Numerical Analysis of One-Dimensional Steady-State Heat Equation with Discontinuity at a Point.

ABRAHAM OFORI

April 2024

Contents

List of Figures

1 Introduction

The study of heat transfer is fundamental to understanding and designing numerous engineering systems, from simple household appliances to complex industrial machinery. One of the primary mathematical tools used to model heat transfer is the heat equation [5]. This project focuses on the one-dimensional steady-state heat equation, particularly in scenarios where there is a discontinuity in the material properties of the medium through which heat is being conducted.

In many practical applications, materials with different thermal conductivities are combined within a single system. These differences can significantly affect the overall heat transfer behavior [1]. Our analysis aims to understand the impact of a discontinuity at a specific point along a one-dimensional rod, where the thermal conductivity changes abruptly from k_1 to k_2. This setup models real-world situations such as composite materials in aerospace engineering, where different sections of a component might consist of distinct materials to optimize performance and weight.

In this report, we will explore how the discontinuity in thermal conductivity affects the temperature distribution along the rod. The left section of the rod has a thermal conductivity of k_1, while the right section has a conductivity of k_2, with $k_1 \neq k_2$. Such a configuration raises interesting questions about the continuity of temperature and heat flux at the interface, which are essential for ensuring the structural integrity and efficiency of the heat transfer process in practical applications [2].

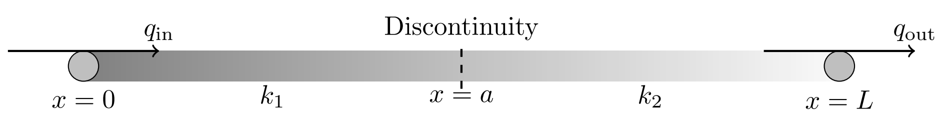
To address these questions, we employ numerical methods to solve the steady-state heat equation. The numerical approach allows for precise control over the parameters defining the discontinuity and offers a detailed view of the temperature profile and heat flux across the boundary. Specifically, we aim to answer the following questions:

- How does the solution change when the ratio $\frac{k_1}{k_2}$ is less than 1, and when it is greater than 1?
- What happens to the temperature and heat flux profiles if the discontinuity is positioned at a computational node versus when it is positioned between nodes?

The findings from this analysis will help in designing more effective thermal management systems by optimizing material properties according to their placement and expected heat loads [1].

By the end of this report, we aim to provide a comprehensive understanding of thermal behavior at discontinuity, offering insights that are critical for engineers and designers in optimizing material choices and configurations in various thermal applications.

2 Formulation of Discontinuous Heat Equation:



The general heat equation in one dimension with zero heat source is given by

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = 0$$

.

Where k, is the thermal conductivity of the material. The steady state is when the time component in the PDE is 0. That is the equation we are interested in is

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = 0$$

[5]

2.1 Boundary Conditions

In order to accurately model the heat transfer along the one-dimensional rod and ensure the mathematical soundness and physical relevancy of our analysis, appropriate boundary conditions must be specified. The boundary conditions for this problem are a combination of Dirichlet and Neumann conditions, which are essential for defining the behavior of the system at the boundaries and the interface between different materials.

1. **Dirichlet Boundary Conditions**: These conditions specify the temperature directly at the boundaries of the domain. For our analysis:
   - At the left end of the rod(where $x = 0$), the temperature is fixed at zero, denoted as $u_1(x = 0) = 0$. This condition could represent, for instance, a rod end maintained at a constant reference temperature.
   - At the right end of the rod (where $x = L$), the temperature is set to T_o, expressed as $u_2(x = L) = T_o$. This might model the rod end exposed to a constant temperature environment or in thermal equilibrium with another system.
2. **Neumann Boundary Condition**: This type specifies the heat flux across a boundary. In our model, the continuity of heat flux at the discontinuity point, where the thermal conductivity changes, is given by:
   - $k_1 \frac{\partial u_1}{\partial x} = k_2 \frac{\partial u_2}{\partial x}$ at the interface. This condition ensures that the heat flux is conserved across the discontinuity, reflecting a physically realistic scenario where no heat is lost or gained at the interface
   - At $x = a$, $u_1(a) = u_2(a)$. That is the continuity of temperature at $x = a$

   These boundary conditions are crucial for the development of a realistic and robust model. They help in simulating real-world scenarios accurately and provide a comprehensive understanding of how discontinuities in material properties affect heat transfer in composite systems. By implementing these conditions, we ensure that our numerical solutions reflect both the physical laws of conservation and the practical constraints of engineering applications.

3 Analytical/Exact Solution

To solve the problem of heat conduction in a one-dimensional rod with discontinuous thermal conductivity analytically, we consider the steady-state heat equation without internal heat generation [5]:

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = 0$$

Given the thermal conductivity changes at a certain point, say $x = a$, where $0 < a < L$, we have two separate conductivities k_1 and k_2. The problem can be divided into two.

- Region 1: $0 \leq x \leq a$, with thermal conductivity of k_1
- Region 2: $a < x \leq L$, with thermal conductivity of k_2

Since k is constant, our equation can be written as $k\frac{d^2u}{dx^2} = 0$.
With Region 1: having a solution as $u_1(x) = c_1x + c_2$
Region 2: having a solution as $u_2(x) = c_3x + c_4$
Applying the Dirichlet and Neumann conditions discussed above, we have:
$c_1 = \frac{T_0}{a + \alpha(1-a)}$, and $c_2 = 0$, with $\alpha = \frac{k_1}{k_2}$

$c_3 = \dfrac{\alpha T_0}{a+\alpha(1-a)}$, and $c_4 = \dfrac{aT_0(1-\alpha)}{a+\alpha(1-a)}$, with $\alpha = \dfrac{k_1}{k_2}$
Hence the Exact Solution is given by

$$u(x) = \begin{cases} \dfrac{T_0}{a+\alpha(1-a)}x & \text{if } 0 \leq x \leq a, \\ \\ \dfrac{\alpha T_0}{a+\alpha(1-a)}x + \dfrac{aT_0(1-\alpha)}{a+\alpha(1-a)} & \text{if } a < x \leq L. \end{cases}$$

Let's Consider an example where $\dfrac{k_1}{k_2} < 1$

- $L = 1$
- $k_1 = 1$
- $k_2 = 5$
- $a = 0.4$
- $u_1(0) = 0$
- $T_0 = u_2(L) = 10$

$c_1 = \dfrac{250}{13}$, $c_3 = \dfrac{50}{13}$, and $c_4 = \dfrac{80}{13}$
So

$$u(x) = \begin{cases} \dfrac{250}{13}x & \text{if } 0 \leq x \leq 0.4, \\ \\ \dfrac{50}{13}x + \dfrac{80}{13} & \text{if } 0.4 < x \leq 1. \end{cases}$$

Below is the graph representing the Analytical solution:

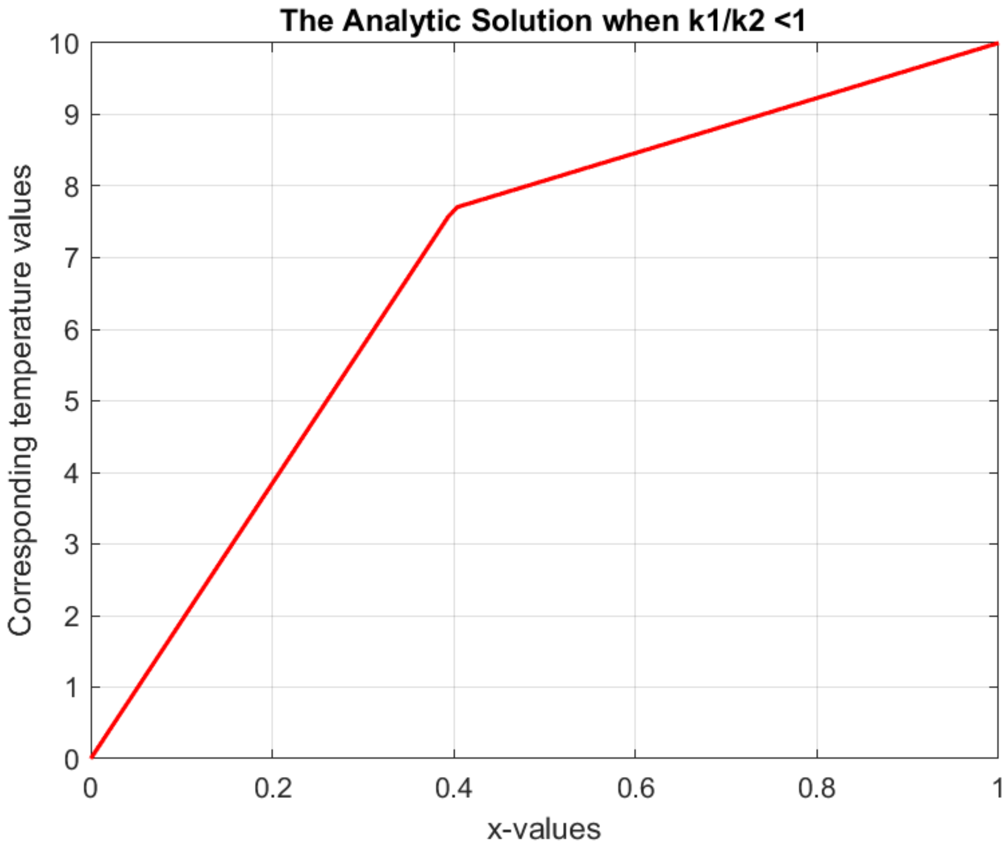


Figure 1: Graph of the analytical solution

4 Numerical Solution

To numerically solve the one-dimensional steady-state heat equation with discontinuity in thermal conductivity, we can consider several numerical methods. The most appropriate for capturing the discontinuity effectively include the Finite Difference Method **(FDM)**, Finite Element Method **(FEM)**, and the Finite Volume Method **(FVM)**. The Finite Difference Method (FDM) is a widely used technique for solving partial differential equations, particularly when dealing with problems involving discontinuities in material properties [4]. Here, we'll focus on discussing the Finite Difference Method due to its straightforward implementation for problems involving heat conduction in one dimension [3].

4.1 Implementation Of The Finite Difference Method

Finite Difference Method is particularly useful due to its simplicity and effectiveness in handling problems with boundary conditions and discontinuities [4]. The method involves discretizing the spatial domain into a grid of points and approximating the derivatives in the differential equation using differences between these points.

Steps to Implement FDM for this Problem:

1. Discretize the Spatial Domain: Divide the rod into $N-1$ segments with N points, including the boundaries. Ensure that the point of discontinuity $x = a$ falls exactly on one of these grid points to accurately capture the change in thermal conductivity [3].

2. Set Up the Difference Equations:

- Approximate the second derivative using the central difference:

$$\frac{d^2u}{dx^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

- The steady-state heat equation without internal heat generation simplifies to

$$k_j \frac{d^2u}{dx^2} = 0 \implies k_j \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$

- For each region, apply the thermal conductivity k_1 or k_2 accordingly. That is $j = 1, 2$ and $i = 1, 2, ..., N$

3. Apply Boundary Conditions:

   - At $x = 0 : u_0 = 0$ (Dirichlet Condition).
   - At $x = L : u_N = T_0$

4. Handle The Discontinuity:

   - At $x = a$, ensure the continuity of temperature and the conservation of heat flux:

$$k_1 \frac{u_a - u_{a-1}}{h} = k_2 \frac{u_{a+1} - u_a}{h}$$

$$u_1(a) = u_2(a)$$

   - This requires careful handling of indices to ensure that u_{a-1}, u_a, and u_{a+1} fall on the grid points straddling the discontinuity [3].

5. Solve the System of Linear Equations: The discretization leads to a system of linear equations. Solve this system using a numerical solver suitable for solving systems of equation, like Gaussian Elimination, etc.

4.1.1 Advantages of FDM for This Problem

- Simple Implementation: Easy to implement for grid-based discretizations [3].
- Handles Discontinuity: Effectively captures the discontinuity by aligning it with a grid point
- Flexibility: Easy to adjust resolution by changing the number of grid points.

Based on the graph, our method seems to solve the problem exactly as the analytical approach [4].

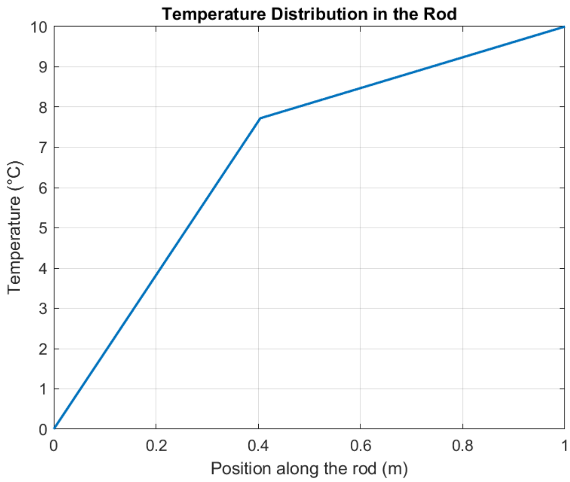


Figure 2: Numerical Profile for $k1/k2 < 1$

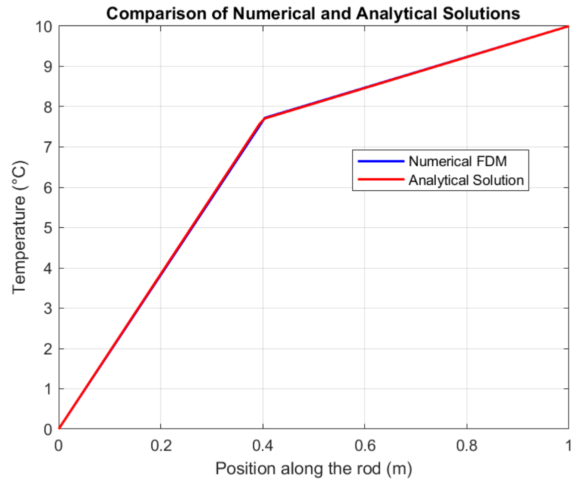


Figure 3: Numerical vs Analytical for $\frac{k_1}{k_2} < 1$

4.1.2 Analytical/Exact Solution For $\dfrac{k_1}{k_2} > 1$

With the same example above but here $k_1 = 5$ and $k_2 = 1$

$c_1 = \dfrac{50}{17}$, $c_3 = \dfrac{250}{17}$, and $c_4 = -\dfrac{80}{17}$

So

$$u(x) = \begin{cases} \dfrac{50}{17}x & \text{if } 0 \leq x \leq 0.4, \\ \\ \dfrac{250}{17}x - \dfrac{80}{17} & \text{if } 0.4 < x \leq 1. \end{cases}$$

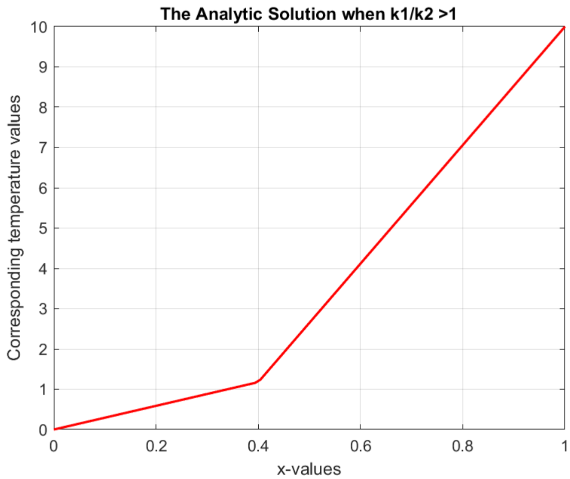


Figure 4: Profile of Analytical Solution

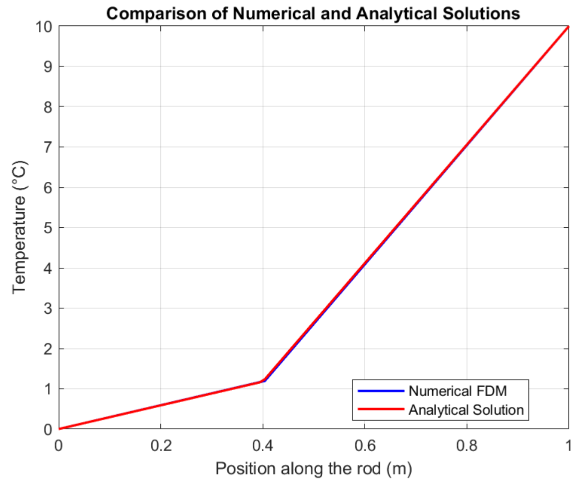


Figure 5: Profile of Numerical vs Analytical

4.1.3 Interpretation of Results

Scenario when $k_1/k_2 < 1$ In the case where the thermal conductivity ratio k_1/k_2 is less than one, the numerical and analytical solutions demonstrate close agreement across the rod's length, affirming the accuracy of both modeling approaches. The noticeable change in the temperature gradient at the discontinuity (around $x = 0.4$ m) reflects the lower conductivity in the first segment of the rod, leading to a slower temperature increase compared to the second segment. This scenario illustrates the reduced effectiveness in heat transfer where k_1 is less than k_2, indicative of the material's lower capacity to conduct heat.

Scenario when $k_1/k_2 > 1$ Conversely, when k_1/k_2 is greater than one, the temperature profile exhibits a sharper increase in the first segment of the rod, where k_1 exceeds k_2. This behavior highlights the enhanced thermal conductivity of the first material, facilitating a more rapid temperature rise under consistent heat flow conditions. After the point of discontinuity, the temperature increase moderates, aligning with the second material's lower conductivity. This result is crucial for material selection in engineering applications, such as in designing heat exchangers or managing thermal stresses in composite materials, as it underscores the importance of selecting appropriate materials based on their thermal conductive properties [1].

4.2 Discontinuity at Computational Node vs Between Nodes

Here we consider $a = x(49)$ and $a = (x(49) + x(50))/2$

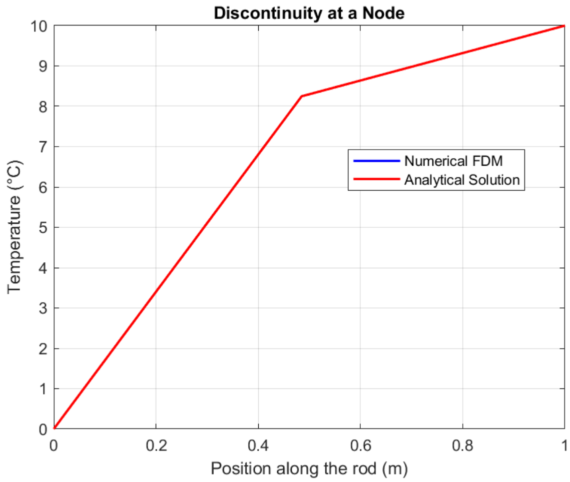


Figure 6: At a Node

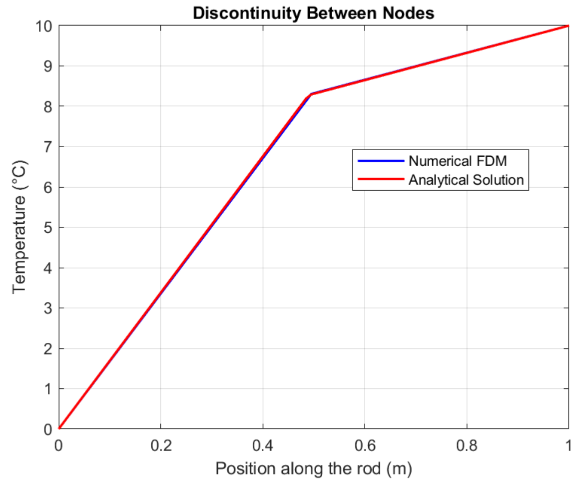


Figure 7: Between Nodes

4.2.1 Discontinuity at a Computational Node

The scenario where the discontinuity is precisely at a computational node exhibited a distinct and sharp transition in the temperature profile at the point of discontinuity. This abrupt change is consistent with what one would expect when a sudden shift in material properties is modeled without the need for interpolation between grid points.

Key observations:

- **Sharp Transition:** The temperature profile shows a marked and abrupt change exactly at the discontinuity, highlighting the model's sensitivity to abrupt changes in input parameters [3].
- **Solution Agreement:** There is a strong alignment between the numerical and analytical solutions, demonstrating the accuracy of both methods in capturing sharp material changes [4].
- **Engineering Applications:** This modeling approach is suitable for applications where precise material boundaries are known and can be aligned with computational nodes, such as in layered materials or structured composites [1].

4.2.2 Discontinuity Between Computational Nodes

In contrast, when the discontinuity is positioned between computational nodes, the temperature transition is smoother. This scenario more realistically models situations where material properties change over a finite distance rather than instantaneously.

Key observations:

- **Smooth Gradient Transition:** The temperature profile gradually changes across the discontinuity, suggesting that the numerical method effectively interpolates the changing material properties [3].
- **Interpolative Modeling:** The close match between numerical and analytical results across the discontinuity confirms the method's capability to handle gradual changes in material properties effectively.
- **Real-World Relevance:** This approach is especially relevant for modeling real-world scenarios where material properties do not abruptly change but vary over a range, such as in graded materials or thermal barriers [1].

4.2.3 Comparative Analysis

Both scenarios accurately capture the respective physical behaviors associated with different types of material discontinuities. While the discontinuity at a node provides precise modeling of abrupt changes, the discontinuity between nodes offers a more nuanced portrayal of gradual material transitions. The choice of modeling approach should therefore be guided by the specific nature of the material discontinuity in the application of interest [2].

5 Conclusion

5.1 Observations and Final Thoughts

The study has explored the behavior of a one-dimensional steady-state heat equation with discontinuities in thermal conductivity under various conditions. These conditions include differences in thermal conductivity ratios ($k_1/k_2 < 1$ and $k_1/k_2 > 1$) and the spatial alignment of these discontinuities with computational nodes [3].

Key Observations

- **Thermal Conductivity Ratios:**
  - When $k_1/k_2 < 1$: The temperature gradient shows a significant increase after the discontinuity, highlighting enhanced heat transfer efficiency where k_2 is greater [1].
  - When $k_1/k_2 > 1$: A more rapid temperature increase is observed in the section with higher k_1, indicating faster heat diffusion due to higher conductivity.
- **Discontinuity Placement:**
  - At a Computational Node: The transition at the discontinuity is sharp and clearly defined, suitable for modeling systems with distinct material boundaries. And Numerical solutions perfectly match with analytical solution [3]
  - Between Computational Nodes: The temperature profile transitions more smoothly, potentially offering a more realistic depiction of material properties that vary gradually.

Final Thoughts

The investigation demonstrates the significant influence of both the conductivity ratio and the precise location of conductivity changes on the thermal behavior of the rod. Understanding these dynamics is crucial for accurate modeling and simulation in thermal engineering applications. The study highlights the importance of careful consideration in the setup of numerical models to capture essential physical phenomena accurately [2]. Continued exploration and refinement of these models will enhance their utility in addressing complex real-world engineering problems.

Appendices

Below are the MATLAB codes used for analyzing the one-dimensional steady-state heat equation with varying thermal conductivities and discontinuities:

A MATLAB Code for Thermal Conductivity Analysis when $k1/k2 < 1$

```


% Number of grid points and domain specifications
N = 100;                  % Total number of grid points
L = 1;                    % Length of the rod
x = linspace(0, L, N)';   % Spatial grid
dx = x(2) - x(1);         % Grid spacing

% Thermal conductivities
k1 = 1;                   % Thermal conductivity in the first segment
k2 = 5;                   % Thermal conductivity in the second segment
a = 0.4;                  % Location of discontinuity

% Initialize the temperature vector
T = zeros(N, 1);

% Boundary conditions
T(1) = 0;                 % Temperature at x=0
T(N) = 10;                % Temperature at x=L (T0)

% Assemble the coefficient matrix A
A = zeros(N, N);
for i = 2:N-1
    if x(i) <= a
        k = k1;
    else
        k = k2;
    end
    A(i, i-1) = k;
    A(i, i) = -2*k;
    A(i, i+1) = k;
end

% Adjust for thermal conductivity discontinuity at 'a'
index = round(a / L * (N-1)) + 1;  % Index at discontinuity
A(index, index-1) = k1;            % Applying k1 on the left side of the discontinuity
A(index, index) = -(k1 + k2);      % Sum of both conductivities at the node
A(index, index+1) = k2;            % Applying k2 on the right side of the discontinuity

% Apply Dirichlet boundary conditions
A(1, :) = 0; A(1, 1) = 1;          % First row for the left boundary
A(N, :) = 0; A(N, N) = 1;          % Last row for the right boundary

% Solve the system of equations
T = A \ T;

C3 = 50 / 13;
C1 = 5 * C3;
T_analytic = zeros(N, 1);
for i = 1:N
    if x(i) <= a
        T_analytic(i) = C1 * x(i);
    else
        T_analytic(i) = C3 * x(i) + 80 / 13;
    end
end
% Plotting both numerical and analytical solutions
figure;
plot(x, T, 'b', 'LineWidth', 1.5);
```

```
hold on;
plot(x, T_analytic, 'r', 'LineWidth', 1.5);
title('Comparison of Numerical and Analytical Solutions');
xlabel('Position along the rod (m)');
ylabel('Temperature ( C )');
legend('Numerical FDM', 'Analytical Solution', 'Location', 'best');
grid on;
```

Listing 1: MATLAB code for analyzing the heat equation.

B MATLAB Code for Thermal Conductivity Analysis when $k1/k2 > 1$

```
% Number of grid points and domain specifications
N = 100;                  % Total number of grid points
L = 1;                    % Length of the rod
x = linspace(0, L, N)';   % Spatial grid
dx = x(2) - x(1);         % Grid spacing

% Thermal conductivities
k1 = 5;                   % Thermal conductivity in the first segment
k2 = 1;                   % Thermal conductivity in the second segment
a = 0.4;                  % Location of discontinuity

% Initialize the temperature vector
T = zeros(N, 1);

% Boundary conditions
T(1) = 0;                 % Temperature at x=0
T(N) = 10;                % Temperature at x=L (T0)

% Assemble the coefficient matrix A
A = zeros(N, N);
for i = 2:N-1
    if x(i) <= a
        k = k1;
    else
        k = k2;
    end
    A(i, i-1) = k;
    A(i, i) = -2*k;
    A(i, i+1) = k;
end

% Adjust for thermal conductivity discontinuity at 'a'
index = round(a / L * (N-1)) + 1;  % Index at discontinuity
A(index, index-1) = k1;            % Applying k1 on the left side of the discontinuity
A(index, index) = -(k1 + k2);      % Sum of both conductivities at the node
A(index, index+1) = k2;            % Applying k2 on the right side of the discontinuity

% Apply Dirichlet boundary conditions
A(1, :) = 0; A(1, 1) = 1;          % First row for the left boundary
A(N, :) = 0; A(N, N) = 1;          % Last row for the right boundary

% Solve the system of equations
T = A \ T;

C3 = 250 / 17;
C1 = 50/17;
T_analytic = zeros(N, 1);
for i = 1:N
    if x(i) <= a
        T_analytic(i) = C1 * x(i);
    else
        T_analytic(i) = C3 * x(i) - 80 / 17;
    end
end
```

```
% Plotting both numerical and analytical solutions
figure;
plot(x, T, 'b', 'LineWidth', 1.5);
hold on;
plot(x, T_analytic, 'r', 'LineWidth', 1.5);
title('Comparison of Numerical and Analytical Solutions');
xlabel('Position along the rod (m)');
ylabel('Temperature ( C )');
legend('Numerical FDM', 'Analytical Solution', 'Location', 'best');
grid on;
```

Listing 2: MATLAB code for analyzing the heat equation.

C MATLAB Code for Thermal Conductivity Analysis with Discontinuity at a Node

```

% Number of grid points and domain specifications
N = 100;                  % Total number of grid points
L = 1;                    % Length of the rod
x = linspace(0, L, N)';   % Spatial grid
dx = x(2) - x(1);         % Grid spacing

% Thermal conductivities
k1 = 1;                   % Thermal conductivity in the first segment
k2 = 5;                   % Thermal conductivity in the second segment
a = x(49) % Location of discontinuity

% Initialize the temperature vector
T = zeros(N, 1);

% Boundary conditions
T(1) = 0;                 % Temperature at x=0
T(N) = 10;                % Temperature at x=L (T0)

% Assemble the coefficient matrix A
A = zeros(N, N);
for i = 2:N-1
    if x(i) <= a
        k = k1;
    else
        k = k2;
    end
    A(i, i-1) = k;
    A(i, i) = -2*k;
    A(i, i+1) = k;
end

% Adjust for thermal conductivity discontinuity at 'a'
index = round(a / L * (N-1)) + 1;  % Index at discontinuity
A(index, index-1) = k1;            % Applying k1 on the left side of the discontinuity
A(index, index) = -(k1 + k2);      % Sum of both conductivities at the node
A(index, index+1) = k2;            % Applying k2 on the right side of the discontinuity

% Apply Dirichlet boundary conditions
A(1, :) = 0; A(1, 1) = 1;          % First row for the left boundary
A(N, :) = 0; A(N, N) = 1;          % Last row for the right boundary

% Solve the system of equations
T = A \ T;
alpha = k1/k2;
C1 =T(N)/(a+alpha*(1-a)) ;
C3 = alpha*T(N)/(a+alpha*(1-a));
C4 = a*T(N)*(1-alpha)/(a+alpha*(1-a));
T_analytic = zeros(N, 1);
```

```matlab
for i = 1:N
    if x(i) <= a
        T_analytic(i) = C1 * x(i);
    else
        T_analytic(i) = C3 * x(i) + C4;
    end
end
% Plotting both numerical and analytical solutions
figure;
plot(x, T, 'b', 'LineWidth', 1.5);
hold on;
plot(x, T_analytic, 'r', 'LineWidth', 1.5);
title('Discontinuity at a Node');
xlabel('Position along the rod (m)');
ylabel('Temperature ( C )');
legend('Numerical FDM', 'Analytical Solution', 'Location', 'best');
grid on;
```

Listing 3: MATLAB code for analyzing the heat equation.

D MATLAB Code for Thermal Conductivity Analysis with Discontinuity between Nodes

```matlab

% Number of grid points and domain specifications
N = 100;                  % Total number of grid points
L = 1;                    % Length of the rod
x = linspace(0, L, N)';   % Spatial grid
dx = x(2) - x(1);         % Grid spacing

% Thermal conductivities
k1 = 1;                   % Thermal conductivity in the first segment
k2 = 5;                   % Thermal conductivity in the second segment
a = (x(49)+x(50))/2 % Location of discontinuity

% Initialize the temperature vector
T = zeros(N, 1);

% Boundary conditions
T(1) = 0;                 % Temperature at x=0
T(N) = 10;                % Temperature at x=L (T0)

% Assemble the coefficient matrix A
A = zeros(N, N);
for i = 2:N-1
    if x(i) <= a
        k = k1;
    else
        k = k2;
    end
    A(i, i-1) = k;
    A(i, i) = -2*k;
    A(i, i+1) = k;
end

% Adjust for thermal conductivity discontinuity at 'a'
index = round(a / L * (N-1)) + 1;  % Index at discontinuity
A(index, index-1) = k1;            % Applying k1 on the left side of the discontinuity
A(index, index) = -(k1 + k2);      % Sum of both conductivities at the node
A(index, index+1) = k2;            % Applying k2 on the right side of the discontinuity

% Apply Dirichlet boundary conditions
A(1, :) = 0; A(1, 1) = 1;          % First row for the left boundary
A(N, :) = 0; A(N, N) = 1;          % Last row for the right boundary

% Solve the system of equations
```

```
T = A \ T;

C3 = (0.2*10)/(0.4899+0.2*(1-0.4899));
C1 = 10/(0.4899+0.2*(1-0.4899));
C4 = (0.4899*10*(1-0.2))/(0.4899+0.2*(1-0.4899));
T_analytic = zeros(N, 1);
for i = 1:N
    if x(i) <= a
        T_analytic(i) = C1 * x(i);
    else
        T_analytic(i) = C3 * x(i) + C4;
    end
end
% Plotting both numerical and analytical solutions
figure;
plot(x, T, 'b', 'LineWidth', 1.5);
hold on;
plot(x, T_analytic, 'r', 'LineWidth', 1.5);
title('Discontinuity Between Nodes');
xlabel('Position along the rod (m)');
ylabel ('Temperature ( C )');
legend('Numerical FDM', 'Analytical Solution', 'Location', 'best');
grid on;
```

Listing 4: MATLAB code for analyzing the heat equation.

References

[1] I. Oziashvili and V. Vardanyan. Thermal conductivity of composite materials. *Mechanics of Composite Materials*, 53(2):251–264, 2017.

[2] Suhas V Patankar. *Numerical Heat Transfer and Fluid Flow*. CRC Press, 1980.

[3] John C. Strikwerda. *Finite Difference Schemes and Partial Differential Equations*. Society for Industrial and Applied Mathematics, 2004.

[4] Lloyd N Trefethen. *Spectral Methods in MATLAB*. Society for Industrial and Applied Mathematics, 2000.

[5] M. Necati Özisik. *Heat Conduction*. John Wiley & Sons, 1993.