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THE SPACE OF SIMPLEXWISE LINEAR HOMEOMORPHISMS OF A CONVEX 2-DISK

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§1. INTRODUCTION

LET K^n be a finite simplicial complex, whose underlying space $|K^n|$ is a combinatorial n-dimensional disk in $\mathbb{R}^n$. Let k be the number of interior vertices of K^n, and let $L(K^n)$ be the space of homeomorphisms of $|K^n|$ that are affine linear on each simplex of K^n, and the identity on the boundary of K^n. Each such simplexwise linear homeomorphism is determined by its values on the interior vertices, and $L(K^n)$ is naturally identified with an open subspace of $\mathbb{R}^{nk}$.

Our main result is that if $n = 2$, and $|K^2|$ is convex, then $L(K^2)$ is homeomorphic to $\mathbb{R}^{2k}$. In particular $L(K^2)$ is contractible. This answers a question, for $n = 2$, of R. Thom (in (5.2) of[13] 1958). Our proof is direct, and does not make use of any of the results mentioned below. (A manifold being homeomorphic to $\mathbb{R}^{2k}$ is slightly stronger than being contractible. For $n > 5$, if a manifold of dimension n is contractible and "simply-connected at infinity" then it is homeomorphic to $\mathbb{R}^n$. See Stallings[12]. Whitehead[14] has examples of contractible manifolds that are not homeomorphic to $\mathbb{R}^n$.)

Interest in $L(K^n)$ started with papers of Cairns ([2], [3] and [4]) where he showed, among other things:

THEOREM (Cairns)[3]. *If the boundary of K^2 is a triangle* (*with* 3 *vertices*), *then* $\pi_0 L(K^2) = 0$.

He ([2], p. 808) and Whitehead[15], 1961, showed that this and related results imply that every combinatorial 4-manifold has a smooth structure. See Kuiper[9] for an alternate treatment. Later Ho[5] showed the following:

THEOREM (Ho). *If the boundary of K^2 is a triangle* (*with* 3 *vertices*), *then* $\pi_1 L(K^2) = 0$.

Bing and Starbird[1] subsequently showed:

THEOREM (Bing, Starbird). *If $|K^2|$ is starlike, and K^2 has no spanning* 1-*simplices, then* $\pi_0 L(K^2) = 0$.

Our Theorem does not imply Bing and Starbird's theorem, but some condition on the shape of $|K^2|$ is necessary for both their result and ours, since Bing and Starbird have an example ([1], Example 4) of a K^2 such that $L(K^2)$ is not even connected.

For $n \geq 2$, Ho[6] has one positive result:

THEOREM (Ho). *If $k \leq 2$, then $L(K^n)$ is contractible.*

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On the other hand, Kuiper[9] has shown, using the existence of an exotic smooth 7-sphere:

THEOREM (Kuiper). *For at least one of $n = 3, 4, 5$ or 6, there exists K^n, whose boundary is the boundary of an n-simplex, and $\pi_{6-n}L(K^n)$ is non-trivial.*

Also, Starbird[11] has a convex K^3, with no spanning simplices, such that $\pi_0 L(K^3)$ is not trivial. Thus our methods necessarily do not generalize to all $n \geq 3$.

Using Kuiper's result (5.3) in [9], our Theorem implies the following result of Smale[10]:

COROLLARY (Smale). *The space of diffeomorphisms of a smooth 2-disk, fixed on the boundary, is contractible.*

On the other hand, it is not clear how to use the above Corollary to obtain our result. If K_1 is a subdivision of K, then $L(K)$ naturally becomes a subspace of $L(K_1)$. The direct limit over all subdivisions of K is a space L_∞. The Corollary implies that L_∞ is contractible, but says nothing about any $L(K)$.

Our idea is to fibre the space $L(K)$ with convex disks, and this allows us to see how to build up the product structure of $L(K)$ inductively. Section 2 discusses the general fibering lemmas needed; §3 contains the basic lemma that allows us to know that subsets exist down or up along fibres; §4 applies the basic lemma of §3 and sets up the proof of the main theorem; §5 is the proof of the main theorem.

§2. CONVEX DISK DECOMPOSITIONS

Let X be a locally compact subset of $\mathbb{R}^d$. Let $\mathscr{D}$ be a continuous decomposition of X into compact convex sets of the same dimension n. We call such a $\mathscr{D}$ a *convex disk decomposition* (of X) of dimension n.

LEMMA 2.1. *Let $\mathscr{D}$ be a convex disk decomposition of X. Then there is a continuous section s: $X/\mathscr{D} \to X$ such that for all $D \in \mathscr{D}$, $s(D)$ is a point in the relative interior of D. (By relative interior we mean the interior relative to the n-dimensional affine linear subspace of $\mathbb{R}^d$ containing d.)*

Proof. Let $s(D) =$ center of gravity of $D = (1/\int_D|\mathrm{d}x|)\int_D x\ \mathrm{d}x$. s is continuous since $\mathscr{D}$ is a continuous collection and the measure is always the n-dimensional measure of the decomposition sets.

LEMMA 2.2. *Let $X \subset Y$ be locally compact subsets of $\mathbb{R}^d$ with convex disk decompositions $\mathscr{D}_X$, $\mathscr{D}_Y$ respectively, both of dimension n, such that $\mathscr{D}_X = \{X \cap D_Y | D_Y \in \mathscr{D}_Y\}$. If $X \cap D_Y$ is non-empty for each D_Y, then X is homeomorphic to Y (with fibre relative interiors going to fibre relative interiors).*

Proof. Let s: $X/\mathscr{D} \to X$ be the section of Lemma 2.1. Since the decompositions are continuous and of constant dimension, the boundaries form continuous decompositions. Hence the radial homeomorphism from $s(D_X)$, pushing linearly each ray in D_X to D_Y, is continuous (see Fig. 1).

LEMMA 2.3. *Let $X \subset Y$ be locally compact subsets of $\mathbb{R}^d$ with convex disk decompositions $\mathscr{D}_X$, $\mathscr{D}_Y$ respectively of dimensions $n-1$ and n, such that $\mathscr{D}_X = \{X \cap D_Y | D_Y \in \mathscr{D}_Y\}$. If each*

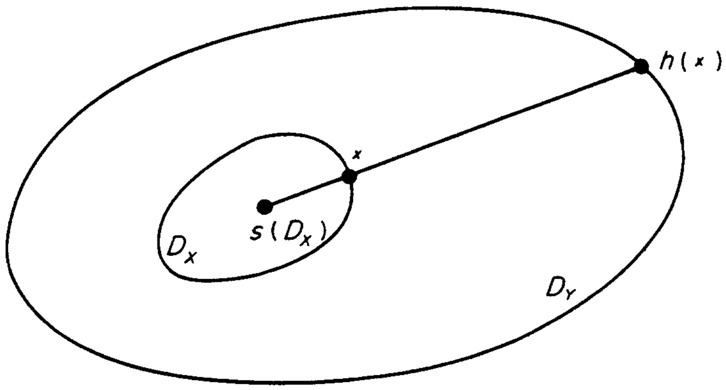

Fig. 1.

D_Y is separated by $X \cap D_y$ (which is thus non-empty) and X is the common boundary of a separation of $X - Y$ into two disjoint open sets, then Y is homeomorphic to $X \times I$.

Proof. Let Z be the closure in Y of one of the open sets of the separation of $Y - X$. Then $\mathscr{D}_Z = \{Z \cap D_Y | D_Y \in \mathscr{D}_Y\}$ is a convex disk decomposition of Z of dimension n. Let s: $Z/\mathscr{D}_Z \to Z$ be the section of Lemma 2.1. For $x \in X$, let $[x] \in \mathscr{D}_Z$ be such that $x \in [x]$. Define f: $X \times [0, (1/2)] \to Y$ by

$$f(x, t) = ts([x]) + (1 - t)x.$$

(See Fig. 2.)

Let $W = f(X \times [0, 1/2])$. Lemma 2.2 applied to W and Y completes the proof.

§3. SIMPLEXWISE LINEAR HOMEOMORPHISMS

Let K be a finite geometric simplicial complex (in the sense of Hudson [8]). Thus K has an underlying space $|K|$ which we will always assume to be a subset of $\mathbb{R}^N$ for some N, so the simplices of K have affine linear structures compatible with the ambient space. With this in mind we say a map f: $|K| \to \mathbb{R}^2$ is *simplexwise linear* (or *SL*) if $f|\sigma$ is affine linear for all simplices $\sigma \in K$. We write f: $K \to \mathbb{R}^2$.

From this point on we suppose further that $|K|$ is homeomorphic to a 2-disk, and that K is oriented. Note that $|K|$ need not be a convex subset of the plane, or even in the plane for that matter. Let ∂K denote the boundary of K. let $T \subset \partial K$ be an arc, where $t_0, \ldots, t_n$

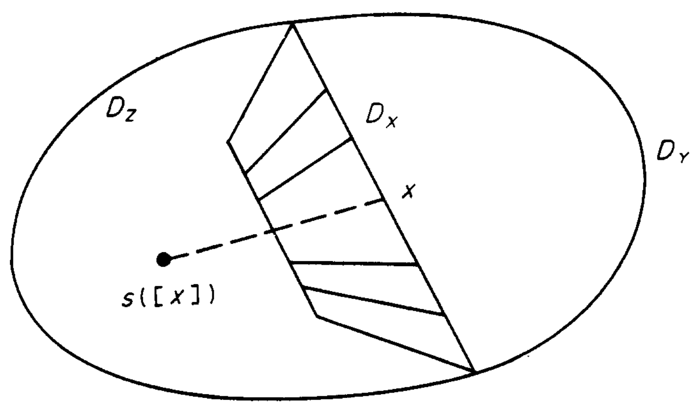

Fig. 2.

are the vertices of T in order, clockwise, in the orientation of K. We call T the *top* of K. Let $b_0 = t_0, \ldots, b_m = t_n$ be the remaining vertices of ∂K in counterclockwise order. We call these vertices the *bottom* of K. Let B_F be some subset of $\{b_0, \ldots, b_m\}$ with b_0, $b_m \in B_F$.

In $\mathbb{R}^2$ let π_x, π_y denote the projections onto the x and y axes, respectively. We say that a function f: $T \cup B_F \to \mathbb{R}^2$ (linear on each 1-simplex) is in *standard position* (with respect to T and B_F) if:

(i) $\pi_x f(t_0) \leq \pi_x f(t_1) \leq \ldots \leq \pi_x f(t_n)$,
(ii) $f|T$ is convex (down),
(iii) $\pi_x f(b_i) \leq \pi_x f(b_j)$ for $i < j$, b_i, $b_j \in B_F$,
(iv) $f|B_F$ is convex (up).

(See Fig. 3.)

Let $\langle v_1, v_2, v_3 \rangle = \sigma$ denote a positively oriented 2-simplex of K. If f: $\sigma \to \mathbb{R}^2$ is affine linear, we write

$$\det(f|\sigma) = \det\begin{pmatrix} 1\, f(v_1) \\ 1\, f(v_2) \\ 1\, f(v_3) \end{pmatrix}$$

$\det(f|\sigma)$ can be regarded as twice the signed area of $f(\sigma)$, and is independent of the order of the vertices, v_1, v_2, v_3, as long as the order is compatible with the orientation of K.

Define: $R = \{f\colon K \to \mathbb{R}^2 | f \text{ is } SL, \det(f|\sigma) \geq 0\ \forall \sigma \in K^2\}$, $E = \{f\colon K \to \mathbb{R}^2 | f \text{ is } SL$, orientation preserving and one-to-one$\}$.

It is easy to see that $E \subset R$; on the other hand, even if $f \in R$ and $f|\partial K$ is one-to-one, it is not necessary that $f \in \operatorname{cl} E$, where cl here denotes the closure operator in the space of all simplexwise linear maps (which was identified as a Euclidean space in the Introduction). Consider the example of a simplexwise linear map $f \in R$ in Fig. 4.

It is convenient in the following lemma to have the final map in R instead of cl E. Let K^i denote the set of i-simplices.

Let p: $K \to \mathbb{R}^1$ be simplexwise linear. We say p is *well situated* if:

(i) for all $v \in K^0$, $1k(v, k) \cap p^{-1}p(v)$ is 0, 1 or 2 points;
(ii) for $M \subset K$ a subcomplex, and ∂M its (mod 2) boundary, then $p(M) = p(\partial M)$;
(iii) for $i = 0, 1, \ldots, m-1$, $p(b_i) < p(b_{i+1})$, b_i in B.

Note that if $f \in E$, then $p = \pi_x \circ f$ is well situated.

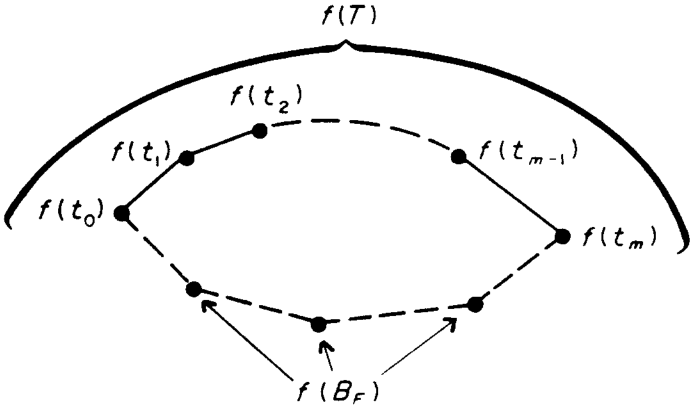

Fig. 3.

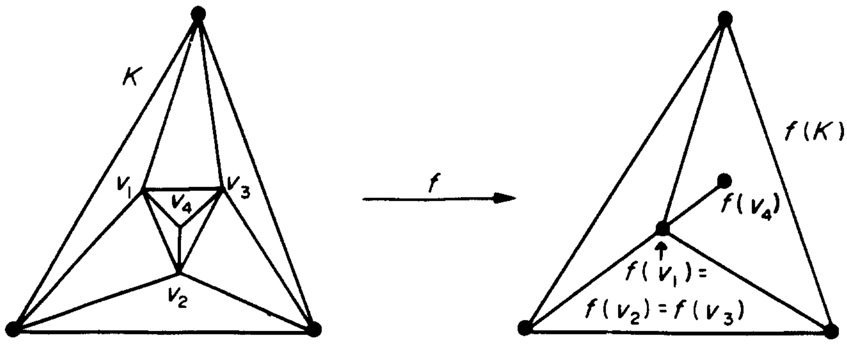

Fig. 4.

LEMMA 3.1. *Let K, with T and B_F as above, be given with (simplexwise linear) f: $T \cup B_F \to \mathbb{R}^2$ in standard position. Let p: $K \to \mathbb{R}^1$ be such that $\pi_x \circ f = p|T \cup B_F$, and K is well situated with respect to p. Then f extends to $\hat{f}$: $K \to \mathbb{R}^2$ such that*

(i) $p = \pi_x \circ \hat{f}$,

(ii) $\hat{f} \in R$,

(iii) *for each $v \in B - B_F$, with $p(b_i) < p(v) < p(b_j)$ for some b_i, $b_j \in B_F$, $i < j$; then $\pi_y \circ \hat{f}(v) \geq t\pi_y \circ \hat{f}(b_i) + (1-t)\pi_y \circ \hat{f}(b_j)$, with strict inequality when $f(T \cup B_F)$ has a 2-dimensional affine span, where $p(v) = tp(b_i) + (1-t)p(b_{i+1})$. (The last condition simply says that $p(v)$ lies above the line segment from $\hat{f}(b_i)$ to $\hat{f}(b_{i+1})$.)*

Proof. We proceed by induction on the number of 2-simplices of K. When this number is one the result is clear, since there is at most one vertex to extend to.

For each edge of T, $\sigma_i = \langle t_{i-1}, t_i \rangle$, for $i = 1, \ldots, n$, let $v_i \in K$ be the unique vertex of K such that $v_i\sigma_i \in K$ (i.e. $v_i = \mathrm{l}k(\sigma_i, K)$). Label σ_i + if $p(v_i) > p(t_i)$ and label σ_i − if $p(v_i) < p(t_{i-1})$. Clearly σ_1 is labeled + or is unlabeled, and σ_n is labeled − or is unlabeled, by conditions (ii), (iii) of being well-situated and condition (i) of being in standard position. If, for some i, σ_i is labeled + and σ_{i+1} is labeled −, then consider the link $\mathrm{l}k(t_i, K)$. Since $p(v_{i+1}) < p(t_i) \leq p(t_{i+1})$ and $p(t_{i-1}) \leq p(t_i) < p(v_i)$, $p^{-1}p(t_i)$ intersects both $\langle t_{i-1}, v_i \rangle$ and $\langle v_{i+1}, t_{1+1} \rangle$. If these are distinct 1-simplices of K, then $p^{-1}p(t_i)$ intersects $\mathrm{l}k(t_i, k)$ in these two 1-simplices in two points, and it is also intersects another point in $\mathrm{l}k(t_i, k)$ from v_i to v_{i+1}, contradicting condition (i) of being well situated. Hence either some 1-simplex of T is unlabeled, or $\langle t_{i-1}, t_{i+1} \rangle \in K$ for some i.

Case 1. $\langle t_{i-1}, t_{i+1} \rangle \in K$. Replace T by $(T - \{\sigma_{i-1}, \sigma_i\}) \cup \langle t_{i-1}, t_{i+1} \rangle$. Replace K by $K - \{\sigma_{i-1}, \sigma_i, \langle t_{i-1}, t_i, t_{i+1} \rangle\}$, where B and B_F remain the same. Obtain $\hat{f}$ for the new K by induction, and extend to t_i in the unique possible way.

Now, if Case 1 does not hold, then some σ_i is unlabeled, so $p(t_{i-1}) \leq p(v_i) \leq p(t_i)$.

Case 2. $v_i \in T$. Here $p(v_i) = p(t_{i-1})$ or $p(t_i)$, so $v_i = t_j$. Say $p(t_i) = p(t_j)$. If $t_j \neq t_{i+1}$, then consider the cycle $\langle t_i, t_{i+1}, \ldots, t_j, t_i \rangle$ in K. This cycle bounds a subcomplex $M \subset K$ such that $p(M) = p(t_i) = \ldots = p(t_i)$ by condition (ii) of being well situated. Define $\hat{f}(M) = f(t_i)$, and proceed by induction on the rest of K.

Case 3. $v_i \in K - \partial K$. If $p(t_{i-1}) < p(t_i)$, then define $\hat{f}(v_i) = rf(t_{i-1}) + (1-r)f(t_i)$, where

$$r = \frac{p(v_i) - p(t_i)}{p(t_{i-1}) - p(t_i)};$$

otherwise, define $\hat{f}(v_i) = f(t_{i-1}) = f(t_i)$. (See Fig. 5.)

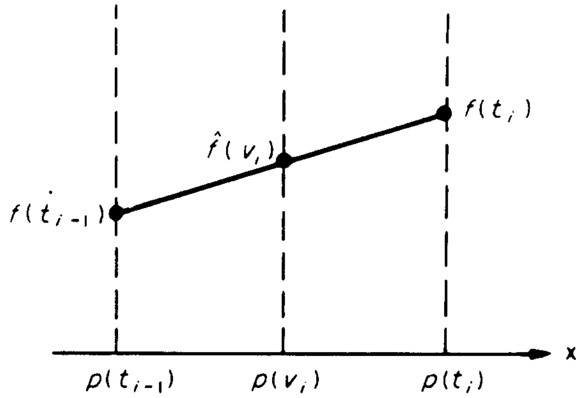

Fig. 5

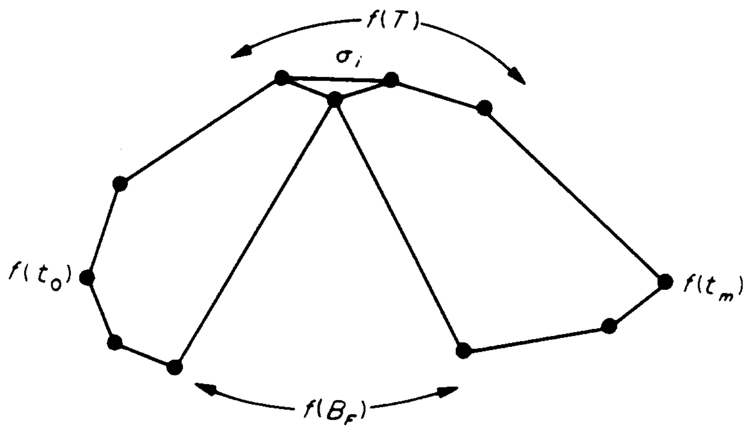

Fig. 6

:move $\tau = \langle t_{i-1}, t_i, v_i \rangle$ and $\langle t_{i-1}, t_i \rangle$ from K, where v_i is inserted between t_{i-1}, t_i in Tfine $\hat{f}$ on this new K, and extend linearly on τ. Note $\det(\hat{f}|\tau) = 0$, so all we know is th'$\in R$.

ıse 4. $v_i \in B - B_F$. Define $\hat{f}(v_i)$ as in Case 3, but we observe that $\hat{f}(v_i)$ is above the lin‹om $f(t_0)$ to $f(t_n)$, as in the last part of the conclusion. Strictly speaking, in this case K ıroken up into two disks with v_i as a single point in common. (See Fig. 6.)

ıse 5. $v_i \in B_F$. Define $\hat{f}(v_i) = f(v_i)$ as one must. Again K breaks up into two disks with v_i as‹ single vertex in common. This ends the proof of the lemma.

§4. MAPPING SPACES

Let$K$ be as in §3 and let $v_1, \ldots, v_n$ be the vertices of ∂K in clockwise order. For each $v \in \partial K$,let C_v be a convex subset of the plane $\mathbb{R}^2$. (In our applications every C_v, except possibl› one, will be a single point.) Let $\mathscr{C}$ be the collection of all C_v's; such a collection is callec a *proper collection* of convex sets if:

(i) for every choice of $v_i' \in C_{v_i}$, $i = 1, \ldots, n$, $(v_1', \ldots, v_1')$ is a simple closed polygonal curve in $\mathbb{R}^2$ in clockwise order,

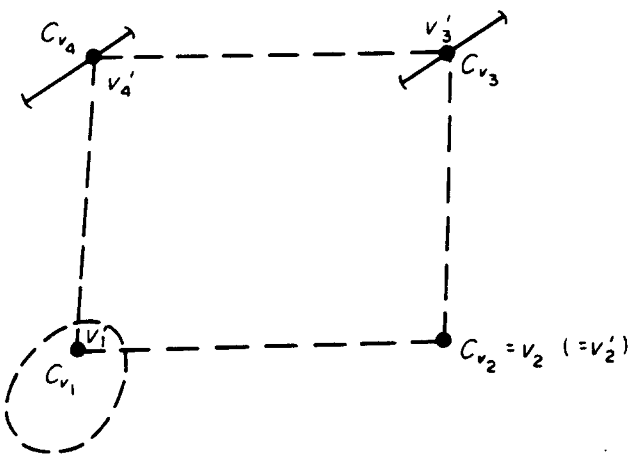

Fig. 7.

(ii) each C_v is relatively open (in its affine linear span).
(See Fig. 7.)

For any collection $\mathscr{C}$ of convex sets, define: $L(K, \mathscr{C}) = \{f: K \to \mathbb{R}^2 | f \text{ is } SL, f(v) \in C_v \, \forall v \in \partial K\}$; $E(K, \mathscr{C}) = E \cap L(K, \mathscr{C})$; $R(K, \mathscr{C}) = R \cap L(K, \mathscr{C})$.

It is not immediately clear that $E(K, \mathscr{C})$ or $R(K, \mathscr{C})$ is non-empty. Part of the effect of what is to follow wil be to show that for certain $\mathscr{C}$'s they are non-empty.

$L(K, \mathscr{C})$ is not to be confused with $L(K)$ of the introduction. If $|K|$ is assumed to lie in $\mathbb{R}^2$, and $\mathscr{C}$ is the (proper) collection of points $C_v = \{v\}$, then $E(K, \mathscr{C}) = L(K)$. Since this is such a frequent situation in what is to follow, we shall abuse notation slightly and replace $\mathscr{C}$ by ∂K. Thus if $\mathscr{C}$ is a proper collection of points, we will write $E(K, \partial K)$, $R(K, \partial K)$ instead of $E(K, \mathscr{C})$, $R(K, \mathscr{C})$, $L(K, \mathscr{C})$, respectively, where $\mathscr{C}$ will have been defined beforehand. Note that even if $|K| \subset \mathbb{R}^2$, C_v need not necessarily be $\{v\}$, as in the definition of $L(K)$.

LEMMA 4.1. *If $\mathscr{C}$ is a proper collection of convex sets, then*

$$E(K, \mathscr{C}) = \{f: K \to \mathbb{R}^2 | f \text{ is } SL, \det f(\sigma) > 0 \ \forall \sigma \in K^2\} \cap L(K, \mathscr{C}).$$

Proof. (See also Ho[7], 1981.) The inclusion "$\subset$" is trivial. For "$\supset$" let $f: K \to \mathbb{R}^2$ be such that $f \in L(K, \mathscr{C})$ and $\det(f|\sigma) > 0 \ \forall \sigma \in K^2$. Let S be the 2-sphere obtained by adjoining the cone on ∂K to $|K|$ in some Euclidean space, and consider $\mathbb{R}^2$ to be contained in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Extend f to map $\hat{f}: S \to S^2$ by sending ∞ to ∞ and coning over ∂K and $f(\partial K)$; by the definition of $\mathscr{C}$ being proper, $\hat{f}$ is a map of 2-spheres of topological degree 1. By the Hopf criterion, if $y \in f(K) - f(K^1)$, (that is, y is a "regular" point) then $f^{-1}(y)$ is a finite number of points and $\deg \hat{f} = \#\hat{f}^{-1}(y) = \#f^{-1}(y)$ counted with multiplicity. ($\# X$ is the number of points in X). But all multiplicities are 1 by the determinant condition, so there is exactly one point inverse. This concludes the proof.

Let $\mathbb{R}^{2n}$ be regarded as $2n$-tuples $(x_1, y_1, \ldots, x_n, y_n)$, with half the coordinates as x-coordinates, and the other half as y-coordinates. Let

$$\Pi_x: \mathbb{R}^{2n} \to \mathbb{R}^n, \ \Pi_y: \mathbb{R}^{2n} \to \mathbb{R}^n$$

be projections onto the x and y coordinates, respectively. Let $\pi_x: \mathbb{R}^2 \to \mathbb{R}^1$, $\pi_y: \mathbb{R}^2 \to \mathbb{R}^1$ be the same projections for $n = 1$. Let $X \subset \mathbb{R}^{2n}$ be any set. Define

$$\mathrm{cl}_x X = \cup \{\mathrm{cl}(\Pi_x{}^{-1}(p) \cap X) | p \in \mathbb{R}^n\}$$

and

$$\mathrm{cl}_y X = \cup \{\mathrm{cl}(\Pi_y{}^{-1}(p) \cap X) | p \in \mathbb{R}^n\},$$

where cl is the usual closure operator in $\mathbb{R}^{2n}$.

Thinking of $\Pi_x{}^{-1}(p) \cap X$ as fibres in X, cl_x simply takes the closure in each fibre, and similarly for cl_y. Note that the space of all simplexwise linear maps $K \to \mathbb{R}^2$ is naturally identified with $\mathbb{R}^{2n}$, where n is the total number of vertices of K. The other spaces defined above are regarded as subspaces of this $\mathbb{R}^{2n}$.

Let $\mathrm{cl}\,\mathscr{C} = \{\mathrm{cl}\, C_v | v \in (\partial K)^0\}$.

LEMMA 4.2. *Let C be a proper collection of convex sets. Then*

$$\mathrm{cl}_x E(K, \mathscr{C}) = R(K, \mathrm{cl}\,\mathscr{C}) \cap \Pi_x{}^{-1}\Pi_x E(K, \mathscr{C}).$$

Proof. The inclusion "$\subset$" is immediate, using Lemma 4.1, and the facts that $\mathrm{cl}_x X \subset \mathrm{cl}\, X$, and the determinant function is continuous.

For "$\supset$", let $f \in R(K, \mathrm{cl}\,\mathscr{C}) \cap \Pi_x{}^{-1}\Pi_x E(K, \mathscr{C})$. Let $g \in E(K, \mathscr{C})$ be such that $\Pi_x f = \Pi_x g$. For $0 \leq t \leq 1$, define $f_t = tg + (1-t)f$, so $f_0 = f, f_1 = g$. Note that $\Pi_x f_t = \Pi_x f = \Pi_x g$ for all $0 \leq t \leq 1$. Since $g \in E$, $\det(g|\sigma) > 0$ for all $\sigma \in K^2$; also, $\det(f_t|\sigma)$ is a linear function of t, since f_0 and f_1 have the same x-coordinates. (Note that

$$\det(f_t|\sigma) = \det\begin{pmatrix} 1 & \pi_x f_t(a) & \pi_y f_t(a) \\ 1 & \pi_x f_t(b) & \pi_y f_t(b) \\ 1 & \pi_x f_t(c) & \pi_y f_t(c) \end{pmatrix}$$

where $\sigma = \langle a, b, c \rangle \in K^2$ has positive orientation.) Thus $\det(f_t|\sigma) > 0$ for $0 < t \leq 1$. Similarly, $f_t(v) \in C_v$ for $0 < t \leq 1$. Hence $f_t \in E(K, \mathscr{C})$ for $0 < t \leq 1$ by Lemma 4.1; $f = f_0 = \lim_{t \to 0} f_t$, and thus $f_t \in E(K, \mathscr{C})$ and $\Pi_x f_t = \Pi_x f_0$ together imply $f \in \mathrm{cl}_x E(K, \mathscr{C})$. This finishes the lemma.

Remark. Note that the lemma is true if x is replaced by y.

One must be aware of the example in §3, where we see that $R(K, \mathrm{cl}\,\mathscr{C})$ is not the same as $\mathrm{cl}\, E(K, \mathscr{C})$. To avoid this problem the restriction by means of the x-coordinates is essential.

In what follows we see that the fibres defined by the determinant conditions are convex. This (and in Lemma 3.1) is where the idea of "pushing" back and forth along "parallel tracks" is very useful. The idea of "pushing" along tracks goes back to Cairns.

LEMMA 4.3. $\mathscr{D} = \{\Pi_x{}^{-1}(f) \cap \mathrm{cl}_x E(K, \mathscr{C}) | f \in \Pi_x E(K, \mathscr{C})\}$ *is a convex disk decomposition of* $\mathrm{cl}_x E(K, \mathscr{C})$ *of dimension* $k + m$, *where* $\mathscr{C}$ *is a proper collection of bounded convex sets,* k *is the number of interior vertices of* K, *and* m *is the number of boundary vertices* v *such that* $\Pi_x | C_v$ *is not one-to-one.*

Proof. By Lemma 4.2 each element of $\mathscr{D}$ is of the form

$$D = \Pi_x{}^{-1}(f) \cap R(K, \mathrm{cl}\,\mathscr{C}) = \{g: K \to \mathbb{R}^2 | g \text{ is } SL,\ \det(g|\sigma) \geq 0$$

$$\forall \sigma \in K^2, \Pi_x g = f, g(v) \in \mathrm{cl}\, C_v\ \forall v \in (\partial K)^0\},$$

for some $f \in \Pi_x E(K, \mathscr{C})$. Since all maps in D have the same x-coordinates, the determinant conditions on each $g|\sigma$ are linear inequalities in the y-coordinate, as in Lemma 4.2; it is clear that D is closed and convex. D is bounded (and thus compact) since each C_v is bounded, and so $E(K, \mathscr{C})$ is bounded.

In order to calculate the dimension of D, let S be the affine linear subspace of $\mathbb{R}^{2n}$ (the space of SL maps) defined by

$$S = \{g\colon K \to \mathbb{R}^2 | g \text{ is } SL,\ \Pi_x g = f,\ g(v) \in \text{affine linear span of } C_v\ \forall v \in (\partial K)^0\}.$$

Clearly $D \subset S$. Since $f \in \Pi_x E(K, \mathscr{C})$, there is an $h \in E(K, \mathscr{C})$ such that $\Pi_x h = f$. By Lemma 4.1 $\det(h|\sigma) > 0$ for all $\sigma \in K^2$. Thus h is in the interior of D relative to S, since each C_v is a relatively open convex set. Thus the dimension of D is the same as the dimension of S. But clearly

$$S = \{g\colon K \to \mathbb{R}^2 | g \text{ is } SL,\ \Pi_x g = f,\ g(v) \in (\text{affine span of } C_v) \cap \Pi_x{}^{-1} f(v)\ \forall v \in (\partial K)^0\}$$

(f is regarded as a map $f\colon K \to \mathbb{R}^1$ using the x-coordinates.) The last condition in this definition of S is now independent of the projection condition, and the set it describes is 0- or 1-dimensional depending on whether $\Pi_x|C_v$ is one-to-one or not. Thus the dimension of S is just the number of vertices that are allowed to move (in each "track" $= \pi_x{}^{-1}\pi_x(p), p \in \mathbb{R}^2$), which is $k + m$.

We now show $\mathscr{D}$ is continuous. Suppose $f_i \to f$, for $f_i, f \in \Pi_x E(K, \mathscr{C})$. We need to show that $D_i = \Pi_x{}^{-1}(f_i) \cap \mathrm{cl}_x E(K, \mathscr{C}) \to D = \Pi_x{}^{-1}(f) \cap \mathrm{cl}_x E(K, \mathscr{C})$ in the Hausdorff metric (on compact subsets of Euclidean space).

First we show $\lim D_i \subset D$. Let $g_i \in D_i$, $g_i \to g$. By Lemma 4.2, $g_i \in R(K, \mathrm{cl}\,\mathscr{C})$. Since the determination function is continuous, and each $\mathrm{cl}\, C_v$ is closed ($v \in (\partial K)^0$), then

$$g \in R(K, \mathrm{cl}\,\mathscr{C}) \cap \Pi_x{}^{-1}(f) = \mathrm{cl}_x E(K, \mathscr{C}) \cap \Pi_x{}^{-1}(f) = D.$$

Next we show $\lim D_i \supset D$. Let $g \in D$. We wish to show that there are $g_i \in D_i$ such that $g_i \to g$. Since the relative interior of D is dense in D, it is enough to assume that g is in the relative interior of D, namely $g \in E(K, \mathscr{C}) \cap \Pi_x{}^{-1}(f)$ (see Lemma 4.2). Let $v \in (\partial K)^0$. Since $f_i \in \Pi_x E(K, \mathscr{C})$, there is at least one point in $\pi_x{}^{-1} f_i(v) \cap C_v$ (note that if $\pi_x|C_v$ is one-to-one there is no choice here). Let $g_i(v)$ be the *nearest* point to $g(v)$ in $\pi_x{}^{-1} f_i(v) \cap$ (affine span of C_v). (See Fig. 8.)

For i sufficiently large $g_i(v) \in C_v$, since $g(v) \in C_v$, and $g_i(v) \to g(v)$. For $v \in K - (\partial K)$, define $g_i(v) = (f_i(v), \pi_y g(v))$. Clearly $g_i(v) \to g(v)$ for such v. Thus for i sufficiently large $g_i \in E(K, \mathscr{C}) \cap \Pi_x{}^{-1}(f)$, (continuity of the determinant function), and $g_i \to g$. Hence we have shown $\lim D_i \supset D$, completing the proof of the lemma.

Let K be a triangulation of a 2-disk in $\mathbb{R}^2$, and let $|K|$ be its underlying point set. Let $h\colon |K| \to \mathbb{R}^2$ be the restriction of a projective homeomorphism of the projective plane such that $h(|K|)$ does not intersect the line at infinity. Recall ∂K is regarded as a finite set of vertices, so that ∂K is a proper collection of convex sets (for all points: $v = \{v\}$, $C_{h(v)} = \{h(v)\}$).

LEMMA 4.4. *$E(K, \partial K)$ is homeomorphic to $E(h(K), h(\partial K))$.*

Proof. For $f \in E(K, \partial K)$ define $h_*(f)\colon h(K) \to \mathbb{R}^2$ by $h_*(f)(h(v)) = h(f(v))$, for $v \in K^0$, and extend linearly on 1- and 2-simplices. $h_*(f)$ is a homeomorphism since $h_*(f)(\sigma) = h(f(\sigma))$ $\forall \sigma \in K$ (h being a projective homeomorphism). $h_*\colon E(K, \partial K) \to E(h(K), h(\partial K))$ is the required homeomorphism, with inverse $(h_*)^{-1} = (h^{-1})_*$.

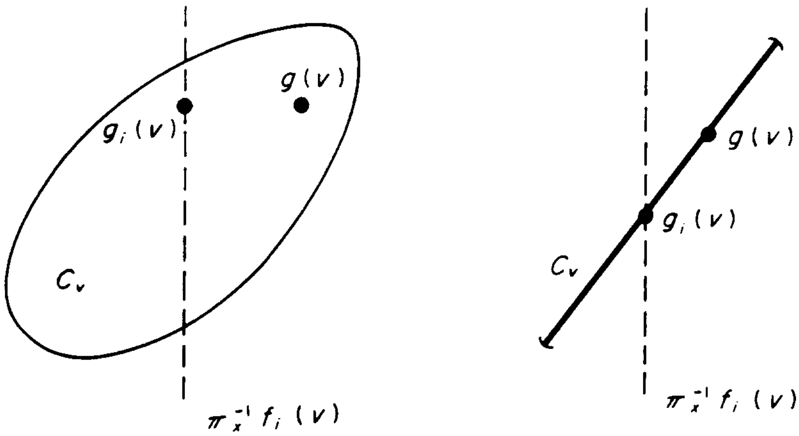

Fig. 8.

§5. THE MAIN RESULT

Let K be as in §3. Assume ∂K is embedded in $\mathbb{R}^2$ (with i: $\partial K \to \mathbb{R}^2$), and that it bounds a convex disk D such that for every vertex $v \in (\partial K)^0$, there is a line l in $\mathbb{R}^2$ such that $l \cap D = \{i(v)\}$. Such an $i(\partial K)$ is called *strictly convex*. The situation when $i(\partial K)$ is just convex will be covered in a remark at the end of this section in Corollary 5.3. As in §4 we regard $i(\partial K)$ as a proper collection of convex sets, all points ($C_v = \{i(v)\}$). Let k be the number of interior vertices of K. A *spanning* 1-*simplex* of K is a 1-simplex $T \in K - \partial K$ whose vertices are in ∂K.

THEOREM 5.1. *For K as above $E(K, \partial K)$ is homeomorphic to* $\mathbb{R}^{2k}$.

Proof. We will reduce the general case to the case where K has no spanning 1-simplices. Assume that the theorem has been proved for all K with no spanning 1-simplices, and then the general case follows by induction on the number of 2-simplices of K. If K has one 2-simplex, then it has no interior vertices, and $E(K, \partial K)$ is a single point, i.e. it is homeomorphic to $\mathbb{R}^0$. Now assume the theorem holds in the general case with less than n 2-simplices, and that K has n 2-simplices. If K has no spanning 1-simplex, then the result holds by assumption, so suppose K has a spanning 1-simplex. Then this 1-simplex divides K into subcomplexes K_1 and K_2, and clearly $E(K, \partial K)$ is homeomorphic to $E(K_1, \partial K_1) \times E(K_2, \partial K_2)$. K_1 and K_2 both have fewer than n 2-simplices, so the theorem follows immediately.

We will now prove that the theorem holds for all K with no spanning 1-simplices by induction on k. If $k = 0$, then K having no spanning 1-simplices implies that K has a single 2-simplex, and $E(K, \partial K)$ is a single point. Now assume the theorem holds whenever there are $k - 1$ interior vertices.

Let τ be a 1-simplex of ∂K. Let σ be the unique 2-simplex of K with τ as a face, and let $v \in K^0$ be the vertex of σ not in τ. Note that $v \notin \partial K$, since K has no spanning 1-simplices. Let L be the subcomplex of K obtained by removing σ and τ (one elementary collapse).

Using Lemma 4.4 we may assume that $i(\partial K)$ is in standard position with $\partial K - \tau$ as T, and τ as B, with τ lying on the x-axis. Note that the homeomorphism h in Lemma 4.4 keeps $h(i(\partial K))$ strictly convex, since it preserves lines. (To find the required h, project the line l (in Fig. 9) to the line at infinity, and then adjust by an affine linear homeomorphism to get τ on the x-axis.)

Recall the notation of §3; $\tau = \langle t_0, t_m \rangle$ where $t_0, \ldots, t_m$ are the vertices of $T \subset \partial K$ (see Fig. 10).

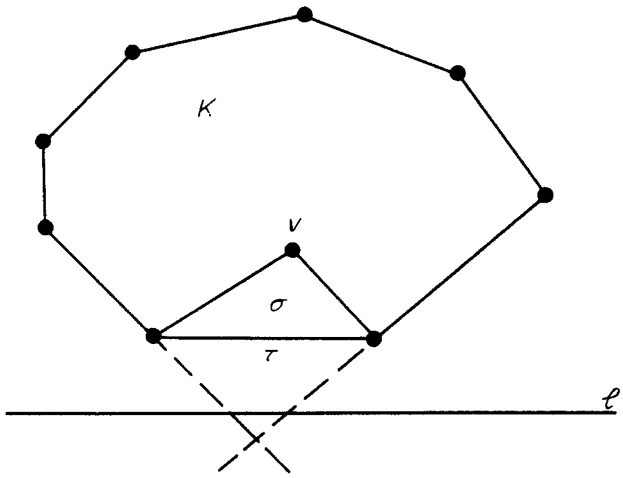

Fig. 9.

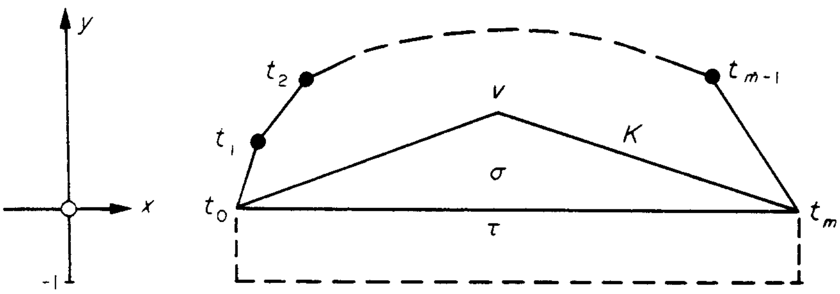

Fig. 10.

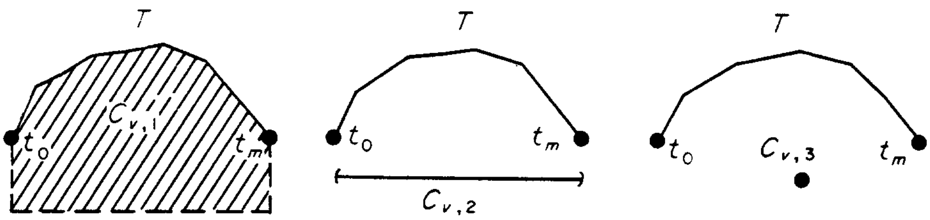

Fig. 11.

We will replace $\pi_x i$ with π_x in the following notation.

Let $\mathscr{C}_i = \{C_{v,i}\}$ be the proper collections of convex subsets for L, $i = 1, 2, 3$ defined as follows (see Fig. 11)

$C_{t_j,i} = \{t_j\}$ for $j = 1, \ldots, m$ and $i = 1, 2, 3$.
$C_{v,1} = (\pi_x t_0, \pi_x t_m) \times (-1, 0) \cup \operatorname{int}|K|$, ((,) denotes an interval in $\mathbb{R}^1$).
$C_{v,2} = (\pi_x t_0, \pi_x t_m) \times \{-1/2\}$, ((,) denotes an interval in $\mathbb{R}^1$).
$C_{v,3} = ((\pi_x t_0 + \pi_x t_m/2), -1/2)$, ((,) denotes a point in $\mathbb{R}^2$).

With abuse of notation we will write $\mathscr{C}_3$ as ∂L, as if v were originally at $C_{v,3}$.

We have the following sequence of inclusions:

$$E(K, \partial K) \subset E(L, \mathscr{C}_1) \supset E(L, \mathscr{C}_2) \supset E(L, \partial L),$$

and we intend to show the following (where $\approx$ denotes "homeomorphic to"):

$$E(K, \partial K) \overset{h_1}{\approx} E(L, \mathscr{C}_1) \overset{h_2}{\approx} E(L, \mathscr{C}_2) \times \mathbb{R}^1 \overset{h_3}{\approx} E(L, \partial L) \times \mathbb{R}^1 \times \mathbb{R}^1.$$

Since L has $k-1$ interior vertices, $E(L, \partial L) \approx \mathbb{R}^{2k-2}$ by induction, so we will be done once the three homeomorphisms are established. They are similarly obtained.

The homeomorphism h_1. Define the convex disk decompositions

$$\mathscr{D}_1 = \{\Pi_x{}^{-1}(f) \cap \mathrm{cl}_x E(L, \mathscr{C}_1) | f \in \Pi_x E(L, \mathscr{C}_1)\},$$

$$\mathscr{D}_0 = \{\Pi_x{}^{-1}(f) \cap \mathrm{cl}_x E(K, \partial K) | f \in \Pi_x E(K, \partial K)\}.$$

Be Lemma 4.3 $\mathscr{D}_1$ and $\mathscr{D}_0$ are both convex disk decomposition of dimension k. In order to apply Lemma 2.2 we need only show that every element D of $\mathscr{D}_1$ intersects $E(K, \partial K)$. Let $g \in D \in \mathscr{D}_1$, g in the relative interior of D, i.e. $g \in E(L, \mathscr{C}_1)$. By Lemma 3.1 (with $p = \pi_x g$, $B_F = \{t_0, t_m\}$), there is a $g' \in \Pi_x{}^{-1}\Pi_x(g) \cap R(L, \mathscr{C}_1)$ with $\pi_y g'(v) > 0 = \pi_y(t_0) = \pi_y(t_m)$. Identifying $R(K, \partial K)$ as a subset of $R(L, \mathscr{C}_1)$ we get $g' \in \Pi_x{}^{-1}\Pi_x(g) \cap R(K, \partial K)$. Then for $0 < t \leq 1$, and t sufficiently small, $g = (1-t)g' + tg \in \Pi_x{}^{-1}\Pi_x(g) \cap E(K, \partial K)$ by Lemma 4.2, since $\Pi_x{}^{-1}\Pi_x(g) \cap R(K, \partial K)$ is convex by Lemma 4.3.

Thus Lemma 2.2 implies that

$$\mathrm{cl}_x E(K, \partial K) \approx \mathrm{cl}_x E(L, \mathscr{C}_1).$$

Since the homeomorphism of Lemma 2.2 takes fibres to fibres, the relative interior of each fibre in one must be mapped onto the relative interior of a fibre in the other. Thus we have the homeomorphism h_1.

The homeomorphism h_2. Define

$$\mathscr{D}_2 = \{\Pi_x{}^{-1}(f) \cap \mathrm{cl}_x E(L, \mathscr{C}_2) | f \in \Pi_x E(L, \mathscr{C}_2)\},$$

which is, as before, a convex disk decomposition of $\mathrm{cl}_x E(L, \mathscr{C}_2)$, but now of dimesnion $k-1$ since $\pi_x | C_{v,2}$ is one-to-one. As before we need to show that every element D of $\mathscr{D}_1$ intersects $E(L, \mathscr{C}_2)$ as a hyperplane, and that $\mathscr{D}_2$ separates $\mathscr{D}_1$ globally. The only extra condition for $g \in E(L, \mathscr{C}_1)$ to be in $E(L, \mathscr{C}_2)$ is that $\pi_y g(v) = -1/2$, so we need only show that the intersection is non-empty. Let $g \in D \in \mathscr{D}_1$, g in the relative interior of D, i.e. $g \in E(L, \mathscr{C}_1)$. Now apply Lemma 3.1 with $p = \pi_x g$, $B_F = \{t_0, v, t_m\}$ to obtain a $g'' \in \Pi_x{}^{-1}\Pi_x(g) \cap R(L, \mathscr{C}_1)$ with $\pi_y g''(v) = -3/4$. Let $\bar{g}$ be as in the argument for homeomorphism h_1, where $\bar{g} \in \Pi_x{}^{-1}\Pi_x(g) \cap E(K, \partial K)$, so $\pi_y \bar{g}(v) > 0$. Then there is a $0 < t \leq 1$ such that

$$t\bar{g} + (1-t)g'' \in \Pi_x{}^{-1}\Pi_x(g) \cap E(L, \mathscr{C}_1),$$

(i.e. $t\pi_y \bar{g}(v) + (1-t)\pi_y g''(v) = -1/2$), using Lemma 4.2.

The condition $\pi_y g(v) = -1/2$ clearly implies that $E(L, \mathscr{C}_2)$ separates $E(L, \mathscr{C}_1)$ globally. Thus Lemma 2.3 implies that

$$\mathrm{cl}_x E(L, \mathscr{C}_1) \approx \mathrm{cl}_x E(L, \mathscr{C}_2) \times I.$$

Again, this homeomorphism restricted to fibre relative-interiors yields the homeomorphism h_2.

The homeomorphism h_3. We show that

$$E(L, \mathscr{C}_2) \approx E(L, \partial L) \times \mathbb{R}^1.$$

Here we fibre in the y-direction. Namely, define

$$\mathscr{D}_2' = \{\Pi_y^{-1}(f) \cap \mathrm{cl}_y E(L, \mathscr{C}_2) | f \in \Pi_y E(L, \mathscr{C}_2)\}$$

$$\mathscr{D}_3' = \{\Pi_y^{-1}(f) \cap \mathrm{cl}_y E(L, \partial L) | f \in \Pi_y E(L, \partial L)\};$$

$\mathscr{D}_2'$ is a convex disk decomposition of dimension k, since $\pi_y|C_{v,2}$ is not one-to-one, and $\mathscr{D}_3$ is a convex disk decomposition of dimension $k-1$. Again we need to show that every element D of $\mathscr{D}_2'$ intersects $E(L, \partial L)$ as a hyperplane, and that $\mathscr{D}_3'$ separates $\mathscr{D}_2'$ globally; as before it suffices to show that the intersection is non-empty. Let $g \in D' \in \mathscr{D}_2'$, with $g \in E(L, \mathscr{C}_2) - E(L, \partial L)$. Let x_0 be in the opposite side of $(\pi_x t_0 + \pi t_m/2)$ from $\pi_x g(v)$ in the interval $(\pi_x t_1, \pi_x t_m)$. (regard $\pi_x i$ as π_x). Apply Lemma 3.1, lastly, where the top and bottom are appropriately redefined, to find $g''' \in \Pi_y^{-1}\Pi_y(g) \cap R(L, \mathscr{C}_2)$ such that $\pi_x g'''(v) = x_0$. Then again there is a $0 \leq t \leq 1$ such that

$$tg''' + (1-t)g \in \Pi_y^{-1}\Pi_y(g) \cap E(L, \partial L),$$

i.e.

$$t\pi_x g'''(v) + (1-t)\pi_x g(v) = \frac{\pi_x t_0 + \pi_x t_m}{2}.$$

The condition

$$\pi_x g(v) = \frac{\pi_x t_0 + \pi_x t_m}{2}$$

separates $E(L, \mathscr{C}_2)$, so that

$$\mathrm{cl}_x E(L, \mathscr{C}_2) \approx \mathrm{cl}_y E(L, \partial L) \times I.$$

Restricting to fibre relative-interiors, and then taking the product with the identity map on $\mathbb{R}^1$, yields the homeomorphism h_3, thus completing the proof of the theorem.

Remark. If $i(\partial K)$ is not strictly convex, but just convex, then technically Theorem 5.1 may not be true. For instance, if $\langle v_1, v_2, v_3\rangle \in K$ and $i(v_1)$, $i(v_2)$, $i(v_3)$ are collinear in $i(\partial K)$. Then $E(K, \partial K)$ is empty. However, this is the only difficulty. For $i(\partial K)$ we say an arc $v_1, v_2, \ldots v_p \in \partial K$ ($\langle v_i, v_{i+1}\rangle \in \partial K$) is a *natural edge* if $i(v_1,), \ldots, i(v_p)$ is a maximal collinear set. (See Fig. 12.)

ADDENDUM 5.2. *If* i: $\partial K \to \mathbb{R}^2$ *is convex, with no spanning* 1-*simplices between vertices of a natural edge, then* $E(K, \partial K)$ *is homeomorphic to* $\mathbb{R}^{2k}$.

We then obtain the result mentioned in the introduction.

COROLLARY 5.3. *If* K *is a triangulation of a convex disk in* $\mathbb{R}^2$, *then* $E(K, \partial K)$ *is homeomorphic to* $\mathbb{R}^{2k}$.

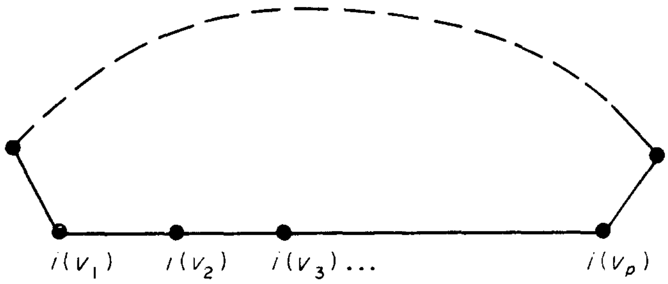

Fig. 12.

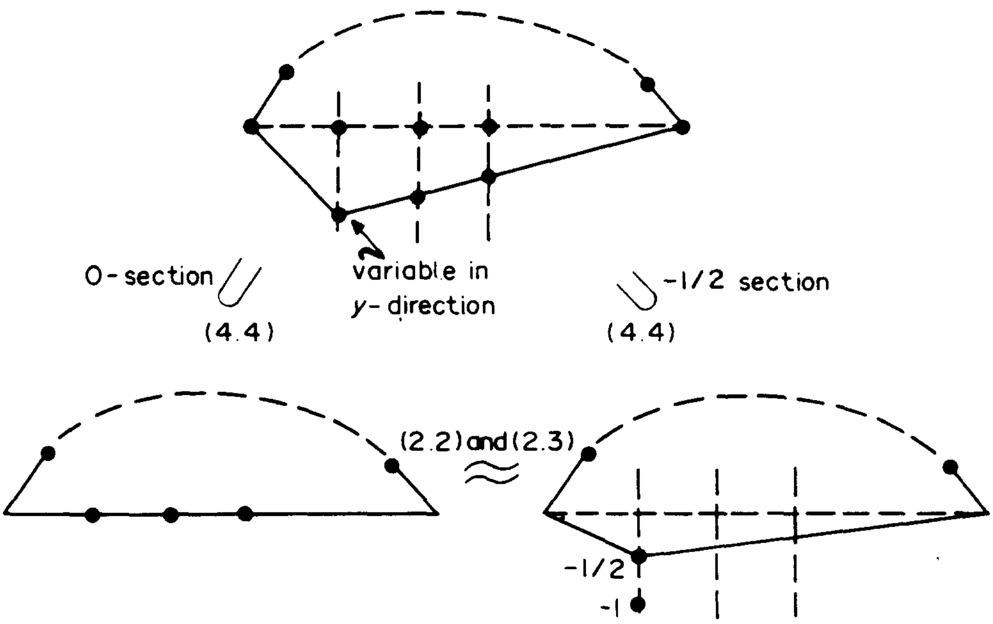

Fig. 13.

Sketch of a proof of the addendum. We reduce this situation to the case of Theorem 5.1. We first put $i(\partial K)$ in standard position by Lemma 4.4 with a natural edge as B. We then pull the vertices in the interior of the natural edge down one at a time using Lemma 3.1, and we see that the spaces so obtained are homeomorphic by the proof of Lemmas 2.2 and 2.3. By doing this to all the natural edges, we obtain a new, strictly convex $i(\partial K)$ (see Fig. 13).

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