Taylor Series

- Recall the geometric series:

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

- A general power series is an infinite sum

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

that represents a function $f(x)$ when $|x| < R$.

- R is called the <u>radius of convergence</u>. In particular, when $|x| < R$, $|a_n x^n| \to 0$ as $n \to \infty$. On the other hand, for $|x| > R$, $|a_n x^n|$ does not go to 0 as $n \to \infty$.
- For example, in the case of the geometric series, all of the a_n are equal to 1. Thus, if $x = \frac{1}{2}$, then $|a_n x^n| = |\frac{1}{2}|^n$. The higher order terms therefore become increasingly negligible as $n \to \infty$ when $x = \frac{1}{2}$.

Ex. If $X=-1$, then in the geometric series, $|a_n X^n| = 1$ does not tend to 0.

The infinite sum $1-1+1-1+\cdots$ bounces back and forth between 0 and 1.

- When $|x|>1$, the geometric series diverges.

- Basic tools

Rules of polynomials apply within the radius of convergence

Since $\dfrac{1}{1-x} = 1+x+x^2+\cdots$

Ex: Substitution $X=-u$

$$\frac{1}{1+u} = 1-u+u^2-u^3+\cdots$$

Ex: Substitution $X=-v^2$

$$\frac{1}{1+v^2} = 1-v^2+v^4-v^6+\cdots$$

Ex: Term by term multiplication

- $\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) = (1+x+x^2+\cdots)(1+x+x^2+\cdots)$

$$= 1 + 2x + 3x^2 + \cdots$$

- Remember that x here is some number like $\frac{1}{2}$. As you take higher and higher powers of x, the terms get smaller and smaller.

Ex Term by term differentiation

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\left[1+x+x^2+x^3+\cdots\right]$$

$$= 1 + 2x + 3x^2 + \cdots$$

(agrees with previous example)

Ex Term by term integration

- $\int \frac{du}{1+u} = \int (1-u+u^2-u^3+\cdots)\,du = C + u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots$

- $\ln(1+x) = \int_0^x \frac{du}{1+u} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Ex $$\int \frac{dv}{1+v^2} = \int (1 - v^2 + v^4 - v^6 + \dots)\,dv$$

$$= C + v - \frac{v^3}{3} + \frac{v^5}{5} - \frac{v^7}{7} + \dots$$

- $$\tan^{-1} x = \int_0^x \frac{dv}{1+v^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Taylor's Series and Taylor's Formula

- If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, we want to figure out what all of the coefficients are.
- Differentiating term by term, we have:
  - $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$
  - $f''(x) = (2)(1)a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \cdots$
  - $f'''(x) = (3)(2)(1)\, a_3 + (4)(3)(2)\, a_4 x + \cdots$
- Plugging in $x=0$, we have:

$$f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2a_2, \quad f'''(0) = 3!\, a_3$$

- In general, Taylor's formula holds:

$$\boxed{\begin{aligned} f^{(n)}(0) &= n!\, a_n, \\ a_n &= \frac{1}{n!} f^{(n)}(0) \end{aligned}}$$

Ex: $f(x) = e^x$

- $f'(x) = e^x$
- $f''(x) = e^x$

$\vdots$

- $f^{(n)}(x) = e^x$

$\Rightarrow f^{(n)}(0) = 1$

- By Taylor's formula, $\boxed{a_n = \frac{1}{n!}}$.

Hence, $e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

- In a more compact form:

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

- Now we can calculate e to any desired degree of accuracy:

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Ex:
- $f(x) = \cos x$
- $f'(x) = -\sin x$
- $f''(x) = -\cos x$
- $f'''(x) = \sin x$
- $f^{(4)}(x) = \cos x$

$\vdots$

- $f(0) = \cos(0) = 1$
- $f'(0) = -\sin(0) = 0$
- $f''(0) = -\cos(0) = -1$
- $f'''(0) = \sin(0) = 0.$

- Only even coefficients are non-zero and their signs alternate. Therefore,

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

- To find the Taylor series for $\sin x$, we can either proceed as in the case of $\cos x$, or alternatively, just differentiate the series for $\cos x$:

  - $\sin x = \frac{d}{dx}\cos x = 0 - 2\left(\frac{1}{2!}\right)x + 4\cdot\left(\frac{1}{4!}\right)x^3 - 6\cdot\left(\frac{1}{6!}\right)x^5 + \cdots$

    $$\Rightarrow \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

- Let's compare these Taylor series with the quadratic approximation from earlier in the semester:

$$\cos x \approx 1 - \frac{1}{2}x^2 \quad , \quad \sin x \approx x$$

- In a compact form, we can write the Taylor series for $\cos x$ and $\sin x$ as:

  - $$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k = \frac{(-1)^0 \cdot x^0}{0!} + \frac{(-1)^1 x^2}{2!} + \cdots = 1 - \frac{1}{2}x^2 + \cdots$$
  - $$\sin x = \sum_{k=0}^{\infty} \frac{x^{2k+1}(-1)^k}{(2k+1)!} = \frac{(-1)^0 x^1}{1!} + \frac{(-1)^1 x^3}{3!} + \cdots = x - \frac{x^3}{3!} + \cdots$$

Ex: Taylor Series With Another Base Point

- A Taylor Series with a base point at $x=b$ (instead of $x=0$) looks like

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f^{(3)}(b)}{3!}(x-b)^3 + \cdots$$

- Ex $f(x)=\sqrt{x}$, $b=1$. (It is a bad idea to use $b=0$ because $f(x)$ is not differentiable at 0).

$$x^{\frac{1}{2}} = 1 + \frac{1}{2}(x-1) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x-1)^2 + \cdots$$

Ex: Binomial Expansion $f(x) = (1+x)^a$

- $(1+x)^a = 1 + \frac{a}{1}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \dots$