- Top 3 math Classes for Scientists (reflects personal bias):
  1. Calculus
  2. Linear algebra
  3. Diff EQ

- Why is Calculus important?
  1. Many "fundamental" "laws" of nature are expressed in terms of the <u>rate of change</u> of one variable with respect to another. And rate of change $\longleftrightarrow$ derivatives $\updownarrow$ differential Calculus.

  2. Many "empirical models" are also expressed in terms of rates of change

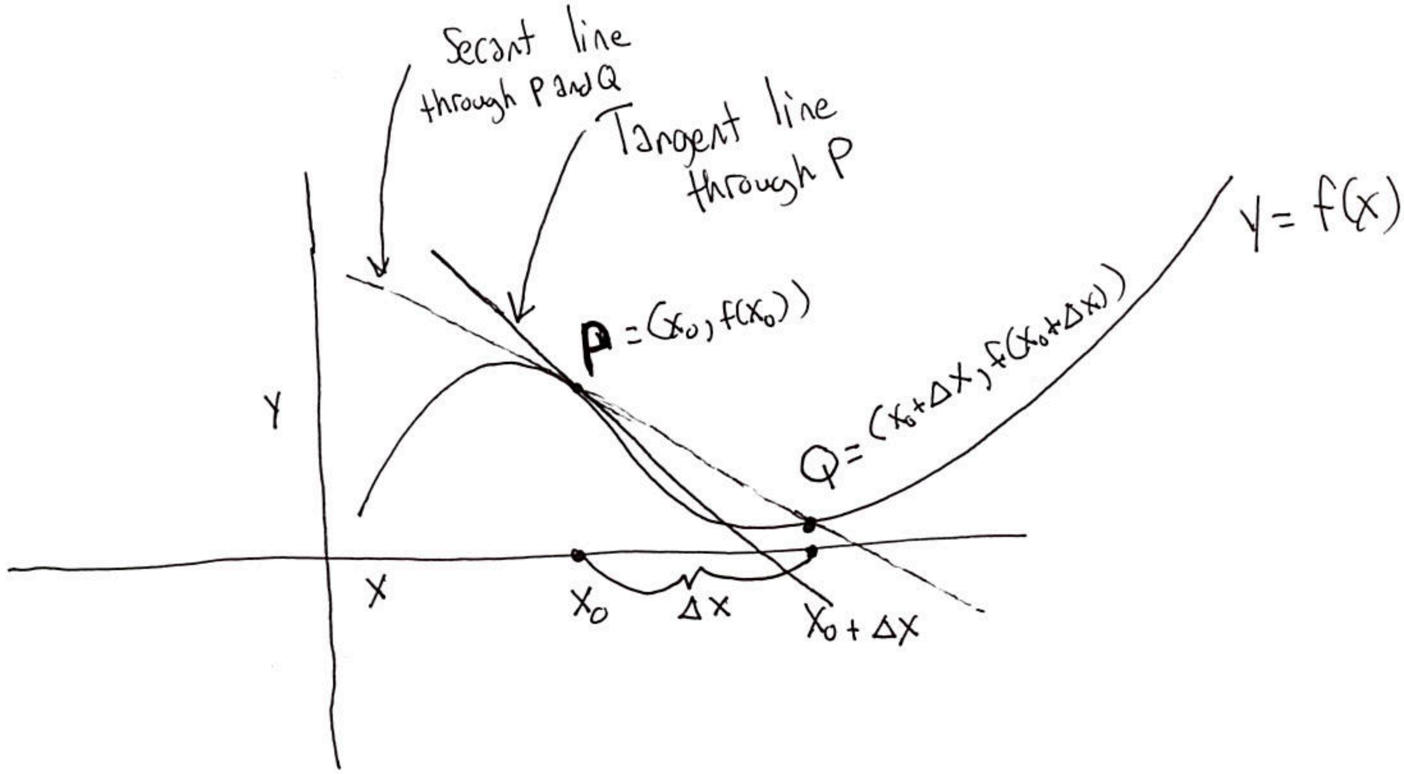
     $\to$ a model based on experimental data that does not claim to be a fundamental law of nature.

Examples to come...

Today: Derivatives

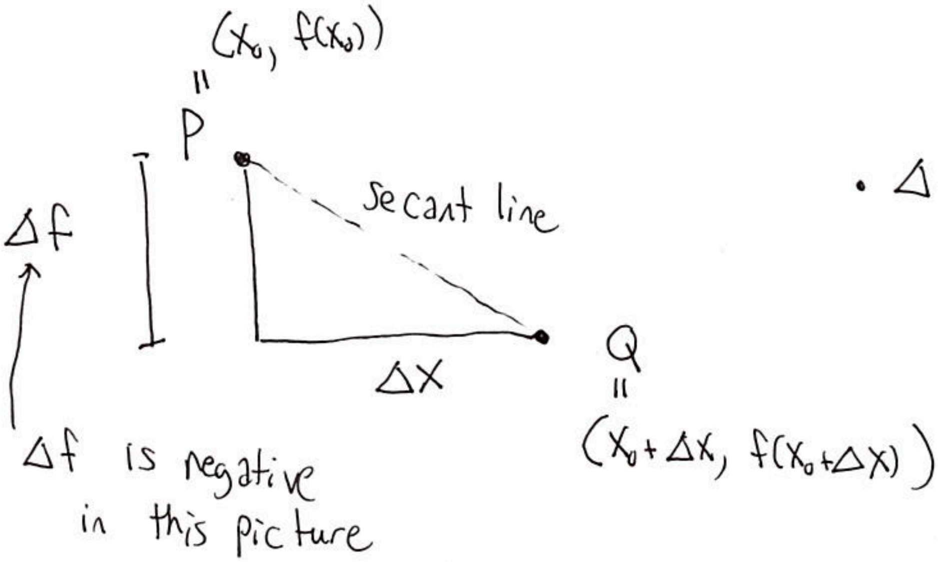
- I) Geometric interpretation
- II) Mathematical (analytic) definition - limiting procedure
- III) How to compute (in principle) the derivative of any function $f(x)$
- IV) Physical interpretation

- I) Geometric interpretation



Geometric Definition of the derivative of f at x_0: the slope of the tangent line through $P = (x_0, f(x_0))$

- The slope of the tangent line at P is the <u>limit of the slopes</u> of the secant lines PQ as $Q \to P$ (P stays fixed)



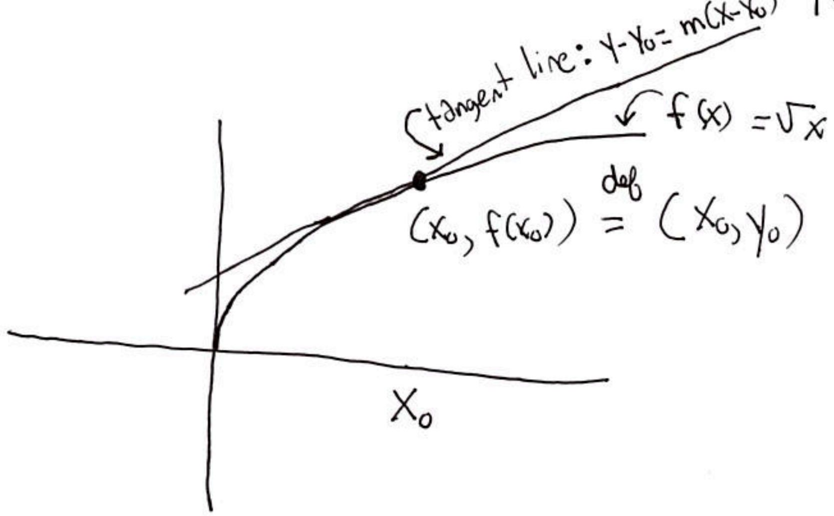
- $\Delta f \overset{\text{def}}{=} f(x_0+\Delta x) - f(x_0)$

- $$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \underbrace{\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}}_{\text{"difference quotient"}} \overset{\text{def}}{=} \underbrace{f'(x_0)}_{\text{"derivative of } f \text{ at } x_0\text{"}}$$

this is II): the mathematical (analytic) definition of the derivative

- Consider $f(x) = \sqrt{x}$ for $x \geq 0$.

  Given $x_0 > 0$, find the equation of the tangent line at the point $(x_0, f(x_0))$



- High school algebra: the line through (x_0, y_0) of slope m can be expressed as

$$y - y_0 = m(x - x_0)$$

- $y_0 = \sqrt{x_0}$
- m = slope of tangent line at x_0 $= f'(x_0)$

  $\Rightarrow$ we need to compute $f'(x_0)$.

- To compute: we use the analytic definition:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \underbrace{\left(\frac{\sqrt{x_0 + \Delta x} + \sqrt{x_0}}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}}\right)}_{1} \left(\frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x}\right) = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{(\sqrt{x_0 + \Delta x} + \sqrt{x_0}) \cdot \cancel{\Delta x}} = \boxed{\frac{1}{2\sqrt{x_0}}}$$

Simplified version using the identity $(\sqrt{x_0 + \Delta x} + \sqrt{x_0})(\sqrt{x_0 + \Delta x} - \sqrt{x_0})$

- Thus, $m = f'(x_0) = \dfrac{1}{2\sqrt{x_0}}$.

- Conclusion: the tangent line is (whenever $x_0 > 0$)

$$\boxed{y - \underbrace{\sqrt{x_0}}_{y_0} = \underbrace{\frac{1}{2\sqrt{x_0}}}_{m}(x - x_0)}$$

- Let's show: the y intercept of this line is positive when $x_0 > 0$.

- Recall: the y intercept is the "y-value" when $x = 0$:

$$y_{int} = \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(0 - x_0) = \sqrt{x_0} - \frac{1}{2}\sqrt{x_0} = \boxed{\frac{1}{2}\sqrt{x_0} > 0}$$

- (Lots of) Notation
  - Assume that $y = f(x)$ is a function
  - Change in $y = \Delta y = \Delta f = f(x) - f(x_0)$
    $= f(x_0 + \Delta x) - f(x_0)$
  - Difference quotient $= \dfrac{\Delta y}{\Delta x} = \dfrac{\Delta f}{\Delta x}$
  - Derivative (limit as $\Delta x \to 0$):
    - $\dfrac{\Delta y}{\Delta x} \to \dfrac{dy}{dx}$ (Leibniz) ← Evaluation at $x = x_0$ is not explicitly indicated
    - $\dfrac{\Delta f}{\Delta x} \to f'(x_0)$ (Newton)
  - Other derivative notation: $\dfrac{df}{dx}$, f', Df, y'

Example: $f(x) = x^n$ $\quad n = 1, 2, 3, \ldots$

- Compute $\frac{d}{dx} x^n$
- The difference quotient: $\dfrac{\Delta y}{\Delta x} = \dfrac{(x_0 + \Delta x)^n - x_0^n}{\Delta x}$
  - For simplicity, let's just write "x" instead of "x_0"
    This is a reasonable simplification, as long as you remember what you are doing.
  - Then $\dfrac{\Delta y}{\Delta x} = \dfrac{(x + \Delta x)^n - x^n}{\Delta x}$
  - Expand: $(x+\Delta x)^n = \overbrace{(x+\Delta x)(x+\Delta x)\ldots(x+\Delta x)}^{n \text{ times}}$

$$= x^n + n(\Delta x)x^{n-1} + O((\Delta x)^2)$$

  - Above, "$O((\Delta x)^2)$" is an abbreviation for all of the terms involving $(\Delta x)^2$, $(\Delta x)^3$, and higher order
  - The fully detailed expansion formula is called the Binomial Theorem. See your book for the details.
- Now $\dfrac{\Delta y}{\Delta x} = \dfrac{(x+\Delta x)^n - x^n}{\Delta x} = \dfrac{\cancel{x^n} + n(\Delta x)x^{n-1} + O((\Delta x)^2) - \cancel{x^n}}{\Delta x} = n x^{n-1} + O((\Delta x))$.
- Take the limit: $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = n x^{n-1}$. • Thus, $\boxed{\frac{d}{dx} x^n = n x^{n-1}}$

- Note the additive and constant multiple properties of differentiation:

  - $\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x) + g'(x)$
  - $\frac{d}{dx}\left(c\,f(x)\right) = c\,f'(x)$ when c is a constant.

You can now compute the derivatives of all polynomials.

Ex: $\frac{d}{dx}(x^7+x) = \frac{d}{dx}x^7 + \frac{d}{dx}x = 7x^6+1.$