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# 1

(a)

$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$
$$1 = A(1-x) + B(1+x)$$

Thus $A = B = 1/2$.

(b)

$$\int \frac{1}{1-x^2} dx$$
$$= \int (\frac{1}{2(1+x)} + \frac{1}{2(1-x)}) dx$$
$$= \frac{1}{2}(\ln|1+x| - \ln|1-x|) + C$$
$$= \ln\sqrt{|\frac{1+x}{1-x}|} + C$$

(c)
If $y = \tanh^{-1} x$, then $x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$. This implies $e^{2y} = \frac{1+x}{1-x}$. Thus $y = \ln\sqrt{\frac{1+x}{1-x}}$, $-1 < x < 1$. The two functions are the same up to a constant when $-1 < x < 1$.

## 2

We use the method of substitution. Set $u = x^2 + 1$.

$$\begin{aligned}
&\int xe^{-x^4-2x^2-1}dx \\
=&\frac{1}{2}\int e^{-u^2}du \\
=&\frac{1}{2}E(u) \\
=&\frac{1}{2}E(x^2+1)
\end{aligned}$$

## 3

(1)

$$\begin{aligned}
\frac{dP}{dt} &= -\sqrt{P} \\
-\frac{1}{\sqrt{P}}dP &= dt \\
-2\sqrt{P} + C &= t
\end{aligned}$$

When $t = 0$, $P = 676$. Thus $C = 52$. When $P = 0$, $t = C = 52$. This means in 52 weeks they will all be dead.

(2)
Suppose his stake is $S$ dollars. Then we have

$$\begin{aligned}
\frac{dS}{dt} &= -\frac{1}{3}S \\
-\frac{3dS}{S} &= dt \\
-3\ln S + C &= t
\end{aligned}$$

Thus -$3\ln S(0) + 3\ln S(1) = t(0) - t(1)$. We know $S(1) = \frac{1}{2}S(0)$ and $t(0) = 0$. Thus $t(1) = 3\ln 2$

(3)

$$\begin{aligned}
\frac{dN}{dt} &= kN^{1+\epsilon} \\
\frac{dN}{kN^{1+\epsilon}} &= dt \\
\frac{1}{k(-\epsilon)}N^{-\epsilon} &= t + C
\end{aligned}$$

If $N = N_0$ when $t = 0$, then $C = -\frac{N_0^{-\epsilon}}{k\epsilon}$. Thus $t = \frac{N_0^{-\epsilon}}{k\epsilon} - \frac{N^{-\epsilon}}{k\epsilon}$. When $t$ goes to $\frac{N_0^{-\epsilon}}{k\epsilon}$, $\frac{N^{-\epsilon}}{k\epsilon}$ will go to 0, hence $N$ will go to $\infty$.

## 4

(a)

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{\kappa}{m}v^2 + g \\
\frac{dv}{-\frac{\kappa}{m}v^2 + g} &= dt \\
\frac{dv}{2g}\left(\frac{1}{1+\sqrt{\frac{\kappa}{gm}}v} + \frac{1}{1-\sqrt{\frac{\kappa}{gm}}v}\right) &= dt \\
\frac{1}{2g}\sqrt{\frac{gm}{\kappa}}\ln\frac{1+\sqrt{\frac{\kappa}{gm}}v}{1-\sqrt{\frac{\kappa}{gm}}v} &= t + C
\end{aligned}$$

When $t = 0$, $v = 0$. Thus $C = 0$. So we have $v = \sqrt{\frac{gm}{\kappa}}\frac{e^{2t\sqrt{\frac{g\kappa}{m}}}+1}{e^{2t\sqrt{\frac{g\kappa}{m}}}-1}$

(b)
The limit exists and is equal to $\sqrt{\frac{gm}{\kappa}}$. This result makes sense because when $v = \sqrt{\frac{gm}{\kappa}}$, the right hand side of the differential equation is 0, which means the speed does not change.

(c)
The answer will not change. If $v_0$ is not 0, then the only change is that we will replace $t$ with $t + C$ where $C$ is determined by $v_0$. When $t$ goes to infinity, the limit of $v$ is still the same as in (b).

## 5

(a)

Take $m = 1/2$, $n = k$ and $\theta = x$. Then the formula gives us $\sin\frac{1}{2}x\cos kx = \frac{1}{2}(\sin(k+\frac{1}{2})x + \sin(\frac{1}{2}-k)x) = \frac{1}{2}(\sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x)$. This is what we want.

(b)

$$\begin{aligned}
&2\sin\frac{1}{2}x(\cos x + \cos 2x + ... + \cos nx)\\
=&(\sin(1+\frac{1}{2})x - \sin(1-\frac{1}{2})x) + (\sin(2+\frac{1}{2})x - \sin(2-\frac{1}{2})x) + ... + (\sin(n+\frac{1}{2})x - \sin(n-\frac{1}{2})x)\\
=&\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x
\end{aligned}$$

This proves the proposition.

(c)

$$\begin{aligned}
\sum_{k=1}^{n}\cos kx &= \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\\
&= \frac{2\sin\frac{1}{2}(n+\frac{1}{2}-\frac{1}{2})\cos\frac{1}{2}(n+\frac{1}{2}+\frac{1}{2})}{2\sin\frac{1}{2}x}\\
&= \frac{\sin\frac{1}{2}nx\cos\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}
\end{aligned}$$

(d)

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\cos x dx &= \lim_{n\to\infty}\frac{\pi}{2n}\sum_{i=1}^{n}\cos\frac{i\pi}{2n}\\
&= \lim_{n\to\infty}\frac{\pi}{2n}\frac{\sin(\frac{1}{2}\frac{\pi}{2})\cos(\frac{1}{2}\frac{(n+1)\pi}{2n})}{\sin\frac{\pi}{4n}}\\
&= \sin\frac{\pi}{4}\cos\frac{\pi}{4}\lim_{n\to\infty}\frac{\frac{\pi}{2n}}{\sin\frac{\pi}{4n}}\\
&= \frac{1}{2}\cdot 2\\
&= 1
\end{aligned}$$

## 6

$\int_0^a x^n dx + \int_0^{a^n} y^{\frac{1}{n}} dy = a^{n+1}$, $a > 0$.

Now let's check the above equation.

$$
\begin{aligned}
&\int_0^a x^n dx + \int_0^{a^n} y^{\frac{1}{n}} dy \\
=&\frac{1}{n+1} x^{n+1}|_0^a + \frac{1}{\frac{1}{n}+1} y^{\frac{1}{n}+1}|_0^{a^n} \\
=&\frac{a^{n+1}}{n+1} + \frac{n}{n+1} a^{n+1} \\
=&a^{n+1}
\end{aligned}
$$