Solution to PSet 6

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Part II 1

a)

By second fundamental theorem,

$$\mathrm{Si}'(x) = \mathrm{sinc}(x).$$

For $x \neq 0$,

$$\mathrm{Si}'' = \mathrm{sinc}'(x) = \frac{x\cos x - \sin x}{x^2}.$$

For $x = 0$,

$$\mathrm{Si}'' = \mathrm{sinc}'(0) = \lim_{x \to 0} \frac{\sin x - x}{x^2} = 0,$$

where the last step is due to the second order approximation of $\sin x$ around 0.

b)

$\mathrm{Si}'(x) = \mathrm{sinc}(x) = 0$ gives

$$\sin(x) = 0, x \neq 0.$$

Therefore $x = k\pi$ where k is a non-zero integer.

Since $\mathrm{Si}''(k\pi) = \frac{\cos(k\pi)}{k\pi}$ for $k \neq 0$,

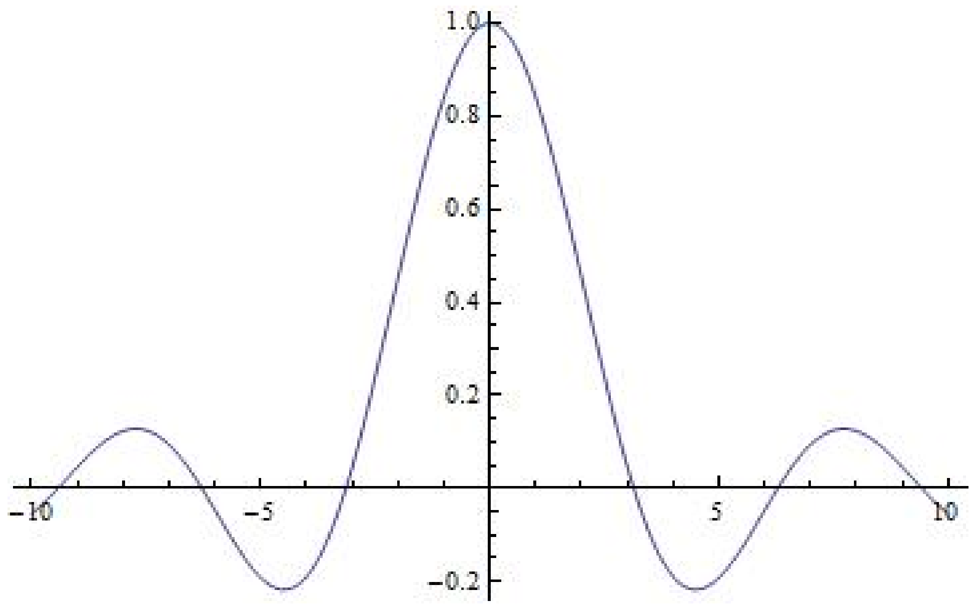
1. if k is an non-zero positive even integer or a negative odd integer, $k\pi$ is a local minimum;

2. if k is an non-zero negative even integer or a positive odd integer, , $k\pi$ is a local maximum.

c)

See Figure Si′ and Figure Si″.

Si′



d)

Please see the Figure Si in the last page.

e)

By second fundamental theorem,

$$h'(x) = \mathrm{Si}'(x^r)x^{r-1} = \frac{1}{r}\left(\mathrm{Si}(x^r)\right)'.$$

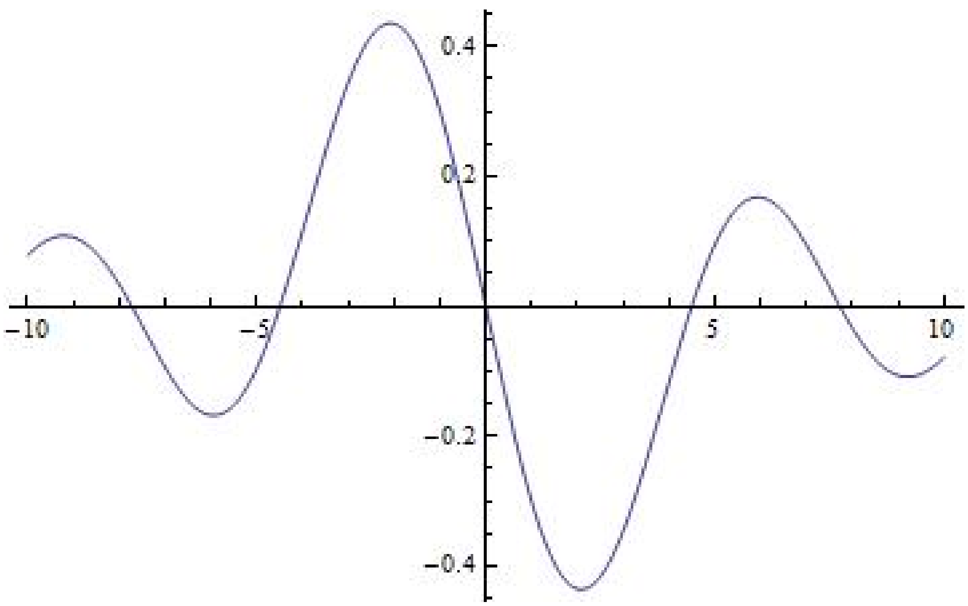
Therefore

$$h(x) = \frac{1}{r}\mathrm{Si}(x^r) + C.$$

Since $h(0) = 0$ and $\mathrm{Si}(0) = 1$, we have $C = -\frac{1}{r}$. Therefore

$$h(x) = \frac{\mathrm{Si}(x^r) - 1}{r}$$

Si''



f)

$$\lim_{x\to 3}\frac{x^2}{x-3}\int_3^x \operatorname{sinc}(t)dt = \lim_{x\to 3} x^2\frac{\operatorname{Si}(x)-\operatorname{Si}(3)}{x-3} = 9\operatorname{Si}'(3) = 9\operatorname{sinc}(3) = 3\sin 3.$$

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a)

By chain rule,

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}.$$

By second fundamental theorem, $\frac{dV}{dh} = A(h)$. Therefore

$$-cA(h) = A(h)\frac{dh}{dt},$$

which means

$$\frac{dh}{dt} = -c.$$

b)

From a)

$$h = h_0 - ct.$$

Therefore it will take time h_0/c.

Part II 3

Fix z and consider the horizontal slice, the area is given by $\frac{1}{2}z^2 - \frac{1}{2}z^8$. By the constraint we have, $z < 1$, the volume is

$$\int_0^1 \frac{1}{2}z^2 - \frac{1}{2}z^8 dz = \frac{1}{9}.$$

Part II 4

a)

Let $R - \ell/2 \le r \le R + \ell/2$. Then for each r there is a cylinder shell for the solid with radius r. For $R - \ell/2 \le r \le R$, the volume of the shell is

$$2\pi r \times \sqrt{3}(r - R + \ell/2)dr.$$

For $R \le r \le R + \ell/2$, the volume of the shell is

$$2\pi r \times \sqrt{3}(R - r + \ell/2)dr.$$

So the integral is

$$\int_{R-\ell/2}^{R} 2\pi r \times \sqrt{3}(r - R + \ell/2)dr + \int_{R}^{R+\ell/2} 2\pi r \times \sqrt{3}(R - r + \ell/2)dr$$

b)

The integral is

$$\int_{R-\ell/2}^{R+\ell/2} 2\pi r \times \sqrt{3} \times \ell/2 dr + 2\sqrt{3}\pi \left(\int_{R-\ell/2}^{R} r^2 dr - \int_{R}^{R+\ell/2} r^2 dr \right)$$
$$= \sqrt{3}\pi \frac{\ell}{2} \left[(R + \frac{\ell}{2})^2 - (R - \frac{\ell}{2})^2 \right] + \frac{2\sqrt{3}\pi}{3} \left(2R^3 - (R + \frac{\ell}{2})^3 - (R - \frac{\ell}{2})^3 \right)$$
$$- \sqrt{3}\pi R \left(2R^2 - (R + \frac{\ell}{2})^2 - (R - \frac{\ell}{2})^2 \right)$$
$$= \frac{\sqrt{3}\pi R \ell^2}{2}.$$

c)

By disk method, for $0 \le h \le \sqrt{3}\ell/2$, each slice is a annulus with volume

$$\pi(R + \frac{\ell}{2} - \frac{\sqrt{3}h}{3})^2 - \pi(R - \frac{\ell}{2} + \frac{\sqrt{3}h}{3})^2 dr.$$

Therefore the total volume is

$$\int_0^{\sqrt{3}\ell/2} \pi(R + \frac{\ell}{2} - \frac{\sqrt{3}h}{3})^2 - \pi(R - \frac{\ell}{2} + \frac{\sqrt{3}h}{3})^2 dh$$
$$= \int_0^{\sqrt{3}\ell/2} 2\pi R\ell - \frac{4\pi h R}{\sqrt{3}} dh$$
$$= \frac{\sqrt{3}\pi R \ell^2}{2}.$$

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The line passes $(r, 0)$ and $(0, h)$. For $0 \le x \le r$, there is a shell with volume

$$2\pi x \times h(1 - \frac{x}{r}) dx.$$

Therefore the total volume of the cone is

$$\int_0^r 2\pi x \times h(1 - \frac{x}{r}) dx = \frac{\pi r^2 h}{3}$$

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Assume that the sphere has radius a, the cylinder has radius r and the height is h. We want to first compute the volume of the hole removed. There is cylinder of height h and radius r inside the hole separating the hole into an upper piece and lower piece. This cylinder has volume

$$\pi h r^2 = \pi h (a^2 - \frac{h^2}{4})$$

Now we use the disk method to compute the volume of the upper piece. For $h/2 \le y \le a$, there is a disk of height y of volume

$$\pi (a^2 - y^2) dy$$

Therefore the volume is

$$\int_{h/2}^{a} \pi (a^2 - y^2) dy = \pi a^2 (a - h/2) - \frac{1}{3}\pi a^3 + \frac{1}{3}\pi (h/2)^3 = .$$

Therefore the hole has volume

$$\pi h (a^2 - \frac{h^2}{4}) + 2 \times \left(\pi a^2 (a - h/2) - \frac{1}{3}\pi a^3 + \frac{1}{3}\pi (h/2)^3 \right) = \frac{\pi}{6}(8a^3 - h^3).$$

Therefore the volume of the spherical ring is

$$\frac{4}{3}\pi a^3 - \frac{\pi}{6}(8a^3 - h^3) = \frac{\pi}{6} h^3.$$

Si