Analysis of Algorithms

Homework 3

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December 1, 2020

1 The Kruskal and Prim Algorithms

We are given the following weighted, undirected graph $G = (V, E)$ where V is the set of vertices, E is a set of edges, and $w_{u,v}$ is the weight from vertices u to v.

$$V = \{a, b, c, d, e, f, g, h, i\}$$
$$E = \{\{a, b\}, \{a, h\}, \{a, i\}, \{b, c\}, \{b, f\}, \{c, d\}, \{d, e\},$$
$$\{d, g\}, \{d, h\}, \{e, f\}, \{f, g\}, \{h, i\}\}$$
$$w_{a,b} = 4, w_{a,h} = 10, w_{a,i} = 6, w_{b,c} = 7, w_{b,f} = 12, w_{c,d} = 8, w_{d,e} = 3,$$
$$w_{d,g} = 5, w_{d,h} = 2, w_{e,f} = 11, w_{f,g} = 1, w_{h,i} = 9)$$

1.1 Kruskal

We are to run the Kruskal algorithm on this graph, showing intermediate stages. We begin with the set of edges to return F, initialised to ϕ. We also begin with a disjoint set D, starting with each vertex in its own set; i.e $D = \{\{v\} : v \in V\}$.

The set the algorithm returns as a minimum spanning forrest is

$$F = \{\{a, b\}, \{a, i\}, \{b, c\}, \{c, d\}, \{d, e\}, \{d, g\}, \{d, h\}, \{f, g\}\}$$

as shown by the intermediate steps in table 1.

Prim

Using the same example we are to run Prim's algorithm which returns the same thing. We start with an set of vertices V_0 initialised to an arbitrary vertex (here we choose a), which we insert vertices into one at a time. We also use a set E_0 which is the subset of edges to return, initialised to ϕ. By the end,

$$E_0 = \{\{a, b\}, \{a, i\}, \{b, c\}, \{c, d\}, \{d, e\}, \{d, g\}, \{d, h\}, \{f, g\}\}$$

Step	Variable	Value
1	F	$\{\{f,g\}\}$
	D	$\{\{a\},\{b\},\{c\},\{d\},\{e\},\{f,g\},\{h\},\{i\}\}$
2	F	$\{\{d,h\},\{f,g\}\}$
	D	$\{\{a\},\{b\},\{c\},\{d,h\},\{e\},\{f,g\},\{i\}\}$
3	F	$\{\{d,e\},\{d,h\},\{f,g\}\}$
	D	$\{\{a\},\{b\},\{c\},\{d,e,h\},\{f,g\},\{i\}\}$
4	F	$\{\{a,b\},\{d,e\},\{d,h\},\{f,g\}\}$
	D	$\{\{a,b\},\{c\},\{d,e,h\},\{f,g\},\{i\}\}$
5	F	$\{\{a,b\},\{d,e\},\{d,g\},\{d,h\},\{f,g\}\}$
	D	$\{\{a,b\},\{c\},\{d,e,f,g,h\},\{i\}\}$
6	F	$\{\{a,b\},\{a,i\},\{d,e\},\{d,g\},\{d,h\},\{f,g\}\}$
	D	$\{\{a,b,i\},\{c\},\{d,e,f,g,h\}\}$
7	F	$\{\{a,b\},\{a,i\},\{b,c\},\{d,e\},\{d,g\},\{d,h\},\{f,g\}\}$
	D	$\{\{a,b,c,i\},\{d,e,f,g,h\}\}$
8	F	$\{\{a,b\},\{a,i\},\{b,c\},\{c,d\},\{d,e\},\{d,g\},\{d,h\},\{f,g\}\}$
	D	$\{\{a,b,c,d,e,f,g,h,i\}\}$

Table 1: The steps taken during the execution of the Kruskal algorithm

Step	Variable	Value
1	V_0	$\{a\}$
	E_0	ϕ
2	V_0	$\{a,b\}$
	E_0	$\{\{a,b\}\}$
3	V_0	$\{a,b,i\}$
	E_0	$\{\{a,b\},\{a,i\}\}$
4	V_0	$\{a,b,c,i\}$
	E_0	$\{\{a,b\},\{a,i\}\},\{b,c\}$
5	V_0	$\{a,b,c,d,i\}$
	E_0	$\{\{a,b\},\{a,i\}\},\{b,c\},\{c,d\}$
6	V_0	$\{a,b,c,d,h,i\}$
	E_0	$\{\{a,b\},\{a,i\}\},\{b,c\},\{c,d\},\{d,h\}$
7	V_0	$\{a,b,c,d,e,h,i\}$
	E_0	$\{\{a,b\},\{a,i\}\},\{b,c\},\{c,d\},\{d,e\},\{d,h\}$
8	V_0	$\{a,b,c,d,e,g,h,i\}$
	E_0	$\{\{a,b\},\{a,i\}\},\{b,c\},\{c,d\},\{d,e\},\{d,g\},\{d,h\}$
9	V_0	$\{a,b,c,d,e,f,g,h,i\}$
	E_0	$\{\{a,b\},\{a,i\},\{b,c\},\{c,d\},\{d,e\},\{d,g\},\{d,h\},\{f,g\}\}$

Table 2: The steps taken during the execution of the Prim algorithm

2 Minimal Spanning Trees and Shortest Paths

It is not necessarily true that the path between any two vertices on a minimal spanning tree of a graph is also a shortest path between these two vertices on this graph. As a counter-example, suppose we have the following graph.

$$V = \{a, b, c\}$$
$$E = \{\{a, b\}, \{b, c\}, \{a, c\}\}$$
$$w_{a,b} = 2, w_{b,c} = 2, w_{a,c} = 3$$

Here, the shorted path between a and c is the single edge $\{a, c\}$ of weight 3, however the path between them in the minimal spanning tree is of weight 4.

3 Finding the Maximal Spanning Tree

We are to write an algorithm to find the maximal spanning tree of a graph. The algorithm presented below is a slight modification of Kruskal's algorithm, where instead of sorting the edges by increasing weight, we sort them by decreasing weight.

```
function MaximalSpanningTree(V, E)
    F := φ
    D := DisjointSet(V)
    Sort E by decreasing weight
    for (u, v) ∈ E do
        if D.FindSet(u) ≠ D.FindSet(v) then
            F := F ∪ {(u, v)}
            D.Union(D.FindSet(u), D.FindSet(v))
    return F
```

The proof of its correctness is the same logic as that for Kruskal's algorithm, and its complexity is identical, namely $O(E\alpha(V))$, where α is the inverse Ackermann function. It uses $O(V)$ extra space because D uses $O(V)$ space, and the maximal spanning tree contains $|V| - 1$ edges.

4 Propositions on Minimal Spanning Trees

For each of the following propositions we are to prove or disprove them. Let $G = (V, E)$ be a connected undirected weighted graph.

1. G has only one minimal spanning tree.

Disproof. This statement is false, as shown in the counter-example in figure 1. Since all three edges are the same weight, and any two of them form a spanning tree, there are three distinct spanning trees, which are all minimal.

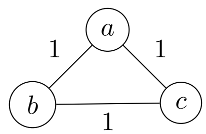

Figure 1: A graph with three different minimal spanning trees

2. If all the weights in G are different, then there exists only one minimal spanning tree.

   Proof. This proposition is true. We shall prove it by contradiction. Suppose the contrary; i.e. there exist two minimal spanning trees S and T such that $S \neq T$. Let e be the edge with the smallest weight in S which is not in T, or vice versa. (Without loss of generality, suppose $e \in S$.)

   If we add e to T, we obtain a graph T_1 with a cycle. This cycle must contain some other edge f which is not in S, and we know that $w_e < w_f$, since we chose e to be the smallest edge which was not common to S and T, and all the edges are of different sizes.

   Therefore we can define T_2 as the graph obtained if we remove f from T_1. This breaks the only cycle in T_1, meaning T_2 is a tree. It spans the same vertices that T_1 spans, which are the same vertices that T spans, so since T spans the entire graph, so does T_2. Finally, the total weight of the edges in T_2 is smaller than that of T, since T_2 only differs by having e instead of f, which has a smaller weight.

   Thus we have shown that there exists a smaller spanning tree than T, which contradicts T's minimality. Therefore we can conclude that there cannot be two minimal spanning trees if the weights are all different.

3. If G has two edges of the same weight, then there must be at least two minimal spanning trees.

   Disproof. This can be disproven with a counter-example as shown in figure 2.

5 More Proofs on Spanning Trees

- Each connected undirected graph $G = (V, E)$ has a spanning tree.

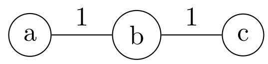

Figure 2: A graph with two edges of the same weight and only one minimal spanning tree

Proof. A spanning tree is a connected subgraph which contains every vertex but no cycles. G itself is connected (thus spans all the vertices), but may contain cycles. We will show that it is possible to remove all cycles from G without causing the graph to become disconnected.

Suppose G has a path $a_0, a_1, \ldots, a_n, a_0$ which is a cycle.