COMP 252: Summary

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This is a condensed scribing of the lectures given by Luc Devroye in the Winter semester of 2018 for the Honours Data Structures and Algorithms class (COMP 252, McGill University). Subjects covered: Divide-and-Conquer algorithm examples, Dynamic Programming, various Abstract Data Types and implementations.

Definitions & Theorems

Introduction

Definition 1. An **algorithm** is a finite set of instructions which takes in a finite number of inputs, performs a task exactly, produces a finite output and exits. (c.f. Church-Turing Thesis.)

Definition 2. We examine two **cost models**:

1. *Uniform cost model:* every operation takes 1 time unit.
2. *Bitwise (logarithmic) cost model:* every *bit* operation takes 1 time unit (i.e. basic operations like reading, shifting, etc.).

Complexity

Definition 3. An **oracle** is a black-box type device capable of solving some computational problem in a single time unit: takes in input(s) $x_1, \cdots, x_n$ and provides output(s) $y_1, \cdots, y_k$.

Definition 4. O **- Big O**: upper bound. We say that $a_n = O(b_n)$ if $\exists c > 0$, $n_0 > 0$ such that $a_n \leq cb_n$, $\forall n \geq n_0$.

Definition 5. Ω **- Big Omega**: lower bound. We say that $a_n = \Omega(b_n)$ if $\exists c > 0, n_0 > 0$ such that $a_n \geq cb_n$, $\forall n \geq n_0$.

Lower Bounds chapter: if the lower bound for solving a problem with input of size n is f_n, it implies that $\forall$ algorithms (most efficient), $\exists$ an input $x_1, \cdots, x_n$ (worst-case) s.t. $T(\text{algorithm}, x_1, \cdots, x_n) \geq f_n$.

Definition 6. Θ **- Big Theta**: precise asymptotic behaviour. We say that $a_n = \Theta(b_n)$ if $a_n = O(b_n) = \Omega(b_n)$.

Recurrences

Theorem 7 (Master Theorem)**.** *Given a recurrence* $T_n = aT_{\lfloor n/b \rfloor} + f(n)$,

(a) $n^{\log_b a}/f(n) > n^{\epsilon} \;\; \forall n \geq n_0 \implies T_n = O(n^{\log_b a})$. [1]

(b) $f(n)/n^{\log_b a)} > n^{\epsilon} \;\; \forall n \geq n_0 \implies T_n = O(f(n))$. [2]

(c) $f(n) = \Theta(n^{\log_b a}) \implies T_n = \Theta(n^{\log_b a} \cdot \log_b n)$.

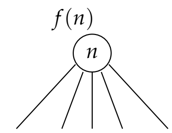
[1] If $n^{\log_b a}/f(n) = \Theta(n^c) \;\; \forall n \geq n_0$, then $T_n = \Theta(n^{\log_b a})$ instead.

[2] This case holds under the technical assumption

$$\lim_{n\to\infty} \left(\frac{f(n)}{a \cdot f(n/b)} \right) > 1.$$

Definition 8. An algorithm's **recursion tree** is an a-ary tree used to visualize its recurrence given as $T_n = a \cdot T_{n/b} + f(n)$. Calling T_n costs $f(n)$ units of "work" and results in a new problem with worst-case time $T_{n/b}$ being called "a" times. Each node in the tree represents an algorithm call, with the root node referring to the first call on a problem of size n. Given the recursion tree for some algorithm, the **time complexity** of the algorithm is the sum of the required work over all of its nodes.

The node representing a problem of time n is denoted by



where $f(n)$ is the work required, and each of the "a" lines lead to a problem of size n/b.

Lower Bounds

c.f. Devroye's printed notes for theory.

Data Structures

Definition 9. An **endogenous** data structure stores values in the structure, while an **exogenous** data structure maintains pointers to externally stored values.

1 *Divide-and-Conquer*

LECTURE 01/09

Example 1.1. Fast Exponentiation: compute x^n efficiently.

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = M^2 \begin{bmatrix} x_{n-2} \\ x_{n-3} \end{bmatrix} = \cdots = M^{n-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

Compute M^n, where $M = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$:

MATRIX(n)

if $n = 0$ **return** $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$
if $n = 1$ **return** M
else
  if n even **return** MATRIX($\frac{n}{2}$)
  if n odd **return** MATRIX($\frac{n-1}{2}$) $\cdot M$

TIME COMPLEXITY:

RAM model:

$$T_n = T_{\frac{n}{2}} + 1$$
$$\Rightarrow T_n = \Theta(\log_2 n)$$

Bit model:

$$T_n = T_{\frac{n}{2}} + n^2$$
$$\Rightarrow T_n = \Theta(n^2)$$

LECTURE 01/11 [3]

[3] Scribed: 2018.

Example 1.2. Chip Testing: determine state (G/B) of all chips, where tester oracle has two chips evaluate each other, i.e. 3 possible outputs: GG (both from $\mathscr{G}$ or both from $\mathscr{B}$), GB and BB (at least one from $\mathcal{B}$).[4]

[4] Given $|\mathscr{G}| > |\mathscr{B}|$ where:
$\mathscr{G}$: set of good chips;
$\mathscr{B}$: set of bad chips

ALGORITHM(n)

GOOD = FINDGOODCHIP(n)
do $n-1$ tests against GOOD to determine state of all chips.

FINDGOODCHIP(n)

if $n = 1$ **return** the one chip.
if n even
  pair all chips and test ($n/2$ pairs).
  eliminate all non-GG pairs.
  eliminate the second chip of each GG pair.
  FINDGOODCHIP(remaining chips)
else if n odd
  take a random chip x and test against all other chips.
  take a majority vote:
  if $\geq \frac{n-1}{2}$ G votes: $x \in \mathscr{G}$
    return x
  else $x \in \mathscr{B}$
    eliminate x
    FINDGOODCHIP(remaining chips with n even)

TIME COMPLEXITY of FINDGOODCHIP(n):

$$\begin{cases} T_n \leq \frac{3n}{2} + T_{\lfloor \frac{n}{2} \rfloor} \\ T_1 = 0,\ T_2 = 0 \end{cases} \Rightarrow T_n = \Theta(n)$$

Example 1.3. Karatsuba Multiplication: multiply two n-bit numbers a_n and b_n efficiently.

$$\begin{cases} a_n = ([\underbrace{010...}_{\alpha_2}][\underbrace{...001}_{\alpha_1}]) = \alpha_1 + \alpha_2 \times 2^{n/2} \\ b_n = ([\overbrace{000...}^{\beta_2}][\overbrace{...110}^{\beta_1}]) = \beta_1 + \beta_2 \times 2^{n/2} \end{cases}$$

$$a_n \times b_n = \alpha_1\beta_1 + \alpha_2\beta_2 + (\alpha_1\beta_2 + \alpha_2\beta_1) \times 2^{n/2}$$

where $\alpha_1\beta_2 + \alpha_2\beta_1 = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2) + \alpha_1\beta_1 + \alpha_2\beta_2$.

```
MULTIPLY(a_n, b_n)
ONE = MULTIPLY(α1, β1)
TWO = MULTIPLY(α2, β2)
THREE = MULTIPLY(α1 − α2, β1 − β2)
return ONE + TWO + (ONE + TWO + THREE) × 2^(n/2) (shifting)
```

TIME COMPLEXITY:

$$T_n = 3T_{\frac{n}{2}} + 2n + 10n \quad \Rightarrow \quad T_n = \Theta(n^{\log_2 3})$$

TOOM-COOK OPTIMIZATION:

$$T_n = 5T_{n/3} + o(n) = \Theta(n^{\log_3 5})$$

Example 1.4. Convex Hull: find C.H. in order given n points in $\mathbb{R}^2$.

```
FINDUPPERHULL(1 ··· n)
FINDUPPERHULL(1, ..., n/2).
FINDUPPERHULL(n/2 + 1, ..., n).
MERGE:
    progress by x-coordinate from left to right,
    eliminate points with angle < 180°.
```

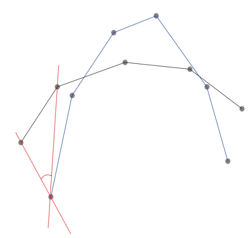


Figure 1: Diagram of the merging step. The blue and black lines are the two upper hulls.

TIME COMPLEXITY:

$$T_n = 2T_{\frac{n}{2}} + n \quad \Rightarrow \quad T_n = \Theta(n \log n)$$

LECTURE 01/16 [5]

[5] Scribed: 2018, Adam.

Example 1.5. Strassen Matrix Multiplication: matrix multiply two $n \times n$ matrices A and B.

$$A \times B = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \times \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_2 & A_1B_2 + A_2B_4 \\ \hline A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{array}\right]$$

Use algebraic trick to reduce the 8 multiplications to 7, yielding:

TIME COMPLEXITY:

$$T_n = n^2 + 7T_{n/2} \Rightarrow \; T_n = \Theta(n^{\log_2 7}) \approx \Theta(n^{2.807})$$

LECTURE 01/18 [6]

[6] Scribed: 2018, Olivia

Example 1.6. Selection Problem: finding the kth smallest number in a set $S = \{x_1, \cdots, x_n\}$.

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Example 1.7. Mergesort. Prove inductively that $T_n = \Theta(n \log n)$.
$T_n = n - 1 + T_{\lfloor \frac{n}{2} \rfloor} + T_{\lceil \frac{n}{2} \rceil}$, where n: number of comparisons.

Proof. Hypothesis $P(n)$: $\exists c > 0$ s.t. $T_n \leq cn \log_2 n$.

- **Base Case** $n = 0$: OK.
- **Inductive Step**: Assume $P(k)$ true up to $n - 1$.

n even:

$$\begin{aligned} T_n &\leq n - 1 + 2c\left(\frac{n}{2}\right)\log_2\left(\frac{n}{2}\right) \\ &= n - 1 + cn\log_2 n - cn \\ &\leq cn \log_2 n \qquad \text{for all } c \geq 1 \end{aligned}$$

n odd:

$$\begin{aligned} T_n &\leq n - 1 + c\left(\frac{n+1}{2}\right)\log_2\left(\frac{n+1}{2}\right) + c\left(\frac{n-1}{2}\right)\log_2\left(\frac{n-1}{2}\right) \\ &= n - 1 + c\frac{n}{2}\log_2\left(\frac{n^2-1}{4}\right) + \frac{1}{2}c\log_2\left(\frac{n+1}{n-1}\right) \\ &\leq n - 1 + c\frac{n}{2}\log_2\left(\frac{n^2}{4}\right) + \frac{1}{2}c\log_2(2) \\ &= n - 1 + c\frac{n}{2}(2\log_2 n - 2) + \frac{1}{2}c \\ &= n - 1 - c(n - \frac{1}{2}) + cn\log_2 n \\ &\leq cn\log_2 n \qquad \text{for all } c \geq 0 \end{aligned}$$

□

Example 1.8. Selection Problem Take II. Finding the k^{th} smallest element in a set of size n.

2 *Dynamic Programming*

Example 2.1. Binomial Coefficient.

Example 2.2. Travelling Salesman Problem.

LECTURE 01/25 [7]

[7] Scribed: 2018.

Example 2.3. LCS.

Example 2.4. Knapsack Problem.

Example 2.5. Optimal Binary Search Tree

- **Context**: Static trees are simpler versions of normal dynamic search trees: no adding, removing, etc. They are used for example in CPU for parsing programs (reading key words).
  The cost of the tree is defined as: $C = \sum_{i=1}^{n} w_i l_i$ where w_i are is weights of each key and l_i its level (distance from the root)
- **Goal**: Given a static set of n ordered keys with weights assigned to each, find the minimizing tree (minimize C).

ALGORITHM to find $C[1, n]$:

$C[i, j]$ = minimal optimal cost for problem with items $i \cdots j$.

$\implies$ Will need $W[i, j] = \sum_{k=1}^{j} w_k$ = weights of $i \cdots j$.[8]

[8] This can be done in $\Theta(n^2)$ time:

WEIGHTS(w_i)
```
for i = 1 to n
    W[i,i] = w_i
    for j = i+1 to n
        W[i,j] = W[i,j-1] + w_j
```

DYNAMIC PROGRAM

```
// Initialization
for i = 1 to n
    C[i,i] = w_i        // tree with only one key
    if j < i        // invalid tree (interval with end index > start index)
        C[i,j] = 0
// Recursive Step
for size = 2 to n        // size of interval
    for i = 1 to n        // starting index of interval
        j = i + size - 1
        if j ≤ n        // to ensure a valid interval (last index max n)
            C[i,j] = min_{i≤k≤j} (C[i,k-1] + C[k+1,j] + W[i,j])
            root[i,j] = one of the r's that minimizes C[i,j]
return C, root
```

$$
\begin{aligned}
C[i,j] &= \sum_{\substack{i=1 \\ i \neq k}}^{j} w_i(l_i + 1) + w_{root} \\
&= \sum_{\substack{i=1 \\ i \neq k}}^{j} w_i l_i + \sum_{\substack{i=1 \\ i \neq k}}^{j} w_i + w_{root} \\
&= \sum_{i=1}^{k-1} w_i l_i + \sum_{s=k+1}^{j} w_s l_s + \sum_{t=i}^{j} w_t \\
&= C[i,k-1] + C[k+1,j] + W[i,j]
\end{aligned}
$$

$\implies$ The complexity of this algorithm is $O(n^3)$ since we have $\sum_{k=2}^{n} \sum_{m=1}^{n}$ *size* computations/runs through the loop.[9]

[9] Knuth reduced this to n^2: look up and add info.

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Example 2.6. Matching Problem.

Example 2.7. Job Scheduling.

LECTURE 02/01 - LOWER BOUNDS [10]

[10] c.f. Devroye's printed notes.

3 *Data Structures*

LECTURE 02/06

3.1 *Lists*

LECTURE 02/08

3.2 *Trees*

types of trees
traversals & transforming btw storing representations
lung diagram

LECTURE 02/13

dictionaries

3.3 Red-Black Trees

LECTURE 02/20 [11]

[11] Scribed: 2018, Akshal

LECTURE 02/22 [12]

[12] c.f. CLRS

3.4 Augmenting Data Structures

Example 3.1. RB tree + Linked List.

Example 3.2. Order Statistic Tree.

Example 3.3. Interval Tree.

Example 3.4. Level Linked RB Tree (for fast browsing).

LECTURE 02/27 [13]

[13] Scribed: 2017 notes.

3.5 Cartesian Trees

Definition 10. A **cartesian tree** on n nodes (Figure 3.5) is a binary tree storing n tuples $(x_i, y_i)\ :\ (key,\ timestamp)$: we build a BST by inserting the nodes by increasing timestamps. The cartesian tree is therefore a *binary search tree* with respect to keys x_i, and a *heap* with respect to timestamps y_i.

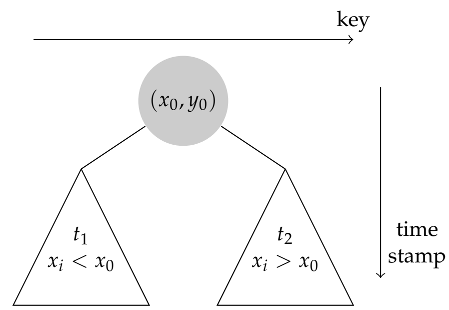

Figure 2: Diagram of a cartesian tree: (BST) keys are increasing when sweeping from left to right, and (heap) timestamps are increasing on any path from a root to a leaf.

Example 3.5. BST: inserting $x_1, \cdots, x_n$ in sequence, i.e. a cartesian tree for $(x_1, 1), (x_2, 2), \cdots, (x_n, n)$.

Example 3.6. RBST: randomly permuting all elements before adding them, i.e. a cartesian tree for $(x_1, \tau_1), (x_2, \tau_2), \cdots, (x_n, \tau_n)$ where $(\tau_1, \cdots, \tau_n)$ is a random permutation of $(1, \cdots, n)$.

Example 3.7. Treap: same concept as RBST, i.e. a cartesian tree for $(x_1, u_1), (x_2, u_2), \cdots, (x_n, u_n)$ where $u_i \in [0, 1] \in \mathbb{R}$ random.

Atomic Operations on Cartesian Trees:

i.e. FIX-ing operations: the input is (v, t), a Cartesian tree t in which only one node, v, does not satisfy the heap property.

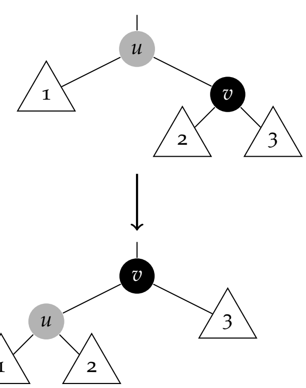

Figure 3: tree rotation done during FIX-UP and FIX-DOWN. In FIX-UP, the black node v is misplaced, it should be above the gray node u. In FIX-DOWN, the gray node is misplaced, it should be below the black one (only difference is which we are pointing to).

FIX-UP(*node*)

```
while node ≠ root && node.y < node.parent.y
    if node == node.parent.right then
        node.parent.right = node.left
        node.left = node.parent
        node.parent.left = node.right
        node.right = node.parent
```

```
FIX-DOWN(node)
while node is not leaf && (node.y > node.left.y || node.y > node.right.y)
    if node.right.y > node.left.y then
        node.leftt = node.left.right
        node.left.right = node
        node.right = node.right.left
        node.right.left = node
```

Other Operations on Cartesian Trees:

Exercise 3.8. Study how to do these (recursively) without using FIX-UP and FIX-DOWN.

1. INSERT(t,(x, y)): Inputs are a Cartesian tree t and a node (x, y) to be added to the tree. First, insert the node (x, ∞) just as you would do it in a binary search tree, implying the node will be a leaf since the heap property must be satisfied. Second, change the node to (x, y) and use FIX-UP to fix the Cartesian tree.

2. DELETE(t, *node*): Inputs are a Cartesian tree t and a pointer *node* to a node that will be removed. First, change *node.y* to ∞ and use FIX-DOWN to move the node down the tree. It will end up as a leaf so the second step is just to remove the node.

3. JOIN(t_1,t_2): Inputs are two Cartesian trees t_1 and t_2 that need to be joined into a single tree, given that all keys in t_1 are smaller than all keys in t_2. First, create a temporary node $(k, -\infty)$ such that $MAX_X(t_1) < k < MIN_X(t_2)$ and let its left child be the root of t_1 while its right child is the root of t_2. Second, delete the node and the tree will fix itself up.

4. SPLIT(t,k): Inputs are a Cartesian tree t and a value k that will split the tree into two trees that have x coordinates smaller than k and bigger than k, respectively. First, insert a temporary node $(k, -\infty)$ (it will end up as a root) and after the procedure, the left subtree and right subtree of the node are the trees we are looking for.

Expected Value Analysis for Treaps:

Define D_k to be the depth of the node with rank k, namely (k, T_k). Since the depth of a node is equivalent to the number of ancestors it has, let

$$D_k = \sum_{j \neq k} X_{jk} \text{ where } X_{jk} = \begin{cases} 1, & \text{if } j \text{ ancestor of } k \\ 0, & \text{otherwise} \end{cases}$$

So next we wish to calculate the expected value for D_k.

$$\begin{aligned}
\mathbb{E}[D_k] &= \sum_{j \neq k} \mathbb{P}((j, T_j) \text{ is an ancestor of } (i, T_i)) \\
&= \sum_{j \neq k} \mathbb{P}(T_j \text{ is the smallest of } T_j, T_{j+1}, \cdots, T_k) \\
&= \sum_{j<k} \frac{1}{k-j+1} + \sum_{j>k} \frac{1}{j-k+1} \\
&= \left(\frac{1}{2} + \cdots + \frac{1}{k}\right) + \left(\frac{1}{2} + \cdots + \frac{1}{n-k}\right) \\
&= (H_k - 1) + (H_{n-k+1} - 1) \\
&\leq 2\ln(n) \simeq 1.39 \log_2(n).
\end{aligned}$$

We will denote H_k to be the harmonic number, $\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. It can be approximated by $\int_i^k \frac{1}{x} dx \approx \ln(k)$ but more importantly, it can be bounded as $H_k \in [\ln(k+1), \ln(k)+1]$.

Example 3.9. Quicksort Tree: the QUICKSORT procedure is equivalent to building a Treap (or RBST): same number of comparisons needed.

QUICKSORT(*list*, *low*, *high*)

i = random index between *low* and *high*
a = *list*[i]
if *high* > *low* **then**
    Partition *list*[*low*...*high*] so elements before index j are smaller than a and elements after index j are greater than a
    QUICKSORT(*list*, *low*, $j-1$)
    QUICKSORT(*list*, $j+1$, *high*)

EXPECTED COMPLEXITIES FOR QUICKSORT:

1. Let C_n = # of comparisons $= \sum_{i=1}^{n} D_i$.

$$\begin{aligned}
\mathbb{E}[C_n] &= \sum_{i=1}^{n} E[D_i] \;=\; \sum_{i=1}^{n}(H_i + H_{n-i+1} - 2) \;=\; 2\sum_{i=1}^{n}(H_i - 1) \\
&= 2\sum_{i=1}^{n} H_i - 2n \;=\; 2\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{j} - 2n \;=\; 2\sum_{j=1}^{n}\frac{1}{j}\sum_{i=j}^{n} 1 - 2n \\
&= 2\sum_{j=1}^{n}\frac{1}{j}(n-j+1) - 2n \;=\; 2(n+1)H_n - 4n \\
&\simeq 2n\ln(n) \;=\; 1.386294\ldots n\log_2(n)
\end{aligned}$$

2. Operations INSERT, DELETE, JOIN and SPLIT are all $O(\log n)$.

3.6 Priority Queues

4 Strings

4.1 Information Theory

LECTURE 03/13