

General Position Stresses

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Abstract

Let G be a graph with n vertices, and d be a target dimension. In this paper we study the set of rank $n - d - 1$ matrices that are equilibrium stress matrices for at least one (unspecified) d -dimensional framework of G in general position. In particular, we show that this set is algebraically irreducible. Likewise, we show that the set of frameworks with such equilibrium stress matrices is irreducible. As an application, this leads to a new and direct proof that every generically globally rigid graph has a generic framework that is universally rigid.

1 Introduction

Equilibrium stresses are an essential tool in the study of graph rigidity. Stresses can carry information about local, global and universal rigidity for specific and generic frameworks of a graph. An equilibrium stress of a framework (G, \mathbf{p}) is a vector ω in the cokernel of its rigidity matrix $R(\mathbf{p})$. Once \mathbf{p} is fixed, the equilibrium stresses are given by the solutions to

$$\omega^t R(\mathbf{p}) = 0 \tag{1}$$

and form a linear space. The dimension s of the space of stresses determines static, and so kinematic, rigidity of (G, \mathbf{p}) via the Maxwell index theorem

$$s - f = m - dn + \binom{d+1}{2} \tag{2}$$

where f is the dimension of the space of non-trivial infinitesimal flexes¹ and n and m are the number of vertices and edges of the graph G .

On the other hand, one can fix a set of edge weights ω and reformulate the equilibrium condition (1) as the vector equation

$$\sum_{j \neq i} \omega_{ij} (\mathbf{p}_i - \mathbf{p}_j) = 0 \quad (\text{at all vertices } i) \tag{3}$$

where we have $\omega_{ij} = \omega_{ji}$ and $\omega_{ij} = 0$ for non-edges. Holding ω fixed and treating \mathbf{p} as variable, the l.h.s. of (3) defines a linear map on 1-dimensional configurations that has as its matrix Ω a symmetric matrix with zeros on the non-edges and the all ones vector in its kernel.

In recent work on global and universal rigidity [7, 10, 11], it has been useful instead to let \mathbf{p} vary and study the variety of all the equilibrium stresses that a graph G in d -dimensions. This variety is simply the set of symmetric n -by- n matrices Ω of rank $\leq n - d - 1$, that have the all ones vector in their kernel and zeros in the entries corresponding to non edges of G .

For our applications, described below, we are most interested in knowing when this (real) “stress variety” is irreducible. Importantly, irreducibility implies that any subset that satisfies an

¹Formally, this is the quotient of the space of all flexes by the subspace of trivial flexes.

extra algebraic condition must be of strictly lower dimension. In turn, this means that various properties can be shown to either hold almost everywhere, or to hold almost nowhere. Similarly, this irreducibility means that we can talk about “generic” stresses in this variety and how they behave.

The general question of when a linear section of a determinantal variety by coordinate hyperplanes is irreducible has been studied in the algebro-geometric community [4, 8, 9, 13, 16]. Unfortunately for us, many of the known irreducibility results only apply for matrices of very high or very low rank and, so, do not suffice for our applications.

Another related question studied in the literature is the following: given a graph G , which frameworks (G, \mathbf{p}) support an unexpectedly high-dimensional spaces of stresses? Works such as [22] and [15] approach the problem from the point of view of the Grassmann-Cayley algebra (see, e.g., [3]). These only give an implicit description, as opposed to the kind of structural results we are after. Universality results for stresses, such as [20], suggest that the limitations of the above-mentioned results arise for a fundamental reason, namely that there are pairs (G, d) for which the d -dimensional stress variety of G can be very complicated. This would seem to rule out an explicit description for all graphs.

Results and guide to reading In this paper we take a different approach. Instead of looking to parameterize *all* stresses supported by some d -dimensional framework, we restrict our attention to the equilibrium stresses for G that have rank exactly $n - d - 1$ and such that any set of $n - d - 1$ columns is linearly independent. We call these “general position stresses” (or, for short, “Gstresses”) and denote the set of them by Gstr . For our applications, a good understanding of such Gstr is usually enough.

Let us start with the special case in which G is a $(d + 1)$ -connected graph that contains a K_{d+1} subgraph. A consequence of the “rubber band” theorem of Linial–Lovász–Widgerson [17] is that we can put any generic edge weights on the edges of G outside the distinguished K_{d+1} , and there will be a unique assignment of weights to the edges of this K_{d+1} so that we will obtain a general position stress of rank $n - d - 1$ for G . Analyzing the resulting map more closely, we obtain a parameterization of the general position stresses for G .

Theorem 1.1. *Let G be a $(d + 1)$ -connected graph with m edges and suppose that H is a K_{d+1} subgraph of G . Then Gstr is irreducible, of dimension $m - \binom{d+1}{2}$, and parameterized by a Zariski open subset of $\mathbb{R}^{m - \binom{d+1}{2}}$, representing the weights on the edges of $G \setminus H$, under a rational map.*

The existence of a K_{d+1} subgraph is important as a hypothesis if we want to use rubber bands: if we were to put generic weights on *every* edge, we will obtain an equilibrium stress matrix with rank $n - 1$, which correspond to frameworks with all points on top of each other. On the other hand, if we pick some arbitrary set of $m - \binom{d+1}{2}$ edges and give them generic weights, the rubber band construction no longer applies.

To deal with the general setting, where there may not be a K_{d+1} subgraph, we will replace rubber bands by “orthogonal representations”, which were introduced by Lovász–Saks–Schrijver [18], as the basis of a parameterization of Gstr .

Our main result is that Gstr is also well behaved in this general setting. If G has m edges and is $(d + 1)$ -connected then Gstr is an irreducible, quasi-projective variety of dimension $m - \binom{d+1}{2}$. If G is not $(d + 1)$ -connected, then Gstr is empty.

As an immediate application of this result, we can obtain a new direct proof the main result from [7], namely that a graph that is generically globally rigid in d -dimensions must have a generic d -dimensional framework that is super stable (and thus universally rigid).

Next, we can turn things around and look at the set of d -dimensional frameworks that are general position and have a stress of rank $n - d - 1$. We call these the “Gstressable” frameworks. Indeed, we will see that the Gstressable frameworks are exactly those frameworks with a Gstress. Our main result on this topic is that the set of Gstressable frameworks is an irreducible constructible set of configurations.

We will see that when G is not $d + 1$ connected, the Gstressable set is empty, and when G is generically globally rigid, this set includes almost all d -dimensional frameworks. So the most interesting case is for graphs in-the-middle of these two extremes. As we will see, for such graphs, if a framework is Gstressable, it must be the case that its rigidity matrix has dropped rank. For example if G is generically locally rigid, then a Gstressable framework must be infinitesimally flexible. Still, not every framework with a deficient rigidity matrix is Gstressable.

Finally, we show that the requirement of general position can be relaxed somewhat without sacrifice. In particular we define a framework as “Fstressable” if it has a stress of rank $n - d - 1$, and each vertex has a full dimensional affine span. We then show that any Fstressable can be approximated by Gstressable frameworks. This places the Fstressable set in the Euclidean and thus Zariski closure of the Gstressable set, rendering it irreducible as well.

As mentioned above, we are most interested in the property of irreducibility, but along the way, we will take care to prove what we can about the algebraic object-types of the sets we encounter. The nicest objects that we will deal with are “algebraic”. Such an object is cut out of \mathbb{R}^N using algebraic equalities. The next type is “quasi-projective”. Such an object is cut out using algebraic equalities and non-equalities (\neq). The next larger class is “constructible”. This class allows for finite unions of quasi-projective objects, or alternatively, a finite number of Boolean operations over algebraic sets. These objects are less uniform, but when irreducible, still contain almost all of the points in their Zariski-closure. Finally there are the semi-algebraic sets. Such an object is cut out using algebraic equalities non-equalities (\neq), and inequalities. Semi-algebraic objects can comprise very small portions of their Zariski-closures.

This paper relies essentially on the construction of GORs from [18], the construction of PSD stresses from GORs from [1], and the detailed study of the combination from [7]. The main innovation of this paper is to use GORs in the complex setting. This will allow us to construct and study the full set of Gstresses (real, but of unconstrained signature).

2 Background

2.1 Stresses

In this paper, we will always fix some dimension d and some graph G with n vertices and m edges.

Let \mathbf{p} be a *configuration* of n points in \mathbb{R}^d . Let $\omega = (\dots, \omega_{ij}, \dots)$ be an assignment of a real scalar $\omega_{ij} = \omega_{ji}$ to each edge, $\{i, j\}$ in G . (We have $\omega_{ij} = 0$, when $\{i, j\}$ is not an edge of G .) We say that ω is an *equilibrium stress vector* for (G, \mathbf{p}) if the vector equation

$$\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \tag{4}$$

holds for all vertices i of G . The equilibrium stress vectors of (G, \mathbf{p}) form the co-kernel of its rigidity matrix $R(\mathbf{p})$.

We associate an n -by- n *equilibrium stress matrix* Ω of (G, \mathbf{p}) to a stress vector ω of (G, \mathbf{p}) , by setting the i, j th entry of Ω to $-\omega_{ij}$, for $i \neq j$, and the diagonal entries of Ω are set such that the row and column sums of Ω are zero.

For each of the d spatial dimensions, if we define a vector \mathbf{v} in \mathbb{R}^n by collecting the associated coordinate over all of the points in \mathbf{p} , we have $\Omega\mathbf{v} = 0$. Thus if the dimension of the affine span of the vertices \mathbf{p} is d , then the rank of Ω is at most $n - d - 1$, but it could be less.

We define the equilibrium stress matrices of G in dimension d to be the union of the sets of equilibrium stress matrices for (G, \mathbf{p}) over all \mathbf{p} in \mathbb{R}^d . These are the symmetric n -by- n matrices of rank $n - d - 1$ or less, with the all ones vector in its kernel, and zero entries corresponding to non-edge pairs of distinct vertices.

In what follows we will drop the word “equilibrium” as well as the dimension for brevity, and just call these stress matrices for G .

2.2 Kernel frameworks

We will also go in the reverse direction. We will start with a stress matrix and find its associated frameworks. To this end, we introduce homogeneous coordinates for a configuration \mathbf{p} of n points in \mathbb{R}^d by $\hat{\mathbf{p}}_i = (\mathbf{p}_i; 1)$; i.e., appending a 1 in the last dimension. Denote by $\hat{\mathbf{p}}$ the resulting vector configuration in \mathbb{R}^{d+1} . We note for later that \mathbf{p} is in *affine general position* (no subset of $k \leq d + 1$ points of \mathbb{R}^d on a $(k - 2)$ -dimensional affine space) if and only if $\hat{\mathbf{p}}$ is in *linear general position* (no subset of $k \leq d + 1$ vectors of \mathbb{R}^{d+1} in a $(k - 1)$ -dimensional linear space).

Now, fixing a stress matrix Ω for G , we say that (G, \mathbf{p}) is a *kernel framework* for Ω if $\Omega P = 0$, where P has as its rows the vectors $\hat{\mathbf{p}}_i^t$. If \mathbf{p} has r -dimensional affine span and is a kernel framework for Ω , then Ω has rank at most $n - r - 1$, but it might be lower.

2.3 Rigidity

The following definitions are standard.

Two frameworks (G, \mathbf{p}) and (G, \mathbf{q}) are *equivalent* if

$$\|\mathbf{p}_j - \mathbf{p}_i\| = \|\mathbf{q}_j - \mathbf{q}_i\| \quad (\text{all edges } \{i, j\} \text{ of } G)$$

They are *congruent* if there is a Euclidean motion T of \mathbb{R}^d so that

$$\mathbf{q}_i = T(\mathbf{p}_i) \quad (\text{all verts. } i \text{ of } G)$$

A framework (G, \mathbf{p}) is *rigid* if there is a neighborhood $U \ni \mathbf{p}$ of configurations in \mathbb{R}^d so that if $\mathbf{q} \in U$ and (G, \mathbf{q}) is equivalent to (G, \mathbf{p}) , then \mathbf{q} is congruent to \mathbf{p} . A framework (G, \mathbf{p}) is *globally rigid* if *any* (G, \mathbf{q}) in \mathbb{R}^d equivalent to (G, \mathbf{p}) is congruent to it. A framework (G, \mathbf{p}) is *universally rigid* if *any* (G, \mathbf{q}) , in any dimension, equivalent to (G, \mathbf{p}) is congruent to it.

A configuration is *generic* if its coordinates are algebraically independent over \mathbb{Q} .

Theorem 2.1 ([2, 11]). *Let d be a dimension and G a graph. Then either every generic framework (G, \mathbf{p}) in dimension d is (globally) rigid or no generic framework is (globally) rigid.*

We summarize this theorem by saying that rigidity [2] and global rigidity [11] are *generic properties*. If every generic framework (G, \mathbf{p}) in dimension d is (globally) rigid, we say that G is *generically (globally) rigid* in dimension d .

Rigidity Certificates One important way to certify that a framework is rigid is via the stronger property of infinitesimal rigidity: The *rigidity matrix* $R(\mathbf{p})$ of a framework (G, \mathbf{p}) is the matrix of the linear system

$$\langle \mathbf{p}_j - \mathbf{p}_i, \mathbf{p}'_j - \mathbf{p}'_i \rangle = 0 \quad (\text{all edges } \{i, j\} \text{ of } G)$$

where the vector configuration \mathbf{p}' is variable. The kernel of $R(\mathbf{p})$ comprises the *infinitesimal flexes* of (G, \mathbf{p}) . When G is a graph with $n \geq d$ vertices, a d -dimensional framework (G, \mathbf{p}) is called *infinitesimally rigid* when $R(\mathbf{p})$ has rank $dn - \binom{d+1}{2}$. Infinitesimal rigidity implies rigidity [2].

One important way to certify that a framework is universally rigid is via the still stronger property of super stability. A framework with d -dimensional affine span is *super stable* if it has a positive semidefinite (PSD) equilibrium stress matrix Ω of rank $n - d - 1$ and its edges directions are not on a conic at infinity (a technical property discussed next).

The *edge directions* of a framework (G, \mathbf{p}) is the configuration \mathbf{e} of $|E|$ points at infinity $\mathbf{e}_{ij} := \mathbf{p}_j - \mathbf{p}_i$. A framework *has its edge directions on a conic at infinity* if there is a quadric surface \mathcal{Q} at infinity containing all of \mathbf{e} . Notably, if a framework is infinitesimally rigid, then its edge directions cannot be on a conic at infinity.

The main connection between these concepts is due to Connelly.

Theorem 2.2 ([5]). *If (G, \mathbf{p}) is super stable, then it is universally rigid.*

Generic Global Rigidity Connelly [6] proved the following sufficient condition for generic global rigidity of a graph.

Theorem 2.3. *If some generic framework (G, \mathbf{p}) in \mathbb{R}^d has an (even indefinite) equilibrium stress matrix of rank $n - d - 1$, then the graph G is generically globally rigid in \mathbb{R}^d .*

Gortler Healy and Thurston [11] proved the following:

Theorem 2.4. *If some generic framework (G, \mathbf{p}) in \mathbb{R}^d does not have equilibrium stress matrix of rank $n - d - 1$, then the graph G is not generically globally rigid in \mathbb{R}^d .*

The following is the easy half of a theorem by Hendrickson [12], which we will need below.

Theorem 2.5. *If G is generically globally rigid in \mathbb{R}^d , then it must be $(d + 1)$ -connected.*

3 PSD general position stresses (mostly review)

Definition 3.1. *Fix a dimension d . Let G be a graph with $n \geq d + 2$ vertices. Let **PGstr** be the real semi-algebraic set of n -by- n positive semidefinite general position d -dimensional stress matrices for G . Specifically, this is the set of real PSD symmetric matrices that have 0 entries corresponding to non-edges of G , have the all-ones vector in its kernel, have rank equal to $n - d - 1$, and such that every subset of $n - d - 1$ columns is a linearly independent set.*

The main theorem of this section is the following

Theorem 3.2. *Fixing a dimension d , let G be $(d + 1)$ -connected. Then the set of positive semidefinite general position (real) stress matrices for G is an irreducible real semi-algebraic set of dimension $m - \binom{d+1}{2}$.*

If G is not $(d + 1)$ -connected then this set is empty.

This result is mostly just a repackaging of a result from [7], which uses a construction of Alfakih's [1], that is, in turn, based on the work of [18]. Let's work through this backwards.

Before going on, we recall that Gale duality is the following linear algebra statement.

Lemma 3.3. *Let A be an $m \times n$ matrix of rank $n - r$ with entries in any field and X an $n \times r$ matrix with columns that span the kernel of A . Then a subset of r rows of X is linearly independent if and only if the complementary set of $n - r$ columns of A is linearly independent.*

See [21] for a nice proof.

Let us explicitly restate a general position result used in [1] to relate general position in stresses to general position kernel frameworks.

Lemma 3.4. *Let G be $(d+1)$ -connected. If $\Omega \in \text{PGstr}$ for G , and \mathbf{p} affinely spans \mathbb{R}^d and is in the kernel of Ω , then \mathbf{p} is in affine general position. If \mathbf{p} is in affine general position, then any rank $n-d-1$ PSD stress matrix Ω with \mathbf{p} in its kernel is in PGstr.*

Proof. This follows from Gale duality (see Lemma 3.3). We give the details for completeness.

Let \mathbf{p} be an affinely spanning configuration of n points in \mathbb{R}^d and Ω a stress matrix of rank $n-d-1$ with \mathbf{p} in its kernel. By general position, the vector configuration $\hat{\mathbf{p}}$ is in linearly general position. If the matrix P has the vectors $\hat{\mathbf{p}}_i^t$ as its rows then $\Omega P = 0$, and so the columns of X span the kernel of Ω . Gale duality then implies that any $n-d-1$ columns Ω are linearly independent. Hence Ω is a general position stress.

Going in the other direction, suppose that Ω is a general position stress matrix. Let P be any matrix with columns spanning the kernel of Ω where the last column is all ones. This is possible because Ω is a stress matrix. Gale duality now implies that the rows of P are in linearly general position. Interpreted as homogeneous coordinates, we see that P corresponds to a configuration \mathbf{p} in affine general position. Since every affinely spanning kernel framework arises this way they are all in general position. \square

3.1 GORs and connectivity

In [18], Lovász Saks and Schriver define a concept called a (GOR) general position orthogonal representation of a graph G in \mathbb{R}^D .

Definition 3.5. *Let G be a graph and let $D \geq 1$ be a fixed dimension. An (OR) orthogonal representation of G in \mathbb{R}^D is a vector configuration \mathbf{v} indexed by the vertices of G in \mathbb{R}^D with the following property: \mathbf{v}_i is orthogonal to the vectors associated with each non-neighbor of vertex i . The set of ORs is an algebraic set.*

A (GOR) general position orthogonal representation of G in \mathbb{R}^D is an OR in \mathbb{R}^D with the added property that the \mathbf{v}_i are in general linear position. The set of GORs is a quasi-projective variety.

The relevant results from [18] are the following.

Theorem 3.6. *Let $n > D$. Let G , a graph on n vertices, be $(n-D)$ -connected for some D . Then G must have a GOR in \mathbb{R}^D [18, Theorem 1.1]. Moreover, the set of all such GORs of G is an irreducible quasi-projective variety [18, Theorem 2.1]. If G is not $(n-D)$ -connected, then it cannot have a GOR in \mathbb{R}^D .*

In our terminology, we will set $D := n - d - 1$ where d is fixed, and will consider graphs that are $(d+1)$ -connected. These graphs have at least $d+2$ vertices, so $D \geq 1$. With this notation, the theorem tells us that we need $(d+1)$ -connectivity to obtain GORs in \mathbb{R}^{n-d-1} .

Definition 3.7. *Let G be a $(d+1)$ -connected graph with n vertices, for some d . Denote by D_G the dimension of the set its GORs in \mathbb{R}^{n-d-1} .*

The following is from [7].

Corollary 3.8. *Let G be a $(d+1)$ -connected graph with n vertices and m edges. Then the dimension D_G is $n(n-d) - \binom{n+1}{2} + m$.*

3.2 LSS stresses

Because of the orthogonality property of a GOR, its Gram matrix has the right zero/non-zero pattern to be a stress matrix. Alfakih [1] builds on this. First we set some notation.

Definition 3.9. Let G be a $(d+1)$ -connected graph and \mathbf{v} a GOR of G in dimension $n-d-1$. Since G is $(d+1)$ -connected, $n \geq d+2$, so $n-d-1 > 0$.

The $(n-d-1) \times n$ matrix X with the \mathbf{v}_i as its columns is the configuration matrix of \mathbf{v} . We denote the Gram matrix $X^t X$ of \mathbf{v} by Ψ . Note that Ψ is, by construction, PSD and has rank $n-d-1$, (as \mathbf{v} is in general position).

A GOR \mathbf{v} is called centered if its barycenter is the origin. We define GOR^0 to be the set of centered GORs.

The Gram matrix Ψ is a stress matrix (which we will call Ω) if and only if \mathbf{v} is centered. (Recall that the extra condition is that the all-ones vector is in the kernel.) Such an Ω is PSD and of rank $n-d-1$.

We define the set LSS of Lovász-Saks-Schrijver stresses to be the collection of stress matrices Ω arising as the Gram matrices of centered GORs. Denote its dimension by D_L .

A full rank centering map of a GOR \mathbf{v} is a set of non-zero scalars α_i so that the configuration

$$\{\alpha_1 \mathbf{v}_1, \dots, \alpha_n \mathbf{v}_n\}$$

has its barycenter at the origin. Alfakih's main result in [1] is the following.

Theorem 3.10 ([1]). Let G be $(d+1)$ -connected. Then any GOR in \mathbb{R}^{n-d-1} for G has a full rank centering map. This gives rise to stress matrix Ω in LSS. Moreover, any framework (G, \mathbf{p}) with d -dimensional affine span, that has Ω as an equilibrium stress matrix, must be in affine general position and be super stable.

The following is from [7].

Corollary 3.11. If G is $(d+1)$ -connected then $D_L = m - \binom{d+1}{2}$ and LSS is an irreducible semi-algebraic set. If G is not $(d+1)$ -connected, then LSS is empty.

3.3 Finishing the Proof

We are nearly ready for the proof of Theorem 3.2.

Lemma 3.12. PGstr is equal to LSS.

Proof. Every matrix in LSS has all of these properties. In particular, the general position of the vector configurations in GOR^0 gives rise to the required linear independence of column subsets.

Going the other way, every matrix Ω in PGstr can be shown to be in LSS. By assumption, Ω is of rank $n-d-1 > 0$. It can be Takagi factored, $\Omega = X^t X$, with the columns of X describing a real vector configuration in \mathbb{R}^{n-d-1} . This configuration must have all of the properties of a centered OR. The linear independence of each set of $n-d-1$ columns from Ω places the columns of X in general position. \square

Proof of Theorem 3.2. We simply combine Corollary 3.11 with Lemma 3.12. \square

4 General position stresses

With the necessary background in place, we define our main new concept.

Definition 4.1. Fix a dimension d . Let G be a graph with $n \geq d + 2$ vertices. Let Str be the real quasi projective variety of real symmetric n -by- n matrices that have 0 entries corresponding to non-edges of G , have the all-ones vector in its kernel, and have rank equal to $n - d - 1$.

Let Gstr be the real quasi projective variety of n -by- n general position d -dimensional stress matrices for G . Specifically, this is the matrices in Str such that every subset of $n - d - 1$ columns is linearly independent. We call such a matrix a Gstress .

The following is the central result of this paper.

Theorem 4.2. Fixing a dimension d , let G be $(d + 1)$ -connected. Then the set of general position stress matrices for G is an irreducible real quasi-projective variety of dimension $m - \binom{d+1}{2}$.

If G is not $(d + 1)$ -connected, then this set is empty.

Again, we have a lemma that allows us to transfer general position between stresses and their kernel frameworks.

Lemma 4.3. Let G be $(d + 1)$ -connected. If $\Omega \in \text{Gstr}$ for G , and \mathbf{p} affinely spans \mathbb{R}^d and is in the kernel of Ω , then \mathbf{p} is in affine general position. If \mathbf{p} is in affine general position, then any rank $n - d - 1$ stress matrix Ω with \mathbf{p} in its kernel is in Gstr .

Proof. The proof of Lemma 3.4 did not use the signature of Ω so it works verbatim here. \square

The plan is to first study GORs in the complex setting, then study LSS stress matrices in the complex setting. We will obtain results for the complex setting that are, essentially, the same as those from the real PSD one. Finally we take the real locus of this set of complex stress matrices, which will be Gstr .

4.1 Complex GORs

Definition 4.4. Let G be a graph and let $D \geq 1$ be a fixed dimension. An (COR) complex orthogonal representation of G in \mathbb{C}^D is a vector configuration \mathbf{v} indexed by the vertices of G in \mathbb{C}^D with the following property: if $\mathbf{v}_i = (\alpha_1, \dots, \alpha_D)$ and i is not a neighbor of j and $\mathbf{v}_j = (\beta_1, \dots, \beta_D)$, then

$$\sum_{k=1}^D \alpha_k \beta_k = 0$$

Notice that this ‘‘algebraic dot product’’ is defined without conjugation. The set of CORs form a complex algebraic set.

A (CGOR) general position complex orthogonal representation of G in \mathbb{C}^D is a COR in \mathbb{C}^D with the added property that the \mathbf{v}_i are in general linear position. The set of CGORs form a complex quasi-projective variety set.

Theorem 4.5. Let $n > D$. Let G be a $(n - D)$ -connected graph with n vertices. Then the set of CGORs in \mathbb{C}^D is non-empty and is an irreducible quasi projective variety. If G is not $(n - D)$ -connected, then it cannot have a CGOR in \mathbb{C}^D , or a COR in which the non-neighbors of every vertex are represented by linearly independent vectors.

Proof. First we suppose that G is $(n - D)$ -connected. From Theorem 3.6, we have the existence of a GOR, which is also a CGOR. Thus the set of CGOR is non-empty. The proof of the irreducibility of the GORs from [18] carries over identically to the complex setting, thus the set of CGORs is irreducible as well. This gives us the first half of theorem.

Because orthogonality under our bilinear form does not imply linear independence in the complex setting, we need to deal with the low-connected case slightly differently than [18, Theorem 1.1']. Suppose that G is not $(n - D)$ -connected. The graph G is the union of vertex-induced subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ so that $V_1 \cap V_2$ has at most $n - D - 1$ vertices. For convenience, set $X_1 = V_1 \setminus V_2$ and $X_2 = V_2 \setminus V_1$. Both of these sets are non-empty. We also have $|X_1| + |X_2| \geq D + 1$.

Now we consider a COR \mathbf{v} for G in \mathbb{C}^D . Suppose first that the vectors $\{\mathbf{v}_i : i \in X_1\}$ have a D -dimensional span. From the bilinearity of the complex algebraic dot product, for each $j \in X_2$, we have the linear constraints

$$\mathbf{v}_j \cdot \mathbf{v}_i = 0 \quad (\text{all } i \in X_1)$$

The $|X_2|$ vectors of \mathbf{v} corresponding to vertices in X_2 are, therefore all zero vectors. We conclude that \mathbf{v} is not in general position. Moreover, the non-neighbors of any vertex in X_1 contain all of X_2 , so there is a vertex with its non-neighbors linearly dependent.

The same is true if the X_2 vectors have a D -dimensional span. So going forward, let us assume that neither the X_1 or X_2 vectors have a D -dimensional span.

Suppose that there was a linear dependency in either the X_1 or X_2 vectors. Since neither set has full span, this would rule out general position. Moreover, the non-neighbors of any vertex in X_i contain all of X_j , so there is a vertex with its non-neighbors linearly dependent.

So going forward, let us assume that both the X_1 and the X_2 vectors are linearly independent.

The returning to the constraints, for each $j \in X_2$, the linear constraints

$$\mathbf{v}_j \cdot \mathbf{v}_i = 0 \quad (\text{all } i \in X_1)$$

are linearly independent. The $|X_2|$ vectors are constrained to lie in a subspace of dimension at most

$$D - |X_1|$$

and by linear independence, have a cardinality with this same bound. But this would contradict $|X_1| + |X_2| \geq D + 1$. \square

In our terminology, we will set $D := n - d - 1$ where d is fixed, and will consider graphs that are $(d + 1)$ -connected. These graphs have at least $d + 2$ vertices, so $D \geq 1$. With this notation, the theorem tells us that we need $(d + 1)$ -connectivity to obtain CGORs in \mathbb{C}^{n-d-1} .

Definition 4.6. Let G be a $(d + 1)$ -connected graph with n vertices, for some d . Denote by D_{CG} the complex dimension of the set its CGORs in \mathbb{C}^{n-d-1} .

Theorem 4.7. Let G be a $(d + 1)$ -connected graph with n vertices and m edges. Then the set of CGORs has dimension, D_{CG} , equals D_G .

The proof for the dimension count, is then no different from that of Corollary 3.8 (which is done in [7]).

4.2 Complex LSS

We now apply the LSS construction to the complex setting.

Definition 4.8. Let G be a $(d+1)$ -connected graph and \mathbf{v} a CGOR of G in dimension $n-d-1$. The $(n-d-1) \times n$ matrix X with the \mathbf{v}_i as its columns is the configuration matrix of \mathbf{v} . We denote the algebraic-Gram matrix $X^t X$ (with no conjugation) of \mathbf{v} by Ψ . Note that Ψ has rank $n-d-1$, (as \mathbf{v} is in general position and using Lemma 4.10).

A CGOR \mathbf{v} is called centered if its barycenter is the origin. We define CGOR^0 to be the complex quasi-projective variety of centered CGORs.

The Gram matrix Ψ is a (complex) stress matrix (which we will call Ω) if and only if \mathbf{v} is centered. (Recall that the extra condition is that the all-ones vector is in the kernel.)

We define the set CLSS to be the collection of stress matrices Ω arising as the Gram matrices of centered CGORs. Denote its complex dimension by D_{CL} .

Theorem 4.9. If G is $(d+1)$ -connected, then $D_{\text{CL}} = D_L$ and CLSS is irreducible.

If G is not $(d+1)$ -connected then CLSS is empty.

Proof. This follows using the proofs of Theorem 3.10 from [1] and Corollary 3.11 from [7] work verbatim in the complex setting. \square

4.3 Finishing the proof

We will obtain Gstr using a Gram matrix construction. There is a technical issue, which is that, for complex matrices A , we do not necessarily have $\text{Rank}(A^t A) = \text{Rank}(A)$. However, if A has linearly independent rows, this does hold.

Lemma 4.10. Let A be a complex D -by- n matrix with rank D . Then $A^t A$ has rank D .

This follows from the fact that A^t must have linearly independent columns by assumption. Meanwhile, left multiplication by a matrix with linearly independent columns does not change the rank.

Lemma 4.11. Gstr is equal to the real locus of CLSS .

Proof. Let X be the real locus of CLSS and let $\Omega \in X$ be given. By definition, there is a $\mathbf{v} \in \text{CGOR}^0$ so that Ω is the Gram matrix of \mathbf{v} . The configuration matrix of \mathbf{v} has $n-d-1$ rows and n columns in linearly general position, so Lemma 4.10 implies that Ω has rank $n-d-1$. General position of \mathbf{v} also implies that the columns of Ω are in linearly general position, so Ω is a Gstress. Hence $X \subseteq \text{Gstr}$.

Now let Ω be a general position stress. By assumption, Ω is of rank $n-d-1 > 0$. It can be Takagi factored, $\Omega = X^t X$, with the columns of X describing a vector configuration in \mathbb{C}^{n-d-1} . Because Ω is a stress matrix, X will correspond to a centered orthogonal representation of G .

To conclude, we need to check that this orthogonal representation is in general position. Suppose the contrary. Then we have some $k \leq n-d-1$ columns of X that are linearly dependent. Let X' denote the submatrix of X obtained by throwing away the other columns. Then $X'^t X'$ corresponds to k columns of Ω that are linearly dependent, contradicting that Ω is a general position stress. Hence X is in general position, and, thus, defines an element of CGOR^0 . \square

Now we compute the dimension of the GStresses.

Lemma 4.12. If G is $(d+1)$ -connected, then Gstr has dimension equal to D_L

Proof. Gstr includes PGstr and so must be of real dimension at least D_L .

On the other hand the as the real locus of a complex quasi-projective variety (Lemma 4.11), its real dimension cannot be larger than the complex dimension D_{CL} , which is equal to D_L [23]. \square

Lemma 4.13. *If G is $(d+1)$ -connected, then Gstr is an irreducible real quasi-projective variety.*

Proof. The lemma follows from Lemma A.4 in light of lemmas 4.11 and 4.12 and Theorem 4.9. \square

Proof of Theorem 4.2. This combines Lemmas 4.12 and 4.13. \square

4.4 An easier special case

We have used CGORs to describe the general position stresses of any $(d+1)$ -connected graph G with m edges. In the special case that G also has K_{d+1} as a subgraph, we can obtain the same characterization of its general position stresses without resorting to GOR technology. In particular, we can parameterize Gstr by a Zariski open subset of $\mathbb{R}^{m - \binom{d+1}{2}}$ using a more elementary result of Linial, Lovász and Widgerson [17].

Lemma 4.14. *Let G be a $(d+1)$ -connected graph with m edges and suppose that H is a K_{d+1} subgraph of G . Then, for a Zariski open subset of assignments $\mathbf{w}_{G \setminus H}$ of weights to the edges of G not contained in H , there is a unique general position stress Ω that agrees with $\mathbf{w}_{G \setminus H}$ on the edges of $G \setminus H$.*

Proof. It is well-known (e.g., [17]), that for $\mathbf{w}_{G \setminus H}$ consisting of positive real numbers and fixed, and affinely independent positions \mathbf{p}_H for the vertices of H , there is a unique, configuration \mathbf{p} in dimension d that agrees with \mathbf{p}_H on the vertices of H with the additional property that the vertices of $G \setminus H$ are in equilibrium with the weights $\mathbf{w}_{G \setminus H}$. This \mathbf{p} can be found as the unique solution to a non-singular linear system depending on G , \mathbf{p}_H , and $\mathbf{w}_{G \setminus H}$ ². Since matrix singularity is an algebraic condition, this existence and uniqueness is true over a Zariski open subset even when allowing negative numbers.

Once we have $\mathbf{p}_{G \setminus H}$, we can use it to complete $\mathbf{w}_{G \setminus H}$ to an equilibrium stress as follows. The resultant forces³ on H are an equilibrium load by Lemma B.3. Since (H, \mathbf{p}_H) is statically rigid and independent, this load has a unique resolution \mathbf{w}_H by Lemma B.1. Combining \mathbf{w}_H and $\mathbf{w}_{G \setminus H}$, we get equilibrium at every vertex. So this process gives us a unique Ω for fixed $\mathbf{w}_{G \setminus H}$ and \mathbf{p}_H .

As a stress for \mathbf{p} , Ω must have rank at most $n - d - 1$. Given \mathbf{p}_H , the rest of \mathbf{p} was uniquely determined by the equilibrium condition of our linear system, so Ω must have rank equal to $n - d - 1$. Moreover its kernel frameworks must be the affine transforms of \mathbf{p} .

Notice that in our linear system, if we change the positions of \mathbf{p}_H , a process that can be modeled as an affine transform, then the solution \mathbf{p} will undergo the same affine transform. Ω will also be a stress for this affinely transformed \mathbf{p} , and so must be the unique stress completion that will be obtained from our construction. Thus $\mathbf{w}_{G \setminus H}$ has a unique stress completion.

Finally we need to establish the general position property. It is shown in [17, Theorem 2.4], that for $\mathbf{w}_{G \setminus H}$ consisting of generic positive real numbers and fixed, affinely independent positions \mathbf{p}_H for the vertices of H , that the resulting \mathbf{p} is in affine general position in dimension d . This result also holds generically when allowing negative numbers, and so holds over a Zariski open subset.

²This system arises as the gradient of a convex energy function, but we don't need that here. All that is important is that it can be solved.

³Computing these is where we need to know the vertex positions.

Since \mathbf{p} is in affine general position Lemma 4.3 says the stress we obtained is a general position stress. \square

Definition 4.15. Let G be a $(d+1)$ -connected graph with a K_{d+1} subgraph H . We define the map $\rho : X \rightarrow \text{Str}$ to be the one from Lemma 4.14 where $X \subseteq \mathbb{R}^{m - \binom{d+1}{2}}$ is the implicit Zariski open domain where the linear system is non-singular.

The map ρ is clearly injective. Now we argue that it is surjective onto Gstr .

Lemma 4.16. Let G be a $(d+1)$ -connected graph with m edges and suppose that H is a K_{d+1} subgraph of G . Let Ω be a general position stress. Let $\mathbf{w}_{G \setminus H}$ be its values on the edges not contained in H . Then $\rho(\mathbf{w}_{G \setminus H})$ is well defined and equal to Ω .

Proof. By our Gstr assumption, any affinely spanning \mathbf{p} in the kernel of Ω (which must be in general position from Lemma 4.3) is a non-singular affine image of a unique kernel configuration \mathbf{p}_0 with the vertices of H the standard simplex. Meanwhile, due to the rank of Ω , there is only one \mathbf{p} in the kernel, up to an affine transform. Thus the linear system used to define the map ρ is non-singular, so the map is well defined here. Hence we can use the construction of Lemma 4.14 to uniquely complete the weights on the edges of $G \setminus H$ coming from Ω . This unique completion must be Ω , since Ω is a completion. \square

We summarize this section by the following.

Theorem 4.17. Let G be a $(d+1)$ -connected graph with m edges and suppose that H is a K_{d+1} subgraph of G . Then Gstr is irreducible, of dimension $m - \binom{d+1}{2}$, and parameterized by a Zariski open subset of $\mathbb{R}^{m - \binom{d+1}{2}}$, representing the weights on the edges of $G \setminus H$, under a rational map.

This result agrees with Theorem 4.2, while circumventing the machinery of orthogonal representations. The proof also shows that the dimension $m - \binom{d+1}{2}$ appearing in Theorem 4.2 has a natural interpretation when there is a K_{d+1} subgraph: we can freely assign weights to the rest of the edges and then compensate on edges of the K_{d+1} . That this dimension stays the same without a K_{d+1} subgraph is interesting and somewhat deeper.

5 General position stressable frameworks

Now we turn the tables and look at frameworks that are in equilibrium under a Gstress .

Definition 5.1. Let d be a dimension and let G be a graph with $n \geq d+2$ vertices. We define the **Gstressable frameworks** to be the set of frameworks (G, \mathbf{p}) in \mathbb{R}^d such that \mathbf{p} is in general affine position, and has an equilibrium stress of rank $n - d - 1$.

Lemma 5.2. A framework is Gstressable iff it has full span and a stress from Gstr .

Proof. See Lemma 4.3. \square

The main result of this section is the following

Proposition 5.3. The Gstressable frameworks form a real irreducible constructible set.

The basic idea is to create a rational map taking each stress in Gstr to a kernel framework. To make this map well defined, we mod out by affine transforms. In particular we select $d + 1$ vertices and pin them to canonical locations. Then for each stress, we can set up a linear system which will determine a kernel framework. This linear system will be non-singular, unless the stress Ω is of rank $< n - d - 1$, or when the kernel framework of Ω has these selected vertices with deficient span. Neither of these can happen for a Gstress .

Using this approach, we can immediately conclude that the Gstressable frameworks, as the image of an irreducible semi-algebraic set under a rational map, is irreducible and is semi-algebraic. But with a bit more work, we can upgrade this conclusion to real-constructible.

Proof. We will work over the complex general position stresses. In the complex setting, we know that the image of an irreducible constructible set under a rational map is constructible. In our case, the image, which we denote by X , consists of complex frameworks (in pinned position) with complex general position stresses.

Next we enlarge our set to include all non-singular affine images of each framework in X . To do this we build an irreducible bundle over X where the points are (\mathbf{p}, A) with A a non-singular affine transformation. Then we apply the polynomial map $(\mathbf{p}, A) \mapsto (A(\mathbf{p}_i))_{i=1}^n$. Hence, the affine orbit of X is constructible and irreducible.

This gives us the irreducible constructible set S of complex frameworks in general position with complex stresses of rank $n - d - 1$.

Now we take the real locus of S the set of real frameworks in general position with complex general stresses of rank $n - d - 1$. This must be a real constructible set.

Finally, we need to show that every framework in this real locus also has a real stress of rank $n - d - 1$. This is done in the next lemma, which completes the proof. \square

Lemma 5.4. *Let S be the set of complex frameworks in general position with complex stresses of rank $n - d - 1$. Denote by $\text{Re}(S)$ the real locus of S . Then $\text{Re}(S)$ is equal to the set of real frameworks in general position with a real stress of rank $n - d - 1$.*

Proof. One direction is trivial: by definition every Gstressable framework has a stress of rank $n - d - 1$ and so is in $\text{Re}(S)$.

For the other direction, every framework in $\text{Re}(S)$ is in general position, so we just need to show a real stress of rank $n - d - 1$. Fixing a $\mathbf{p} \in \text{Re}(S)$, its complex stress space is just the co-kernel of its rigidity matrix. Since the rigidity matrix is real valued, this complex space has a basis of real vectors. This makes the complex stress space equal to the complexification of the real stress space. This, in turn, implies that a generic stress from the real stress space is also generic in the complex stress space. Since every generic stress in the complex stress space has rank $n - d - 1$, we are done. \square

6 Locally Spanning

General position is a strong restriction on a framework or a GOR. In this section, we relax general position to a condition on vertex neighborhoods that is enough to get results similar to those in Section 5.

Definition 6.1. *Let G be a graph with $n \geq d + 2$ vertices. We define the **Fstressable** frameworks to be the set of d -dimensional frameworks (G, \mathbf{p}) such that each vertex neighborhood⁴ in \mathbf{p} has a full d -dimensional affine span and such that \mathbf{p} has an equilibrium stress of rank $n - d - 1$.*

⁴This includes the points corresponding to a vertex and its neighbors.

Our main results in this section will be the following, which say, informally, that the Fstressable frameworks can be approximated by Gstressable ones.

Theorem 6.2. *The set of Gstressable frameworks is standard-topology dense in the set of Fstressable frameworks.*

Theorem 6.3. *Fstressable is a contractible set. It is irreducible and has the same dimension as Gstressable.*

To this end we start with the next definition. It may seem a bit unnatural, but it will prove to be exactly what we need when we dualize and look at frameworks.

Definition 6.4. *A local full spanning orthogonal representation (for short, a FOR) \mathbf{v} of G in \mathbb{R}^D is an OR in \mathbb{R}^D with the added following property: For each vertex $i \in V(G)$, there are neighbors i_1, \dots, i_{n-D-1} of i so that the set of vectors*

$$\{\mathbf{v}_j : j \in \{1, \dots, n\} \setminus \{i, i_1, \dots, i_{n-D-1}\}\}$$

are linearly independent.

Any FOR necessarily spans \mathbb{R}^D .

We point out for later that any FOR has the property that the non-neighbors of any vertex are represented by linearly independent vectors.

Lemma 6.5. *Every GOR is a FOR.*

Proof. In a GOR, the configuration \mathbf{v} is in general position. Thus any D of the \mathbf{v}_j will be linearly independent, ensuring we can satisfy the FOR condition. \square

The definition of an FOR relaxes that of a GOR, but not so much that Theorem 3.10 becomes false. Here is the first part of it for FORs.

Lemma 6.6. *Any FOR has a full rank centering map.*

Proof. The main step of Alfaikih's proof that every GOR has a full rank centering map in [1] is that in a GOR, for each \mathbf{v}_i , there is a linear dependence among the vectors in \mathbf{v} with a non-zero coefficient on \mathbf{v}_i . This holds for a FOR because, from the definition, associated with each \mathbf{v}_i is a set of D vectors \mathbf{v}_j which span the D -dimensional space. The rest of the proof in [1] goes through unmodified. \square

If we require only that the non-neighbors of every vertex are linearly independent, then the conclusion of Lemma 6.6 becomes false, and also, the span of such an OR may be less than D -dimensional. We need to rule out both of these cases for our intended application.

We also want to generalize the signature of our metric.

Definition 6.7. *We pick of a set of D signs $s_i = \pm 1$. The choice of signs gives us a symmetric bilinear form*

$$\langle x, y \rangle := \sum_{i=1}^D s_i x_i y_i$$

With the signature fixed, we say that we are in a pseudo-Euclidean setting. In this setting, we call two vectors x and y orthogonal if $\langle x, y \rangle = 0$.

In any pseudo-Euclidean setting we can define the same notion of an OR, GOR, and FOR.

Results and proofs from [18] imply that the set of FORs has nice algebraic and geometric properties.

Lemma 6.8. *In each of the the real, or pseudo-Euclidean settings, the set of FORs is irreducible and the set of GORs is (standard topology) dense in the FORs. If G is not $(n-D)$ -connected, then the set of FORs is empty.*

Proof. In the real setting, it is explicitly stated in [18, Remark on page 448] that the GORs are dense in the class of ORs in which the non-neighbors every vertex are represented by linearly independent vectors. This follows from the proof of [18, Theorem 2.1]. We can apply the same proof for any pseudo-Euclidean setting by an appropriate sprinkling of negative signs.

The FORs are a subset of this larger class. The GORs are a subset of FORs from Lemma 6.5, and hence the GORs are dense in FOR as well.

For the less connected case, Theorem 4.5 shows that G cannot have a FOR in dimension D , since a FOR implies that the non-neighbors of every vertex are represented by linearly independent vectors. \square

In our terminology, we will set $D := n - d - 1$ where d is fixed, and will consider graphs that are $(d+1)$ -connected. These graphs have at least $d+2$ vertices, so $D \geq 1$. With this notation, the theorem tells us that we need $(d+1)$ -connectivity to obtain FORs in dimension $n - d - 1$.

At this point, we need to describe how to obtain a stress matrix from an FOR using Alfakih's construction from [1]. Given a centered FOR \mathbf{v} in some real or pseudo-Euclidean setting of G in dimension $n - d - 1$, the $(n - d - 1) \times n$ matrix X with the \mathbf{v}_i as its columns is the *configuration matrix* of \mathbf{v} . Let Ψ be the matrix $X^t S X$, where S is the diagonal matrix representing the pseudo-Euclidean setting we are in. Then Ψ will have rank $n - d - 1$, and its number of positive and negative eigenvalues will be determined their quantity in S . By the orthogonality property of an FOR, Ψ will have zeros corresponding to the non-edges of G . Because \mathbf{v} is centered, Ψ is a stress matrix.

Definition 6.9. *The set FLSS is the set of $n \times n$ stress matrices for a graph G that arise from applying the above construction to a centered FOR in the real or some pseudo-Euclidean setting, for a graph G in dimension d .*

We are ready to define the analogue to Gstr in the FOR setting.

Definition 6.10. *Let G be a graph with $n \geq d+2$ vertices. Let FST be the real quasi-projective variety of n -by- n locally full spanning d -dimensional stress matrices for G . Specifically, this is the set of real symmetric matrices Ω that have 0 entries corresponding to non-edges of G , have the all-ones vector in the kernel, rank $n - d - 1$, and the following additional property: For each vertex $i \in V(G)$, there are neighbors $\{i_1, \dots, i_d\}$ of i so that the columns of Ω corresponding to the vertices in*

$$V(G) \setminus \{i, i_1, \dots, i_d\}$$

are linearly independent.

Lemma 6.11. *Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are vectors that span \mathbb{R}^D , and let Ψ be the Gram matrix of the \mathbf{v}_i under any pseudo-Euclidean bilinear form. Then the columns of Ψ corresponding to any spanning subset of the \mathbf{v}_i are independent.*

Proof. Suppose that $B = \{\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_D}\}$ are independent, and hence span \mathbb{R}^D . Then the $D \times D$ Gram matrix Ψ' of these vectors represents the bilinear form in the basis B . Since any pseudo-Euclidean form is non-degenerate (there is no vector orthogonal to all vectors), Ψ' is non-singular. Since Ψ' appears as a sub-matrix of Ψ , the associated columns are linearly independent in Ψ . \square

As in the Gstress setting, the FStresses turn out to be exactly the ones arising from the FLSS construction.

Lemma 6.12. *The sets FLSS and FST are equal.*

Proof. First we suppose that Ω is in FLSS. All the properties of FST except for independence of complementary sets columns to a (subset of) each vertex neighborhood follow in the same was as in the proof of Lemma 3.12. To establish the last property, we use the defining properties of an FOR and then Lemma 6.11 to show that independence of spanning subsets of the FOR give rise to independent columns in Ω .

For the other direction, we can factor Ω as $X^t S X$, where S is the diagonal ± 1 matrix giving the pseudo-Euclidean signature and X is $n - d - 1 \times n$ and real. Lemma 6.11 and the definition of a locally general position stress matrix show that the vector configuration from the columns of X have the properties of an FOR. \square

At this point, we are ready to dualize back to frameworks.

Lemma 6.13. *A framework is Fstressable iff it has stress from Fstr.*

Proof. First suppose that (G, \mathbf{p}) is a framework in dimension d that is Fstressable. Let Ω be a stress matrix for (G, \mathbf{p}) of rank $n - d - 1$. For each vertex i of G , with neighbors i_1, \dots, i_k , the points

$$\mathbf{p}_i, \mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_k}$$

are affinely spanning. After relabeling, we may as well assume that

$$\mathbf{p}_i, \mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_d}$$

affinely span \mathbb{R}^d . By Gale duality, we conclude that the $n - d - 1$ columns of Ω corresponding to

$$V(G) \setminus \{i, i_1, \dots, i_d\}$$

are linearly independent. Hence $\Omega \in \text{Fstr}$, as desired.

Now suppose that (G, \mathbf{p}) is an affinely spanning framework in \mathbb{R}^d with a stress $\Omega \in \text{Fstr}$. By the definition of Fstr, (G, \mathbf{p}) has a stress of rank $n - d - 1$. Let a vertex $i \in V(G)$ be given. By the definition of a locally full spanning stress, there are neighbors i_1, \dots, i_d of i so that the columns of Ω corresponding to

$$V(G) \setminus \{i, i_1, \dots, i_d\}$$

are linearly independent. By Gale duality, the points

$$\mathbf{p}_i, \mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_d}$$

affinely span \mathbb{R}^d . As i was arbitrary, (G, \mathbf{p}) is Fstressable. \square

And now we can show that, in a sense, Fstressable frameworks are degenerations of Gstressable frameworks.

Theorem 6.14. *Let $\Omega \in \text{Fstr}$ be a locally full spanning equilibrium stress of a graph G . Then every standard topology neighborhood of Ω contains a Gstress.*

As a consequence, Fstr is irreducible and of the same dimension as Gstr.

Proof. As $\Omega \in \text{Fstr}$, by Lemma 6.12, Ω is the Gram matrix of a real or pseudo-Euclidean FOR \mathbf{v} that has rank D and is centered. In particular, the configuration matrix V of \mathbf{v} has the all ones vector in its kernel.

Let $\varepsilon > 0$ be given. By continuity of the map that sends a configuration to its Gram matrix, there is a $\delta > 0$ so that if \mathbf{v}' is a configuration with

$$\max_{1 \leq i \leq n} \|\mathbf{v}_i - \mathbf{v}'_i\| < \delta$$

then

$$\|\Omega - \Omega'\| < \varepsilon$$

where Ω' is the Gram matrix of \mathbf{v}' and the norm on the lhs is the operator norm (or any equivalent to it).

By Lemma 6.8, for any $r > 0$, there is a GOR \mathbf{v}'' so that

$$\max_{1 \leq i \leq n} \|\mathbf{v}_i - \mathbf{v}''_i\| < r$$

Because \mathbf{v} is a FOR and \mathbf{v}'' is a GOR, the configuration matrices V and V'' have the same rank for any choice of r and \mathbf{v}'' . If we let $x(r)$ be the orthogonal projection of the all ones vector onto the kernel of V'' , then as $r \rightarrow 0$, $x(r)$ converges to the all ones vector.

It follows that we can find an $r > 0$ and a GOR \mathbf{v}'' , so that

$$\max_{1 \leq i \leq n} \|\mathbf{v}_i - \mathbf{v}''_i\| < r < \delta/2$$

and, for each coordinate α_i of $x(r)$,

$$|1 - 1/\alpha_i| \|\mathbf{v}''_i\| < 2|1 - 1/\alpha_i| \|\mathbf{v}_i\| < \delta/2$$

We now set, for each $1 \leq i \leq n$

$$\mathbf{v}'_i = \frac{1}{\alpha_i} \mathbf{v}''_i$$

This is possible for sufficiently small δ because it forces all of the α_i to be non-zero. By construction

$$\alpha_1 \mathbf{v}''_1 + \cdots + \alpha_n \mathbf{v}''_n = 0$$

so

$$\mathbf{v}'_1 + \cdots + \mathbf{v}'_n = 0$$

which implies that \mathbf{v}' is centered. Since none of the α_i are zero, by [1, Lemma 3.2], \mathbf{v}' will also be a GOR. Since

$$\|\mathbf{v}_i - \mathbf{v}'_i\| \leq \|\mathbf{v}_i - \mathbf{v}''_i\| + |1 - 1/\alpha_i| \|\mathbf{v}''_i\| < \delta$$

we have proven that Gstr is Euclidean-topology dense in Fstr .

For the second statement, we simply note that as Gstresses are standard topology dense in the locally full spanning stresses, they are also Zariski dense. \square

Now we are ready to prove Theorem 6.2. The basic idea is to setup a map from stresses to kernel frameworks. The only complication is that we do not have general position, so we cannot fix the pinned vertices globally. Rather we do so locally in a Zariski neighborhood of an appropriate Ω .

Proof of Theorem 6.2. Let (G, \mathbf{p}) be an Fstressable framework, with stress Ω from Fstr . We select $d+1$ vertices that have a full span in \mathbf{p} and pin them in their places. Then we build a map from stress space to framework space as in the proof of Proposition 5.3. This rational map is continuous, and well defined over a Zariski-open neighborhood of Ω . By Theorem 6.14, this Zariski neighborhood contains has a Euclidean dense subset of Gstressses. By continuity of the parameterization, we can, for any Euclidean neighborhood U of (G, \mathbf{p}) , find a Gstress Ω' has a Gstressable kernel framework (G, \mathbf{p}') in U . \square

Proof of Theorem 6.3. For the first statement, we will apply the same (complexified) strategy used to prove Proposition 5.3. We will select $d+1$ vertices and pin to the canonical simplex and use a rational map from Fstr to Fstressable frameworks.

The only complication is that in our current setting, such a rational map may be undefined over some subvariety V of Fstr , where, in the equilibrium framework, the selected vertices would have a deficient span. To deal with this we can always pick a different set of vertices to pin, giving rise to a different rational map, with its own undefined locus W . Taking the union of the images over a finite number of such maps we can obtain all Fstressable frameworks.

The irreducibility and dimension then follows from the density of the Gstressable frameworks in the Fstressable frameworks (Theorem 6.2). \square

7 Stressed Corank

In this section we explore the typical rank of the rigidity matrix of a Gstressable framework. We will assume that G is $d+1$ -connected.

Definition 7.1. Let $\text{corank}(G)$ denote the minimum of corank of the rigidity matrix $R(\mathbf{p})$ over all d -dimensional frameworks (G, \mathbf{p}) . The quantity $\text{corank}(G)$ is the corank of the rigidity matrix for every generic framework (G, \mathbf{p}) .

Definition 7.2. Let $\text{corank}(G, \mathbf{p})$ be the corank of the rigidity matrix for (G, \mathbf{p}) . Let $\text{stressedCorank}(G)$ denote the minimum of $\text{corank}(G, \mathbf{p})$ over all Gstressable frameworks.

By definition $\text{stressedCorank}(G) \geq \text{corank}(G)$.

Because Gstressable is irreducible, we can talk about generic frameworks and their behaviour.

Lemma 7.3. $\text{stressedCorank}(G)$ is equal to the dimension of the space of equilibrium stresses for every generic Gstressable framework.

Proof. Since the stress count is the dimension of the left kernel of the rigidity matrix, and the rank of a matrix drops only when all its minors of some order vanish, the rank only drops on Zariski-closed subsets. \square

The next statement gives us another way to think about stressedCorank

Proposition 7.4. Suppose Ω is generic in Gstr . Let \mathbf{p} be a kernel framework of Ω with a d -dimensional span. Then $\text{corank}(G, \mathbf{p}) = \text{stressedCorank}(G)$.

Proof. The basic principle is that the image of a generic point of an irreducible algebraic object under a rational map, defined over \mathbb{Q} , is a generic point in the image. For this proposition, we build a map that takes a stress Ω in Gstr and a non-singular affine transform A to a kernel framework \mathbf{p} . The image of this map is the Gstressable frameworks (Lemma 4.3).

When Ω is generic in Gstr , we can find an A so that (Ω, A) is generic as a point of a bundle. Then from Lemma 7.3, $\text{corank}(G, \mathbf{p}) = \text{stressedCorank}(G)$. \square

We are now in a position to relate the dimension of the set of Gstressable frameworks of a graph G to the stressed corank of G .

Proposition 7.5. *For every graph G and dimension d :*

$$\begin{aligned}\dim(G\text{stressable}) - d(d+1) &= m - \binom{d+1}{2} - \text{stressedCorank}(G) \\ \dim(G\text{stressable}) &= m + \binom{d+1}{2} - \text{stressedCorank}(G)\end{aligned}$$

Proof. In an irreducible algebraic setting, the dimension of the image of a rational map plus the dimension of a generic fiber equals the dimension of the domain (See e.g. [7, Theorem A.12]).

On the first line, left hand side, we have the image of a map from stresses to pinned frameworks. The $d(d+1)$ accounts for the d -dimensional affine transforms. \square

7.1 Characterizing stressedCorank

Lemma 7.6. *If G is generically globally rigid in \mathbb{R}^d , then any generic framework (G, \mathbf{p}) in \mathbb{R}^d must have an equilibrium stress matrix in Gstr , and thus be Gstressable.*

Proof. Let \mathbf{p} be generic. By Theorem 2.4 (G, \mathbf{p}) must be in the kernel of some stress matrix Ω of rank $n - d - 1$. As \mathbf{p} is generic, it is in affine general position. Lemma 4.3 then tells us that $\Omega \in \text{Gstr}$. \square

Theorem 7.7. *Suppose that G is generically globally rigid in \mathbb{R}^d . Then $\text{stressedCorank}(G) = \text{corank}(G)$.*

Proof. Let \mathbf{p} be a generic framework. We have $\text{corank}(G, \mathbf{p}) = \text{corank}(G)$. From Lemma 7.6, \mathbf{p} is Gstressable. Since \mathbf{p} is a generic framework, \mathbf{p} is also generic in Gstressable. Thus, using Lemma 7.3 $\text{corank}(G, \mathbf{p}) = \text{stressedCorank}(G)$. \square

A much more interesting question arises when G is $(d+1)$ -connected but not generically globally rigid.

In the next proof, we will use the following. See [7, Lemma B.4, inter alia]

Lemma 7.8. *Let (G, \mathbf{p}) be a framework in dimension d , so that the rank of the rigidity matrix at \mathbf{p} is maximum over all configurations. If there is a stress matrix Ω for (G, \mathbf{p}) that is of rank $n - d - 1$, then every sufficiently small neighborhood U of \mathbf{p} contains only frameworks \mathbf{q} that have a stress matrix of rank $n - d - 1$. (Moreover, if Ω is PSD, then all the frameworks in U have a PSD stress matrix of rank $n - d - 1$.)*

Proposition 7.9. *Suppose that G is $(d+1)$ -connected but not generically globally rigid. Then $\text{stressedCorank}(G) > \text{corank}(G)$.*

Before giving the proof, we observe the following easy case. Suppose that G is generically stress free. Certainly $\text{stressedCorank}(G) > 0 = \text{corank}(G)$.

Proof. Suppose there were a Gstressable framework \mathbf{p} with only $\text{corank}(G)$ stresses. This means that its rigidity matrix displays the maximal possible rank for G and d . Then from Lemma 7.8, all configurations \mathbf{q} in a sufficiently small neighborhood of \mathbf{p} have a stress matrix of rank $n - d - 1$. In this neighborhood, there must be a generic \mathbf{q} , where the stress matrix would give a certificate of generic global rigidity as per Theorem 2.3. A contradiction. \square

One might be tempted to expect that for a non generically globally rigid graph, we would have $\text{stressedCorank}(G) = \text{corank}(G) + 1$. But this is not true. In the example of Figure 1, we have $\text{corank}(G) = 0$, but it can be shown that $\text{stressedCorank}(G) = 2$. So this presents the following question: Is there a characterization that gives us $\text{stressedCorank}(G)$? Perhaps the simplest relevant case is for G a non generically globally rigid, 3-connected, planar graph with at least one triangular face.

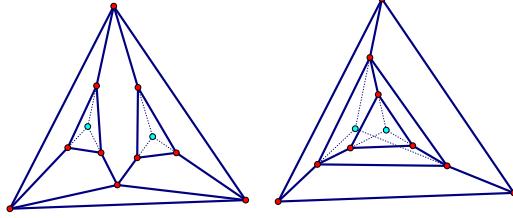


Figure 1: This shows two frameworks such that when the configuration is generic, they only have the 0 stress and they are infinitesimally rigid, so by the count of edges and vertices, the number of independent stresses is equal to the number of independent infinitesimal flexes. But when the stress coefficients on all the inner members is chosen generically, say with positive numbers, and the stress coefficients on the external triangle are chosen to create equilibrium (necessarily with negative stresses), then the dimension of the stress space is two. The green vertices are not vertices of the framework, but the intersection of three lines that are extensions of edges in the framework needed to insure the equilibrium condition for our assumed stress. In both cases there are two independent infinitesimal flexes that are infinitesimal rotations, each on a triangle about its center green point.

8 Generic universal rigid framework

The following is the main theorem of [7].

Theorem 8.1. *If G is generically globally rigid in \mathbb{R}^d , then there exists a framework (G, \mathbf{p}) in \mathbb{R}^d that is infinitesimally rigid in \mathbb{R}^d and super stable. Moreover, every framework in a small enough neighborhood of (G, \mathbf{p}) will be infinitesimally rigid in \mathbb{R}^d and super stable. This neighborhood must include some generic framework.*

Here we give a much simpler proof based on Gstr.

Proof of Theorem 8.1. Let GC be the configurations that are in general position. This is quasi-projective and irreducible. Let GIF be the general position frameworks that are infinitesimally flexible. This is a Zariski-closed subset of GC (ie. defined by the vanishing of a non-trivial polynomial). Let Gstr(GIF) be the stresses in Gstr that have a kernel framework in GIF (and so no kernel frameworks that are infinitesimally rigid). By Lemma 7.6, $\text{Gstr}(\text{GIF}) \subsetneq \text{Gstr}$. Now we will show that $\dim(\text{Gstr}(\text{GIF})) < \dim(\text{Gstr})$.

We (again) consider our rational map that maps from a matrix $\Omega \in \text{Gstr}$ to the framework in its kernel with a d -dimensional affine span and with $d+1$ chosen vertices pinned in canonical locations. Due to general position, this map is defined everywhere over Gstr. Let W be the preimage of GIF under this map. This is a Zariski closed subset of Gstr. We have $W \subseteq \text{Gstr}(\text{GIF}) \subsetneq \text{Gstr}$. Since GIF is a Zariski-closed subset of the range, its preimage, W , lies in a Zariski closed strict subset of Gstr.

From Theorem 2.5, G must be $(d + 1)$ -connected. Thus from Theorem 4.2, Gstr must be irreducible and of dimension D_L . Since this Zariski closed subset W is strict and Gstr is irreducible, then it must be of strictly lower dimension.

Now, any framework in GIF is related, by a non-singular affine transform to framework with its $d + 1$ chosen vertices in the canonical position. This canonically transformed framework must also be in GIF , as infinitesimal flexibility is invariant to non-singular affine transforms. Thus $\text{Gstr}(GIF) = W$. So we have shown $\dim(\text{Gstr}(GIF)) < D_L$.

Since $\text{Gstr}(GIF)$ has dimension less than D_L , and PGstr is dimension D_L by Corollary 3.11, we must have $\text{Gstr}(GIF) \not\supset \text{PGstr}$. Any stress in PGstr and not in $\text{Gstr}(GIF)$ must have its full-span kernel frameworks be infinitesimally rigid. This means we must have a PSD max rank stress matrix Ω with a infinitesimally rigid kernel (G, \mathbf{p}) . This framework must be super stable.

To finish the proof, we note infinitesimal rigidity is an open property will be preserved in a small enough neighborhood of \mathbf{p} . Lemma 7.8, which applies by infinitesimal rigidity of (G, \mathbf{p}) then gives us a neighborhood U of \mathbf{p} in which every framework (G, \mathbf{q}) has a PSD stress matrix of rank $n - d - 1$. This gives us super stability in this neighborhood.

Finally, any open neighborhood contains a generic point. □

A Algebraic geometry background

We quickly review some algebraic geometry.

Definition A.1. An *real algebraic set* V is a subset of \mathbb{R}^n that can be defined by a finite set of algebraic equations.

A *Zariski open subset* of some set $S \in \mathbb{R}^n$ is a subset of S defined by removing an algebraic set from S . A *Zariski closed subset* of some set S is the intersection of S and an algebraic set.

A *quasi-projective variety* is a Zariski open subset of an algebraic set. (Ie. we can cut out some subvariety of a variety).

A *constructible set* is the finite union of quasi-projective varieties.

A *semi-algebraic set* S is a subset of \mathbb{R}^n that can be defined by a finite set of algebraic equalities and inequalities as well as a finite number of Boolean operations.

Any semi-algebraic set is constructible. Any constructible set is quasi-projective. Any quasi-projective variety is algebraic.

A *real semi-algebraic set* S has a *real dimension* $\dim(S)$, which we will define as the largest t for which there is an open subset of S , in the Euclidean topology, that is a smooth t dimensional smooth sub-manifold of \mathbb{R}^n .

The *Zariski-closure* of a semi-algebraic set S is the smallest real algebraic set V containing S . V will have the same dimension as S .

An algebraic set called *irreducible* if it is not the union of two proper algebraic subsets.

A semi-algebraic set is called irreducible if its Zariski closure is irreducible.

Any Zariski-closed strict subset of an irreducible semi-algebraic set must be of strictly lower dimension.

The image of a real algebraic, quasi-projective, constructible, or semi-algebraic set set under a polynomial or rational map is semi-algebraic. If the domain is irreducible, then so too is the image.

Definition A.2. In the complex setting \mathbb{C}^n , we can also define algebraic sets, quasi-projective varieties and constructible sets accordingly. (There is no notion of semi-algebraic sets.)

The image of a complex algebraic, quasi-projective, or constructible set set under a polynomial or rational map is constructible. If the domain is irreducible, then so too is the image.

Definition A.3. Given an irreducible object S of any one of the above types, we call a point $\mathbf{x} \in S$ generic if \mathbf{x} does not satisfy any polynomial equations with coefficients in \mathbb{Q} that do not vanish on all of S .

A rational map acting on S , and defined using coefficients in \mathbb{Q} , will map generic points of the domain to generic points in the image.

Lemma A.4. If the real locus of an irreducible complex quasi-projective variety has a real dimension matching that of the complex quasi-projective variety, then it too must be irreducible.

Proof Sketch. Let V be a real algebraic set and V^* be the smallest complex algebraic set that contains V (called its complexification). The complex dimension of V^* must equal the real dimension of V [23]. If V is reducible, then so too is V^* [23].

Now suppose that W is an irreducible complex algebraic set and V its real locus. V is a real algebraic set with real dimension less than or equal to the complex dimension of W . Suppose that the real dimension of V equals the complex dimension of W . Then V^* has the same dimension of W . By construction, V^* must be contained in W . Since W is irreducible, any strict algebraic subset must be of lower dimension, so in fact $W = V^*$. From the previous paragraph, if V were reducible, then so too would be V^* and W , forming a contradiction. Thus V must be irreducible.

We can apply the same argument to the quasi-projective setting. We start with T , a complex quasi-projective variety, and write its Zariski closure as \overline{T} . (The Zariski closure of a quasi-projective variety does not affect dimension or reducibility.) We let S be the real locus of T and let \overline{S}^* be the smallest complex algebraic set containing S . Again, by construction $\overline{S}^* \subseteq \overline{T}$. Then we can prove, using dimension and irreducibility as above, that $\overline{S}^* = \overline{T}$. Again, reducibility of S would imply reducibility for \overline{S}^* and therefore for T . \square

B Statics

In this appendix we state some highly specific statics facts. See a reference like [24] or [14].

A load \mathbf{f} on a framework (G, \mathbf{p}) is an assignment of vectors \mathbf{f}_i to each vertex $i \in V(G)$. A load \mathbf{f} is said to be *resolvable* by (G, \mathbf{p}) if there are numbers ρ_{ij} so that, at each vertex i

$$\sum_{j \in N(i)} \rho_{ij} (\mathbf{p}_j - \mathbf{p}_i) = -\mathbf{f}_i$$

An *equilibrium* load is any load resolvable by (K_n, \mathbf{p}) . A framework is called *statically rigid* if every equilibrium load is resolvable. The important classical fact is.

Lemma B.1. A framework (G, \mathbf{p}) is statically rigid if and only if it is kinematically rigid. If (G, \mathbf{p}) is minimally statically rigid, there is a unique resolution for every equilibrium load.

Proof. This follows from linear algebra duality. The row rank of the rigidity matrix is the same as the dimension of the space of resolvable equilibrium loads. For K_n , this is $dn - \binom{d+1}{2}$, so if (G, \mathbf{p}) is statically rigid, its rigidity matrix has row rank, and hence column rank this large. \square

It's useful to have a different characterization of equilibrium loads, which is more extrinsic. Informally, it says there is no net force and no net torque.

Lemma B.2. A load \mathbf{f} on (K_n, \mathbf{p}) is an equilibrium load if and only if

$$\sum_{i=1}^n \mathbf{f}_i = 0 \quad \text{and} \quad \sum_{i=1}^n \mathbf{f}_i \wedge \mathbf{p}_i = 0$$

Proof. First suppose that \mathbf{f} is an equilibrium load. Then we have for appropriate edge weights $\rho_{ij} = \rho_{ji}$

$$-\sum_{i=1}^n \mathbf{f}_i = -\sum_{i=1}^n \sum_{j \neq i} \rho_{ij}(\mathbf{p}_j - \mathbf{p}_i) = -\sum_{i < j} \rho_{ij}[(\mathbf{p}_j - \mathbf{p}_i) + (\mathbf{p}_i - \mathbf{p}_j)] = 0$$

and also

$$\sum_{i=1}^n \mathbf{f}_i \wedge \mathbf{p}_i = \sum_{i=1}^n \left(\sum_{j \neq i} \rho_{ij}(\mathbf{p}_j - \mathbf{p}_i) \right) \wedge \mathbf{p}_i = \sum_{i=1}^n \sum_{j \neq i} \rho_{ij}(\mathbf{p}_j \wedge \mathbf{p}_i) = \sum_{i < j} \rho_{ij}(\mathbf{p}_i \wedge \mathbf{p}_j + \mathbf{p}_j \wedge \mathbf{p}_i) = 0$$

For the other direction we use linear algebra duality. The equilibrium loads are the image of the transposed rigidity matrix $R^t(\mathbf{p})$. If \mathbf{p} has d -dimensional affine span, this space has dimension $dn - \binom{d+1}{2}$, since complete graphs are always infinitesimally rigid if spanning. On the other hand, the two equations in the statement of the lemma impose $d + \binom{d}{2}$ linear constraints on a possible load \mathbf{f} . Hence the spaces are the same. \square

Immediately, we get.

Lemma B.3. *Suppose that \mathbf{f} is an equilibrium load on a framework (G, \mathbf{p}) that is supported only on $S \subseteq V(G)$. Then \mathbf{f} is resolvable by $(K_{|S|}, \mathbf{p}_S)$.*

Proof. Throwing away zero vectors doesn't change the constraints in Lemma B.2. \square

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