Metric Estimation from Trajectory Data

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Presentation Outline

1. Motivation
2. Sub-Problem
3. Methods
4. Results
5. Future Work
6. Questions ?

Motivation

1. Vid 1
2. Vid 2

Sub-Problem

Let $\mathcal{T} = \{\gamma_i(t)\}_{i=1}^N$ be a finite set of trajectories $\in \mathbb{R}^D$ arising from unknown Riemannian manifold (M, g) equipped with metric g, $\mathcal{P} = \{\phi_i^0\}_{i=1}^N$ be the set of initial conditions (starting point and tangent vector) for observed trajectory $\gamma_i(t)$, and $\mathcal{G}$ be the set of possible Riemannian metric tensors. Furthermore, denote $\hat{\gamma}_i(t)$ as the geodesic trajectory generated on M equipped with estimated metric $\hat{g}$ and initial conditions ϕ_i^0.

We wish to minimize:
$L(\hat{g}) = \frac{1}{N} \sum_{i=1}^N ||\gamma_i(t) - \hat{\gamma}_i(t)||_2^2 + \lambda_1 J(\hat{g}) + \lambda_2 K(\hat{g})$

Estimation Considerations (Methods)

1. We want our estimate of our metric tensor to be smooth (since Riemannian metrics are infinitely differentiable)
2. We want our estimate to be robust to reasonable levels of noise
3. We want to be able to encode prior information (equivariance to some group(s))

Metric Tensor Inference (Methods)

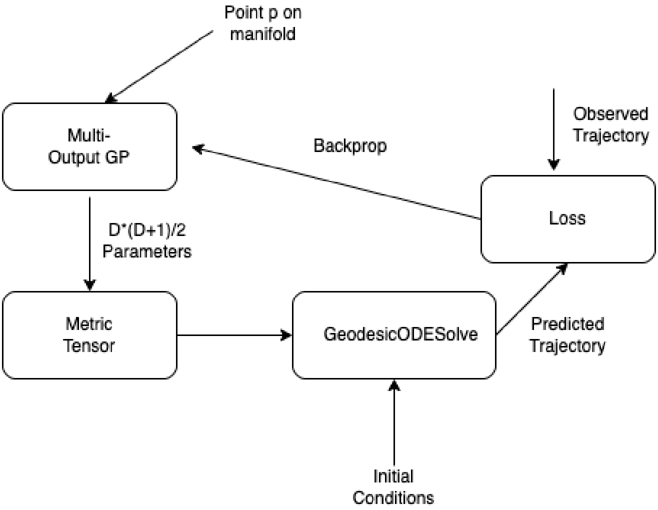


Figure 1: Workflow for Tensor Estimation

Geodesic Equation (Methods)

1. Christoffel Symbols
   a. $\Gamma^i_{jk} = \frac{1}{2} g^{il} [\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}]$
2. Geodesic Equation in 2D
   a. $\ddot{x} + \Gamma^x_{xx} \dot{x}^2 + 2\Gamma^x_{xy} \dot{x}\dot{y} + \Gamma^x_{yy} \dot{y}^2 = 0$
   b. $\ddot{y} + \Gamma^y_{xx} \dot{x}^2 + 2\Gamma^y_{xy} \dot{x}\dot{y} + \Gamma^y_{yy} \dot{y}^2 = 0$

Encoding Prior Information (Methods)

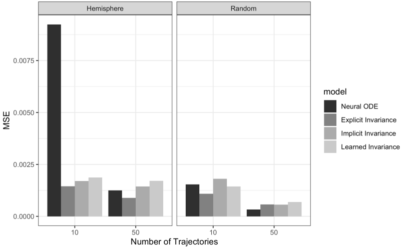
1. For a manifold (M, g) embedded in $\mathbb{R}^n$, start with a representation of $\theta \in \mathrm{Iso}(M, g)$: $\rho(\theta) \in GL(\mathbb{R}^n)$.
2. Identify $\rho(\theta)$ with the flow of a KVF h^K_θ.
3. Create the integral curve of K which starts at the point $p \in M$: $\gamma(\theta) = h^K_\theta(p)$ such that $\gamma(0) = h^K_0(p) = p$.
4. Solve for the KVF which amounts to solving the ODE:

   $\frac{d\gamma}{d\theta}|_{\theta=0} = K|_{\gamma(0)=p}$
5. Substitute K into Killing's equation
6. Use Killing's equation to constrain the Christoffel symbols as in equation.

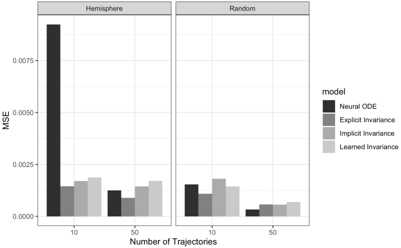
Encoding Prior Information: $SO(2)$ (Methods)

1. $\rho(\theta \in SO(2)) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
2. $h_\theta^X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
3. $\gamma(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$
4. $K^x = -y;\ K^y = x$
5. $\Gamma^x_{xx} y = \Gamma^y_{xx} x;\ \Gamma^x_{yy} y = \Gamma^y_{yy} x;\ \Gamma^x_{xy} y = \Gamma^y_{xy} x$

Results (Circle)



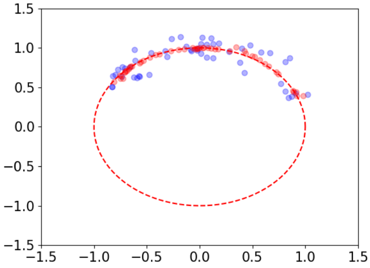
(a) Noise



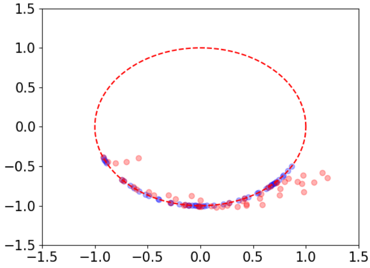
(b) N

Figure 2: Empirical Model Performance

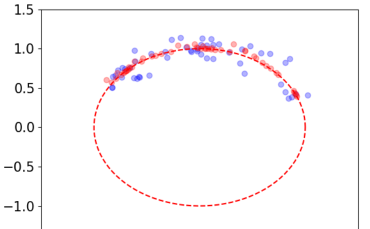
Results (Circle)

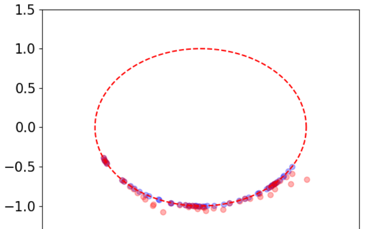


(a) NNODE (Train)



(b) NNODE (Validation)





Results (Fisher Rao Metric)

See Notebook

Future Work

1. Analysis of Asymptotic Properties
2. Uniqueness of Recovered Metric
3. Extend to pseudo-Riemannian metrics
4. Use real world data & extract Ricci Curvature, Laplacian etc.

Questions ?