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APPLICATIONS OF STRESS THEORY:

REALIZING GRAPHS AND KNESER-POULSEN

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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BIOGRAPHICAL SKETCH

Maria Teresa Belk was born Maria Teresa Sloughter in Kennewick, Washington, on June 27, 1977. She grew up in Greenville, South Carolina and attended Carleton College in Northfield, Minnesota, graduating Magna Cum Laude with distinction in Mathematics in 1999. She arrived at Cornell in September of 1999, where she met her future husband, James M. Belk. She was married on January 15, 2005, and she received a Ph.D. in August 2005. She will be moving to College Station, Texas, in August 2005 to begin a postdoctoral position at Texas A&M University.

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Chapter 1

Introduction

This chapter will introduce the two questions being addressed in this thesis. Section 1 will introduce the question of realizing graphs, and section 2 will introduce the Kneser-Poulsen conjecture in hyperbolic space.

1.1 Realizing Graphs

A *graph* G is a finite set of vertices $V(G) = \{1, \ldots, n\}$ and a finite set of edges $E(G)$, where each edge is a set containing exactly two vertices. The graphs we consider do not contain loops or multiple edges. The standard way to draw a graph is to draw a point for each vertex, and to draw a line segment between two vertices for each edge. The *complete graph on n vertices*, denoted by K_n, is the graph with n pairwise adjacent vertices. A good reference on graph theory is [Di00].

A *realization* of a graph G is a function which assigns to each vertex i of G a point p_i in some Euclidean space. When we draw a realization, we generally also draw the edges between vertices as straight lines. Note that a realization is different from an embedding, since the word embedding is usually reserved for the case when there are no self-intersections. For example, two vertices may be assigned to the same point in a realization, and edges may intersect and even overlap.

We say a graph G is *d-realizable* if, given any realization $p_1, \ldots, p_n$ of the graph in some finite dimensional Euclidean space, there exists a realization $q_1, \ldots, q_n$ in $\mathbb{E}^d$ with the same edge lengths: $|p_i - p_j| = |q_i - q_j|$ for all $\{i, j\} \in E(G)$. Note that this definition of d-realizability is a property of graphs — for a graph to be

d-realizable, *every* realization of the graph must have a realization in $\mathbb{E}^d$.

Also note that we allow edges to have length zero. It turns out that allowing edges of zero length does not change which graphs are d-realizable.

Examples:

1. A path is 1-realizable, because we can arrange the vertices in order on a line with the appropriate distance between any two points.

2. Similarly, a tree (a connected graph containing no cycles) is also 1-realizable.

3. A triangle is not 1-realizable, because the triangle with all edge lengths 1 can only be realized in $\mathbb{E}^2$ but not in $\mathbb{E}^1$.

The following is a standard definition from graph theory.

Definition 1.1.1. *A* minor *of a graph G is any graph obtained from G by a sequence of*

- *edge deletions and*
- *edge contractions (identify the two vertices belonging to an edge and then remove any loops or multiple edges)*

The 1-realizable and 2-realizable graphs are classified in [BC05]:

Theorem 1.1.1 (Connelly). *A graph G is 1-realizable if and only if it does not have K_3 as a minor (i.e., G is a forest).*

Theorem 1.1.2 (Belk, Connelly). *A graph G is 2-realizable if and only if it does not have K_4 as a minor.*

The main realizability result in this thesis is a classification of 3-realizable graphs:

Theorem 1.1.3 (Belk, Connelly). *A graph G is 3-realizable if and only if it does not have either K_5 or $K_{2,2,2}$ as a minor.*

1.2 Kneser-Poulsen Conjecture in Hyperbolic Space

Kneser and Poulsen independently conjectured the following.

Conjecture (Kneser 1955, Poulsen 1954). *Let $p_1, \ldots, p_n$ and $q_1, \ldots q_n$ be two configurations in $\mathbb{E}^d$ and let $B(p_1), \ldots, B(p_n)$ and $B(q_1), \ldots, B(q_n)$ be unit balls in $\mathbb{E}^d$ with centers p_i and q_i. If*

$$|p_i - p_j| \leq |q_i - q_j|$$

for all i and j, then:

$$Vol\left(\bigcup B(p_i)\right) \leq Vol\left(\bigcup B(q_i)\right)$$

The configuration $\mathbf{q} = (q_1, \ldots q_n)$ is called an *expansion* of the configuration $\mathbf{p} = (p_1, \ldots, p_n)$. An expansion is a *continuous expansion* in $\mathbb{E}^d$ if there exists a continuous motion $\mathbf{p}(t) = (p_1(t), \ldots, p_n(t)), 0 \leq t \leq 1, p_i(t) \in \mathbb{E}^d$ with $\mathbf{p}(0) = \mathbf{p}$, $\mathbf{p}(1) = \mathbf{q}$, and with $\mathbf{p}(t)$ being an expansion of $\mathbf{p}(s)$ whenever $s < t$.

It is a well-known fact that if $\mathbf{q}$ is an expansion of $\mathbf{p}$ in $\mathbb{E}^2$, then there is a continuous expansion in $\mathbb{E}^4$. Bezdek and Connelly used this fact to prove the conjecture for the case $d = 2$ [BezC04].

One could ask whether the Kneser-Poulsen conjecture holds in hyperbolic space. Csikós has proven that if $\mathbf{q}$ is an expansion of $\mathbf{p}$ in $\mathbb{H}^2$ and there is a continuous expansion in $\mathbb{H}^4$, then the area of the union of circles does not decrease from $\mathbf{p}$ to $\mathbf{q}$. However, it is unknown if every expansion in $\mathbb{H}^2$ is a continuous expansion in $\mathbb{H}^4$.

Question. *If* $\mathbf{q} = (q_1, \ldots, q_n)$ *is an expansion of* $\mathbf{p} = (p_1, \ldots, p_n)$ *in* $\mathbb{H}^2$*, is there a continuous expansion from* $\mathbf{p}$ *to* $\mathbf{q}$ *in* $\mathbb{H}^N$ *for some* $N \geq 2$*?*

This is equivalent to asking if there is a continuous expansion in $\mathbb{H}^{n-1}$, since n points in $\mathbb{H}^N$ are contained in a copy of $\mathbb{H}^{n-1}$. I have obtained the following:

Theorem 1.2.1. *If* $\mathbf{q} = (q_1, q_2, q_3, q_4)$ *is an expansion of* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *in* $\mathbb{H}^2$*, then there is a continuous expansion from* $\mathbf{p}$ *to* $\mathbf{q}$ *in* $\mathbb{H}^N$ *for some* $N \geq 2$*.*

As commented above, the continuous expansion is actually in $\mathbb{H}^3$.

Chapter 2

Tensegrities and Stress Theory

This chapter will give the important definitions and theorems that will be used to attack the two questions from chapter 1. Most of the results in this chapter can be found in [RW81], [Co82], or [BezC99].

In chapter 5, we will generalize much of this chapter to hyperbolic space. Sections 1 and 3 will generalize in obvious ways. Section 2 will be harder to generalize, and, in particular, we will not have a generalization of Theorem 2.2.2— this theorem provides a useful way to show that that a tensegrity in Euclidean space is universally globally rigid (that is, that there is no other configuration satisfying certain distance constraints). As far as I know, it is an open question whether a suitable generalization of Theorem 2.2.2 to hyperbolic space exists.

2.1 Tensegrities and Rigidity

Definition 2.1.1. *A* tensegrity, *denoted $G(\mathbf{p})$ is a configuration $\mathbf{p} = (p_1, \ldots, p_n)$ with $p_i \in \mathbb{E}^d$ and a graph G, where each edge of the graph is labelled as a cable, strut, or bar, and where each vertex is labelled as being pinned or unpinned.*

The idea is that cables are allowed to decrease in length (or stay the same length), but not to increase in length. Struts are allowed to increase in length (or stay the same length), but not to decrease in length. Bars are forced to remain the same length. When drawing a tensegrity, we will denote bars by a single line between vertices, struts by a double line between vertices, and cables by a dotted line between vertices.

Pinned vertices are forced to remain where they are. Usually, the tensegrities we will consider will have only unpinned vertices. Unless explicitly stated otherwise, vertices are assumed to be unpinned.

Definition 2.1.2. *Fix a graph G. Let $\mathbf{p}$ and $\mathbf{q}$ be two configurations of G. We say that $G(\mathbf{p})$ dominates $G(\mathbf{q})$, which we will denote by $G(\mathbf{p}) \succeq G(\mathbf{q})$, if for every pinned vertex i, $p_i = q_i$, and for every edge $\{i, j\}$*

$$|p_i - p_j| \left\{ \begin{array}{c} \geq \\ = \\ \leq \end{array} \right\} |q_i - q_j| \text{ if } \{i,j\} \text{ is a } \left\{ \begin{array}{c} \textit{cable} \\ \textit{bar} \\ \textit{strut} \end{array} \right\}.$$

A tensegrity $G(\mathbf{p})$ dominates a tensegrity $G(\mathbf{q})$ if $G(\mathbf{q})$ is a tensegrity that satisfies the strut and cable conditions of $G(\mathbf{p})$.

In general, if we are given a tensegrity $G(\mathbf{p})$, we are interested in other tensegrities that $G(\mathbf{p})$ dominates (tensegrities that satisfy the strut and cable conditions). Is there a configuration $\mathbf{q}$ in $\mathbb{E}^d$ which is not congruent to $\mathbf{p}$ such that $G(\mathbf{q})$ satisfies the cable-strut conditions? If not, then $G(\mathbf{p})$ is *globally rigid.* Is there a continuous motion of the vertices $\mathbf{p}(t)$ such that $G(\mathbf{p}(t))$ satisfies the cable-strut conditions? If not then, $G(\mathbf{p})$ is *rigid.* We will make these definitions precise.

Definition 2.1.3. *A tensegrity $G(\mathbf{p})$ in $\mathbb{E}^d$ is* globally rigid in $\mathbb{E}^d$ *if for all configurations $\mathbf{q}$ in $\mathbb{E}^d$, $G(\mathbf{p}) \succeq G(\mathbf{q}) \Rightarrow \mathbf{p} \cong \mathbf{q}$ ($\mathbf{p}$ is congruent to $\mathbf{q}$). A tensegrity is* universally globally rigid *if it is globally rigid for all d.*

Thus, a globally rigid tensegrity is the only way to satisfy the cable-strut conditions in dimension d, and a universally globally rigid tensegrity is the only way to satisfy the cable-strut conditions in any dimension. In this thesis, we will usually be concerned with universal global rigidity rather than global rigidity.

Figure 2.1 shows four universally globally rigid tensegrities. The three points

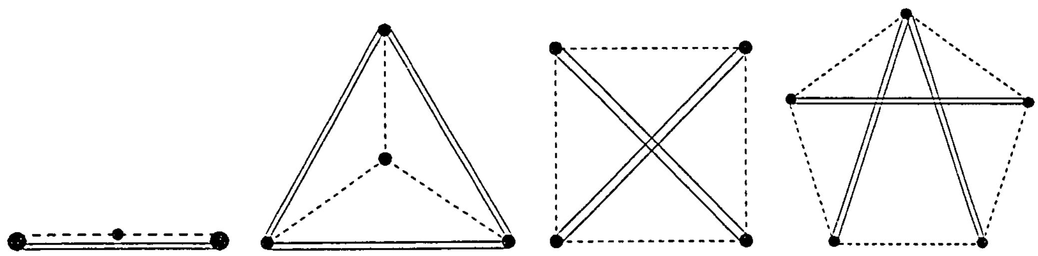

Figure 2.1: Four universally globally rigid tensegrities. The first tensegrity is one dimensional — the three points are collinear. The others are two dimensional.

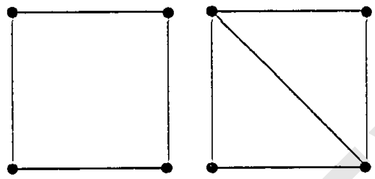

Figure 2.2: Two bar frameworks. The first is not rigid in the plane. The second is rigid in the plane, but not rigid in three dimensions since it can fold along the diagonal.

in the first tensegrity are collinear, and it should be clear that there is no other configuration of three points satisfying the cable and strut conditions.

A path $\mathbf{p}(t) = (p_1(t), \ldots, p_n(t)), 0 \le t \le 1$ with $\mathbf{p}(0) = \mathbf{p}$ is called a *flex* of $G(\mathbf{p})$ if $G(\mathbf{p}) \succeq G(\mathbf{p}(t))$ for all t. A flex $\mathbf{p}(t)$ is a *trivial flex* if $\mathbf{p}(t)$ is congruent to $\mathbf{p}$ for all t.

Definition 2.1.4. *A tensegrity $G(\mathbf{p})$ is* rigid *if every continuous flex $\mathbf{p}(t)$ of $G(\mathbf{p})$ is trivial.*

Thus, a rigid tensegrity cannot be continuously moved. Figure 2.2 shows two example bar frameworks (tensegrities with just bars, no cables or struts) to illustrate rigidity. In the plane, the first flexes while the second is rigid. In three dimensions, they both flex.

The following theorem gives two alternative definitions for a rigid tensegrity. The theorem can be found in [RW81] — the important thing is that the configu-