Understanding The Co-Existence of Twin Birds Inside and Outside a Cage

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Abstract

This project aims to explain and replicate some of the mathematics and simulations behind the journal article, "Twin birds inside and outside the cage" by Jafari *et al.* This article explores the dynamics of the chaotic system of equations described below, which has the unique property of having two coexisting strange attractors, one entirely inside, and one entirely outside a sphere of radius 4.

$$\begin{aligned} \dot{\rho} &= 4\rho\varphi - 16\varphi \\ \dot{\theta} &= \rho^2 + 3\varphi - 8\rho + 15 \\ \dot{\varphi} &= -a\theta - \varphi \end{aligned} \tag{1}$$

1 Background

Before launching into an analysis of the system of equations, I want to briefly talk about context. By giving the reader context, I hope to explain why I chose a system of equations that, although it may have engineering applications, was not created to model any specific phenomenon. First I summarize the context given in the article. In recent years, there has been a trend of examining chaotic systems with unusual features. In particular, chaotic systems with hidden attractors. Jafari *et al.* describe a hidden attractor as, "An attractor that has a basin of attraction not intersecting with any small neighborhood of any equilibrium point." What makes the hidden attractors that emerges out of their system of equations unique is their co-existence on the interior and exterior of an impassable sphere. To the best of the authors' knowledge, this is the only such system. Moving on the how this article fits into the context of what we have learned in our Non-linear Dynamics module, I will elliptically mention the topics and techniques that the article and our module have in common. Both explore a chaotic system, and our exploration of the strange attractor in the Lorenz Equations can be seen as an analogue to the hidden attractors in the article. In addition, the methods used to analyze the nature of the equilibrium points in the article are similar to those that we use to determine the nature of fixed points. Finally, I think it worth mentioning that this system of equations appeals to me aesthetically.

2 Analysis

We start our analysis by trying to find the fixed points of the system of equations. We do this by setting $\dot{\rho}$, $\dot{\theta}$ and $\dot{\varphi}$ equal to zero, and then finding which values of ρ, θ, φ and a satisfy these equations.

$$\begin{aligned} 4\rho\varphi - 16\varphi &= 0 \\ \rho^2 + 3\varphi - 8\rho + 15 &= 0 \\ -a\theta - \varphi &= 0 \end{aligned} \tag{2}$$

From the first line of (2) we get the condition that either $\rho = 4$ or $\varphi = 0$ in order for it to be zero. Substituting $\varphi = 0$ into the second line, we obtain $\rho^2 - 8\rho + 15 = 0$. Solving the quadratic, we get that either $\rho = 5$ or $\rho = 3$. Substituting $\varphi = 0$ into the third line, we obtain $-a\theta = 0$. Therefore, θ must also equal zero. Thus we have determined out first two fixed points, $(\rho^*, \theta^*, \varphi^*) = (3, 0, 0)$ and $(\rho^*, \theta^*, \varphi^*) = (5, 0, 0)$. To find the final fixed point, we substitute $\rho = 4$ into our second line, which gives us $\varphi = 1/3$. Substituting this result into our final line, we obtain $\theta = -1/3a$. Thus our third fixed point is $(\rho^*, \theta^*, \varphi^*) = (4, -1/3a, 1/3)$, where $a \neq 0$.

In order to determine the stability of the fixed points we determine the Jacobian of our system of equations.

$$J(\rho^*, \theta^*, \varphi^*) = \begin{pmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \varphi} \\ \frac{\partial g}{\partial \rho} & \frac{\partial g}{\partial \theta} & \frac{\partial g}{\partial \varphi} \\ \frac{\partial h}{\partial \rho} & \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} 4\varphi^* & 0 & 4\rho^* - 16 \\ 2\rho^* - 8 & 0 & 3 \\ 0 & -a & -1 \end{pmatrix}$$

We can now determine the characteristic equation of our Jacobian

$$\begin{vmatrix} 4\varphi^* - \lambda & 0 & 4\rho^* - 16 \\ 2\rho^* - 8 & -\lambda & 3 \\ 0 & -a & -1 - \lambda \end{vmatrix}$$

$$\begin{aligned} &= a(3(4\varphi^* - \lambda) - (4\rho^* - 16)(2\rho^* - 8)) + (-1 - \lambda)(4\varphi^* - \lambda)(-\lambda) \\ &= \lambda^3 + (1 - 4\varphi^*) + \lambda(3a - 4\varphi^*) + 4a[2(\rho^* - 4)^2 - 3\varphi^*] \end{aligned} \tag{3}$$

We now substitute our fixed points into the characteristic equation:

$(\rho^*, \theta^*, \varphi^*)$	Characteristic Equation
$(3, 0, 0)$	$\lambda^3 + \lambda^2 + 3a\lambda + 4a$
$(5, 0, 0)$	$\lambda^3 + \lambda^2 + 3a\lambda + 4a$
$(4, -1/3a, 1/3)$	$\lambda^3 - \frac{\lambda^2}{3} + (3a - \frac{4}{3})\lambda - 4a$

Unfortunately, our module primarily deals with the stability of two-dimensional systems, so my expertise in stability analysis of three-dimensional systems is limited. However, the article refers to Routh-Hurwitz stability criterion as a means of determining the stability. In order for a third-order polynomial of the form $P(s) = s^3 + a_2 s^2 + a_1 s + a_0$ to be stable, it has to satisfy the conditions:

$$a_2, a_0 > 0, a_2 a_1 > a_0 \tag{4}$$

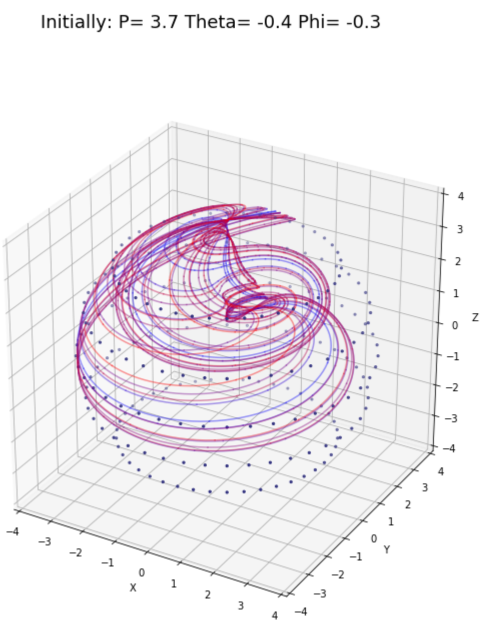
For the corresponding characteristic equations of $(3, 0, 0)$ and $(5, 0, 0)$ we have $a_2 = 1 > 0$ and $a_0 = 4a > 0$ only if $a > 0$. However, $a_2 a_1 = 3a > a_0 = 4a$ only if $a < 0$. Since a cannot simultaneously be greater and less than zero, the equations cannot satisfy the criterion and the fixed points $(3, 0, 0)$ and $(5, 0, 0)$ cannot be stable.

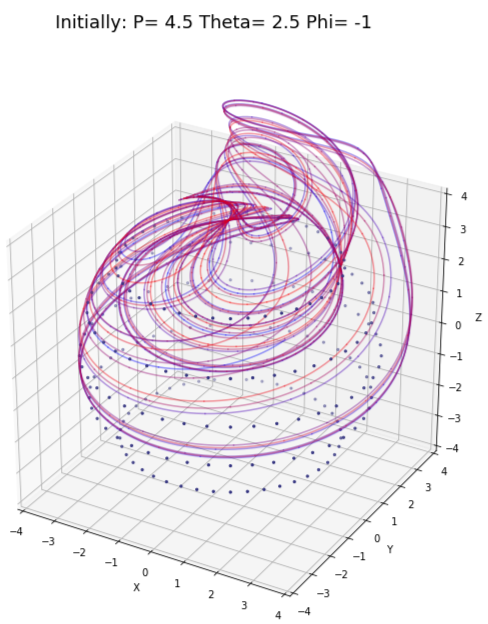
The characteristic equation of $(4, -1/3a, 1/3)$ similarly fails the criterion since $a_2 = -1/3 < 0$.

3 Simulation

In order to try and reproduce the figures presented in the article, I used a python-based programming environment to run simulations based on the system of equations and initial conditions given. The general method here involved breaking the system of differential equations into a system of update equations and then making the time step between each update significantly minute. One of the challenges that I faced was having to convert the equations from spherical co-ordinates to Cartesian co-ordinates. Matplotlib does not easily work with spherical co-ordinates, thus the easiest remedy to the problem was to write a function that converts ndarrays of the spherical co-ordinates into ndarrays of their corresponding Cartesian co-ordinates.

To give the simulations a sense of the passage of time, as well as to give them a more striking visual appearance, I varied the colour of the path as time progresses, starting out as blue, and ending off as red. Finally, in order to give the simulations a context in terms of their relationship to the impassable sphere, centred at the origin, with radius four, I plotted a minimalist 'sphere' composed of five rings of dots. The reason for this minimalist aesthetic is to make the sphere's presence apparent, but unobtrusive. Being able to visualize the full path of the simulation was prioritized. Below are two of the simulations I ran, both setting the value of $a = 1$. The first exhibits the bird in the cage and the second, the bird outside. Then on the following page, I have plotted the projections of the simulations onto the X-Y, X-Z and Y-Z planes. Figures 1-3 belong to the first simulation and 4-6 to the second.





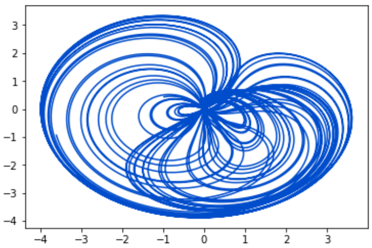


Figure 1: X-Y Plane

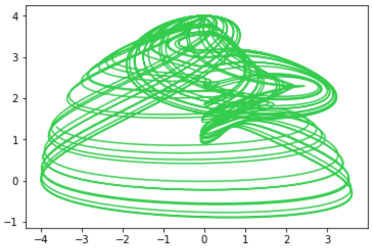


Figure 2: X-Z Plane

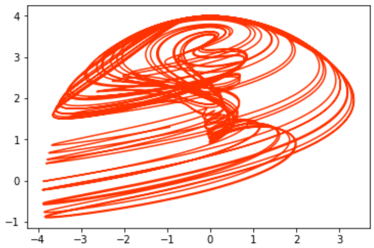


Figure 3: Y-Z Plane

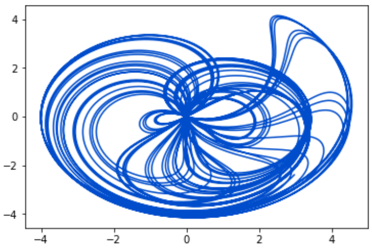


Figure 4: X-Y Plane

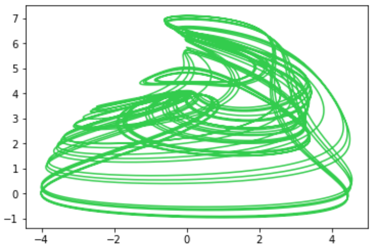


Figure 5: X-Z Plane

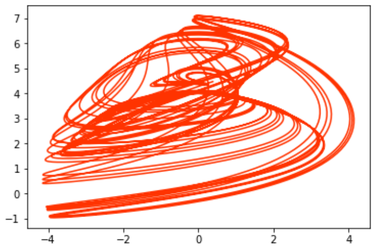


Figure 6: Y-Z Plane

4 Conclusion

This project has sought to unpack the intriguing article, "Twin birds inside and outside the cage," and has taken a title that hopefully conveys some of that intrigue. Unpacking the mathematics of the article involved finding the fixed points of the system, and then analyzing the behaviour at them by using the Routh-Hurwitz stability criterion. This project has then gone on to reveal the hidden attractors visually by running simulations of the system. Given how easily these equations lend themselves to the visual medium, I have put more effort than I care to admit in tweaking colour parameters in the simulations. Hopefully the audience will be receptive to the resulting graphs. Moving forward, I would like to be able to understand how field-programmable gate arrays work, and how they can be used simulate chaotic systems such as this one. The authors allude to potential engineering applications of the system of equations. Given more time, I would be very eager to explore these potential applications, and not just in engineering. Perhaps there are still unthought of ways that the system of equations could be applied to real world contexts.

5 Epilogue

I must admit that upon undertaking this project, I underestimated the amount of work that would be involved in finding a recent article that uses mathematics accessible to a second year. This is not to say that I sought to avoid all models that made use of more advance mathematical models than we are accustomed to. Rather that in order to try and apply as much of what we have learned in nonlinear dynamics as possible, I sought an article that largely used mathematics within the framework of our course. I hope that the article I finally settled on reflects a balance between using concepts familiar and foreign.