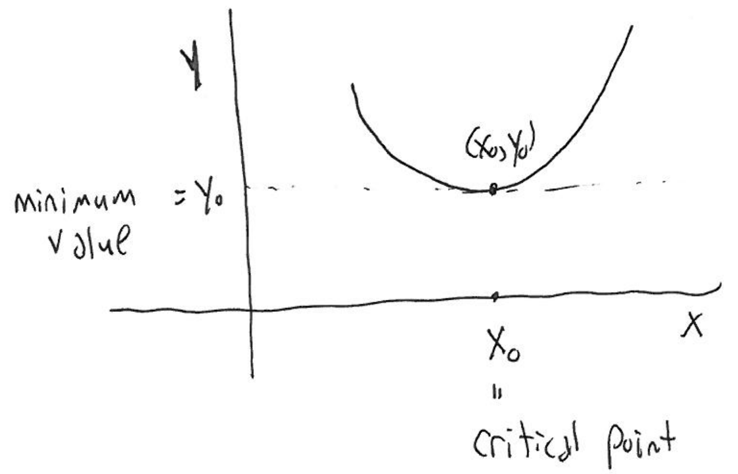
Curve Sketching

- Goal: Qualitatively draw the graph of $f(x)$ using knowledge of $f'(x)$ and $f''(x)$.

- Typical picture of a minimum:



Note: $f'(x_0) = 0$

- $f'(x) < 0$ when $x < x_0$; f is decreasing to left of x_0
- $f'(x) > 0$ when $x > x_0$; f is increasing to right of x_0

- Curve sketching recipe
  - Plot discontinuities (and watch out for $\pm\infty$)
  - Find the critical pts, i.e., the pts. x with $f'(x) = 0$.
  - (a) Plot critical points (+ critical values if possible)
    (b) Decide on the sign in between critical points
  - Plot the pts. x with $f(x) = 0$ (i.e., the zeros of f)
  - Determine how f behaves at the endpoints (which may be $x = \pm\infty$).

Ex: $y = 3x - x^3 = f(x)$

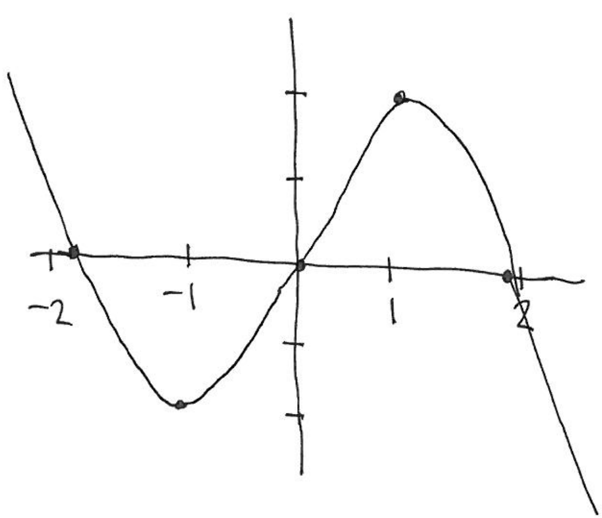
- No discontinuities
- $y' = 3 - 3x^2 = 3(1 - x^2)$. Thus, $y' = 0$ when $x = \pm 1$.
- (a) $f(1) = 2$, $f(-1) = -2$. Plot these points.

  (b) f' is $-$ $x < -1$

  $+$ $-1 < x < 1$

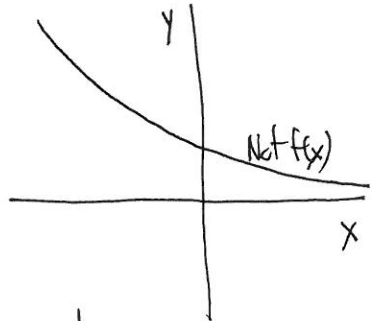
  $-$ $x > 1$
- Zeros of f: $3x - x^3 = x(3 - x^2) = 0$. $x = 0, \pm\sqrt{3}$
- As $x \to \infty$, $y \to -\infty$

  $x \to -\infty$, $y \to \infty$.



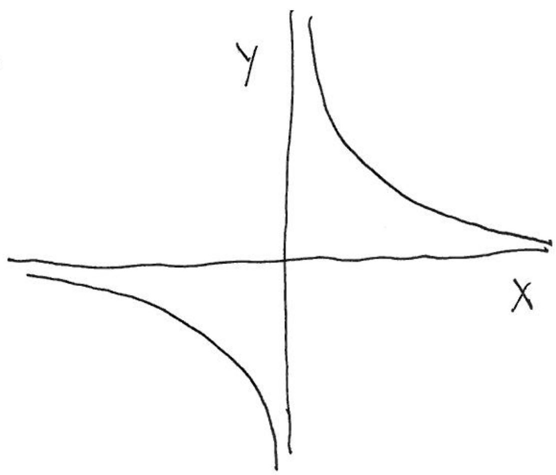
Ex: $y = \frac{1}{x} = f(x)$ $y' = \frac{-1}{x^2}$

Rmk: Even though $y' < 0$, the graph of f is not



• Reason: discontinuity at $x=0$

Correct Picture:



Ex: • $y = x^3 - 3x^2 + 3x$

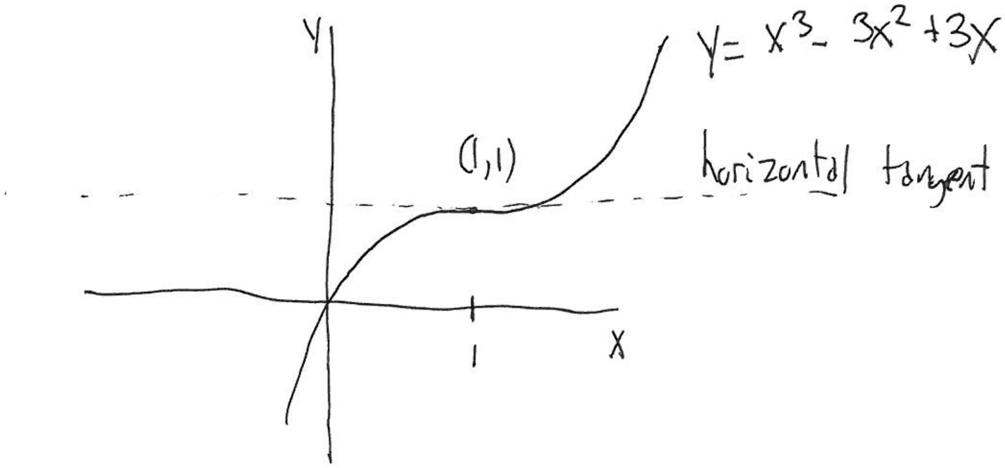
• $y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$.

• Critical point: $y' = 0$ when $x = 1$.

y' is $+$ when $x < 1$

$+$ when $x > 1$

$\Rightarrow$ y is always increasing



Ex: $y = \frac{\ln x}{x} = f(x)$ $x > 0$ only.

- what happens as $x \to 0^+$?

  Set $x = 2^{-n}$. Then $y = \frac{\ln 2^{-n}}{2^{-n}} = -n \ln 2 \cdot 2^n$

  $\to -\infty$

  as $n \to \infty$.

  Thus, $y \to -\infty$ as $x \to 0^+$

- Critical pts.: $y' = \frac{x \overbrace{\frac{d}{dx}\ln x}^{\frac{1}{x}} - \ln x \cdot \overbrace{\frac{d}{dx}x}^{1}}{x^2} = \frac{1 - \ln x}{x^2}$.

  $y' = 0 \Rightarrow \ln x = 1 \Rightarrow \boxed{x = e}$

- $f(e) = \frac{\ln e}{e} = \frac{1}{e}$ = critical value.

  - $y' > 0$ when $0 < x < 1$
  - $y' < 0$ when $x > 1$

- Zeros: $y = 0 \Rightarrow \ln x = 0 \Rightarrow x = e^0 = 1$.

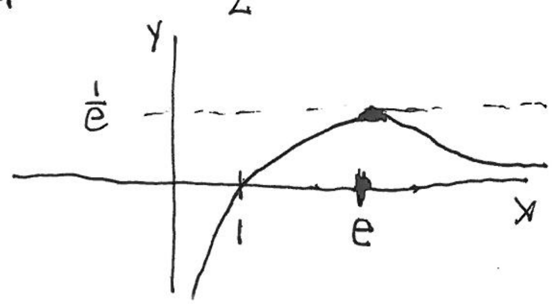
- What happens as $x \to \infty$?

  Set $x = 2^n$ and let $n \to \infty$

  $y = \frac{\ln(2^n)}{2^n} = \frac{n \ln 2}{2^n} \approx \frac{.693n}{2^n} \xrightarrow{\text{as } n \to \infty} 0$

  Thus $y \to 0$ as $x \to \infty$

- In total:



· Two uses for Second derivative

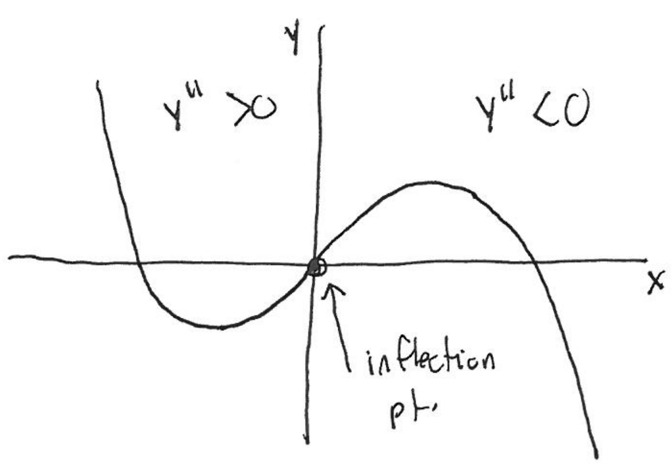
① Second derivative test for $f(x_0)$

Let x_0 be a critical pt. Then:

$f'(x_0)$	$f''(x_0)$	Critical pt. is
0	+	min
0	−	max
0	0	Need more info: Could be a min/max/neither

② Graphing

- The points where $f'' = 0$ are called <u>inflection points</u>.
- Usually the graph changes from concave up to down, or vice versa.



- $y = 3x - x^3$
- $y' = 3(1-x^2)$
- $y'' = -6x$

- $y'' = 0$ when $x = 0$
- $y'' > 0$ when $x < 0$
- $y'' < 0$ when $x > 0$