Determining the Convexity of any Polygon

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1 Abstract

This paper presents and proves the correctness of an algorithm which determines if a sequence of points in three-dimensional space forms a convex polygon in the order that they are given or not. We will discuss simple convex and concave polygons as well as complex (self intersecting) polygons, and ultimately even non-planar polygons.

2 Definitions

Turn between vectors The turn between two three-dimensional vectors $\vec{u}$ and $\vec{v}$, which we shall denote henceforth as $T(\vec{u}, \vec{v}) \in [0, \pi]$, is defined as follows.

$$T(\vec{u}, \vec{v}) = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

For any two vectors there are two angles between them, and the function T always gives us the smaller of the two.

Polygon A polygon is a sequence of at least three points (also called vertices) in three-dimensional space where each point is connected to the next by a straight line segment called an edge. There is also an edge between the last and first points. The edges form a closed path.

Most definitions of a polygon are in fact what we will refer to as a simple polygon, but since we are interested in processing any given sequence of points, and to simplify our language in this paper, we shall extend our definition of a polygon to include any such sequences.

Planar polygon A planar polygon is a polygon whose vertices all share a common plane.

Non-planar polygon A non-planar polygon is simply a polygon which is not planar.

Simple polygon A simple polygon is a planar polygon whose bounding edges do not intersect with one another. That is, their edges to not touch each other except for each edge touching exactly one edge at each of its end points.

Convex polygon A polygon is convex if it is simple and all its interior angles are less than π radians.

Concave polygon A polygon is concave if it is simple and not convex. That is, if it has an interior angle greater than π radians.

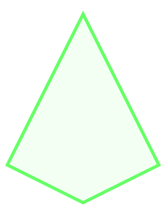
Complex polygon A polygon is complex if it is planar, but not simple. That is, if its edges intersect with each other.

Turn of a polygon at a vertex If A, B, and C are three adjacent vertices of a polygon P such that B is in between A and C, the turn of the polygon at the point B is $T_P(B) = T(\overrightarrow{AB}, \overrightarrow{BC})$.

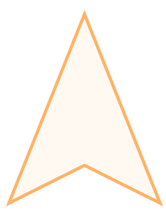
Note, for polygons for which the term 'exterior angle' is defined, the turn at a vertex is equal to the absolute value of the exterior angle at that vertex.

Total turn of a polygon The total turn of a polygon P, is the sum of the turns of all its vertices. We shall use the following shorthand notation to denote this.

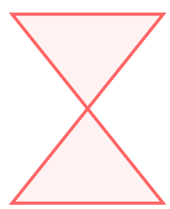
$$\sum T_P = \sum_{v \in P} T_P(v)$$



(a) Convex



(b) Concave



(c) Complex

Figure 1: Examples of different types of polygons.

3 The Algorithm

We will begin by explaining how the algorithm works, then presenting the algorithm at the end of this section.

3.1 Input

The algorithm accepts a sequence of points P in three-dimensional space. The points represent a polygon formed by constructing an edge between all adjacent vertices in the sequence and an additional edge between the first and last vertices.

3.2 Output

The algorithm returns True if the input forms a convex polygon, and False if they form any other type of polygon.

3.3 Processing

First, if at least three consecutive points lie consecutively on a single line, the middle points may be removed as they contribute nothing to the polygon and are thus insignificant. The check necessary to do this is simple enough and has been omitted for conciseness. From now on we may assume that any such cases are corrected before any further processing is performed.

The algorithm then boils down to a single check. A sequence of vertices forms a simple convex polygon if and only if the sum of the turns is equal to 2π. Formally, the algorithm checks if the following equation holds true.

$$\sum T_P = 2\pi$$

3.4 Pseudo Code

The function IsConvex takes as input a sequence of points P and returns whether or not the points form a convex polygon.

```
function IsConvex(P)
    Remove insignificant vertices from P
    if |P| < 3 then return False
    Σ := 0                                   ▷ The running sum of the turn.
    for i from 0 to |P| − 1 do
        A := P_{i−1 mod |P|}                 ▷ The previous point.
        B := P_i                             ▷ The current point.
        C := P_{i+1 mod |P|}                 ▷ The next point.
        Σ := Σ + T(→AB, →BC)                 ▷ Add the turn at point B.
    return Σ = 2π
```

4 Proof of Correctness

4.1 Claim

A sequence of vertices P forms a convex polygon if and only if the total turn $\sum T_P$ is equal to 2π.

In order to prove our claim, we must prove all of the following cases.

1. If a polygon is convex then the total turn is equal to 2π. (See §4.2.1.)
2. If a polygon is concave then the total turn is greater than 2π. (See §4.2.2.)
3. If a polygon is complex then the total turn is greater than 2π. (See §4.2.3 and §4.2.4.)
4. If a polygon is non-planar, then the total turn is greater than 2π. (See §4.2.5.)

4.2 Proof

4.2.1 Convex Polygons Have a Total Turn of Precisely 2π

It is known that the exterior angles of a simple polygon sum to precisely 2π. An exterior angle is defined as π minus the interior angle. However since by definition of a convex polygon all its interior angles are less than π, the exterior angles must be positive.

Therefore for a convex polygon P, we have

$$\sum T_P = \sum_{v \in P} T_P(v) = \sum_{v \in P} |\operatorname{ext}(v)| = \sum_{v \in P} \operatorname{ext}(v) = 2\pi$$

where $\operatorname{ext}(v)$ is the exterior angle at vertex v.

4.2.2 The Total Turn of a Concave Polygon Exceeds 2π

A concave polygon must have at least one negative exterior angle. This is because it must have, by definition, at least one interior angle greater than π radians. Since $\operatorname{ext}(v) = \pi - \operatorname{int}(v)$, where $\operatorname{int}(v)$ is the interior angle, $\operatorname{ext}(v)$ must be negative for some vertex v.

Therefore for a concave polygon P, we have the following inequality.

$$\sum T_P = \sum_{v \in P} T_P(v) = \sum_{v \in P} |\operatorname{ext}(v)| > \sum_{v \in P} \operatorname{ext}(v) = 2\pi$$

4.2.3 A Complex polygon's Total Turn Exceeds 2π

For any complex polygon P, we are able to find a sequence of consecutive vertices of P which when appended to an intersection point of P forms a simple polygon E in the following way.

First we must assume that no edge doubles back on the previous edge. This means that $\forall v \in P,\ T_P(v) < \pi$. For cases where this assumption does not hold, see §4.2.4. This will allow us to assert that E is simple.

We choose any arbitrary point on the polygon's bounding path. Then from that point we trace the bounding path, marking any intersection points we encounter. Once we arrive at an intersection point which we've previously encountered, we stop. Let this intersection point be called I.

The closed path which we traced from the first encounter with I to our second encounter with I is a simple polygon (see figure 2).

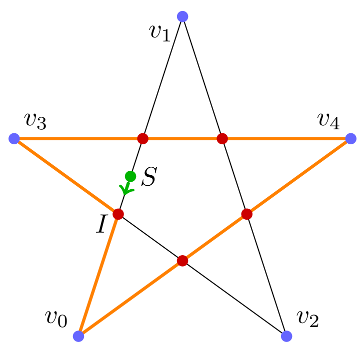


Figure 2: Extracting a simple polygon from a complex one. This complex polygon is a pentagram with vertices v_0 through v_4. S is the arbitrarily chosen starting point. The orange arrowhead marks the bounding edges of a simple polygon $E = (v_0, v_4, v_3, I)$ embedded in the complex pentagram. I is the first (and necessarily the only) intersection point of the pentagram which we encountered twice.

The embedded polygon E found with this procedure must be simple since all of the following hold:

- E must enclose some area since (in accordance with out assumption at the beginning of this section) no edges double back on their previous neighbour. (If not for our assumption, if we encountered a vertex whose turn was π, we would encounter a visited intersection point an infinitesimal distance after said vertex. This would cause E to be ill-defined, and for all intents and purposes, a single point, which is certainly not a simple polygon.)
- E must have no intersection points since all of P's intersection points are touched only once by E, except for point I which is the start and end point, so is not an intersection point of E either.

Let $\alpha \in (-\pi, \pi)$ be the exterior angle of E at vertex I. Since E is a simple polygon it can't have exterior angles of $\pm\pi$, else the bounding edges would go back on themselves contradicting E's simplicity and our earlier assumption.

Suppose now we were to 'walk' a cycle through the edges of P starting at the midpoint of the line segment formed by I and one of the two vertices adjacent to I which is common to both P and E (either v_0 or v_3 in figure 2). Suppose we start walking in the direction away from I and are only allowed to walk forward and turn.

The first time we reach I, we will have already turned all the turns of E but one — $T_E(I)$, which is equal to $|\alpha|$. We immediately pass I and enter a section of the path which is no longer part of E. Let us analyze the current position.

Since I is an intersection point of P, we will in future be forced to encounter it again before arriving finally at the start. However, since we've just passed I, it is positioned a small distance directly behind us. This means that in order to reach it in future, our path must force us to make some turns which sum to at least π radians.

This now allows us to formulate our first inequality.

$$\sum T_P \geq \sum T_E - |\alpha| + \pi \tag{1}$$

Since we know that $\sum T_E \geq 2\pi$ due to the simplicity of E, and that $|\alpha| \in [0, \pi)$, we can from inequality 1 come to the following conclusion.

$$\sum T_P \geq 3\pi - |\alpha| > 2\pi \tag{2}$$

4.2.4 Complex Polygons Containing a Turn of π Radians

Although we've already shown our claim to be true for most complex polygons, we've conveniently ignored a family of cases where the previous proof cannot apply. We shall now address those.

Firstly, if there are more than two edges which double back on their previous neighbour, then since each consists of a turn of π radians, there must be more than 2π radians in the total turn of the polygon.

Secondly, if there are precisely two such cases at vertices v_i and v_j then either these are the only two significant vertices and can discard this as invalid input as done in the algorithm, or we have at least one other (non-zero) turn at another significant vertex v_k; in which case the total turn is 2π for the two turns at v_i and v_j, plus the turn at v_k, in total exceeding 2π.

Finally if there is only one such case of a vertex v with turn of π radians, then we can apply the same procedure to extract a simple polygon from our complex one with the sole exception that the starting point which would otherwise have had the freedom to be arbitrarily chosen must be at v.

With such a starting point it is guaranteed that the extracted polygon will be well defined and simple since the first intersection point to be encountered twice (I) will either not lie on the overlapping edges created by the turn of π radians at v (this occurs in the case where other intersection points exist), otherwise it is the point furthest from v to lie on both edges adjacent to v with the extracted polygon E being the original polygon minus the vertex v. Once we have our extracted simple polygon, the remainder of the proof is the same as with the other complex polygons.

4.2.5 Non-planar Polygons Have a Total Turn Greater Than 2π

The proof for this can be found in Lemma 2.1 of Sullivan, Curves of Finite Total Curvature.[1] Here's a paraphrased version of the lemma.

> Suppose P is a polygon in $\mathbb{R}^3$. If P' is obtained from P by deleting one vertex v_n then $\sum T_{P'} \leq \sum T_P$. We have equality here if $v_{n-1}v_nv_{n+1}$ are colinear in that order or if $v_{n-2}v_{n-1}v_nv_{n+1}v_{n+2}$ lie convexly in some plane, but never otherwise.

For a non-planar polygon P, removing vertices v_n until only a triangle Q is left defines a sequence of polygons whose total turn is decreasing. In at least one pair of the sequence (corresponding to where some $v_{n-2}v_{n-1}v_nv_{n+1}v_{n+2}$ is not planar), the total turn must be strictly decreasing. Since $\sum T_Q = 2\pi$ by virtue of Q being a triangle, we conclude that $\sum T_P > \sum T_Q = 2\pi$.

References

[1] John M Sullivan. Curves of finite total curvature. In *Discrete differential geometry*, pages 137–161. Springer, 2008.