Spring 2007 Math 510 HW4 Solutions

Section 5.8

3. Consider the sum of the binomial coefficients along the diagonals of Pascal's triangle running upward from the left. The first few are: 1, 1, $1+1=2$, $1+2=3$, $1+3+1=5$, $1+4+3=8$. Computer several more of these diagonal sums and determine how these sums are related.

Solution. Let us set $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$, ... with a_n representing the sum starting with $\binom{n}{0}$. By the examples given, one should be able to see the relation $a_n = a_{n-1} + a_{n-2}$. (Do a few more list if you can not see the pattern). How the observation from the first few numbers does not always guarantee that the relation always holds. A proof must be given. Note that diagonal sums in the question is $a_n = \sum_{r=0}^{n} \binom{n-r}{r}$. One should note that $\binom{k}{r} = 0$ if $r > k \geq 0$ or $k < 0$ are all integers. By Pascal's formula, we have

$$\begin{aligned}
a_n &= \sum_{r=0} (\binom{n-r-1}{r} + \binom{n-r-1}{r-1}) \\
&= \sum_{r=0} \binom{n-1-r-}{r} + \sum_{k=0}^{n-1} \binom{n-2-k}{k} \\
&= a_{n-1} + a_{n-2}.
\end{aligned}$$

Here in the second sum, we have changed the index by $k = r - 1$.

5. Expand $(2x-y)^7$ using the binomial coefficient theorem.

Solution.

$$\begin{aligned}
(2x-y)^7 &= \binom{7}{0}(2x)^7 + \binom{7}{1}(2x)^6(-y) + \binom{7}{2}(2x)^5(-y)^2 + \binom{7}{3}(2x)^4(-y)^3 \\
&\quad + \binom{7}{4}(2x)^3(-y)^4 + \binom{7}{5}(2x)^2(-y)^5 + \binom{7}{6}(2x)^1(-y)^6 + \binom{7}{7}(-y)^7 \\
&= \binom{7}{0}2^7x^7 - \binom{7}{1}2^6x^6y + \binom{7}{2}2^5x^5y^2 - \binom{7}{3}2^4x^4y^3 + \binom{7}{4}2^3x^3y^4 \\
&\quad - \binom{7}{5}2^2x^2y^5 + \binom{7}{6}2x^1y^6 - \binom{7}{7}y^7.
\end{aligned}$$

8. Use the binomial theorem to prove that

$$2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.$$

Proof. In the binomial theorem, we set $x = 3$ and $y = -1$. Then the we have

$$\begin{aligned}
2^n = (x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\
&= \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-1)^k.
\end{aligned}$$

12. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array} \right)^2 = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ ((-1)^m \left(\begin{array}{c} 2m \\ m \end{array} \right) & \text{if } n = 2m. \end{cases}$$

Proof. Let us consider identity of polynomials $(1+x)^n(1-x)^n = (1-x^2)^n$ and then use the binomial theorem to expand both sides.

$$(1-x^2)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (-x^2)^k = \sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) x^{2k}$$

$$\begin{aligned} (1-x)^n(1+x)^n &= \left(\sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) (-x)^i \right) \left(\sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) x^j \right) \\ &= \sum_{k=0}^{2n} \left(\sum_{i=0}^{k} \left(\begin{array}{c} n \\ i \end{array} \right) (-1)^i \left(\begin{array}{c} n \\ k-i \end{array} \right) \right) x^k. \end{aligned}$$

We now compare the coefficient of x^n in both expansions. The coefficient in the second expansion is

$$\sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) (-1)^i \left(\begin{array}{c} n \\ n-i \end{array} \right) = \sum_{i=0}^{n}(-1)^i \left(\begin{array}{c} n \\ i \end{array} \right)^2,$$

which is exactly the left hand side of the identity to be proved. If n is odd, then the coefficient of x^n in the first expansion is 0. If $n = 2m$ is even, then the coefficient of x^n in the first expansion is $(-1)^m \left(\begin{array}{c} n \\ m \end{array} \right)$ by $2k = n = 2m$. This proves the given identity.

16. By integrating the binomial expansion, prove that for a positive integer n,

$$1 + \frac{1}{2}\left(\begin{array}{c} n \\ 1 \end{array} \right) + \frac{1}{3}\left(\begin{array}{c} n \\ 2 \end{array} \right) + \cdots + \frac{1}{n+1}\left(\begin{array}{c} n \\ n \end{array} \right) = \frac{2^{n+1}-1}{n+1}.$$

Proof. Consider the polynomial function

$$(1+x)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k.$$

Take the definite integral on the interval $[0, 1]$ we have

$$\int_0^1 (1+x)^n dx = \frac{1}{n+1}(1+x)^{n+1}|_0^1 = \frac{1}{n+1}((1+1)^{n+1} - (1+0)^{n+1} = \frac{2^{n+1}-1}{n+1}.$$

This is exactly the right hand side of the identity. The integration on $[0, 1]$ to the expansion is

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \int_0^1 x^k dx = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{k+1} = \sum_{k=0}^{n} \frac{1}{k+1} \left(\begin{array}{c} n \\ k \end{array} \right)$$

which is just the left hand side of the identity. Thus the both sides of the identity holds for all positive integer n.

22. Prove that for all real numbers r and all integers k and m,

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

Proof. . First by convention, we have $\binom{m}{k} = 0$ if $k > m \geq 0$ are integers or $k < 0$ is an integer.

If $k < 0$ then both sides are zero. If $k \geq 0$ and $m < k$ then both sides are zero. Thus we only need to consider cases when $m \geq k \geq 0$ are integers. In this case

$$\begin{aligned}
\binom{r}{m}\binom{m}{k} &= \frac{r(r-1)\cdots(r-m+1)}{m!}\frac{m!}{k!(m-k)!} \\
&= \frac{r(r-1)\cdots(r-m+1)}{k!(m-k)!} \\
&= \frac{r(r-1)\cdots(r-k+1)}{k!}\frac{(r-k)(r-k-1)\cdots(r-m+1)}{(m-k)!} \\
&= \frac{r(r-1)\cdots(r-k+1)}{k!}\frac{(r-k)(r-k-1)\cdots(r-k-(m-k)+1)}{(m-k)!} \\
&= \binom{r}{k}\binom{r-k}{m-k}.
\end{aligned}$$