MATH 454

Riemann integral. $[a,b]$ closed, bded. $f:[a,b]\to\mathbb{R}$ bdded. f Riemann integrable if $\underline{\int_a^b} f = \overline{\int_a^b} f := \int_a^b f = \int_a^b f(x)\,dx$ where

$\underline{\int_a^b} f = \sup\{\sum_{i=1}^{n} (\inf_{(x_{i-1},x_i)} f)(x_i - x_{i-1}) : a = x_0 < \dots < x_n = b\}$ $L(f,P)$

$\overline{\int_a^b} f = \inf\{\sum_{i=1}^{n} (\sup_{(x_{i-1},x_i)} f)(x_i - x_{i-1}) : a = x_0 < \dots < x_n = b\}$ $U(f,P)$

Ch1: Lebesgue outer measure

$I \subseteq \mathbb{R}$ interval $\Rightarrow l(I) = \begin{cases} b-a & \text{if } I = [a,b], (a,b), [a,b), (a,b] \\ \infty & \text{unbdded} \end{cases}$

$A \subseteq \mathbb{R}$ set $\to$ outer measure $m^*(A)$

Def $m^*(A) = \inf\{\sum_{k=1}^{\infty} l(I_k) : (I_k)_{k\in\mathbb{N}}$ open, bded intervals s.t. $A \subseteq \bigcup I_k\} \in [0,\infty]$

Prop A countable $\Rightarrow m^*(A) = 0$.
Pf $\exists (x_n)_{n\in\mathbb{N}}$ st. $A = \{x_n : n\in\mathbb{N}\}$. $\varepsilon>0$. Let $I_{k\varepsilon} = (x_k - \varepsilon 2^{-k}, x_k + \varepsilon 2^{-k})$ $A \subseteq \bigcup I_{k\varepsilon}$
$\therefore m^*(A) \le \sum l(I_{k\varepsilon}) = \varepsilon \to 0$ as $\varepsilon \to 0$.

Prop $A \subseteq B \Rightarrow m^*(A) \le m^*(B)$
Pf $A \subseteq B$ $\therefore$ $B \subseteq \bigcup_{k=1}^\infty I_k \Rightarrow A \subseteq \bigcup_{k=1}^\infty I_k$ every cover of B also covers A.
$\therefore \{(I_k)$ open bded st. $B \subseteq \bigcup I_k\}$ $\therefore \inf_A \le \inf_B$
$\subseteq \{$ " " $A \subseteq \bigcup I_k\}$ $\therefore m^*(A) \le m^*(B)$

Prop $\forall I \subseteq \mathbb{R}$ interval, $m^*(I) = l(I)$
Pf ① $I = [a,b]$, $a<b \in \mathbb{R}$.
- $I \subseteq (a-\varepsilon, b+\varepsilon) \Rightarrow m^*(I) \le b-a+2\varepsilon \to l(I)$ ✓
- let $(I_k)_{k\in\mathbb{N}} = (a_k,b_k)$ st. $I \subseteq \bigcup I_k$. $[a,b]$ compact $\therefore$ finite subcover $(I_k)_{k=1}^{k_0}$ reorder finite subcover st.
$l([a,b]) = b-a = (b - b_{k_{N-1}}) + (b_{k_{N-1}} - b_{k_{N-2}}) + \dots + (b_{k_1} - a) \le \sum_{j=1}^{k_0} l(I_{kj}) \le \sum_{k=1}^\infty l(I_k)$ ✓

② $I = (a,b)$, $[a,b)$ or $(a,b]$. $[a+\varepsilon, b-\varepsilon] \subseteq I \subseteq [a,b]$
③ $I = [a,\infty)$, (a,∞). $\varepsilon>0$. $[a+\varepsilon, a+\frac{1}{\varepsilon}] \subseteq I$ $m^*($ " $) = \frac{1}{\varepsilon} - \varepsilon \to \infty$
④ $I = (-\infty,b)$, $(-\infty,b]$. $[b-\frac{1}{\varepsilon}, b-\varepsilon] \subseteq I$.
⑤ $I = \mathbb{R}$. $(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}) \subseteq I$.

Prop $\forall A \subseteq \mathbb{R}$, $y\in\mathbb{R}$, $m^*(A+y) = m^*(A)$ translat^n invariance
Pf $I_k = (a_k,b_k)$ $\{x+y : x\in A\}$
$A \subseteq \bigcup I_k \Rightarrow A+y \subseteq \bigcup I_k + y = \bigcup(I_k+y) = \bigcup(a_k+y, b_k+y)$
$m^*(A+y) \le \sum l((a_k+y, b_k+y)) = \sum l((a_k,b_k))$
$\overset{\inf}{\Rightarrow} m^*(A+y) \le m^*(A)$
$A = (A+y)-y \Rightarrow m^*(A) \le m^*(A+y)$

Prop $\forall (A_k)_{k\in\mathbb{N}}, A_k \subseteq \mathbb{R} \Rightarrow m^*(\bigcup_{k=1}^\infty A_k) \le \sum_{k=1}^\infty m^*(A_k)$
countable subadditivity
Pf • if $\exists k_0 \in \mathbb{N}$ st $m^*(A_{k_0}) = \infty \Rightarrow \sum m^*(A_k) = \infty$.
• $\forall k\in\mathbb{N}$, $m^*(A_k) < \infty$ then $\forall \varepsilon>0$ $\exists (I_{kj})_{j\in\mathbb{N}}$ open bded st. $m^*(A_k) > \sum_{j=1}^\infty l(I_{kj}) - \varepsilon 2^{-k}$
$\bigcup_{k=1}^\infty A_k \subseteq \bigcup_{k=1}^\infty \bigcup_{j=1}^\infty I_{kj} \Rightarrow m^*(\bigcup A_k) \le \sum_{k=1}^\infty \underbrace{\sum_{j=1}^\infty l(I_{kj})}_{m^*(A_k)+\varepsilon 2^{-k}}$
$\xrightarrow{\varepsilon\to 0} m^*(\bigcup A_k) \le \sum_{k=1}^\infty m^*(A_k)$

Def A mbl iff $\boxed{\forall B \subseteq \mathbb{R}}$ $\boxed{m^*(B) = m^*(B\cap A) + m^*(B\setminus A)}$
then $m^*(A) \equiv m(A)$ Lebesgue measure.
$\therefore A$ mbl iff $\forall B \subseteq \mathbb{R}$, $m^*(B)<\infty$ $m^*(B) \ge m^*(B\cap A) + m^*(B\setminus A)$

Prop $m^*(A) = 0 \Rightarrow A$ mbl
Pf $m^*(B\cap A) = 0$, $m^*(B\setminus A) \le m^*(B)$ ✓

Prop $A \subseteq \mathbb{R}$ mbl $\Rightarrow \mathbb{R}\setminus A$ mbl.

Prop $(A_k)_{k\in\mathbb{N}}$ mbl $\Rightarrow \bigcup_{k=1}^\infty A_k$ and $\bigcap_{k=1}^\infty A_k$ mbl and countable additivity: if $A_i \cap A_j = \emptyset$ $\forall i\neq j$, then $m(\bigcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty m(A_k)$.
$\Rightarrow$ excision prop if $A_2 \subseteq A_1$, $m(A_2)<\infty$ then $m(A_1\setminus A_2) = m(A_1) - m(A_2)$
Pf: ① finite prove for $n=2$, induct $\Rightarrow n\ge 3$.
② countable. def $A'_1 = A_1$, $A'_n = A_n \setminus (\bigcup_{k=1}^{n-1} A_k)$
$\bigcup_{k=1}^\infty A'_k = \bigcup_{k=1}^\infty A_k$. $A'_i \cap A'_j = \emptyset$ $\forall i\ne j$.
$\Rightarrow m^*(B) = m^*(B \cap (\bigcup_{k=1}^\infty A_k)) + m^*(B \setminus \bigcup_{k=1}^\infty A_k)$

Prop continuity of measure. $(A_k)_{k\in\mathbb{N}}$ mbl.
(i) if $A_k \subseteq A_{k+1}$ $\forall k\in\mathbb{N} \Rightarrow m(\bigcup_{k=1}^\infty A_k) = \lim_{k\to\infty} m(A_k)$
(ii) $A_k \supseteq A_{k+1}$, $m(A_1)<\infty \Rightarrow m(\bigcap_{k=1}^\infty A_k) = \lim_{k\to\infty} m(A_k)$
Pf (i) if $\exists k_0\in\mathbb{N}$ $m(A_{k_0}) = \infty \Rightarrow$ LHS = RHS = ∞.
otw: Def $A'_k = A_k \setminus A_{k-1}$ $\forall k\ge 2$, $A'_1 = A_1$. disjoint.
$m(\bigcup_{k=1}^\infty A_k) = m(\bigcup A'_k) = \sum A'_k = \sum_{k=2}^\infty (m(A_k) - m(A_{k-1})) + m(A_1)$
$= \lim_{n\to\infty} [\sum_{k=2}^n m(A_k) - m(A_{k-1})] + m(A_1) = \lim_{n\to\infty} m(A_n)$.
(ii) $A_k \subseteq A_1$ $\bigcap_{k=1}^\infty A_k = A_1 \setminus (\bigcup_{k=1}^\infty (A_1\setminus A_k))$
$A_k \supseteq A_{k+1} \Rightarrow A_1\setminus A_k \subseteq A_1 \setminus A_{k+1}$ increasing complement
$\therefore m(\bigcup_{k=1}^\infty (A_1\setminus A_k)) = \lim_{k\to\infty} m(A_1\setminus A_k) = \lim_{k\to\infty}(m(A_1) - m(A_k))$
$m(\bigcap A_k) = m(A_1) - m(\bigcup(A_1\setminus A_k)) = \lim_{n\to\infty} m(A_n)$

Prop translation invariance $A \subseteq \mathbb{R}$ mbl, $y\in\mathbb{R}$ $\Rightarrow A+y$ mbl.
Pf use translat^n inv of m^*.

Prop (i) $\forall$ interval of $\mathbb{R}$ is mbl
(ii) $\forall$ open set & closed set is mbl.
Pf let $B \subseteq \mathbb{R}$, $m^*(B)<\infty$. $\forall \varepsilon>0$ $(I_k)_{k\in\mathbb{N}}$ open bded intervals st. $B \subseteq \bigcup I_k$, $\sum l(I_k) < m^*(B)+\varepsilon$.
$\Rightarrow m^*(B\cap I) \le m^*(\bigcup_{k=1}^\infty (I_k \cap I))$, $m^*(B\setminus I) \le \sum m^*(I_k\setminus I)$
$m^*(B\cap I) + m^*(B\setminus I) \le \sum m^*(l(I_k\cap I) + l(I_k\setminus I)) < m^*(B)+\varepsilon$ $\xrightarrow{\varepsilon\to 0} m^*(B)$

Outer & Inner Approx

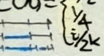
Thm $A \subseteq \mathbb{R}$. then TFAE (i) A mbl
(ii) $\forall\varepsilon>0$ $\exists O_\varepsilon \subseteq \mathbb{R}$ open st. $A \subseteq O_\varepsilon$, $m^*(O_\varepsilon\setminus A) < \varepsilon$
(iii) $\exists (O_n)_{n\in\mathbb{N}}$ open st. $A \subseteq G$, $m^*(G\setminus A) = 0$, $G = \bigcap_{n=1}^\infty O_n$ G_δ
(iv) $\forall\varepsilon>0$ $\exists F_\varepsilon \subseteq \mathbb{R}$ closed st. $A \supseteq F_\varepsilon$, $m^*(A\setminus F_\varepsilon) < \varepsilon$
(v) $\exists (F_n)_{n\in\mathbb{N}}$ closed st. $A \supseteq F$, $m^*(A\setminus F) = 0$, $F = \bigcup_{n=1}^\infty F_n$
Pf (i)⇒(ii) case $m(A)<\infty$ $\sum l(I_k) < m(A)+\varepsilon$.
$m(A) = \infty$: $A = \bigcup A_k$, $A_k = A \cap [k,k+1)$.

Thm $\forall A \subseteq \mathbb{R}$, if $m^*(A)>0$ then $\exists B \subseteq A$ non-mbl.
Pf $A = \bigcup_{k\in\mathbb{Z}} A_k$, $A_k = A \cap [k,k+1)$. $\exists k_0$ st. $m^*(A_{k_0})>0$.
② $\forall x\in A_{k_0}$ def $S_x = (x+\mathbb{Q}) \cap A_{k_0}$, $\Omega = \{S_x : x\in A_{k_0}\}$
axiom of choice: $\exists f:\Omega\to\bigcup S$ st. $\forall S\in\Omega$, $f(S)\in S$.
↪ can pick an elem^t from each S_x.
③ $A_{k_0} \subseteq \bigcup_{q\in\mathbb{Q}\cap[-1,1]} (f(\Omega)+q)$, $\{f(\Omega)+q\}_{q\in\mathbb{Q}\cap[-1,1]}$ disjoint.
④ assume by contradiction $f(\Omega)$ mbl then show $m^*(A_{k_0}) = 0$ ✗. $m(\bigcup f(\Omega)+q) = 0$ or ∞ but ≤ 3 $\therefore = 0$ ✗.

Def (Cantor set) $C = \bigcap_{k=1}^\infty C_k$ $C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$ and $\forall k\ge 2$ $C_k = \bigcup_{j=1}^{2^{k-1}} I_{kj}$, $j\in\{1,2,\dots,2^{k-1}\}$. $I_{k,2j-1}$ and I_{k2j} are the 1st and final 3rd of $I_{k-1,j}$.
Thm C closed, uncountable and $m^*(C) = 0$.

Def a collect^n of sets $\mathcal{C}$ is a σ-algebra if (i) $\mathbb{R}\in\mathcal{C}$, (ii) $C_1,C_2\in\mathcal{C} \Rightarrow C_2\setminus C_1\in\mathcal{C}$ (iii) $\forall (C_k)\subseteq\mathcal{C} \Rightarrow \bigcup_{k=1}^\infty C_k \in \mathcal{C}$.
Prop any intersect of σ-algs is a σ-alg.
Def Borel set: set in intersect of all σ-algs which contain the open sets. ↪ Borel σ-alg.
Prop $\exists$ subset D of C (cantor) that is not Borel.

Def (Cantor-Lebesgue funct^n / staircase) $\mathcal{C}: [0,1]\to[0,1]$
$\mathcal{C}(x) = \begin{cases} \frac{1}{2} & \text{if } x\in(\frac{1}{3},\frac{2}{3}) = [0,1]\setminus C_1 \\ \frac{1}{4} \text{ if } x\in(\frac{1}{9},\frac{2}{9}), & \frac{3}{4} \text{ if } x\in(\frac{7}{9},\frac{8}{9}) \quad [0,1]\setminus C_2 \\ \frac{i}{2^k} & \text{if } x\in J_{ki}: i^{th} \text{ interval in } [0,1]\setminus C_k \ i\in\{1,\dots,2^k-1\} \end{cases}$



$\mathcal{C}(0) = 0$, $\forall y\in C\setminus\{0\}$, $\mathcal{C}(y) = \sup\{\mathcal{C}(x) : x\in[0,y)\setminus C\} \neq \emptyset$
$\Rightarrow \mathcal{C}(1) = 1$, since $\frac{i}{2^k} < 1$ $\forall k\in\mathbb{N}$, $i\in\{1,\dots,2^k-1\}$, & $\frac{2^k-1}{2^k}\to 1$.
Prop $\mathcal{C}$ is increasing and cts.
Pf (for $\exists$ of $D\subseteq C$): define $\psi(x) = \mathcal{C}(x)+x$ $\psi:[0,1]\to[0,2]$
strictly ↑, cts, bijective since {strict monot ⇒ injective; interm. value ⇒ surject.
Note if f strictly ↑ and def'd on an interval, f^{-1} cts
$\therefore \psi^{-1}$ cts $\therefore C$ closed $\Rightarrow (\psi^{-1})^{-1}(C) = \psi(C)$ closed $\therefore$ mbl.
$\Rightarrow m(\psi(C)) = 1 > 0$ $\therefore \exists$ subset $E \subseteq \psi(C)$ non-mbl.
$\therefore$ let $D = \psi^{-1}(E) \subseteq C$ $\therefore m^*(D) = 0$ mbl.
but D can't be Borel since ψ^{-1} cts and $\boxed{\text{cts funct}^s \text{ map Borel to Borel}}$

Ch2: Lebesgue mbl funct

Prop $A\subseteq\mathbb{R}$ mbl, $f: A\to\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,\infty\}$. then TFAE:
(i) $\forall c\in\mathbb{R}$ $f^{-1}((c,+\infty])$ mbl (iii) $\forall c\in\mathbb{R}$ $f^{-1}([-\infty,c))$ mbl
(ii) " $f^{-1}([c,+\infty])$ mbl (iv) " $f^{-1}([-\infty,c])$ mbl.
$\Rightarrow$ **Def** f mbl if one of (i)–(iv) satisfied. (aka all).
Pf (prop) (i)⇒(ii) $f^{-1}([c,+\infty]) = f^{-1}(\bigcap_{n=1}^\infty (c-\frac{1}{n},+\infty]) = \bigcap_{n=1}^\infty f^{-1}((c-\frac{1}{n},+\infty])$ mbl.
(ii)⇒(iii) $f^{-1}([-\infty,c)) = f^{-1}(\overline{\mathbb{R}}\setminus[c,+\infty]) = A\setminus f^{-1}([c,+\infty])$ mbl.
(iii)⇒(iv) $f^{-1}([-\infty,c]) = \bigcap_{n=1}^\infty f^{-1}([-\infty,c+\frac{1}{n}))$ mbl
(iv)⇒(i) $f^{-1}((c,\infty]) = A\setminus f^{-1}([-\infty,c])$.

Prop $A\subseteq\mathbb{R}$ mbl, $f:A\to\overline{\mathbb{R}}$
(i) if f mbl then $\forall B\subseteq\mathbb{R}$ Borel $\Rightarrow f^{-1}(B)$ mbl.
(ii) if f finite: $f(A)\subseteq\mathbb{R}$: f mbl $\Leftrightarrow \forall B\subseteq\mathbb{R}$ Borel, $f^{-1}(B)$ mbl.
Pf (i) def $\mathcal{C} = \{B\subseteq\mathbb{R} : f^{-1}(B)$ mbl$\}$, show $\mathcal{C}$: σ-alg. and contains the opens.
(ii) $f^{-1}((c,+\infty]) = f^{-1}((c,\infty))$, Borel. for ⇐ show f mbl.

Rem△ $\exists D$ mbl, f mbl (cts!) st. $f^{-1}(D)$ non mbl.
e.g. $D\subseteq C$ non-Borel, $E = \psi(D)\subseteq\psi(C)$ non-mbl. $\psi^{-1} = f$ is cts. $f^{-1}(D) = \psi(D) = E$ non-mbl.

Def $A\subseteq\mathbb{R}$ mbl, $P(x)$ statement depending on $x\in A$. $P(x)$ true a.e. in A: $m(\{x\in A : P(x)$ false$\}) = 0$
Rem△ $(P_n(x))_{n\in\mathbb{N}}$. $\forall n\in\mathbb{N}$, for a.e. $x\in A$, $P_n(x)$ true $\Leftrightarrow$ for a.e. $x\in A$, $\forall n\in\mathbb{N}$, $P_n(x)$ true
Prop $f:A\to\overline{\mathbb{R}}$ mbl, $g:A\to\overline{\mathbb{R}}$ st. $f=g$ a.e. in A $\Rightarrow g$ mbl.
Prop $(A_n)_{n\in\mathbb{N}}$, $A = \bigcup_{n=1}^\infty A_n$. $f_n: A_n\to\overline{\mathbb{R}}$ mbl $\therefore f: x\mapsto f_n(x)$ if $x\in A_n$ mbl.

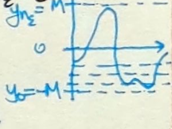
Def (characteristic funct^n) of B: $\chi_B: \begin{cases} A\to\mathbb{R} \\ x\mapsto \begin{cases} 1 & \text{if } x\in B \\ 0 & \text{otw.} \end{cases}\end{cases}$
Def (simple function) $f:A\to\mathbb{R}$ st. $f(A)$ finite: $\exists a_1<a_2<\dots<a_N\in\mathbb{R}$ st. $f(A) = \{a_1,\dots,a_N\}$.
$\therefore$ canonical rep: $\boxed{f = \sum_{k=1}^N a_k\chi_{A_k}}$ where $A_k = f^{-1}(\{a_k\})$.

Prop $A\subseteq\mathbb{R}$ mbl.
(i) $\forall B\subseteq A$ mbl, $f:A\to\overline{\mathbb{R}}$ mbl, $f|_B$ mbl. $A\xrightarrow{g}B\xrightarrow{f}\mathbb{R}$
(ii) $\forall B\subseteq\mathbb{R}$ Borel, $f:B\to\mathbb{R}$ cts, $g:A\to B$ mbl $\Rightarrow f\circ g$ mbl.
(iii) $\forall f:A\to\overline{\mathbb{R}}$, $g:A\to\mathbb{R}$ mbl $\Rightarrow f+g$ mbl (not ∞ to avoid $\infty-\infty$)
(iv) $\forall f,g: A\to\mathbb{R}$, $f\cdot g$ mbl
(v) $\forall f_1,\dots,f_n: A\to\overline{\mathbb{R}}$ mbl, $\max(f_1,\dots,f_n)$, $\min(f_1,\dots,f_n)$ mbl.
Rem△ $\exists f,g$ mbl s.t. $f\circ g$ not mbl: $D\subseteq C$ non-borel and mbl. $E = \psi(D)$ non-mbl. χ_D mbl, ψ^{-1} cts $\therefore$ mbl but $((\chi_D\circ\psi^{-1})^{-1}(\frac{1}{2},\infty] = \psi(\underbrace{\chi_D^{-1}(\frac{1}{2},\infty]}_{D}) = E$ non-mbl.
$\therefore \chi_D\circ\psi^{-1}$ non-mbl.
Pf: (iii) $\forall c\in\mathbb{R}$, $x\in A$ $(f+g)(x)<c \Leftrightarrow f(x)<c-g(x)$
$\Leftrightarrow \exists q\in\mathbb{Q}$ $f(x)<q<c-g(x)$ $\therefore (f+g)^{-1}([-\infty,c)) = \bigcup_{q\in\mathbb{Q}} f^{-1}([-\infty,q))\cap g^{-1}([-\infty,c-q))$
(iv) $f\cdot g = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$ all mbl.
(v) $(\max(f_1,\dots,f_n))^{-1}([-\infty,c)) = \bigcap_{k=1}^n f_k^{-1}([-\infty,c))$ $(\min(f_i))^{-1}([-\infty,c)) = \bigcup_{k=1}^n f_k^{-1}([-\infty,c))$

Def $(f_n)_{n\in\mathbb{N}}$ $f_n: A\to\overline{\mathbb{R}}$, $f:A\to\overline{\mathbb{R}}$.
- converges ptwise: $\forall x\in A$, $\lim_{n\to\infty} f_n(x) = f(x)$
- uniformly: $|f|<\infty$ in A and $\lim_{n\to\infty}\sup_A |f_n - f| = 0$.

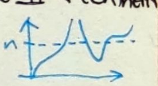
Prop $f_n\to f$ ptwise a.e. in A; $f_n: A\to\overline{\mathbb{R}}$, $f:A\to\overline{\mathbb{R}}$ then f mbl.
Pf $N = \{x\in A : \lim_{n\to\infty} f(x) \ne f(x)\}$.
$\forall c\in\mathbb{R}$, $f^{-1}([-\infty,c]) = \underbrace{(f^{-1}([-\infty,c])\cap N)}_{\subseteq N \therefore \text{ mbl}} \cup \overbrace{(f^{-1}([-\infty,c])\cap(A\setminus N))}^{B}$
$B = \bigcap_{k\in\mathbb{N}}\bigcup_{N=1}^\infty\bigcap_{n=N}^\infty f_n^{-1}([-\infty,c+\frac{1}{k}])$ ↳ mbl. $\lim_{n\to\infty} f_n(x)\le c \Leftrightarrow \forall k\in\mathbb{N}$ $\exists N\in\mathbb{N}$ st. $\forall n\ge N$, $f_n(x)\le c+\frac{1}{k}$.

Simple Approx Lemma $f:A\to\mathbb{R}$ mbl, bded ($|f|\le M$) then $\forall\varepsilon>0$ $\exists \varphi_\varepsilon,\psi_\varepsilon: A\to\mathbb{R}$ simple s.t. $\varphi_\varepsilon\le f\le\psi_\varepsilon<\varphi_\varepsilon+\varepsilon$ and $\varphi_\varepsilon,\psi_\varepsilon\to f$ uniformly in A.



Pf choose $n_0\in\mathbb{N}$ st. $\frac{2M}{n_\varepsilon}<\varepsilon$. $y_k = M(\frac{2M}{n_\varepsilon}-1)$ st
$-M = y_0 < y_1 < \dots < y_{n_\varepsilon} = M$, $y_k = y_{k-1} + \frac{2M}{n_\varepsilon}$ $\forall k\in\{1,\dots,n_\varepsilon\}$.
Let $A_k = f^{-1}([y_{k-1}, y_k))$ $\therefore A = \bigcup A_k$, $A_k\cap A_{k'} = \emptyset$, $k\ne k'$.
$= f^{-1}((-\infty,y_k))\cap f^{-1}([y_{k-1},\infty))$ mbl.
Def $\varphi_\varepsilon = \sum_{k=1}^{n_\varepsilon} y_{k-1}\chi_{A_k}$, $\psi_\varepsilon = \sum_{k=1}^{n_\varepsilon} y_k\chi_{A_k} = \varphi_\varepsilon + \frac{2M}{n_\varepsilon} < \varphi_\varepsilon + \varepsilon$.

Simple Approx Thm $f:A\to\overline{\mathbb{R}}$ on $A\subseteq\mathbb{R}$. f mbl iff $\exists (\varphi_n)_{n\in\mathbb{N}}$ simple st. (i) $(\varphi_n)_{n\in\mathbb{N}}\to f$ ptwise in A
(ii) $|\varphi_n|\le|f|$ in A $\forall n\in\mathbb{N}$.



Pf (⇒) ① case $f\ge 0$ def $f_n = \min(f,n)$ $\forall n\in\mathbb{N}$.
$\Rightarrow \exists \varphi_n \le f_n \le \varphi_n + \varepsilon$, $\forall\varepsilon>0$ in A.
def $\tilde{\varphi}_n = \max(0,\varphi_1,\dots,\varphi_n)$ st. $\tilde{\varphi}_n\ge 0$, $\tilde{\varphi}_{n+1}\ge\tilde{\varphi}_n$ $\forall n\in\mathbb{N}$ $\le f_n$ $\therefore$ (ii) ✓
& for (i):
* $f(x)<\infty$: $\forall n\ge f(x)$, $f_n(x) = f(x)$ $\therefore 0\le f(x)-\varphi_n(x)<\frac{1}{n}$ $\therefore \lim_{n\to\infty}\varphi_n(x) = f(x)$
* $f(x) = \infty$: $\forall n\in\mathbb{N}$, $f_n(x) = n \Rightarrow \varphi_n(x) > n-\frac{1}{n}\to\infty$ ✓
② general $f = f_+ - f_-$: $f_+ = \max(f,0)$, $f_- = \max(-f,0)$
$\Rightarrow \exists \varphi_{n,\pm}\to f_\pm$, $0\le\varphi_{n,\pm}\le f_\pm \Rightarrow (\varphi_{n+} - \varphi_{n-})\to f_+ - f_- = f$
(⇐) simple ⇒ mbl and $|\varphi_n|\le\varphi_{n+}+\varphi_{n-}\le f_+ + f_- = |f|$.
ptwise lim of mbl ⇒ mbl.

Egoroff thm $A\subseteq\mathbb{R}$, $m(A)<\infty$. $(f_n)_{n\in\mathbb{N}}: A\to\mathbb{R}$ mbl, conv ptwise to $f:A\to\mathbb{R}$. then $\forall\varepsilon>0$ $\exists F_\varepsilon\subseteq A$ closed s.t.
(i) $f_n\to f$ uniformly on F_ε (ii) $m(A\setminus F_\varepsilon)<\varepsilon$.

Lush Thm $f:A\to\mathbb{R}$ mbl. then $\forall\varepsilon>0$, $\exists F_\varepsilon\subseteq A$ closed st.
(i) f cts on F_ε (ii) $m(A\setminus F_\varepsilon)<\varepsilon$.

Ch3: Lebesgue Integral

→ simple funct^s: $\psi: A\to\mathbb{R}$ simple, $m(A)<\infty$, $\psi = \sum_{k=1}^N a_k\chi_{A_k}$
↳ integral $\int_A\psi = \int_A\psi(x)\,dx = \sum_{k=1}^N a_k m(A_k)$
↳ $\forall B\subseteq A$ mbl, $\int_B\psi = \int_B \psi|_B$.
→ mbl and bded functions $f:A\to\mathbb{R}$, $A\subseteq\mathbb{R}$ mbl, $m(A)<\infty$. f Lebesgue integrable ⇔
$\underline{\int_A} f = \overline{\int_A} f := \int_A f$, where $\begin{cases}\underline{\int_A} f = \sup\{\int_A\varphi : \varphi \text{ simple}, \varphi\le f \text{ on } A\} \\ \overline{\int_A} f = \inf\{\int_A\varphi : \varphi \text{ simple}, \varphi\ge f \dots\}\end{cases}$ $\neq\emptyset$ since f bded.

Prop $f:[a,b]\to\mathbb{R}$ Riemann integrable ⇒ f Lebesgue integrable.
Thm $f:A\to\mathbb{R}$ mbl and bded on A, $m(A)<\infty$ ⇒ f integrable
Pf: Simple approx lemma: $\exists(\varphi_n)_{n\in\mathbb{N}}$ st. $\varphi_n\le f\le\varphi_n+\frac{1}{n}$ $\forall n\in\mathbb{N}$
$\therefore \int_A\varphi_n\le\underline{\int_A} f\le\overline{\int_A} f\le\int_A(\varphi_n+\frac{1}{n})$ $\varphi_n = \sum_{k=1}^{k_n} a_{kn}\chi_{A_{kn}} \Rightarrow \varphi_n + \frac{1}{n} = \sum_{k=1}^{k_n}(a_{kn}+\frac{1}{n})\chi_{A_{kn}}$
$\Rightarrow \int_A \varphi_n + \frac{1}{n} = \sum(a_{kn}+\frac{1}{n})m(A_{kn}) = \int_A\varphi_n + \underbrace{\frac{1}{n}m(A)}_{\to 0 \text{ as } n\to\infty}$

Prop $f,g: A\to\mathbb{R}$ mbl and bded, then
(i) $\forall\alpha,\beta\in\mathbb{R}$, $\alpha f+\beta g$ mbl bded. $\int_A\alpha f+\beta g = \alpha\int_A f+\beta\int_A g$
(ii) if $f\le g$ on A ⇒ $\int_A f\le\int_A g$
(iii) $|f|$ mbl and bded, and $|\int_A f|\le\int_A|f|$
(iv) $\forall B\subseteq A$ mbl, $f\chi_B$ mbl bded, and $\int_A f\chi_B = \int_B f$
(v) $\forall A_1,A_2\subseteq A$ mbl disjoint, $\int_{A_1\cup A_2} f = \int_{A_1} f + \int_{A_2} f$. if $m(A_2) = 0$ then $\int_{A_1\cup A_2} f = \int_{A_1} f$.
Pf: simple funct^s ⇒ bded mbl funcs.

Lem $n\in\mathbb{N}$, $a_1,\dots,a_n\in\mathbb{R}$; $A_1,\dots,A_n\subseteq A$, $m(A)<\infty$ [mbl disjoint] then $\int\sum a_k\chi_{A_k} = \sum\int a_k\chi_{A_k} = \sum a_k m(A_k)$

Thm (Bounded Convergence) Let $A \subseteq \mathbb{R}$ mbl, $m(A) < \infty$

Let $(f_n)_{n \in \mathbb{N}} : A \to \mathbb{R}$ mbl on A s.t.

(i) $\exists M > 0$ s.t. $\forall n \in \mathbb{N}$, $|f_n| \leq M$ on A

(ii) $\exists f : A \to \mathbb{R}$ s.t. $\forall x \in A$, $\lim_{n \to \infty} f_n(x) = f(x)$

then: f bded mbl, $\lim_{n \to \infty} \int_A f_n = \int_A \lim_{n \to \infty} f_n = \int_A f$

Pf: f_n mbl $\therefore$ $f = \lim_{n \to \infty} f_n$ mbl and $|f| \leq M$.

by Egoroff $\forall \varepsilon > 0$ $\exists F_\varepsilon \subseteq A$ mbl s.t. $f_n \to f$ uniformly on F_ε, $m(A \setminus F_\varepsilon) < \varepsilon$: $\left|\int_A f_n - \int_A f\right| = \left|\int_A f_n - f\right| \leq \int_A |f_n - f| = \int_{A \setminus F_\varepsilon} |f_n - f| + \int_{F_\varepsilon} |f_n - f|$

bound both by $\sup|f_n - f|$: $\to 0$ on F_ε, $\leq 2M$ on $A \setminus F_\varepsilon$.

$\Rightarrow \limsup_{n \to \infty} \left|\int_A f_n - \int_A f\right| \leq 2M\varepsilon \to 0$ as $\varepsilon \to 0 \Rightarrow \lim_{n \to \infty} \int_A f_n = \int_A f$