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MINIMAL TRANSLATION COVERS FOR SETS OF DIAMETER 1

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Abstract

In this note we prove that the smallest perimeter convex domain that is a translation cover for the collection of all planar sets of diameter ≤ 1 is the circle of diameter $2/\sqrt{3}$.

1. Introduction

Let K be a convex body in the d-dimensional Euclidean space $\mathbf{E^d}$, that is let K be a compact convex set with nonempty interior in $\mathbf{E^d}$. Among the balls that contain K, there exits exactly one with minimal radius $R(K)$. It is called the circumscribed ball and $R(K)$ is the circumscribed radius of K. The diameter $D(K)$ of K is the maximum of the distances between any two points of K. Jung's theorem ([2], p. 84-85) asserts that $R(K) \leq D(K) \cdot \sqrt{d/(2d+2)}$, where the equality holds for the regular simplex as well as for every K that contains the regular d-simplex of edge length $D(K)$.

Let $\mathbf{K_1^d}$ be the collection of all sets of diameter ≤ 1 in $\mathbf{E^d}$. We say that the set C of $\mathbf{E^d}$ is a translation cover for $\mathbf{K_1^d}$ if each member of $\mathbf{K_1^d}$ lies in some translate of C. Jung's theorem asserts that the smallest circumscribed radius of the convex bodies of $\mathbf{E^d}$ that are translation covers for $\mathbf{K_1^d}$ is $\sqrt{d/(2d+2)}$.

For a convex body K in $\mathbf{E^d}$ the support function h_K is defined by

$$h_K(u) = \sup\{\langle x, u\rangle \,|\, x \in K\}$$

for any vector u in $\mathbf{E^d}$, where $\langle x, u\rangle$ denotes the standard inner product of the vectors x and u. Let $\mathbf{S^{d-1}}$ be the unit sphere of $\mathbf{E^d}$ centered at the origin that is $\mathbf{S^{d-1}} = \{x \in \mathbf{E^d} \,|\, \langle x, x\rangle = 1\}$. The number

$$w_K(u) = h_K(u) + h_K(-u)$$

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for $u \in \mathbf{S^{d-1}}$ is called the width of K in the direction u. Finally, the mean value of the width function w_K over $\mathbf{S^{d-1}}$ is called the mean width $w(K)$ of K. Thus,

$$w(K) = (2/\omega_d) \cdot \int_{\mathbf{S^{d-1}}} h_K(u) d\mathbf{L^{d-1}}(u),$$

where $\mathbf{L^{d-1}}$ denotes the $(d-1)$–dimensional spherical Lebesgue measure computed in $\mathbf{S^{d-1}}$ and $\omega_d = \mathbf{L^{d-1}}(\mathbf{S^{d-1}}) = 2\pi^{(d/2)}/\Gamma(d/2)$ is the surface area of $\mathbf{S^{d-1}}$. As $w(K) \leq 2 \cdot R(K)$the following statement is a stronger form of Jung's theorem.

THEOREM 1. *The smallest mean width of the convex bodies of* $\mathbf{E^d}$ *that are translation covers for the collection of all sets of diameter* ≤ 1 *in* $\mathbf{E^d}$ *is* $\sqrt{2d/d+1}$.

Helly's theorem ([6]) easily implies (see [1]) that the set C of $\mathbf{E^d}$ is a translation cover for $\mathbf{K^d_1}$ if and only if any set of $d+1$ points having diameter 1 in $\mathbf{E^d}$ is a subset of some translate of C. For $d = 2$ this observation and Cauchy's surface area formula [4] (according to which if K is a convex domain in $\mathbf{E^2}$, then the perimeter of K $Per(K) = \pi \cdot w(K)$) imply the following stronger form of Theorem 1. This has been claimed in [1] without proof. In this note we provide a detailed proof of Theorem 2 based on the proper Fourier expansion of the support function.

THEOREM 2. *The smallest perimeter convex domain that is a translation cover for the collection of all planar sets of diameter* ≤ 1 *is the circle of diameter* $2/\sqrt{3}$.

Recall that the Reuleaux triangle of constant width 1 is defined as the intersection of the 3 disks of unit radius centered at the vertices of a regular triangle of side length 1. Then notice that any set of 3 points of diameter ≤ 1 in $\mathbf{E^2}$ is a subset of some Reuleaux triangle of constant width 1. Thus, Helly's theorem easily implies that the set C of $\mathbf{E^2}$ is a translation cover for $\mathbf{K^2_1}$ if and only if any Reuleaux triangle of constant width 1 is a subset of some translate of C. Finally, notice that the Reuleaux triangle of constant width 1 can be placed in a square Q of unit side length, and can then be turned freely within Q while maintaining contact with all 4 sides of Q. Hence, the covered part of Q of area $\pi/6 + 2\sqrt{3} - 3 = 0.98770039...$ (for more details see [3]) is a translation cover for $\mathbf{K^2_1}$. Thus, searching for the smallest area convex domain that is a translation cover for $\mathbf{K^2_1}$ the following problem arises rather naturally.

PROBLEM. Prove or disprove that the smallest area translation cover for the collection of all planar sets of diameter ≤ 1 is the truncated unit square of area $\pi/6 + 2\sqrt{3} - 3 = 0.98770039....$

Although the above problem has been raised independently by B. C. Rennie (1977) [5], K. Bezdek and R. Connelly (1989) [1] several years ago, it is still a challanging unsolved question of discrete geometry. For related results on translation covers see [1].

Finally, we mention as an independent result that the proof of Theorem 2 gives the following characterization of circles.

REMARK. The circular disk is the only convex domain for which all tangential regular triangles meet the domain at the midpoints of their sides.

2. Proof of Theorem 1

Let $v_1, v_2, \ldots, v_{d+1}$ be the vertices of a regular d-simplex of edge length 1 in $\mathbf{E^d}$ the center of which is the origin. Let C be a translation cover for the collection of all sets of diameter ≤ 1 in $\mathbf{E^d}$. Without loss of generality we may assume that v_i belongs to C for all $1 \leq i \leq d+1$. As C is a translation cover for the collection of all sets of diameter ≤ 1 in $\mathbf{E^d}$ there must be a vector t such that the points $t - v_1, t - v_2, \ldots, t - v_{d+1}$ (which are the vertices of a regular d-simplex of edge length 1) belong to C. Let u_i be the unit vector of $\mathbf{E^d}$ having the same direction as $v_i, 1 \leq i \leq d+1$. Obviously, $\langle v_i, u_i \rangle = \sqrt{d/(2d+2)}, 1 \leq i \leq d+1$ and $\Sigma_{i=1}^{d+1} u_i = o$. Let K be the convex hull of the points $v_1, v_2, \ldots, v_{d+1}, t - v_1, t - v_2, \ldots, t - v_{d+1}$. As $K \subset C$ Theorem 1 follows from the inequality $\Sigma_{i=1}^{d+1} w_K(u_i) \geq (d+1)\sqrt{2d/(d+1)}$. Finally, the inequality needed can be obtained as follows: $\Sigma_{i=1}^{d+1} w_K(u_i) = \Sigma_{i=1}^{d+1}(h_K(u_i) + h_K(-u_i)) \geq \Sigma_{i=1}^{d+1}(\langle v_i, u_i \rangle + \langle t - v_i, -u_i \rangle) = \Sigma_{i=1}^{d+1}(2\langle v_i, u_i \rangle - \langle t, u_i \rangle) = 2 \cdot \Sigma_{i=1}^{d+1} \langle v_i, u_i \rangle - \langle t, \Sigma_{i=1}^{d+1} u_i \rangle = 2(d+1)\sqrt{d/(2d+2)} = (d+1)\sqrt{2d/(d+1)}$. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let K be the convex domain that is the smallest perimeter translation cover for the collection of all sets of diameter ≤ 1 in $\mathbf{E^2}$. Without loss of generality we may assume that the origin o is an interior point of K. Assume that a cartesian (x_1, x_2)-coordinate system is given. Then there is a one-to-one correspondence (modulo 2π) between the vectors $u \in \mathbf{S^1}$ and the angle, say ω, between u and the positive x_1-axis; in other words: $u = u(\omega) = (\cos\omega, \sin\omega)$. Consequently, we write the restricted support function h_K of K (defined on $\mathbf{S^1}$) as a periodic function $h_K(\omega)$ of the angle ω. As $h_K(\omega)$ is absolutely continuous it has a Fourier expansion, say

$$\sum_{k=0}^{\infty}(a_k \cos k\omega + b_k \sin k\omega),$$

where the coefficients are defined by

$$a_0 = (1/2\pi)\int_0^{2\pi} f(\omega)d\omega, \quad a_k = (1/\pi)\int_0^{2\pi} f(\omega)\cos(k\omega)d\omega \quad (k = 1, 2, \ldots),$$

$$b_k = (1/\pi)\int_0^{2\pi} f(\omega)\sin(k\omega)d\omega \qquad (k = 0, 1, \dots),$$

and then the Fourier expansion of $h'_K(\omega)$ is

$$\sum_{k=0}^{\infty}(kb_k \cos k\omega - ka_k \sin k\omega).$$

(For more details see for example [4].)

Let v_1^*, v_2^*, v_3^* be the vertices of a positively oriented regular triangle of side length 1 in $\mathbf{E}^2$. As K is the smallest perimeter translation cover for $\mathbf{K}_1^d$ the proof of Theorem 1 and the formula $Per(K) = \pi w(K)$ imply that there exists a translate $v_1v_2v_3$ of the triangle $v_1^*v_2^*v_3^*$ such that the vertices v_1, v_2, v_3 all lie on the boundary of K in positive (i.e. anti-clockwise) order moreover, the line through v_i parallel to the side opposite to v_i in the triangle $v_1v_2v_3$ is a supporting line to K at the point v_i for each $1 \le i \le 3$. Without loss of generality we may assume that angle assigned to the outer unit normal vector of the supporting line at v_1 is ω and so $v_1 = h_K(\omega)u(\omega) + h'_K(\omega)u'(\omega)$, where $u(\omega) = (\cos\omega, \sin\omega)$ (for more details see for example [4]). As a result we get that $v_2 = h_K(\omega + (2\pi/3))u(\omega + (2\pi/3)) + h'_K(\omega + (2\pi/3))u'(\omega + (2\pi/3))$. Finally, as the intersection point of the supporting lines to K at the points v_1 and v_2 lies at distance 1 from v_1 as well as v_2 the following equation must hold for almost all ω:

$$h_K(\omega)u(\omega) + [1 + h'_K(\omega)]u'(\omega) = h_K(\omega + (2\pi/3))u(\omega + (2\pi/3)) + [-1 + h'_K(\omega + (2\pi/3))]u'(\omega + (2\pi/3)).$$

Taking the inner product of both sides of the above equation with the vector $u(\omega)$ and then with the vector $u'(\omega)$ we get the following two equations:

$$h_K(\omega) = -(1/2)h_K(\omega + (2\pi/3)) - (\sqrt{3}/2)h'_K(\omega + (2\pi/3)) + (\sqrt{3}/2),$$

$$h'_K(\omega) = (\sqrt{3}/2)h_K(\omega + (2\pi/3)) - (1/2)h'_K(\omega + (2\pi/3)) - (1/2).$$

Multiplying the second equation with $\sqrt{3}$ and subtracting from the first equation we obtain the following single equation that must hold for almost all ω:

$$h_K(\omega + (2\pi/3)) = (\sqrt{3}/2)h'_K(\omega) - (1/2)h_K(\omega) + (\sqrt{3}/2).$$

Thus, the functions on the two sides of the above equation must have the same Fourier expansion. Computing the Fourier coefficients on both sides based on the proper Fourier expansions of $h_K(\omega)$ and of $h'_K(\omega)$ we get the following equations:

$$a_0 = -(1/2)a_0 + (\sqrt{3}/2),$$

$$a_k\cos(k(2\pi/3)) + b_k\sin(k(2\pi/3)) = -(1/2)a_k + (\sqrt{3}/2)kb_k, \qquad k = 1, 2, \dots,$$

$$-a_k\sin(k(2\pi/3)) + b_k\cos(k(2\pi/3)) = -(\sqrt{3}/2)ka_k - (1/2)b_k, \qquad k = 1, 2, \dots.$$

Finally, these equations imply that the Fourier expansion of $h_K(\omega)$ is of the form

$$(\sqrt{3}/3) + a_1\cos\omega + b_1\sin\omega$$

that is K is a circle of radius $\sqrt{3}/3$. This completes the proof of Theorem 2.

REFERENCES

[1] K. BEZDEK and R. CONNELLY, Covering curves by translates of a convex set, *Amer. Math. Month.* **96** (1989), 789–806.

[2] T. BONNESEN und W. FENCHEL, *Theorie der konvexen Körper*, Springer, Berlin, 1934. English translation: BCS Associates, Moscow, Idaho, 1987.

[3] G. F. D. DUFF, On universal covering sets and translation covers in the plane, *James Cook Mathematical News* **2** (1980), 109–120.

[4] H. GROEMER, *Geometric applications of Fourier series and spherical harmonics*, Cambridge University Press, 1996.

[5] V. KLEE and S. WAGON, Old and new unsolved problems in plane geometry and number theory, *The Dolciani Mathematical Expositions*, MAA, 1991.

[6] S. R. LAY, *Convex sets and their applications*, John Wiley, New York, 1982.

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