

# Global rigidity of complete bipartite graphs

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## 1. Introduction

In this note, we prove the following.

**Theorem 1.1.** *Let  $d \in \mathbb{N}$  and  $m, n \geq d + 1$ , with  $m + n \geq \binom{d+2}{2} + 1$ . Then the complete bipartite graph  $K_{m,n}$  is generically globally rigid in dimension  $d$ .*

This statement has appeared in [5, Theorem 63.2.2], but a proof hasn't yet been circulated.

## 2. Setup and background

We start by introducing the necessary concepts and definitions.

### 2.1. Rigidity

**Frameworks** A framework  $(G, p)$  a graph  $G$  with  $n$  vertices and a *configuration*  $p : V \rightarrow \mathbb{E}^d$ , mapping the vertex set  $V$  of  $G$  to a  $d$ -dimensional point set in Euclidean space.

By picking an origin arbitrarily, we identify points  $x \in \mathbb{E}^d$  with affine coordinates of the form  $\hat{x} := (\dots, 1) \in \mathbb{R}^{d+1}$ . Thus, we may identify a configuration with a vector in  $(\mathbb{R}^d)^n$  or its affine counterpart  $\hat{p} \in (\mathbb{R}^{d+1})^n$ . We can also write this as an  $n \times (d + 1)$  *configuration matrix*  $\hat{P}$ .

Fix a dimension  $d$  and a graph  $G$ . Two frameworks  $(G, p)$  and  $(G, q)$  are *equivalent* if

$$\|p(j) - p(i)\| = \|q(j) - q(i)\| \quad (\text{all edges } \{i, j\} \text{ of } G)$$

They are *congruent* if there is a Euclidean motion  $T$  of  $\mathbb{E}^d$  so that

$$q(i) = T(p(i)) \quad (\text{all verts. } i \text{ of } G)$$

A framework  $(G, p)$  is *rigid* if there is a neighborhood  $U \ni p$  so that if  $q \in U$  and  $(G, q)$  is equivalent to  $(G, p)$ , then  $q$  is congruent to  $p$ . A framework  $(G, p)$  is *globally rigid* if any  $(G, q)$  equivalent to  $(G, p)$  is congruent to it.

Rigidity [1] and global rigidity [6] are *generic properties*. A configuration is *generic* if its coordinates are algebraically independent over  $\mathbb{Q}$ .

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**Theorem 2.1** ([1, 6]). Let  $d$  be a dimension and  $G$  a graph. Then either every generic framework  $(G, p)$  in dimension  $d$  is (globally) rigid or no generic framework is (globally) rigid.

If every generic framework  $(G, p)$  in dimension  $d$  is globally rigid, we say that  $G$  is *generically globally rigid (GGR)* in dimension  $d$ .

**Infinitesimal rigidity** The *rigidity matrix*  $R(p)$  of a framework  $(G, p)$  is the matrix of the linear system

$$\langle p(j) - p(i), p'(j) - p'(i) \rangle = 0 \quad (\text{all edges } \{i, j\} \text{ of } G)$$

where the vector configuration  $p'$  is variable. The kernel of  $R(p)$  comprises the *infinitesimal flexes* of  $(G, p)$ . When  $G$  is a graph with  $n \geq d$  vertices, a  $d$ -dimensional framework  $(G, p)$  is called *infinitesimally rigid* when  $R(p)$  has rank  $dn - \binom{d+1}{2}$ . Infinitesimal rigidity implies rigidity [1].

**Generic global rigidity** The main tool we will use in this paper to prove that a graphs are GGR is the following:

**Theorem 2.2** ([6]). Let  $G$  be a graph and  $d$  a dimension. Suppose that there is a framework  $(G, p)$  that is infinitesimally rigid and globally rigid in dimension  $d$ . Then  $G$  is GGR in dimension  $d$ .

To construct frameworks that are globally rigid, we use the stronger property of universal rigidity. A framework  $(G, p)$  is *universally rigid* if any equivalent framework in any dimension is congruent. One important way to certify that a constructed framework is universally rigid is via the still stronger property of super stability. To define this we need a bit more terminology.

**Equilibrium stresses** For a graph  $G$ , define the space  $S(G)$  of *graph supported matrices* to be the symmetric  $n \times n$  matrices that have zeros in the off-diagonal entries indexed by non-edges  $\{i, j\}$ . A matrix  $\Omega \in S(G)$  is a *stress matrix* if it has the vector of all ones in its kernel. A stress matrix  $\Omega$  is an *equilibrium stress matrix* of a framework  $(G, p)$  if  $\Omega \hat{P} = 0$ . A computation shows that for each vertex  $i$

$$\sum_{j \neq i} \Omega_{ij} [p(j) - p(i)] = 0$$

if and only if a stress matrix  $\Omega$  is an equilibrium stress matrix for  $(G, p)$ . Thus, equilibrium stress matrices are obtained by re-arranging *equilibrium stresses* of  $(G, p)$ , which are vectors  $\omega$  in the cokernel of  $R(p)$ .

Suppose that  $n \geq d$ , let  $(G, p)$  be a  $d$ -dimensional framework and denote by  $m$  the number of edges,  $r$  the rank of the rigidity matrix,  $s$  the dimension of the space of equilibrium stresses and  $f$  the dimension of infinitesimal flexes. Linear algebra duality gives us the Maxwell index theorem:

$$m - r = s - f + \binom{d+1}{2} \tag{1}$$

We then see that  $(G, p)$  is infinitesimally rigid if and only if  $s = m - dn + \binom{d+1}{2}$ .

**Super stability** The *edge directions* of a framework  $(G, p)$  is the configuration  $e$  of  $|E|$  points at infinity  $e(i, j) := p(j) - p(i)$ . A framework *has its edge directions on a conic at infinity* if there is a quadric surface  $\mathcal{Q}$  at infinity containing all of  $e$ . A framework with  $d$ -dimensional affine span is *super stable* if it has a positive semidefinite (PSD) equilibrium stress matrix  $\Omega$  of rank  $n - d - 1$  and its edges directions are not on a conic at infinity.

The main connection between these concepts is due to Connelly.

**Theorem 2.3** ([3]). *If  $(G, p)$  is super stable, then it is universally rigid.*

## 2.2. Bipartite graphs and partitioned point sets

We are interested in graphs  $G$  that are simple and bipartite, with vertex partition  $U, V$  and edge set  $E$ . We denote by  $u$  and  $v$  the size of  $U$  and  $V$ , respectively, and the total number of vertices by  $n := u + v$ . The number of edges is  $m := |E|$ .

For notational convenience, we denote a configuration of the vertices of a bipartite graph by a pair of mappings  $p : U \rightarrow \mathbb{E}^d$  and  $q : V \rightarrow \mathbb{E}^d$ , and a framework on a bipartite graph by  $(G, p, q)$ . All the other definitions discussed in the previous section for point configurations extend naturally to partitioned point configurations  $(p, q)$ .

## 2.3. General position and quadric separability

We say that a point set  $p$  of at least  $d + 1$  points in dimension  $d$  is in *(affine) general position* if any  $d + 1$  of the points are affinely independent. This is equivalent to any  $d + 1$  of the vectors in  $\hat{p}$  being linearly independent.

Fix a dimension  $d \in \mathbb{N}$  and set  $D = \binom{d+2}{2} - 1$ . Let  $\mathcal{V} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{D+1}$  denote the degree 2 homogeneous *Veronese map*, which is defined by  $x \mapsto xx^\top$ . This is well-defined, since the image is a subset of symmetric  $(d + 1) \times (d + 1)$  matrices, which are naturally identified with  $\mathbb{R}^{D+1}$  by a suitable choice of coordinates. We give this  $\mathbb{R}^{D+1}$  the trace inner product

$$\langle X, Y \rangle = \text{Tr}(XY)$$

We define the action of  $\mathcal{V}$  on  $x \in \mathbb{E}^d$  by  $\mathcal{V}(x) = \hat{x}\hat{x}^\top$ . This extends to an action  $\mathcal{V}(p)$  on point configurations including partitioned point configurations. For  $x \in \mathbb{E}^d$ ,  $\mathcal{V}(x)$  is a matrix with a 1 in the bottom right corner. Thus, we can view  $\mathcal{V}$  as mapping  $\mathbb{E}^d$  to a  $D$ -dimensional affine space, which we denote by  $\mathbb{A}^D$ .

The inner product described above identifies  $(\mathbb{R}^{D+1})^*$  with quadratic polynomials on  $\mathbb{E}^d$ , since, if  $Q$  is a  $(d + 1) \times (d + 1)$  symmetric matrix,

$$\hat{x}^\top Q \hat{x} = \text{Tr}((\hat{x}\hat{x}^\top)Q) = \langle \mathcal{V}(x), Q \rangle$$

We note that the identification implies the following, which we need later:

**Lemma 2.1.** *Let  $d \in \mathbb{N}$  be a dimension. Then  $\mathcal{V}(\mathbb{E}^d)$  affinely spans  $\mathbb{A}^D$ .*

*Proof.* If  $\mathcal{V}(\mathbb{E}^d)$  has defective affine span, then there is a non-zero quadratic polynomial vanishing on all of  $\mathbb{E}^d$ , which is impossible.  $\square$

A partitioned point configuration  $(p, q)$  is *strictly quadratically separable* if  $\mathcal{V}(p)$  and  $\mathcal{V}(q)$  are strictly separable by an (affine) hyperplane in  $\mathbb{A}^D$ . This is equivalent to there being a quadric surface  $\mathcal{Q}$  in  $\mathbb{E}^d$  strictly separating the points of  $p$  from those of  $q$ .

We recall that  $\mathcal{V}(p)$  and  $\mathcal{V}(q)$  are not strictly separable by a hyperplane in  $\mathbb{A}^D$ , if and only if their convex hulls  $\text{conv}(\mathcal{V}(p))$  and  $\text{conv}(\mathcal{V}(q))$  have non-empty intersection.

### 3. Super-stable realizations of $K_{d+1,d+1}$

In this section we construct super stable realizations of  $K_{d+1,d+1}$  in dimension  $d$  with some additional properties.

**Proposition 3.1.** *For each  $d$ , there are  $(p, q)$  such that: (a)  $p$  and  $q$  both have  $d$ -dimensional affine span in  $\mathbb{E}^d$ ; (b)  $\mathcal{V}(p, q)$  has  $2d + 1$ -dimensional linear span in  $\mathbb{R}^{D+1}$ ; (c)  $(K_{d+1,d+1}, p, q)$  is super stable.*

The key rigidity theoretic tool we need is a result of Connelly and Gortler.

**Theorem 3.1** ([4]). *Let  $u \geq v \geq 1$ . If  $\text{conv}(\mathcal{V}(p))$  and  $\text{conv}(\mathcal{V}(q))$  intersect in their relative interiors, then  $(K_{u,v}, p, q)$  is super stable. If  $\text{conv}(\mathcal{V}(p))$  and  $\text{conv}(\mathcal{V}(q))$  are disjoint, then  $(K_{u,v}, p, q)$  is not universally rigid.*

The rest of this section builds the proof of Proposition 3.1 in stages.

**Not quadratically separable** Let  $\mathcal{C}$  be a curve in  $\mathbb{E}^d$ . We assume there is a parameterization  $f_{\mathcal{C}}(t)$ . Suppose that  $(p, q)$  is on  $\mathcal{C}$ , and define  $(p, q)$  to be *alternating on  $\mathcal{C}$*  if there are  $s_i$  and  $t_j$  such that  $s_1 < t_1 < s_2 < \dots$ , with  $p(i) = f_{\mathcal{C}}(s_i)$  and  $q(j) = f_{\mathcal{C}}(t_j)$ .

**Lemma 3.2.** *If  $u \geq v \geq d + 1$ , and  $(p, q)$  is alternating on a degree  $d$  curve  $\mathcal{C}$ , then  $(p, q)$  is not strictly quadratically separable.*

*Proof.* The alternating property implies that any separating quadric  $\mathcal{Q}$  would have to intersect  $\mathcal{C}$  transversely in at least  $2d + 1$  points. Since  $\mathcal{C}$  is degree  $d$ , Bezout's Theorem implies that, for any quadric  $\mathcal{Q}$ , either  $|\mathcal{C} \cap \mathcal{Q}| \leq 2d$  or  $\mathcal{Q}$  contains a component of  $\mathcal{C}$ . Either case contradicts strict separation.  $\square$

**General position** We will place  $(p, q)$  so that it alternates along the *rational normal curve*  $\mathcal{C}_d$ , which is the projective counterpart to the moment curve. For  $s, t$  real,

$$[s : t] \mapsto [t^d : t^{d-1}s : \dots : ts^{d-1} : s^d].$$

The part of  $\mathcal{C}_d$  in  $\mathbb{E}^d$  is the moment curve, obtained by setting  $s = 1$ .

The rational normal curve  $\mathcal{C}_d$  is characterized by the property that any  $d + 1$  points on it are affinely independent. We need a similar statement for the re-embedded curve  $\mathcal{V}(\mathcal{C}_d)$ , which has the parameterization

$$[s : t] \mapsto \mathcal{V}([t^d : t^{d-1}s : \dots : ts^{d-1} : s^d])$$

It is immediate that the degree of  $\mathcal{V}(\mathcal{C}_d)$  is  $2d$ . Looking more closely, we see that, in fact,  $\mathcal{V}(\mathcal{C}_d)$  is a rational normal curve in its affine span.<sup>1</sup> The  $ij$ th entry of  $\mathcal{V}([t^d : t^{d-1}s : \dots : ts^{d-1} : s^d])$  is

$$t^{d-i+1}s^{i-1}t^{d-j+1}s^{j-1} = t^{2d-i-j+2}s^{i+j-2}$$

Since every entry is determined by  $i + j$ , there are  $2d + 1$  distinct entries, and any set of these parameterizes a rational normal curve of degree  $2d$ . From this, we see that any  $2d + 1$  points on  $\mathcal{V}(\mathcal{C}_d) \cap \mathbb{A}^D$  are affinely independent.

**Lemma 3.3.** *Suppose that that  $(p, q)$  is alternating along the  $d$ -dimensional rational normal curve and each of  $p$  and  $q$  have  $d + 1$  points in  $\mathbb{E}^d$ . Then  $p$  and  $q$  are in affine general position in  $\mathbb{E}^d$  and, under the Veronese map, any  $2d + 1$  points of  $(p, q)$  have  $2d$ -dimensional affine span in  $\mathbb{A}^D$ .*

**Proof of Proposition 3.1** Parts (a) and (b) are Lemma 3.3.

For part (c), we know from Lemma 3.2 and the alternating pattern, that  $P = \text{conv}(\mathcal{V}(p))$  and  $Q = \text{conv}(\mathcal{V}(q))$  have non-empty intersection. To conclude super-stability from Theorem 3.1, we need that  $P$  and  $Q$  intersect in their relative interiors.

Suppose the contrary for a contradiction. Let  $P'$  and  $Q'$  be maximal faces of  $P'$  and  $Q'$  that meet in their relative interiors. Necessarily,  $P'$  and  $Q'$  are proper, so they span at most  $2d$  points in total. Writing any  $x \in P' \cap Q'$  as a convex combination of vertices of  $P'$  and  $Q'$ , respectively, shows that these points are affinely dependent in  $\mathbb{A}^D$ . This contradicts Lemma 3.3. Hence we conclude that  $P$  and  $Q$  intersect in their relative interiors, as desired.  $\square$

## 4. The proof

For the rest of this section let  $d, m$  and  $n$  be as in the statement. We refer to the subgraph induced by the first  $d + 1$  vertices in each part as the *core* of  $K_{m,n}$ , and denote by  $U_1$  and  $V_1$  the vertices of the core. For notational convenience, we also denote by  $p(U')$  and  $q(V')$  the sub-configurations of  $p$  and  $q$  indexed by  $U' \subseteq U$  and  $V' \subseteq V$ .

The proof strategy is to start from Proposition 3.1 to produce a configuration  $(p, q)$  such that  $(G, p, q)$  is infinitesimally rigid and globally rigid. The desired statement then follows from Theorem 2.2.

### 4.1. The Bolker-Roth stress decomposition

We will use results of Bolker and Roth [2] on the stresses of complete bipartite graphs. Denote by  $p^\vee$  the *Gale dual* (see, e.g., [8, Section 6.3]) of the vector configuration  $\hat{p}$ . The *rank* of  $p^\vee$  is equal to the dimension of the cokernel of the configuration matrix  $\hat{P}$ . We similarly define the rank of  $(\mathcal{V}(p, q))^\vee$  in terms of the images in  $\mathbb{R}^{D+1}$  of  $p$  and  $q$  under  $\mathcal{V}$ .

**Theorem 4.1** ([2]). *Suppose that  $p$  and  $q$  both have full affine span in  $\mathbb{E}^d$ . The the dimension of the space of equilibrium-stresses of  $(K_{u,v}, p, q)$  is given by  $\text{rank}(p^\vee)\text{rank}(q^\vee) + \text{rank}((\mathcal{V}(p, q))^\vee)$ .*

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<sup>1</sup>We thank Jessica Sidman for suggesting this approach.

## 4.2. Trilateration and global rigidity

A graph  $H$  is a *trilateration in dimension  $d$*  of a graph  $G$ , if  $H$  is obtained from  $G$  by adding a new vertex to  $G$  and connecting it to at least  $d + 1$  neighbors.

Global and universal rigidity are very well-behaved with respect to trilateration: it is preserved no matter where the new vertex is placed, so long as the neighbors affinely span  $\mathbb{E}^d$ . This seems to be a folklore result.

**Lemma 4.1.** *Let  $(G, p)$  be a globally (resp universally) rigid framework in dimension  $d$ . Let  $H$  be a graph obtained by trilaterating  $G$  and  $U$  the set of at least  $d + 1$  neighbors. If  $p(U)$  affinely spans  $\mathbb{R}^d$ , then for any placement  $p(v_0)$  of the new vertex  $v_0$ , the resulting framework on  $H$  is globally (resp universally) rigid.*

*Sketch.* This follows from the fact that  $K_{d+2}$  is universally rigid in dimension  $d$ , and gluing two globally (or universally) rigid frameworks along  $d + 1$  affinely independent vertices in dimension  $d$  preserves global (or universal) rigidity.  $\square$

We note that an immediate consequence is that trilaterating a generically globally rigid graph yields another generically globally rigid graph.

## 4.3. Proof of Theorem 1.1

By Lemma 4.1, it is sufficient to prove that  $K_{m,n}$  is generically globally rigid when  $m, n \geq d + 1$  and  $m + n = \binom{d+2}{2} + 1 = D + 2$ . From now on, we make this assumption.

**The construction** Now we describe the construction of  $(p, q)$ . Realize the core of  $K_{m,n}$  using Proposition 3.1. Then, at each trilateration step, place the new vertex generically.

**Infinitesimally rigid** Using  $m + n = \binom{d+2}{2} + 1$ , we have, by direct computation

$$mn - (m - d - 1)(n - d - 1) - 1 = d(m + n) - \binom{d+1}{2}$$

The Maxwell index theorem (Equation (1)) then tells us that if the dimension of the space of equilibrium stresses of  $(K_{m,n}, p, q)$  has dimension

$$(m - d - 1)(n - d - 1) + 1$$

then  $(K_{m,n}, p, q)$  is infinitesimally rigid.

The desired equilibrium stress space dimension follows from Theorem 4.1 and two observations:

- Since the core is realized in general position and the rest of the points are placed generically,  $(p, q)$  is in general position. Hence  $\text{rank}(p^\vee) \text{rank}(q^\vee) = (m - d - 1)(n - d - 1)$ .
- The core is realized (non-generically) so that  $\text{rank}((\mathcal{V}(p(U_1), q(V_1)))^\vee) = 1$ . This is because the  $2d + 2$  points of  $\mathcal{V}(p(U_1), q(V_1))$  have  $2d$ -dimensional affine span in  $\mathbb{A}^D$  by Proposition 3.1. (And so have one non-trivial affine dependency in  $\mathbb{A}^D$ .)

We add  $D + 2 - 2(d + 1)$  additional generic points in  $\mathbb{E}^d$  during the trilateration phase. The images of these under  $\mathcal{V}$  are generic in  $\mathcal{V}(\mathbb{E}^d)$ , which has  $D$ -dimensional affine span in  $\mathbb{A}^D$  (Lemma 2.1), so no new affine dependencies appear during the trilateration phase. Hence, the  $D + 2$  points of  $\mathcal{V}(p, q)$  affinely span  $\mathbb{A}^D$ , which implies that  $\text{rank}((\mathcal{V}(p, q))^\vee) = 1$ .

**Globally rigid** Proposition 3.1 implies that the core is super stable and thus universally rigid. Lemma 4.1 then implies that  $(G, p, q)$  is universally rigid and thus globally rigid.  $\square \quad \square$

## 5. Concluding remarks

The hypotheses of Theorem 1.1 are also necessary. We need  $m, n \geq d + 1$  to avoid contradicting Hendrickson's necessary conditions [7] for a graph to be GGR. The classification of equilibrium stresses in complete bipartite graphs by Bolker and Roth [2] implies that, if  $m + n < \binom{d+2}{2} + 1$  any equilibrium stress matrix of a generic framework  $(K_{m,n}, p, q)$  has all zeros on its diagonal and is thus necessarily of deficient rank.

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