The Stability of Tensegrity Frameworks

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ABSTRACT: For a tensegrity framework with bars, cables (unilateral tension members) and struts (unilateral compression members), this paper presents a sequence of four mathematical concepts for detecting its rigidity. In order from the strongest to the weakest, these concepts are called infinitesimal (or static) rigidity, prestress stability, second-order rigidity and rigidity. Emphasis is placed on pre-stress stability, which lies between infinitesimal rigidity and second-order rigidity.

INTRODUCTION

Tensegrity frameworks can be subtle, complicated structures with interesting geometrical and mechanical properties.

We can separate two related points of view in our study of these structures. One is purely mathematical: What are the geometric properties of a tensegrity? In particular, when is the mathematical system, inspired by the tensegrity, rigid? Here there is no concern given to the strength of materials, for instance. However, we must be careful to define what we mean by "rigid". There are many inequivalent, reasonable definitions.

The other point of view is scientific or applied. If we are handed a structure with its relevant physical data, is it "safe" for the purpose it was designed? For example, will it stand up under all reasonable loadings? Many — but not all — of the mathematical definitions of rigidity are inspired by different approaches to this applied problem. All the approaches we discuss are independent of the details of the material and the loads. Our definitions will be based on the geometry and the elementary concepts of whether a distance between points is (a) constant (a bar); (b) at its maximum (a cable); or (c) at its minimum (a strut).

Drawing on mathematical studies of the past 15 years, we survey four distinct mathematical concepts of "rigidity", with examples, statements of theorems and some motivational remarks. Figure 1 shows the basic relationship among the four concepts. (The figure numbers in the picture refer to examples of plane frameworks, in the paper, which fit into precisely the indicated box in the scheme.)

This paper focuses on the intermediate concept of 'pre-stress stability'. Roughly speaking, a framework is pre-stress stable if there are tensions and compressions in the members of the framework such that work must be done to deform the framework from its given configuration. These

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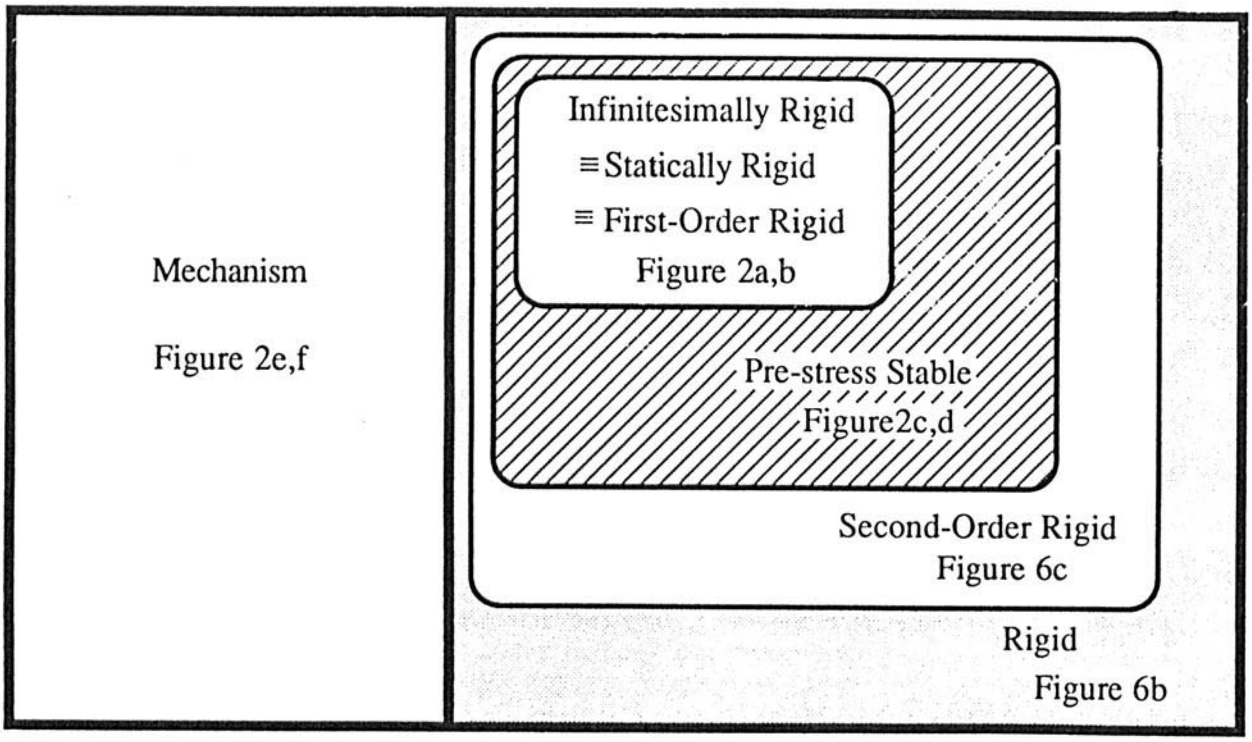


Fig. 1.

tensions and compressions are what we call a self stress for the framework, since there is a vector equilibrium at each joint of the framework. We say that this (self) stress 'stabilizes the framework'.

It is a simple matter to show that if a framework is pre-stress stable then it is mathematically rigid. It also seems reasonable that such an ideal framework can be physically constructed with the appropriate materials to supply the required stress-strain characteristics needed for the stability. We will see that many examples of "rigid" tensegrity frameworks share this geometric property.

On the other hand, we also discuss the notion of 'second-order rigidity'. If a framework is a mechanism, then the configuration of points must move in some non-trivial way. The first and second derivatives of any such motion must satisfy certain natural equations that come from differentiating the member lengths (or more conventially, the squares of their lengths). These conditions give rise to certain algebraic relations on the vectors that represent any proposed first and second derivatives at the joints. If there are only trivial solutions (solutions corresponding to rigid motions) then we say that the framework is 'second-order rigid'. (If we take only first derivatives, the algebraic conditions on the vectors lead to 'first-order' rigidity — which another term for infinitesimal rigidity, and is equivalent to static rigidity.)

It is a non-trivial mathematical result that the above kinematic definition of second-order rigidity is equivalent to saying that for each possible loading of the framework, there is a self-stress that 'stabilizes' the framework for at least this loading. However, if the framework is not pre-stress stable, different loadings will require different distributions of the tensions and compressions in the members. Alternately, if the ideal framework is build with a physical strain on the members, some motion will require no (or negative) work. Hence it seems difficult to construct such a framework and have it hold its shape. Nevertheless the framework is mathematically rigid.

1. THE DEFINITION OF A FRAMEWORK

We model a tensegrity framework as follows. The joints or vertices are just labeled points $\mathbf{p}_1, ..., \mathbf{p}_v$ in Euclidean d-dimensional space $\mathbb{R}^d$. Here $\mathbb{R}^d$ will usually be $\mathbb{R}^2$ or $\mathbb{R}^3$. For purposes of doing calculations and for simplicity we regard each $\mathbf{p}_i$ as a vector and the whole collection of joints as $\mathbf{p} = (\mathbf{p}_1, ..., \mathbf{p}_v)$ called the configuration.

The next ingredient in the definition is purely combinatorial. Each pair of labeled vertices p_i, p_j (or just $\{i,j\}$) will be called either a member or a non-member, and each member will be designated as a cable, bar, or strut. We call the collections of cables, bars, and struts the graph G of the (tensegrity) framework, and this graph together with a configuration p is denoted G(p). Note that distinct vertices may coincide, and if we regard the members as line segments connecting their vertices they may intersect in a point (or points) that are not in the configuration. We depict this information graphically as follows:

Vertices are denoted by o.
Cables are denoted by o - - - o.
Bars are denoted by o———o.
Struts are denoted by o═══o.

Figure 2 shows some examples of tensegrity frameworks in $\mathbb{R}^2$.

2. FLEXES AND THE DEFINITIONS OF RIGIDITY

To study the rigidity of a framework G(p), we consider motions of the points of the configurations p subject to constraints given by the graph G. We say a continuous path $p(t) = (p_1(t), \ldots p_v(t)), 0 < t < 1$, of configurations is a *flex* of G(p) if the following conditions hold

(a) $p(0) = p$

(b) $$|p_i(t) - p_j(t)| \begin{cases} <|p_i - p_j| & \text{if } \{i,j\} \text{ is a cable} \\ =|p_i - p_j| & \text{if } \{i,j\} \text{ is a bar} \\ >|p_i - p_j| & \text{if } \{i,j\} \text{ is a strut} \end{cases}$$

Here $|X|$ is the Euclidean length of the vector X. So condition (b) says that cables cannot increase in length, bars must stay the same length, and struts cannot decrease in length. Clearly, since no points are tied down in our definition, we can flex the framework as a rigid body. That is, any continuous combination of translations and rotations will be a flex of any G(p). If these are the only flexes, we say G(p) is *rigid*. For example, the tensegrity frameworks of Figures 2a, 2b, 2c and 2d are rigid. The frameworks of 2e and 2f are not. Often such non-rigid frameworks are called *mechanisms*. We use the simple word rigid to mean that the framework is not a mechanism.

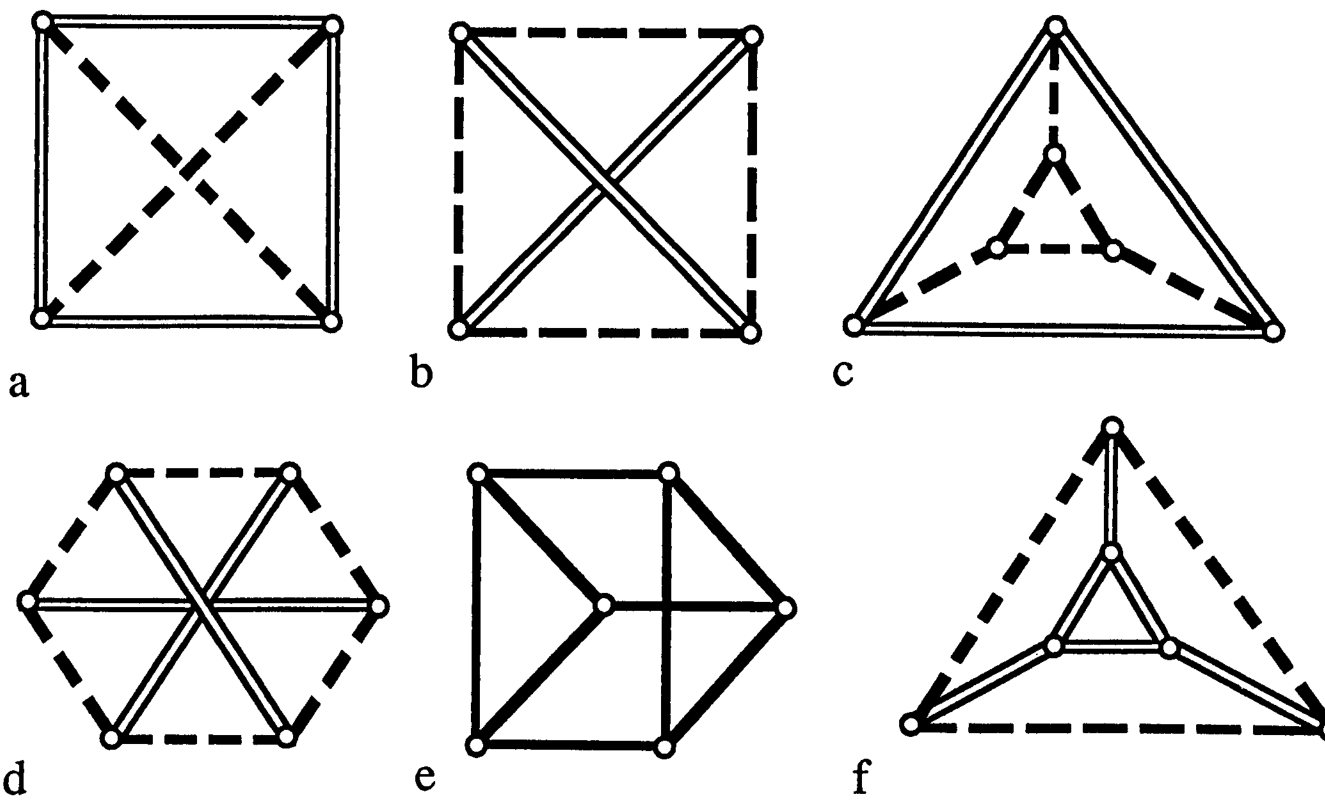


Fig. 2.

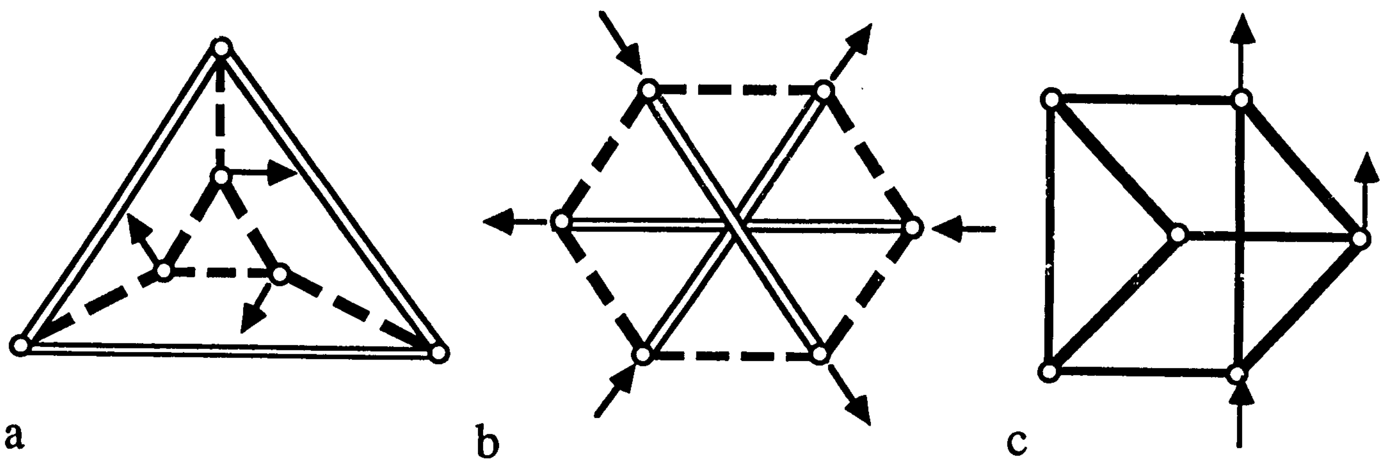


Fig. 3.

This is the weakest reasonable notion of rigidity of a framework we can imagine. Several other simple mathematical definitions turn out to be equivalent. For example, we could ask that all 'near by' frameworks with bars the same length, with cables no longer and struts no shorter, be congruent to the original ('local uniqueness'). On the other hand, we could only ask that there are no analytic flexes which are not congurences. As [5,14] show, these alternatives are equivalent to our definition of rigidity (i.e. (i) a converging sequence of non-congruent frameworks with proper member lengths implies (ii) a continuous flex of non-congruent frameworks implies (iii) an analytic flex of non-congruent frameworks).

Many structures built with dowel rods representing struts or bars, and tensed wire representing cables, may be rigid in this geometric sense, but they may feel very "loose" when built. All of the other concepts we present will imply rigidity as defined here.

3. INFINITESIMAL RIGIDITY

We next define a stronger property than rigidity. This essentially "linearizes" the rigidity defined above. If we differentiate the constraints of (b) with respect to t, and the path is differentiable, we obtain constraints on the derivatives of the vertices. With this in mind, let $\mathbf{p}' = (p_1', \ldots, p_v')$ be another configuration of vectors associated to the original configuration $\mathbf{p}$. We say $\mathbf{p}'$ is an infinitesimal flex for $G(\mathbf{p})$ if the following equations, linear in $\mathbf{p}'$, are valid:

$$\text{(c)}\quad \begin{array}{ll} (p_i-p_j)\cdot(p_i'-p_j')<0, & \text{for each } \{i,j\} \text{ a cable.} \\ (p_i-p_j)\cdot(p_i'-p_j')=0, & \text{for each } \{i,j\} \text{ a bar.} \\ (p_i-p_j)\cdot(p_i'-p_j')>0, & \text{for each } \{i,j\} \text{ a strut.} \end{array}$$

Again, since we have not pinned any of our vertices, there are always certain "trivial" infinitesimal flexes that every framework has. These are obtained by differentiating smooth trivial flexes. Explicitly in dimension 2 or 3, these trivial infinitesimal flexes are of the form:

$$p_i' = r \times p_i + t,\ i = 1, \ldots, v,$$

where r, t are fixed vectors in $\mathbb{R}^3$ and $X \times Y$ is the usual cross-product of vectors. See [1], [5] or [9] for further discussion. It is easy to check that condition (c) (for bars and therefore for cables and struts) holds for any trivial infinitesimal flex with this form.

We say a tensegrity framework is infinitesimally rigid, (or first-order rigid) if all the infinitesimal flexes are trivial. The tensegrity frameworks of Figures 2a and 2b are infinitesimally rigid, but the frameworks of Figures 2c, 2d and 2e are not. Figure 3 shows non-trivial infinitesimal flexes of those frameworks.

When an arrow p_i' is missing, it is assumed that $p_i' = 0$.

Notice that Figures 3a and 3b are rigid but not infinitesimally rigid. So our two definitions are not the same. A fundamental connection between the two concepts is:

Theorem 1. If G(p) is infinitesimally rigid, then it is rigid.

See [5] or [13] for a proof. Note that G(p) can be any tensegrity framework, not just a bar framework.

One must be careful not to confuse these two concepts. According to Grünbaum and Shephard [11] it is a quite common mistake to ignore such distinctions.

4. STRESSES AND STATIC RIGIDITY

The form of this linearization of rigidity, infinitesimal rigidity, is not the usual way most structural engineers regard rigid structures. In order to discuss their approach, we define a *stress* ω for a framework G(p), as an assignment of a scalar $\omega_{ij} = \omega_{ji}$ for each member {i,j} of G. Roughly speaking, this is the magnitude of the force, either in compression or tension, that is "in" the member connecting p_i to p_j. From our mathematical point of view, we are not directly interested in the actual strength or physical properties of this member and so our definition does not mention any physical properties of the member. In structural analysis, stress is regarded as the magnitude of the force per cross-sectional area of the member. For our analysis, we will ignore this and simply regard ω_{ij} as a scalar.

We will also combine all these scalars into one vector $\omega = (..., \omega_{ij}, ...)$. We say ω is a *proper stress* if $\omega_{ij} > 0$ for a cable and $\omega_{ij} < 0$ for a strut. Note the stress for a bar can be either positive, negative, or zero.

Define a *load* F on a framework G(p) as just a collection of vectors $F_1, ... F_v$, one for each vertex of G. We say that a stress ω *resolves* the load F if for every vertex i, the following equilibrium equation holds:

$$F_i + \sum_j \omega_{ij}(p_j - p_i) = 0$$

The sum is taken over all j adjacent to i. Unfortunately since our frameworks are not tied down, most loads cannot be resolved by any framework. We say $F = (F_1, ..., F_v)$ is an equilibrium load if

$$\sum_j F_i = 0$$

and

$$\sum_j F_i \times p_i = 0$$

for d = 3(or 2). (This condition can be generalized to higher dimensions — see [6] or [13].) The first equation essentially says that the linear momentum of the load is O, and the second equation says that the angular momentum is O. We say that G(p) is *statically rigid* if every equilibrium load can be resolved by a proper stress. Figure 4 shows two frameworks, one statically rigid, with a non-equilibrium load (a) and an equilibrium load it can resolve (b) and a framework with an equilibrium load it cannot resolve (c) and thus is not statically rigid.

Suppose G(p) has an infinitesimal flex p′. Then by adding an appropriate trivial infinitesimal flex to p′ we may assume that p′ satisfies the condition to be an equilibrium load. If this p′ is not trivial, then it is a good exercise to show that this p′ cannot be resolved by any proper self stress. Thus if G(p) is infinitesimally flexible, then it is not statically rigid. In fact, the following equivalence is true.

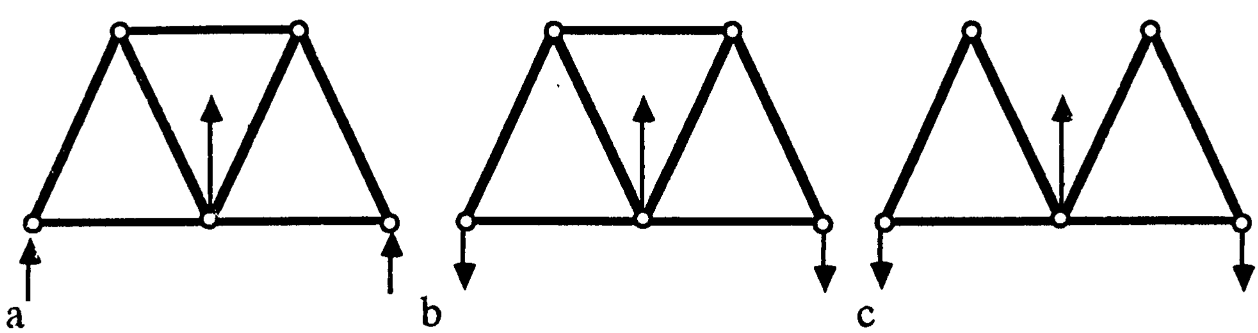
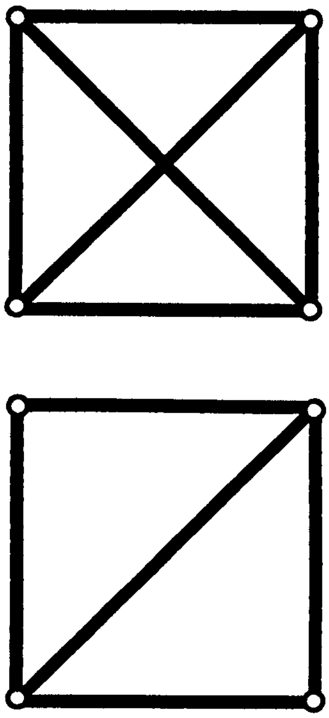


Fig. 4.

Theorem 2. A tensegrity framework is statically rigid if and only if it is infinitesimally rigid.

See [5] or [13] for proofs of this very basic duality result.

Note that our approach of dealing with equilibrium loads and trivial motions would not be necessary if we always had a reasonable subset of the vertices that were automatically pinned, that is fixed to the ground. In our formulation these pinned vertices supply all the appropriate forces so that any load applied at the other vertices will combine to create an equilibrium load. Our approach simply makes the mathematics easier and more general although occasionally it is useful to have pinned vertices as part of the description of rigidity.



a

b

Fig. 5.

5. LIMITS OF STATIC RIGIDITY

This linear approach to the rigidity of frameworks is adequate for many situations. For example, convex triangulated surfaces built of bars are statically rigid and it turns out that they can resolve any equilibrium load in only one way (see [2], [10] or [14]). There is one and only one stress that resolves any given equilibrium load. In general, we call such a framework isostatic. If one is going to build a structure and verify its ability to resolve any range of conceivable loads, one should know how the loading forces are distributed throughout the structure.

However, there are many frameworks that are rigid and even statically rigid, but not isostatic. For instance, it may be very desirable to build some redundancy into the framework. A simple example is in Figure 5a, commonly seen in many structures.

This is certainly statically rigid, but not isostatic. It is redundant in the sense that if any member is removed, it remains (statically) rigid (Figure 5b). This redundancy in a framework shows up as a proper self stress — a stress on the framework which resolves the O load (i.e. internal forces in the members which reach an internal equilibrium at each vertex). Such self stresses play a central role in the rigidity of frameworks with cables or struts.

Theorem 3. A tensegrity framework G(p) is statically rigid if and only if the underlying bar framework, with bars replacing all members, is statically rigid and there is a self stress which has scalars $\omega_{ij} > O$ for all cables $\{i,j\}$ and $\omega_{ij} < O$ for all struts $\{i,j\}$.

See [13] for a proof of this characterization. Figures 2(a) and (b) show tensegrity frameworks related to the self stresses in Figure 5(a).

So if a load is applied to a redundant framework, how does one determine how it is resolved? If one is interested only in whether or not a framework is rigid, or even statically rigid — yes or no —, the question of how the load is resolved is not so important. Otherwise the properties of the materials come into play.

6. ENERGY METHODS

Even for the yes-or-no problem there are also reasonable situations where the statics alone cannot determine rigidity. For example, Figures 2c and 2d are rigid (in our geometric sense) and it would be helpful to understand why.

A natural approach is to introduce energy or potential functions that provide "stability" for the given framework. For member $\{i,j\}$, suppose H_{ij} is a real-valued function of one real variable. Regard H_{ij} as the potential energy stored in member $\{i,j\}$ when the square of the length of a member (in some configuration) is at a given value. More precisely for a given configuration p, the total potential energy at p is

$$H(p) = \sum_{ij} H_{ij}(|p_i - p_j|^2)$$

where the sum is taken over all members $\{i,j\}$ of G. We have written this potential function as a function of the squares of the edge lengths (instead of the edge lengths themselves directly) for technical reasons to make later calculations easier. The important point is that it is a function only of the lengths of the members of G(p). In particular, if all members have the same length in another configuration, the energy is unchanged. Note that this can be a purely mathematical 'energy function' — with no claim to physical accuracy. Any such function may be used to prove the geometric property of rigidity.

For a particular framework G(p), the functions H_{ij} may be given by the structural characteristics of the members $\{i,j\}$ when the problem is the physical verification of a structure. In our yes-or-no question about the rigidity of the mathematical framework, we will create the H_{ij} that serve our purpose. In either case, for cables, bars or struts, the H_{ij} must satisfy certain natural restrictions. For $\{i,j\}$ a cable, H_{ij} must be monotone increasing near $(p_i-p_j)^2$; for $\{i,j\}$ a bar, H_{ij} must have a local minimum at $(p_i-p_j)^2$; and for $\{i,j\}$ a strut, H_{ij} must be monotone decreasing near (p_i-p_j).

If the H_{ij} satisfy the above conditions, we have the following energy principle, which implicitly is Castigliano's principle of least work:

Energy Principle. If, for all configurations q sufficiently near p, H(q) has strict minimum at p, modulo rigid motions, then G(p) is rigid.

This is clear, since if there is any configuration q, near p, where all of the member constraints are satisfied, then each $H_{ij}(|p_i-p_j|^2) > H_{ij}(|q_i-q_j|^2)$ and H(p)>H(q). Since H(p) is a strict minimum then H(q) = H(p) and p must be congruent to q. since the member constraints imply congruence, the framework is rigid.

Another interpretation of this is that as p is moved to q, the energy H increases and the gradient of the energy applies a 'force' tending to restore the framework back to p.

7. STRESSES AND ENERGY

If possible, we check this minimum by the second derivative test. Stresses play an important role in this analysis of the energy H. If H does have a local minimum at the configuration p, for each member $\{i,j\}$, let $\omega_{ij} = \frac{1}{2}H'_{ij}(|p_i-p_j|^2)$. It then turns out (see [8]) that this stress $(..., \omega_{ij}, ...) = \omega$ resolves the O load — i.e. is a self stress for G(p). Note that for a framework with only bars, this self stress coming from H is always O. Note also that p is a critical point of H (not necessarily a minimum) if and only if ω is a self stress.

A usual approach to structural analysis is to regard H'_{ij} as the fundamental function defining the characteristics of the member $\{i,j\}$. This is regarded as the usual stress-strain curve. For a given change in length Δx, the strain, there is a characteristic response $H'_{ij}[(|p_i-p_j| + \Delta x)^2]$, regarded as a "stress", or restoring force. Notice that we are not directly assuming that the stress-strain curve is linear.

8. MATRICES FROM ENERGY

Let p be any configuration for G. Regard p′ as a long dv by 1 matrix and let R(p) be that e by dv matrix, called the rigidity matrix, so that

$$R(p)p' = \begin{bmatrix} \vdots \\ (p_i-p_j)\cdot(p'_i-p'_j) \\ \vdots \end{bmatrix}$$

where e is the number of members of G. (Note that when G has all bars and p′ is an infinitesimal flex at G(p), then R(p)p′ = 0 — see [1,2] or [14]).

Let $\omega = (..., \omega_{ij}, ...)$ be any stress for G. Then define the stress matrix as that symmetric v by v matrix Ω where the (i,j)-th entry is:

$$\Omega_{ij} = \begin{cases} -\omega_{ij} = -\omega_{ji} & \text{if } i \neq j \\ \sum_{k\neq i}\omega_{ik} & \text{if } i = j \end{cases}$$

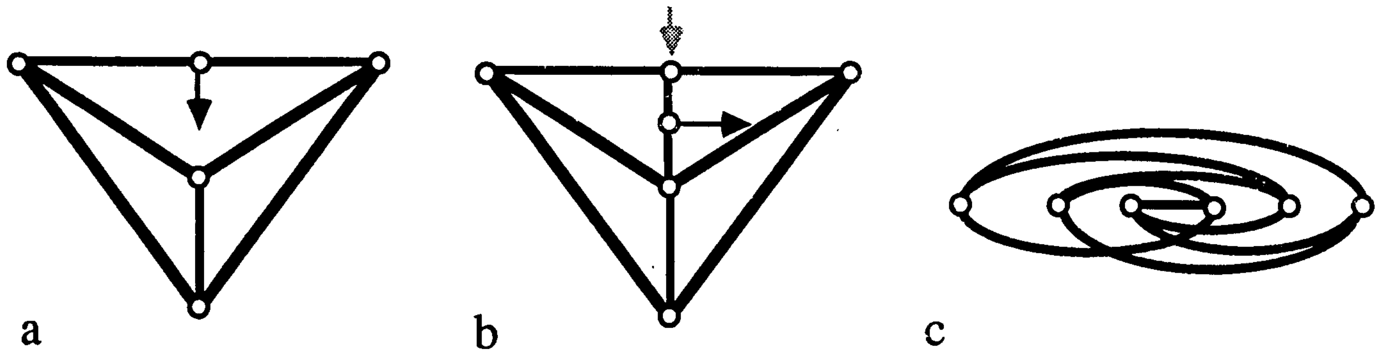


Fig. 6.

Note that the row and column sums are 0.

We also enlarge Ω, so that it can be applied to configurations, in a very simple way. Define $\tilde{\Omega}$ as that dv by dv matrix, where each entry Ω_{ij} is replaced by a d by d block $\Omega_{ij}I_d$, where I_d is the d by d identity matrix.

Now suppose we start with H(q) as defined before. To check for a local strict minimum by the second derivative test, we first have a critical point. We have already defined the self stress ω when p is a critical point for H. The second derivative test then uses the Hessian of second derivatives. To define the Hessian, let $c_{ij} = \frac{1}{4}H''(|p_i - p_j|^2)$, the second derivative, and let D be the e by e diagonal matrix where the diagonal entries are c_{ij}. Then the Hessian of H at p is the following dv by dv symmetric matrix.

$$\tilde{\Omega} + R(p)^T DR(p) = S$$

See [8] for this calculation. The matrix $R(p)^T DR(p)$ is called the *stiffness* matrix and if the $c_{ij} > 0$, it is always positive semi-definite.

We call the constants c_{ij} the stiffness coefficients and we shall assume that they are always *positive*. In the next section we apply the energy principle to prove the rigidity of some tensegrity frameworks.

9. PRE-STRESS STABILITY

We now say that G(p) is *pre-stress stable with stabilizing pre-stress* ω, if ω is proper and S is positive semi-definite with only the trivial infinitesimal flexes in its kernel. In other words $q^T Sq > 0$ for all q in $\mathbb{R}^{dv}$, and if $qSq = 0$, q is a trivial infinitesimal flex of **p**. (This is precisely the condition to apply the energy principle.)

Note that the stiffness coefficients and the self stress can be part of the given data, in which case there is nothing else to do. However, one can imagine *changing* the self stress ω in such a way that S changes from being not positive semi-definite, to being pre-stress stable, without changing the stiffness coefficients at all.

In any case, if S is the Hessian for some H and all unstressed members are deleted for any choice of a proper w and stiffness coefficients, and G(p) is pre-stress stable, then it is rigid by the energy principle. Thus when we say G(p) is pre-stress stable, we mean that there is such a stabilizing self stress. It turns out that if there is a stabilizing self stress for one set of positive stiffness coefficients, then, for any other positive stiffness coefficients, there is a stabilizing self stress as well.

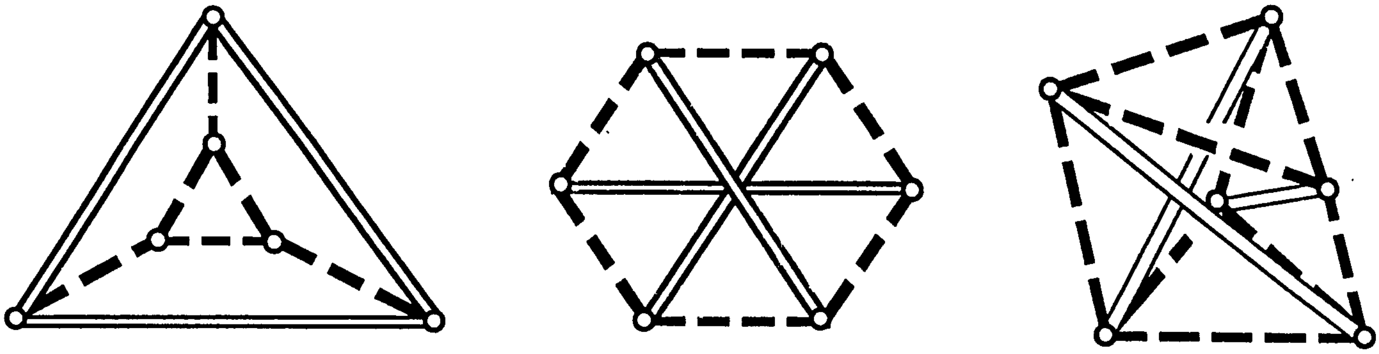
Now it is easy to show the following:

Theorem 4. If a tensegrity framework G(p) is statically rigid (or infinitesimally rigid), then G(p) is pre-stress stable.

Note that the converse certainty does not hold. For example Figures 6a and 6b are pre-stress stable (even in $\mathbb{R}^3$) but not statically rigid (even in $\mathbb{R}^2$). In fact many of the tensegrity structures of Kenneth Snelson and Buckminster Fuller (see [9] or [10]) have just these properties. See Figure 6c for a classic example in $\mathbb{R}^3$. See [4,7,8] for proofs of the pre-stress stability and related properties of these examples.

In fact, if a tensegrity framework has e members, at least one cable or strut, v vertices, and G(p) is statically rigid in $\mathbb{R}^d$ then using a simple count from linear algebra it must be true that

$$dv - \frac{d(d+1)}{2} + 1 < e$$



a b c

Fig. 7.

Many of the most popular tensegrities do not satisfy this inequality, and so they need something else to explain their rigidity, such as pre-stress stability. In Figures 6a and 6b we have $2v - 3 + 1 = 10$ and $e = 9$, while Figure 6c has $3v - 6 + 1 = 13$ and $e = 12$. See [3,13,17,18] for discussions of these counts.

10. SECOND-ORDER RIGIDITY

There is one further form of 'rigidity', between pre-stress stability and rigidity, which has been investigated in recent studies. Recall the definition of infinitesimal (or first-order rigidity).

$(p_i-p_j)\cdot(p_i'-p_j') < 0,$ for each $\{i,j\}$ a cable;
$(p_i-p_j)\cdot(p_i'-p_j') = 0,$ for each $\{i,j\}$ a bar;
$(p_i-p_j)\cdot(p_i'-p_j') > 0,$ for each $\{i,j\}$ a strut.

Motivated by taking second derivatives, we get additional conditions:

for each $\{i,j\}$ a cable, either

$(p_i-p_j)\cdot(p_i'-p_j') < 0,$ or
$(p_i-p_j)\cdot(p_i'-p_j') = 0,$ and $|(p_i'-p_j')|^2=(p_i-p_j)\cdot(p_i''-p_j'')<0;$

for each $\{i,j\}$ a bar:

$(p_i-p_j)\cdot(p_i'-p_j') = 0,$ and $|(p_i'-p_j')|^2=(p_i-p_j)\cdot(p_i''-p_j'')=0;$

for each $\{i,j\}$ a strut, either

$(p_i-p_j)\cdot(p_i'-p_j') > 0,$ or
$(p_i-p_j)\cdot(p_i'-p_j') = 0,$ and $|(p_i'-p_j')|^2=(p_i-p_j)\cdot(p_i''-p_j'')>0.$

This pair $\mathbf{p}', \mathbf{p}''$ is called a *second-order motion*. A tensegrity framework is *second-order rigid* if every second-order motion has a trivial infinitesimal flex $\mathbf{p}'$. Figure 7a shows a framework which is second order rigid, but not first-order rigid (an infinitesimal flex is shown). Figure 7b shows a framework which is rigid, but not second-order rigid (a second-order flex is shown with $\mathbf{p}''$ shaded).

The following result is true, but non-trivial for tensegrity frameworks (see [8] for a proof).

Theorem 5. If a framework G(p) is second-order rigid then it is rigid.

If $\mathbf{p}'$ is a compatible set of first derivatives ('velocities'), then $\mathbf{p}''$ is a compatible set of second derivatives ('accelerations'). Of course there may be no path with these derivatives — and the second order flex may not extend to a flex.

We state a central result for testing second-order rigidity [8], drawn from linear programming.

Theorem 6. A framework G(p) is second-order rigid if and only if for every no-trivial infinitesimal motion $\mathbf{p}'$ there is a proper self stress ω with stress matrix Ω such that
$(\mathbf{p}')^T\tilde{\Omega}\mathbf{p}'>0$

This indicates a basic connection to pre-stress stability.

Corollary 7. If a tensegrity framework G(p) is prestress rigid then it is second-order rigid.

Figure 7c shows a framework which is second-order rigid, but not pre-stress stable. (The curved lines represent collinear bars — the framework has the underlying graph of Figure 2d.) Of course it has several different non-trivial infinitesimal motions and several different self stresses to block them. In this case, no single self stress will block all infinitesimal motions. If we build an actual model of this framework with an initial 'pre-stress' inside the model, some infinitesimal motion will 'lower the energy' — and the model will be unstable. The geometry of rigidity predicts 'stability' for the model only if the rest lengths of all the members are exactly the lengths realized in the model!

However, if a tensegrity framework has either a 1-dimensional cone of infinitesimal motions, modulo congruences, or a 1-dimensional cone of self stresses, then it is second-order rigid if and only if it is pre-stress stable. Many standard examples have these particular characteristics (see [12] and [8]). For such frameworks, we can construct a 'stable' model by modifying the rest lengths to 'build in' the pre-stress. Thus a stable model of Figure 2(b) can be built by 'tightening' an exterior cable — producing the desired pre-stress.

11. MORE EXAMPLES

Even bar frameworks can be usefully shown to be rigid by these concepts. Assume we obtain first-order rigidity for a bar framework with a self stress. Each member which is non-zero in this self stress can be regarded as a cable or strut depending on the sign of the self stress. However on the bar framework, the self stress can be reversed (multiply by -1), so the arrangement of cables and struts can also be reversed (recall Figures 2a, b).

Assume that we have a bar framework stabilized by a self stress ω. Again, each member which is non-zero in this self stress can be regarded as a cable or strut depending on the sign of the self stress. In this situation, we cannot necessarily reverse the self stress (recall Figures 2c and 2f) — unless we actually have first-order rigidity, or have multiple stabilizing self stresses.

We note that there are elegant heuristics (and theorems) for self stresses in plane framework with planar graphs, drawn from work on polyhedral pictures (Whiteley [15]).

In conclusion, we cite a simple example in $\mathbb{R}^3$, without proof. Suppose we triangulate a flat triangle in $\mathbb{R}^3$ (Figure 8a). A non-trivial infinitesimal flex into 3-space is indicated. However, it turns out that there is an appropriate self stress ω for any of these frameworks, and this can be made part of a stabilizing pre-stress. Thus, not only is the framework of Figure 8a pre-stress stable in $\mathbb{R}^3$, but so is the tensegrity framework of Figure 8b.

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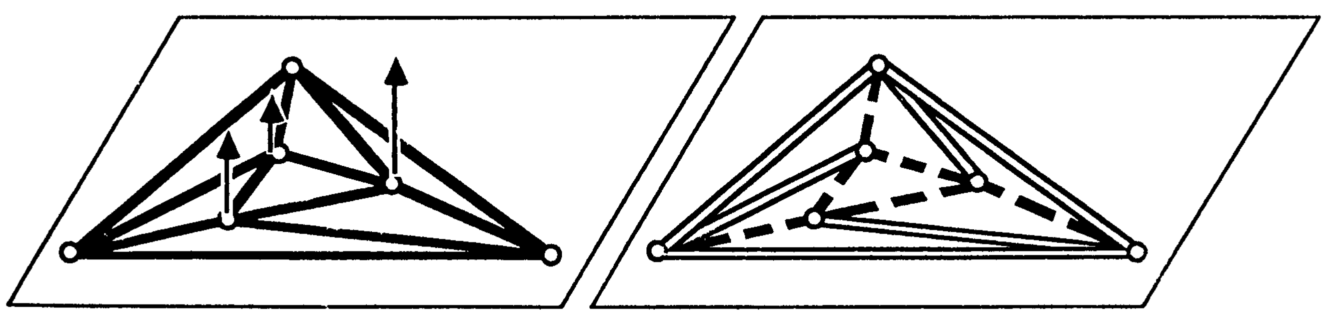

a b

Fig. 8.

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