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TWO-DISTANCE PRESERVING FUNCTIONS FROM EUCLIDEAN SPACE

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Abstract

Let $0 < c < s$ be fixed real numbers such that $c/s \leq (\sqrt{5}-1)/2$, and let $f : \mathbb{E}^2 \to \mathbb{E}^d$ for $d \geq 2$ be a function such that for every $p, q \in \mathbb{E}^2$ if $|p-q| = c$, then $|f(p) - f(q)| \leq c$, and if $|p-q| = s$, then $|f(p) - f(q)| \geq s$. Then f is a congruence. This result depends on and expands a result of Rádo et. al. [9], where a similar result holds, but for $\sqrt{3}/3$ replacing $(\sqrt{5}-1)/2$. We also present a further extensions where $\mathbb{E}^2$ is replaced by $\mathbb{E}^n$ for $n > 2$ and where the range of c/s is enlarged.

1. Introduction and background

A function $f : \mathbb{E}^n \to \mathbb{E}^m$ is *unit distance preserving* if for all $p, q \in \mathbb{E}^n$, $|p-q| = 1$ implies $|f(p) - f(q)| = 1$. We use vector notation throughout, and $|\dots|$ denotes the standard Euclidean norm. We say f is a *congruence* if all distances are preserved, i.e. for all $p, q \in \mathbb{E}^n$, $|p-q| = |f(p) - f(q)|$. In 1953 [3] Beckmann and Quarles showed that for $n \geq 2$, any unit distance preserving function $\mathbb{E}^n \to \mathbb{E}^n$, is a congruence. On the other hand in 1985 [6] Dekster showed that there is a function $f : \mathbb{E}^2 \to \mathbb{E}^6$ that is unit distance preserving, but f is NOT a congruence. A natural question then was what sort of distance constraints would imply that any function $f : \mathbb{E}^n \to \mathbb{E}^m$ satisfying them is a congruence. An excellent result in this direction was the following:

THEOREM 1. (Rádo, Andreason and Válcan [9]) *Suppose $n \geq 2$, $0 < c < s$, $0 < c/s \leq 1/\sqrt{3}$, and $f : \mathbb{E}^n \to \mathbb{E}^m$ is such that for all $p, q \in \mathbb{E}^n$ the following holds:*
(1) *If $|p-q| = c$, then $|f(p) - f(q)| \leq c$.*
(2) *If $|p-q| = s$, then $|f(p) - f(q)| \geq s$.*
Then f is a congruence.

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Note that with all of these results, there is no assumption of continuity on the function f. Continuity follows, of course, when f is a congruence. The example in Dekster's result [6] is definitely not continuous.

We will use the theory of tensegrity structures as well as the statement of Theorem 1 itself to improve the constant $1/\sqrt{3}$ in Theorem 1 to the golden ratio $(\sqrt{5}-1)/2$. We can also improve the constant even further allowing a larger set of ratios c/s as the dimension n of the domain Euclidean space increases.

Let

$$F_k(c,s) = \{f : \mathbb{E}^n \to \mathbb{E}^m \mid (1) \text{ and } (2) \text{ hold for all } n \geq k\}.$$

Then the set that we want to identify is the following.

$$X_k = \{r \in \mathbb{E}^1 \mid \text{If } c/s = r, \text{ then } f \in F_k(c,s) \text{ is a congruence}\}$$

So Rádo et al.'s Theorem says simply that the interval $(0, 1/\sqrt{3}] \subset X_2$. It is easy to show that $X_2 \subset X_3 \subset \ldots$. Our theorem is the following.

THEOREM 2. *The interval* $(0, (\sqrt{5}-1)/2) \subset X_2$, *and* $limsup_{k\to\infty} X_k = 1$.

COROLLARY. *If* $f : \mathbb{E}^n \to \mathbb{E}^m$, $n \geq 2$ *is a function that preserves lengths* c *and* s, *where* $0 < c/s < (\sqrt{5}-1)/2$, *then* f *is a congruence.*

REMARK 1. Dekster and Wilker in [7] attempt, unsuccessfully, to construct two-distance functions from the plane that are not congruences, where the ratio between the two distances is $1/3$ in one case and $2\sqrt{2}/5 = 0.565\ldots$ in another case. Both of these ratios are in the range where the results of [9] imply that no such two-distance function exist. Although they made it clear that their method did not work for the examples they tried, in this case, Dekster and Wilker's method is doomed to failure. However, there may be hope for this method when the ratio of distances is $(\sqrt{5}-1)/2$ or larger. Also in the case when the ratio is $1/N$ for $N > 1$, it is shown by Benz and Berens in [2] that f is a congruence, and this is somewhat easier than in [9].

The improvement here is from $1/\sqrt{3} = 0.577\ldots$ to $(\sqrt{5}-1)/2) = 0.618\ldots$. The proof of Theorem 1 is quite complicated. Indeed, H. Lenz [8] in a reveiw for Zentralblat said "... The proof is not easy (alone Lemma 6 needs the distinction of 25 cases). The reviewer tried in vain to simplify it." Also, the proceedings that contains the result of Rádo et al. [9] is not readily available. There seems to be only one copy available in the United States. If we do not use Rádo et al.'s Theorem 1 above, then we can still easily show that there is an infinite set in X_2 and that $limsup_{n\to\infty} X_n = 1$. With the theory of tensegrity structures, there are a wide range of tools that can be used to show what ratios are in each X_n, extending what was used in [9]. But for X_2, for instance, it is clear that even with the extended methods used here, there seems to be no easy way to show $(\sqrt{5}-1)/2$ is in X_2. We do not know if any X_n for $n \geq 2$ is a connected interval. However, it is clear that it is easy to extend what is known to be in any X_n for $n > 2$. It just seems to be time consuming to do the calculations. The difficult part of [9] is showing that the interval

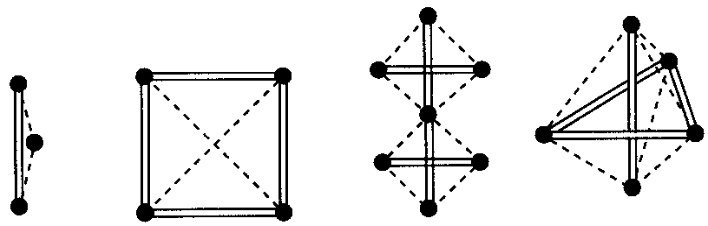

Figure 1. Some tensegrities

$[1/3, 1/2] \subset X_2$. It is not so hard to show that this implies the interval $(0, 1/2] \subset X_2$, and then our techniques will show that the interval $(0, (\sqrt{5}-1)/2) \subset X_2$.

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2. Tensegrity techniques

Suppose that we have a configuration of k labeled points $p = (p_1, \ldots, p_k)$, each $p_i \in \mathbb{E}^n$. Furthermore suppose that we have a graph G, without loops or multiple edges, whose vertices correspond to those k points, and whose edges are designated as cables or struts. We call such a graph a *tensegrity graph.* Together we call the graph G and the configuration p a *tensegrity* and we denote it by $G(p)$. There are many interesting rigidity properties of tensegrities, but we will need only a very weak property. Suppose that $q = (q_1, \ldots, q_k)$, each $q_i \in \mathbb{E}^m$, is another configuration with corresponding vertices for any $m = 0, 1, 2, \ldots$. We say that the tensegrity $G(q)$ *satisfies the tensegrity constraints of* $G(p)$ if the following holds:

Cable: For $\{i, j\}$ a cable, $|q_i - q_j| \leq |p_i - p_j|$.

Strut: For $\{i, j\}$ a strut, $|q_i - q_j| \geq |p_i - p_j|$.

Fix the tensegrity $G(p)$ in $\mathbb{E}^n$. It is called *unyielding* if for every other configuration q in $\mathbb{E}^m$ such that $G(q)$ satisfies the tensegrity constraints, then for all $\{i, j\}$ a cable or a strut of G, $|q_i - q_j| = |p_i - p_j|$. In other words, the inequality cable and strut constraints with respect to $G(p)$ force equality constraints. (We thank Branko Grünbaum for this descriptive name.) Note also that in our definition of unyielding, the target configuration can be in a Euclidean space of any dimension.

When we represent tensegrities in figures, we will denote the vertices of a configuration as small circular disks, cables as dashed line segments between its vertex end points, and struts as double line segments between its end points. Figure 1, shows some examples of tensegrities.

Note that the line segments representing tensegrities can intersect, and that the intersection point may or may not correspond to a vertex of the configuration. It also can happen that a tensegrity may be unyielding and yet not be rigid, as in Figure 1c. We say $G(p)$ is *rigid in* $\mathbb{E}^n$ if every continuous family $q(t)$, $0 \leq t \leq 1$

Figure 2. Some unyielding tensegrities

of configurations satisfying the cable and strut constraints, has every pair of its vertices stay at a fixed distance. But the questions of rigidity do not directly of concern us here, although many of the ideas here and in [5] can be used to show rigidity results.

There are many interesting examples of unyielding tensegrities (for example see [5]) but for our purposes the simplest classes are the only ones that seem to be useful. One case is when the configuration consists of just three collinear, but distinct points, $p = (p_1, p_2, p_3)$, where p_2 lies between p_1 and p_3. Define the tensegrity graph to have cables $\{1,2\}, \{2,3\}$ and strut $\{1,3\}$ as in Figure 1a. Then it is easy to see that this tensegrity is unyielding.

A simple, but more interesting, tensegrity is the following. Let $p = (p_1, p_2, p_3, p_4)$ be the four vertices, in cyclic order, of a convex quadrilateral in $\mathbb{E}^2$. Define the external edges $\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}$ to be cables and the diagonals $\{2,4\}, \{1,3\}$ to be struts for the tensegrity graph G. Then by a basic result of [3], $G(p)$ is unyielding. Later we will provide a seperate self-contained proof and generalize this tensegrity to higher dimensional analogues. Figure 2 shows some examples of these unyielding tensegrities. The lengths of the cables and struts are indicated for later use. Example 2a was used extensively in [9]. Example 2b is what we will use here to extend the result of [9].

3. Implied cable and strut lengths

Consider any fixed $f : \mathbb{E}^n \to \mathbb{E}^m$, where $n \geq 2$. Define

$$C_f = \{c \in \mathbb{E}^1 \mid \text{for all } p, q \in \mathbb{E}^n, \text{ where } |p - q| = c, \text{ then } |f(p) - f(q)| \leq c\}$$
$$S_f = \{s \in \mathbb{E}^1 \mid \text{for all } p, q \in \mathbb{E}^n, \text{ where } |p - q| = s, \text{ then } |f(p) - f(q)| \geq s\}$$

We regard C_f as those lengths that are shortened or fixed by f, and S_f are those lengths that are lengthened or fixed by f. A (positive) real number could be in C_f or S_f by hypothesis or it could be because of other reasons. In any case C_f and S_f both include the set of all positive real numbers if and only if f is a congruence. We call C_f the *cable lengths (for f)* and S_f the *strut lengths (for f)*.

Suppose that some set of positive real numbers C is known to be in C_f and another set S is known to be in S_f. If it is then true that for f some other real number c is in C_f or s is in S_f, then we say that c is an *implied cable length for f* or s is an *implied strut length for f* respectively.

For example, if c_1 and c_2 are cable lengths for f, then $c_1 + c_2$ is an implied cable length for f. This is quite easy to see, but this and many more complicated relationships for implied cable and strut lengths can be seen from the following Lemma.

LEMMA 1. *Suppose $f : \mathbb{E}^n \to \mathbb{E}^m$ is a function and $G(p)$ is an unyielding tensegrity in $\mathbb{E}^n$, with cables, whose lengths are $c_1, c_2, \dots$, and with struts, whose lengths are $s_1, s_2, \dots$. Suppose further that $c_2, \dots \in C_f$ and $s_1, s_2, \dots \in S_f$. Then the length c_1 is an implied strut length for f. Similarly, if $c_1, c_2 \cdots \in C_f$, and $s_2, \dots \in S_f$, then s_1 is an implied cable length for f.*

PROOF. We will show the first case when $c_2, \dots \in C_f$ and $s_1, s_2, \dots \in S_f$. (The other case is similar.) We wish to show that c_1 is a strut length for f. Suppose not. Suppose that there are points $u, v \in \mathbb{E}^n$, with $|u-v| = c_1$, but $|f(u)-f(v)| < c_1$. Construct a configuration $\hat{p}$ congruent to p in $\mathbb{E}^n$, where for some a, b, $\hat{p}_a = u$, and $\hat{p}_b = v$. So $|\hat{p}_a - \hat{p}_b| = c_1$. Define $q = f(\hat{p}) = (f(\hat{p}_1), \dots, f(\hat{p}_k))$, where k is the number of vertices in G. So for c_1,

$$|p_a - p_b| = |\hat{p}_a - \hat{p}_b| = |u - v| = c_1 > |f(u) - f(v)| = |q_a - q_b|$$

and for the other cables $\{i, j\}$ in G, since the cable lengths of G are in C_f, we have

$$|p_i - p_j| = |\hat{p}_i - \hat{p}_j| \geq |f(\hat{p}_i) - f(\hat{p}_j)| = |q_i - q_j|,$$

and for struts $\{i, j\}$ in G, since the strut lengths of G are in S_f, we have

$$|p_i - p_j| = |\hat{p}_i - \hat{p}_j| \leq |f(\hat{p}_i) - f(\hat{p}_j)| = |q_i - q_j|.$$

Hence $G(q)$ satisfies all the cable and strut conditions for the tensegrity $G(p)$, but the strict inequality contradicts the unyielding property of $G(p)$. It must be that our assumption is wrong and for every $u, v \in \mathbb{E}^n$, with $|u - v| = c_1$, $|f(u) - f(v)| \geq c_1$. This finishes the proof.

If we have positive real numbers x and y, an unyielding tensegrity of Figure 1a can be constructed with strut length $x+y$ and cable lengths x and y. So if $x, y \in C_f$ then $x + y$ is an implied cable in C_f as well. And if $x \in C_f$ and $x + y \in S_f$, then y is an implied strut S_f. These properties, equivalent to the triangle inequality, are very simple consequences of Lemma 1.

We include the following Lemma for completeness and contrast. It was the workhorse for [9].

LEMMA 2, [9]. *Let c_1, c_2, s_2 be positive real numbers such that $c_1 \leq c_2$, ${c_1}^2 + {s_2}^2 \leq {c_2}^2$, $c_1, c_2 \in C_f$ and $s_2 \in S_f$, $n \geq 2$, where $f : \mathbb{E}^n \to \mathbb{E}^m$ is a function. Then*

$$s_1 = 2\sqrt{{c_2}^2 - \left(\frac{{c_2}^2 - {s_2}^2 + {c_1}^2}{2s_2}\right)^2}$$

is an implied cable length in C_f.

The inequality conditions are what is needed to insure that they will correspond to the lengths of a convex quadriateral tensegrity in Figure 2a. Then Lemma 1 implies that s_1 is an implied cable length for f.

We need the following Lemma for our improvement of [9].

LEMMA 3. *Let* c_2, s_1 *be positive real numbers such that* $c_2 \leq s_1 \leq 2c_2$, $c_2 \in C_f$, *and* $s_1 \in S_f$, $n \geq 2$, *where* $f : \mathbb{E}^n \to \mathbb{E}^m$ *is a function. Then*

$$c_1 = \frac{{s_1}^2 - {c_2}^2}{c_2}$$

is and implied strut length in S_f.

PROOF. Let $G(p)$ be the tensegrity in Figure 2b. Consider the isosceles triangle determined by two edges, both of length c_2, and base length s_1. This is possible since $s_1 \leq 2c_2$. Put two of them together as in Figure 2b. These four points will form a trapezoid with the s_1 length for its diagonals if and only if $c_2 \leq s_1$. The equality cases correspond to limiting cases when the configuration becomes collinear (when $s_1 = 2c_2$), and when the two vertices at the bottom in Figure 2b coincide (when $c_2 = s_1$). It is an elementary calculation to see that under these assumptions, $c_1 = ({s_1}^2 - {c_2}^2)/c_2$.

By the results in [5], the tensegrity graph of Figure 2b is unyielding. Since the c_1 length is a cable in Figure 2b, by Lemma 1 it is is an implied strut length for any function $f : \mathbb{E}^n \to \mathbb{E}^m$.

Notice that when $s_1 = 2c_2$, then $c_1 = 3c_2$, and this corresponds to (an unyielding) tensegrity whose configuration lies on a line. When $s_1 = c_2$, then $c_1 = 0$. This says nothing about the function f.

4. Improving ratios

We need the following Lemma in order to improve the ratio in Rádo et al.'s Theorem. Let $\tau = (\sqrt{5} + 1)/2$.

LEMMA 4. *Let* $\rho : \mathbb{E}^1 \to \mathbb{E}^1$ *be defined by* $\rho(t) = t^2 - 1$, *with domain the interval* $[1, 2]$, *and range the interval* $[0, 3]$. *Then for any* $t_0 \in (\tau, 2]$ *some finite iterate of* t_0 *by* ρ *lies in the interval* $[2, 3]$.

PROOF. It is easy to see that ρ is strictly monotone increasing, continuous throughout its domain, and $\rho^{-1}(t) = \sqrt{t+1}$. It is also clear that the only fixed point for ρ (and ρ^{-1}) is the solution to the quadratic equation $t = t^2 - 1$, when $t \geq 1$. Thus the unique fixed point is τ. It is also easy to see that τ is a repulsive fixed point for ρ, and an attractive fixed point for ρ^{-1}. Indeed for any t in the domain of ρ, $t > \tau$, then $\rho(t) > t$. Since for $t > \tau$, $\rho(t) > t$, $\rho(\tau) = \tau$ and ρ is monotone increasing, it follows that for $t > \tau$, $\tau < \rho^{-1}(t) < t$. Define $\alpha_0 = 3$, and inductively $\alpha_i = \rho^{-1}(\alpha_{i-1})$ for $i = 1, 2, \ldots$. So $\alpha_1 = 2, \alpha_2 = \sqrt{3}, \alpha_3 = \sqrt{\sqrt{3}+1}, \ldots$. By the argument above, each α_i for $i = 1, 2, \ldots$ is a well-defined decreasing sequence, where each $\alpha_i > \tau$. So this sequence must converge to a fixed point of ρ. But τ is the only fixed point in the domain of ρ. Hence $\lim_{i\to\infty} \alpha_i = \tau$.

Considering maps of intervals we then have under ρ^{-1}:

$$[2,3] \to [\sqrt{3}, 2] \to [\sqrt{\sqrt{3}+1}, \sqrt{3}] \to \ldots$$

In other words, $\rho^{-1}([\alpha_i, \alpha_{i-1}]) = [\rho^{-1}(\alpha_i), \rho^{-1}(\alpha_{i-1})] = [\alpha_{i+1}, \alpha_i]$, for $i = 1, 2 \ldots$. But this implies the conclusion of the Lemma, because the union of the intervals

$$\bigcup_{i=1}^{\infty} [\alpha_i, \alpha_{i-1}] = (\tau, 3].$$

We can now prove the first part of the conclusion of Theorem 2.

PROPOSITION 1. *The interval* $(0, \tau^{-1}) \subset X_2$.

PROOF. Suppose that $0 < c < s$, $c/s \in (0, \tau^{-1})$, $f : \mathbb{E}^n \to \mathbb{E}^m$ for $n \geq 2$, and $f \in F_2(c, s)$. In other words, c is a cable length for f, and s is a strut length for f. By rescaling, if necessary, we assume, without loss of generality, that $c = 1$. If $c/s \leq 1/2 < 1/\sqrt{3}$ then $1/s \leq 1/2$ and $s \geq 2$. Hence, Theorem 1 applies, and f is a congruence. If $1/2 \leq c/s = 1/s < \tau^{-1}$, then $\tau < s \leq 2$. By Lemma 4, some finite iterate of s by ρ lies in the interval $[2, 3]$, where $\rho(t) = t^2 - 1$. Let $s_0 = s$, and $s_i = \rho(s_{i-1})$ for $i = 1, 2, \ldots$. By Lemma 3, if $s_{i-1} \in S_f$, then $s_i = s_{i-1}^2 - 1 = \rho(s_{i-1}) \in S_f$ as well. Hence for some k, s_k is an implied strut in S_f, and $s_k \in [2, 3]$. We conclude that f is a congruence by Theorem 1. This means that $s \in X_2$, which is what we needed to show.

Although this particular result was not part of Rádo et al. [9], the general method with the sequences of implied lengths was used. The key difference is that they used the tensegrity of Figure 2a, which creates implied cable lengths instead of implied strut lengths from the tensegrity of Figure 2b, used here. But in order to show that the interval $[1/3, 1/2] \subset X_2$, as was the case in [9], the method was to show that there was a sequence of implied cable lengths converging to 0. It is not clear how a sequence of increasing implied strut lengths, say using Figure 2b, could be used to show that f is a congruence.

5. Creating unyielding tensegrities using stresses

For our higher dimensional results, we need more tensegrities that have the unyielding property. To do that we use the idea of an equilibrium stress and stress matrices. Let G be a tensegrity graph with designated cables and struts and k vertices. Let $\omega_{i,j} = \omega_{j,i}$, defined for all $i \neq j$, $i, j = 1, 2, \dots, k$, be real numbers such that when $\{i,j\}$ is a cable for G, then $\omega_{i,j} > 0$, and when $\{i,j\}$ is a strut for G, then $\omega_{i,j} < 0$. When $\{i,j\}$ is not a cable or strut for G, then $\omega_{i,j} = 0$. Let ω denote the vector of all such scalars. We call ω a *proper stress* for G. (The adjective proper refers to the sign conditions. Without those inequality constraints, ω is simply called a *stress*.) We identify $\mathbb{E}^{nk}$ as the space of all configurations since it is determined by the n sets of k coordinates of each point of the configuration. Define the *stress energy function* $E_\omega : \mathbb{E}^{nk} \to \mathbb{E}^1$ for ω from the space of all configurations by the following:

$$E_\omega(q) = \sum_{1 \leq i < j \leq k} \omega_{i,j} |q_i - q_j|^2, \tag{5-1}$$

where $q = (q_1, \dots, q_k)$ is a configuration in $\mathbb{E}^n$, for some n. This function can be used to show that a given tensegrity is unyielding. It uses what we call the principle of least work, which is a part of the basic principle used in determining the stability of frameworks in structural engineering.

PROPOSITION 2. *Suppose that $G(p)$ is a tensegrity in $\mathbb{E}^n$, and $\omega = (\dots, \omega_{i,j}, \dots)$ is a proper stress for G such that p is a minimum point for the associated stress energy function E_ω. Then $G(p)$ is unyielding.*

PROOF. Suppose the contrary, that $G(p)$ is not unyielding. This means that there is some configuration $q = (q_1, \dots, q_k)$ in $\mathbb{E}^n$ and that for some $\{a,b\}$, a cable or strut of G, $|q_a - q_b| < |p_a - p_b|$ if $\{a,b\}$ is a cable, or $|q_a - q_b| > |p_a - p_b|$ if $\{a,b\}$ is a strut. Furthermore these inequalities hold for all the cables and struts of G, but with the possibility of equality. We wish to find a contradiction. Since ω is a proper stress for G, in both the cable and strut case, $\omega_{i,j}|q_i - q_j|^2 \leq \omega_{i,j}|p_i - p_j|^2$, with strict inequality when $\{i,j\} = \{a,b\}$. Thus by (5-1), we have $E_\omega(q) < E_\omega(p)$. This contradicts p being a minimum for E_ω. Thus $G(p)$ is unyielding, as desired.

In order to be able use this proposition, it is helpful to be able to regard the stress energy function as a quadratic form on the space of configurations and to use matrix notation in general. We first calculate its matrix. Suppose that $\omega = (\dots, \omega_{i,j}, \dots)$ is a proper stress for G, a tensegrity graph. Suppose that we have a configuration $q = (q_1, \dots, q_k)$, where each point q_i is in $\mathbb{E}^n$. Write each $q_i = (x_i, y_i, z_i, \dots)$ in terms of its coordinates. Then

$$E_\omega(q) = \sum_{1 \leq i < j \leq k} \omega_{i,j}(x_i - x_j)^2 + \sum_{1 \leq i < j \leq k} \omega_{i,j}(y_i - y_j)^2 + \sum_{1 \leq i < j \leq k} \omega_{i,j}(z_i - z_j)^2 + \dots. \tag{5-2}$$

This shows that the quadratic form E_ω is a sum of quadratic forms, each for one of the coordinates and each using the same stress ω. Consider one term of the sum (5-2), and write it in matrix form as follows:

$$(5-3) \qquad \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1 - x_2)^2.$$

With this in mind, we define what we call the *stress matrix* Ω in terms of the stress ω as follows.

(1) "(i)" For $i \neq j$ the i,j entry of Ω is $-\omega_{i,j}$.
(2) "(ii)" The diagonal entry of Ω for the i−th row and i−th column is $\sum_j \omega_{i,j}$.

The sum in (ii) does not include $\omega_{i,i}$ (which has not been defined anyway). So the row and column sums of Ω are 0. The relation (5-3) also shows how to associate a quadratic form to this symmetric matrix. It is easy to check that with this definition,

$$(5-4) \qquad \sum_{1 \leq i < j \leq k} \omega_{i,j}(x_i - x_j)^2 = \begin{bmatrix} x_1 & x_2 & \dots & x_k \end{bmatrix} \Omega \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}.$$

Of course, a similar relation holds for the other coordinates of the configuration. The following is easy to see.

LEMMA 5. *Any symmetric matrix Ω that has row and column sums* 0 *is the stress matrix associated to a stress ω given by (i) and (ii).*

Recall that a quadratic form, such as E_ω is *positive semi-defininte* if $E_\omega(q) \geq 0$, for all q. From this discussion, the following is now clear.

LEMMA 6. *The quadratic form E_ω is positive semi-definite if and only if the quadratic form associated to the symmetric matrix Ω by (5-4) is positive semi-definite.*

REMARK 2. The ideas of this section and the definition of the stress matrix Ω are in [5]. We have repeated them here for the convenience of the reader, to make this paper self-contained, and since the definitions are simple. Also in [5] the concern is mostly with the rigidity of the configuration, and the unyielding property, which is most relevant here, is not explicitly defined.

6. Small unyielding tensegrities

We define particular tensegrities for higher dimensions that will be used for the second half of Theorem 2.

LEMMA 7. *Suppose that an a-dimensional simplex σ_1 and a b-dimensional simplex σ_2 have a point that is in the relative interior of both simplices. Create a configuration consisting of the vertices of σ_1 and σ_2, and a tensegrity graph G consisting of struts corresponding to all the edges of σ_1 and all the edges of σ_2 and cables connecting each vertex of σ_1 to each of σ_2. This tensegrity $G(p)$ has a proper stress, non-zero on each cable and strut, such that p is a minimum point for the associated quadratic form for the corresponding stress matrix Ω. Thus $G(p)$ is unyielding.*

PROOF. Let $(p_1, \dots, p_{a+1})$ be the vertices of σ_1, and let $(p_{a+2}, \dots, p_{a+b+2})$ be the vertices of σ_2. Since they share a point each in their relative interiors, there are scalars, all positive, $\lambda_1, \lambda_2, \dots, \lambda_{(a+1)+(b+1)}$ such that

$$(6-1) \qquad \sum_{i=1}^{a+1} \lambda_i p_i = \sum_{i=(a+1)+1}^{(a+1)+(b+1)} \lambda_i p_i,$$

and

$$\sum_{i=1}^{a+1} \lambda_i = 1 = \sum_{i=(a+1)+1}^{(a+1)+(b+1)} \lambda_i.$$

The configuration of the vertices of both simplices is $p = (p_1, \dots, p_{(a+1)+(b+1)})$. Define a stress matrix as

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{a+1} \\ -\lambda_{(a+1)+1} \\ \vdots \\ -\lambda_{(a+1)+(b+1)} \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{a+1} & -\lambda_{(a+1)+1} & \dots & -\lambda_{(a+1)+(b+1)} \end{bmatrix} = \Omega.$$

From this we see that the quadratic form corresponding to Ω is positive semi-definite and that the stress coefficients $\omega_{i,j} = \lambda_i \lambda_j$ if i and j are vertices of the same simplex, and $\omega_{i,j} = -\lambda_i \lambda_j$ if i and j are vertices of different simplices. (Recall that the off diagonal entries of Ω are the corresponding stress coefficients but with the opposite sign by condition (i).) It is clear that Ω is a symmetric matrix, so by Lemma 5, we only need to check that row and column sums are 0. The row sum can be calculated by multipying Ω on the right by column vector of all one's. But

$$\begin{bmatrix}\lambda_1 & \lambda_2 & \dots & \lambda_{a+1} & -\lambda_{(a+1)+1} & \cdots & -\lambda_{(a+1)+(b+1)}\end{bmatrix}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

$$=\sum_{i=1}^{a+1}\lambda_i-\sum_{i=(a+1)+1}^{(a+1)+(b+1)}\lambda_i=1-1=0.$$

Hence Ω is a stress matrix for ω by Lemma 5. To show that p is a minimum point for E_ω, again we calculate $E_\omega(p)$. We already know that the quadratic form for Ω is positive semi-definite, and thus E_ω is positive semi-definite. Hence p is a minimum point for E_ω if and only if $E_\omega(p)=0$. Let $(x_1,x_2,\dots,x_{a+b+2})$ be the first coordinates of each point of p. Then from (6-1) we have

$$\begin{bmatrix}\lambda_1 & \lambda_2 & \dots & \lambda_{a+1} & -\lambda_{(a+1)+1} & \cdots & -\lambda_{(a+1)+(b+1)}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\dots\\x_{a+b+2}\end{bmatrix}$$

$$=\sum_{i=1}^{a+1}\lambda_i x_i-\sum_{i=(a+1)+1}^{(a+1)+(b+1)}\lambda_i x_i=0.$$

Applying a similar argument to the other coordinates we see that $E_\omega(p)=0$. Thus p is a minimum point for E_ω, and $G(p)$ is unyielding by Proposition 2.

For example, the configurations for both tensegrities of Figure 2 are unyielding since their struts intersect on their relative interiors and all the other pairs of points are connected with cables. Incidentally, in the plane, another example of an unyielding tensegrity using Lemma 7, is a triangle with three struts for its edges and a point in its interior connected with cables to the three vertices of the triangle. However, this particular tensegrity does not seem to be useful for a proof of Theorem 2. We will see other examples in higher dimensions later.

The use of stresses here is also has similarities to an idea of Bárány in [1] which was used for a different problem.

7. High-dimensional unyielding tensegrities

In order to do the calculations for the tesegrities to come we need some Lemmas about the circumcenter of regular simplices. Recall that an n-dimensional simplex has $n+1$ affine independent vertices; its *circumcenter* is the point equidistant to all of these vertices; and an *altitude* goes from a vertex to the nearest point on the hyperplane through the opposite face.

LEMMA 8. *For a regular n-dimensional simplex with side length 1, let r_n be the length of the radius of the circumcenter, and a_n the length of the altitude. Then*

$$r_n^2 = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) \tag{7-1}$$

$$a_n^2 = \frac{1}{2}\left(1 + \frac{1}{n}\right). \tag{7-2}$$

PROOF. Without loss of generality, let the circumcenter be the origin 0, and let $p = (p_1, \dots, p_{n+1})$ be the vertices of the regular simplex. Since the edges of the simplex have unit length when $i \neq j$, $1 = (p_i - p_j) \cdot (p_i - p_j) = 2r_n^2 - 2p_i \cdot p_j$, and so $p_i \cdot p_j = r_n^2 - \frac{1}{2}$. Then by symmetry, $\sum_{i=1}^{n+1} p_i = 0$, and $(\sum_{i=1}^{n+1} p_i)^2 = (n+1)r_n^2 - [(n+1)^2 - (n+1)]p_i \cdot p_j = 0$. Solving for r_n^2 gives (7-1).

To calculate a_n observe that it is the length of the line from the circumcenter of an $(n-1)$-dimensional face to the opposite vertex. It is also perpendicular to any such radius, whose length is r_{n-1}. Hence, $a_n^2 = 1 - r_{n-1}^2$, and (7-2) follows.

The calculations of Lemma 8 are well-known, but we simply include them for completeness.

LEMMA 9. *Let $r_n(\alpha)$ be the length of the circumradius of the n-dimensional simplex with all edges of length 1, except one which is of length $\alpha \leq 1$. Then*

$$r_n(\alpha)^2 = \frac{1}{n^2\left(\left(\frac{2}{\alpha}\right)^2 - 2\left(1 - \frac{1}{n}\right)\right)} + \frac{1}{2}\left(1 - \frac{1}{n}\right) \tag{7-3}$$

PROOF. Let $p = (p_1, \dots, p_{n+1})$ be the vertices of the simplex σ, where $|p_1 - p_2| = \alpha$, and all the other edges have length 1. Let σ^{n-2} be the $(n-2)$-dimensional simplex determined by $p = (p_3, \dots, p_{n+1})$, and let $\hat{\sigma}^{n-2}$ be the circumcenter of σ^{n-2}, $\hat{\sigma}^{n-1}$ the circumcenter of the simplex determined by $(p_2, \dots, p_{n+1})$, and $\hat{\sigma}^n$ the circumcenter of σ. By the symmetry of σ, $\hat{\sigma}^n$ lies in altitude on the base of the isosceles triangle determined by p_1, p_2, and $\hat{\sigma}^{n-2}$, while $\hat{\sigma}^{n-1}$ lies on the edge from p_1 to $\hat{\sigma}^{n-2}$. It is clear that $|p_2 - \hat{\sigma}^n| = r_n(\alpha)$, and $|p_2 - \hat{\sigma}^{n-1}|^2 = r_{n-1}^2 = \frac{1}{2}(1 - \frac{1}{n})$, by Lemma 7, and using the notation of Lemma 7. Since $\hat{\sigma}^n, \hat{\sigma}^{n-1}, p_1$ form a right triangle at $\hat{\sigma}^{n-1}$,

$$|\hat{\sigma}^n - \hat{\sigma}^{n-1}|^2 = r_n(\alpha)^2 - r_{n-1}^2.$$

Also $|\hat{\sigma}^{n-2} - \hat{\sigma}^{n-1}| = a_{n-1} - r_{n-1} = \frac{1}{n}a_{n-1}$ by Lemma 7, where a_{n-1} is the length of the altitude of the $(n-1)$-dimensional unit simplex. See Figure 3. By similar triangles we get

$$\frac{|\hat{\sigma}^n - \hat{\sigma}^{n-1}|}{a_{n-1}/n} = \frac{\alpha/2}{\sqrt{a_{n-1}^2 - (\alpha/2)^2}}.$$

Combining these formulas and using the definitions of (7-1) and (7-2) we get (7-3), as desired.

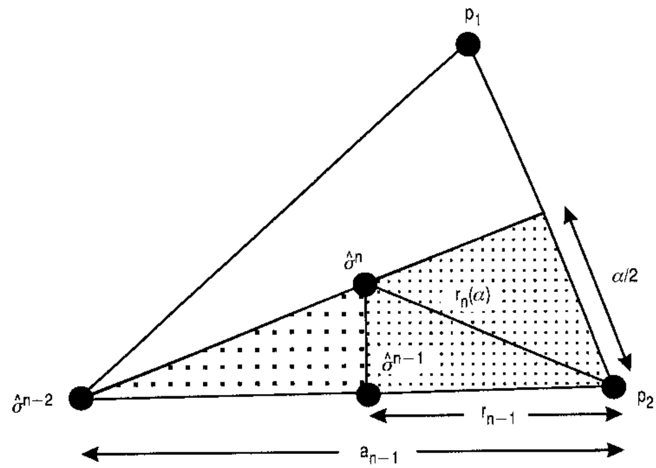

Figure 3. The cross-section triangle

It is clear from Figure 3 that the simplex exists with the length α if and only if $0 \leq \alpha \leq 2a_{n-1}$, and $1 < 2a_{n-1}$.

LEMMA 10. *Let $n = a + b$ be positive integers, $0 < \alpha < 1$, and define*

$$(7-4) \qquad c_{a,b}^2(\alpha) = \frac{1}{a^2\left(\left(\frac{2}{\alpha}\right)^2 - 2\left(1 - \frac{1}{a}\right)\right)} + \frac{1}{2}\left(1 - \frac{1}{a}\right) + \frac{1}{2}\left(1 - \frac{1}{b+1}\right).$$

Then there is an unyielding tensegrity in $\mathbb{E}^n$ with one strut of length of α, all the other struts of length 1, and all cables of length $c_{a,b}(\alpha)$.

PROOF. Let σ_α^a be an a-dimensional simplex with one edge of length α, and others of length 1, by Lemma 9. In a complimentary orthogonal b-dimensional hyperplane, let σ^b be a b-dimensional simplex, where $\sigma_\alpha^a \cup \sigma^b$ is the circumcenter of both simplices. By Lemma 9 the circumradius of σ_α^a is $r_a(\alpha)$ as in (7-3). By Lemma 7 the circumradius of σ^b is r_b as in (7-1). Since σ_α^a and σ^b lie in orthogonal subspaces, the distance between any vertex of σ_α^a and any vertex of σ^b is $\sqrt{r_a(\alpha)^2 + r_b^2} = c_{a,b}(\alpha)$. Since $\sigma_\alpha^a \cup \sigma^b$ lies in the relative interior of both σ_α^a and σ^b, Lemma 7 implies that any pair of vertices from different simplices are cables and all the other pairs of vertices form struts in an unyielding tensegrity. The lengths of the cables are all $c_{a,b}(\alpha)$, and the strut lengths come from the construction of the simplices. This finishes the proof.

Figure 4 shows an example of such a tensegrity in dimension 3, with $a = 2$ and $b = 1$.

We now can show how to improve the ratios of cables to struts, so that we know that we have a congruence, if the domain space is higher dimensional. Recall

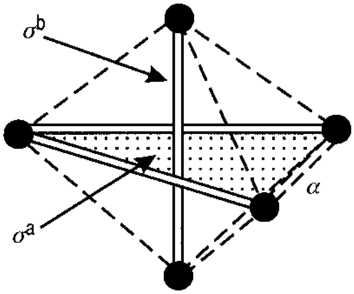

Figure 4. A three-dimensional unyielding tensegrity

that X_k is the set of ratios, where it is true that any function, for $n \geq k$, $f : \mathbb{E}^n \to \mathbb{E}^m$ is a congruence when it has a cable to strut ratio in X_k. The following proposition is the second part of Theorem 2.

PROPOSITION 2. $\limsup_{k\to\infty} X_k = 1$

PROOF. Let $f : \mathbb{E}^n \to \mathbb{E}^m, n \leq k$ be any function with cable length c and strut length s, such that $c_{a,b}(1/2) = c/s$, and $a+b = n$, where $c_{a,b}$ is defined by (7-4). By Lemma 10, there is an unyielding tensegrity that has all cable lengths $c_{a,b}(1/2)$, one strut length $1/2$ and all the other strut lengths 1. By Lemma 1, $1/2$ is an implied cable length for f. By Theorem 1 (but is easily shown directly), this implies that f is a congruence. From the formula for (7-4) it is clear that $lim_{a,b\to\infty} c_{a,b}(1/2) = 1$. Thus $limsup_{k\to\infty} X_k = 1$. This finishes the proof.

REMARK 3. In dimension 3, we can apply the tensegrity of Lemma 10, and the technique of Lemma 4 to get a sequence of ratios in X_3 converging to

$$\sqrt{(17-\sqrt{161})/8} = 0.734, \ \ldots$$

which is outside the interval $(0, \tau)$. But for any fixed dimension k, the techniques do not seem to be able to show that there are ratios in X_k converging to 1.

8. Conjectures and challenges

The following are some natural questions that beg to be addressed.

CHALLENGE 1. *Find a proof of Theorem 1 independent from and preferably simpler than that described in* [9].

Perhaps some of the additional tensegrities described in this paper could help find a systematic way of proving the more complicated Lemmas in [9].

CHALLENGE 2. *Find some* $0 < \rho < 1$ *and* $2 < m$ *find a two-distance preserving function* $f : \mathbb{E}^2 \to \mathbb{E}^m$ *where the ratio between the distances is* ρ.

We can be even more adventurous.

CONJECTURE 1. *The set of cable-strut ratios* X_2 *which guarantee that a function* f *from* $\mathbb{E}^2$ *with any of those ratios is exactly the interval* $(0, 1/\tau)$.

In other words this conjecture states that for any any positive ratio ρ not in the interval $(0, 1/\tau)$, for some m there is a function $f : \mathbb{E}^2 \to \mathbb{E}^m$ that is not a congruence but has cable-strut ratio ρ. This is easy to show for $\rho \geq 1$. Indeed, it could happen that there is a two-distance preserving function under the same conditions. The methods of [7] could prove useful here.

CONJECTURE 2. *The set of cable-strut ratios* X_3 *which guarantee that a function* f *from* $\mathbb{E}^3$ *with any of those ratios is exactly the interval* $(0, \sqrt{(17-\sqrt{161})/8})$.

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