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CONTRACTIBLE, NON-COLLAPSIBLE PRODUCTS WITH CUBES

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§1. INTRODUCTION

A COMPACT polyhedron X is said to be *q-collapsible* if $X \times I^q$ is PL collapsible, where I^q denotes the unit cube with its usual PL structure. We shall let $\mathscr{C}_n$ denote the class of compact contractible n-dimensional polyhedra.

E. C. Zeeman[18] initiated the study of q-collapsibility when he pointed out that the 3-dimensional Poincaré conjecture is true if every element $X \in \mathscr{C}_2$ which embeds in a 3-manifold is 1-collapsible. On the other hand, Cohen[2] proved that the 3-dimensional Poincaré conjecture is false if there exists an $X \in \mathscr{C}_2$ such that X embeds in a 3-manifold, but X is not 3-collapsible.

The positive collapsibility results, which have been achieved thus far, are that every element of $\mathscr{C}_2$ is 6-collapsible and every polyhedron in $\mathscr{C}_n (n \geq 3)$ is $2n$-collapsible. These results are consequences of the fact [2] that any spine of I^q is q-collapsible. Better lower bounds for q-collapsibility have been achieved in the special cases where X is constructed by attaching 2-cells to circles a and b by the words $a^p b^q$, $a^r b^s$, with $ps - qr = \pm 1$. See [5, 8, 10, 11, 14–17]. (It follows from [13] and [2] that all such complexes are 3-collapsible.)

The only non-collapsibility result heretofore has been the fact [3] that for all $n \geq 3$ there is an element $X_n \in \mathscr{C}_n$ which is not 1-collapsible. In this paper we greatly enlarge the known class of non-collapsibility phenomena by proving the following theorems.

THEOREM 1. *For every integer $m \geq 5$ there is a polyhedron $\mathscr{B}^m$, which is topologically an m-ball, such that $\mathscr{B}^m$ is not $(m-4)$-collapsible, while $\mathscr{B}^m$ is $(m+1)$-collapsible.*

The class of highly non-collapsible polyhedra for which we come closest to completely specifying the integers k for which k-collapsibility occurs is described in

THEOREM 2. *Suppose that Σ^n is a non-simply connected PL homology n-sphere ($n \geq 3$). Suppose $X = X^{n-1}$ is a spine of Σ^n-(Int B^n), for some PL ball B^n in Σ^n. Then $S^0 * X$ is not $(n-2)$-collapsible. But $S^0 * X$ is $(n+2)$-collapsible if Σ^n bounds a contractible PL manifold.*

(A *PL homology n-sphere* is a closed PL n-manifold with the homology of S^n. When $n \geq 4$ they always bound contractible manifolds, by [9].)

The above theorems will be derived from the following Main Proposition (Theorem 1 using R. D. Edwards work on non-combinatorial triangulations of S^n) and the fact that spines of I^q are q-collapsible.

MAIN PROPOSITION. *Suppose Σ^n is a non-simply connected PL homology n-sphere. Let S^p denote the standard PL p-sphere ($p \geq 0$) and let B be a PL $(n+p+1)$-ball in $(S^p * \Sigma^n) - S^p$. Let $\mathscr{B}^{n+p+1} = (S^p * \Sigma^n) - (\text{Int } B)$. Then $\mathscr{B}^{n+p+1}$ is not $(n-2)$-collapsible.*

Some remarks on stable collapsibility for arbitrarily triangulated topological balls, and a generalization of the first part of Theorem 1 by the third author, will be given in §5.

§2. PROOF OF THE MAIN PROPOSITION

Let $S^p (p \geq 0)$, $\Sigma^n (n \geq 3)$, and $\mathscr{B} = \mathscr{B}^{n+p+1}$ be as in the Main Proposition. We shall prove this proposition via two lemmas. The first lemma gives a homotopy-theoretic consequence of the geometric hypothesis that $\mathcal{B}$ is k-collapsible. The second lemma, a purely homotopy-theoretic result, shows that this consequence cannot occur if $k \leq n-2$.

LEMMA 1. *If $\mathcal{B}$ is k-collapsible, then $S^p \times \Sigma^n$ embeds in a complex $(S^p \times \Sigma^n) \cup Y^{p+k+1}$ such that there is a homotopy equivalence*

$$(S^p \times \Sigma^n) \cup Y^{p+k+1} \simeq \Sigma^n \vee S^{p+n}$$

and such that each inclusion map $i_a (a \in S^p)$ induces an isomorphism

$$i_{a*}: \pi_1(a \times \Sigma^n) \xrightarrow{\cong} \pi_1((S^p \times \Sigma^n) \cup Y^{p+k+1}).$$

Proof. By assumption $\mathcal{B} \times I^k$ is collapsible. We triangulate so that it is simplicially collapsible and so that $S^p \times I^k$ is a subcomplex. We collapse in order of decreasing dimension until all of the $(p+k+2)$-simplexes and some of the $(p+k+1)$-simplexes are gone, and then we stop before any $(p+k)$-simplexes are deleted. Hence

$$\mathcal{B} \times I^k \searrow (S^p \times I^k) \cup Y_1^{p+k+1} \tag{1}$$

for some subcomplex Y_1^{p+k+1}. We now use the fact ((3.1) of [3]) that, if $P \searrow Q \supset Q_0$, then $P \searrow$(a second derived neighborhood of Q_0) $\cup Q$. Using this, (1) implies

$$\mathcal{B} \times I^k \searrow N(S^p \times I^k) \cup Y_2^{p+k+1}. \tag{2}$$

Here, $N(S^p \times I^k)$ is the regular neighborhood of $S^p \times I^k$ in $\mathcal{B} \times I^k$ and $Y_2 = Y_1 - \text{Int}\, N(S^p \times I^k)$. Ignoring Int $N(S^p \times I^k)$, which is not affected by these collapses, we have

$$(\mathcal{B} \times I^k) - \text{Int}\, N(S^p \times I^k) \searrow \partial N(S^p \times I^k) \cup Y_2^{p+k+1}. \tag{3}$$

Let $N(S^p)$ denote the regular neighborhood of S^p in $\mathcal{B}$. Then $N(S^p \times I^k) = N(S^p) \times I^k$ (see (6.1) of [2]). But, since $\mathcal{B} = (S^p * \Sigma^n) - (\text{Int}\, B)$ (where B is a PL ball missing S^p), $N(S^p)$ is also a regular neighborhood of S^p in $S^p * \Sigma^n$. Hence $N(S^p) \cong S^p \times v\Sigma^n$, $\partial N(S^p) \cong S^p \times \Sigma^n$ and $\mathcal{B} - \text{Int}\, N(S^p) \cong (cS^p \times \Sigma^n) - (\text{Int}\, B)$ (with B a PL ball in $\text{Int}(cS^p \times \Sigma^n)$). Thus (3) may be rewritten as

$$[(cS^p \times \Sigma^n) - (\text{Int}\, B)] \times I^k \searrow (S^p \times \Sigma^n) \times I^k \cup Y_2^{p+k+1} \simeq (S^p \times \Sigma^n) \cup Y^{p+k+1}. \tag{4}$$

The last homotopy equivalence is the one which is naturally induced by the projection $\pi: S^p \times \Sigma^n \times I^k \to S^p \times \Sigma^n$, each attaching map g for a cell of Y_2 being replaced by the attaching map πg for a cell of Y.

Note that, for each $a \in S^p$, the inclusion map induces isomorphisms

$$\pi_1(a \times \Sigma^n) \to \pi_1(cS^p \times \Sigma^n) \to \pi_1(cS^p \times \Sigma^n - \text{Int}\, B).$$

So the assertion that i_{a*} is an isomorphism follows from (4).

We may, however, collapse $[(cS^p \times \Sigma^n) - (\text{Int}\, B)] \times I^k$ in another manner, without even using the hypothesis that $\mathcal{B}$ is k-collapsible. Since B is an $(n+p+1)$-ball in the interior of the PL manifold $cS^p \times \Sigma^n$, we may assume that B is a small ball meeting $\Sigma^n (= \{c\} \times \Sigma^n)$ at a single point and that $cS^p \times \Sigma^n \searrow \Sigma^n \vee B$. (Such balls certainly exist. By homogeneity of manifolds we may assume that B has these properties.) Excising the interior of B and writing $\partial B = S^{p+n}$, we get

$$[(cS^p \times \Sigma^n) - (\text{Int}\, B)] \times I^k \searrow [(cS^p \times \Sigma^n) - (\text{Int}\, B)] \searrow \Sigma^n \vee S^{p+n} \tag{5}$$

Combining (4) and (5) we get a homotopy equivalence $f: (S^p \times \Sigma^n) \cup Y^{p+k+1} \to \Sigma^n \vee S^{p+n}$. □

The Main Proposition will now follow directly from

LEMMA 2. *If, as in the conclusion of Lemma 1, $f: (S^p \times \Sigma^n) \cup Y^{p+k+1} \to \Sigma^n \vee S^{p+n}$ is a homotopy equivalence such that $f|(a \times \Sigma^n)$ induces an isomorphism on fundamental groups for all $a \in S^p$, then $k \geq n-1$.*

Remark. Under this hypothesis it can occur that $k = n - 1$. For, let $Y = cS^p \times X^{n-1}$ and consider $(S^p \times \Sigma^n) \cup (cS^p \times X^{n-1}) \subset cS^p \times \Sigma^n$, where $X = X^{n-1}$ is a spine of Σ^n − Int(PL n-ball). Let $N(X)$ be a regular neighborhood of X in Σ^n, $B_0 = \mathrm{Cl}(\Sigma^n - N(X))$, $B_1 = (1/2)(cS^p)$ (a smaller concentric cone) and $B = B_0 \times B_1$. One easily checks that

$$(S^p \times \Sigma^n) \cup (cS^p \times X) \nearrow (cS^p \times \Sigma^n) - \mathrm{Int}\, B \searrow \Sigma^n \vee S^{p+n}.$$

(In fact the second collapse was given in the proof of Lemma 1.) Define f to be the composition of this expansion and collapse.

Proof of Lemma 2. Suppose, on the contrary, that $k \leq n - 2$. Consider the lift of f to universal covering spaces

$$\begin{array}{ccc} \tilde{Z} = \widetilde{(S^p \times \Sigma^n) \cup Y^{p+k+1}} & \xrightarrow[\simeq]{\tilde{f}} & \widetilde{\Sigma^n \vee S^{p+n}} \\ \Big\downarrow q & & \Big\downarrow q' \\ Z = (S^p \times \Sigma^n) \cup Y^{p+k+1} & \xrightarrow[\simeq]{f} & \Sigma^n \vee S^{p+n} \end{array}$$

Since $k \leq n - 2$, Y^{p+k+1} has no cells of dimension greater than $(p + n - 1)$. So $H_{p+n}(\tilde{Z}) = H_{p+n}(q^{-1}(S^p \times \Sigma^n))$, where we set $Z = (S^p \times \Sigma^n) \cup Y^{p+k+1}$. However, $q^{-1}(S^p \times \Sigma^n)$ is a $(p + n)$-manifold. It is connected if $p > 0$ and has at most two components if $p = 0$. [This is because the composition $\pi_1(a \times \Sigma^n) \xrightarrow{i_*} \pi_1(S^p \times \Sigma^n) \xrightarrow{j_*} \pi_1(Z)$ is an isomorphism if i and j are inclusion maps. So $\pi_1(S^p \times \Sigma^n)$ goes onto $\pi_1(Z)$ if $p > 0$ and $\pi_1(a_i \times \Sigma^n)$ goes onto $\pi_1(Z)(i = 1, 2)$ if $S^0 = \{a_1, a_2\}$.] Thus $H_{p+n}(\tilde{Z})$ is a free abelian group of rank at most two. But $H_{p+n}(\tilde{Z}) \cong H_{p+n}(\widetilde{\Sigma^n \vee S^{p+n}})$, where the latter is a free abelian group of rank equal to the cardinality of $\pi_1\Sigma^n$, if $p \neq 0$ or $\pi_1\Sigma^n$ is infinite, and to $1 +$ [cardinality of $\pi_1\Sigma^n$] otherwise. Since $\pi_1\Sigma^n$ is a non-trivial group with trivial abelianization this rank is certainly greater than two! Having reached a contradiction, we conclude that $k \geq n - 1$.

□

§3. PROOF OF THEOREM 1

THEOREM 1. *For every integer $m \geq 5$ there is a polyhedron $\mathscr{B}^m$, which is topologically an m-ball, such that $\mathscr{B}^m$ is not $(m-4)$-collapsible, while $\mathscr{B}^m$ is $(m + 1)$-collapsible.*

Proof. Set $n = m - 2$. Let Σ^n be a non-simply connected PL homology n-sphere such that $\Sigma^n = \partial C^{n+1}$ for some PL manifold C^{n+1} with $C^{n+1} \times I \overset{PL}{\cong} I^m$. Such Σ^n are well known to exist for all $n \geq 3$. (See[12] and [9] and use the high-dimensional Poincaré conjecture.) R. D. Edwards[6] (see also[7]) has proved that $S^1 * \Sigma^n$ is a topological m-sphere. Thus, by the Generalized Schonflies Theorem[1], $\mathscr{B}^m = (S^1 * \Sigma^n) - \mathrm{Int}\, B$ is a topological ball, if B is a PL m-ball in $(S^1 * \Sigma^n) - S^1$. The Main Proposition asserts that this ball $\mathscr{B}^m$ is not $(m-4)$-collapsible.

To see that $\mathscr{B}^m$ is $(m + 1)$-collapsible we show that it is a spine of I^{m+1} and apply[2]: Since $C^{n+1} \times I$ is an $(n + 2)$-ball we may think of S^{n+1} as the double of C^{n+1}. Thus $S^{n+1} = C_1 \cup C_2$ where $C_1 \cong C_2 \cong C^{n+1}$ and $C_1 \cap C_2 = \Sigma^n$. We consider $S^1 * \Sigma^n \subset S^1 * S^{n+1}$. Define the *exterior* of one polyhedron in another to be the closure of the complement of its regular neighborhood. Let $C_i' = C_i$-(half-open collar) be the exterior of Σ^n in C_i $(i = 1, 2)$. The exterior, E_i, of $S^1 * \Sigma^n$ in $S^1 * C_i$ is PL equivalent to the ball $I^2 \times C_i'$ (Fig. 1). [To see this, note that the exterior of S^1 in $S^1 * C_i$ is PL equivalent to $cS^1 \times C_i$. Enlarging to a regular neighborhood of $S^1 * \Sigma^n$ subtracts $cS^1 \times$ (collar) from the exterior, leaving $cS^1 \times C_i'$]. Therefore the exterior of $S^1 * \Sigma^n$ in $S^1 * S^{n+1}$ is the disjoint union of 2-balls E_1 and E_2. Connect these two balls by a tube T which meets $S^1 * \Sigma^n$ in an m-ball B_0; i.e., by a regular neighborhood in $S^1 * S^{n+1}$ of an arc in S^{n+1} which is transverse to Σ^n (Fig. 2). Then $E_1 \cup T \cup E_2$ is the exterior of $\mathscr{B}^m = (S^1 * \Sigma^n) - \mathrm{Int}\, B$ where $B = B_0 \cup$collar. Clearly $E_1 \cup T \cup E_2$ is an $(m + 1)$-ball. By

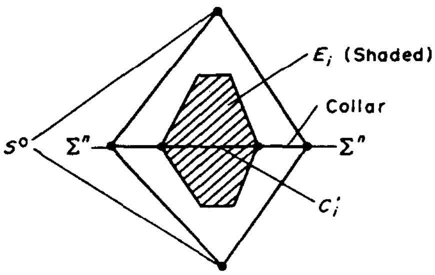

Fig. 1.

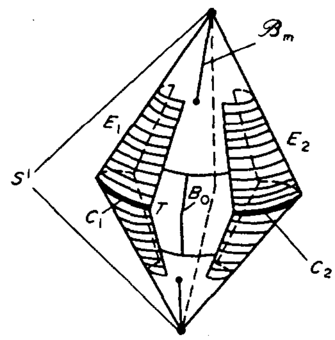

Fig. 2.

Newman's theorem its closed complement—the regular neighborhood of $\mathscr{B}^m$ in $S^1 * S^{n+1} = S^{n+3}$—is also a $PL(m+1)$-ball. Thus $\mathscr{B}^m$ is $(m+1)$-collapsible.

□

§4. PROOF OF THEOREM 2

THEOREM 2. *Suppose that Σ^n is a non-simply connected PL homology n-sphere ($n \geq 3$). Suppose $X = X^{n-1}$ is a spine of $\Sigma^n - (\mathrm{Int}\, B^n)$ for some PL ball B^n in Σ^n. Then $S^0 * X$ is not $(n-2)$-collapsible. But $S^0 * X$ is $(n+2)$-collapsible if Σ^n bounds a contractible PL $(n+1)$-manifold.*

Proof. Let $\mathscr{B}^{n+1} = (S^0 * \Sigma^n) - (\mathrm{Int}\ Q)$ where $Q = (1/2)(cS^0) \times B^n \subset S^0 * \Sigma^n$. By the Main Proposition, $\mathscr{B}^{n+1}$ is not $(n-2)$-collapsible. But one sees easily that

$$\begin{aligned}\mathscr{B}^{n+1} &= (S^0 * \Sigma^n) - \mathrm{Int}\ Q) \\ &\searrow (S^0 * \Sigma^n) - \mathrm{Int}(S^0 * B^n) \\ &\searrow S^0 * X.\end{aligned}$$

Therefore $\mathscr{B}^{n+1} \times I^{n-2} \searrow (S^0 * X) \times I^{n-2}$. Since $\mathscr{B}^{n+1}$ is not $(n-2)$-collapsible it follows that $S^0 * X$ is not $(n-2)$-collapsible.

If Σ^n bounds a contractible manifold then, by the proof of Theorem 1, $\mathscr{B}^{n+1}$ is the spine of a $PL(n+2)$-ball. Since $\mathscr{B}^{n+1} \searrow S^0 * X$, $S^0 * X$ is also the spine of a $PL(n+2)$-ball and is thus $(n+2)$-collapsible.

□

§5. GENERALIZATION TO ARBITRARY TRIANGULATED BALLS

The first part of Theorem 1 has been generalized by the third author to the

THEOREM. *Let $\mathscr{B}^m$ be any complex homeomorphic to an m-ball. Suppose there is a subcomplex $X \subset \mathrm{Int}\, \mathscr{B}^m$ such that $p = \dim X \leq m-3$ and $\pi_1(\mathscr{B}^m - X) \neq 1$. Then $\mathscr{B}^m$ is not $(m-p-3)$-collapsible.*

Remarks. (1). The subcomplex X in this theorem is "wild" in the same way that S^p was "wild" in Theorem 1: They have codimension at least three and non-simply connected complements. Notice that, by topological general position, this could not happen for any subcomplex each of whose simplexes is locally flat. But failure of some simplex of $\mathscr{B}^m$ to be locally flat does not by itself imply non-collapsibility. For example, if B^p ($p \geq 2$) is a triangulated PL ball and Σ^n is a non-simply connected PL homology sphere which bounds a contractible manifold, then $B^p * \Sigma^n \overset{PL}{\cong} v * (\partial B^p * \Sigma^n)$ is a PL collapsible topological ball for which any p-simplex in Int B^p fails to be locally flat since it has a non-simply connected link. Notice however that such a simplex does have a simply connected complement.

(2). The proof of the theorem above is similar to the proof of Theorem 1. X plays the role of S^p and the boundary of a regular neighborhood of X in $\mathscr{B}^m$ plays the role of $S^p \times \Sigma^n$. In the analogue of Lemma 1, however, we only know (from Van Kampen's Theorem) that the homotopy equivalence, f, gives a map into $S^{m-1} \vee K^{m-1}$ for some complex K^{m-1}, and when restricted to some component of the boundary this map induces a non-trivial homomorphism of fundamental groups. Fortunately, this is all that is needed in the analogue of Lemma 2 to show that the lift of f to universal covering spaces does not induce a $(\mathbf{Z}\pi_1)$ isomorphism of the top dimensional homology modules.

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