MATHEMAGICAL FORMULAS FOR SYMMETRIC FUNCTIONS

Contents

The Bibliography

[1] François Bergeron. *Algebraic combinatorics and coinvariant spaces*. CMS Treatises in Mathematics. Canadian Mathematical Society, Ottawa, ON ; A K Peters, Ltd., Wellesley, MA, 2009.

[2] James Haglund. *The q,t-Catalan numbers and the space of diagonal harmonics*, volume 41 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. With an appendix on the combinatorics of Macdonald polynomials.

[3] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[4] I.G. Macdonald. A new class of symmetric functions. *Séminaire Lotharingien de Combinatoire*, 20(B20a) :131–171, 1988.

[5] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[6] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

See also

[a] `https://en.wikipedia.org/wiki/Partition_(number_theory)`

[b1] `https://en.wikipedia.org/wiki/Symmetric_polynomial`

[b2] `https://en.wikipedia.org/wiki/Ring_of_symmetric_functions`

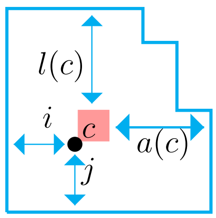
[c] `https://en.wikipedia.org/wiki/Young_tableau`

1. Basic Notations

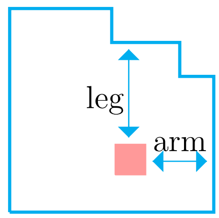
Partitions

$\mu \vdash n$ iff $\mu = \mu_1, \cdots, \mu_k$; $\mu_1 \geq \cdots \geq \mu_k > 0$; and $n = \sum \mu_j := |\mu|$. $\ell(\mu) = k$.

μ :

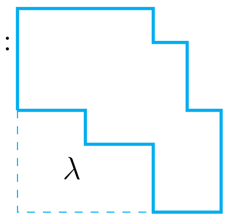


μ' :



$1^n = \underbrace{1, \cdots, 1}_{n \text{ times}}$:

μ/λ :



$$l_\mu(c) = l(c) := {\mu'}_{i+1} - (j+1); \quad a_\mu(c) = a(c) := \mu_{j+1} - (i+1); \quad h_\mu(c) = h(c) := a(c) + l(c) + 1$$

(1) $z_\mu := \prod_{i=1} i^{d_i} d_i!$ for $\mu = 1^{d_1} \cdots n^{d_n}$ (2) $n(\mu) := \sum_{c \in \mu} l(c) = \sum_{(i,j) \in \mu} j$ and $n(\mu') := \sum_{c \in \mu} a(c) = \sum_{(i,j) \in \mu} i$

Example 1 clic here Wikipedia page on this

Tableaux

$\mu = \mu_1, \cdots, \mu_k \vdash n$ iff $\mu \subset \mathbb{N} \times \mathbb{N}$; $\mu = \{\, c \mid c = (i,j),\ 0 \leq j \leq \ell(\mu) - 1;\ 0 \leq i \leq \mu_{j+1} - 1 \,\}$; $\sum \mu_j = n$.

Tableau
$\tau : \mu \to \{1, 2, \cdots, n\}$

Semi-Standard Tableau
$\tau(a,j) < \tau(b,j) \Rightarrow a \leq b$
and $\tau(i,c) < \tau(i,d) \Rightarrow c < d$

Standard Tableau $\mathbf{f}^\mu$
τ bijection, $\tau(a,j) < \tau(b,j) \Rightarrow a < b$
and $\tau(i,c) < \tau(i,d) \Rightarrow c < d$

Hook lenght formula :

(3) $$f^\mu = \frac{n!}{\prod_{c \in \mu} h(c)}$$

(4) $$\sum_{\mu \vdash n} (f^\mu)^2 = n!$$

Determinental formula :

(5) $$f^\mu = n! \det \left(\left(\frac{1}{(\mu_i - i + j)!} \right)_{i,j} \right)$$

Example 2 clic here Wikipedia page on this

2. Classical Basis of $\mathbf{\Lambda}$

$\mathbf{\Lambda} = \mathbf{\Lambda}_\mathbb{Q}$: ring of symmetric functions; $\boldsymbol{x} := \{x_1, x_2, x_3, \cdots\}$. Basis are indexed by partitions, $g = g(\boldsymbol{x})$.

Monomial symetric functions

(6) $$m_\mu := \sum_{\substack{i_1, \cdots, i_k \in \mathbb{N}^* \\ \text{distinct}}} x_{i_1}^{\mu_1} x_{i_2}^{\mu_2} \cdots x_{i_k}^{\mu_k}$$

Power sum symmetric functions :

(7) $$p_n := \sum_{i \in \mathbb{N}} x_i^n = m_{(n)} \quad \text{and} \quad p_\mu := p_{\mu_1} \cdots p_{\mu_k}$$

Complete homogeneous symmetric functions

(8) $$h_n := \sum_{\lambda \vdash n} m_\lambda \quad \text{and} \quad h_\mu := h_{\mu_1} \cdots h_{\mu_k}$$

where $h_0 = e_0 = 1$ and $e_k = h_k = 0$ for all $k < 0$.

Elementary symmetric functions

(9) $$e_n := \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} = m_{1^n} \text{ and } e_\mu := e_{\mu_1} \cdots e_{\mu_k}$$

Schur symmetric functions (Jacobi-Trudi determinant formulas)

(10) $$s_\mu = \det\left((h_{\mu_i - i + j})_{i,j}\right); \quad s_{(n)} = h_n$$

(11) $$s_{\mu'} = \det\left((e_{\mu_i - i + j})_{i,j}\right); \quad s_{1^n} = m_{1^n} = e_n$$

Example 3 clic here Wikipedia page on this

3. Generating Functions and Identities

Generating functions

$$(12)\qquad E(t) := \sum_{r\geq 0} e_r t^r = \prod_{i\geq 1}(1+x_i t)$$

$$(13)\qquad H(t) := \sum_{r\geq 0} h_r t^r = \prod_{i\geq 1}\frac{1}{(1-x_i t)} = \frac{1}{E(-t)}$$

$$\Rightarrow \quad (14)\qquad \sum_{r=0}^{n}(-1)^r e_r h_{n-r} = 0$$

$$(15)\quad P(t) := \sum_{r\geq 0} p_r t^r = \frac{\mathrm{d}}{\mathrm{d}t}\log H(t) = \frac{H'(t)}{H(t)} \Leftrightarrow H(t) = e^{P(t)} \qquad \Rightarrow \quad (16)\qquad nh_n = \sum_{r=1}^{n} p_r h_{n-r}$$

$$(17)\quad P(-t) := \sum_{r\geq 0} p_r t^r = \frac{\mathrm{d}}{\mathrm{d}t}\log E(t) = \frac{E'(t)}{E(t)} \Leftrightarrow E(t) = e^{-P(-t)} \qquad \Rightarrow \quad (18)\quad ne_n = \sum_{r=1}^{n}(-1)^{r-1} p_r e_{n-r}$$

Wikipedia page on this

Changing basis

$$(15) \Rightarrow \quad (19)\quad h_n(\boldsymbol{x}) = \sum_{\mu\vdash n}\frac{p_\mu(\boldsymbol{x})}{z_\mu} \qquad (17) \Rightarrow \quad (20)\quad e_n(\boldsymbol{x}) = \sum_{\mu\vdash n}\frac{(-1)^{n-\ell(\mu)}p_\mu(\boldsymbol{x})}{z_\mu},$$

see also 29 and 30

The $\boldsymbol{\omega}$ linear map

$$\left.\begin{aligned}\omega : \boldsymbol{\Lambda} &\to \boldsymbol{\Lambda},\\ p_n &\mapsto (-1)^{n-1}p_n\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\omega^2(g(\boldsymbol{x})) &= g(\boldsymbol{x}),\ \forall g\in\boldsymbol{\Lambda};\\ \omega(s_\mu) &= s_{\mu'};\end{aligned}\right.$$

Changing basis

$$(21)\qquad \omega(h_n) = e_n;$$

(22) $\omega(m_\mu) = f_\mu, \{f_\mu\}$ is the **forgotten base**

Scalar procduct

$$(23)\qquad \langle p_\mu, p_\lambda\rangle := z_\mu\delta_{\mu,\lambda}$$

$$\langle g, d\rangle = \langle\omega(g),\omega(d)\rangle\ \forall d,g\in\boldsymbol{\Lambda}.$$

Cauchy Kernel

$$(24)\qquad \Omega(\boldsymbol{xy}) := \prod_{i\geq 1}\frac{1}{(1-x_i y_j)}$$

$\{f_\mu\}$ and $\{g_\mu\}$ dual basis iff

$$\langle d_\mu, g_\lambda\rangle = \delta_{\mu,\lambda} \quad \text{or}$$

$$\Omega(\boldsymbol{xy}) = \sum_\mu d_\mu(\boldsymbol{x})g_\mu(\boldsymbol{y})$$

$$(25)\quad \Omega(\boldsymbol{xy}) = \sum_n h_n(\boldsymbol{xy}) = \sum_\mu s_\mu(\boldsymbol{x})s_\mu(\boldsymbol{y}) = \sum_\mu h_\mu(\boldsymbol{x})m_\mu(\boldsymbol{y}) = \sum_\mu e_\mu(\boldsymbol{x})f_\mu(\boldsymbol{y}) = \sum_\mu p_\mu(\boldsymbol{x})\frac{p_\mu(\boldsymbol{y})}{z_\mu}$$

4. Frobenius transform and Hilbert series

Cyclic structure

$$\lambda(\sigma) = \lambda_1,\cdots,\lambda_k \text{ iff } \sigma = (\sigma_1,\cdots,\sigma_{\lambda_1})\cdots(\sigma_{\lambda_1+\cdots+\lambda_{k-1}+1},\cdots,\sigma_{|\lambda|})$$

Cyclic type

$$\sigma = \sigma_\mu \text{ iff } \lambda(\sigma) = \mu$$

Class functions

$$(26)\quad R(\mathbb{S}_n) := \{\chi : \mathbb{S}_n \to \mathbb{C} \mid \chi(\sigma) = \chi(\tau\sigma\tau^{-1}),\ \forall\tau\in\mathbb{S}_n\};$$

Characters

$$\chi_V + \chi_W = \chi_{V\oplus W} \quad \text{and} \quad \chi_V\chi_W = \chi_{V\otimes W}$$

Frobenius transform $\mathcal{F}$ (of an $\mathbb{S}_n$-module V)

$$(27)\quad \mathcal{F}(V) = \mathcal{F}(\chi_V) := \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n}\frac{1}{n!}\chi_V(\sigma)p_{\lambda(\sigma)} = \sum_{\lambda\vdash n}\frac{1}{z_\mu}\chi_V(\sigma_\lambda)p_\lambda \qquad \Rightarrow \qquad \mathcal{F}(V\oplus W) = \mathcal{F}(V)+\mathcal{F}(W)$$

Changing basis

(28) $$\mathcal{F}(\chi^\mu) = \sum_{\lambda\vdash n} \frac{1}{z_\mu}\chi^\mu(\sigma_\lambda)p_\lambda = s_\mu,$$

where χ^μ is an irreductible character.

(29) $$\mathcal{F}(\chi_{1_{\mathbb{S}_n}}) = \sum_{\lambda\vdash n} \frac{1}{z_\mu}p_\lambda = h_n,$$

where $1_{\mathbb{S}_n}$ is the trivial representation.

(30) $$\mathcal{F}(\chi_{\mathrm{Sign}_{\mathbb{S}_n}}) = \sum_{\lambda\vdash n} \frac{1}{z_\mu}\chi_{\mathrm{Sign}_{\mathbb{S}_n}}(\sigma_\lambda)p_\lambda = \sum_{\mu\vdash n}\frac{1}{z_\mu}(-1)^{n-\ell(\mu)}p_\mu = e_n,$$

where $\mathrm{Sign}_{\sim_n}$ is the sign representation.

Note 29 is equivalent to 19 and 30 is equivalent to 20.

Graded Frobenius characteristic

(31) $$\mathrm{Frob}_q(V) := \sum_{n\geq 1}\mathcal{F}(V_n)q^n,$$

where $V = \oplus_{n\geq 1}V_n$ is a graded $\mathbb{S}_n$-module.

Bigraded Frobenius characteristic

(32) $$\mathrm{Frob}_{q,t}(V) := \sum_{n\geq 1}\mathcal{F}(V_{n,k})q^n t^k,$$

where, $V = \oplus_{n,k\geq 1}V_{n,k}$ is a bigraded $\mathbb{S}_n$-module.

Hilbert series (poincaré series)

(33) $$\mathrm{Hilb}_q(V) := \sum_{n\geq 1}\dim(V_n)q^n,$$

where $V = \oplus_{n\geq 1}V_n$ is a graded space.

Bigraded Hilbert series

(34) $$\mathrm{Hilb}_{q,t}(V) := \sum_{n\geq 1}\dim(V_{n,k})q^n t^k,$$

where, $V = \oplus_{n,k\geq 1}V_{n,k}$ is a bigraded space.

5. Plethysm (λ-rings)

plethysm is defined by :

(35) $p_n[\boldsymbol{x}+Y] = p_n[\boldsymbol{x}] + p_n[Y],$

(36) $p_n[\boldsymbol{x}Y] = p_n[\boldsymbol{x}]p_n[Y]$

(37) $p_n[x] = x^n$ therefor $p_n[p_k(\boldsymbol{x})] = p_{nk}(\boldsymbol{x}),$

(38) $p_n[c] = c,$ if c is a constant,

$p_n[q\boldsymbol{x}] = q^n p_n(\boldsymbol{x})$

$p_n[t\boldsymbol{x}] = t^n p_n(\boldsymbol{x})$

Example 4 clic here

6. Macdonald symmetric functions

More scalar product

... for original Macdonald polynomials

(39) $$\langle p_\mu, p_\lambda\rangle_{q,t} = z_\mu\delta_{\lambda,\mu}\prod_{i=1}^{\ell(\mu)}\frac{1-q^{\mu_i}}{1-t^{\mu_i}}$$

...for combinatorial Macdonal polynomials

(40) $$\langle p_\mu, p_\lambda\rangle_{*} = (-1)^{|\mu|-\ell(\mu)}z_\mu\delta_{\lambda,\mu}\prod_{i=1}^{\ell(\mu)}(1-q^{\mu_i})(1-t^{\mu_i})$$

(41) $$\langle H_\mu, H_\lambda\rangle_* = \mathcal{E}_\mu(q,t)\mathcal{E}'_\mu(q,t)\delta_{\lambda,\mu}, \text{ where } \mathcal{E}_\mu(q,t) = \prod_{c\in\mu}(q^{a(c)} - t^{l(c)+1}) \text{ and } \mathcal{E}'_\mu(q,t) = \prod_{c\in\mu}(t^{l(c)} - q^{a(c)+1})$$

Cauchy formula for the H_μ

(42) $$e_n\left[\frac{\boldsymbol{x}\boldsymbol{y}}{(1-q)(1-t)}\right] = \sum_{\mu\vdash n}\frac{H_\mu(\boldsymbol{x};q,t)H_\mu(\boldsymbol{y};q,t)}{\mathcal{E}_\mu(q,t)\mathcal{E}'_\mu(q,t)}$$

Original Macdonald polynomials
(Gram-Schmidt of the monomial basis in respect to $\langle\cdot,\cdot,\rangle_{q,t}$)

(43) $$P_\mu(\boldsymbol{x};q,t) = m_\mu + \sum_{\gamma\prec\mu} u_\gamma(q,t) m_\gamma$$

Combinatorial Macdonald polynomials

(44) $$H_\mu(\boldsymbol{x};q,t) = P_\mu\left[\frac{\boldsymbol{x}}{1-t};q,t^{-1}\right]\prod_{c\in\mu}(q^{a(c)}-t^{l(c)+1})$$

Example 5 clic here

(45) $$H_\mu(\boldsymbol{x};q,1) = \prod_i H_{\mu_i}(\boldsymbol{x};q,1)$$

(46) $$H_\mu(\boldsymbol{x};q,t) = H_{\mu'}(\boldsymbol{x};t,q)$$

(47) $$H_n(\boldsymbol{x};q,1) = h_n\left[\frac{\boldsymbol{x}}{1-q}\right]\prod_{i=1}^n(1-q^i)$$

(48) $$H_n(\boldsymbol{x};q,1) = e_n\left[\frac{\boldsymbol{x}}{1-q}\right]\prod_{i=1}^n(1-q^i)$$

(49) $H_\mu(\boldsymbol{x};q,t) = \mathrm{Frob}_{q,t}(\mathcal{M}_\mu)$, where $\mathcal{M}_\mu = \mathbb{C}\{\delta\boldsymbol{x}^\alpha\delta\boldsymbol{y}^\beta\Delta_\mu(\boldsymbol{x},\boldsymbol{y}) \mid \alpha,\beta\in\mathbb{N}^n\}$ is a Garcia-Haiman module and $\Delta_\mu = \det(x_k^i y_k^j)_{\substack{1\le k\le n\\ (i,j)\in\mu}}$.

Specialisation

(50) $$H_\mu(\boldsymbol{x};0,0) = s_n$$

(51) $$H_\mu(\boldsymbol{x};0,1) = h_\mu$$

(52) $$H_\mu(\boldsymbol{x};1,1) = s_{1^n}$$

(q,t)**-Kostka polynomials** $\boldsymbol{K_{\lambda,\mu}(q,t)}$

(53) $H_\mu(\boldsymbol{x};q,t) = \sum_{\lambda\vdash|\mu|} K_{\lambda,\mu}(q,t)s_\lambda(\boldsymbol{x})$, where $K_{\lambda,\mu}(q,t)\in\mathbb{N}[q,t]$.

(54) $$K_{\lambda,\mu}(q,t) = K_{\lambda,\mu'}(t,q)$$

(55) $$K_{\lambda,\mu}(q,t) = q^{n(\mu')}t^{n(\mu)}K_{\lambda',\mu}(q^{-1},t^{-1})$$

(56) $$K^{-1}_{\lambda,\mu}(t,q) = K^{-1}_{\lambda',\mu}(q,t)$$

7. Macdonald Operators

The ∇ operator

(57) $$\nabla(H_\mu) := q^{n(\mu')}t^{n(\mu)}H_\mu$$

(58) $\nabla(\boldsymbol{\Lambda}_{\mathbb{Z}[q,t]}) \subseteq \boldsymbol{\Lambda}_{\mathbb{Z}[q,t]}$ and $\nabla^{-1}(\boldsymbol{\Lambda}_{\mathbb{Z}[q,t]}) \subseteq \boldsymbol{\Lambda}_{\mathbb{Z}[q,t,1/q,1/t]}$

(59) $\nabla(e_n) = \mathrm{Frob}_{q,t}(\mathcal{DH}_n)$, where $\mathcal{DH} = \{f\in\mathbb{C}[\boldsymbol{x},\boldsymbol{y}] \mid p_{h,k}(\delta\boldsymbol{x},\delta\boldsymbol{y})f(\boldsymbol{x},\boldsymbol{y}) = 0, \forall h,k \text{ s.t. } h+k>0\}$ is the diagonal harmonic space.

(60) $\nabla(e_n)|_{t=1} = \sum_{\gamma\in\mathcal{D}_{n,n}} q^{\mathrm{area}(\gamma)}e_{\rho(\gamma)}$, see figure 2.

(61) $$\langle\nabla(e_n), en\rangle = C_n(q,t)$$

Example 6 clic here

The $\boldsymbol{\Delta}_F$ operators

(62) $$B_\mu := \sum_{(i,j)\in\mu} q^i t^j$$

(63) $$\Delta_F H_\mu(\boldsymbol{x};q,t) := F[B_\mu]H_\mu(\boldsymbol{x};q,t)$$

(64) $$\Delta_F(\boldsymbol{\Lambda}_{\mathbb{Z}[q,t]}) \subseteq \boldsymbol{\Lambda}_{\mathbb{Z}[q,t]}$$

(65) $$\Delta_{FG} = \Delta_F\circ\Delta_G$$

(66) $$\Delta_{F+G} = \Delta_F + \Delta_G$$

(67) $\Delta_{cG} = c\Delta_G$, for $c\in\mathbb{Q}$

ω^* and ω

(68) $$\omega^*(F(\boldsymbol{x};q,t)) := \omega\left(F\left(\boldsymbol{x};q^{-1},t^{-1}\right)\right)$$ (69) $$\omega^*(H_\mu(\boldsymbol{x};q,t)) = q^{-n(\mu')}t^{-n(\mu)}H_\mu(\boldsymbol{x};q,t)$$

(70) $$\omega^*\nabla\omega^*(H_\mu(\boldsymbol{x};q,t)) = \nabla^{-1}(H_\mu(\boldsymbol{x};q,t))$$ (71) $$\omega(H_\mu(\boldsymbol{x};q,t)) = q^{n(\mu')}t^{n(\mu)}H_\mu(\boldsymbol{x};q^{-1},t^{-1})$$

(72) $$\langle\nabla_{e_{d-1}}(e_n),F\rangle = \langle\nabla_{\omega F}(e_d),s_d\rangle, \forall F\in\Lambda^n \Rightarrow \quad \forall\mu\vdash n, \langle\nabla_{e_{d-1}}(e_n),s_\mu\rangle = \langle\nabla_{s_{\mu'}}(e_d),s_d\rangle,$$

The operator that multiplies by $\boldsymbol{e_1}$

(73) $$\begin{aligned}\underline{e_1} : \boldsymbol{\Lambda}^d_{\mathbb{Q}(q,t)} &\to \boldsymbol{\Lambda}^{d+1}_{\mathbb{Q}(q,t)}\\ H_\mu &\mapsto \sum_{\mu\lessdot\lambda} d_{\lambda,\mu}(q,t)H_\lambda\end{aligned}$$

The operator that differentiates by $\boldsymbol{e_1}$

(74) $$\begin{aligned}\delta_{e_1} : \boldsymbol{\Lambda}^d_{\mathbb{Q}(q,t)} &\to \boldsymbol{\Lambda}^{d-1}_{\mathbb{Q}(q,t)}\\ H_\mu &\mapsto \sum_{\lambda\lessdot\mu} c_{\lambda,\mu}(q,t)H_\lambda\end{aligned}$$

$d_{\lambda,\mu}(q,t) = \prod_{c\in\mathcal{R}_{\lambda,\mu}} \frac{q^{a_\mu(c)}-t^{l_\mu(c)+1}}{q^{a_\lambda(c)}-t^{l_\lambda(c)+1}} \prod_{c\in\mathcal{C}_{\lambda,\mu}} \frac{t^{l_\mu(c)}-q^{a_\mu(c)+1}}{t^{l_\lambda(c)}-q^{a_\lambda(c)+1}}$, $c_{\lambda,\mu}(q,t) = \prod_{c\in\mathcal{R}_{\mu,\lambda}} \frac{t^{l_\mu(c)}-q^{a_\mu(c)+1}}{t^{l_\lambda(c)}-q^{a_\lambda(c)+1}} \prod_{c\in\mathcal{C}_{\mu,\lambda}} \frac{q^{a_\mu(c)}-t^{l_\mu(c)+1}}{q^{a_\lambda(c)}-t^{l_\lambda(c)+1}}$

$\mathcal{R}_{\lambda,\mu}$ is the set of cells in the same row as λ/μ $\quad$ $\mathcal{C}_{\lambda,\mu}$ is the set of cells in the same column as λ/μ

It is proven that $\underline{e_1}\left[\frac{\boldsymbol{x}}{(1-q)(1-t)}\right]$ is the adjoint of δ_{e_1} in respect to $\langle\cdot,\cdot\rangle_*$.

8. Combinatorial aspects

The Schur symmetric functions :

(75) $$s_\mu := \sum_{\tau:\mu\to\boldsymbol{x}} x_\tau, \tau \text{ semi-standard and } x_\tau = \prod_{c\in\mu} x_{\tau(c)}$$

Example 7 clic here

Pieri formula :

(76) $$h_n s_\mu = \sum_{\substack{\theta\vdash n+|\mu|\\ \theta/\mu \text{ ia a } n\text{-horizontal strip}}} s_\theta.$$ (77) $$e_n s_\mu = \sum_{\substack{\theta\vdash n+|\mu|\\ \theta/\mu \text{ ia a } n\text{-vertical strip}}} s_\theta.$$

Example 8 clic here

The **Kostka numbers $\boldsymbol{K_{\mu,\lambda}}$**

(78) $$K_{\mu,\lambda} := \#\{\text{ Semi-standard tableaux of shape } \mu \text{ fillings of } \lambda\}$$

(79) $$s_\mu = \sum_{\lambda\vdash n} K_{\mu,\lambda}m_\lambda,$$ (80) $$h_\mu = \sum_{\lambda\vdash n} K_{\lambda,\mu}s_\lambda,$$ (81) $$e_\mu = \sum_{\lambda\vdash n} K_{\lambda,\mu}s_{\lambda'}.$$

Example 9 clic here

domino tabloid, $\boldsymbol{d_{\lambda,\mu}}$

(82) $$d_{\lambda,\mu} = \#\{\text{ domino tableaux of shape } \lambda \text{ and type } \mu\}$$

(83) $$e_\lambda = \sum_{\mu\vdash n}(-1)^{|\mu|-\ell(\mu)}d_{\lambda,\mu}h_\mu,$$ (84) $$h_\lambda = \sum_{\mu\vdash n}(-1)^{|\mu|-\ell(\mu)}d_{\lambda,\mu}e_\mu,$$

$\chi^\mu(\lambda)$

(85) $$\chi^\mu(\lambda) := \sum_T (-1)^{ht(T)},$$ summed over all border-strip tableaux, T, of shape μ and type λ,

(86) $$ht(T) = \prod ht(T_{\lambda_i})$$

(87) $$s_\mu = \sum_{\lambda \vdash n} \frac{1}{z_\mu} \chi^\mu(\lambda) p_\lambda$$

$w_{\lambda,\mu}$ and $v_{\lambda,\mu}$

(88) $$w_{\lambda,\mu} = \#\{ \text{ Matrices of zeros and ones, with row sums } \lambda \text{ and column sums } \mu \ \}$$

(89) $$v_{\lambda,\mu} = \#\{ \text{ Matrices of non negative integers, with row sums } \lambda \text{ and column sums } \mu \ \}$$

(90) $$e_\lambda = \sum_{\mu \vdash n} w_{\lambda,\mu} m_\mu,$$

(91) $$h_\lambda = \sum_{\mu \vdash n} v_{\lambda,\mu} m_\mu,$$

Example 10 clic here

9. q-analogs (see [Ber2009] for more on this)

(92) $$[n]_q := 1 + q + q^2 + q^3 + \cdots + q^{n-1}$$

(93) $$[n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q$$

(94) $$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

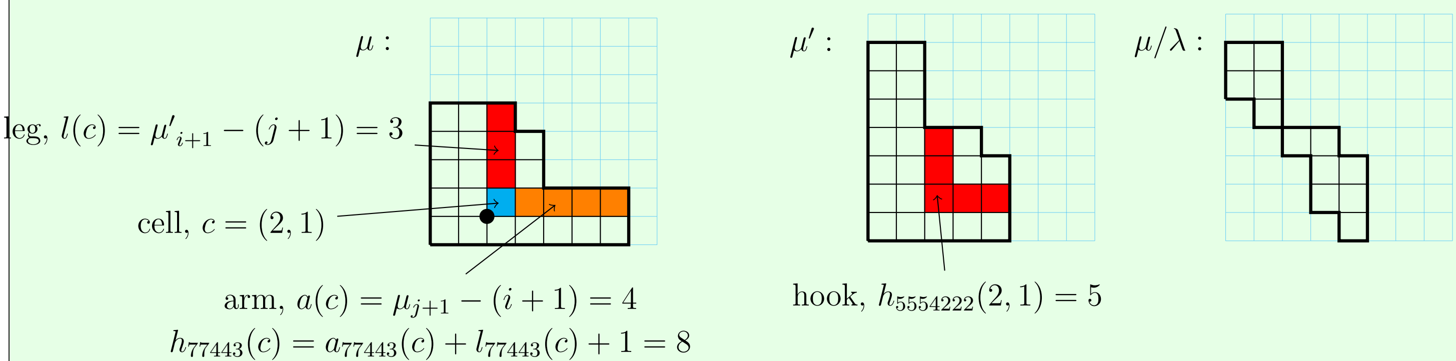
(95) $$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{[2n]_q [2n-1]_q \cdots [n+3]_q [n+2]_q}{[n]_q [n-1]_q \cdots [2]_q [1]_q}$$

(96) $$e_k(1, q, q^2, \cdots, q^{n-1}) = q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

(97) $$h_k(1, q, q^2, \cdots, q^{n-1}) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$$

Example 11 clic here

Example 1 : $\mu = 77443 \vdash 25$, $|77443| = 25$, $\ell(77443) = 5$, $\mu' = 5554222$ *and* $\lambda = 43321$ *:*



$z_\mu = z_{77443} = 7^2 \cdot 4^2 \cdot 3 \cdot 2!2!1! = 9408$; $n(\mu) = n(77443) = 39$; $n(\mu') = n(5554222) = 57$.

go back

Example 2 :

Tableau

5	
3	63
111	9

shape 221
No order

Semi-Standard Tableau

3	
2	3
1	1

shape 221 and filling 212
(i.e. filled by $\{1^2, 2, 3^2\}$)

Semi-Standard Tableau

3	
2	2
1	1

shape 221 and filling 221
(i.e. filled by $\{1, 1, 2, 2, 3\}$)

All standard tableaux of shape 221 :

3	
2	5
1	4

5	
2	4
1	3

5	
3	4
1	2

4	
2	5
1	3

4	
3	5
1	2

$f^{221} = \frac{5!}{\prod_{c\in 221} h(c)} = \frac{5\cdot4\cdot3\cdot2\cdot1}{4\cdot2\cdot3\cdot1\cdot1} = 5\,;$ $5! = \sum_{\mu\vdash 5} (f^\mu)^2 = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 120$

go back

Example 3 :

$$\begin{aligned}
m_{21}(x,y,z) &= x^2y+x^2z+xy^2+xz^2+y^2z+yz^2\\
h_{21}(x,y,z) &= h_2h_1=(m_2+m_{11})m_1=(x^2+y^2+z^2+xy+xz+yz)(x+y+z)\\
e_{21}(x,y,z) &= (xy+xz+yz)(x+y+z)\\
p_{21}(x,y,z) &= (x^2+y^2+z^2)(x+y+z)\\
s_{21}(x,y,z) &= x^2y+x^2z+xy^2+xz^2+y^2z+yz^2+2xyz
\end{aligned}$$

For $n=4$:

$$\begin{aligned}
h_4(x,y,z) &= m_{1111}+m_{211}+m_{22}+m_{31}+m_4\\
h_{31}(x,y,z) &= 4m_{1111}+3m_{211}+2m_{22}+2m_{31}+m_4\\
h_{22}(x,y,z) &= 6m_{1111}+4m_{211}+3m_{22}+2m_{31}+m_4\\
h_{211}(x,y,z) &= 12m_{1111}+7m_{211}+4m_{22}+3m_{31}+m_4\\
h_{1111}(x,y,z) &= 24m_{1111}+12m_{211}+6m_{22}+4m_{31}+m_4
\end{aligned}$$

$$\begin{aligned}
s_4(x,y,z) &= h_4 & s_4(x,y,z) &= e_{1111}-3e_{211}+e_{22}+2e_{31}-e_4\\
s_{31}(x,y,z) &= h_{31}-h_4 & s_{31}(x,y,z) &= e_{211}-e_{22}-e_{31}+e_4\\
s_{22}(x,y,z) &= h_{22}-h_{31} & s_{22}(x,y,z) &= e_{22}-e_{31}\\
s_{211}(x,y,z) &= h_{211}-h_{22}-h_{31}+h_4 & s_{211}(x,y,z) &= e_{31}-e_4\\
s_{1111}(x,y,z) &= h_{1111}-3h_{211}+h_{22}+2h_{31}-h_4 & s_{1111}(x,y,z) &= e_4
\end{aligned}$$

go back

Example 4 :

a) Let $f=(q+t)p_k$, then : $f\left[\frac{5q\boldsymbol{x}}{1-t}\right]=(q+t)p_k\left[\frac{5q\boldsymbol{x}}{1-t}\right]=(q+t)\frac{5q^n}{1-t^n}p_k(\boldsymbol{x})$.

b) $p_n[p_1(\boldsymbol{x})]=p_n[x_1+x_2+\cdots]=p_n(\boldsymbol{x})=\sum_{i\in\mathbb{N}}p_n[x_i]=\sum_{i\in\mathbb{N}}x_i^n$

c) $p_n[p_k(\boldsymbol{x})]=p_n\left[\sum_{i\in\mathbb{N}}x_i^k\right]=\sum_{i\in\mathbb{N}}p_n[x_i^k]=\sum_{i\in\mathbb{N}}x_i^{kn}=p_{nk}(\boldsymbol{x}) \quad\Rightarrow\quad p_n[f(\boldsymbol{x})]=f[p_n(\boldsymbol{x})]\forall f\in\Lambda$

d) Let $g=p_3(\boldsymbol{x})+p_{111}(\boldsymbol{x})$ and $f=p_{11}(\boldsymbol{x})+p_2(\boldsymbol{x})$, then :

$$\begin{aligned}
g[f(\boldsymbol{x})] &= (p_3+p_{111})[f(\boldsymbol{x})]\\
&= p_3[p_{11}(\boldsymbol{x})+p_2(\boldsymbol{x})]+p_{111}[p_{11}(\boldsymbol{x})+p_2(\boldsymbol{x})]\\
&= p_3[p_{11}(\boldsymbol{x})]+p_3[p_2(\boldsymbol{x})]+(p_1[p_{11}(\boldsymbol{x})+p_2(\boldsymbol{x})])^3\\
&= p_3[p_1(\boldsymbol{x})]p_3[p_1(\boldsymbol{x})]+p_6(\boldsymbol{x})+(p_{11}(\boldsymbol{x})+p_2(\boldsymbol{x}))^3\\
&= p_6(\boldsymbol{x})+p_{33}(\boldsymbol{x})+p_{222}(\boldsymbol{x})+3p_{2211}(\boldsymbol{x})+3p_{21111}(\boldsymbol{x})+p_{1^6}(\boldsymbol{x})
\end{aligned}$$

and

$$\begin{aligned}
f[g(\boldsymbol{x})] &= (p_{11}+p_2)[g(\boldsymbol{x})]\\
&= (p_1[p_3(\boldsymbol{x})+p_{111}(\boldsymbol{x})])^2+p_2[p_3(\boldsymbol{x})+p_{111}(\boldsymbol{x})]\\
&= p_6(\boldsymbol{x})+p_{33}(\boldsymbol{x})+2p_{3111}(\boldsymbol{x})+p_{222}(\boldsymbol{x})+p_{1^6}(\boldsymbol{x})
\end{aligned}$$

Therefor $g[f(\boldsymbol{x})]\neq f[g(\boldsymbol{x})]$.

go back

Example 5 :

$$H_4(x,y,z) = q^6 s_{1111} + \left(q^5+q^4+q^3\right) s_{211} + \left(q^4+q^2\right) s_{22} + \left(q^3+q^2+q\right) s_{31} + s_4$$
$$H_{31}(x,y,z) = q^3 s_{1111} + \left(q^3t+q^2+q\right) s_{211} + \left(q^2t+q\right) s_{22} + \left(q^2t+qt+1\right) s_{31} + ts_4$$
$$H_{22}(x,y,z) = q^2 s_{1111} + \left(q^2t+qt+q\right) s_{211} + \left(q^2t^2+1\right) s_{22} + \left(qt^2+qt+t\right) s_{31} + t^2 s_4$$
$$H_{211}(x,y,z) = q s_{1111} + \left(qt^2+qt+1\right) s_{211} + \left(qt^2+t\right) s_{22} + \left(qt^3+t^2+t\right) s_{31} + t^3 s_4$$
$$H_{1111}(x,y,z) = s_{1111} + \left(t^3+t^2+t\right) s_{211} + \left(t^4+t^2\right) s_{22} + \left(t^5+t^4+t^3\right) s_{31} + t^6 s_4$$

go back

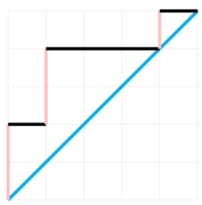


FIGURE 1.
Dyck path, $\gamma \in \mathcal{D}_5$
with riser $\rho(\gamma) = 221$

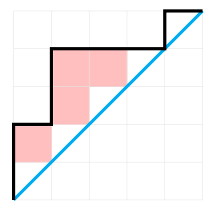
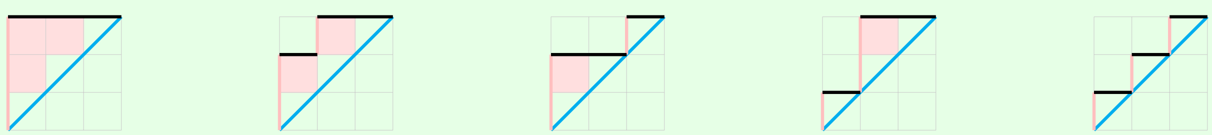


FIGURE 2.
Dyck path of area 4
$C_n := \#D_n$

Example 6 :

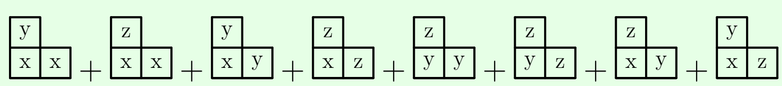


$$\nabla|_{t=1}(e_3) = \quad q^3e_3 \quad + \quad q^2e_{21} \quad + \quad qe_{21} \quad + \quad qe_{21} \quad + \quad e_{111}$$

go back

Example 7 :

$s_{21}(x,y,z) =$

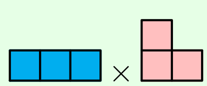


$$= x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 + 2xyz$$

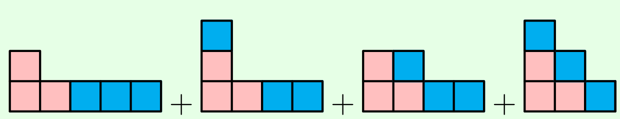
go back

Example 8 :

$h_3 s_{21} =$



$=$



$$= s_{51} + s_{411} + s_{42} + s_{321}$$

go back

Example 9 :

$$K_{221,5}=0,\ K_{221,41}=0,\ K_{221,32}=0,\ K_{221,311}=0,\ K_{221,221}=1,\ K_{221,2111}=2,\ K_{221,11111}=5$$
$$s_{221}=m_{221}+2m_{2111}+5m_{11111}$$
$$K_{5,221}=1,\ K_{41,221}=2,\ K_{32,221}=2,\ K_{311,221}=1,\ K_{221,221}=1,\ K_{2111,221}=0,\ K_{11111,221}=0$$
$$h_{221}=s_5+2s_{41}+2s_{32}+s_{311}+s_{221}$$
$$e_{221}=s_{11111}+2s_{2111}+2s_{221}+s_{311}+s_{32}$$

go back

Example 10 :

$$w_{21,3}=\#\{\}=0,\ w_{21,21}=\#\left\{\begin{bmatrix}1&1\\1&0\end{bmatrix}\right\}=1,\ w_{21,111}=\#\left\{\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix},\ \begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix},\ \begin{bmatrix}0&1&1\\1&0&0\\0&0&0\end{bmatrix}\right\}=3$$

$$e_{21}=m_{21}+3m_{111}$$

$$v_{21,3}=\#\left\{\begin{bmatrix}2&0\\1&0\end{bmatrix}\right\}=1,\ v_{21,21}=\#\left\{\begin{bmatrix}1&1\\1&0\end{bmatrix},\ \begin{bmatrix}2&0\\0&1\end{bmatrix}\right\}=2,\ v_{21,111}=w_{21,111}=3$$

$$h_{21}=m_3+2m_{21}+3m_{111},$$

go back

Example 11 :

$$[5]_q=q^4+q^3+q^2+q+1$$

$$[4]!_q=(q^3+q^2+q+1)(q^2+q+1)(q+1)(1)$$

$$\begin{bmatrix}8\\4\end{bmatrix}_q=\frac{[8]_q[7]_q[6]_q[5]_q[4]_q[2]_q[2]_q[1]_q}{[4]_q[2]_q[2]_q[1]_q[4]_q[2]_q[2]_q[1]_q}$$

$$C_4(q):=\frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q=\frac{[8]_q[7]_q[6]_q}{[4]_q[2]_q[2]_q[1]_q}$$

$$=1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

go back