MATH 18.01 - MIDTERM 1 - SOME REVIEW PROBLEMS WITH SOLUTIONS

18.01 Calculus, Fall 2014	Professor: Jared Speck

Problem 1. Compute the second derivative of the function $f(x) = \arctan x$.

Problem 2. Compute the derivative of the function $f(x) = \sin(x)x^{x^x}$.

Problem 3. Compute $\lim_{x\to\pi/2} \frac{\cos(3x)}{\cos(x)}$. At this point in the course, you are forbidden from using L'Hôpital's rule.

Problem 4. Compute the derivative $\frac{dy}{dx}$ for the curve $x^2 + y^3 + xy = 7$ at the point $(x, y) = (2, 1)$.

Problem 5. Find the equation for the tangent line to $y = \ln(x/3)$ at $x = 3e$.

Problem 6. Consider the function

$$f(x) = \begin{cases} -\frac{\sin x}{2x} & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \frac{\cos x - 1}{x^2} & \text{if } x > 0. \end{cases}$$

Is $f(x)$ continuous at $x = 0$? If not, then is the discontinuity removable?

Problem 7. Prove the quotient rule $(u/v)' = (u'v - uv')/v^2$ using only the definition of a derivative.

Problem 8. Let r be a real number. Compute $\lim_{h\to 0} \frac{(1+2h)^r - 1}{h}$ by interpreting this limit as a derivative.

Solutions

Problem 1. Compute the second derivative of the function $f(x) = \arctan x$.

Solution:

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \text{(you should know how to prove this)},$$
$$\frac{d^2}{dx^2}\arctan x = \frac{d}{dx}\frac{1}{1+x^2} = \frac{-2x}{(1+x^2)^2}.$$

Problem 2. Compute the derivative of the function $f(x) = \sin(x)x^{x^x}$.

Solution:
We first compute the derivative of x^x using logarithmic differentiation:

$$\begin{aligned}
y &= x^x, \\
\ln y &= \ln(x^x) = x\ln x, \\
\frac{y'}{y} &= 1 + \ln x, \\
y' &= (1+\ln x)x^x.
\end{aligned}$$

We now compute the derivative of x^{x^x} :

$$\begin{aligned}
z &= x^{x^x}, \\
\ln z &= \ln(x^{x^x}) = x^x \ln x, \\
\frac{z'}{z} &= \ln x \frac{d}{dx}x^x + x^x\frac{d}{dx}\ln x = \ln x(1+\ln x)x^x + x^x\frac{1}{x}, \\
z' &= x^{x^x}\left\{\ln x(1+\ln x)x^x + x^x\frac{1}{x}\right\}.
\end{aligned}$$

Finally, we compute the derivative of $\sin(x)x^{x^x}$ using the product rule:

$$\begin{aligned}
\frac{d}{dx}\left(\sin(x)x^{x^x}\right) &= x^{x^x}\frac{d}{dx}\sin(x) + \sin(x)\frac{d}{dx}x^{x^x} \\
&= x^{x^x}\cos x + \sin(x)x^{x^x}\left\{\ln x(1+\ln x)x^x + x^x\frac{1}{x}\right\}.
\end{aligned}$$

Problem 3. Compute $\lim_{x\to\pi/2}\frac{\cos(3x)}{\cos(x)}$. At this point in the course, you are forbidden from using L'Hôpital's rule.

Solution:

$$
\begin{aligned}
\lim_{x\to\pi/2}\frac{\cos(3x)}{\cos(x)} &= \lim_{x\to\pi/2}\frac{\cos(3x)-\cos(3\pi/2)}{\cos(x)-\cos(\pi/2)} \\
&= 3\lim_{x\to\pi/2}\frac{\cos(3x)-\cos(3\pi/2)}{3x-3\pi/2}\frac{x-\pi/2}{\cos(x)-\cos(\pi/2)} \\
&= 3\lim_{h\to 3\pi/2}\frac{\cos(h)-\cos(3\pi/2)}{h-3\pi/2}\times\frac{1}{\lim_{h\to\pi/2}\frac{\cos(h)-\cos(\pi/2)}{h-\pi/2}} \\
&= 3\frac{d}{du}\cos(u)|_{u=3\pi/2}\times\frac{1}{\frac{d}{du}\cos(u)|_{u=\pi/2}}, \\
&= 3(-\sin(3\pi/2))\times\frac{1}{-\sin(\pi/2)} \\
&= 3(1)(-1) = -3.
\end{aligned}
$$

Problem 4. Compute the derivative $\frac{dy}{dx}$ for the curve $x^2+y^3+xy=7$ at the point $(x,y)=(2,1)$.

Solution:

$$
\begin{aligned}
2x+3y^2y'+xy'+y=0, \\
y' = -\frac{2x+y}{3y^2+x} = -\frac{5}{5} = -1.
\end{aligned}
$$

Problem 5. Find the equation for the tangent line to $y=f(x)=\ln(x/3)$ at $x=3e$.

Solution:

$$
\begin{aligned}
f(3e) &= \ln e = 1, \\
f'(x) &= \frac{1}{x/3}\times\frac{1}{3} = \frac{1}{x}, \\
f'(3e) &= \frac{1}{3e}.
\end{aligned}
$$

Tangent line (with $x_0 = 3e$):

$$
\begin{aligned}
y - f(x_0) &= f'(x_0)(x-x_0), \\
y - 1 &= \frac{1}{3e}(x-3e).
\end{aligned}
$$

Problem 6. Consider the function

$$
f(x) = \begin{cases} -\frac{\sin x}{2x} & \text{if } x<0, \\ 0 & \text{if } x=0, \\ \frac{\cos x - 1}{x^2} & \text{if } x>0. \end{cases}
$$

Is $f(x)$ continuous at $x = 0$? If not, then is the discontinuity removable?

Solution:
We recall the following important limits, which have been previously investigated in this course:

$$\lim_{x\to 0^-} -\frac{\sin x}{2x} = -\frac{1}{2},$$
$$\lim_{x\to 0^+} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

Therefore, since $\lim_{x\to 0} f(x) = -\frac{1}{2} \neq f(0)$, $f(x)$ is not continuous at $x = 0$.

However, if we redefine f by setting $f(0) = -\frac{1}{2}$, then for the new f, we have $\lim_{x\to 0} f(x) = f(0)$. Thus, the redefined f is continuous (and hence the original f has a removable discontinuity at $x = 0$).

Problem 7. Prove the quotient rule $(u/v)' = (u'v - uv')/v^2$ using only the definition of a derivative.

Solution:
Given a point x and a small number Δx, we define

$$\Delta u = u(x + \Delta x) - u(x), \qquad \Delta v = v(x + \Delta x) - v(x).$$

Since u, v are differentiable (and therefore continuous), we have

$$\lim_{\Delta x\to 0} \Delta u = 0,$$
$$\lim_{\Delta x\to 0} \Delta v = 0,$$
$$\lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x} = u',$$
$$\lim_{\Delta x\to 0} \frac{\Delta v}{\Delta x} = v'.$$

Using the above facts, we have that

$$\begin{aligned}
\left(\frac{u}{v}\right)' &= \lim_{\Delta x\to 0} \frac{\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\frac{(u+\Delta u)v-(v+\Delta v)u}{v(v+\Delta v)}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\frac{v\Delta u-u\Delta v}{v(v+\Delta v)}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x}\frac{v}{v(v+\Delta v)} - \lim_{\Delta x\to 0} \frac{\Delta v}{\Delta x}\frac{u}{v(v+\Delta v)} \\
&= \frac{u'v}{v^2} - \frac{v'u}{v^2} \\
&= \frac{u'v - uv'}{v^2}.
\end{aligned}$$

Problem 8. Let r be a real number. Compute $\lim_{h\to 0} \frac{(1+2h)^r-1}{h}$ by interpreting this limit as a derivative.

Solution:

$$\lim_{h\to 0} \frac{(1+2h)^r-1}{h} = \frac{d}{dx}(1+2x)^r|_{x=0} = r(1+2x)^{r-1} \times 2|_{x=0} = 2r.$$