Linear Algebra 2

Homework 1 – Complex Numbers

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2. We are tasked with solving the following equations for z.

   (b)
$$
\begin{aligned}
z^2 &= -10 + 20i \\
(a+bi)^2 &= -10 + 20i \\
a^2 + 2abi - b^2 &= -10 + 20i \\
a^2 - b^2 &= -10 \\
2ab &= 20 \\
b &= \frac{10}{a} \\
a^2 - \left(\frac{10}{a}\right)^2 &= -10 \\
a^2 - \frac{100}{a^2} &= -10 \\
a^4 + 10a^2 - 100 &= 0 \\
\text{Let } t &= a^2 \\
t^2 + 10t - 100 &= 0 \\
t &= -5 + 5\sqrt{5} \\
a = \pm\sqrt{-5 + 5\sqrt{5}} &= \pm\sqrt{5}\sqrt{\sqrt{5} - 1} \\
b = \pm\frac{10}{\sqrt{5}\sqrt{\sqrt{5}-1}} &= \pm\frac{2\sqrt{5}}{\sqrt{\sqrt{5}-1}} \\
z_1 &= \sqrt{5}\sqrt{\sqrt{5}-1} + \frac{2\sqrt{5}}{\sqrt{\sqrt{5}-1}}i \\
z_2 &= -\sqrt{5}\sqrt{\sqrt{5}-1} - \frac{2\sqrt{5}}{\sqrt{\sqrt{5}-1}}i
\end{aligned}
$$

(d)

$$\begin{aligned}
z^2 + |z|^2 &= 2 - 4i \\
(a+bi)^2 + |a+bi|^2 &= 2 - 4i \\
(a+bi)^2 + \sqrt{a^2+b^2}^2 &= 2 - 4i \\
a^2 - b^2 + 2abi + a^2 + b^2 &= 2 - 4i \\
2a^2 + 2abi &= 2 - 4i \\
2a^2 &= 2 \\
2ab &= -4 \\
a^2 &= 1 \\
b &= -\frac{2}{a} \\
a &= \pm 1 \\
b &= \mp 2 \\
z_1 &= 1 - 2i \\
z_2 &= -1 + 2i
\end{aligned}$$

3. For the following properties of the complex conjugates and absolute values we must prove each property, using either Cartesian or polar form.

(a)

$$\begin{aligned}
\overline{z_1} + \overline{z_2} &= \overline{z_1 + z_2} \\
\text{Let } z_1 &= a_1 + b_1 i \\
\text{Let } z_2 &= a_2 + b_2 i \\
\overline{z_1} + \overline{z_2} &= a_1 - b_1 i + a_2 - b_2 i \\
&= (a_1 + a_2) - (b_1 + b_2)i \\
&= \overline{(a_1 + a_2) + (b_1 + b_2)i} \\
&= \overline{a_1 + b_1 i + a_2 + b_2 i} \\
&= \overline{z_1 + z_2}
\end{aligned}$$

(d)

$$\begin{aligned}
\overline{\overline{z}} &= z \\
\text{Let } z &= a + bi \\
\overline{\overline{z}} &= \overline{\overline{a + bi}} \\
&= \overline{a - bi} \\
&= a + bi \\
&= z
\end{aligned}$$

4. We are asked to write each of the following complex numbers in their polar representation:

(b)
$$\begin{aligned}
1-i &= r(\cos\theta + i\sin\theta)\\
r &= |1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}\\
\theta &= -\arctan(1) = -\frac{\pi}{4}\\
1-i &= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\\
&= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right)
\end{aligned}$$

(d)
$$4-4i = 4(1-i) = 4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

5. (c) We are to express the following number in cartesian form.

$$\left(\frac{1+2i}{-2+i}\right)^{2048} = \frac{1+2^{2048}}{2^{2048}+1} = 1$$

6. (b) We seek to solve the following equation using the polar representation of complex numbers.

$$\begin{aligned}
z^3 &= -2+2i = |-2+2i|e^{\frac{3\pi i}{4}} = 2\sqrt{2}e^{\frac{3\pi i}{4}}\\
z_1 &= \left(2\sqrt{2}e^{\frac{3\pi i}{4}}\right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{\pi}{4}i} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i\\
z_2 &= \left(2\sqrt{2}e^{\left(\frac{3\pi}{4}+2\pi\right)i}\right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{11\pi i}{12}}\\
&= \sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)i\\
z_3 &= \left(2\sqrt{2}e^{\left(\frac{3\pi}{4}+4\pi\right)i}\right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{19\pi i}{12}}\\
&= \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2} - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)i
\end{aligned}$$