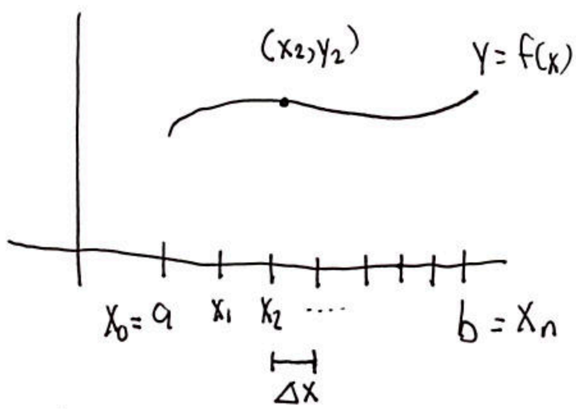
- Work, Average Value, Probability

- How to compute the average value of a function



- Average value of f over $[a, b]$

$$\approx \frac{y_1 + \cdots + y_n}{n}, \text{ where } y_1 = f(x_1),\ y_2 = f(x_2), \ldots$$

- The length Δx of each subinterval is $\Delta x = \frac{b-a}{n}$ (equal lengths)

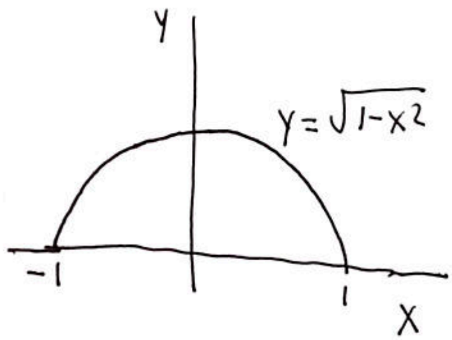
- $\displaystyle\int_a^b f(x)\,dx$ = limit of Riemann Sums = $\displaystyle\lim_{n\to\infty} (y_1 + \cdots + y_n)\overbrace{\left(\frac{b-a}{n}\right)}^{\Delta x}$

- Thus: $$\lim_{n\to\infty} \underbrace{\frac{y_1 + \cdots + y_n}{n}}_{n\text{th average}} = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{"continuous average"}}$$

Ex: The average of a constant c is equal to c:

$$\frac{1}{b-a}\int_a^b 31\,dx = 31$$

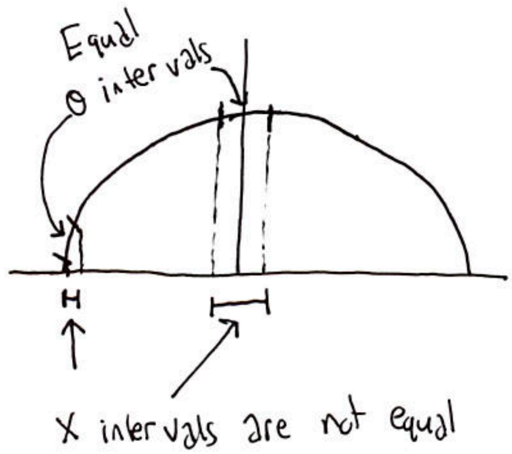
Ex:



The average height of $y=\sqrt{1-x^2}$ on the interval $-1 \le x \le 1$ is

$$\frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\times \text{Area of Semicircle} = \frac{\pi}{4}$$

Ex: Find the average height y on a semicircle with respect to <u>arc length</u> (use $d\theta$, not dx)

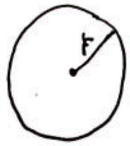


This is an averaage computed using a <u>different weight</u> than in the previous example

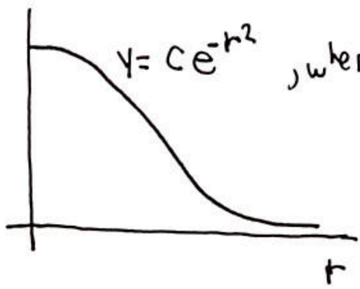
- Along the circle, $y = \sin\theta$, $0 \le \theta \le \pi$

Thus, Average $= \frac{1}{\pi} \int_0^{\pi} \sin\theta \, d\theta = \frac{1}{\pi} \left[-\cos\theta \right]_0^{\pi} = \frac{1}{\pi} \left[-\cos(\pi) + \cos(0) \right]$

$= \frac{2}{\pi}$

Ex: Dart board: 

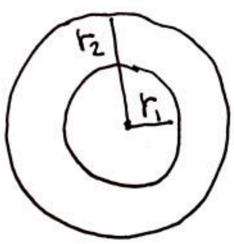
- You aim for the center, but your aim is not perfect.
- Let r denote the distance from the center
- Let's assume that your accuracy is "normally distributed":



$y = ce^{-r^2}$, where c is a constant

- The number of hits within a given ring with $r_1 < r < r_2$ is

$$c \int_{r_1}^{r_2} e^{-r^2} (2\pi r)\, dr$$



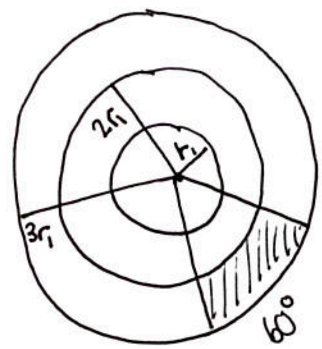
- c is a constant such that $c \int_0^\infty e^{-r^2} (2\pi r) dr = 1$

"The total probability is 1"

Above: • ce^{-r^2} = "probability density" of hitting some fixed point at a distance r from the center

- $2\pi r\, dr$ = area of a ring of width dr and radius r
- $ce^{-r^2}(2\pi r) dr$ = probability of hitting somewhere within the ring of width dr and radius r

Ex



shaded region makes up $\frac{1}{6}$ of the area in between $2r_1$ and $3r_1$

Let's find the probability of hitting the shaded region

- Probability $= \dfrac{\text{part}}{\text{whole}}$

$$= \frac{\frac{1}{6}\int_{2r_1}^{3r_1} ce^{-r^2}(2\pi r)\,dr}{\int_0^{\infty} ce^{-r^2}(2\pi r)\,dr}$$

- $\displaystyle\int_a^b re^{-r^2}dr = -\frac{1}{2}e^{-r^2}\Big]_a^b = \frac{1}{2}\left(e^{-a^2}-e^{-b^2}\right)$

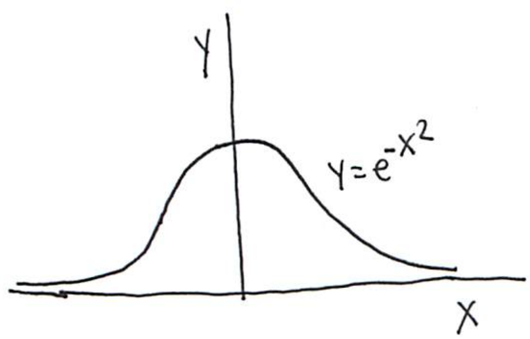
- $\displaystyle\int_0^{\infty} re^{-r^2}dr = -\frac{1}{2}e^{-r^2}\Big]_0^{R\to\infty} = \underbrace{-\frac{1}{2}e^{-R^2}}_{\to 0 \text{ as } R\to\infty}\Big|_{R\to\infty} + \frac{1}{2}e^{-0^2} = \frac{1}{2}$

- Thus, probability $= \dfrac{\frac{1}{6}\int_{2r_1}^{3r_1} e^{-r^2}r\,dr}{\int_0^{\infty} e^{-r^2}r\,dr} = \dfrac{1}{3}\displaystyle\int_{2r_1}^{3r_1} e^{-r^2}r\,dr = \frac{1}{6}\left(e^{-(2r_1)^2} - e^{-(3r_1)^2}\right)$

$= \frac{1}{6}\left(e^{-4r_1^2} - e^{-9r_1^2}\right)$

Ex: Compute $$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

- This integral represents the area under the curve $y = e^{-x^2}$ for $-\infty < x < \infty$:

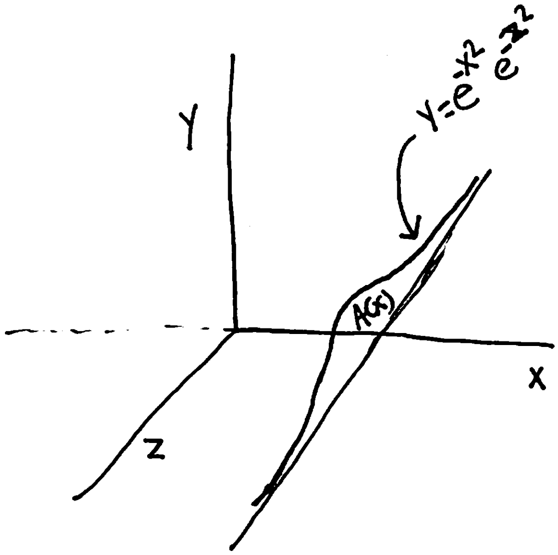


- This is one of the most important integrals in calculus

- Let's first revolve this graph about the Y-axis and find the volume using the shell method:

$$V = \int_0^\infty \underbrace{e^{-x^2}}_{\text{Shell height}}\ \underbrace{2\pi x}_{x\,=\,\text{shell radius}}\ \overbrace{dx}^{\text{Shell thickness}} = -\pi\, e^{-x^2}\Big]_0^{R\to\infty}$$

$$= \pi\left[e^0 - e^{-R^2}\right]_{R\to\infty} = \pi$$



- Let's now find the volume of the same solid by using slices of constant x values.
- Let $A(x)=$ area of the slice
- $dx=$ thickness of slice

- Use polar coordinates $r^2=x^2+z^2$ in the (x,z) plane
- $y=e^{-r^2}=e^{-(x^2+z^2)}=e^{-x^2}e^{-z^2}$
- Note that $A(x)=\int_{z=-\infty}^{\infty} e^{-r^2}\,dz = e^{-x^2}\int_{z=-\infty}^{\infty} e^{-z^2}\,dz = e^{-x^2} I$

It follows that

$$\pi \overset{\text{from before}}{=} V = \int_{-\infty}^{\infty} A(x)\,dx = \int_{-\infty}^{\infty} e^{-x^2} I\,dx = I\int_{-\infty}^{\infty} e^{-x^2}\,dx = I^2$$

- Thus, $I=\sqrt{\pi}=\int_{-\infty}^{\infty} e^{-x^2}\,dx$.
- Equivalently: $\dfrac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$.

- Here is a rescaled version of the above formula (it can be derived by u substitution):

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\, dx = 1$$

- $\frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-x^2}{2\sigma^2}}$ is known as a "normal distribution" with standard deviation σ