THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 74, No. 12 DECEMBER 15, 1948

On the Relative Abundance of the Elements*

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(Received September 1, 1948)

The theory of the neutron capture process for the formation of the elements has been reformulated, taking into account the radioactive decay of neutrons as well as the expansion of the universe. However, the calculations have been carried out for the case of a static universe. Satisfactory agreement between this theory and observed relative abundances has been obtained with the assumption that the density of matter was 5×10^{-9} g/cm^3 at the start of the capture process.

I. INTRODUCTION

THE problem of explaining the observed relative abundance of the elements is of interest both in itself and because it may give information concerning the structure and evolution of the universe. The failure of equilibrium theories[1] to give a reasonably simple explanation of the observed relative abundances, together with the changes in physical conditions which must have accompanied the early stages of the expansion of the universe, led Gamow[2] to the suggestion that a rapid non-equilibrium process was responsible. Recently a non-equilibrium theory of the building up of nuclei based on the successive capture of neutrons was suggested by Alpher, Bethe, and Gamow,[3] and the detailed results of this theory were developed by one of us.[4]

The purpose of this paper is to reformulate the theory of the neutron capture process by taking into account explicitly the radioactive decay of neutrons. The effect of the expansion of the universe on this theory is indicated in the formulation but is not included in the calculations described. This work revises previous estimates as to the physical conditions during the early stages of the expansion. Finally, some of the features of the abundance data resulting from variations in neutron capture cross section are discussed.

II. FORMULATION OF THE PROBLEM

It is suggested that the elements were formed in the following manner. Early in the expansion of the universe, the ylem consisted of neutrons only. During the early stages the universe

* The work described in this paper was supported by the Bureau of Ordnance, U. S. Navy, under Contract NOrd-7386.

[1] The following is a partial list of papers on the equilibrium theory: T. E. Sterne, Mon. Not. R. Astr. Soc. **73**, 736, 767, 770 (1938). C. von Weizsäcker, Phys. Zeits. **39**, 633 (1938). S. Chandrasekhar and L. R. Henrich, Astrophys. J. **95**, 288 (1942). G. Wataghin and P. S. de Toledo, Phys. Rev. **73**, 79 (1948). G. B. van Albada, Bull. Astr. Inst. Neth. Vol. X, No. 374, Sept. 12, 1946. See also Astrophys. J. **105**, 393 (1947). O. Klein, G. Beskow and L. Treffenberg, Ark. Mat. Astr. Fys. **33A**, No. 1 (1946). G. Beskow and L. Treffenberg, Ark. Mat. Astr. Fys. Pt. I, **34A**, No. 13 (1947); Pt. II, **34A**, No. 17 (1947). J. Géhéniau, I. Prigogine, and M. Demeurs, Physica, 's Grav. **13**, 429 (1947). F. Hoyle, Mon. Not. R. Astr. Soc. **106**, 343 (1946).

[2] G. Gamow, Phys. Rev. **70**, 572 (1946).

[3] R. A. Alpher, H. A. Bethe, and G. Gamow, Phys. Rev. **73**, 803 (1948).

[4] R. A. Alpher, Phys. Rev. **74**, 1577 (1948).

TABLE I.*

j	P_j	$(\xi_j/\xi_1)_{max}$
1	5.01	1.000
2	4.62	0.782
3	4.72	0.442
4	4.88	0.204

i	Π_i	$(\xi_i/\xi_1)_{max}$	i	Π_i	$(\xi_i/\xi_1)_{max}$
7	1.16	4.27×10^{-2}	72	97.0	1.57×10^{-10}
12	1.60	3.98×10^{-3}	77	137	8.82×10^{-11}
17	2.21	4.08×10^{-4}	82	193	5.25×10^{-11}
22	3.11	4.73×10^{-5}	87	273	3.27×10^{-11}
27	4.37	6.47×10^{-6}	92	385	2.11×10^{-11}
32	6.16	1.05×10^{-6}	104.5	134	1.96×10^{-11}
37	8.70	2.03×10^{-7}	124.5	134	1.44×10^{-11}
42	12.3	4.66×10^{-8}	144.5	134	1.10×10^{-11}
47	17.3	1.27×10^{-8}	164.5	134	8.38×10^{-12}
52	24.4	4.09×10^{-9}	184.5	134	6.34×10^{-12}
57	34.4	1.52×10^{-9}	204.5	134	
62	48.6	6.41×10^{-10}	224.5	134	
67	68.7	3.03×10^{-10}	244.5	134	

* The index i designates the center element of the group of five or twenty which the element represents.

expanded at a rate controlled by the density of radiation since this was very high compared to the density of the ylem. As the expansion proceeded, the temperature eventually dropped sufficiently so that thermally stable nuclei could be formed. Neutrons underwent radioactive decay into protons and electrons. The capture of neutrons by these protons led to deuterons, neutron capture by deuterons led to tritons, etc.

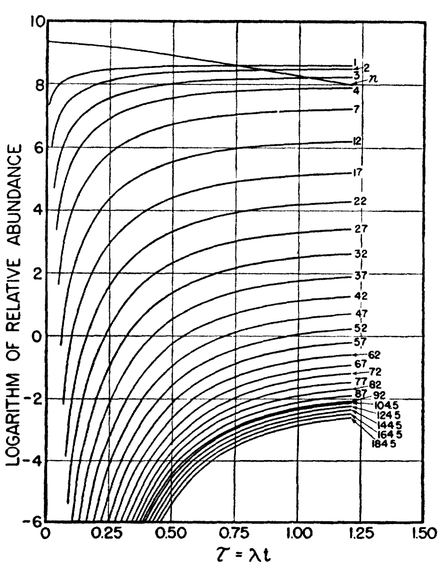

FIG. 1. Logarithm of the relative abundance *versus* $\tau = \lambda t$. The numbers on the right hand side refer to the nuclear species or group. The curve labeled n shows the change in neutron concentration.

Successive neutron captures led to the heavier elements with intervening β-decay processes adjusting the neutron-proton ratios toward stable values. If this sequence of events was the process responsible for forming the elements, it seems clear that the capture cross sections for fast neutrons of the elements must have played an important role in determining the present relative abundances. The reduction of neutron concentration as a result of their decay into protons and the dilution of all matter by the universal expansion must have controlled and terminated the element-forming process.

In the previous development of a neutron capture theory,[4] it was assumed that the effect of the expansion could be neglected, that the rate of deuteron formation was constant, and that all processes other than radiative capture of neutrons could be ignored. It was implicitly assumed that there was sufficient time between successive neutron captures for the necessary adjustment of nuclei by β-decay. In this paper the neutron-capture process is reformulated. The assumptions made in the previous paper[4] are retained with the exception that neutron decay and the universal expansion are explicitly introduced. In calculations of this reformulated theory, however, the expansion once again has been neglected because of computational difficulties.

Let V be any finite volume element in the universe, and let N_j be the number of nuclei of species j in that volume, where j represents the number of nucleons. Let n_j be given by $n_j = N_j/V$. It is assumed that the composition of the universe is always homogeneous. Then, we have

$$\frac{dn_j}{dt} = \frac{1}{V}\frac{dN_j}{dt} - \frac{n_j}{V}\frac{dV}{dt}. \tag{1}$$

It has been shown[4] that in a cosmological model consisting of a homogeneous, isotropic, perfect fluid the density of matter ρ_m varies, during the early stages of the expansion, as

$$\rho_m \cong K t^{-\mu}. \tag{2}$$

If the universe contains matter only, the exponent μ is found to be 2; if there is a trace of matter in a universe consisting principally of radiation, the exponent is $\frac{3}{2}$. Since $\rho_m = M/V$

where M is the mass of matter in the volume element V, we have

$$\frac{1}{\rho_m}\frac{d\rho_m}{dt}=-\frac{1}{V}\frac{dV}{dt}=-\frac{\mu}{t}. \quad (3)$$

Substituting Eq. (3) into Eq. (1) we have

$$dn_j/dt=(1/V)(dN_j/dt)-(\mu n_j/t). \quad (4)$$

The change in the total number of neutrons, N_n, with time is given by

$$\frac{dN_n}{dt}=-\lambda N_n-\sum_{j=1}^{J}p_jn_nN_j, \quad (5)$$

where λ is the neutron decay constant, p_j is the effective neutron capture volume swept out per second by nuclei of species j and where J is the total number of nuclear species differing only in atomic weight A. In Eq. (5) the first term represents the radioactive decay of neutrons, while the summation term represents the rate at which neutrons are captured to form the various nuclei other than protons. Substituting Eq. (5) into Eq. (4), using $\tau=\lambda t$ as the independent variable and dividing the resulting equation by n_0, which is the particle concentration at time τ_0, we obtain,

$$\frac{d\xi_n}{d\tau}=-\left(1+\frac{\mu}{\tau}\right)\xi_n-\sum_{j=1}^{J}P_j\xi_n\xi_j, \quad (6)$$

where

$$\xi_j=n_j/n_0, \quad (6a)$$

and

$$P_j=p_jn_0/\lambda. \quad (6b)$$

The equation describing proton formation is given by,

$$d\xi_1/d\tau=\xi_n-P_1\xi_n\xi_1-(\mu\xi_1/\tau), \quad (7)$$

where it is seen that protons are formed by neutron decay and are used up only in deuteron formation by neutron capture. In general

$$(d\xi_j/d\tau)=P_{j-1}\xi_n\xi_{j-1}-P_j\xi_n\xi_j-(\mu\xi_j/\tau). \quad (8)$$

Equation (8) states that nuclei of species j are formed through the capture of neutrons by nuclei of the preceding species $j-1$, and that nuclei of species j are changed into the succeeding species $j+1$ by still another neutron capture.

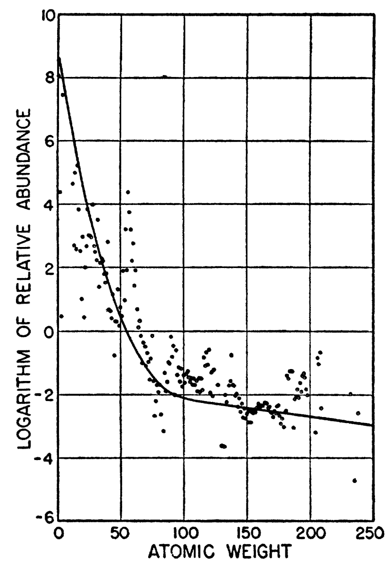

FIG. 2. Comparison of theoretical and observed relative abundances *versus* atomic weight. The observed data are those given by Harrison S. Brown. The theoretical curve corresponds to a matter density of 5×10^{-9} g/cm^3 at the start of the element forming process.

In addition, the concentration of all species is diminished by the expansion of the universe according to the term $-\mu\xi_j/\tau$. Equations (6), (7) and (8) constitute a set of $J+1$ equations which should be solved simultaneously.

The computational difficulties associated with the above set of $J+1$ equations has led us to simplify these equations in the following manner:

$$\begin{aligned}
&d\xi_n/d\tau=-\xi_n-P_1\xi_n\xi_1-P_2\xi_n\xi_2-P_3\xi_n\xi_3-P_4\xi_n\xi_4,\\
&d\xi_1/d\tau=\xi_n-P_1\xi_n\xi_1,\\
&d\xi_2/d\tau=P_1\xi_n\xi_1-P_2\xi_n\xi_2,\\
&d\xi_3/d\tau=P_2\xi_n\xi_2-P_3\xi_n\xi_3,\\
&d\xi_4/d\tau=P_3\xi_n\xi_3-P_4\xi_n\xi_4 \qquad (9)\\
&\quad\vdots\\
&d\xi_j/d\tau=P_{j-1}\xi_n\xi_{j-1}-P_j\xi_n\xi_j.
\end{aligned}$$

In Eqs. (9) the terms of the type $-\mu\xi_j/\tau$, which represent the dilution of matter resulting from the expansion of the universe, have been neglected. In order to simplify the computations, only the first five of Eqs. (9) were solved simultaneously and the remaining equations were grouped in sets representing five or twenty nuclear species each and solved in succession.

Neglecting the summation term

$$\sum_{j=5}^{J} P_j \xi_n \xi_j$$

in the neutron equation does not materially affect the remainder of the computation because of the almost exponential decrease in relative abundance with increasing atomic weight. The significance of the grouping into sets of say q nuclear species is that the representative species of the ith group must capture on the average q neutrons to form the $(i+1)$st group. The equation for the change in concentration, ξ_i, of the representative element of the ith group containing q species is given by

$$d\xi_i/d\tau = \Pi_{i-1}\xi_n\xi_{i-1} - \Pi_i\xi_n\xi_i, \qquad (10a)$$

where

$$\Pi_i = \sum{}' P_j/q^2, \qquad (10b)$$

in which the prime indicates summation over the q species in the ith group. In the calculations the group size q was taken to be 5 for $4 < A < 96$ and 20 for $A \geqq 96$. The quantities p_j are given by[4]

$$p_j = p_A = 1.4\times 10^{-19+0.03A}[(1+A)/A]^{\frac{1}{2}}\ \text{cm}^3/\text{sec.}, \quad A<100, \qquad (11a)$$

and

$$p_A = 1.4\times 10^{-16}\ \text{cm}^3/\text{sec.}, \quad A>100, \qquad (11b)$$

where the capture cross sections of the elements, which do not appear explicitly, are given by[4]

$$\log_{10}(\sigma E^{\frac{1}{2}}) = 0.03A - 1.00, \quad A<100, \qquad (12a)$$

and

$$\log_{10}(\sigma E^{\frac{1}{2}}) = 2, \quad A>100. \qquad (12b)$$

In Eqs. (12), σ is in barns, and E is the collision energy in ev. It is evident from the exponential increase in σ with increasing A in Eqs. (12) that using an arithmetic average in computing the Π_i, given by Eq. (10b), is at best a first approximation. Having the p_j from Eq. (11), the P_j may be determined from Eq. (6b) by using the present value of λ, the neutron decay constant, and appropriately choosing n_0, the concentration of particles at the start of the process. The neutron decay constant is given by the Fermi theory[5] as 15 minutes, whereas recent experiments indicate it is probably of the order of $\frac{1}{2}$ hour.[6] The latter figure has been used in our calculations. There does not seem to be any basis on which to choose a value of n_0 other than the fit of theory to the observed relative abundance data, and, as a matter of fact, n_0 should be regarded as an arbitrary parameter. As will be discussed later, a value of $n_0 = 3.23\times 10^{15}$ cm^{-3} appears to give a good fit. Values of the P_j for $j = 1, \cdots, 4$ and the Π_i are given in Table I.

As initial conditions for the integration, it has been assumed that the process started at $\tau_0 = 0$, with $\xi_n = 1$, and $\xi_j = 0$ for $j \neq n$. It is to be understood that $\tau_0 = 0$ is considered the starting time of the element-forming process, and is not the starting time of the expansion. It can be shown that on the latter time scale the process probably started at about 250 seconds.

III. RESULTS AND DISCUSSION

Integration of Eqs. (9), with grouping according to Eq. (10b), has been carried out with several choices of the parameter n_0 to sufficiently high atomic weights to enable the choice of a best n_0 value. The integrations were done by the lineal element method employing intervals in $\tau = \lambda t$ of 0.01. As described previously, the neutron equation together with the equations for the first four nuclear species were solved simultaneously. The best value of n_0 was found to be $n_0 = 3.23\times 10^{15}$ cm^{-3} and is therefore to be regarded as the initial concentration of neutrons. The integration with this value of n_0 was then carried to the group $i = 184.5$. The integrations were not performed for higher atomic weights because it was found that a plot of the logarithms of the relative abundances *versus* atomic weight was a straight line beyond $A\sim 100$. It was found in the integration that a slight irregularity resulted in the vicinity of the change in group size from 5 to 20, which has been ignored. The results are plotted as the logarithm of the relative abundance *versus* τ in Fig. 1. It should be noted that ξ_n, ξ_1, ξ_2, ξ_3, ξ_4 are the result of simultaneous solution of the first 5 of Eqs. (9), while ξ_7, ξ_{12}, etc. are the group representative elements. The indices 7, 12, etc. are the center

[5] See H. A. Bethe, *Elementary Nuclear Physics* (John Wiley and Sons, Inc., New York, 1947), p. 105.

[6] A. H. Snell and L. C. Miller, Bull. Am. Phys. Soc. **23**, 21 (1948).

points of the groups of elements. It may be seen in Fig. 1 that the individual ξ_j reach limiting values $(\xi_j)_{\max}$ after a τ of about 1.2. The curves attain the limiting values $(\xi_j)_{\max}$ given in Table I, simply because the neutrons are used up by radioactive decay and by capture. As the atomic weight increases, the time, τ, required to reach $(\xi_j)_{\max}$ becomes progressively greater. However, the error involved in taking $(\xi_j)_{\max}$ at a τ of about 1.2 does not exceed several percent for the elements of highest atomic weight

Values of $(\xi_j)_{\max}$ taken at $\tau=1.21$ from Fig. 1 are, after appropriate normalization, the relative concentrations or relative abundances, as computed according to the neutron capture theory. These limiting values have been plotted as $[\log(\xi_j)_{\max}+\text{const}]$ *vs* atomic weight, $j=A$, in Fig. 2, for comparison with the observed relative abundance data.[7] A smooth curve is drawn in Fig. 2 rather than individually computed points, in order to avoid confusion with the abundance data. The theoretical abundances computed in this paper give a better representation of the observed data than those previously reported[4] in that for atomic weights greater than 100 the theoretical values do not drop off as rapidly with increasing atomic weight. This results from the present calculations leading to abundances at limiting values rather than at an arbitrarily chosen time prior to the attainment of limiting values. In judging the goodness of representation it should be noted that the observed abundance data may be no better than plus or minus about one unit on the logarithmic plot of Fig. 2.

In a previous paper[4] it was pointed out that a correlation existed between neutron capture cross sections for given elements and the relative abundances of these elements. This correlation may be expressed as

$$\log\sigma = C_1 \log\alpha + C_2, \tag{13}$$

where α is relative abundance. If one combines Eq. (13) with Eqs. (12), one can derive for $A<100$ a relationship of the form

$$\log\alpha = C_3 A + C_4, \tag{14a}$$

and for $A>100$

$$\log\alpha = \text{constant}. \tag{14b}$$

If the constant C_1 in Eq. (13) is evaluated from the observed data, then one arrives at values for the constants in Eqs. (14) such that the two lines given by Eqs. (14) are a satisfactory first approximation to the relative abundances of those elements involved in obtaining the constants of Eq. (13).

Some of the detailed features of the observed abundances are probably obscured by the scatter of data. Certain details are nevertheless evident. These are the low abundance of the Li, Be and B isotopes,[7] the large abundance in the vicinity of iron, and the apparent series of peaks for those elements whose atomic weights correspond to the "magic number" nuclei.[4,8] The anomalous behavior of Li, Be and B is probably the result of high cross sections for proton reactions at relatively low temperatures.[4,9] No reasonable explanation has as yet been found for the large abundance peak in the vicinity of iron. The "magic number" nuclei show markedly smaller neutron capture cross sections than other nuclear species, and, in terms of a neutron-capture theory, should exhibit relatively high abundance. This is because they would tend to form only a small amount of the succeeding nuclei. Quantitative examination of this behavior indicates that the element with small σ does indeed pile up, but the abundances of the succeeding elements

[7] Harrison S. Brown, Rev. Mod. Phys. (in press). With the exception of those nuclei below oxygen in atomic weight, these data are relative abundances in the universe as a whole, stated as the number of nuclei of a given atomic weight per 10,000 atoms of silicon. Possibly because of thermonuclear reactions in stars, there appear to be differences from star to star in the relative abundances of isotopes among the lighter elements. As a result, Brown has given only elemental abundances for these lighter elements, and the precision of these data is much less than for elements above oxygen. Despite their uncertainty, these elemental abundances have been converted to nuclear abundances by using the isotopic abundance ratios of the Segrè table, and are included in Fig. 2. Relative abundances of the noble gases A, Ne, Kr, and Xe are not sufficiently well fixed, and are not given by Brown. Brown has not reported relative abundance data on Li, Be, and B. However, relative abundance data previously given by Goldschmidt (reference 4) indicate these elements to be very much less abundant than elements of neighboring atomic weights. In those cases where Brown has given upper and lower limits for the relative abundance of a particular species, the average value has been used.

With regard to the precision of the data, Brown has stated that with the exception of the light elements, the relative abundance data are good to within a factor of about 4. Fluctuations for certain individual elements may exceed a factor of 10 because of experimental difficulties in the measurement of their relative abundance.

[8] M. G. Mayer, Phys. Rev. **74**, 235 (1948).

[9] H. A. Bethe, Phys. Rev. **55**, 434 (1939).

are greatly depressed. The depression is so large that probably the abundances did not derive entirely in this manner. It is suggested that while certain elements exhibiting small σ may have piled up, the succeeding elements may have been built up in large part through neutron capture by nuclei of normal cross section which were isobars of those nuclei exhibiting small σ.

The value of $n_0 \cong 3\times 10^{15}$ cm^{-3} corresponds to a density of matter at the start of the process of the order of 5×10^{-9} g/cm^3. While it has not been necessary to specify explicitly the temperature at which the process of successive neutron captures started, the temperature must have been sufficiently low such that thermal dissociation of nuclei could be neglected. On the other hand, the temperature must have been sufficiently high such that resonance effects in the capture of neutrons would not occur. The latter statement follows from the fact that small abundances are not observed for nuclei having large resonance cross sections. A temperature of about 10^5 ev$\cong 10^9$°K is suggested.[4]

The value of n_0 which has been obtained in this study should be considered as a lower limit because, (1) the introduction of small cross sections for certain elements in the calculations would depress the computed abundance values, and therefore require a higher value of n_0, and (2) the introduction of the universal expansion would also require a higher starting value of n_0 to overcome the dilution of the ylem. It is the nature of the solutions of Eqs. (9) that too high a value of n_0 leads to theoretical relative abundances that are too high and vice versa.

We are undertaking the solution of Eqs. (6)–(8), which include the expansion, in the hope that a better value of the starting density of matter may be obtained and that through this knowledge of the early state and evolution of the universe may be improved.

If the early universe consisted of matter only, then a starting density of matter as low as $\rho_0 \cong 5\times 10^{-9}$ g/cm^3 would have been attained quite late in the expansion, in fact, at a time very long compared to the neutron life time. With this state of affairs there would not have been a sufficient concentration of neutrons available to build the elements according to the process which has been described. Thus, the starting time of the neutron capture process must have been small compared to the neutron life time. We have seen that the temperature at the start of the process must have been of the order of 10^9°K and it appears that the density of matter must have been no more than several orders of magnitude greater than 10^{-9} g/cm^3. Since 10^9°K corresponds to a density of radiation of the order 1 g/cm^3 the behavior of the universe during the times of interest here must have been controlled by radiation. For a universe of essentially radiation only a temperature of about 10^9°K is reached in the expansion at about 250 seconds, which is satisfactorily short compared to the neutron life time.

Calculations of the relative abundances of the elements according to the neutron capture theory lead to an estimate of the temperature and density of matter and radiation at a specific epoch in the evolution of the universe. It is noteworthy that recently Gamow[10] has shown that a knowledge of these conditions yields a reasonable picture of the manner in which the galaxies were formed and has estimated their size, separation and density.

IV. ACKNOWLEDGMENTS

We wish to thank Miss Kathryn E. Pace for invaluable assistance on the calculations and Miss Shirley L. Thomas for aid in preparation of the manuscript. We are indebted to Dr. G. Gamow for many stimulating discussions. We also wish to thank Dr. Harrison S. Brown for his cooperation in giving us his relative abundance data prior to publication.

[10] G. Gamow, Nature (in press).