MATH 240: Discrete Structures

Anna Brandenberger, Binyuan Sun

May 11, 2018

This is a transcript of the lectures given by Prof. Ben Seamone during the winter semester of the 2017-2018 academic year (01-04 2018) for the Discrete Structures class (MATH 240). **Subjects covered** are: Mathematical foundations of logical thinking and reasoning; Mathematical language and proof techniques; Quantifiers; Induction; Elementary number theory; Modular arithmetic; Recurrence relations and asymptotics; Combinatorial enumeration; Functions and relations; Partially ordered sets and lattices; Introduction to graphs, digraphs and rooted trees.

1 Logic and Fundamentals

1.1 Propositional Logic

Basic Terms

Definition 1.1. A **proposition** or statement is something which can be verified as being either TRUE or FALSE.

Examples of propositions and non propositions:

Example	
$x = 5$	YES
$x + y$	NO
"This statement is false."	NO

Definition 1.2. A **sentence** is a formula consisting of

- ATOMS: simple propositions
- CONNECTORS: operations joining atoms

Sometimes parentheses are used to denote order of operations (if necessary.)

Definition 1.3. The three **connectors** are:

- AND: "P and Q" = $P \wedge Q$ (both are true)
- OR: "P or Q" = $P \vee Q$ (one or the other is true)
- NOT: "not P" = $\neg P$ (takes on the opposite truth value of P)

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Table 1: Basic Truth Table

Order of operations:

1. Parentheses
2. $\neg$
3. $\vee$ and $\wedge$
4. etc.

Definition 1.4. A sentence is a:

1. **Tautology** if it is always true.
2. **Contradiction** if it is always false.
3. **Contingency** otherwise.

Example 1.1. Determine if each sentence is a tautology, contradiction or contingency.

1. $P \wedge \neg P$: Contradiction (c.f. Table 2).
2. $\neg(P \wedge Q) \vee (\neg P \vee Q)$: Tautology (c.f. Table 3).

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Table 2: Table illustrating a Contradiction.

P	Q	$\neg P$	$P \wedge Q$	$\neg P \vee Q$	Final
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	F	T	T
F	F	T	F	T	T

Table 3: Table illustrating a Tautology.

CAN WE be more efficient in our logic? An **identity** is a statement that two sentences have the same truth values for the same truth values of their atoms.

SOME simple ones:[1]

[1] Use $\mathbb{T}$ for a statement that is always true, $\mathbb{F}$ for a statement that is always false.

- Identity: $\begin{cases} P \wedge \mathbb{T} \equiv P \\ P \vee \mathbb{F} \equiv P \end{cases}$
- Idempotent: $\begin{cases} P \wedge P \equiv P \\ P \vee P \equiv P \end{cases}$
- Complement: $\begin{cases} P \wedge \neg P \equiv \mathbb{F} \\ P \vee \neg P \equiv \mathbb{T} \end{cases}$
- Double negation: $\neg(\neg P) \equiv P$
- DeMorgan's Laws: $\begin{cases} \neg(\neg P \vee \neg Q) \equiv P \wedge Q \\ \neg(\neg P \wedge \neg Q) \equiv P \vee Q \end{cases}$
- Associative: $\begin{cases} P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R \\ P \vee (Q \vee R) \equiv (P \vee Q) \vee R \end{cases}$
- Distributive: $\begin{cases} P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) \end{cases}$

Conditional Statements

Definition 1.5. "P implies Q" written $P \implies Q$.

Equivalent Phrasings to "P implies Q":

- "If P, then Q".
- "Q if P".
- "P only if Q".
- "Q whenever P".
- "Whenever Q, then also P."
- "P is sufficient for Q."
- "Q is necessary for P."

Example 1.2. P = "the sun is shining", Q = "teeth are purple". The statement given is $P \implies Q$. To show this is false, you must show P is true and Q false.

$$\begin{aligned} \neg(P \implies Q) &\equiv P \wedge \neg Q \\ (P \implies Q) &\equiv \neg(P \wedge \neg Q) && \text{double negation} \\ &\equiv \neg P \vee \neg(\neg Q) && \text{DeMorgan's Law} \\ &\equiv \neg P \vee Q \end{aligned}$$

P	Q	$\neg P \vee Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 4: Table illustrating Example 1.2.

Examples 1.3. using identities:

1. Verify $P \Rightarrow (Q \vee \neg Q)$ is a tautology.

$$\begin{aligned} P \Rightarrow (Q \vee \neg Q) &\equiv P \Rightarrow \mathbb{T} && \text{("complement")} \\ &\equiv \neg P \vee \mathbb{T} && \text{("implies")} \\ &\equiv \mathbb{T} && \text{(identity)} \end{aligned}$$

2. Determine if the following sentence is a tautology, contradiction, or contingency.

$$\begin{aligned} [\neg P \vee (\neg P \wedge Q)] \wedge P &\equiv [\neg P \wedge P] \vee [(\neg P \wedge Q) \wedge P] \\ &\equiv \mathbb{F} \vee [\neg P \wedge P \wedge Q] \\ &\equiv \mathbb{F} \vee [\mathbb{F} \wedge Q] \\ &\equiv \mathbb{F} \vee \mathbb{F} \equiv \mathbb{F} \end{aligned}$$

Definition 1.6. The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

Proof. Show that $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$.

$$\begin{aligned} P \Rightarrow Q &\equiv \neg P \vee Q \\ &\equiv Q \vee \neg P \\ &\equiv \neg(\neg Q) \vee \neg P \\ &\equiv \neg Q \Rightarrow \neg P \end{aligned}$$

□

S = this shape is a square
R = this shape is a rectangle
$S \Rightarrow R$ means if the shape is a square then it is a rectangle. $\neg R \Rightarrow \neg S$ means if the shape is not a rectangle then it is not a square.

The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$. In general $P \Rightarrow Q$ & its converse need not be related.

Biconditional Statements

Definition 1.7. $P \Leftrightarrow Q \equiv$ "P and Q are equivalent"

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 5: Table illustrating biconditional statements.

Two new identities:

- $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- $P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$

Updated order of operations:

1. Parentheses
2. $\neg$
3. $\vee$ and $\wedge$
4. $\implies$ and $\Leftrightarrow$

Ambiguous statements:

1. $A \wedge B \vee C$
   $(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$
   $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

2. $P \Rightarrow Q \Rightarrow R$
   Check that:
   $(P \Rightarrow Q) \Rightarrow R \not\equiv P \Rightarrow (Q \Rightarrow R)$
   $\neg(\neg P \vee Q) \vee R \not\equiv \neg P \vee (\neg Q \vee R)$

PHRASING of $\Leftrightarrow$:

$$\begin{aligned} P \Leftrightarrow Q &\equiv \text{P if and only if Q} \\ &\equiv \text{P if Q, and conversely} \\ &\equiv \text{P precisely when Q} \\ &\equiv \text{P is necessary and sufficient for Q} \end{aligned}$$

NECC: $Q \Rightarrow P$, SUFF: $P \Rightarrow Q$

A note: $P \equiv Q$ and $P \Leftrightarrow Q$ essentially mean the same thing.

Symbolization

Examples 1.4.

1. If $\boxed{\text{you study}}$ then, then $\boxed{\text{you will pass}}$ [2]: $S \Rightarrow P$
2. The $\boxed{\text{ground gets wet}}$ whenever $\boxed{\text{it rains}}$ [3]: $R \Rightarrow W$
3. The $\boxed{\text{sun is out}}$ but $\boxed{\text{it is raining}}$ [4]: $S \wedge R$
4. You cannot $\boxed{\text{scuba dive}}$ unless you $\boxed{\text{have had lessons}}$ [5]:
   $S \Rightarrow L$ or $\neg L \Rightarrow \neg S$
5. If $\boxed{\text{you only study under pressure}}$, then $\boxed{\text{you do not learn}}$ [6]:
   $(P \Leftrightarrow S) \Rightarrow \neg L$

[2] S: you study
P: you pass

[3] W: the ground gets wet
R: it rains

[4] W: the sun is out
R: it is raining

[5] W: can scuba dive
R: had lessons

[6] S: you study
P: you are under pressure
L: you learn

1.2 *Predicate logic*

Predicate logic = 1st order logic

Definition 1.8. We introduce two quantifiers:

1. $\forall$ = "for all" = universal quantifier
2. $\exists$ = "there exists" = "existential quantifier"

Justification:
Consider a statement like: "Every positive integer is the sum of two primes".
To construct a truth table, we would need infinitely many columns (since this statement contains infinitely many atoms). So this can't be done in propositional logic.

Predicates

Definition 1.9. A **predicate** is a sentence containing variables that is either true or false.

Examples 1.5. of symbolization:

1. Every even integer n greater than 2 can be expressed as the sum of two primes.

$$\begin{aligned} Z(n) &= \text{n is a positive integer} \\ P(n) &= \text{n is the sum of 2 primes} \end{aligned}$$

$\forall n(Z(n) \Rightarrow P(n))$

2. "Not everything $\boxed{\text{is about you}}$"[7]: $\neg(\forall x Y(x))$

3. "When $\boxed{\text{it rains}}$, everything $\boxed{\text{gets wet}}$[8]": $R \Rightarrow (\forall x W(x))$

4. "Every person who $\boxed{\text{wants wine}}$ gets $\boxed{\text{asked for ID}}$ by the bartender"[9]: $\forall x(W(x) \Rightarrow I(x,b))$

[7] $Y(x)$ = "x is about you"

[8] R: "it rains"
$W(x)$: "x gets wet"

[9] $W(x)$ = x wants wine;
$I(x,y)$ = y asks x for ID;
b = the bartender

We can use equality to symbolize:

Example 1.6. "Sue is the only person who knows how this works":
$K(x)$: x knows how this works
$\forall x(K(x) \Rightarrow x = \text{Sue})$

When using multiple quantifiers, order matters (read L to R):

Example 1.7. $F(x,y)$ = "x and y are friends":
$\forall x \exists y F(x,y) \rightarrow$ every person has a friend
$\exists x \forall y F(x,y) \rightarrow$ someone is friends with everyone

Functions, or variables which depend on other variables are allowed.

Example 1.8. "Matt brings a friend":
$B(x,y)$: x brings y
$f(x)$: a friend of x
m : Matt

Updated Order of Operations

1. Parentheses
2. $\neg$
3. $\forall, \exists$ (L to R)
4. $\wedge, \vee$
5. $\Rightarrow, \Leftrightarrow$

Theorem 1.1. ***Negation of predicates:***

- $\neg(\forall x P(x)) \equiv \exists x \neg P(x)$
- $\neg(\exists x P(x)) \equiv \forall x \neg P(x)$

Examples 1.9.

- "Not everything is about you".[10]
  claim: Everything is about you: $\forall x Y(x)$
  negation: There is a thing that is not about you: $\neg(\forall x Y(x)) \equiv \exists x \neg Y(x)$

- "All cardinals are red".[11]
  claim: $\forall x(C(x) \Rightarrow R(x))$
  negation: $\exists x \neg(C(x) \Rightarrow R(x))$
  $\equiv \exists x \neg(\neg C(x) \vee R(x))$
  $\equiv \exists x(C(x) \wedge \neg R(x))$

- "There is an even number greater than three which is prime".[12]
  claim: $\exists x(E(x) \wedge G(x) \wedge P(x))$
  negation: $\forall x \neg((E(x) \wedge G(x) \wedge P(x))$
  $\equiv \forall x[\neg E(x) \vee \neg G(x) \vee \neg P(x)]$

[10] $Y(x)$: x is about you

[11] $C(x)$: x is a cardinal
$R(x)$: x is red

[12] $E(x)$: x is even
$G(x)$: x is greater than 3
$P(x)$: x is prime

A SENTENCE OF the form $\forall x(P(x) \Rightarrow Q(x))$ is called vacuously true if $P(x)$ is $\mathbb{F}$ $\forall x$. This is because $P \Rightarrow Q$ is true whenever P is false. The rule holds because its never given a chance to fail.

Example 1.10. Every $\boxed{\text{McGill prof}}$ with $\boxed{\text{200 years of experience}}$ is a $\boxed{\text{great golfer}}$.[13]

[13] $M(x)$: McGill prof
$O(x)$: 200 years of experience
$G(x)$: great golfer

Proposition 1.2. ***To disprove a claim*** *$\forall x(P(x) \Rightarrow Q(x))$, we show $\exists x$ for which $P(x) \wedge \neg Q(x)$ is true. This x is called a* ***counterexample*** *to the claim.*

Arguments

Definition 1.10. Rule of inference

An **argument** is a finite collection of statements $A_1, A_2, \ldots, A_n$ (called predicate, or hypothesis) followed by statement B called the conclusion. *An argument is valid* if B is true whenever all $A_1, A_2, \ldots, A_n$ are true.

Proposition 1.3. ***To validate an argument****:*

1. *Check a truth table (is B true when $A_1, A_2, \ldots, A_n$ are all true?)*
2. *Show $A_1 \wedge A_2 \wedge \ldots, A_n \implies B$ is a tautology*

Some Rules of Inference
(more posted on MyCourses)

1. Modus ponens
   Arguments: $p \implies q$
   Conclusion: $\frac{p}{q}$
2. Modus tollens
   $P \implies Q$
   $\frac{\neg Q}{\neg P}$

1.3 *Set Definitions and Operations*

Definition 1.11. A set is a collection of distinct objects, called elements or members.

NOTATION

$a \in S$ means a is an element of the set S
$a \notin S$ means a is not an element of the set S

BASIC SETS

$\mathbb{N}$: the natural numbers
$\mathbb{Z}$: the integers
$\mathbb{R}$: the real numbers
$\emptyset$: the empty set

WRITING DOWN SETS:

- list the elements, in any order inside $\{\}$
- if the elements follow a pattern, we sometimes use "..." to denote that the elements continues

Proposition 1.4. *Set builder notation:*
$\{element \mid rule\ the\ element\ obeys\ to\ be\ in\ the\ set\}$

Examples 1.12.
1. $\mathbb{Q}$ the rational numbers
   $\{\frac{m}{n} | m \in \mathbb{Z}, n \neq 0\}$
2. $\mathbb{C}$ the complex numbers
   $\{a + b | a \in R, b \in R, i^2 = -1\}$

Theorem 1.5. *Two sets are equal if and only if they either contain the same elements or are both empty.*

Examples 1.13.

1. $\{x \in \mathbb{R} | x^2 - 3 = 0\} = \{\sqrt{3}, -\sqrt{3}\}$
2. $\{x \in \mathbb{N} | x^2 - 3 = 0\} = \emptyset$
3. $\{n \in \mathbb{Z} | n^2 + 1 = 0\} = \{x | x > 0 \land x < 0\}$
4. $Q : \{\emptyset\} \stackrel{?}{=} \emptyset$ no! LHS contains an element $\emptyset$

Definition 1.12. We write $A \subseteq B$ and say A is a subset of B if every $x \in A$ also obeys $x \in B$.

Example 1.14. $\emptyset \subset X$ for every set X
$\mathbb{N} \subseteq \mathbb{R} \subseteq \mathbb{C}$
$\{1,2\} \subseteq \{1,2,\pi\}$

Proposition 1.6. *If $A \subseteq B$ & $A \neq B$, we can write as above and say A is a proper subset of B.*

NOTATION $\{1,2\} \not\subseteq \{1,2\}$
$\{1,2\} \subset \{1,2,\pi\}$

Examples 1.15.
$\{a,b\} \in \{a,b,c\}$? F
$\{a,b\} \subseteq \{a,b,c\}$? T
$\{a,b\} \in \{a,b,\{a,b\}\}$? T
$\{a,b\} \subseteq \{a,b,\{a,b\}\}$? T

Proposition 1.7. $A = B \Leftrightarrow A \subseteq B$ *and* $B \subseteq A$

Definition 1.13. $\mathcal{U}$ = the universal set = the set containing everything (that we are interested in)

Venn diagrams

- UNION: $A \cup B = \{x \mid x \in A \lor x \in B\}$
- INTERSECTION: $A \cap B = \{x \mid x \in A \land x \in B\}$
- DIFFERENCE: $A \setminus B = \{x \mid x \in A \land x \notin B\}$
- COMPLEMENT: $\overline{A} = \{x \mid x \notin A\} = U \setminus A$
- SYMMETRIC DIFFERENCE: $A \oplus B = \{x \mid (x \in A \lor x \in B) \land x \notin A \cap B\}$
  $= \{x \mid x \in A \cup B \land x \notin A \cap B\}$

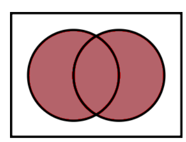


Figure 1: $A \cup B$

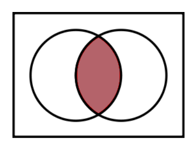


Figure 2: $A \cap B$

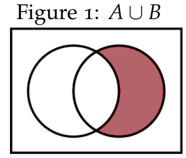


Figure 3: $B \setminus A$

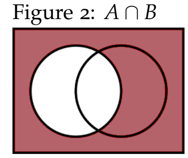


Figure 4: $\overline{A}$

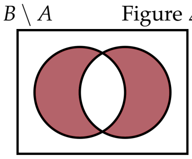


Figure 5: $A \oplus B$

Definition 1.14. Power set of A: $\mathcal{P}(A) = \{B \mid B \subseteq A\}$

Example 1.16. $\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

Proof Methods

Most statements to prove are of the form $P \Rightarrow Q$ or $P \Leftrightarrow Q$.

Proposition 1.8. ***Proof methods*** *for $P \Rightarrow Q$:*

1. ***Direct proof***: *Assume P is true. Show Q is true, usually with some application of the transitive rule of inference.*
2. ***Contrapositive proof***: *Prove the contrapositive ($\neg Q \Rightarrow \neg P$).*

Examples 1.17. $P \Rightarrow R_1 \Rightarrow R_2 \Rightarrow Q$

Set Identities (to be posted)

- COMPLEMENT LAW: $A \cup \overline{A} = U, \quad A \cap \overline{A} = \emptyset$
- DEMORGAN: $\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$
- DISTRIBUTIVE: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example 1.18. Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Recall: $X = Y \Leftrightarrow X \subseteq Y$ and $Y \subseteq X$. [14]

[14] We must show each of the sets in the statement is contained in the other as a subset.

Proof.

1. Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:
   Let $x \in A \cap (B \cup C)$

$$\begin{aligned}
&\Rightarrow x \in A \;\&\; x \in (B \cup C) \\
&\Rightarrow x \in A \;\&\; (x \in B \mid\mid x \in C) \\
&\Rightarrow (x \in A \;\&\; x \in B) \text{ or } (x \in A \& x \in C) \\
&\Rightarrow x \in A \cap B \mid\mid x \in A \cap C \\
&\Rightarrow x \in (A \cap B) \cup (A \cap C)
\end{aligned}$$

2. Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$:
   Let $y \in (A \cap B) \cup (A \cap C)$
   Either $y \in A \cap B$ or $y \in A \cap C \Rightarrow y \in A$ and either $y \in B$ or $y \in C$

$$\begin{aligned}
&\Rightarrow y \in (B \cup C) \\
&(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \\
&A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
\end{aligned}$$

$\square$

Definition 1.15. Another operation on sets is the **Cartesian product** of A and B denoted $A \times B = \{(a,b) \mid a \in A, b \in B\}$

Proposition 1.9. *Two elements in $A \times B$ are equal precisely when they are in both coordinates: $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_2 = b_1$*

Example 1.19. $A = \{1,2\}, B = \{3,4\}$
$A \times B = \{(1,3),(2,3),(1,4),(2,4)\}$
$B \times A = \{(3,1),(3,2),(4,1),(4,2)\}$

Example 1.20. Prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Proof. $\Rightarrow x \in A$ and $y \in B \cup C$
If $y \in C$, then $(x,y) \in A \times C$; if $y \in B$, then $(x,y) \in A \times B$
So either $(x,y) \in A \times B$ or $(x,y) \in A \times C$
$\Rightarrow (x,y) \in (A \times B) \cup (A \times C)$
$\Rightarrow A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ $\square$

1.4 Relations and Equivalence Relations

Definition 1.16. A **binary relation** $\mathcal{R}$ from A to B is a subset of $A \times B$.

Using the example above, $R = \{(1,3),(1,4)\}$ is a binary relation from A to B (one of many).

Proposition 1.10. *A binary relation on A is a subset of A^2.*

Definition 1.17. **Useful properties**: Let R be a binary relation on A.

R is	Definition
reflexive	$\forall a \in A, (a,a) \in R$
symmetric	$(a,b) \in R \Rightarrow (b,a) \in R$
antisymmetric	$(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$
transitive	$(a,b),(b,c) \in R \Rightarrow (a,c) \in R$

Example 1.21.
$\{(x,y) | y = x^2\}$ is a bin. rel. on $\mathbb{R}$.
$\{(x,y) | x = y^2\}$ is a bin. rel. on $\mathbb{R}$.
$\{(m,n) | \frac{m}{n} \in \mathbb{Z}\}$ is a bin rel on $\mathbb{Z}$.

Example 1.22. Determine if the relation has any of the 4 properties defined above: $R = \{(x,y) \in \mathbb{R}^2 | x \leq y\}$

– **Reflexive**: $\forall x \in \mathbb{R}, x \leq x \Rightarrow (x,x) \in R$

– **Symmetric**: $(1,2) \in R$ and $(2,1) \notin R$

– **Antisymmetric**: If $(x,y) \in R$ and $(y,x) \in R$, then $x \leq y$ and $y \leq x \Rightarrow x = y$

– **Transitive**: If $(x,y),(y,z) \in R$, then $x \leq y \cap y \leq z \Rightarrow (x,z) \in R$ $\square$

Definition 1.18. A binary relation on A is a **partial order** on A is it is reflexive, antisymmetric, and transitive

Example 1.23. $R = \{(a,b) \in \mathbb{Z}^2 \mid a - b \text{ is even}\}$

Note: n is even $\Leftrightarrow n = 2k$ for some $k \in \mathbb{Z}$

– **Reflexive**: $\forall x \in \mathbb{Z}, a - a = 0$ is even $\Rightarrow (a,a) \in R$

– **Symmetric**: $(a, b) \in R \Rightarrow a - b$ even $\Rightarrow b - a$ even $\Rightarrow (b, a) \in R$

– **Antisymmetric**: $(16, 8) \in R$ & $(8, 16) \in R$ but $16 \neq 8$

– **Transitive**: If $(a, b) \in R$ and $(b, c) \in R$, then $\begin{cases} a - b = 2k \\ b - c = 2l \end{cases}$

$\Rightarrow a - c = 2k + 2l = 2(k + l)$ is even.

Definition 1.19. A binary relation on A is an **equivalence relation** on A if it is reflexive, symmetric, and transitive.

Definition 1.20. A **total order on** A is a partial order R where $\forall a, b \in A$, either (a, b) or (b, a) is in R. (Linear order.)

Remark 1. Not every partial order is a total order.

Example 1.24. Let A be some set $R = \{(x, y) | x, y \in P(A), x \subseteq y\}$
Why is this not a total order?
$A = \{1, 2, 3\}, \quad M = \{1\}, \quad N = \{2, 3\}$
$(x, y) \notin R \because M \not\subseteq N \wedge N \not\subseteq M$
Show it is a partial order.

– **Reflexive**: $\forall x \subseteq A$, is $(x, x) \in R$?

– **Antisymmetric**: $(x, y) \in R \wedge (y, x) \in R$
$\Rightarrow X \subseteq Y \wedge Y \subseteq X$
$\Rightarrow X = Y$

– **Transitive**: $(X, Y) \in R, (Y, Z) \in R$
$\Rightarrow X \subseteq Y, Y \subseteq Z$
$\Rightarrow X \subseteq Z$
$\Rightarrow (X, Z) \in R$

Equivalence Relations

Recall: Definition 1.19: an **equivalence relation** is a relation R which is:

- Reflexive
- Symmetric
- Transitive

Definition 1.21. We say a is related to b in a equivalent relation R if $(a, b) \in R$, denoted as follows:

Notation 1.11. a is related to b in a equivalent relation R:

$$a\mathcal{R}b \qquad a \sim_{\mathcal{R}} b \qquad a \sim b \text{ (if } \mathcal{R} \text{ is understood)}$$

Definition 1.22. If R is an equivalence relation, the **equivalence class** of an element $a \in R$ is denoted $[a]$ or $\bar{a}$, is $[a] = \{x | x \sim a\}$

Example 1.25. 1. Show $R = \{(x, y) \in \mathbb{Z}^2 | x - y = 3k \text{ for some } k \in \mathbb{Z}\}$ is an equivalence class.

2. What are $[0], [1], [2], [3]$?

1. R: $x - x = 0 \Rightarrow (x, x) \in R$
S: $x - y = 3k \Rightarrow y - x = 3k \Rightarrow (x, y) \in R \Rightarrow (y, x) \in R$
I: $(x, y) \in R, \quad (y, z) \in R$
$\Rightarrow x - y = 3k, y - z = 3l \Rightarrow x - z = 3(k + l) \in \mathbb{Z} \Rightarrow (x, z) \in R$

2. $[0] = \{x | x \sim 0\} = \{x | x - 0 = 3k\} = \{x | x = 3k\} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$
$[1] = \{x | x \sim 1\} = \{x | x - 1 = 3k\} = \{x | x = 3k + 1, k \in \mathbb{Z}\}$
$[2] = \{x | x \sim 2\} = \{x | x - 2 = 3k\} = \{x | x = 3k + 2, k \in \mathbb{Z}\}$
$[3] = \{x | x \sim 2\} = \{x | x - 3 = 3k\} = \{x | x = 3k + 3, k \in \mathbb{Z}\} = \{x | x = 3(k + 1), k \in \mathbb{Z}\} = \{x | x = 3l, l \in \mathbb{Z}\} = [0]$

Theorem 1.12. *Let $a \in A$, R an equivalence relation to A. For any $x \in A$, $[x] = [a] \Leftrightarrow x \sim a$.*

To prove $P \Leftrightarrow Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$

Proof. $(\Leftarrow)$ Suppose $[x] = [a] \because x \in [x] \Rightarrow x \in [a] \Rightarrow x \sim a$

$(\Rightarrow)$ Suppose $x \sim a$, We will show $\underbrace{[x] \subseteq [a]}_{\text{(a)}}$ and $\underbrace{[a] \subseteq [x]}_{\text{(b)}}$

(a) Let $y \in [x] \Rightarrow y \sim x$.
Since $x \sim a$, by transitivity, $y \sim a \Rightarrow y \in [a] \Rightarrow [x] \subseteq [a]$

(b) $z \in [a]$
$\Rightarrow z \sim a$
$\Rightarrow z \sim x$ (since $x \sim a$)
$\Rightarrow z \in [x]$
$\Rightarrow [a] \subseteq [x]$

Together, this gives $[a] = [x]$ □

Theorem 1.13. *Let $a, b \in A$, R an equivalence relation of A.*
$[a] \neq [b] \Leftrightarrow [a] \cap [b] = \emptyset$

Proof. $(\Rightarrow)$ If $[a] \cap [b] = \emptyset$ then since $a \in [a]$, $a \notin [b] \Rightarrow [a] \neq [b]$.

$(\Leftarrow)$ We will prove the contrapositive: show $[a] \cap [b] \neq \emptyset \Rightarrow [a] = [b]$.
$[a] \cap [b] \neq \emptyset \Rightarrow \exists x, x \in [a] \cap x \in [b] \Rightarrow x \sim a \cap x \sim b \Rightarrow [x] = [a] \cap [x] = [b] \Rightarrow [a] = [b]$

□

Recap: $P \Rightarrow Q$:Direct proof, Contrapositive (direct proof of $\neg Q \Rightarrow \neg P$)
$P \Leftrightarrow Q : P \Rightarrow Q \wedge Q \Rightarrow P, \quad P \Rightarrow Q \wedge \neg P \Rightarrow \neg Q, \quad [\neg Q \Rightarrow \neg P \wedge \neg P \Rightarrow \neg Q]$

1.5 *Proof by Contradiction*

PRINCIPLE: If your assumption leads, by logical steps, to something false (a contradiction), then the assumption must have been wrong and thus the opposite statement is right.
Following are some proofs using this method:

Theorem 1.14. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$

$$\Rightarrow \sqrt{(2)} = \frac{m}{n}, \quad m, n \in \mathbb{Q}$$

We may assume, without loss of generality (WLOG), that $\frac{m}{n}$ is in reduced form, i.e they share no common factors.

$$\begin{aligned}
&\sqrt{(2)} = \frac{m}{n} \Rightarrow 2 = \frac{m^2}{n^2} \\
&\Rightarrow m^2 = 2n^2 \\
&\Rightarrow m \text{ is even} \quad \Rightarrow \quad m = 2k \text{ for some } k \in \mathbb{Z} \\
&\Rightarrow (2k)^2 = 2n^2 \quad \Rightarrow \quad 4k^2 = 2n^2 \quad \Rightarrow \quad 2k^2 = n^2 \\
&\Rightarrow n \text{ is even}
\end{aligned}$$

$\Longrightarrow$ m and n have a common factor of 2. $\Rightarrow\Leftarrow$
Thus our assumption $\sqrt{2} \in \mathbb{Q}$ was wrong. □

Theorem 1.15. ***Fundamental Theorem of Arithmetic**. Every $n \in N$ can be written as a product of primes that is unique up to order.*

Proof. given later (Section 2). □

Theorem 1.16. *There are infinitely many primes.*

Proof. Suppose there are finitely may primes $p_1, p_2, \ldots, p_n$

Let $q = p_1 p_2 \ldots p_n + 1$. Theorem 1.15 (Fundamental Theorem of Algebra) states that q is divisible by some prime: WLOG, $p_1 \Rightarrow q = p_1 k$ for some $k \in \mathbb{Z}$.

$$\begin{aligned}
q - p_1 p_2 \ldots p_n &= 1 \\
p_1 k - p_1 p_2 \ldots p_n &= 1 \\
\underbrace{p_1}_{\geq 2} \underbrace{(k - p_2 \ldots p_n)}_{\in \mathbb{N}} &= 1 \qquad \Rightarrow\Leftarrow
\end{aligned}$$

Thus our assumption that there are finitely many primes was wrong. □

Proposition 1.17. ***Proving Existential Statements** $(\exists x P(x))$*

1. *Give the example.*
2. *When you can't give the example, a "non constructive proof" can sometimes be found.*

Example 1.26. 1. "There exists an even prime". It's 2.

2. "There is a real solution to $x^2 - x - 1 = 0$". It's $\frac{1 \pm \sqrt{5}}{2}$.

Example 1.27. "There are irrational numbers x, y such that $x^y \in \mathbb{Q}$.

Proof. We know $\sqrt{2} \notin \mathbb{Q}$.

Consider $\sqrt{2}^{\sqrt{2}}$.

If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$, then $x = \sqrt{2}, y = \sqrt{2}$ is our example.

If $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$, then consider $\sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$: $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} * \sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$.

In either case, we ave an example that proves such x, y exist. $\square$

2 Number Theory

The study of properties of integers.

2.1 Division

Theorem 2.1. ***Division Algorithm***
If $a, b \in \mathbb{Z}$, $b \neq 0$, then $\exists q, r \in \mathbb{Z}$ such that:
$a = qb + r$, with $0 \leq r < |b|$.
Furthermore, q and r are unique. t

The proof of this algorithm uses the following as main tool:

Lemma 2.2. ***Well Ordering Principle***. *If $A \subseteq \mathbb{N}$, then A has a smallest element.*

Proof. of the Division Algorithm (Theorem 2.1)
Consider the case where $a, b > 0$. Look at the following multiples of $b : 0, b, 2b, 3b, 4b, \cdots$. There is some multiple of $b > a$. Let $B = \{kb | k \in \mathbb{N}, kb > a\}$

Example 2.1. $(2a)b = (2b)a \geq 2a > a$

By the *Well Ordering Principle (Lemma 2.2)*, B has a smallest element. Call it $(q+1)b$.
$qb \leq a < (q+1)b$
Let $r = a - qb$
Then
$a = qb + r$
And:
$0 \leq a - qb < (q+1)b - qb$
$0 \leq r < b$
WHAT ABOUT UNIQUENESS?
Assume $\exists q_1, q_2, r_1, r_2$ such that $a = q_1 b + r_1$, $a = q_2 b + r_2$. We will show $q_1 = q_2$ and $r_1 = r_2$, and so distinct solutions are impossible.
$0 = (q_1 - q_2)b + (r_1 - r_2)$, $(q_1 - q_2)b = r_2 - r_1$.
But

$$\begin{aligned} 0 &\leq r_1 < b \\ -b &< -r_2 \leq 0 \\ \Rightarrow -b &< r_1 - r_2 < b \end{aligned}$$

Since $r_1 - r_2$ is a multiple of b, we have $r_1 - r_2 = (0)(b) \Rightarrow r_1 = r_2$ and $(q_1 = q_2)b = 0 \Rightarrow q_1 = q_2$. Therefore q and r are unique. □

A note about proving the remaining cases: you can use the $a > 0, b > 0$ case.

Definition 2.1. b divides $a \Leftrightarrow \exists q$ such that $a = qb$. Write $b \mid a$ if b divides a and $b \nmid a$ if b does not divide a.

Proposition 2.3. *g is a divisor of a if $g \mid a$ and $g \mid b$ and if g is the largest such integer, we call g the greatest common divisor or a and b; we write $g = gcd(a, b)$.*

Let's prove some things about divisors.

Lemma 2.4. *If $c \mid a$ and $c \mid b$ then, $c \mid xa + yb\ \forall x, y \in \mathbb{Z}$.*

Proof. $\exists k, l \in \mathbb{Z}$ such that $a = kc, b = lc \Rightarrow xa + yb = x(kc) + y(lc) = (xk + yl)c \Rightarrow c \mid (xk + yl)c = c \mid (xa + yb)$ □

Example 2.2. $\underbrace{87}_{a} = 4(\underbrace{21}_{b}) + \underbrace{3}_{r}$

$gcd(a, b) = 3$
$gcd(b, r) = 3$

Lemma 2.5. *If $a = qb + r$, then $gcd(a, b) = gcd(b, r)$.*

Proof. If $a = b = 0$ or $b = r = 0$, then the gcd's are undefined. If $a \neq 0$, then $gcd(a, 0) = a$. Let $g_1 = gcd(a, b)$ and $g_2 = gcd(b, r)$. We have that $g_1 \mid a$ and $g_1 | b$.

If $a \neq 0$ then $gcd(a, 0) = a$.

$\Rightarrow g_1 \mid a - qb$
$\Rightarrow g_1 \mid r$
$\Rightarrow g_1$ divides r and b
$\Rightarrow g_1 \leq g_2$
Similarly, $g_2 \mid b$ and $g_2 \mid r$
$\Rightarrow g_2 \mid qb + r$ (by lemma above)
$\Rightarrow g_2 \Rightarrow g_2 \leq g_1$ (because $g_2 \mid a$ & $g_2 \mid b$)
$g_1 = g_2$ □

Proposition 2.6. ***Euclidean Algorithm***
Suppose $a > b \quad (a, b \in \mathbb{N})$. Then write:

$$\begin{aligned} a &= q_1 b + r_1 && (0 \leq r_1 < b) \\ b &= q_1 r_1 + r_2 && (0 \leq r_2 < r_1) \\ r_1 &= q_2 r_2 + r_3 && (0 \leq r_3 < r_2) \end{aligned}$$

and continue until the remainder becomes 0, $r_{k+1} = 0$. Then $r_k = gcd(a, b)$

Example 2.3. Find $gcd(630, 196) = 14$

$$\begin{aligned} 630 &= 3(196) + 42 \\ 196 &= 4(42) + 28 \\ 42 &= 1(28) + 14 \\ 28 &= 2(14) + 0 \end{aligned}$$

Proof.
DOES THIS TERMINATE?
This set has a smallest element by W.O.P., this is r_k.
DOES IT WORK?
By previous lemma,
$gcd(a, b) = gcd(b, r_1) = gcd(r_1, r2) = gcd(r_k, 0) = r_k$ □

As a consequence of the Euclidean Algorithm, $\exists x, y$ such that $gcd(a, b) = xa + yb$:

$$\begin{aligned} 14 &= (42) - 1(28) \\ &= (42) - 1[196 - 4(42)] \\ &= 5(42) - 1(196) \\ &= [630 - 3(196)] - 1(196) \\ \Rightarrow 14 &= 5(630) - 16(196) \end{aligned}$$

Definition 2.2. $a, b \in \mathbb{N}$ are **relatively prime** if $gcd(a, b) = 1$.

Example 2.4. Show 91 and 8 are relatively prime:

$$\begin{aligned} 91 &= 11(8) + 3 \\ 8 &= 2(3) + 2 \\ 3 &= 1(2) + 1 \\ 2 &= 2(1) + 0 \end{aligned}$$

Again, $\exists s, t \in \mathbb{Z}$ s.t. $91s + 8t = 1$

$$\begin{aligned} 1 &= (3) - 1(2) \\ &= 3 - 1(8 - 2(3)) \\ &= 3(3) - 1(8) \\ &= 3(91 - 11(8)) - 1(8) \\ &= \underbrace{3}_{s}(91) - \underbrace{34}_{t}(8) \end{aligned}$$

Theorem 2.7. *If a & b are relatively prime, then $\forall n \in \mathbb{N}$, $\exists x, y \in \mathbb{Z}$ s.t.*

$$n = xa + yb$$

Proof.

$$\begin{aligned} &gcd(a, b) = 1 \\ \Rightarrow &\exists s, t \; sa + tb = 1 \\ \Rightarrow &(sn)a + (tn)b = n \\ \Rightarrow &xa + yb = n \end{aligned}$$

$\square$

Corollary 2.8. *If $gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.*

Proof.

$\exists x, y$ such that $ax + by = 1$

$\Rightarrow c = cax + cby$

Since $a \mid a$ and $a \mid bc$

$a \mid a(xc) + bc(y) \Rightarrow a \mid c(ax + by)$

$\Rightarrow a \mid c$ $\square$

Definition 2.3. $p \in \mathbb{N}$ is called prime if $d \mid p \Rightarrow d = 1$ or p

Lemma 2.9. *$\forall n \in \mathbb{N}, n \geq 2, \exists p \mid n,$ p prime.*

Proof. Suppose n has no prime divisor. The set of all positive integers with no prime divisors would be non-empty. By WOP (Lemma 2.2), it has a smallest element. Call this m.[15]. Since $m \mid m$, m is not prime. Thus $\exists d, 1 < d < m$ such that $d \mid m$. By minimality of m, d has a prime divisor, say p. $p \mid d$ and $d \mid m$

$\Rightarrow p \mid m$ $\Rightarrow\Leftarrow$ $\square$

[15] Note: This variation on a proof by contradiction uses a minimal counterexample

Lemma 2.10. *If $n \in \mathbb{N}, n$ composite, $n \geq 2$, then $\exists p \in \mathbb{N}$ such that $p \mid n$, p prime and $p \leq \sqrt{n}$.*

Proof. n composite $\Rightarrow n = ab$ for some $a, b \in \mathbb{N}$. WLOG, $a \leq b$.

Claim: $a \leq \sqrt{n}$.

If not, $a > \sqrt{n}$ and $b \geq a > \sqrt{n} \Rightarrow ab > \sqrt{n}\sqrt{n} = n$ $\Rightarrow\Leftarrow$

$\exists p \mid a$, p prime $\Rightarrow p \leq a \leq \sqrt{n}$ $(\exists p \mid a, a \mid n \Rightarrow p \mid n)$ $\square$

Lemma 2.11. *$p \mid ab,\ p$ prime $\implies p \mid a$ or $p \mid b$*

Proof. $p \mid a \implies$ done.
$p \nmid a \implies gcd(a,p) = 1 \implies p \mid b$. □

Corollary 2.12. *$p \mid a_1 \dots a_k \Rightarrow p \mid a_i$ for some i*

Theorem 2.13. ***Fundamental Theorem of Arithmetic.***
For an $n \in \mathbb{N}, n \geq 2, n = p_1 p_2 \dots p_k$ for some primes $p_1, p_2, \dots, p_k$. This representation is unique.

Proof. Suppose $n = p_1 p_2 \dots p_k$ and $n = q_1 q_2 \dots q_l$ (all p_i, q_i prime). $\Rightarrow p_1 p_2 \dots p_k = q_1 q_2 \dots q_l$. Cancel any common prime factors. If there's anything left, p_i, then $p_i \mid q_i \Rightarrow p_i = q_i$. $\Rightarrow\Leftarrow$
The two representations must be identical. □

2.2 *Congruence and Modular Arithmetic*

Definition 2.4. Let $n \in \mathbb{N}$. We say $a, b \in \mathbb{Z}$ are **congruent modulo** n iff $n \mid (a - b)$. We write $a \equiv b \pmod{n}$.

Example 2.5.

$$13 \equiv 83 \pmod{10}$$
$$736 \equiv -1044 \pmod{2}$$

Theorem 2.14. *$R_n = \{(a,b) \in \mathbb{Z}^2 \mid n|(a-b)\}$ is an equivalence relation.*

Proof. To be proved. □

Lemma 2.15. *For $n \in \mathbb{N}$ if $a = qn + r$ for some $q \in \mathbb{Z}$, $0 \leq r < n$, then $a \sim r$.*

Proof.

$$\begin{aligned} & a = qn + r \\ \Rightarrow & a - r = qn \\ \Rightarrow & n \mid a = r \\ \Rightarrow & (a, r) \in R_i \quad \square \end{aligned}$$

Corollary 2.16. *$[a] = [r]$ in $\mathcal{R}_n$ where r is the remainder of a after division by n.*

Corollary 2.17. *The equivalence classes of $\mathcal{R}_n = \{(a,b) \in \mathbb{Z}^2 \mid n \mid (a,b)\}$ are $[0], [1], \dots, [n-1]$.*

Example 2.6. Find an integer $0 \leq k \leq 17$ such that

1. $18 \equiv k \pmod{17}$
   $\Rightarrow k \equiv 1 \pmod{17}$
2. $-18 \equiv \pmod{17}$
   $-18 \equiv -1 \equiv 16 \pmod{17}$

EQUIVALENT STATEMENTS:

1. $a \equiv b \pmod n$
2. $n \mid (a-b)$
3. $a - b \equiv 0 \pmod n$
4. $a \in [b] \in \mathcal{R}_n$
5. $b \in [a] \in \mathcal{R}_n$
6. $[a] = [b] \in \mathcal{R}_n$

Operations

Lemma 2.18. *If* $a \equiv b \pmod n$ *and* $x \equiv y \pmod n$ *then:*

1. $(a \pm x) \equiv (b \pm y) \pmod n$
2. $ax \equiv by \pmod n$

Proof.

1. $n \mid (a-b)$ and $n \mid (x-y)$
   $\Rightarrow n \mid (a \pm x) - (b \pm y)$
   $\Rightarrow (a \pm x) \equiv (b \pm y) \pmod n$
2. Note that $ax - by = ax - ay + ay - by = a(x-y) + y(x-y)$
   $\because n \mid (x-y) \wedge n \mid (a-b)$

$$\begin{aligned} &\Rightarrow n \mid [a(x-y) + y(a-b)] \\ &= n \mid (ax - by) \\ &= ax \equiv by \pmod n \end{aligned}$$

□

Example 2.7. Solve $\begin{cases} 2x + 3y & \equiv 1 \pmod 6 \\ x + 3y & \equiv 5 \pmod 6 \end{cases}$

$$\begin{aligned} 3x + 6y &\equiv 6 \pmod 6 \\ \Rightarrow 3x &\equiv 0 \pmod 6 \\ \implies x &= 0, 2, 4 \pmod 6 \text{ are the possibilities.} \end{aligned}$$

If $x \equiv 0 \bmod 6 \Rightarrow 3y \equiv 5 \pmod 6$

But $3 \nmid 3y - 5 \Rightarrow 6 \nmid 3y - 5$

So $3y \not\equiv 5 \bmod 6 \quad \forall y \in \mathbb{Z}$

If $x \equiv 2 \text{ mod } 6 \Rightarrow 2 + 3y \equiv 5 \text{ mod } 6$

$$\Rightarrow 3y \equiv 3 \mod 6$$
$$\Rightarrow y \equiv 1 \mod 2$$
$$\text{Check: } y \equiv \cancel{0}, 1, \cancel{2}, 3, \cancel{4}, 5$$
$$ka \equiv kb \mod kn \Leftrightarrow a \equiv b \mod n$$

If $x \equiv 4 \text{ mod } 6 \Rightarrow 4 + 3y \equiv 5 \text{ mod } 6$

$$\Rightarrow 3y \equiv 1 \mod 6$$
$$\text{But } 3 \nmid 3y - 1 \Rightarrow 6 \nmid 3y - 1$$
$$\text{So no solutions.}$$

$$\Longrightarrow \text{ Final Solutions: } \begin{cases} x \equiv 2 \mod 6 \\ y \equiv 1, 3, 5 \mod 6 (y \equiv 1 \mod 2) \end{cases}$$

Proposition 2.19. *If* $ac \equiv bc \mod n$ *and* $gcd(c, n) \equiv 1$, *then* $a \equiv b \mod n$.

Proof.

$$ac \equiv bc \mod n$$
$$\Rightarrow n \mid ac - bc$$
$$\Rightarrow n \mid c(a - b)$$
$$\Rightarrow n \mid a - b \text{ since } gcd(c, n) = 1 \Rightarrow a \equiv b \mod n$$

□

Example 2.8. Solve $2x \equiv 1 \mod 9$

$$2x \equiv 1 \mod 9$$
$$\Rightarrow 2x \equiv 10 \mod 9$$
$$\text{since } gcd(2, 9) = 1$$
$$\Rightarrow x \equiv 5 \mod 9$$

Theorem 2.20. *Let* $n \in \mathbb{N}, a \in \mathbb{Z}$. *Then* $gcd(a, n) = 1 \Leftrightarrow \exists s \in \mathbb{Z}$ *such that* $sa \equiv 1 \mod n$, *where we call s the* ***multiplicative inverse***.

Proof.

$(\Rightarrow)$: $gcd(a, n) = 1 \Rightarrow \exists s, t \in \mathbb{Z}$ such that $sa + tn = 1$
$\Rightarrow n \mid (sa - 1)$
$\Rightarrow sa \equiv 1 \mod n$

$(\Leftarrow)$: If $\exists s, sa \equiv 1 \mod n \Rightarrow \exists q$ such that $sa - 1 = qn$
$\Rightarrow 1 = sa - qn$
If $\exists d$ s.t. $d \mid a$ and $d \mid n$:
$\Rightarrow d \mid 1 \Rightarrow gcd(a, n) = 1$

□

Example 2.9. Find the inverse of $91 \mod 190$.

$$\begin{aligned}
190 &= 2(91) + 8 \\
91 &= 11(8) + 3 \\
8 &= 2(3) + 2 \\
3 &= 1(2) + 1 \\
2 &= 2(1) + 0
\end{aligned}$$

Backtracking gives:...

$$\begin{aligned}
1 &= (71)(91) - (34)(190) \\
(71)(91) - 1 &= \underbrace{(34)(190)}_{\equiv 0 \mod 190} \\
&\Rightarrow (71)(91) \equiv 1 \mod 190 \\
&\Rightarrow 91^{-1} \equiv 71 \mod 190 \\
&\Leftrightarrow 71^{-1} \equiv 91 \mod 190
\end{aligned}$$

since $190 \mid (1 - (71)(91))$

Example 2.10. Find x such that $2x \equiv 1 \mod 9$ (cont. Example 2.8), aka find $2^{-1} \mod 9$.

$$\begin{aligned}
9 &= 4(2) + 1 \Rightarrow 1 = 9 - 4(2) \\
2 &= 2(1) + 0 \\
\Rightarrow (-4)(2) &\equiv 1 \mod 9 \\
2^{-1} &\equiv -4 \equiv 5 \mod 9
\end{aligned}$$

So:

Not using inverse:

$$\begin{aligned}
2x &\equiv 1 \mod 9 \\
\Rightarrow 10x &\equiv 5 \mod 9 \\
\Rightarrow (1)x &\equiv 5 \mod 9 \\
\Rightarrow x &\equiv 5 \mod 9
\end{aligned}$$

Using inverse:

$$\begin{aligned}
2x &\equiv 1 \mod 9 \\
\Rightarrow x &\equiv (1)(2^{-1}) \mod 9 \\
\Rightarrow x &\equiv 5 \mod 9
\end{aligned}$$

Why no solutions for $g \nmid b$? (recall $g = gcd(a, n)$)

$$\begin{aligned}
ax &\equiv b \mod n \\
&\Rightarrow ax - b = kn \\
\Rightarrow \underbrace{b}_{g \nmid b} &= \underbrace{ax - kn}_{\text{divisible by } g} \\
&\Rightarrow \text{no solution.}
\end{aligned}$$

SOLVING ALGORITHM $(ax \equiv b \mod n)$

```
if gcd(a,n) = 1
    find a^{-1} mod n backtracking from Euclidian Algorithm,
    return x ≡ a^{-1} b mod n
    // simplify to x = z mod n
    // all solutions: x = z + nk, k ∈ ℤ
else gcd(a,n) = g ≠ 1
    if g | b
        return SOLVING ALGORITHM ((a/g) x ≡ b/g mod n/g)
        // solution has g equivalence classes mod n
        // i.e. x ≡ z + (n/g) k mod n where x < n
    else g ∤ b
        return no solution.
```

Example 2.11. Solve if possible:

1. $91x \equiv 10 \mod 190$
   $gcd(91, 190) = 1 \Rightarrow$ there is a solution. Use $91^{-1} \equiv 71 \mod 190$.

$$\begin{aligned} 91x &\equiv 10 \mod 190 \\ \Rightarrow x &\equiv (10)(91^{-1}) \mod 190 \\ &\equiv (10)(71) \mod 190 \\ &\equiv 710 \mod 190 \\ &\equiv 3(190) + 140 \mod 190 \\ \Rightarrow x &\equiv 140 \mod 190 \end{aligned}$$

2. $92x \equiv 10 \mod 196$
   $gcd(92, 166) = 4$ using Euclidian Algorithm.

$$\begin{aligned} 92 &= 2^2 \cdot 23 \\ 196 &= 2^2 \cdot 7^2 \end{aligned} \qquad 4 \nmid 10 \implies \text{no solution}$$

3. $92x \equiv 12 \mod 196$
   $gcd(92, 196) = 4$: divide both sides of equation and modulus by 4.
   $23x \equiv 3 \mod 49$
   $gcd(23, 49) = 1$
   $\Rightarrow 23^{-1}$ exists. Using Euclidian Algorithm:

$$\begin{aligned} 49 &= 2(23) + 3 \\ 23 &= 7(2) + 2 \\ 3 &= 1(2) + 1 \\ 2 &= 2(1) + 0 \\ 1 &= 3 - 1(2) \end{aligned}$$

Backtracking:

$$\begin{aligned} 1 &= 3 - 1(23 - 7(3)) \\ &= 8(3) - 1(23) \\ &= 8(49 - 2(23)) - (23) \\ &= 8(49) - 17(23) \\ 1 &\equiv \underbrace{8(49)}_{\equiv 0} - 17(23) \mod 49 \\ 1 &\equiv (-17)(23) \mod 49 \\ \Rightarrow 23^{-1} &\equiv -17 \mod 49 \\ &\equiv 32 \mod 49 \end{aligned}$$

So:

$$\begin{aligned} 92x \equiv 12 \mod 196 \Leftrightarrow 23x &\equiv 3 \mod 49 \\ x &\equiv 3(23)^{-1} \mod 49 \\ x &\equiv 3(32) \mod 49 \\ \Leftrightarrow x &\equiv 96 \mod 49 \\ \Leftrightarrow x &\equiv 47 \mod 49 \end{aligned}$$

What if we wanted all solutions modulo 196?
Every solution is $x = 47 + 49k, k \in \mathbb{Z}$
For $k = 0, 1, 2, 3$: $x = 47, 96, 145, 194$. These correspond to the solution's equivalence classes mod 196.
$x \equiv 47, 96, 145, 194 \mod 196$

Theorem 2.21. ***Fermat's Little Theorem***: *If p is a prime and a is any integer, then $a^{p-1} \equiv 1 \mod p$. [A consequence $a^p \equiv a \mod p$] We can use this to quickly simplify large powers $\mod p$.*

Proof. Later (posted). □

Simplifying Algorithm ($a^x \mod p$)

```
if p prime
    Rewrite x = q(p − 1) + r
    // q ∈ ℤ, 0 ≤ r < p
    return a^x ≡ a^{q(p−1)+r}
              ≡ (a^{p−1})^q a^r mod p
              ≡ a^r mod p
```

Example 2.12. Find $2^{39674} \mod 523$

We can check 523 is prime.

Rewrite 39674 by dividing by 522: $39674 = 76(522) + 2$.

$$\begin{aligned} 2^{39674} &\equiv 2^{(76)(522)+2} \mod 523 \\ &\equiv (\underbrace{2^{522}}_{\equiv 1 \text{ by FLT}})^{76} * 2^2 \mod 523 \\ &\equiv (1)^{76} 4 \mod 523 \equiv 4 \mod 523 \end{aligned}$$

Example 2.13.

$$\begin{aligned} 4762^{5377} \mod 13 &\equiv [336(13) + 4]^{5367} \\ &\equiv 4^{5367} \mod 13 \\ &\equiv 4^{(447)(12)+3} \mod 13 \\ &\equiv (\underbrace{(4)^{12}}_{\equiv 1 \text{ by FLT}})^{447} * 4^3 \mod 13 \\ &\equiv 64 \mod 13 \\ &\equiv 12 \mod 13 \end{aligned}$$

Example 2.14. Show $x^{97} - x - 1 \equiv 0 \mod 97$ has no solution.

$$\begin{aligned} x^{97} &\equiv x \mod 97 \quad \text{(FLT)} \\ x - x + 1 &\equiv 0 \mod 97 \implies 1 \equiv 0 \mod 97 \\ 1 &\not\equiv 0 \mod 97 \quad \Rightarrow\Leftarrow \\ &\implies \text{no solution} \end{aligned}$$

END OF MATERIAL FOR MIDTERM 1

2.3 *RSA Encryption*

Notes posted.

3 Combinatorics

Glorified counting. How do we count?

3.1 Proof by Induction

Definition 3.1. Induction is used to prove a statement $S(n)$ is true for all $n \in \mathbb{N}$ by proving:

1. Base case: $S(1)$ true,
2. Inductive step: $S(n) \Rightarrow S(n+1)$.

WHY DOES this work — ? $S(1) \Rightarrow S(2) \Rightarrow S(3) \Rightarrow \dots$

Example 3.1. Prove $1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$

Proof. Base Case: $n=1$: $1 = \frac{(1)(2)}{2} \checkmark$

Inductive Step: Assume $1+2+...+n = \frac{n(n+1)}{2}$ (hypothesis).
Show $1+2+\cdots+(n+1) = \frac{(n+1)(n+2)}{2}$

$$\begin{aligned} 1+2+\cdots+(n)+(n+1) &= \frac{n(n+1)}{2}+(n+1) \\ &= (n+1)(\frac{n}{2}+1) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

□

Example 3.2. Prove that the sum of odd integers $\sum_{i=1}^{n}(2i-1) = n^2$.

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \end{aligned}$$

Proof. Base case: $n=1$: $\sum_{i=1}^{1}(2i-1) = 1$

Inductive step :

Assume $\sum_{i=1}^{n}(2i-1) = n^2$

Show $\implies \sum_{i=1}^{n+1}(2i-1) = (n+1)^2$

$$\begin{aligned} \sum_{i=1}^{n}(2i-1) = n^2 &= \sum_{i=1}^{n}(2i-1) \\ &= n^2 + [2(n+1)+1] \\ &= n^2+2n+1 \\ &= (n+1)^2 \end{aligned}$$

□

Example 3.3. Prove $\sum_{i=0}^{n}(i)(i!) = (n+1)!-1$

Note: this is a statement taking $n \geq 0$ (an integer) as the variable.

Proof. Base case: $n=0$ $\sum_{i=0}^{n}(i)(i!) = 0 \checkmark$

Inductive step: Assume $\sum_{i=0}^{n}(i)(i!) = (n+1)!-1$

Show $\implies \sum_{i=0}^{n+!}(i)(i!) = (n+2)!-1$

$$\begin{aligned} \sum_{i=0}^{n+1}(i)(i!) &= \sum_{i=0}^{n}(i)(i!) + (n+1)(n+1)! \\ &= [(n+1)!-1]+(n+1)(n+1)! \\ &= (n+1)![1+(n+1)]-1 \\ &= (n+1)!(n+2)-1 \\ &= (n+2)!-1 \end{aligned}$$

□

Example 3.4. $n! \geq 2^n \ \forall\ n \geq 4$.

Proof. Base case: $n=4$: $24 > 16 \checkmark$

Inductive step: Assume $n! > 2^n$

$$\begin{aligned} (n+1)! = (n!)(n+1) &> (n!)(2) \\ = (n!)(n+1) &> (2^n(2)) = 2^{n+1} \end{aligned}$$

□

Example 3.5. Prove $5 \mid 8^n - 3^n$ for all $n \geq 0$.

Proof. Base case: $n=0$: $5 \mid 8^0 - 3^0 \checkmark$

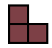
Inductive Step: Assume $5 \mid 8^n - 3^n$

$$\begin{aligned} 8^{n+1}-3^{n+1} &= 8(8^n)-3(3^n) \\ &= (5+3)(8^n)-3(3^n) \\ &= 5(8^n)+3(8^n-3^n) \end{aligned}$$

$5 \mid 5(8^n) \wedge 5 \mid 3(8^n-3^n)$

$\Rightarrow 5 \mid 8^{n+1} - 3^{n+1}$

□

Example 3.6. Call a rotation of  a "tromino". Take a $(2^n \times 2^n)$ checkerboard ($n \geq 1$) and delete any tile (square). Prove you can tile it with trominos.

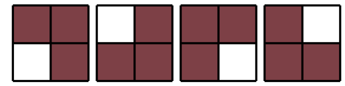


Figure 6: Base case.

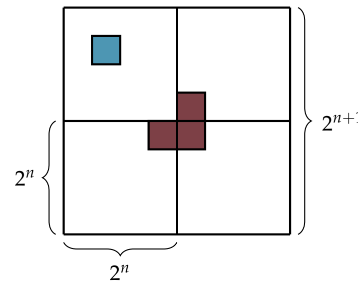


Figure 7: Inductive step.

Proof.

Base Case: $n = 1$:

With a square removed from a (2×2) board, the remaining part is exactly one tromino piece. ✓

Inductive Step: Assume true for n, show true for $n+1$:

Consider a $(2^{n+1} \times 2^{n+1})$ board. Place a tromino in the middle (one block covered - i.e. removed - in 3 of the 4 quadrants). Remove one block from the fourth quadrant. Now each of the 4 $2^n \times 2^n$ boards can be tiled (by Inductive Hypothesis).

□

Example 3.7. Prove that a set with n elements has 2^n subsets.

Proof. Base case: $n = 0$: $\emptyset$ has 1 subset, $1 = 2^0$. ✓

Inductive Step: Assume true for n. Let A be a set with $n+1$ elements (write $|A| = n+1$). Let $x \in A, B = A \setminus \{x\}$. B has 2^n subset by IH. For each $X \subseteq B$, X and $\cup\{x\}$ are distinct subsets of A.
$\Rightarrow 2^n + 2^n = 2(2^n) = 2^{n+1}$ subsets. □

Common Mistakes

1. *Finding incorrect patterns.*
   CLAIM: $n^2 + n + 41$ is prime for all $n \leq 0$. The proof works until $n = 39$ but at $n = 40, 40^2 + 40 + 41 = 41^2$

2. *Forgetting the base case.*
   CLAIM: $n(n+1)$ is odd for all $n \geq 1$.
   Assume true for n.
   $(n+1)(n+2) = \underbrace{n(n+1)}_{odd} + \underbrace{2(n+1)}_{even} \therefore$ ODD
   BUT THIS IS FALSE: We forgot the base case: $n = 1 \Rightarrow (1)(2) = 2$ ~~ODD~~

3. *Making an unstated (wrong) assumption.*
   CLAIM: All cows are the same colour.
   Let C_n denote a set of n cows.

   Base case: $n = 1$: Any set of 1 cow has uniform colour.

Inductive Step: Assume true for any C_n. Look at $\{c_1, c_2, \ldots, c_{n+1}\}$.
By IH, $\{c_1, c_2, \ldots, c_n\}$ all have same colour. By IH, $\{c_1, c_2, \ldots, c_{n+1}\}$ all have same colour.

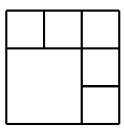
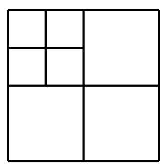
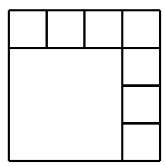
PROBLEM: $S(1) \not\Rightarrow S(2)$ because $\{c_1\} \cap \{c_2\} = \emptyset$ (the two sets share no common cow when $n = 2$).
We have base case $n = 1$ but the statement is false when $n = 2$.

Definition 3.2. Strong Induction is a variation of induction in which we assume that the statement holds for all values preceding n, i.e. one proves:

1. $S(1)$ true
2. $(S(1) \wedge S(2) \wedge \cdots \wedge S(n)) \Rightarrow S(n+1)$

Example 3.8. Prove that any square can be partitioned into $n \geq 6$ non-empty squares.

Proof.

Base case: $n = 6$  $n = 7$  $n = 8$ 

Inductive Step: Assume true for all integers $k, 6 \leq k \leq n \quad (n \geq 8)$.
Show true for $n+1$. By I.H., true for $n-2$. Take one square and quarter it, giving $(n-2) - 1 + 4) = n + 1$. □

Example 3.9. Prove that for any $n \geq 18$, $\exists x, y, \in \mathbb{N} \cup \{0\}$ s.t.

$$n = 4x + 7y$$

Proof. Base case: $n = 18, 19, 20, 21$

$$18 = 4(1) + 7(2)$$
$$19 = 4(3) + 7(1)$$
$$20 = 4(5) + 7(0)$$
$$21 = 4(0) + 7(3)$$

I.H. Assume true $\forall\ 18 \leq k \leq n$, $(n \geq 21)$
By I.H. $\exists x, y, \in \mathbb{N} \cup \{0\}$ s.t. $n - 3 = 4 + 7y \Rightarrow n + 1 = 4(x+1) + 7y$. □

Example 3.10. Prove every $n \in \mathbb{N}$ can be written as

$$n = a_0 2^0 + a_1 2^1 + \cdots + a_k 2^k$$

for some $k \in \mathbb{N} \cup \{0\}$, where $a_i \in \{0, 1\} \forall i$.

i.e. every n can be written in binary.

Proof. Base case: $n = 1 \Rightarrow 1 = (1)2^0$ ✓

Inductive Step: Assume true for all $1 \leq k \leq n - 1$, prove true for n:

1. If n is odd, $n - 1$ is even. Thus $n - 1 = (0)2^0 + a_1 2^1 + \cdots + a^k 2^k$ for some k (by I.H.) $\Rightarrow n = 1(2)^0 + a_1 2^1 + \cdots + a_k 2^k$.
2. If n is even, look at $\frac{n}{2}$. $\frac{n}{2} = a_0 2^0 + \cdots + a_k 2^k$ by (I.H.) $\Rightarrow n = a_0 2^1 + \cdots + a^k 2^{k+1}$. □

3.2 Recursion

Definition 3.3. A sequence $\{a_1, a_2, \dots\}$ is a **recurrence relation** if a_n is a function of $\{a_1, a_2, \dots, a_{n-1}\}$.

NOTATION: The first term can be for any $k \geq 0$. A finite number of **initial values** must be given.

Example 3.11. Let $a_0 = 1, \quad a_n = na_{n-1}, \quad n \geq 1$. Prove $a_n = n!$

$$\begin{aligned} a_0 &= 1 \\ a_1 &= (1)a_2 = 1 \\ a_2 &= (2)a_1 = 2(1) \\ a_3 &= (3)a_2 = (3)(2)(1) \end{aligned}$$

Proof. Base case: $n = 0$ ✓

Inductive Step: Assume $a_n = n!$

$$\Rightarrow a_{n+1} = (n+1)a_n = (n+1)n! = (n+1)! \qquad \square$$

Example 3.12. Let $a_1 = 1, \quad a_n = \sqrt{6 + a_{n-1}}, \quad n \geq 2$. Prove $a_n < 3$.

Proof. Base case : $n = 1 \Rightarrow a_1 = 1 < 3$.

Inductive Step: Assume true for $n \quad (a_n < 3)$ ✓.

$$\begin{aligned} a_{n+1} &= \sqrt{6 + a_n} \\ &< \sqrt{6+3} = 3 \end{aligned} \qquad \square$$

Example 3.13. Prove $a_n = (n+1)2^n$

$$\text{where } \begin{cases} a_0 &= 1 \\ a_1 &= 4 \\ a_n &= 4a_{n-1} - 4_{a-2}, \quad n \geq 2 \end{cases}$$

Proof. Base cases: $n = 0, 1$.

$n = 0$: $(1) = (0+1)2^0$

$n = 1$: $(4) = (1+1)2^1$

Inductive Step: Assume true for all $k, \quad 1 \leq k \leq n$.

$$\begin{aligned} a_{n+1} &= 4a_n - 4n_{n-1} \\ &= 4(n+1)2^n - 4(n)2^{n-1} \\ &= 4 \cdot 2^{n-1}[2(n+1) - n] \\ &= 2^{n+1}(n+2) \end{aligned} \qquad \square$$

Definition 3.4. Fibonacci numbers.

$$f_1 = 1, f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad (n \geq 3)$$

Example 3.14. How many subsets of $\{1, 2, 3, \dots, n\}$ contain no consecutive integers? Let S denote the number of such subsets from $\{1, \dots, n\}$.

$$S_n = (\#\text{ with } n) + (\#\text{ without } n) = S_{n-2} + S_{n-1}$$

$n = 1$: $\{1\}, \quad 2 : \emptyset, \{1\}$

$n = 2$: $\{1, 2\}, \quad 3 : \emptyset, \{1\}, \{2\}$

$n = 3$: $\{1, 2, 3\}, \quad 5 : \emptyset, \{1\}, \{2\}, \{1, 3\}$

S_{n-2} because because if n is in the set, $n - 1$ is not and we are counting subsets from $\{1, \dots, n-2\}$; and S_{n-1} because if n is not in, then we count subsets from $\{1, \dots, n-1\}$. $\implies S_n$ is f_{n+2}.

Some proofs about $\{f_n\}_{n=1}^{\infty}$

Theorem 3.1. $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

Proof. Base case: $n = 1 \Rightarrow f_1 = f_3 - 1 = 2 - 1$ ✓

Inductive Step: Assume true for n. Want $f_1 + f_2 + \cdots + f_n + f_{n+1} = f_{n+3} - 1$

$$\begin{aligned} \boxed{f_1 + f_2 + \cdots + f_n} + f_{n+1} &= \boxed{f_{n+2} - 1} + f_{n+1} \\ &= f_{n+3} - 1 \end{aligned} \qquad \square$$

Theorem 3.2.
$$P(n) : f_{n+1}f_n + f_n f_{n-1} = f_{2n}$$
$$Q(n) : f_n^2 + f_{n-1}^2 = f_{2n-1}$$

Proof. For now, omit base cases: computation.
Assume $P(n)$ and $Q(n)$ true, $\implies$ Prove $Q(n+1)$ true.

$$\begin{aligned}
f_{n+1}^2 + f_n^2 &= (f_n + f_{n-1})^2 + f_n^2 \\
&= f_n^2 + 2f_n f_{n-1} + f_{n-1}^2 + f_n^2 \\
&= (f_n^2 + f_{n-1}^2) + f_n f_{n-1} + f_n f_{n-1} + f_n^2 \\
&= f_{2n-1} + f_n f_{n-1} + f_n(f_{n-1} + f_n) \\
&= f_{2n-1} + f_n f_{n-1} + f_n f_{n+1} \\
&= f_{2n-1} + f_{2n} = f_{2n+1}
\end{aligned}$$

$P(n+1)$ true?

$$\begin{aligned}
f_{n+2}f_{n+1} + f_{n+1}f_n &= (f_{n+1} + f_n)f_{n+1} + (f_n + f_{n-1})f_n \\
&= f_{n+1}^2 + f_n f_{n+1} + f_n^2 + f_n f_{n-1} \\
&= \underbrace{f_{n+1}^2 + f_n^2}_{Q(n+1)} + \underbrace{f_n f_{n+1} + f_n f_{n-1}}_{P(n)} \\
&= f_{2n+1} + f_{2n} \\
&= f_{2n+2}
\end{aligned}$$
□

3.3 Solving Recurrence Relations

Definition 3.5. The **characteristic polynomial** of $a_n = ra_{n-1} + sa_{n-2}$ is $x^2 - rx - s$. Its roots α, β are the characteristic roots of the relation.

Example 3.15. Consider $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 2, a_1 = 1$

$a_0 = 2 \quad a_1 = 1 \quad a_2 = 5 \quad a_3 = 7 \quad a_4 = 17$
$a_5 = 31 \quad a_6 = 65 \quad a_7 = 127$

...

$$a_n = 2^n + (-1)^n \text{ ?}$$

We could prove this by induction, but we would rather have a method more rigorous than "guess and check".

Theorem 3.3. *Consider $a_n = ra_{n-1} + sa_{n-2}$ with specified a_0, a_1. If $x^2 - rx - s$ has 2 distinct roots α, β then*

$$a_n = c_1\alpha^n + c_2\beta^n$$

where c_1 and c_2 solve $a_0 = c_1 + c_2$, $a_1 = c_1\alpha + c_2\beta$.

Proof. Show $a_n = c_1\alpha^n + c_2\beta^n$ solves the relation $a_n = ra_{n-1} + sa_{n-2}$

$$\begin{aligned}
c_1\alpha^n + c_2\beta^n &= r[c_1\alpha^{n-1} + c_2\beta^{n-1}] + s[c_1\alpha^{n-2} + c_2\beta^{n-2}] \\
c_1\alpha^n - rc_1\alpha^{n-1} - sc_1\alpha^{n-2} &= -c_2\beta^n - rc_2\beta^{n-1} - s\beta^{n-2} \\
c_1\alpha^{n-2}[\alpha^2 - r\alpha - s] &= -c_2\beta^{n-2}[\beta^2 - r\beta - s] \\
0 &= 0 \checkmark \\
a_n &= c_1\alpha^n + c_2\beta^n \\
n = 0: \quad a_0 &= c_1 + c_2 \\
n = 1: \quad a_1 &= c_1\alpha + c_2\beta
\end{aligned}$$
□

Example 3.16. (Back to Example 3.15) $a_n = a_{n-1} + 2a_{n-2}$
$\Rightarrow x^2 - x - 2 = 0 = (x+1)(x-2) \Rightarrow$ let $\alpha = 2, \beta = -1$

$$a_n = c_1 2^n + c_2(-1)^n$$
$$n = 0: \quad a_0 = c_1 + c_2 \Rightarrow 2 = c_1 + c_2$$
$$n = 1: \quad a_1 = 2c_1 - c_2 \Rightarrow 1 = 2c_1 - c_2$$

$\Rightarrow c_1 = 1, c_2 = 1 \Rightarrow a_n = 2^n + (-1)^n$

Example 3.17. Fibonacci sequence
$f_n = f_{n-1} + f_{n-2},\ f_0 = 0, f_1 = 1$
Characteristic polynomial: $x^2 - x - 1$

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{(5)}}{2}$$
$$\Rightarrow \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$
$$f_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$$
$$\Rightarrow c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$$
$$\Rightarrow f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

WHAT ABOUT relations of the form $a_n = ra_{n-1} + sa_{n-2} + ta_{n-2}$ (or higher "order")? — Look at a characteristic polynomial whose degree is equal to the order of the recurrence.

Example 3.18. $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}, \quad a_0 = 2, a_1 = 5, a_2 = 15$
C.P. $x^3 - 6x^2 + 11x - 6$ has three roots: $\alpha = 1, \beta = 2, \gamma = 3$
$\Rightarrow a_n = c_1^n + c_2(2)^n + c_3(3)^n$

$$n = 0: \quad 2 = c_1 + c_2 + c_3$$
$$n = 1: \quad 5 = c_1 + 2c_2 + 3c_3$$
$$n = 2: \quad 15 = c_1 + 4c_2 + 9c_3$$

Solve $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 5 \\ 1 & 4 & 9 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -1 \\ c_3 = 2 \end{cases}$

$$\Rightarrow a_n = 1 - 2^n + 2(3^n)$$

Theorem 3.4. *The recurrence relation $a_n = ra_{n-1} + sa_{n-2} + f(n)$ has solution $p_n + q_n$ where p_n is a particular solution of the recurrence (ignoring initial conditions) and q_n is the general solution to $a_n = ra_{n-1} + sa_{n-2}$, where constants c_1, c_2 in q_n are found from initial conditions.*

Note: the method for finding p_n is the equivalent of the constant coefficient method for finding a particular solution to a homogeneous second order ODE, i.e. guess a solution.

Example 3.19. $a_n = 2a_{n-1} + 3a_{n-2} + 5^n \Rightarrow$ guess $p_n = c5^n$

$$c5^n = 2c5^{n-1} + 3c5^{n-2} + 5^n$$
$$25c = 10c + 3c + 25 \qquad \Rightarrow p_n = \frac{25}{12}(5^n)$$
$$12c = 25 \Rightarrow c = \frac{25}{12}$$

$$a_n = 2a_{n-1} + 3a_{n-2}$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$\alpha = 3,\ \beta = -1$$
$$q_n = c_1 3^n + c_2(-1)^n$$
$$\Rightarrow p_n + q_n = \frac{25}{12}(5^n) + c_1 3^n + c_2(-1)^n$$

Solve for c_1 and c_2:

$$\begin{aligned} n=0: \quad -2 &= \frac{25}{12} + c_1 + c_2 \\ n=1: \quad 1 &= \frac{125}{12} + 3c_1 - c_2 \end{aligned} \qquad \Rightarrow \begin{cases} c_1 = -\frac{27}{8} \\ c_2 = -\frac{17}{24} \end{cases}$$

$$a_n = \frac{25}{12}(5^n) - \frac{27}{8}(3^n) - \frac{17}{24}(-1)^n$$

Example 3.20. $a_n = 2a_{n-1} + 3a_{n-2} + 5n, \quad a_0 = 0, a_1 = 1$

As above, $q_n = c_1 3^n + c_2(-1)^n$

Guess $p_n = An + B$

$$\begin{aligned} An + B &= 2A(n-1) + 2B + 3A(n-2) + 3B + 5n \\ 0 &= (-2-6)A + (-1+2+3)B + (-n+2n+3n)A + 5n \\ &= 4B - 8A + 4An + 5n = (4B - 8A) + (4A+5)n \end{aligned}$$

$$4A + 5 = 0 \Rightarrow A = -\frac{5}{4}$$

$$4B - 8A = 0 \Rightarrow B = -\frac{5}{2}$$

$$p_n = -\frac{5}{4}n - \frac{5}{2}$$

$$a_n = -\frac{5}{4}n - \frac{5}{2} + c_1 3^n + c_2(-1)^n$$

$$\begin{aligned} n=0: \quad 0 &= -\frac{5}{2} + c_1 + c_2 \\ n=1: \quad 1 &= -\frac{5}{2} - \frac{5}{4} + 3_c - c_2 \end{aligned} \qquad \Rightarrow \begin{cases} c_1 = \frac{29}{16} \\ c_2 = \frac{11}{16} \end{cases}$$

$$\Rightarrow a_n = -\frac{5}{4}n - \frac{5}{2} + \frac{29}{16}3^n + \frac{11}{16}(-1)^n$$

3.4 *Functions*

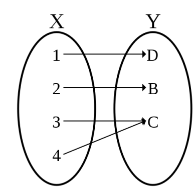


Figure 8: Surjective (non injective) map.

A function f from X to Y, written $f : X \to Y$ is a binary relation $f \subset X \times Y$ such that for every $x \in X$, there is at most one $y \in Y$ such that $(x, y) \in f$. Since y is unique for x we can write $y = f(x)$.

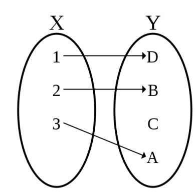


Figure 9: Injective (non surjective) map.

Definition 3.6. A surjection is a relation where $\forall y \in Y \; \exists x \in X$ s.t. $y = f(x)$. (f is "onto".)

Definition 3.7. A injection is a relation where $\forall y \in Y \; \exists$ at most one $x \in X$ s.t. $y = f(x)$. (f is "one to one".)

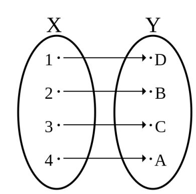


Figure 10: Bijective map.

Definition 3.8. A bijection is a relation where $\forall y \in Y \; \exists! x \in X$ s.t. $y = f(x)$. (f is both surjective and injective.)

RELATION TO COUNTING: Recall $|x|$ denotes the cardinality of X (the number of elements)

1. $\exists$ a surjection, $f : X \to Y \Leftrightarrow |X| \geq |Y|$
2. $\exists$ an injection, $f : X \to Y$ using every element of $X \Leftrightarrow |X| \leq |Y|$
3. $\exists$ a bijection, $f : X \to Y \Leftrightarrow |X| = |Y|$

3.5 *Counting*

Counting objects/structures.

PRINCIPLES OF COUNTING:

1. $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i|$
   if each pair (A_i, A_j) is disjoint ($A_i \cap A_j = \emptyset$)
2. $|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|$

TRANSLATED PRINCIPS OF COUNTING: A_i: ways in which a task can be done.

1. The number of ways which a collection of tasks can be done if no two sets of ways of completing the tasks overlap is the sum of ways of the numbers of ways each task can be done.
2. The number of ways a sequence of tasks can be done is the product of the number of ways each task can be done.

Example 3.21. How many 4 digits number have no repeated digits?

Answer. The tasks are to choose each digit: $9 \times 9 \times 8 \times 7 = 4536$. Since the first digit can't be 0.

Example 3.22. How many 4 digit even numbers have no repeated digits?

Answer. We split into 2 cases:

1. Last digit is 0: $9 \times 8 \times 7 \times 1 = 504$
2. Last digit is not 0: $8 \times 8 \times 7 \times 4 = 1792$

Total: 2296.

SOME QUESTIONS:

1. How many functions are there from an n-set to a k-set? ("m-set" = set with m elements)
   Sequence of events: let $|X| = n, |Y| = k$. For each $x \in X$, choose $y \in Y$ such that $y = f(x)$.
   Number of functions: $\underbrace{k \times k \times \cdots \times k}_{n \text{ times}} = k^n$.
2. How many subsets of an n-set are there?
   Sequence of events: take each element and decide to put it in the subset or not: n tasks with 2 possibilities each.
   Number of subsets: $\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$.

3. How many bijections are there from an n-set to itself?
   1st element has n possible functions values. 2nd element has $n-1$ (since it's a bijection). Etc.
   Number of bijections: $(n)(n-1)(n-1)\ldots(2)(1) = n!$

Definition 3.9. A **permutation** of an n-set is an ordering of its elements. This is precisely the same as a bijection, so there are $n!$ permutations of an n-set.

$X = \{1,2,3,4\}$
BIJ: $\{(1,4),(2,1),(3,2),(4,3)\}$
PERM: 4123

Example 3.23. How many ways can you choose a president, vice president, treasurer, secretary from a group of 7 people?
$P(7,4) = \frac{7!}{3!} = 840$

Definition 3.10. A k-**permutation** of an n-set X is a choice of k elements of X in some order. There are $P(n,k)$ such k-permutations:

4. $P(n,k) = (n)(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$
5. How many subset of size k does an n-set have?
   Count the number of k-permutations again = (# of k-subsets)(k!)
   $P(n,k) = \frac{n!}{(n-k)!} = (\text{\# of } k\text{-subsets})(k!)$
   $\Rightarrow$ number of k-subsets $= \frac{n!}{k!(n-k)!}$

Definition 3.11. $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ pronounced "n choose k"

Proposition 3.5. $\binom{n}{k} = \binom{n}{n-k}$

Proof. 1. $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$

2. Choosing k elements to be in your set is equivalent to choosing the $n-k$ elements in its complement. □

3.6 The Binomial Theorem

Theorem 3.6. *The Binomial Theorem says that if $n \geq 1$ is an integer,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k}$ are called binomial coefficients.

Proof. For a positive integer n, what is the coefficient of $x^k y^{n-k}$ in $(x+y)^n$? $(x+y)(x+y)\cdots(x+y)$

How many ways can we choose k x's from the n factors and $n-k$ y's, choosing one variable from each factor? We only need to count the number of ways to choose x's (get y's for free). Thus the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$. □

Lemma 3.7. $\sum\limits_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. Two methods:

Binomial Theorem: Let $x = y = 1$:

$$(x+y)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k}(1)(1)$$

Combinatorially:

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

where 2^n is the number of subsets of an n-set, and $\binom{n}{i}$ is the number of subsets with cardinality i. □

Lemma 3.8. *An n-set, $n \geq 1$, has the same number of even subsets as odd subsets.*

Proof. Combinatorially: Fix some element x.

$$\begin{aligned} \#\text{ odd subsets with } x &= \#\text{ even without } x \\ \#\text{ odd subsets without } x &= \#\text{ even with } x \\ \hline \#\text{ odd} &= \#\text{ even} \end{aligned}$$

Binomial Theorem: Pick $x = -1, y = 1$.

$$(-1+1)^n = \sum_{k=0}^{n} (-1)^k (1)^{n-k}$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \dots$$
$$\binom{n}{1} + \binom{n}{3} + \dots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$$ □

Lemma 3.9. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Proof. $\binom{n}{k}$ is the number of subsets of size k. Define some k. $\binom{n-1}{k}$ is the number of such subsets without x, and $\binom{n-1}{k-1}$ is the number of such subsets including x. □

Lemma 3.10. $\sum\limits_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Proof. Rewrite what we want to prove:

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \binom{n}{2}\binom{n}{2} + \cdots$$
$$= \binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{k}\binom{n}{n-k} + \cdots$$

$\{1, \ldots, 2n\} = \{1, \ldots, n\} \cup \{1, \ldots, n\}$

$\binom{2n}{n}$: number of n-subsets

$\binom{n}{k}\binom{n}{n-k}$: number of ways to choose k elements from $\{1, \ldots, n\}$ and the rest from $\{n+1, \ldots, 2n\}$. □

Repetitions

1. If order matters:

Proposition 3.11. *Let $a_1, a_2, \cdots, a_n$ be a set of elements that can be decomposed into K subsets of elements that are repeated i_k times ($1 \leq k \leq K$), i.e. each subset has size i_k, $\sum i_k = n$. The number of permutations is $\frac{n!}{\prod_{k=1}^{K} i_k!}$.*

Example 3.24. How many anagrams of EASY are there? 4!

What about CHEESE? Not 6! since we have repetition.

If we take $C_1H_1E_1E_2S_1E_3$ we get $6! = (\text{\# of anagrams of CHEESE})(3!)$,

$\Rightarrow$ # of anagrams of CHEESE $= \frac{6!}{3!}$

For MISSISSIPPI, # of anagrams $= \frac{11!}{(1!)(4!)(4!)(2!)}$

2. If order doesn't matter:

Proposition 3.12. *"Balls & Boxes": The number of ways one can distribute k identical balls between n boxes corresponds to a binary string of length $n+k-1$ with $n-1$ 1's & k 0's, i.e. $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.*

Visualize "Balls & Boxes" as choosing the positions for the $n-1$ separations between boxes, between $n-1+k$ possible positions (positions include elements and separation lines).

Example 3.25. Jim's Morton's offers 30 kinds of Jimbits. How many ways can you choose a dozen?

Answer. $\binom{30+12-1}{29}$

Example 3.26. A sandwich shop offers 3 kinds of bread, 4 kinds of meat and 10 kinds of toppings.

(a) How many possibilities are there if each sandwich has 1 bread, 1 meat and 3 toppings ?

Answer. Number of possible sandwiches: $\binom{3}{1} \times \binom{4}{1} \times \binom{10}{3} = 1440$

How many platters?

Answer. 1440 boxes, 20 balls. $\binom{1440+20-1}{20} \approx 6.89 \times 10^{44}$

(b) What if you may now use at most 1 bread, 2 meats and any of toppings?

Answer. Number of possible sandwiches: $\binom{3}{1} \times [\binom{4}{0} + \binom{4}{1} + \binom{4}{2}] \times 2^{10} = 33972$

Number of platters: $\binom{33782+20-1}{20} \approx 1.55 \times 10^{72}$

Example 3.27. How many non-negative integers solutions are there to the equation: $x_1 + x_2 + x_3 = 48$?

Answer. 48 balls, 3 boxes: $\binom{48+3-1}{3} = 1225$

What if we impose $x_1 \geq 2$ and $x_3 \geq 5$?

Answer. Let $y_1 = x_1 - 2,\ y_2 = x_2,\ y_3 = x_3 - 5$
$\binom{41+3-1}{2} = 903$

Example 3.28. How many (non-negative) solutions are there to the equation: $x_1x_2x_3 = 48$?

Answer. Notice $48 = 2 \times 2 \times 2 \times 2 \times 3$. So we distribute four 2's to three boxes, and distribute one 3 to three boxes. $\binom{4+3-1}{2} \times 3 = 45$.

Counting Elements in Sets

i.e. How to count the amount of elements in sets $|A_1 \cup A_2 \cup \cdots \cup A_n|$ if these sets are not disjoint?

$n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$n = 3$: $$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Example 3.29. 40 students: where 10 in ANTH, 22 in BIOL, 16 in COMP; 6 in A&B, 8 in B&C, 4 in A&C; and 2 in all. How many students are not enrolled in ANY of the 3 courses?

i.e. first find how many in at least one:

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| = (10 + 22 + 16) - (6 + 8 + 4) + 2 = 32$

$\therefore$ There are $40 - 32 = 8$ students in none of the courses.

Theorem 3.13. ***Principle of Inclusion-Exclusion*** *(PIE).*
If $A_1, A_2, \ldots, A_n$ are finite sets, then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1} |\bigcap_{i \in S} A_i|$$

Here $[n]$ represents $[1, \ldots, n]$.

Proof. Must show every $x \in A_1 \cup \cdots \cup A_n$ is counted exactly once on the RHS. Say x appears in k of the sets $A_1 \cup \cdots \cup A_n$. x is counted in collections of 1 set k times, of 2 sets $\binom{k}{2}$ times, of 3 sets $\binom{k}{3}$ times, ...

So x counted $\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \cdots + (-1)^{k-1}\binom{k}{k}$ times in total. Recall

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \cdots = 0$$

Therefore we take

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \cdots + (-1)^{k-1}\binom{k}{k} = 1$$

so each x is counted once. □

Example 3.30. n people put their phones in a box. If every person takes a phone out, how likely is it that no one gets their phone back? The number of permutations of $[n]$ (or bijections from $[n] \to [n]$) where no element is mapped to itself is denoted D_n, they are called derangements.

Let $A_i =$ Set of permuttions which map i to itself.

$$\begin{aligned} D_n &= |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| \\ &= |\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| \\ &= n! - A_1 \cup A_2 \cup \cdots \cup A_n \end{aligned}$$

However,

$$\begin{aligned}
|S| &= A_1 \cup A_2 \cup \cdots \cup A_n \\
&= (|A_1| + |A_2| + \dots) - (|A_1 \cap A_2| + |A_1 \cap A_3| + \dots) + \dots \\
&= n \cdot (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n}{n}(n-n)! \\
&= n! - \frac{n!}{2!} + \frac{n!}{3!} + \cdots + (-1)^{n-1}\frac{n!}{n!} \\
&= n!\left(1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{n!}\right) \\
D_n &= n! - n!\left(1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{n!}\right) \\
&= n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n}}{n!}\right) \\
&= n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n}}{n!}\right) \\
&= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}
\end{aligned}$$

So calculating the probability:

Recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\ x \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}(\text{a permutation is a derangement}) &= \frac{D_n}{\#\text{ permutations}} = \frac{D_n}{n!} \\
&= \sum_{k=0}^{n}\frac{(-1)^k}{k!} = e^{-1}
\end{aligned}$$

So, as $n \to \infty$, $\mathbb{P} \approx e^{-1} \approx 0.36788$, and $D_n \approx \frac{n!}{e}$.

Theorem 3.14. ***Pigeon Hole Principle*** *(PHP). If one distributes $>\ n$ objects to n boxes, some box has more than one object.*

Example 3.31. 22 soccer players on a field measuring $42m \times 98m$. Show there are two players no more than $20m$ apart.

Answer. Cut the field into 21 boxes of size $14 \times 14\ m$. There are 21 boxes and 22 players, therefore 2 players must be in the same square by PHP. $d \le \sqrt{14^2 + 14^2} = \sqrt{196 + 196} < \sqrt{400} = \sqrt{400} = 20$.

Example 3.32. Let S be a set of 9 points in $\mathbb{R}^3$ each with integer coordinates. Show $\exists p_i, p_i \in S$ such that the midpoint of $\overline{p_1 p_2}$ has integer coordinates.

Answer. Let $p_i = (x_i, y_i, z_i)$, $p_j = (x_j, y_j, z_j)$.

Want $\frac{x_i+x_j}{2}, \frac{y_i+y_j}{2}, \frac{z_i+z_j}{2} \in \mathbb{Z} \Leftrightarrow x_i + x_j, y_i + y_j, z_i + z_j$ even.

The possible parity combinations are shown in Table 6.

x	y	z
E	E	E
E	E	O
E	O	E
O	E	E
E	O	O
O	E	O
O	O	E
O	O	O

Table 6: Table for Example 3.32

8 parity combinations, 9 points $\Rightarrow$ 2 have same parity, i.e.

$$x_i \equiv x_j \bmod 2 \qquad y_i \equiv y_j \bmod 2 \qquad z_i \equiv z_j \bmod 2$$

Therefore $x_i + x_j$, $y_i + y_j$, $z_i + z_j$ are all even.

Example 3.33. Show that if $n+1$ numbers are chosen from $[2n]$, then there are

(a) 2 which differ by 1:

Answer. Boxes: $\{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{2n-1, 2n\}$
n boxes, $n+1$ choices, therefore 2 integers in same box by PHP which differ by 1.

(b) 2 which sum to $2n+1$:

Answer. Boxes: $\{1,2n\}, \{2,2n-1\}, \{3,2n-2\}, \ldots$
Each pair sums to $2n+1$. 2 of the $n+1$ choices lie in one box by PHP.

Proposition 3.15. ***Strong version*** *of PHP: If one has $m > n$, m objects, n boxes, then some box receives $\geq \lceil \frac{m}{n} \rceil$ objects.*

Example 3.34. 16 students, in 18 chairs. Show there are 6 consecutive occupied seats.

Answer. Split into 3 blocks of 6 chairs. Distribute 16 people across 3 boxes therefore the same box must gets $\geq \lceil \frac{16}{3} \rceil = 6$ people.

Example 3.35. A student has 37 days to prepare for an exam. If she studies no more than 60 hours, but at least 1 hour each day, show there is some set of consecutive days over which she studied exactly 13 hours (whole hours each day).

Answer. Let $s_i =$ number of hours studied n day i. Let $a_i = s_1 + s_2 + \cdots + s_i$ (number of hours studied from day 1 to day i).

$$1 \leq a_1 < a_2 < \cdots < a_{37} \leq 60$$

$$14 \leq a_1 + 13 < a_2 + 13 < \cdots < a_{37} + 13 \leq 73$$

Consider the list $a_1, a_2, \ldots, a_{37}, a_1 + 13, a_2 + 13, \ldots, a_{37} + 13$. There are 74 integers between 1 and 73 therefore 2 must be the same. $\Rightarrow i, j$ s.t. $a_i = a_j + 13 \Leftrightarrow a_i - a_j = 13 \Leftrightarrow s_{j+1} + s_{j+2} + \cdots + s_i = 13$

4 Graph Theory

4.1 Introduction

Definition 4.1. A **graph** G is a pair (V, E) where V is a set (whose elements are the vertices of the graph) and E is a set of unordered pairs of elements of V (called the edges of G).

We often write $V(G)$ and $E(G)$ for the vertices/edges if $G = (V, E)$. Also, for brevity, we write $uv \in E(G)$ for the edge $\{u, v\}$.

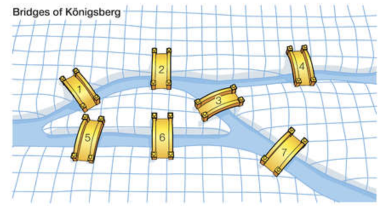


Figure 11: **Bridges of Königsberg**: can you start anywhere in the city, cross every bridge exactly once and finish where you started?

Example 4.1. $V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$

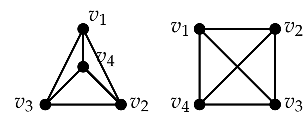


Figure 12: These two figures represent identical graphs (the one in Exercise 4.1).

Graphs are often represented pictorially where $V(G)$ is a set of points or dots, and each edge is a segment or curve between points; but a drawing of a graph is not the graph itself.

USES OF GRAPHS:

1. **Routing problems**: V = locations, E = direct passage between locations.
2. **Social networks**: V = users, E = friends.
3. **Scheduling**: V = events, E = pairs of events which cannot coincide.

Definition 4.2. Given $v \in V(G)$ and $e \in E(G)$, if $v \in e$, then we say that e is **incident** to v. If $uv \in E(G)$, we say that u and v are **adjacent** (neighbours).

Definition 4.3. The **neighbourhood** of $v \in V(G)$ is:
$N(v) = \{u \mid uv \in E(G)\}$

Definition 4.4. The **degree** of $v \in V(G)$ in G is $d(v) = |N(v)|$. We can also write this as $deg(v)$, or $N_G(v)$, $d_G(v)$ which make it clear to which graph one is referring. In general, $0 \leq deg_G(v) \leq |V(G)| - 1$

Definition 4.5. The **degree sequence** of G is a list of the degrees (in increasing order).

Theorem 4.1. *If G is a graph, then $\exists u, v \in V(G)$ s.t. $d(u) = d(v)$.*

Proof. Case 1: Suppose $deg(u) \neq 0 \ \forall u \in V(G)$. Let $|V(G)| = n$. Then $1 \leq deg(n) \leq n - 1$. There are n vertices and $n - 1$ possible degrees. By PHP, two vertices have the same degree.

Case 2: $\exists u$ s.t. $deg(n) = 0$. Then $0 \leq deg(n) \leq n - 2$. $n - 1$ possible degrees & n vertices, so 2 must have the same degree by PHP. $\square$

Lemma 4.2. ***Handshaking Lemma***. *says that in any graph G, there are an even number of vertices with odd degree.*

Theorem 4.3. *If G is a graph, then* $\sum_{v \in V(G)} deg(v) = 2|E(G)|$

Proof. Count the number of pairs (v, e) such that e is incident to v. Each vertex v appears $deg(v)$ times in the set of pairs. Each edge appears exactly twice.

$$\sum_{v \in V(G)} deg(v) = \sum_{e \in E(G)} 2 = 2|E(G)| \qquad \square$$

Proof. *of Handshaking Lemma (Lemma 4.2).*

$$\sum_{\substack{v \in V(G) \\ deg(v)\ odd}} deg(v) + \underbrace{\sum_{\substack{v \in V(G) \\ deg(v)\ even}} deg(v)}_{\text{even}} = \underbrace{2|E(G)|}_{\text{even}} \qquad \square$$

Special graphs:

- empty graph: $E(G) = \emptyset$
- complete graph: $E(G) =$ all possible pairs:
  - denoted K_n if $|V(G)| = n$
  - $|E(K_n)| = \binom{n}{2}$ [in general, $0 \leq |E(G)| \leq \binom{n}{2}$]

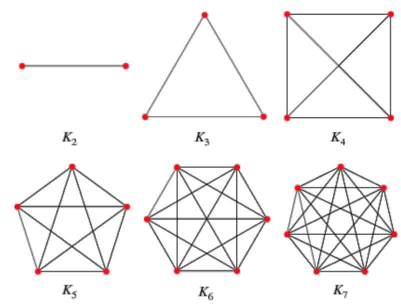


Figure 13: Complete Graphs.

- complete bipartite graph $K_{m,n}$ where :
  - $V(K_{m,n}) = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$
  - $E(K_{m,n}) = \{u_i v_j | 1 \leq i \leq m, 1 \leq j \leq n\}$

  $\implies |V(K_{m,n})| = m + n$

  $\implies |E(K_{m,n})| = m \cdot n$

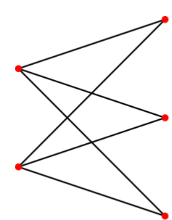


Figure 14: $K_{2,3}$.

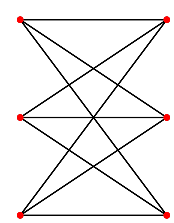


Figure 15: $K_{3,3}$.

- Path P_n: $V(P_n) = \{v_1, \ldots, v_n\}, E(P_n) = \{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n\}$
  $\Rightarrow |E(P_n)| = n - 1$
- Cycle C_n: $V(C_n) = \{v_1, \ldots, v_n\}$
  $E(C_n) = \{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n, v_n v_1\}$
  $\Rightarrow |E(C_n)| = n$

Definition 4.6. We say H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Equivalent: obtain H from G by deleting edges and/or vertices. Note that when we delete $v \in V(G)$, we delete all incident edges as well. If $V(H) = V(G)$ we say that H is a **spanning subgraph**.

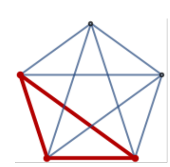


Figure 16: The red part is a subgraph of the entire graph.

Definition 4.7. A **walk** in G is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ where $v_{i-1}v_i \in E(G)\ \forall 1 \leq i \leq k$.

A **path** in a graph is a walk with no repeated vertices. A **closed walk** is a walk where $v_0 = v_k$: v_0 and v_k are the ends of a walk.

A closed path is a **cycle**.

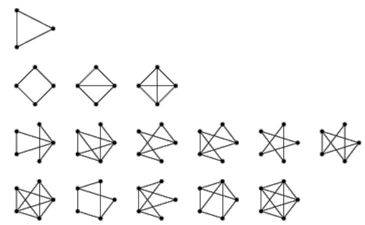


Figure 17: Connected components. Note that a single vertex is a connected component.

Definition 4.8. A graph is **connected** if $\forall u, v \in V(G)$ there is a walk with ends u and v. The maximal connected subgraphs of G are called its **connected components.**

Definition 4.9. A **multigraph** M is a set (of vertices) $V(M)$ and a list of unordered pairs of vertices where repetition is allowed (i.e. $vv \in E(M)$).

Definition 4.10. An **Eulerian walk** in a graph (or multigraph M) is a closed walk which uses every edge exactly once. If G has an Eulerian walk, we call G **Eulerian**.

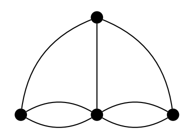


Figure 18: We can recall the Königsberg problem: V are land masses and E are bridges.

Theorem 4.4. *(Euler, 1736). G is Eulerian if and only if it is connected and all vertices have even degree.*

Proof.

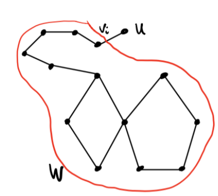
($\Rightarrow$) $\forall\ v \in V(G)$, the number of times an Eulerian walk enters v equals the number of times it exits v. $\implies$ $deg(v) = k + k = 2k$ for some k.

($\Leftarrow$) Suppose G is connected and $deg(v)$ even $\forall\ v \in V(G)$. Let W be a *longest* walk in G with no repeated edges $W = v_0, v_1, \ldots, v_k$.

Suppose W is not Eulerian.

Claim: W is closed.

Proof. Suppose it is not.
$\Rightarrow$ we used odd number of edges incident to v_k
$\Rightarrow$ $\exists$ an unused edge incident v_k
$\Rightarrow$ we can extend W by 1. $\Rightarrow\Leftarrow$ with maximality of W. □



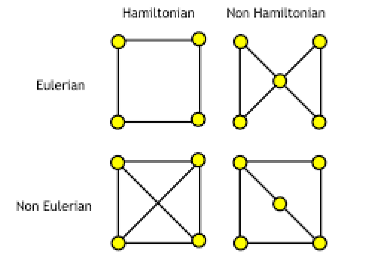
So there is some edge in $E(G)$ not used by W. Because G is connected, there must be an edge uv_i not used by W for some $0 \leq i \leq k$. This contradicts maximality of W: $u, v_i, v_{i+1}, \ldots, v_k = v_0, v_1, \ldots, v_i$ is longer. $\Rightarrow\Leftarrow$. So G is Eulerian. □

This result (Theorem 4.4) holds for multigraphs as well. Note that $deg(v)$ counts a "loop" (edge between the same node) $w \in E(M)$ twice.

Definition 4.11. A path in G is **Hamiltonian** if it has $|V(G)|$ vertices (it "spans" G). A Hamiltonian cycle is a cycle in G with $|V(G)|$ vertices. A graph is Hamiltonian if it has a Hamiltonian cycle.

Finding Hamiltonian cycles, as a rule, is difficult (unless P = NP :').

Are there sufficient conditions for G to be Hamiltonian?



Theorem 4.5. ***(Bondy & Chvátal, 1972)****. If $deg(u) + deg(v) \geq |V(G)|$ $\forall\ u, v \in V(G)$, then G is Hamiltonian $\Leftrightarrow$ $G + uv$ is Hamiltonian where $uv \notin E(G)$.*

Proof. Show G is Hamiltonian $\Leftrightarrow$ $G + uv$ is Hamiltonian where $uv \notin E(G)$.

($\Rightarrow$) Trivial.

($\Leftarrow$) If G is Hamiltonian, then done. Now suppose G is not Hamiltonian. $\Rightarrow \exists$ a Hamiltonian path from u to v in G, say P (Figure 19).

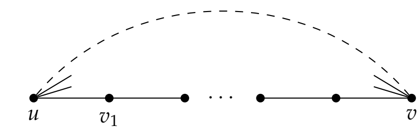
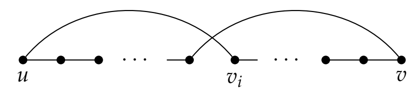


Figure 19: Hamiltonian path P.

Let S be the neighbors of u which lie on P. Let T be the vertices which follow neighbors of v on P. $|S| + |T| \geq |V(G)|$

By PHP, there must be some vertex in $S \cap T$, say v_i.

$\implies$ $v_i, v_{i+1}, \cdots, v, v_{i-1}, v_{i-2}, \cdots, v_1, u, v_i$ is a Hamiltonian cycle of G. $\square$



Theorem 4.6. ***(Ore, 1960)****. If $d(u) + d(u) \geq |V(G)|$ $\forall u, v \in V(G)$ nonadjacent, then G is Hamiltonian. This leads straightforwardly to* ***(Dirac, 1952)****: If $deg_G(v) \geq \frac{|V(G)|}{2}$ then G is Hamiltonian.*

Proof. By adding edges to the graph and applying Theorem 4.5 repeatedly, we get a complete graph, which is Hamiltonian. By the "iff" statements, G is Hamiltonian as well. $\square$

4.2 Trees

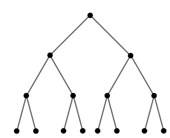


Figure 20: Binary tree. (Any k-ary $K_{1,t}$ is also a tree.)

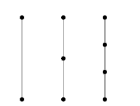


Figure 21: Any path P_n is a tree.

Definition 4.12. A graph is a **tree** if it is connected and has no cycles. A **leaf** of a graph is a vertex of degree 1.

Theorem 4.7. *If $deg(v) \geq 2\ \forall\ v \in V(G)$, then G contains a cycle.*

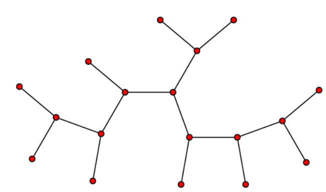


Figure 22: A general tree.

Proof. Let P be a maximal path in G: $P = v_0, v_1, v_2, \ldots, v_k, v$

Since it has $deg(v) \geq 2$, it has some neighbor other than v_k. By the maximality of P, this neighbor must be in P, say v_i. Thus $v_i v_{i+1} \ldots v_k v v_i$ is a cycle. $\square$

Corollary 4.8. *Every tree has a leaf.*

Proof. Contrapositive proof. Left to the reader. $\square$

Theorem 4.9. *A connected graph is a tree if and only if there is a unique path between any 2 vertices.*

Proof. ($\Rightarrow$) Since it is connected, there is at least one path between any two vertices. To prove uniqueness, suppose P_1 and P_2 are distinct uv-paths. Let v_i be the last vertex P_1 and P_2 have in common aside from v. Let v_j be the next vertex in P_1 shared with P_2. The subpaths of P_1 and P_2 from v_i to v_j together thus form a cycle. $\Rightarrow\Leftarrow$

($\Leftarrow$) Let G be a connected graph that is not a tree. The G has a cycle, C. If $u, v \in V(C)$, then there are ≥ 2 uv-paths. □

Theorem 4.10. *Every tree on ≥ 2 vertices has at least 2 leaves.*

Proof. by induction, on the number of vertices.

Base: $n = 2$: the tree on two nodes has 2 leaves.

I.H.: Assume true for $n-1$ vertices. Let T be an arbitrary tree with n vertices. Let x be a leaf of T_i, let $T' = T - x$. T' has $n-1$ vertices, and is a tree because we are removing one node from T, a tree.
If $u, v \in V(T')$, then there exists a unique path in T' between them (the unique path in T could not contain x since $deg(x) = 1$).
Then T' has ≥ 2 leaves, say y, z. Since $deg(x) = 1$, x is adjacent to at most one of y and z.
Then x and at least one of y, z are leaves in T. □

Theorem 4.11. *A connected graph is a tree $\Leftrightarrow$ the deletion of any edge disconnects the graph.*

Proof. c.f. textbook (LPV). □

Theorem 4.12. *An n-vertex tree has $n-1$ edges.*

Proof. Base: $n = 1$: one node, zero edges.

I.H.: Assume true for $n-1$ vertices. Let T be an n-vertex tree, and let x be a leaf. By the I.H., $T - x$ has $n-2$ edges (because it is an $(n-1)$-vertex tree). Then T must have $n-1$ edges. □

Theorem 4.13. *Adding an edge to a tree creates exactly one cycle.*

Proof. Let u, v be non-adjacent vertices. There exists a uv-path, say $P = uv_1 \dots v_k v$. Thus adding uv creates a cycle. Now we must prove that it creates exactly one. Suppose there are more than one.

Any cycle created must contain the edge uv. (Otherwise there is a cycle in the original tree $\Rightarrow\Leftarrow$). But then deleting uv from these cycles gives distinct uv-paths in the tree, contradicting uniqueness (Theorem 4.9.) □

Added clarification: we can let the alleged two cycles created by adding uv be $(u, v, \cdots, a, b, \cdots, u)$ and $(u, v, \cdots, x, y, \cdots, u)$.
Removing uv, we get two paths from v to u: $(v, \cdots, a, b, \cdots, u)$ and $(v, \cdots, x, y, \cdots, u)$. $\Rightarrow\Leftarrow$ with path uniqueness.

Definition 4.13. A **spanning tree** of a graph G is a subgraph T such that $V(T) = V(G)$ and T is a tree.

Theorem 4.14. *Every connected graph has a spanning tree.*

Proof. Base: Trivially true for $n = 1$.

I.H.: Assume true for $|V(G)| = n - 1$.

Delete any vertex v from our graph, which does not disconnect it.[16] The resulting $(n-1)$-vertex graph has a spanning tree. Add the edge vx to the tree for any $x \in N(v)$. The resulting subgraph:

- uses every vertex
- is connected
- contains no cycles (any cycle would contain v which has degree 1 in the subgraph). □

[16] **Exercise**: Prove that any connected graph has a vertex whose deletion does not disconnect the graph.

Counting Trees

Theorem 4.15. ***Cayley's Theorem****. There are n^{n-2} different trees on n* ***labelled*** *vertices.*

Proof.

- See LPV for proof using bijection between trees and "Prufer codes".

- c.f. Kirchhoff's Theorem, using linear algebra. □

There is no theorem for the number of unlabelled trees on n vertices. However, we can find bounds.

Let T_n denote the number of unlabelled n-vertex trees.

$$n^{n-2} \leq n! T_n \tag{1}$$

$$\frac{n^{n-2}}{n!} \leq T_n$$

$$T_n \leq \binom{2n-2}{n-1} \tag{2}$$

Start at root, draw the tree so that no edges cross and vertices at distance i from root are at level i. Walk around the edges and record $\mathcal{D}$ if you move down 1 edge and $\mathcal{U}$ if you move up one edge. Thus the code has length $2|E(T)| = 2(n-1)$ and has $n-1$ $\mathcal{U}$'s and $n-1$ $\mathcal{D}$'s, yielding a code that looks like $UUDUDDUU\ldots$.

Thus the number of combinations of codes is $\binom{2n-2}{n-1}$.

Some trees are counted more than once, but each tree is counted at least once.

$$\implies T_n \leq \binom{2n-2}{n-1}$$

Using Stirling's Approximation, we can say that for large n,

$$\frac{e^n}{\sqrt{2\pi}n^{5/2}} \leq T_n \leq \frac{4^{n-1}}{\sqrt{2\pi n - 1}}$$

Stirling's Approximation:

$$\lim_{n\to\infty} n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

4.3 Graph Colouring

Cliques and Independent Sets

Definition 4.14. Some more terminology:
$\Delta(G)$ = maximum degree of vertices in G
$\delta(G)$ = minimum degree of vertices in G
G is d-regular if $deg_G(v) = d \; \forall \; v \in V(G)$
(G is regular if it is d-regular for some d.)

Definition 4.15. A **clique** in a graph G is a set $X \subseteq V(G)$ such that $uv \in E(G) \; \forall \; u, v \in X, u \neq v$, i.e. X is a complete subgraph of G.
The **clique number** of G is $\mathcal{W}(G)$ = size of largest clique in G.

Definition 4.16. An **independent set** (or stable set) is a set $X \subseteq V(G)$ such that $uv \notin E(G) \; \forall u, v \in X$, i.e. X is an empty subgraph of G.
The **independence number** of G is $\alpha(G)$ = size of largest independent set in G.

Definition 4.17. A **proper vertex k-colouring** is a map $c : V(G) \to [k]$ such that if $uv \in E(G)$ then $c(u) \neq c(v)$. The **chromatic number** of G is $\mathcal{X}(G)$ = the smallest k for which a proper vertex k-colouring exists.

Example 4.2.

G	δ	Δ	$\mathcal{W}$	α	$\mathcal{X}$
P_n	1	2	2	$\lceil \frac{n}{2} \rceil$	2
C_n	2	2	3 if $n = 3$ 2 otherw.	$\lfloor \frac{n}{2} \rfloor$	2 if n even 3 if n odd
K_n	$n-1$	$n-1$	n	1	n
$K_{m,n}$	m	n	2	n	2

Definition 4.18. A graph is bipartite if $V(G)$ can be partitioned into two independent sets.

Proper vertex k-colourings partition $V(G)$ into k independent sets.
G is bipartite $\Leftrightarrow \mathcal{X}(G) \leq 2$

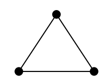
Theorem 4.16. *G is bipartite $\Leftrightarrow$ G contains no odd cycles.*

Lemma 4.17. *If H is a subgraph of G, then $\mathcal{X}(H) \leq \mathcal{X}(G)$.*

Proof. If c is a proper k-coloring of G, then $\forall u, v \in V(H)$ which are adjacent, $c(u) \neq c(v)$, so c is a proper k-colouring of H. □

Lemma 4.18. *Every closed odd walk of a graph contains an odd cycle (length = number of edges).*

Proof. by strong induction on the length of the walk r.

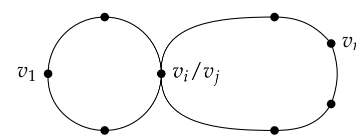


Base: $r = 3$

The only closed walk of length 3 is a cycle of length 3.

I.H. Assume true for all odd integers less than $r = 2t + 1$.

Take a close walk of length r: $v_0, v_1, \ldots, v_{r-1}, v_r = v_0$
If there are no repeated vertices (other than $v_0 = v_r$), then this is an odd cycle.



If there are repeated vertices, i.e. $\exists i, j$ such that $v_i = v_j$ and $0 \leq i < j < r$. Let $W_1 = v_0, v_1, \ldots, v_i, v_{j+1}, \ldots, v_r$ and $W_2 = v_i, v_{i+1}, \ldots, v_j$. W_1 and W_2 together form the original odd closed walk. Since $|W_1| < |W|$ and $|W_2| < |W|$ and each of the two walks is a closed walk, the induction hypothesis guarantees that whichever of W_1 or W_2 has odd length contains an odd cycle. This must be a cycle of W as well. □

Proof. of Theorem 4.16: G is bipartite $\Leftrightarrow$ G contains no odd cycles.

($\Rightarrow$) If G has an odd cycle C, then $3 = \mathcal{X}(C) \leq \mathcal{X}(G) \Rightarrow G$ is not bipartite. $\Rightarrow\Leftarrow$

($\Leftarrow$) Suppose G has no odd cycles. Let $u \in V(G)$. Now define

$$X = \{x \in V(G) \mid u \text{ and } x \text{ have even distance}\}$$
$$Y = \{y \in V(G) \mid u \text{ and } y \text{ have odd distance}\}$$

"distance" is the length of the shortest path between vertices.
Note: $u \in X$ since u is at distance 0 from itself.

We must show any two vertices in X (or in Y) are nonadjacent. Suppose not. Let $x, x' \in X$ and $xx' \in E(G)$. By construction there is an even length path from u to x say P and from u to x' say P'. Then P with xx' with P' is a closed walk of odd length in G, $\Rightarrow G$ has an odd cycle. $\Rightarrow\Leftarrow$
Similarly, if $y, y' \in Y$ and $yy' \in E(G)$, then we get a closed odd walk from the odd uy-path, yy' and the odd uy-path. $\Rightarrow\Leftarrow$ □

General colorings

LOWER BOUNDS ON $\mathcal{G}$

Recall that $\mathcal{X}(G) \geq \mathcal{X}(H)$ $\forall$ subgraph H of G. If H is the largest complete subgraph, then:

$$\mathcal{X}(G) \geq \mathcal{W}(G)$$

Since a proper k-colouring partitions $V(G)$ into independent sets,

$$\mathcal{X}(G) \geq \frac{|V(G)|}{\alpha G}$$

To see this, let $C_i = \{v \in V(G) | c(v) = i\}$ for some proper $\mathcal{X}(G)$-colouring, then:

$$|V(G)| = \sum_{i=1}^{\mathcal{X}(G)} |C_i| \leq \sum_{i=1}^{\mathcal{X}(G)} \alpha(G) = \mathcal{X}(G)\alpha(G)$$

Upper bounds on $\mathcal{X}(G)$

Greedy Colouring Algorithm $(G = (V, E))$

order $V(G)$ arbitrarily $v_1, v_2, \ldots, v_n$
colour $V(G)$ in this order assigning $c(v_i)$ the smallest colour not yet assigned to its neighbours.

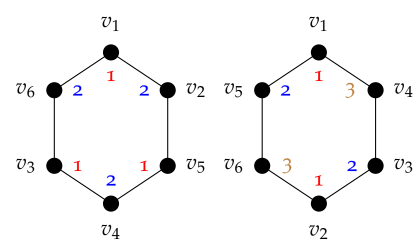


Figure 23: Left is an optimal coloring, right is non optimal. So the output depends on node ordering.

This algorithm **doesn't** guarantee the optimal colouring. At any vertex, the max number of forbidden colours is its degree. So the greedy algorithm gives: $\mathcal{X}(G) \leq \Delta(G) + 1$.

Theorem 4.19. ***Brook's Theorem*** *(1941). If G is not complete or an odd cycle, then $\mathcal{X}(G) \leq \Delta(G)$.*

Proof. Beyond the scope of this course. □

Theorem 4.20. *If G is not regular, then $\mathcal{X}(G) \leq \Delta(G)$.*

Proof. by algorithm:

BFS (Breadth First Search) Algorithm $(G = (V, E))$

Insert any vertex into a first-in-first-out (fifo) queue Q.
while $Q \neq \emptyset$
  Remove the next vertex x from Q and mark it "visited".
  Place any neighbors of x which are not yet marked "visited" into the queue (and ignore other neighbors).

Output: ordering of $V(G)$ in the order they are marked visited.

We can build a "BFS" tree, but we don't need it here.

Choose a vertex v_1 whose degree is not $\Delta(G)$: $deg(v_1) \leq \Delta(G) - 1$. (We can do this since the graph is not regular.)

Now run BFS with v_1 as the initial vertex, yielding an ordering of $V(G)$: $v_1, v_2, \ldots, v_{n-1}, v_n$.

Every v_i, $i \geq 2$ has a neighbour v_j with $j < i$ because for v_i to get placed in the queue, we must have already visited some neighbour v_j. We then colour the vertices in reverse order $v_n, \ldots, v_1$.

For $2 \leq i \leq n$, when we colour v_i, at least one of its neighbours has yet to be coloured (v_j from discussion above), which means that AT MOST $deg(v_i) - 1$ colours have been used on the neighbours of v_i while greedily colouring, $\Rightarrow$ at most $\Delta(G) - 1$ forbidden colours at every step, since $deg(v_1) \leq \Delta(G) - 1$. $\Rightarrow$ if we have $\Delta(G)$ colours, there will always be an available colour: $\therefore$ $\mathcal{X}(G) \leq \Delta(G)$ $\square$

4.4 *Planar Graphs*

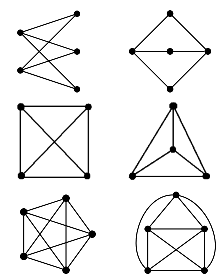


Figure 24: $K_{2,3}$ and K_4 have planar representations. K_5 seems like it does not.

Definition 4.19. A graph is **planar** if it can be drawn in the Euclidian plane so that the edges do not cross (only meet at vertices).

K_5 seems like it cannot be planar since there are vertices that cannot be joined without crossing other edges. But proofs along these lines require the Jordan Curve Theorem which will not be touched in Math 240. We will use an alternate proof.

Definition 4.20. Given a plane drawing G^* of a planar graph G, we define a **face** to be a maximal region bounded by edges.

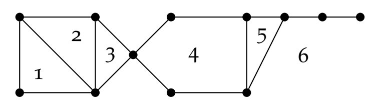


Figure 25: Example of a planar graph. Every drawing has an unbounded face called the outer face.

Theorem 4.21. ***Euler's Formula.***
If G^* is a plane drawing of a connected planar graph G with M vertices, $|E|$ edges, and $|F|$ faces, then: $|V| - |E| + |F| = 2$.

Proof. By induction on $|E|$.

Base: $|F| = 1$. If $|F| = 1$, then G^* has no cycles $\Rightarrow G$ is a tree $\Rightarrow |E| = |V| - 1$ and $|V| - |E| + |F| = |V| - (|V| - 1) + 1 = 2$

Note: The vertices of any face in a drawing can be made to be the vertices of the outer face.
Note: Each edge touches at most 2 faces. If an edge touches exactly one face, deleting it disconnects the graph.
Proofs are omitted since they use topology.

I.H.: Suppose true for $|F| = k - 1$.
Let G^* have $k \geq 2$ faces. Since G is not a tree, it has a cycle C. Let e be an edge of C. Consider $G - e$. It is still planar. It is also still connected any walk that used e can be modified by replacing e with $C - e$. Thus e touches 2 faces since its deletion doesn't disconnect G^* (or G). Therefore deleting e merges the 2 faces into 1. By induction, the new graph with $|V'|, |E'|, |F'|$ satisfies

$$\begin{aligned} |V'| - |E'| + |F'| &= 2 \\ \Rightarrow |V| - (|E| - 1) + (|F| - 1) &= 2 \\ \Rightarrow |V| - |E| + |F| &= 2 \end{aligned}$$ $\square$

This means that $|F|$ is invariant for a graph G: the number of faces does not depend on its drawing.

Theorem 4.22. *If G is a connected planar graph, with $|V| \geq 3$, then*

$$|E| \leq 3|V| - 6$$

Proof. Count the number of pairs (e, f) where e touches f. Call this number T. Since each edge touches at most 2 faces, $T \leq 2|E|$. But every face touches at least 3 edges (no self-touching edge or 2 edges with 2 vertices).

$$\begin{aligned} T &\geq 3|F| \\ 3|F| \leq T &\leq 2|E| \\ |F| &\leq \frac{2}{3}|E| \\ \Rightarrow 2 = |V| - |E| + |F| &\leq |V| - |E| + \frac{2}{3}|E| \\ \Rightarrow \frac{1}{3}|E| &\leq |V| - 2 \\ |E| &\leq 3|V| - 6 \end{aligned}$$

□

Example 4.3. Show K_5 is non-planar:

$$\begin{aligned} |V| &= 5 \\ |E| &= \binom{5}{2} = 10 \\ 10 &\not\leq 3(5) - 6 = 9 \end{aligned}$$

Note: $K_{3,3}$ is also non-planar. But it satisfies the above property.

Theorem 4.23. ***Kuratowski*** *(1930). G planar $\Leftrightarrow$ G has no K_5 or $K_{3,3}$ subdivision.*

Definition 4.21. A **minor** of G is a graph obtained by deleting vertices, deleting edges or contracting edges (Figure 26).

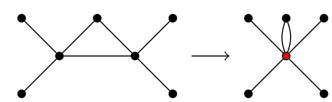


Figure 26: Edge contraction.

Theorem 4.24. ***Wagner*** *(1937). G planar $\Leftrightarrow$ G has no K_5 or $K_{3,3}$ minor.*

Proof. Beyond the scope of this course. ;-; □

Theorem 4.25. *If G is planar with no triangles $\Rightarrow |E(G)| \leq 2|V(G)| - 4$.*

Proof. Count pairs (e, f) where edge e touches face f. Let T be the number of such pairs.

$$\begin{aligned} 4|F| \leq T &\leq 2|E| \\ |F| &\leq \frac{1}{2}|E| \\ \implies 2 = |V| - |E| + |F| &\leq |V| - |E| + \frac{1}{2}|E| \\ \implies |E| &\leq 2|V| - 4 \end{aligned}$$

□

Example 4.4. $K_{3,3}$ is bipartite $\implies$ no odd cycles $\implies$ no triangles.
$K_{3,3}$ planar $\implies 9 \leq 2(6) - 4 = 8$.
Contradiction, implying $K_{3,3}$ non-planar.

Theorem 4.26. *Every planar graph is 4-colourable ($\mathcal{X}(G) \leq 4$).*

Proof. Appel and Haken, 1976. No human readable proof exists. ☐

Theorem 4.27. *Every planar graph is 5-colourable.*

Lemma 4.28. *If G is planar, then $\delta(G) \leq 5$.*

Proof. Suppose $\delta(G) \geq 6$, so $deg(v) \geq 6\ \forall v \in V(G)$.

$$\sum_{v \in V(G)} deg(v) = 2|E|$$
$$\Rightarrow 6|V| \leq 2|E|$$
$$\Rightarrow |E| \geq 3|V| \quad \Rightarrow\Leftarrow \text{ since } |E| \leq 3|V| - 6$$
☐

Proof. *of Theorem 4.27*, by induction on $|V(G)|$.

Base: Trivial if $|V(G)| \leq 5$.

I.H.: Assume true for $|V(G)| = n-1$ and let G be a planar graph with n vertices. We know $\exists v \in V(G)$ with $deg(v) \leq 5$ (by Lemma 4.28).

Case 1: $deg(v) \leq 4$. Delete v and 5-colour $V(G - v)$ by induction. At most 4 colours are used on the neighbours. of v, so there is an available colour for v.

Case 2: $deg(5) = 5$. Delete v and 5-colour $V(G - v)$ If 4 or fewer colours are used on the neighbours of v then we have a colour available for v.
If each neighbour of v got a distinct colour:
Let $N(v) = \{x_1, x_2, x_3, x_4, x_5\}$. Say $c(x_i) = i$. Let G_{13} be the subgraph of G consisting of all vertices coloured 1 or 3 and all edges of G between them.
If x_1 and x_3 are not in the same connected component of G_{13}, then we take the component of G_{13} containing x_1 and "flip" the colours of its vertices (3 to 1 and 1 to 3: Figure 27). This is still a proper colouring of $G - v$ and now we can colour v with 1.
If x_1 and x_3 are in the same connected component: then there is a path P_{13} in G_{13} from x_1 to x_3 (whose vertices are all coloured 1 or 3). $P_{13} \cup \{x_1 v,\ x_3 v\}$ form a closed curve. One of x_2 and x_4 lies inside and one lies outside. Look at the subgraph of $G - v$ with vertices coloured 2 and 4. x_2 and x_4 cannot be in the same connected component, or else there is an $x_2 x_4$-path with vertices coloured 2 and 4 which intersects $P_{13} \cup \{x_1 v,\ x_3 v\}$. So we take the component of G_{24} containing x_2 and swap colours 2 and 4. This leaves the colour 2 available for v. ☐

Proof. Alternate: see LPV p. 208-209. ☐

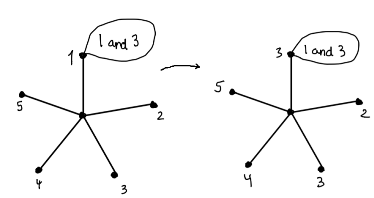


Figure 27: If x_1 and x_3 are not in the same connected component, flip all vertices coloured 1 and 3.

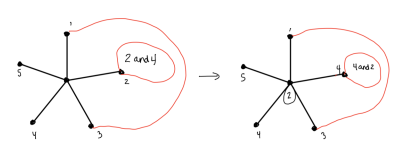


Figure 28: If x_1 and x_3 are in the same connected component, flip the colour of x_2 (and connected components).

4.5 Weighted Graphs

Definition 4.22. A **weighted graph** is a graph G and a function $w: E(G) \to \mathbb{R}$ where $w(e)$ is the weight of e.

Notation 4.29. If H is a subgraph of G, we use $w(H) = \sum_{e \in E(H)} w(e)$.

Types of problems we might wish to solve:

1. Find a uv-path P where $w(P)$ is minimum.
   c.f. Dijkstra's Algorithm.

2. Find a spanning tree T with $w(T)$ minimum.
   c.f. Kruskal's Algorithm and Prim's Algorithm.

3. Find a spanning closed walk W such that $w(W)$ is minimum.
   c.f. Traveling Salesman Problem (difficult, but one can find W such that $w(W) \leq \frac{3}{2} w(W_{optimal})$).

Dijkstra's Algorithm(G): find minimum weight uv-path.

Assign label $(-, 0)$ to u
// The elements in the tuple (a, b) are a: the preceding vertex;
// b: the weight of the minimum path from u.
if v is unlabelled
    if $\exists$ no unlabelled vertex adjacent to a labelled vertex
        return no path.
    else for each $xy \in E(G)$ s.t. x labelled (t, W) and y unlabelled
        compute $W + w(xy)$
    choose xy pair with minimum $W + w(xy)$
    label y $(x, W + w(xy))$
else (v is labelled)
    return the path iteratively obtained by taking v, the vertex from its label, the vertex from that vertex label, etc. (i.e. by backtracking)

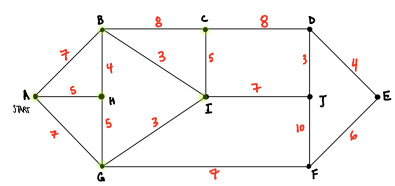


Figure 29: Minimum weight path for this graph is AGFE.

A brief intro to complexity analysis: here we have two types of steps - comparison and addition. At the kth iteration, we have labelled k vertices. Thus there are $\leq k(n-k)$ additions.

Fact: to find the smallest (or largest) element from a set of size t, we need $\leq t - 1$ comparisons. Therefore, $\leq k(n-k) - 1$ comparisons to chose xy. The worst case number of steps:

Recall:

$$\sum_{k=1}^{t} k = \frac{t(t+1)}{2}$$

$$\sum_{k=1}^{t} k^2 = \frac{t(t+1)(2t+1)}{6}$$

$$\begin{aligned}
&\sum_{k=1}^{n-1}[k(n-k)+k(n-k)-1] \\
&=\sum_{k=1}^{n-1}2nk-\sum_{k=1}^{n-1}2k^2-\sum_{k=1}^{n-1}1 \\
&=2n\frac{n(n-1)}{2}-2\frac{(n-1)(n)(2n-1)}{6}-(n-1) \\
&=\frac{1}{3}n^3-\frac{4}{3}n+1
\end{aligned}$$

KRUSKAL'S ALGORITHM(G): find minimum weight spanning tree.

```
G is a connected weight graph.
Find edge with minimum weight e_1.
k ← 1
while k < n
    if ∃e ∈ E(G) such that {e} ∪ {e_1, ··· , e_k} has no cycle
        Set e_{k+1} as the unused edge having least weight
        which does not make a cycle.
        k ← k + 1
    else return e_1, e_2, ..., e_k and exit.
```

Proof. Let T_k be the output. $T_k = \{e_1, \cdots, e_k\}$ satisfies the following:

- has no cycle: by construction.
- is spanning: if $v \notin V(T_k)$ then any edge incident to v could still be added (and the algorithm would not have stopped).
- is connected: let $v, w \in V(G)$. Since G is connected, $\exists$ vw-path in G, say P_{vw}. If P_w is in T_k, then done.
  Else, let $e \in E(P_{vw}) \setminus E(T_k)$. Adding e to T_k creates a cycle. If $e = xy$, then T_k has an xy-path, P_{xy}. Take the path and replace the edge xy with P_{xy} (contained in T_k to get a new vw-walk with fewer edges not in T_k).
  By iterating a finite number of times, we get a vw-walk in G, all of whose edges are in T_k, therefore T_k is connected.

We must now prove that the algorithm outputs the *minimum weight* spanning tree: Let $T_{opt} = MWST$ (and suppose $w(T_k) > w(T_{opt})$)[17].

[17] No need, we didn't end up using contradiction.

$T_k = e_1, e_2, \ldots, e_{n-1}$ in the order added. Let e_k be the first edge which is in T_k and T_{opt}. *Claim*: There is an edge in T_{opt}, say e, such that $T = (T_{opt} - e) + e_k$ is a spanning tree. (To be proved in an assignment.)

$$\begin{aligned} &\implies w(T) \geq w(T_{opt}) \\ \implies \cancel{w(T_{opt})} &- w(e) + w(e_k) \geq \cancel{w(T_{opt})} \\ &\implies w(e_k) \geq w(e) \end{aligned}$$

If $k = 1$, then $w(e_1)$ was edge with minimum weight: $w(e_1) = w(e)$.
If $k > 1$, by choice of k, $e_1, e_2, \ldots, e_{k-1} \in E(T_{opt})$, so the tree $\{e_1, e_2, \ldots, e_{k-1}\} \cup \{e\} \subseteq E(T_{opt}) \Rightarrow$ forms no cycles.

Since the algorithm chose e_k over e, $w(e_k) \leq w(e) \Rightarrow w(e_k) = w(e) \Rightarrow w(T) - w(T_{opt})$. And T shares more edges with T_k than T_{opt} did. Call $T = T'_{opt}$. If $T_k - T'_{opt}$. If $T_k = T'_{opt}$, done. If not, repeat argument to get T''_{opt} which has minimum weight and more edges in common with T_k than T'_{opt} did, and so on. Eventually we stop when we get to T_k, $\Rightarrow w(T_{opt}) = w(T'_{opt}) = w(T''_{opt}) = \cdots = w(T_k)$. $\square$

TRAVELING SALESPERSON PROBLEM: given n points (vertices) to visit and a weight to each pair, i.e. a weighted complete graph, find a minimum weight Hamiltonian cycle C_{opt}.

If the Triangle-Inequality is satisfied: $w(xy)_w(yz) \geq w(xz)\ \forall x,y,z$, then we can find a spanning closed walk W with $w(W) \leq 2w(C_{opt})$.

TSP ALGORITHM(G)

1 Take a MWST.
2 Make a walk by starting at a root and walking down, then up until you can walk down again, etc. Call the walk W.

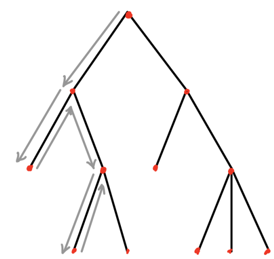


Figure 30: Example of the walk W in gray on the MWST.

Proof. As seen in Figure 30, $w(W) = 2w(T_{opt})$. Also notice that removing any edge from C_{opt} gives a spanning tree (Hamiltonian path).

$$\begin{aligned} w(W) &= 2w(T_{opt}) \\ &\leq 2w(C_{opt} - e) \\ &\leq 2w(C_{opt}) \end{aligned}$$ □

Notes

- LPV stands for the textbook by L. Lovász, J. Pelikán and K. Vesztergombi: *Discrete Mathematics: Elementary and Beyond*.

- Thanks to Suleman Malik for Figures 27 to 30.