MATH 18.01, FALL 2017 - PROBLEM SET # 7

Professor: Jared Speck

Due: by Thursday 4:00pm on 11-9-17
(in the boxes outside of Room 4-174; write your name, recitation instructor, and recitation meeting days/time on your homework)

18.01 Supplementary Notes (including Exercises and Solutions) are available on the course web page: `http://math.mit.edu/~jspeck/18.01_Fall%202017/1801_CourseWebsite.html`. This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

Part I consists of exercises given and solved in the Supplementary Notes. It will be graded quickly, checking that all is there and the solutions not copied.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises given here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

You are encouraged to use graphing calculators, software, etc. to check your answers and to explore calculus. However, (unless otherwise indicated) we strongly discourage you from using these tools to solve problems, perform computations, graph functions, etc. An extremely important aspect of learning calculus is developing these skills. And you will not be allowed to use any such tools on the exams.

Part I (20 points)

Notation: The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Lecture 20. (Thurs., Nov. 2) Work; average value; probability
Read: 7.7 to middle p. 247; Notes AV.
Homework: p. 249: 5, 6, 15 (solutions posted at the end of this pset); 4D: 2, 3, 5.

Material beyond Lecture 20 will not appear on Midterm # 3

Lecture 21. (Fri., Nov. 3) Numerical Integration.
Read: 10.9.
Homework: Notes 3G: 1ad, 4.

Lecture 22. (Tues., Nov. 7) Trigonometric integrals. Direct substitution.
Read: 10.2, 10.3.
Homework: Notes 5B: 9, 11, 13, 16; Notes 5C: 5, 7, 9, 11.

Part II (44 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. *You must show your work; "bare" solutions will receive very little credit.*

iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem **0**." will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (Thurs., Nov. 2; expected value; 6 points) Let $f(x)$ be a probability density function defined for $0 \le x \le 1$. By definition, this means that $f(x) \ge 0$ and

$$\int_0^1 f(x)\,dx = 1.$$

The above formula means that "the total probability is 1." *Remark: In general, probability density functions are defined for an interval $a \le x \le b$, where $-\infty \le a \le b \le \infty$. The values $a = 0$, $b = 1$ studied in this problem represent a special case.*

Let $q(x)$ be another function of x defined for $0 \le x \le 1$. In statistics, one often studies the "average value of q" relative to f, which is defined to be

$$E[q] = \int_0^1 q(x)f(x)\,dx.$$

The following two quantities are especially important:

$$E[x] = \int_0^1 xf(x)\,dx,$$
$$E[x^2] = \int_0^1 x^2 f(x)\,dx.$$

The letter E stands for "expectation," as in "the expected value of x is $E[x]$." It is also common to use the alternate notation $\bar{x} = E[x]$ (note that $\bar{x}$ is just a number equal to a definite integral).

Prove the well-known statistics formula

$$E[(x - \bar{x})^2] = E[x^2] - \bar{x}^2 = E[x^2] - (E[x])^2$$

(the final expression is just a rewriting of the middle expression using the alternate notation). *Remark: The left-hand side is known as the "variance of f." This is a common measure of the "fatness" of a probability density function.*

Solution: Expand the expression $(x-\overline{x})^2 = x^2 - 2x\overline{x} + \overline{x}^2$. Now take the expectation and use the fact that integration is a linear operation:

$$\begin{aligned}
E[(x-\overline{x})^2] &= E[x^2 - 2x\overline{x} + \overline{x}^2] \\
&= \int_0^1 (x^2 - 2x\overline{x} + \overline{x}^2) f(x)\,dx \\
&= \int_0^1 x^2 f(x)\,dx - 2\overline{x}\int_0^1 x f(x)\,dx + \overline{x}^2 \int_0^1 f(x)\,dx \qquad \text{(since } \overline{x} \text{ is just a number)} \\
&= E[x^2] - 2\overline{x}^2 + \overline{x}^2 \qquad \left(\text{since } \int_0^1 f(x)\,dx = 1\right) \\
&= E[x^2] - \overline{x}^2 \\
&= E[x^2] - (E[x])^2.
\end{aligned}$$

2. (Thurs., Nov. 2; 3 + 3 + 3 + 3 + 3 = 15 points)

a) Suppose that the pair (x, y) is randomly chosen from the unit square $0 \le x \le 1$, $0 \le y \le 1$ with equal probability relative to the area. Let c be a constant. What is the probability that $xy \le c$? (Your answer should split into cases depending on c.)

Solution: Case 1: If $c \ge 1$, then the curve $xy = c$ is above the unit square and so

$$\boxed{P(xy \le c) = 1}.$$

Case 2: If $c \le 0$, then the curve is below the unit square and so

$$\boxed{P(xy \le c) = 0}.$$

Case 3: For $0 < c < 1$, we rewrite the curve $xy = c$ as $y = c/x$.

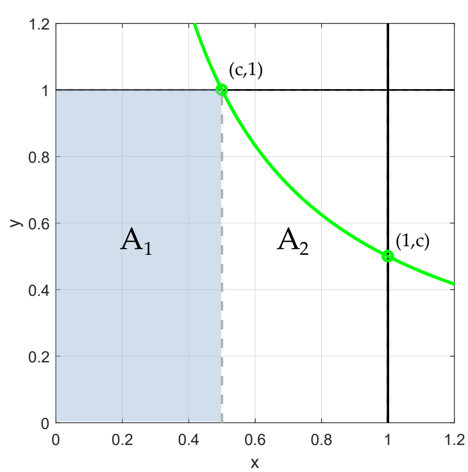

FIGURE 1.

This curve intersects the unit box at the points $(c, 1)$ and $(1, c)$ (as shown in Figure 1). The probability that a point randomly chosen will satisfy $xy \leq c$ can be thought of as the following:

$$P(xy \leq c) = \frac{\text{area under curve } xy \text{ and bounded by square}}{\text{total area of square}} = \frac{A_1 + A_2}{1} = A_1 + A_2$$

where A_1 and A_2 consist of the areas shown in Figure 1. Area A_1 is the rectangle to the left of the point $(c, 1)$ and has been shaded for clarity. Area A_2 is the area under the curve $y = c/x$ from points $(c, 1)$ to $(1, c)$.

Now we compute the two areas. The first is just the area of the rectangle:

$$A_1 = \text{base} \cdot \text{height} = c \cdot 1 = c.$$

The second is the definite integral with appropriate bounds:

$$A_2 = \int_c^1 \frac{c}{x}\, dx = c(\ln(1) - \ln(c)) = -c\ln(c).$$

So the total probability is given by

$$\boxed{P(xy \leq c) = A_1 + A_2 = c - c\ln(c) = c(1 - \ln(c))}.$$

b) Let $a > 0$ be a constant. Evaluate

$$W = \int_0^\infty e^{-at}\, dt = \lim_{N\to\infty} \int_0^N e^{-at}\, dt.$$

This is known as an improper integral because it represents the area of an unbounded region. We are using the letter W to signify "whole."

Solution:

$$W = \int_0^\infty e^{-at}\, dt = \lim_{N\to\infty} \int_0^N e^{-at}\, dt = \lim_{N\to\infty} \left[-\frac{1}{a}\left(e^{-aN} - 1\right)\right] == -\frac{1}{a} \cdot (-1) = \boxed{\frac{1}{a}}$$

The probability that a radioactive particle will decay at some time in the interval $0 \leq t \leq T$ is

$$P([0, T]) = \frac{PART}{WHOLE} = \frac{1}{W} \int_0^T e^{-at}\, dt,$$

where the time variable t carries the units of "hours". Note that $P([0, \infty)) = 1 = 100\%$.

c) The half-life is the time T for which $P([0, T]) = 1/2$. Find the value of a and W for which the half-life is $T = 1$ hour. Suppose that a radioactive particle has a half-life of 1 hour. What is the probability that it survives for 10 hours? Then answer the same question in the case of 100 hours.

Solution: Recall from part (b) that $W = 1/a$. So for the half-life with $T = 1$, we compute

$$P([0, 1]) = \frac{1}{W} \int_0^1 e^{-at}\, dt = a \int_0^1 e^{-at}\, dt = a\left(-\frac{1}{a}\right)(e^{-a} - 1) = 1 - e^{-a}.$$

We know that the half life is defined such that

$$P([0, 1]) = 1/2.$$

This allows us to solve for a

$$\frac{1}{2} = 1 - e^{-a} \implies e^{-a} = \frac{1}{2} \implies -a = \ln(1/2) \implies a = -\ln(1/2) \implies \boxed{a = \ln(2)}$$

and

$$\boxed{W = \frac{1}{a} = \frac{1}{\ln(2)}}.$$

The probability that the particle has decayed within 10 hours is given by $P([0,10])$. The probability that this has *not* happened is $1 - P([0,10])$. Since the particle has a half-life of 1 hours, we use the values for a and W from above to find that the probability of the particle surviving 10 hours is

$$\begin{aligned} 1 - P([0,10]) &= 1 - \ln(2) \int_0^{10} e^{-\ln(2)t}\, dt \\ &= 1 - \ln(2)\left[-\frac{1}{\ln(2)}\left(e^{-10\ln(2)} - 1\right)\right] \\ &= 1 + (e^{2^{-10}} - 1) \\ &= \boxed{2^{-10} \approx 9.766 \times 10^{-4}}. \end{aligned}$$

We use the same idea to find that the probability of the particle surviving 100 hours is

$$\begin{aligned} 1 - P([0,100]) &= 1 - \ln(2) \int_0^{100} e^{-\ln(2)t}\, dt \\ &= 1 - \ln(2)\left[-\frac{1}{\ln(2)}\left(e^{-100\ln(2)} - 1\right)\right] \\ &= 1 + (e^{2^{-100}} - 1) \\ &= \boxed{2^{-100} \approx 7.889 \times 10^{-31}}. \end{aligned}$$

The next two parts illustrate the idea of "fat tails" in statistics. In particular, you will examine how "long-time" probabilities are drastically affected when exponential decay is replaced with polynomial decay.

d) Repeat part b) with the function $1/(1+t^2)$ in place of e^{-at}.

Solution: We get

$$W = \int_0^\infty \frac{1}{1+t^2}\, dt = \lim_{N\to\infty} \int_0^N \frac{1}{1+t^2}\, dt = \lim_{N\to\infty} [\arctan(N) - \arctan(0)] = \lim_{N\to\infty} \arctan(N) = \boxed{\frac{\pi}{2}}.$$

Suppose there is a second (fictitious) radioactive particle known as a Slowon. The probability that the Slowon will decay some time in the interval $0 \le t \le T$ is

$$P([0,T]) = \frac{PART}{WHOLE} = \frac{1}{W} \int_0^T \frac{1}{1+t^2}\, dt,$$

where W is the number that you calculated in part d).

e) What is the probability that the Slowon will survive for 10 hours? With the help of a calculator, give your answer to 4 significant figures. Then answer the same question in the case of 100 hours. How do your answers compare to your answers from part c)?

Solution: By the same argument in (c), the probability that Slowon will survive at least 10 hours is

$$1 - P([0,10]) = 1 - \frac{1}{\pi/2}\int_0^{10} \frac{1}{1+t^2}\,dt = \boxed{1 - \frac{2}{\pi}\arctan(10) \approx 6.345 \times 10^{-2}}$$

The probability that Slowon will survive at least 100 hours is

$$1 - P([0,100]) = 1 - \frac{1}{\pi/2}\int_0^{100} \frac{1}{1+t^2}\,dt = \boxed{1 - \frac{2}{\pi}\arctan(100) \approx 6.366 \times 10^{-3}}$$

In part (c), the particle had a much lower probability of surviving for 10 hours compared to Slowon and an even lower probability of surviving 100 hours compared to Slowon. The probability that the particle in part (c) survies 100 hours dropped *significantly* compared to the probability that the particle survives 10 hours. In the case of Slowon, the probability that the particle survives 100 hours dropped only slightly from the probability that the particle survives 10 hours.

3. (Fri., Nov. 3; justification of Simpson's rule; 8 points) The basis for Simpson's rule is the following formula. Let x_1 be the midpoint of the interval $[x_0, x_2]$, and h denote half of the length of $[x_0, x_2]$. Consider any three points (x_0, y_0), (x_1, y_1), (x_2, y_2). There is a unique parabola $y = f(x) = Ax^2 + Bx + C$ whose graph passes through the three points. Simpsons rule says that the area under the parabola and above $[x_0, x_2]$ is

$$\frac{h(y_0 + 4y_1 + y_2)}{3}.$$

This problem is devoted to proving this formula. It is significant because it illustrates how calculations can be simplified by using symmetry and by looking ahead to see what you need. Since the area will be the same if the parabola is translated to the left or right, we may assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then in terms of the "data" h, y_0, y_1, y_2, make a sketch and determine C. Show, by integrating, that to find the area we only need to find A in terms of the data (more precisely, as you will see, it is actually better to find $2Ah^2$ in terms of the data). Then find $2Ah^2$ in terms of the data. Finally, combine the results to establish the above formula for area.

Solution: Substituting $(x_1, y_1) = (0, y_1)$ into the equation for $f(x)$ gives

$$y_1 = A(0)^2 + B(0) + C \implies \boxed{C = y_1}.$$

Integrating from $[x_0, x_2] = [-h, h]$ gives

$$\begin{aligned}
\int_{-h}^{h} (Ax^2 + Bx + C)\,dx &= \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right)\Bigg|_{-h}^{h} \\
&= \left(\frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch\right) - \left(\frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h)\right) \\
&= \frac{2A}{3}h^3 + 2hC \\
&= \frac{2A}{3}h^3 + 2hy_1 \qquad \text{(since } C = y_1\text{).}
\end{aligned}$$

We see that the term involving B cancels and we really only need to find A. As hinted at in the problem statement, we will find $2Ah^2$ since

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \frac{2A}{3}h^3 + 2hy_1 = 2Ah^2 \cdot \frac{h}{3} + 2hy_1.$$

We will solve the system of two equations we get from substituting x_0 and x_2 into $f(x)$. Namely,

$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + y_1$$
$$y_2 = A(h)^2 + B(h)_C = Ah^2 + Bh + y_1.$$

Adding these gives

$$y_0 + y_2 = 2Ah^2 + 2y_1 \implies \boxed{2Ah^2 = y_0 - 2y_1 + y_2}.$$

Substituting this back into our equation for the area gives

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \frac{2A}{3}h^3 + 2hy_1 = 2Ah^2 \cdot \frac{h}{3} + 2hy_1 = (y_0 - 2y_1 + y_2) \cdot \frac{h}{3} + 2hy_1 = \frac{h(y_0 + 4y_1 + y_2)}{3}$$

as desired.

4. (Fri., Nov. 3; application of Simpson's rule; 6 points) Use a calculator to make a table of values of the integrand and find approximations to the integral

$$\int_0^1 \mathrm{sinc}(x)\,dx,$$

where

$$\mathrm{sinc}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{\sin x}{x} & \text{if } x \neq 0 \end{cases}$$

(see problem 1 from pset 6). Use Simpson's rule with four and eight (equal-length) intervals. (The exact answer to 10 decimal places is .9460830704. Record your approximations to six decimal places to compare).

Solution: Simpson's rule with 4 equal intervals means $h = 1/4$. Define $f(x) = \mathrm{sinc}(x)$. The table of values we need is then

x-value	$f(x)$ value
$x_0 = 0$	$y_0 = 1$
$x_1 = 1/4$	$y_1 = 0.989616$
$x_2 = 1/2$	$y_2 = 0.958851$
$x_3 = 3/4$	$y_3 = 0.908852$
$x_4 = 1$	$y_4 = 0.841471$

Applying Simpson's rule gives

$$\begin{aligned}\int_0^1 \mathrm{sinc}(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\ &\approx \frac{1/4}{3}(1 + 4 \cdot 0.989616 + 2 \cdot 0.958851 + 4 \cdot 0.908852 + 0.841471) \\ &\approx \boxed{0.946087}.\end{aligned}$$

This gives an error E_4 of

$$E_4 \approx |0.946087 - 0.946083| = 4 \times 10^{-6}.$$

to six decimal places.

To approximate using 8 equal intervals, we have $h = 1/8$ and we expand our table to include

x-value	$f(x)$ value
$x_0 = 0$	$y_0 = 1$
$x_1 = 1/8$	$y_1 = 0.997398$
$x_2 = 1/4$	$y_2 = 0.989616$
$x_3 = 3/8$	$y_3 = 0.9767267$
$x_4 = 1/2$	$y_4 = 0.958851$
$x_5 = 5/8$	$y_5 = 0.9361556$
$x_6 = 3/4$	$y_6 = 0.908852$
$x_7 = 7/8$	$y_7 = 0.8771926$
$x_8 = 1$	$y_8 = 0.841471$

Applying Simpson's rule gives

$$\begin{aligned}\int_0^1 \operatorname{sinc}(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8)\\ &\approx \frac{1/8}{3}(1 + 4\cdot 0.997398 + 2\cdot 0.989616 + 4\cdot 0.9767267 + 2\cdot 0.958851 + \cdots\\ &\cdots + 4\cdot 0.9361556 + 2\cdot 0.908852 + 4\cdot 0.8771926 + 0.841471)\\ &\approx \boxed{0.946083}.\end{aligned}$$

This gives an error E_8 of

$$E_8 \approx |0.946083 - 0.946083| = 0$$

to six decimal places.

5. (Tues., Nov. 7; trigonometric integrals; 3 + 3 = 6 points)

a) For any integer $n \geq 0$, use the substitution $\tan^2 x = \sec^2 x - 1$ to show that

$$\int \tan^{n+2} x\,dx = \frac{1}{n+1}\tan^{n+1} x - \int \tan^n x\,dx.$$

Solution: We split the integrand to get

$$\int \tan^{n+2} x\,dx = \int \tan^n x \cdot \tan^2 x\,dx.$$

Now, we replace $\tan^2 x$ by $sec^2 x - 1$ to get

$$\int \tan^{n+2} x\,dx = \int \tan^n x \cdot \tan^2 x\,dx = \int \tan^n x \cdot (\sec^2 x - 1)\,dx = \int \tan^n x \sec^2 x\,dx - \int \tan^n x\,dx.$$

The second integral on the right-hand side is fine since it's part of what we want to show. We'll use u-substitution to compute the first integral on the right-hand side. Choose $u = \tan x$, $du = \sec^2 x\,dx$ to get

$$\int \tan^n x \sec^2 x\,dx = \int u^n\,du = \frac{1}{n+1}u^{n+1} + C = \frac{1}{n+1}\tan^{n+1} x + C.$$

Putting this all together gives

$$\int \tan^{n+2} x\, dx = \int \tan^n x \sec^2 x\, dx - \int \tan^n x\, dx = \frac{1}{n+1} \tan^{n+1} x - \int \tan^n x\, dx$$

where we've allowed the constant term to be absorbed by the second integral.

b) Use part a) to compute $\int \tan^4 x\, dx$.

Solution: Using part (a) with $n = 2$, we get

$$\int \tan^4 x\, dx = \int \tan^{2+2} x\, dx = \frac{1}{3} \tan^3 x - \int \tan^2 x\, dx.$$

Using part (a) again to compute the last term with $n = 0$ gives

$$\int \tan^2 x\, dx = \int \tan^{0+2} x\, dx = 1 \cdot \tan x - \int \tan^0 x\, dx = \tan x - \int 1\, dx = \tan x - x + C.$$

Putting this all together gives

$$\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Solutions to problems on pg. 249:

5) Let h be the height that the bucket has been lifted, and let $s(h)$ denote the weight of the sand left in the bucket at height h. We are told that $s(h)$ is a linear function, i.e., $s(h) = Ah + B$ for some constants A and B. We are given that $s(0) = 60$ pounds and $s(10) = 40$ pounds. Therefore,

$$A(0) + B = 60$$
$$A(10) + B = 40.$$

It follows that $B = 60$ and $A = -2$, which implies that $s(h) = -2h + 60$. The bucket weights 5 pounds, which means that the weight w of the bucket + sand at height h is $w(h) = -2h + 65$.

The amount of work dW done in lifting the bucket through a tiny height dh is therefore $dW = w(h) \times dh$. Therefore, the total work done in lifting the bucket 10 feet is

$$\begin{aligned} W = \int dW = \int_{h=0}^{h=10} w(h)\, dh = \int_{h=0}^{h=10} -2h + 65\, dh \\ = \left[-h^2 + 65h\right]_{h=0}^{h=10} = 550\text{ft.-pounds.} \end{aligned}$$

6) Let y be the distance from the top of the cable down to a tiny chunk of rope of small width dy. The gravitational force on this chunk is $4lbs./ft. \times dy = 4dy$. This chunk will be raised by a distance y. Thus, total amount of work done in raising this chunk is $dW = 4ydy$. Therefore, the total work is

$$Work = \int dW = \int_{y=0}^{y=100} 4ydy = 2y^2 \mid_{y=0}^{y=100} = 20000\text{ft.-pounds.}$$

15) According to Newton's law of gravitation, the magnitude of the total force on each of the two bodies is

$$F = G\frac{Mm}{r^2},$$

where G is the constant of gravitation and r is the distance between them. The amount of work done in moving the particles through a distance dr is $dW = Fdr$. Therefore, the total work done in moving the particles from a distance of a to a distance $2a$ is

$$\begin{aligned} W = \int F\, dr &= \int_a^{2a} G\frac{Mm}{r^2}\, dr \\ &= \left[-GMmr^{-1}\right]_{r=a}^{r=2a} \\ &= -GMm\left(\frac{1}{2a} - \frac{1}{a}\right) \\ &= \frac{GMm}{2a}. \end{aligned}$$