Determining the Convexity of any Polygon

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1 Abstract

This paper presents and proves the correctness of an algorithm which determines if a sequence of points in three-dimensional space forms a convex polygon or not. We will discuss simple concave polygons as well as complex (self intersecting) polygons. As an added bonus, we will be able to detect and reject any input whose vertices do not lie on a common plane.

2 Definitions

Angle between vectors The angle between two three-dimensional vectors $\vec{u}$ and $\vec{v}$, which we shall denote henceforth as $\measuredangle(\vec{u}, \vec{v})$, is defined as follows.

$$\measuredangle(\vec{u}, \vec{v}) = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

It is important to note that $\measuredangle(\vec{u}, \vec{v})$ is always between 0 and π radians. For any two vectors there are two angles between them, and the function $\measuredangle$ always gives us the smaller of the two.

Polygon A polygon is a *plane* figure that is described by a finite number of straight line segments connected to form a closed circuit. The enclosed plane region, as well as the bounding circuit, is what we shall call a polygon.

Simple polygon A simple polygon is a polygon whose bounding line segments do not intersect with each other.

Interior angle For a simple polygon, an angle between two adjacent line segments is called an interior angle (or internal angle) if a point within the angle is in the interior of the polygon. A simple polygon has exactly one internal angle per vertex.

We shall denote the interior angle at vertex Q as $\operatorname{int}(Q)$.

Exterior angle If P, Q, and R are three consecutive points of a sequence of points (such that Q is in between P and R), the exterior angle of the sequence at the point Q is $\operatorname{ext}(Q) = \measuredangle\left(\overrightarrow{PQ}, \overrightarrow{QR}\right)$.

Note, since the range of the function $\measuredangle$ is $[0, \pi]$, by 'exterior angle' we refer always (unless otherwise noted) to a positive angle.

Convex polygon A polygon is convex if it is simple and all its interior angles are less than π radians.

Concave polygon A polygon is concave if it is simple and not convex.

Complex polygon A polygon is complex if it is not simple, i.e. if its edges intersect with each other. There is debate whether or not complex polygons are considered polygons, but for the purposes of this paper we shall refer to them as polygons.

(a) Convex (b) Concave (c) Complex

Figure 1: Examples of different types of polygons.

3 The Algorithm

We will begin by explaining how the algorithm works, then presenting the algorithm at the end of this section.

3.1 Input

The algorithm accepts a sequence of points in three-dimensional space. The points, if they share a common plane, represent a polygon formed by constructing an edge between all adjacent vertices in the sequence and an additional edge between the first and last vertices.

3.2 Output

The algorithm returns True if the input forms a convex polygon, and False otherwise.

Some examples of when the input does not form a convex polygon are

- if the points form a concave polygon,
- if the points form a complex polygon, and
- if the points do not share a common plane, and thus do not form a polygon at all.

3.3 Processing

First, if at least three consecutive points lie consecutively on a single line, the middle points may be removed as they contribute nothing to the sequence and are thus meaningless. The check necessary to do this is simple enough and has been omitted for conciseness. From now on we may assume that any such cases are corrected before any further processing is performed.

The algorithm then boils down to a single check. A sequence of vertices forms a simple convex polygon if and only if the sum of the exterior angles is equal to 2π. Formally, the algorithm checks if the following equation holds true, where V is the input sequence of vertices.

$$\sum_{v \in V} \operatorname{ext}(v) = 2\pi$$

3.4 Pseudo Code

The function IsConvex takes as input a sequence of points V and returns whether or not the points form a convex polygon.

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function IsConvex(V)
  Remove meaningless vertices from V
  if $|V| < 3$ **then return** False
  $s := 0$ ▷ The sum of the exterior angles.
  for i from 0 to $|V| - 1$ **do**
    $P := V_{i-1 \bmod |V|}$ ▷ The previous point.
    $Q := V_i$ ▷ The current point.
    $R := V_{i+1 \bmod |V|}$ ▷ The next point.
    $s := s + \measuredangle\left(\overrightarrow{PQ}, \overrightarrow{QR}\right)$ ▷ Add the exterior angle of Q.
  return $s = 2\pi$

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4 Proof of Correctness

4.1 Claim

A sequence of vertices V forms a simple convex polygon if and only if the sum of the exterior angles is equal to 2π.

In order to prove our claim, we must prove all of the following cases.

1. If a polygon is convex then the sum of its exterior angles is equal to 2π.
2. If a polygon is concave then the sum of its exterior angles is greater than 2π.
3. If a polygon is complex then the sum of its exterior angles is greater than 2π.
4. If all points in a sequence do not share a common plane, then the exterior angles of the points have a sum greater than 2π.

4.2 Proof

4.2.1 Convex Polygons' Exterior Angles Sum to 2π

Suppose a convex polygon is drawn on the ground, and we are standing on one of the edges, facing parallel to the edge with our right foot inside the polygon and our left foot outside. We shall call the direction that we are currently facing the initial direction. Then if we walk forward following the the edges of the polygon, and at each vertex we turn to face the next vertex, turning whichever way (right or left) is a smaller turn. Eventually we would reach the starting edge again.

At each of the vertices we turned toward the right an amount equal to the exterior angle. It must be the case that we always turn to the right, since every interior angle is less than π radians (by definition of convex), and the inside of the polygon is to our right. See figure 2.

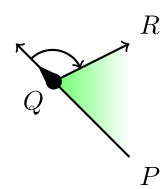

Figure 2: Turns at a vertex of a convex polygon are always right turns.

Since we always turned to our right, and we ended up facing the initial direction, and at no other edge or vertex were we facing the initial direction, it must be that in total we turned precisely one rotation, that is, 2π.

Thus the total amount we turned (2π) must be equal to the sum of the amounts we turned at each vertex ($\sum_{v\in V} \operatorname{ext}(v)$).

4.2.2 A Concave Polygon's Exterior Angles' Sum exceeds 2π

Suppose once again that a polygon is drawn on the ground, but this time let it be concave. Again, suppose we are starting on any edge facing parallel to it with our right foot inside the polygon. This is our initial direction. Suppose we again walk the edges in the same way as we walked the convex polygon.

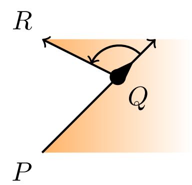

Figure 3: A turn at an vertex Q of a concave polygon where $\mathrm{int}\,(Q) > \pi$, must be a left turn.

At some point, we will reach a vertex whose interior angle is greater than π radians (by definition of concave). When this occurs, we will find that the next vertex is on our left hand side, and we will have to turn in that direction (see figure 3). Since eventually we must reach the starting point and be facing the initial direction, we must at some point turn enough to the right to correct for our leftward deviation. This means that for every interior angle greater than π radians, we will add both the deviation and the correction to the complete turn of 2π which we must eventually make.

The deviation at each of these vertices is $\mathrm{ext}\,(v)$, and the correction at some future turn must match it. Let $V' \subset V$ be the set of vertices whose interior angles are greater than π. Formally,

$$V' = \{v \in V : \mathrm{int}\,(v) > \pi\}. \tag{1}$$

Therefore we have the following.

$$\sum_{v \in V} \mathrm{ext}\,(v) = 2\pi + 2 \sum_{v \in V'} \mathrm{ext}\,(v) \tag{2}$$

And since exterior angles are always positive (see definition above), and in the case of every vertex in V' not equal to zero (otherwise the interior angle would be precisely π contradicting equation 1), equation 2 must be strictly greater than 2π.

4.2.3 A Complex polygon's Exterior Angles' Sum Exceeds 2π

Suppose for a third time that a polygon is drawn on the ground. This time let the polygon have self intersections and thus be complex. If we were to walk the edges like we did before, we would once again have to end facing the initial direction.

Since we started at a point S standing on an edge, this edge touches one vertex F directly in front of us and another vertex B directly behind us. Then let us single out the next vertex after F, which we will label A. Without loss of generality, we assume this point is on our right from the starting position. See figure 4, where the curvy edge between A and B indicates that the path between them could consist of any number of vertices which could be anywhere on the plane and cause self intersections.

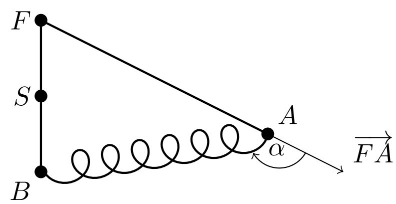

Figure 4: Three consecutive points of a complex polygon.

The smallest turn we could make to change direction from facing in the direction $\overrightarrow{FA}$ at point A to be arrive at point B would be $\alpha = \measuredangle\left(\overrightarrow{FA}, \overrightarrow{AB}\right)$ radians (which must be less than π radians, since B does not lie on the line FA).

If at point A we were to deviate from the complex path drawn on the ground and immediately take a turn of α radians to face B, and subsequently walked straight to point B and finally back to S, we would form a triangle, whose exterior angles we know to sum up to 2π.

We may even, on the path from A to B, turn more than α radians as with a convex polygon with more than three edges, so long as however much we turn now will eventually be saved when we reach point B and then make a turn to face the starting point, and thus keeping our exterior angle total equal to 2π.

However as soon as we turn to the left as with a concave polygon, or turn too far to the right, at least one of which is necessary in order to have self intersections (see figure 5), we raise our exterior angle total past 2π.

Thus if a polygon has self intersecting edges, then the sum of its exterior angles must be greater than 2π.

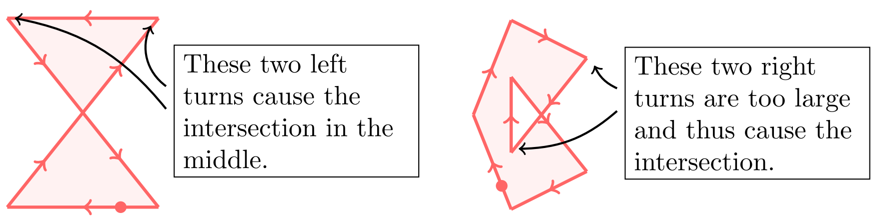

Figure 5: Self intersections by turning too far right or left.

4.2.4 Points which do not share a common plane