

Determining the Convexity of any Polygon

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1 Abstract

This paper presents and proves the correctness of an algorithm which determines if a sequence of points in three-dimensional space forms a convex polygon or not. We will discuss simple concave polygons as well as complex (self intersecting) polygons. As an added bonus, we will be able to detect and reject any input whose vertices do not lie on a common plane.

2 Definitions

Angle between vectors The angle between two three-dimensional vectors \vec{u} and \vec{v} , which we shall denote henceforth as $\angle(\vec{u}, \vec{v})$, is defined as follows.

$$\angle(\vec{u}, \vec{v}) = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

It is important to note that $\angle(\vec{u}, \vec{v})$ is always between 0 and π radians. For any two vectors there are two angles between them, and the function \angle always gives us the smaller of the two.

Polygon A polygon is a *plane* figure that is described by a finite number of straight line segments connected to form a closed circuit. The enclosed plane region, as well as the bounding circuit, is what we shall call a polygon.

Simple polygon A simple polygon is a polygon whose bounding line segments do not intersect with each other.

Interior angle For a simple polygon, an angle between two adjacent line segments is called an interior angle (or internal angle) if a point within the angle is in the interior of the polygon. A simple polygon has exactly one interior angle per vertex.

We shall denote the interior angle at vertex Q as $\text{int}(Q)$.

Exterior angle If P , Q , and R are three consecutive vertices of a polygon (such that Q is in between P and R), the exterior angle of the polygon at a vertex Q is $\text{ext}(Q) = \angle(\overrightarrow{PQ}, \overrightarrow{QR})$.

Note, since the range of the function \angle is $[0, \pi]$, by ‘exterior angle’ we refer always (unless otherwise noted) to a positive angle.

Convex polygon A polygon is convex if it is simple and all its interior angles are less than π radians.

Concave polygon A polygon is concave if it is simple and not convex.

Complex polygon A polygon is complex if it is not simple, i.e. if its edges intersect with each other. There is debate whether or not complex polygons are considered polygons, but for the purposes of this paper we shall refer to them as polygons.

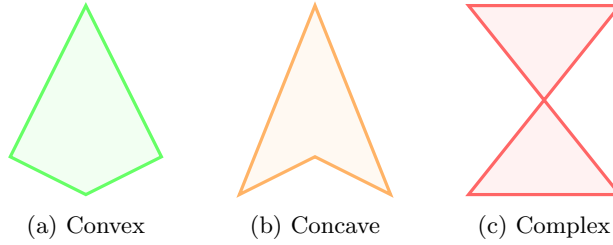


Figure 1: Examples of different types of polygons.

3 The Algorithm

We will begin by explaining how the algorithm works, then presenting the algorithm at the end of this section.

3.1 Input

The algorithm accepts a sequence of points in three-dimensional space. The points, if they share a common plane, represent a polygon formed by constructing an edge between all adjacent vertices in the sequence and an additional edge between the first and last vertices.

3.2 Output

The algorithm returns True if the input forms a convex polygon, and False otherwise.

Some examples of when the input does not form a convex polygon are

- if the points form a concave polygon,
- if the points form a complex polygon, and
- if the points do not share a common plane, and thus do not form a polygon at all.

3.3 Processing

The algorithm boils down to a single check. A sequence of vertices forms a simple convex polygon if and only if the sum of the exterior angles is equal to 2π . Formally, the algorithm checks if the following equation holds true, where V is the input sequence of vertices.

$$\sum_{v \in V} \text{ext}(v) = 2\pi$$

3.4 Pseudo Code

The function `ISCONVEX` takes as input a sequence of points V and returns whether or not the points form a convex polygon.

```

function ISCONVEX( $V$ )
  if  $|V| < 3$  then return False
   $s := 0$                                 ▷ The sum of the exterior angles.
  for  $i$  from 0 to  $|V| - 1$  do
     $P := V_{i-1 \bmod |V|}$                 ▷ The previous point.
     $Q := V_i$                           ▷ The current point.
     $R := V_{i+1 \bmod |V|}$                 ▷ The next point.
     $s := s + \angle(\overrightarrow{PQ}, \overrightarrow{QR})$     ▷ Add the exterior angle of  $Q$ .
  return  $s = 2\pi$ 

```

4 Proof of Correctness

4.1 Claim

A sequence of vertices V forms a simple convex polygon if and only if the sum of the exterior angles is equal to 2π .

4.2 Proof

In order to prove our claim, we must prove (a) that if a polygon is simple then the sum of its exterior angles is equal to 2π , and (b) that if a sequence of points' angles sum to 2π , then it must form a convex polygon. We shall commence with the easier of the two.

4.2.1 Convex Polygons' Exterior Angles Sum to 2π

Lemma 1. If a polygon with n sides is convex, then we know that the sum of the interior angles must be $\pi(n - 2)$. (Proof omitted.)

Lemma 2. For every point on a convex polygon, the sum of the interior and exterior angles must be π radians. (Proof omitted.)

Proof. Given a convex polygon with the sequence of vertices V , by lemma 1 we can formulate $|V|$ equations; one for each vertex $v \in V$.

$$\text{int}(v) + \text{ext}(v) = \pi \quad (1)$$

If we take the sum of all equations in this family of equations, we obtain the following equation.

$$\sum_{v \in V} \text{int}(v) + \sum_{v \in V} \text{ext}(v) = n\pi \quad (2)$$

This can then be rearranged into the next equation.

$$\sum_{v \in V} \text{ext}(v) = n\pi - \sum_{v \in V} \text{int}(v) \quad (3)$$

But by lemma 1, we know the sum of the interior angles of the convex polygon.

$$\sum_{v \in V} \text{ext}(v) = n\pi - \pi(n - 2) = n\pi - n\pi + 2\pi = 2\pi \quad (4)$$

4.2.2 Only Convex Polygons' Angles Sum to 2π