

Linear Algebra 2

Homework 1 – Complex Numbers

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2. We are tasked with solving the following equations for z .

$$(b) \quad z^2 = -10 + 20i$$

$$(a + bi)^2 = -10 + 20i$$

$$a^2 + 2abi - b^2 = -10 + 20i$$

$$a^2 - b^2 = -10$$

$$2ab = 20$$

$$b = \frac{10}{a}$$

$$a^2 - \left(\frac{10}{a}\right)^2 = -10$$

$$a^2 - \frac{100}{a^2} = -10$$

$$a^4 + 10a^2 - 100 = 0$$

$$\text{Let } t = a^2$$

$$t^2 + 10t - 100 = 0$$

$$t = -5 + 5\sqrt{5}$$

$$a = \pm\sqrt{-5 + 5\sqrt{5}} = \pm\sqrt{5}\sqrt{\sqrt{5} - 1}$$

$$b = \pm\frac{10}{\sqrt{5}\sqrt{\sqrt{5} - 1}} = \pm\frac{2\sqrt{5}}{\sqrt{\sqrt{5} - 1}}$$

$$z_1 = \sqrt{5}\sqrt{\sqrt{5} - 1} + \frac{2\sqrt{5}}{\sqrt{\sqrt{5} - 1}}i$$

$$z_2 = -\sqrt{5}\sqrt{\sqrt{5} - 1} - \frac{2\sqrt{5}}{\sqrt{\sqrt{5} - 1}}i$$

$$\begin{aligned}
\text{(d)} \quad & z^2 + |z|^2 = 2 - 4i \\
& (a + bi)^2 + |a + bi|^2 = 2 - 4i \\
& (a + bi)^2 + \sqrt{a^2 + b^2}^2 = 2 - 4i \\
& a^2 - b^2 + 2abi + a^2 + b^2 = 2 - 4i \\
& 2a^2 + 2abi = 2 - 4i \\
& 2a^2 = 2 \\
& 2ab = -4 \\
& a^2 = 1 \\
& b = -\frac{2}{a} \\
& a = \pm 1 \\
& b = \mp 2 \\
& z_1 = 1 - 2i \\
& z_2 = -1 + 2i
\end{aligned}$$

3. For the following properties of the complex conjugates and absolute values we must prove each property, using either Cartesian or polar form.

$$\begin{aligned}
\text{(a)} \quad & \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \\
& \text{Let } z_1 = a_1 + b_1i \\
& \text{Let } z_2 = a_2 + b_2i \\
& \overline{z_1 + z_2} = \overline{a_1 + b_1i + a_2 + b_2i} \\
& = \overline{(a_1 + a_2) + (b_1 + b_2)i} \\
& = \overline{(a_1 + a_2) + (b_1 + b_2)i} \\
& = \overline{a_1 + b_1i + a_2 + b_2i} \\
& = \overline{z_1 + z_2}
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad & \overline{\overline{z}} = z \\
& \text{Let } z = a + bi \\
& \overline{\overline{z}} = \overline{\overline{a + bi}} \\
& = \overline{a - bi} \\
& = a + bi \\
& = z
\end{aligned}$$

4. We are asked to write each of the following complex numbers in their polar representation:

$$\begin{aligned}
\text{(b)} \quad 1 - i &= r(\cos \theta + i \sin \theta) \\
r &= |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
\theta &= -\arctan(1) = -\frac{\pi}{4} \\
1 - i &= \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \\
&= \sqrt{2} \left(\cos \left(\frac{\pi}{4} \right) - i \sin \left(\frac{\pi}{4} \right) \right)
\end{aligned}$$

$$\text{(d)} \quad 4 - 4i = 4(1 - i) = 4\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

5. (c) We are to express the following number in cartesian form.

$$\left(\frac{1 + 2i}{-2 + i} \right)^{2048} = \frac{1 + 2^{2048}}{2^{2048} + 1} = 1$$

6. (b) We seek to solve the following equation using the polar representation of complex numbers.

$$\begin{aligned}
z^3 &= -2 + 2i = |-2 + 2i|e^{\frac{3\pi i}{4}} = 2\sqrt{2}e^{\frac{3\pi i}{4}} \\
z_1 &= \left(2\sqrt{2}e^{\frac{3\pi i}{4}} \right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{\pi i}{4}} = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 1 + i \\
z_2 &= \left(2\sqrt{2}e^{(\frac{3\pi}{4} + 2\pi)i} \right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{11\pi i}{12}} \\
&= \sqrt{2} \left(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) i \\
z_3 &= \left(2\sqrt{2}e^{(\frac{3\pi}{4} + 4\pi)i} \right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{19\pi i}{12}} \\
&= \sqrt{2} \left(\cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12} \right) = \frac{\sqrt{3}}{2} - \frac{1}{2} - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) i
\end{aligned}$$

9. (a) We are to prove or disprove whether or not the z is purely imaginary.

$$\begin{aligned}
z &= \left(\frac{1 + i}{1 - i} \right)^{19} = \left(\frac{\sqrt{2}e^{\frac{i\pi}{4}}}{\sqrt{2}e^{\frac{7i\pi}{4}}} \right)^{19} = \frac{e^{\frac{19}{4}i\pi}}{e^{\frac{133}{4}i\pi}} = e^{-\frac{57}{2}i\pi} \\
&= \cos \frac{57\pi}{2} - i \sin \frac{57\pi}{2} = i
\end{aligned}$$

- (b) We are asked to prove or disprove that the following equation has

five different solutions.

$$\begin{aligned}
 z^5 &= 2 - 2i \\
 &= 2\sqrt{2}e^{\frac{7\pi i}{4}} = 2\sqrt{2}e^{\frac{15\pi i}{4}} = 2\sqrt{2}e^{\frac{23\pi i}{4}} = 2\sqrt{2}e^{\frac{31\pi i}{4}} = 2\sqrt{2}e^{\frac{39\pi i}{4}} \\
 z_1 &= \sqrt[10]{8}e^{\frac{7\pi i}{20}} \\
 z_2 &= \sqrt[10]{8}e^{\frac{4\pi i}{4}} \\
 z_3 &= \sqrt[10]{8}e^{\frac{23\pi i}{20}} \\
 z_4 &= \sqrt[10]{8}e^{\frac{31\pi i}{20}} \\
 z_5 &= \sqrt[10]{8}e^{\frac{39\pi i}{20}}
 \end{aligned}$$

Here, z_1 through z_5 are not equal to each other, since the coefficients of i in the exponents are between 0 and 2π , so none of them are more than a full rotation around the circle they all lie on.

10. (a) Given that $u = \sqrt{3}(1 + i)$ we seek u^4 .

$$\begin{aligned}
 u^4 &= \sqrt{3}^4(1 + i)^4 \\
 &= 9 \left(\binom{0}{4}i^0 + \binom{1}{4}i + \binom{2}{4}i^2 + \binom{3}{4}i^3 + \binom{4}{4}i^4 \right) \\
 &= 9 + 36i - 54 - 36i + 9 \\
 &= -36
 \end{aligned}$$

- (b) We are to solve the equation $z^4 = -36$ and calculate the sum of all of its solutions.