see the graphs below).



Show that the two functions above are solutions to the initial value problem

$$y'(x) = x\sqrt{y}, \qquad y(2) = 1, \qquad \text{in the region } (-\infty, \infty)\,.$$

6. Given the differential equation and initial condition

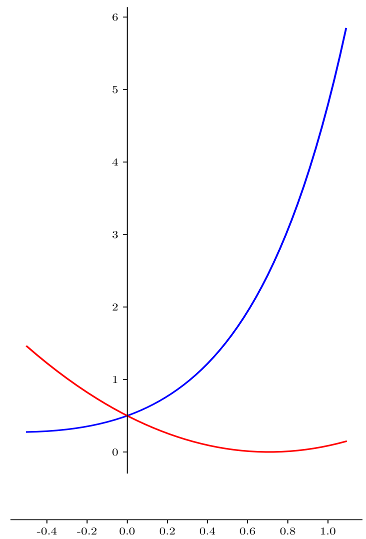
$$y'(x) + 2y = 3x - 6, \qquad y(a) = 0,$$

determine for which values of the parameter a, the curve of the solution has the x axis as its tangent at the point $(a, 0)$.

7. Consider the following differential equation and initial condition:

$$y' = x - 2y, \qquad y(0) = \frac{1}{2}.$$

Wihout solving the equation, determine which curve from the sketch below is the only possible to this initial-value problem. (**Curve needs work to make it look like the hebrew notes**).



Comment: Note that the initial condition starts at the point $\frac{x}{2} < y$, and the function $f(x, y) = x - 2y$.

2 Chapters 5 and 7

Exercises to be submitted:
2,4,6,7,8,10,14: a,b,e,j,k.

The law of existence and uniquess, qualitative analysis of 1st-order differential equations (slope fields)

1. **a**. Consider a differential equation of the form

$$\dot{y} = f(t, y)$$

such that f satisfies the conditions of the law of existence and uniqueness in the entire (t, y) plane. Let the function

$$y_1(t) := 3$$

for all real values of t is a solution to the given differential equation. What can be concluded about the particular solution to the differential equation satisfying the initial condition $y(0) = 1$. (use the law of existence and uniqueness).

b. Similar to the previous exercise, let

$$y_1(t) := t + 2, \qquad y_2(t) := -t^2,$$

for all real values of t be solutions of the given differential equation. What can be concluded about the particular solution to the differential equation satisfying the initial condition $y(0) = 1$. (use the law of existence and uniqueness).

2. Given the differential equation

$$\dot{y} = y\,(y-2)\,(y-3)\,,$$

What can be concluded from the law of existence and uniqueness about the solution that satisfies the initial conditions

$$y(0) = 1 \qquad \text{and} \qquad y(0) = 4?$$

3. Given the differential equation

$$\dot{y} = 2\sqrt{|y|},$$

a. Show that the function defined by

$$y(t) := 0$$

for all real values of t is an equal-weighted solution,

b. find all the solutions to this equation. (Hint: Concentrate on the cases where $y > 0$ and $y < 0$ separately. Split the solutions in the following way:

(i) "$y(t) = \ldots$ when $t > d$" ,

(ii) "$y(t) = \ldots$ when $c \le t \le d$",

(iii) "$y(t) = \ldots$ when $t < c$."

Another way to split solutions defined more compactly is

(i) "$y(t) = \ldots$ when $t \ge c$" ,

(ii) "$y(t) = \ldots$ when $t < c$".

There is another type of solution. What is it?)

c. Given the initial condition $y(1) = 0$ check how many solutions exist. Is there not a contradiction with the law of existence and uniqueness?

4. Consider the following differential equation:

$$\dot{y} = \frac{y}{t^2}$$

a. Show that the function $y_1(t) = 0$ (where $t \neq 0$) is a solution.

b. Show that there are an infinite number of other solutions equivalent to $y_1(t) = 0$ where $t < 0$ but not for $t > 0$, but thet do not tend to zero when $t > 0$. (Hint: Define the function with the form $y(t) = \ldots$ when $t < 0$ and $y(t) = \ldots$ when $t > 0$, or in other words piece-wise.)

c. Why is this example not a contradiction of the uniquness law?

5. Consider the differential equation

$$t\dot{y}(t) = 2y.$$

a. Show that the function $y_1(t) = 0$ is a solution.

b. Show that there are infinitely many other solutions that satisfy the differential equation equivalent to $y_1(t)$ for $t < 0$, that do not tend to zero when $t > 0$.

c. Why is this example not a contradiction of the uniquness law?

6. In the following examples a differential equation is given together with an initial condition:

(i) $\dot{y} = \dfrac{1}{(y+1)(t-2)}, \qquad y(0) = 0.$

(ii) $\dot{y} = \dfrac{1}{(y+2)^2} \quad \text{for} \quad t < 2, \qquad y(0) = 1.$

(iii) $\dot{y} = \dfrac{t}{y}, \qquad y(-1) = 3.$

For each example:

a. Solve the differential equation.

b. Find the domains of the function.

c. Sketch the curve of the solution.

7. Solve the initial value problem

$$yy'(x) = x, \qquad y(0) = 0.$$

Does the solution contradict the law of uniqueness?

8. Solve the initial value problem

$$y^{2/3}y'(x) = x, \qquad y(0) = 0.$$

Is the solution unique?

9. **a**. Formulate the law of existence and uniqueness for a (1st-order) initial value problem.

b. In which set of points (x, y) does the equation

$$xy' = 2y = x^3$$

violate the conditions of the existence and uniqueness law?

c. Find all the solutions to this equation (or prove that there are no solutions), satisfying the initial condition $y(0) = 1$.

d. Find all the solutions to this equation (or prove that there are no solutions), satisfying the initial condition $y(1) = 0$.

10. Prove that the initial value problem

$$x^2y' = y^2, \qquad y(0) = 0,$$

has infinitely many solutions of the form

$$y(x) = \frac{x}{1 + cx}$$

What is the solution when instead $y(2) = 0$? For the same problem, does there exist a unique solution (Hint: What is the equilibrium solution and how is it related to the initial condition $y(2) = 0$?

What is the solution when instead the initial condition is $y(1) = 8$? Does there exist a unique solution to the equation with this initial condition? Is there any contradiction with law of uniqueneuss?

11. Find all solutions to the initial value problem

$$x^2y' = y^2, \qquad y(1) = 0.$$

Is the solution unique? If yes, explain why.

12. For the following initial value problem

$$y' = y^{1/3}, \qquad y(1) = 0,$$

There are two solutions: $\phi_1(x) = 0$ and $\phi_2(x) = \left(\frac{2x}{3}\right)^{3/2}$. Is this a contradiction of the law of uniqueness?

13. Find all solutions to the initial value problem

$$y' = \frac{x\sqrt{y}}{e^{\sqrt{y}}}, \qquad y(0) = 0.$$

How may solutions did you find? If you found more than one, is this not a contradiction of the law of uniqueness? (Hint: Note that while solving the equation, when you divided by the factor $\sqrt{y}$ you assumed that this factor was not equal to zero. Check what happens when $\sqrt{y} = 0$.)

14. For each one of the following differential equation, investigate the behavior of the family of solutions, (namely different solutions for different values of $y(0) = y_0$, where $-\infty < y_0 < \infty$), according to the following steps

- **(i)** Sketch the graph of $f(y)$.
- **(ii)** Find the equilibriums (turning points).
- **(iii)** Find the slope fields and sketch them.
- **(iv)** Sketch the family of different solutions.
- **(v)** For each solution determine if it is stable, stable on one side or unstable.

- **a)** $\dot{y} = ay + by^2, \qquad a > 0,\, b > 0.$
- **b)** $\dot{y} = y\,(y-1)\,(y-2)\,.$
- **c)** $\dot{y} = e^y - 1.$
- **d)** $\dot{y} = e^{-y} - 1.$
- **e)** $\dot{y} = -k\,(y-1)^2\,, \qquad k > 0.$
- **f)** $\dot{y} = y^2\,(y^2-1)\,.$
- **g)** $\dot{y} = y\,(1-y^2)\,.$
- **h)** $\dot{y} = y^2\,(4-y^2)\,.$
- **i)** $\dot{y} = y^2\,(1-y)^2\,.$
- **j)** $\dot{y} = \sin y.$
- **k)** $\dot{y} = \dfrac{1}{y^2-1}.$

Comment: In order to check the slope at a point, etc., calculate y'' directly.

3 Chapter 4

Exercises to be submitted:
1. a), c). 2. c), f). 5. a),c),e), i), l), 6. c),e),g),i).

Linear 2nd-order ODEs, linear nth order ODEs, and linear 2nd-order ODEs whose order can be reduced

1. Find the general solution of the following differential equations:

a) $xy'' = (1 + 2x)\, y'$,

b) $xy'' = x^3 + y'$,

c) $y'' = 9x$,

d) $xy'' - 2y' = x^4$.

2. Solve the following differential equations: Find a general solution, and a particular solution satisfying the initial condition:

a) $y'' + 2y' - 8y = 0, \qquad y(0) = 3, \quad y'(0) = -12.$

b) $y'' - 6y' + 9y = 0, \qquad y(-1) = 3, \quad y'(-1) = 9.$

c) $y'' - 4y' + 4y = 0, \qquad y(1) = 1, \quad y'(1) = 1.$

d) $y'' + 16y' = 0, \qquad y(\pi) = 3, \quad y'(\pi) = -1.$

e) $y'' + y' + y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$

f) $3y'' + 2y' - 5y = 0, \qquad y(0) = 11, \quad y'(0) = -5.$

3. Consider the following differential equation and initial condition:

$$y'' + 3y' + 2y = 0, \qquad y(0) = a, \quad y'(0) = b.$$

a) Prove that the general solution is

$$y(x) = (2a + b)\, e^{-x} - (a + b)\, e^{-2x}.$$

b) Compute the limit.

$$\lim_{x \to \infty} y(x).$$

4. Consider the following differential equation and initial condition:

$$y'' + 3y' - 4y = 0, \qquad y(0) = a, \quad y'(0) = b.$$

Find conditions for the value a and b such that the limit of the solution when $x \to \infty$ will be equal to zero.

5. Find the general solution for each of the following differential equations. Use the method of equating coefficients:

a) $y'' - 2y' - 8y = 32x$.

b) $16y'' - 8y' - 15y = 75x$.

c) $y'' - 4y' + 4y = e^x$.

d) $y'' + 2y' + y = e^{-x}$.

e) $y'' - 4y' + 3y = 12\sin 3x$.

f) $y'' - 4y' + 13y = e^{2x}$.

g) $y'' - 4y' + 13y = x + 2$.

h) $y'' - 4y' + 13y = \sin 3x$.

i) $y''' + y'' - 17y' + 15y = 3x$.

j) $y''' - 3y'' + 3y' - y = e^x$.

k) $y''' + y'' - 17y' - 65y = e^{5x} - xe^{-3x}\sin 2x$.

l) $y''' + 6y'' + 11y' + 6y = 20e^{3x} - xe^{-x}$.

6. Find the general solution of the following differential equations. Use the method of variation of parameters.

a) $y'' - 7y' + 10y = e^{3x}$.

b) $y' + 4y = 4\sin 2x$.

c) $y'' - 4y = e\cosh x$.

d) $y'' - 2y' + y = \dfrac{e^x}{x}, \quad x > 0$.

e) $y'' + 4y = \dfrac{1}{\cos 2x}, \qquad -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$.

f) $y'' - 9y = \dfrac{1}{1 + e^{3x}}$.

g) $y'' - 4y' + 4y = \dfrac{e^{2x}}{x^2}, \qquad x > 0.$

h) $y'' + y = \tan x$ in the region $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ with the initial condition $y(0) = 2$, $y'(0) = 1$.

i) $y'' - 2y' + y = e^x \ln x$ with the initial condition $y(1) = 0$, $y'(1) = 0$.

4 Chapter 3

Exercises to be submitted:
1: a), d). 2: c). 3: b),d). 4: b). 5: b),f).

Systems of linear, 1st-order ODEs

1. Given the matrix $A(t)$, calculate $\dot{A}(t)$ and $\int dt\, A(t)$ in the following examples:

a) $A(t) = \begin{pmatrix} \cos 4t \\ 2\sin 4t \end{pmatrix}.$

b) $A(t) = \begin{pmatrix} e^{-t} & te^{-t} \\ t & t^2 \end{pmatrix}.$

c) $A(t) = \begin{pmatrix} \ln t \\ \\ t \ln t \\ \\ \dfrac{\ln t}{t} \end{pmatrix}.$

d) $A(t) = \begin{pmatrix} e^{2t} & e^{3t} & e^{-t} \\ 2e^{2t} & e^{3t} & e^{-t} \\ 4e^{2t} & -e^{3t} & e^{-t} \end{pmatrix}.$

2. Write the following systems of ODEs in matrix form:

a)

$$\begin{aligned} x_1' &= x_1 + 2x_2, \\ x_2' &= -x_1 + 2x_2. \end{aligned}$$

b)

$$\begin{aligned} x_1' &= x_1 + 3x_2 - e^t, \\ x_2' &= -x_1 + 2x_2 + 2e^t. \end{aligned}$$

c)

$$\begin{aligned} x_1' &= x_1 + 2x_2 - x_3, \\ x_2' &= -x_1 + x_2 + 2x_3, \\ x_3' &= 2x_1 - x_2 - 2x_3. \end{aligned}$$

3. The following are examples of a systems of linear, 1st-order ODEs in the form $X'(t) = AX(t)$. where A is a square matrix with constant components/ Shows that in each example below, the vector given is a solution to the differential equation:

a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad X_1(t) = \begin{pmatrix} -e^t \\ e^t \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}.$

b) $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad X_1(t) = \begin{pmatrix} e^{2t}\sin t \\ -e^{2t}\cos t \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} e^{2t}\cos t \\ -e^{2t}\sin t \end{pmatrix}.$

c) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad X_1(t) = \begin{pmatrix} e^t \\ -2e^t \\ e^t \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix}, \quad X_3(t) = \begin{pmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{pmatrix}.$

d) $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_1(t) = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} te^t \\ e^t \\ 0 \end{pmatrix}, \quad X_3(t) = \begin{pmatrix} \left(2t + \frac{1}{2}t^2\right)e^t \\ te^t \\ e^t \end{pmatrix}.$

4. Solve the system of equations $X'(t) = AX(t)$, where A is a square matrix for the following different cases:

a) $A = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & -1 & -2 \end{pmatrix}.$

b) $A = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}.$

5. Solve the system of equations $X'(t) = AX(t)$, where A is a square matrix for the following different cases:

a) $A = \begin{pmatrix} 2/3 & 1/3 & -2/3 \\ 1/3 & 5/3 & -1/3 \\ 1/3 & -1/3 & 5/3 \end{pmatrix}.$

b) $A = \begin{pmatrix} 1 & 0 & 2 \\ 1/2 & 1/2 & 3/2 \\ -1/2 & 1/2 & 3/2 \end{pmatrix}.$

c) $A = \begin{pmatrix} 12 & 4 \\ -25 & 8 \end{pmatrix}.$

6. Solve the system of equations $X'(t) = AX(t) + F(t)$, where A is a square matrix and $F(t)$ a vector for the following different cases:

a) $A = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}, \quad F(t) = \begin{pmatrix} 14 \\ 35 \end{pmatrix}.$

b) $A = \begin{pmatrix} 6 & 4 \\ 1 & 3 \end{pmatrix}, \quad F(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}.$

c) $A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \quad F(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$

d) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad F(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}.$

e) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad F(t) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$

f) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad F(t) = \begin{pmatrix} -1 + \tan^2 t \\ \tan t \end{pmatrix}.$

5 Chapter 8

Exercises to be submitted:
1: a), c), e).

Laplace transforms

1. Solve the following differential equations (in which the right-hand side is a step function).

a) $y'' + y = \theta\,(t-3)\,, \qquad y(0) = 0, \quad y'(0) = 1.$

b) $z'' + z = \theta\,(t-2) - \theta\,(t-4)\,, \qquad z(0) = 1, \quad z'(0) = 0.$

c) $w'' + w = 3\theta\,(t-2\pi)\sin t, \qquad w(0) = 1, \quad w'(0) = -2.$

d) $y'' + y' + \frac{5}{4}y = \theta\,(t)\sin t - \theta\,(t-\pi)\sin\,(t-\pi)\,, \qquad y(0) = 0, \quad y'(0) = 0.$

e) $y'' + y' + \frac{5}{4}y = \theta(t) - \theta(t-\pi), \qquad y(0) = 0, \quad y'(0) = 0.$

6 Chapter 9

Exercises to be submitted:
1.

Sturn –Liouville theory

1. **a)** Solve the following differential equations inside the region $[0, 6]$:

$$y''(x) + ay(x) = 0, \qquad y(0) = 0, \quad y(6) = 0.$$

Find independent values and functions for the values $a = 1$, $a = -2$ and $a = 0$.

b) Solve the following differential equations inside the region $[0, 2]$:

$$y''(x) + by(x) = 0, \qquad y'(0) = 0, \quad y'(2) = 0.$$

Find independent values and functions for the values $b = 4$, $b = -3$ and $b = 0$. Show that for the value $b = 4$ the independent functions are orthogonal according to the inner product

$$\langle f(x), g(x)\rangle = \int_a^b dx\, f(x)\bar{g}(x).$$

c) Solve the following equation in the region $[0, 8]$:

$$y''(x) + cy(x) = 0, \qquad y(0) = 0, \quad y(8) + y'(8) = 0.$$

Find values of the independent functions for the values $c = 9$, $c = -1$, $c = 0$.