Linear Algebra 2

Homework 1 – Complex Numbers

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2. We are tasked with solving the following equations for z.

   (b)
$$\begin{aligned}
z^2 &= -10 + 20i \\
(a+bi)^2 &= -10 + 20i \\
a^2 + 2abi - b^2 &= -10 + 20i \\
a^2 - b^2 &= -10 \\
2ab &= 20 \\
b &= \frac{10}{a} \\
a^2 - \left(\frac{10}{a}\right)^2 &= -10 \\
a^2 - \frac{100}{a^2} &= -10 \\
a^4 + 10a^2 - 100 &= 0 \\
\text{Let } t &= a^2 \\
t^2 + 10t - 100 &= 0 \\
t &= -5 + 5\sqrt{5} \\
a = \pm\sqrt{-5+5\sqrt{5}} &= \pm\sqrt{5}\sqrt{\sqrt{5}-1} \\
b = \pm\frac{10}{\sqrt{5}\sqrt{\sqrt{5}-1}} &= \pm\frac{2\sqrt{5}}{\sqrt{\sqrt{5}-1}} \\
z_1 &= \sqrt{5}\sqrt{\sqrt{5}-1} + \frac{2\sqrt{5}}{\sqrt{\sqrt{5}-1}}i \\
z_2 &= -\sqrt{5}\sqrt{\sqrt{5}-1} - \frac{2\sqrt{5}}{\sqrt{\sqrt{5}-1}}i
\end{aligned}$$

(d)
$$\begin{aligned}
z^2 + |z|^2 &= 2 - 4i \\
(a+bi)^2 + |a+bi|^2 &= 2 - 4i \\
(a+bi)^2 + \sqrt{a^2+b^2}^2 &= 2 - 4i \\
a^2 - b^2 + 2abi + a^2 + b^2 &= 2 - 4i \\
2a^2 + 2abi &= 2 - 4i \\
2a^2 &= 2 \\
2ab &= -4 \\
a^2 &= 1 \\
b &= -\frac{2}{a} \\
a &= \pm 1 \\
b &= \mp 2 \\
z_1 &= 1 - 2i \\
z_2 &= -1 + 2i
\end{aligned}$$

3. For the following properties of the complex conjugates and absolute values we must prove each property, using either Cartesian or polar form.

(a)
$$\begin{aligned}
\overline{z_1} + \overline{z_2} &= \overline{z_1 + z_2} \\
\text{Let } z_1 &= a_1 + b_1 i \\
\text{Let } z_2 &= a_2 + b_2 i \\
\overline{z_1} + \overline{z_2} &= a_1 - b_1 i + a_2 - b_2 i \\
&= (a_1 + a_2) - (b_1 + b_2)i \\
&= \overline{(a_1 + a_2) + (b_1 + b_2)i} \\
&= \overline{a_1 + b_1 i + a_2 + b_2 i} \\
&= \overline{z_1 + z_2}
\end{aligned}$$

(d)
$$\begin{aligned}
\overline{\overline{z}} &= z \\
\text{Let } z &= a + bi \\
\overline{\overline{z}} &= \overline{\overline{a + bi}} \\
&= \overline{a - bi} \\
&= a + bi \\
&= z
\end{aligned}$$

4. We are asked to write each of the following complex numbers in their polar representation:

(b)
$$\begin{aligned}1-i &= r(\cos\theta + i\sin\theta)\\ r &= |1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}\\ \theta &= -\arctan(1) = -\frac{\pi}{4}\\ 1-i &= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right)\end{aligned}$$

(d)
$$4-4i = 4(1-i) = 4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

5. (c) We are to express the following number in cartesian form.

$$\left(\frac{1+2i}{-2+i}\right)^{2048} = \frac{1+2^{2048}}{2^{2048}+1} = 1$$

6. (b) We seek to solve the following equation using the polar representation of complex numbers.

$$\begin{aligned}z^3 &= -2+2i = |-2+2i|e^{\frac{3\pi i}{4}} = 2\sqrt{2}e^{\frac{3\pi i}{4}}\\ z_1 &= \left(2\sqrt{2}e^{\frac{3\pi i}{4}}\right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{\pi}{4}i} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i\\ z_2 &= \left(2\sqrt{2}e^{\left(\frac{3\pi}{4}+2\pi\right)i}\right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{11\pi i}{12}}\\ &= \sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)i\\ z_3 &= \left(2\sqrt{2}e^{\left(\frac{3\pi}{4}+4\pi\right)i}\right)^{\frac{1}{3}} = \sqrt{2}e^{\frac{19\pi i}{12}}\\ &= \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2} - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)i\end{aligned}$$

9. (a) We are to prove or disprove whether or not the z is purely imaginary.

$$\begin{aligned}z &= \left(\frac{1+i}{1-i}\right)^{19} = \left(\frac{\sqrt{2}e^{\frac{i\pi}{4}}}{\sqrt{2}e^{\frac{7i\pi}{4}}}\right)^{19} = \frac{e^{\frac{19}{4}i\pi}}{e^{\frac{133}{4}i\pi}} = e^{-\frac{57}{2}i\pi}\\ &= \cos\frac{57\pi}{2} - i\sin\frac{57\pi}{2} = i\end{aligned}$$

(b) We are asked to prove or disprove that the following equation has five different solutions.

$$
\begin{aligned}
z^5 &= 2 - 2i \\
&= 2\sqrt{2}e^{\frac{7\pi i}{4}} = 2\sqrt{2}e^{\frac{15\pi i}{4}} = 2\sqrt{2}e^{\frac{23\pi i}{4}} = 2\sqrt{2}e^{\frac{31\pi i}{4}} = 2\sqrt{2}e^{\frac{39\pi i}{4}} \\
z_1 &= \sqrt[10]{8}e^{\frac{7\pi i}{20}} \\
z_2 &= \sqrt[10]{8}e^{\frac{4\pi i}{4}} \\
z_3 &= \sqrt[10]{8}e^{\frac{23\pi i}{20}} \\
z_4 &= \sqrt[10]{8}e^{\frac{31\pi i}{20}} \\
z_5 &= \sqrt[10]{8}e^{\frac{39\pi i}{20}}
\end{aligned}
$$

Here, z_1 through z_5 are not equal to each other, since the coefficients of i in the exponents are between 0 and 2π, so none of them are more than a full rotation around the circle they all lie on.

10. (a) Given that $u = \sqrt{3}(1+i)$ we seek u^4.

$$
\begin{aligned}
u^4 &= \sqrt{3}^4(1+i)^4 \\
&= 9\left(\binom{0}{4}i^0 + \binom{1}{4}i + \binom{2}{4}i^2 + \binom{3}{4}i^3 + \binom{4}{4}i^4\right) \\
&= 9 + 36i - 54 - 36i + 9 \\
&= -36
\end{aligned}
$$

(b) We are to solve the equation $z^4 = -36$ and calculate the sum of all of its solutions.

$$
\begin{aligned}
z_1 &= \sqrt{3}(1+i) \\
&= \sqrt{6}e^{\frac{\pi i}{4}} = \sqrt{6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{3} + \sqrt{3}i \\
z_2 &= \sqrt{6}e^{\frac{3\pi i}{4}} = \sqrt{6}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\sqrt{3} + \sqrt{3}i \\
z_3 &= \sqrt{6}e^{\frac{5\pi i}{4}} = \sqrt{6}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\sqrt{3} - \sqrt{3}i \\
z_4 &= \sqrt{6}e^{\frac{7\pi i}{4}} = \sqrt{6}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{3} - \sqrt{3}i
\end{aligned}
$$

$$
\sum_{k=1}^{4} z_k = 0
$$

14. We are to prove or disprove the following claims.

(b) The set $\{(0,-1),(i,0)\}$ is not a basis of $\mathbb{C}^2$ as a vector space over $\mathbb{R}$.

This is true, since the dimension of $\mathbb{C}^2$ as a vector space over $\mathbb{R}$ is four, but the set has only two elements, thus cannot be a basis.

(c) The set $\{(1-i,1+i),(1+i,2-i)\}$ is linearly independent when $\mathbb{C}^2$ is considered as a vector space over $\mathbb{C}$.

This set is linearly independent since the two vectors are not scalar multiplications of each other. $(1-i)\times i = 1+i$, but $(1+i)\times i \neq 2-i$.

16. We are given the following system of equations.

$$\begin{aligned} cx + (1+i)y &= c \\ (1-i)x - 2cy &= 2-2c \end{aligned}$$

(a) We seek c such that the system of equations has a unique solution.

$$\begin{bmatrix} c & 1+i & c \\ 1-i & -2c & 2-2c \end{bmatrix} = \begin{bmatrix} 1 & \frac{1+i}{c} & c \\ 1-i & -2c & 2-2c \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{1+i}{c} & c \\ 0 & -2c-\frac{2}{c} & 2-3c-ci \end{bmatrix}$$

For there to be a unique solution,

$$\begin{aligned} -2c - \frac{2}{c} &\neq 0 \\ -c^2 - 1 &\neq 0 \\ c^2 &\neq -1 \\ c &\neq \pm i \end{aligned}$$

(b) There are no values of c for which there are infinitely many solutions since no values of c cause the equations to be linearly dependent.

(c) For $c = \pm i$, the equations have no solutions since as shown above, there would be a row in the reduced matrix where all but the last column have a value of 0, thus cannot be satisfied.