FT. SECOND FUNDAMENTAL THEOREM

1. The Two Fundamental Theorems of Calculus

The Fundamental Theorem of Calculus really consists of two closely related theorems, usually called nowadays (not very imaginatively) the First and Second Fundamental Theorems. Of the two, it is the First Fundamental Theorem that is the familiar one used all the time. It is the theorem that tells you how to evaluate a definite integral without having to go back to its definition as the limit of a sum of rectangles.

First Fundamental Theorem *Let $f(x)$ be continuous on $[a, b]$. Suppose there is a function $F(x)$ such that $f(x) = F'(x)$. Then*

$$(1) \qquad \int_a^b f(x)\,dx \;=\; F(b) - F(a) \;=\; F(x)\Big]_a^b \;.$$

(The last equality just gives another way of writing $F(b) - F(a)$ that is in widespread use.) Still another way of writing the theorem is to observe that $F(x)$ is an **antiderivative** for $f(x)$, or as it is sometimes called, an **indefinite integral** for $f(x)$; using the standard notation for indefinite integral and the bracket notation given above, the theorem would be written

$$(1') \qquad \int_a^b f(x)\,dx \;=\; \int f(x)\,dx\Big]_a^b \;.$$

Writing the theorem this way makes it look sort of catchy, and more importantly, it avoids having to introduce the new symbol $F(x)$ for the antiderivative.

In contrast with the above theorem, which every calculus student knows, the Second Fundamental Theorem is more obscure and seems less useful. The purpose of this chapter is to explain it, show its use and importance, and to show how the two theorems are related. To start things off, here it is.

Second Fundamental Theorem. *Let $f(x)$ be continuous, and fix a.*

$$(2) \qquad \text{Set}\quad F(x) \;=\; \int_a^x f(t)\,dt \;; \qquad \text{then}\quad F'(x) = f(x) \;.$$

We begin by interpreting (2) geometrically. Start with the graph of $f(t)$ in the ty-plane. Then $F(x)$ represents the area under $f(t)$ between a and x; it is a function[1] of x. Its derivative — the rate of change of area with respect to x — is the *length of the dark vertical line.* This is what (2) says geometrically.

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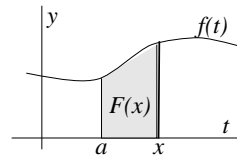
[1]Simmons calls this function $A(x)$ on p. 207 (2nd edition); this section of Simmons is another presentation of much of the material given here.

Example 1. Verify (2) if $f(x) = 2x \sin x^2$ and $a = 0$.

Solution. Here we can integrate explicitly by finding an antiderivative (using the first fundamental theorem):

$$F(x) \;=\; \int_0^x 2t \sin t^2 \, dt \;=\; -\cos t^2 \Big]_0^x \;=\; -\cos x^2 + 1;$$

differentiating by the chain rule, we verify that indeed $F'(x) = 2x \sin x^2$, as predicted by (2). □

Example 2. Let $F(x) \;=\; \int_1^x \dfrac{\sin t}{1 + t} \, dt$. Find $F'(\pi/2)$.

Solution. Neither integration techniques nor integral tables will produce an explicit antiderivative for the function in the integrand. So we cannot use (1). But we can use (2), which says that

$$F'(\pi/2) \;=\; \frac{\sin \pi/2}{1 + \pi/2} \;=\; \frac{1}{1 + \pi/2} \; . \qquad\qquad \square$$

Many students feel the Second Fundamental Theorem is "obvious"; these students are confusing it with the similar-looking

(2') $\qquad\qquad$ Let $\quad F(x) \;=\; \int f(x) \, dx$; $\quad$ then $\quad F'(x) = f(x)$.

Indeed, (2') *is* obvious. The "integral" in it is an indefinite integral, i.e., an antiderivative. So what (2') says is: "Let $F(x)$ be an antiderivative for $f(x)$; then $F(x)$ is an antiderivative for $f(x)$ — a true statement, but not a very exciting one (logicians call it a *tautology*.)

The Second Fundamental Theorem (2) looks almost the same as (2'), but it is actually entirely different, because $F(x)$ is defined as a *definite* integral. The next section, which continues the discussion, should help show the difference.

2. Do functions have antiderivatives?

The First Fundamental Theorem tells us how to calculate $\int_a^b f(x) \, dx$ by finding an anti-derivative for $f(x)$. But the theorem isn't so useful if you can't find an antiderivative. Most calculus students think for example that e^{-x^2} has no antiderivative — "the integral isn't in the tables", "it can't be integrated" are some of the ways they express this. Even for a simple function like $\sqrt{1 - x^2}$, it is not obvious what the antiderivative is. Perhaps it doesn't have any?

The Second Fundamental Theorem provides the answer; it says:

Every continuous function $f(x)$ has an antiderivative: $\quad \int_a^x f(t) \, dt$.

The antiderivative may not be expressible in terms of elementary functions — this is the difficulty with e^{-x^2} — but it always exists.

Example 3. Find an antiderivative for $\dfrac{1}{\sqrt{1+\sqrt{x}}}$.

Solution. This doesn't look so easy to do explicitly. But the Second Fundamental Theorem says the following function is an antiderivative:

$$(3) \qquad\qquad F(x) \;=\; \int_0^x \frac{1}{\sqrt{1+\sqrt{t}}}\,dt$$

Discussion. You may feel that this doesn't represent progress: the formula for the antiderivative is useless. But that's not so: the function $F(x)$ can be calculated by numerical integration. It can be programmed into a calculator so that when you press an x-value, the screen will display the corresponding value of $F(x)$ to 12 decimal digits. Pressing another button will draw the graph of $F(x)$ over any interval on the x-axis that you specify.

Repeating what we said earlier, the integral in (3) should be carefully distinguished from $\displaystyle\int \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$ — this "integral" is just another notation for the antiderivative, and is therefore *not* a solution to the problem. The integral in (3) by contrast is a perfectly definite function, and it does solve the problem of finding an antiderivative.

In this case, it turns out that $F(x)$ does have an expression in terms of elementary functions. It is

$$(4) \qquad\qquad F(x) \;=\; \frac{4}{3}(\sqrt{x}+1)^{3/2} - 4(\sqrt{x}+1)^{1/2} + \frac{8}{3}\;.$$

(You can prove this is correct by differentiating it; the $8/3$ is put in to ensure that $F(0) = 0$, as definition (3) requires.)

The above way of writing $F(x)$ is different from (3). Whether or not it is a better way depends on what you want to know about $F(x)$ and what use you want to make of it. For instance,

$$\text{Is } F(x) > 0 \text{ when } x > 0?$$

The answer is clearly "yes" if we look at the integral (3), since the integrand is positive; it is not at all clear what the answer is if instead we look at (4), because of the $-$ sign. As another example, what is $F'(x)$? From (3) the answer is immediate, whereas from (4) you would have to calculate for a while — as you will know if you took the trouble to check its correctness!

3. Defining new functions: ln(x) and erf(x).

One important use of the Second Fundamental Theorem is to define new functions. Calculus can then be used to study their properties.

To illustrate, we consider first an old function: $\ln x$. Let's pretend we know nothing of logarithms. We do know that

$$\int x^n \, dx \;=\; \frac{x^{n+1}}{n+1}, \qquad n \neq -1 \ .$$

However, we know no explicit formula for an antiderivative of $1/x$, i.e., when $n = -1$.

We therefore use the Second Fundamental Theorem to define an antiderivative of $1/x$, namely

(5) $$L(x) \;=\; \int_1^x \frac{dt}{t} \ .$$

(We use 1 as the lower limit of integration since the integrand is not defined at 0.) What are the properties of this function?

Properties of $L(x) \;=\; \int_1^x \dfrac{dt}{t}$

L-1. $L(x)$ is defined for $x > 0$ (since $1/t$ is continuous for $t > 0$);

L-2. $L(1) = 0$;

L-3. $L'(x) = 1/x$, by the Second Fundamental Theorem;

L-4. $L''(x) = -1/x^2$, by differentiating $1/x$;

L-5. $L(x)$ is increasing for all $x > 0$, since $L'(x) > 0$; its graph is concave (i.e., concave down) since $L''(x) < 0$;

L-6. $L(ab) = L(a) + L(b)$.

Of course, it is this last which is the interesting property; the proof of it is elegant.

Proof of L-6. We break up the integral defining $L(ab)$ into two parts, the first of which is $L(a)$: to do this, we use the interval addition rule (3) in Notes PI.1 .

(6) $$L(ab) \;=\; \int_1^{ab} \frac{dt}{t} \;=\; \int_1^a \frac{dt}{t} \;+\; \int_a^{ab} \frac{dt}{t} \ .$$

Comparing with Property L-6, we see we have to show the last integral on the right above has the value $L(b)$. To see this, make the change of variable $t = au$ and apply the change of variable rule (see (7), p. PI.2 in these notes). You get successively

$$t = au, \qquad dt = a \, du, \qquad \frac{dt}{t} \;=\; \frac{a \, du}{au} \;=\; \frac{du}{u} \ .$$

We have to change the limits on the integral also: $t = a$ and $t = ab$ correspond respectively to $u = 1, u = b$. Thus the rule for changing variable in a definite integral gives

$$\int_a^{ab} \frac{dt}{t} \;=\; \int_1^b \frac{du}{u} \;=\; L(b),$$

which proves **L-6**.   □

Once we have this, the other properties of the logarithm follow in a standard way.

L-7. $L(1/a) = -L(a)$, since $L(1/a) + L(a) = L(\frac{1}{a} \cdot a) = L(1) = 0$.

L-8. $\lim_{x \to \infty} L(x) = \infty$; namely, $L(x)$ is increasing and $L(2^n)$ increases without bound as $n \to \infty$, since $L(2^n) = nL(2)$, by Property 6; note that $L(2) > 0$ since $L(x)$ is increasing. □

In our definition of $L(x)$, the number e appears as the unique number such that

$$L(e) = \int_1^e \frac{dt}{t} = 1 .$$

Such a number exists by the Intermediate Value Theorem,[2] since $L(x)$ is increasing, continuous (since it has a derivative), and gets bigger than 1.

We now turn to a second example of using the Second Fundamental Theorem to define a function $F(x)$ — this time, the function will be genuinely new. It is closely related to an important function in probability and statistics, the error function erf x. (Statistics-oriented calculators have a button for it.) The two functions differ only by a change of scale on the x- and y-axes. There is no simpler or more elementary expression for $F(x)$.

Example 5. Define a function $F(x)$ by $F(x) = \int_0^x e^{-t^2} dt$.

Sketch the graph of $F(x)$, indicating relative maxima and minima, points of inflection, symmetries. Estimate $F(1)$ roughly.

Solution. The graph of $f(t) = e^{-t^2}$ is shown.

$F(x)$ is the indicated shaded area under the graph of $f(t)$.

$F'(x) = e^{-x^2}$ by the Second Fundamental Theorem; since the exponential is always positive, this shows $F(x)$ is increasing for all x, and therefore it has no relative maxima or minima.

$F''(x) = -2xe^{-x^2}$; since $F''(x) < 0$ for $x > 0$, the graph of $F(x)$ is concave (down) when $x > 0$. Similarly, it is convex (concave up) for $x < 0$, and it has a point of inflection at $x = 0$.

$F(x)$ is an odd function. To see this, we note that e^{-t^2} is an even function. As the picture shows, the two shaded areas are equal; the one on the left however must be counted negatively, since the integration is backwards: if $x > 0$, then

$$F(-x) = \int_0^{-x} f(t)dt = -\int_0^x f(t) \, dt = -F(x) .$$

This shows $F(x)$ is an odd function.

$F(1)$ can be estimated as the area under $f(t)$ between 0 and 1; it is roughly comparable to the area of the trapezoid shown, which about .7 .

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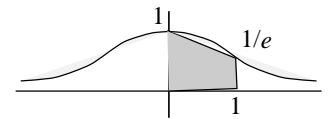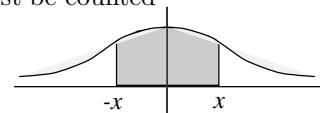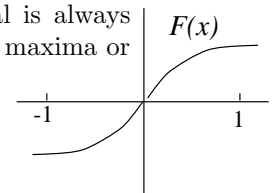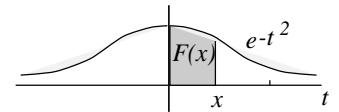
[2]Simmons, p. 78

4. Proof of the two Fundamental Theorems.

We will give an intuitive argument for the Second Fundamental Theorem, and then deduce the First Fundamental Theorem from the Second. Though the argument for the Second Theorem is only suggestive, it has the right ideas in it, and can be easily made rigorous if you have available a precise definition of limit.[3]

Second Fundamental Theorem: Intuitive Argument

We wish to prove that if $f(x)$ is continuous, then

(8) $$F(x) \; = \; \int_a^x f(t)\, dt \qquad \Rightarrow \qquad F'(x) = f(x) \; .$$

We calculate $F'(x)$ using the definition of derivative. Let x change by Δx, and let ΔF be the corresponding change in $F(x)$. From the picture,

$$\Delta F \; = \; F(x + \Delta x) - F(x) \; = \; \int_x^{x+\Delta x} f(t)\, dt$$

(9) $$\approx \; f(x)\Delta x,$$
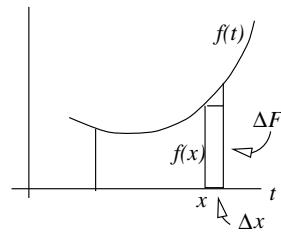
since the area of the vertical strip under the curve is approximately the same as the area of the rectangle. Dividing, we have

$$\frac{\Delta F}{\Delta x} \; \approx \; f(x),$$

where the error in the approximation is bounded by the height of the small curved triangle. Since $f(t)$ is continuous, the error is small compared with $f(x)$, and disappears when we pass to the limit as $\Delta x \to 0$; we get therefore

$$F'(x) \; = \; \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} \; = \; f(x).$$

Note that if $f(t)$ were discontinuous at the point x, the result would not be true; as the picture shows, the approximation in (9) would not be true.

First Fundamental Theorem: Proof[4]

We want to show that if $f(x)$ is continuous and $f(x) = F'(x)$, then

(10) $$\int_a^b f(x)\, dx \; = \; F(b) - F(a).$$

We begin by defining

(11) $$G(x) \; = \; \int_a^x f(t)\, dt; \qquad \text{then}$$

$$G'(x) \; = \; f(x), \qquad \text{by the Second Fundamental Theorem.}$$

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[3]A somewhat fuller argument is given in Simmons, Step 1, p. 206-7.

[4]this same classical reasoning is given in Simmons: Steps 2 and 3, p. 207.

Since $G'(x) = f(x) = F'(x)$, we have $\big(G(x) - F(x)\big)' = 0$, which implies $G(x) - F(x) = C$, i.e.,

$$(12) \qquad G(x) \;=\; F(x) + C, \qquad \text{for some constant } C.$$

To evaluate C, we put $x = a$ in (12); since $G(a) = 0$, we get

$$C \;=\; -F(a) \ .$$

Finally, put $x = b$ in equation (12), and use the above value for C:

$$G(b) \;=\; F(b) - F(a),$$

which is exactly (10), in view of the definition of $G(x)$. $\hfill \square$

Remark. Both fundamental theorems say that differentiation and *definite* integration are inverse operations: each undoes what the other does. In the First Fundamental Theorem we differentiate, then integrate:

$$F(x) \quad \longrightarrow \quad F'(x) \quad \longrightarrow \quad \int_a^x F'(t)\, dt = F(x) - F(a);$$

In the Second Fundamental Theorem, we integrate, then differentiate:

$$f(x) \quad \longrightarrow \quad \int_a^x f(t)\, dt \quad \longrightarrow \quad \frac{d}{dx}\int_a^x f(t)\, dt = f(x) \ .$$

In both cases, the theorem says that we end up essentially where we started – only "essentially" because of the additive constant in $F(x)$.

(Of course, differentiation and *indefinite* integration are also inverse operations, but this is trivial – it's just a restating of the definition of indefinite integration.)

Exercises: Section 3D