Determining the Convexity of any Polygon

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1 Abstract

This paper presents and proves the correctness of an algorithm which determines if a sequence of points in three-dimensional space forms a convex polygon or not. We will discuss simple concave polygons as well as complex (self intersecting) polygons. As an added bonus, we will be able to detect and reject any input whose vertices do not lie on a common plane.

2 Definitions

Angle between vectors The angle between two three-dimensional vectors $\vec{u}$ and $\vec{v}$, which we shall denote henceforth as $\measuredangle(\vec{u}, \vec{v})$, is defined as follows.

$$\measuredangle(\vec{u}, \vec{v}) = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

It is important to note that $\measuredangle(\vec{u}, \vec{v})$ is always between 0 and π radians. For any two vectors there are two angles between them, and the function $\measuredangle$ always gives us the smaller of the two.

Polygon A polygon is a *plane* figure that is described by a finite number of straight line segments connected to form a closed circuit. The enclosed plane region, as well as the bounding circuit, is what we shall call a polygon.

Simple polygon A simple polygon is a polygon whose bounding line segments do not intersect with each other.

Interior angle For a simple polygon, an angle between two adjacent line segments is called an interior angle (or internal angle) if a point within the angle is in the interior of the polygon. A simple polygon has exactly one internal angle per vertex.

We shall denote the interior angle at vertex Q as $\operatorname{int}(Q)$.

Exterior angle If P, Q, and R are three consecutive vertices of a polygon (such that Q is in between P and R), the exterior angle of the polygon at a vertex Q is $\operatorname{ext}(Q) = \measuredangle\left(\overrightarrow{PQ}, \overrightarrow{QR}\right)$.

Note, since the range of the function $\measuredangle$ is $[0, \pi]$, by 'exterior angle' we refer always (unless otherwise noted) to a positive angle.

Convex polygon A polygon is convex if it is simple and all its interior angles are less than π radians.

Concave polygon A polygon is concave if it is simple and not convex.

Complex polygon A polygon is complex if it is not simple, i.e. if its edges intersect with each other. There is debate whether or not complex polygons are considered polygons, but for the purposes of this paper we shall refer to them as polygons.

(a) Convex (b) Concave (c) Complex

Figure 1: Examples of different types of polygons.

3 The Algorithm

We will begin by explaining how the algorithm works, then presenting the algorithm at the end of this section.

3.1 Input

The algorithm accepts a sequence of points in three-dimensional space. The points, if they share a common plane, represent a polygon formed by constructing an edge between all adjacent vertices in the sequence and an additional edge between the first and last vertices.

3.2 Output

The algorithm returns True if the input forms a convex polygon, and False otherwise.

Some examples of when the input does not form a convex polygon are

- if the points form a concave polygon,
- if the points form a complex polygon, and
- if the points do not share a common plane, and thus do not form a polygon at all.

3.3 Processing

The algorithm boils down to a single check. A sequence of vertices forms a simple convex polygon if and only if the sum of the exterior angles is equal to 2π. Formally, the algorithm checks if the following equation holds true, where V is the input sequence of vertices.

$$\sum_{v \in V} \operatorname{ext}(v) = 2\pi$$

3.4 Pseudo Code

The function IsConvex takes as input a sequence of points V and returns whether or not the points form a convex polygon.

```
function IsConvex(V)
    if |V| < 3 then return False
    s := 0                                   ▷ The sum of the exterior angles.
    for i from 0 to |V| − 1 do
        P := V_{i−1 mod |V|}                 ▷ The previous point.
        Q := V_i                             ▷ The current point.
        R := V_{i+1 mod |V|}                 ▷ The next point.
        s := s + ∡(PQ→, QR→)                 ▷ Add the exterior angle of Q.
    return s = 2π
```

4 Proof of Correctness

4.1 Claim

A sequence of vertices V forms a simple convex polygon if and only if the sum of the exterior angles is equal to 2π.

4.2 Proof

In order to prove our claim, we must prove (a) that if a polygon is simple then the sum of its exterior angles is equal to 2π, and (b) that if a sequence of points' angles sum to 2π, then it must form a convex polygon. We shall commence with the easier of the two.

4.2.1 Convex Polygons' Exterior Angles Sum to 2π

Suppose a convex polygon is drawn on the ground, and we are standing on one of the edges, facing parallel to the edge with our right foot inside the polygon and our left foot outside. We shall call the direction that we are currently facing the initial direction. Then if we walk forward following the the edges of the polygon, and at each vertex we turn to face the next vertex, turning whichever way (right or left) is a smaller turn. Eventually we would reach the starting edge again.

At each of the vertices we turned toward the right an amount equal to the exterior angle. It must be the case that we always turn to the right, since every interior angle is less than π radians (by definition of convex), and the inside of the polygon is to our right. See figure 2.

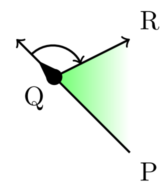

Figure 2: Turning at a vertex of a convex polygon.

Since we always turned to our right, and we ended up facing the initial direction, and at no other edge or vertex were we facing the initial direction, it must be that in total we turned precisely one rotation, that is, 2π.

Thus the total amount we turned (2π) must be equal to the sum of the amounts we turned at each vertex ($\sum_{v \in V} \operatorname{ext}(v)$).

4.2.2 Only Convex Polygons' Angles Sum to 2π

Concave polygons. Suppose once again that a polygon is drawn on the ground, but this time let it be concave. Again, suppose we are starting on any edge facing parallel to it with our right foot inside the polygon. This is our initial direction. Suppose we again walk the edges in the same way as we walked the convex polygon.

At some point, we will reach a vertex whose interior angle is greater than π radians (by definition of concave). When this occurs, we will find that the next vertex is on our left hand side, and we will have to turn in that direction. Since eventually we must reach the starting point and be facing the initial direction, we must at some point turn enough to the right to correct for our leftward deviation. This means that for every interior angle greater than π radians, we will add both the deviation and the correction to the complete turn of 2π which we must eventually make.

The deviation at each of these vertices is $\operatorname{ext}(v)$, and the correction at some future turn must match it. Let $V' \subset V$ be the set of vertices whose interior angles are greater than π. Formally,

$$V' = \{v \in V : \mathrm{int}\,(v) > \pi\}. \tag{1}$$

Therefore we have the following.

$$\sum_{v \in V} \mathrm{ext}\,(v) = 2\pi + 2 \sum_{v \in V'} \mathrm{ext}\,(v) \tag{2}$$

And since exterior angles are always positive (see definition above), and in the case of every vertex in V' not equal to zero (otherwise the interior angle would be precisely π contradicting equation 1), equation 2 must be strictly greater than 2π.

Complex polygons. Suppose for a third time that a polygon is drawn on the ground. This time let the polygon have self intersections and thus be complex. This time we lost the constancy of the fact that the inside of the polygon will always be on our right hand side as we walk the edges. This is because as whenever we walk past an intersection of edges, the filling of the polygon will switch from between being on our right and on our left.