MATH 18.01, FALL 2014 – PROBLEM SET #7

1. We have

$$E[(x-\bar{x})^2] = \int_0^1 (x-\bar{x})^2 f(x)dx = \int_0^1 x^2 f(x)dx + \bar{x}^2 \int_0^1 f(x)dx - 2\bar{x}\int_0^1 xf(x)dx$$
$$= E(x^2) + \bar{x}^2 E(1) - 2\bar{x}E(x) = E(x^2) + \bar{x}^2 - 2\bar{x}^2 = E(x^2) - \bar{x}^2.$$

2. (a) Let R be the region of the unit square where $xy \le c$. The probability that $xy \le c$ is then equal to the area of R divided by the total area of the square, which is 1. The region R is the part of the unit square which lies below the graph of $y = c/x$. If $c \le 1$, then the whole square is below that graph, so R is the whole square and the probability is 1. If $c < 1$, the graph intersects the top side of the square at $x = c$, so the area of the region R is

$$c + \int_c^1 \frac{c}{x}dx = c + c\ln(1) - c\ln(c) = c(1-\ln(c)).$$

(b) $W = \lim_{N\to\infty} \int_0^N e^{-at}dt = \lim_{N\to\infty} \frac{1}{-a}e^{-at}\Big|_{t=0}^{t=N} = \lim_{N\to\infty} \frac{1}{a}(e^0 - e^{-aN}) = \frac{1}{a}.$

(c) By (b), we have

$$P([0,T]) = a\int_0^T e^{-at}dt = 1 - e^{-aT}.$$

If $P([0,T]) = 1/2$, then $e^{-aT} = 1/2$, so the half-life is $T = \frac{1}{a}\ln(2)$ hours. If a particule has a half-life of 1 hour, then $a = \ln(2)$, so

$$P([0,T]) = 1 - 2^{-T} = \frac{2^T - 1}{2^T}.$$

In particular, $P([0,10]) = 1023/1024 = 0.9990234375$ and

$$P([0,100]) = \frac{1,267,650,600,228,229,401,496,703,205,375}{1,267,650,600,228,229,401,496,703,205,376}$$
$$\approx 0.9999999999999999999999999999999.$$

(d) $W = \lim_{N\to\infty} \int_0^N \frac{1}{1+t^2}dt = \lim_{N\to\infty} \arctan(N) = \pi/2.$

(e) By (d),

$$P([0,T]) = \frac{2}{\pi}\int_0^T \frac{1}{1+t^2}dt = \frac{2}{\pi}\arctan(T).$$

Thus, $P([0,10]) \approx 0.9365$ and $P([0,100]) \approx 0.9936$.

3. Assume without loss of generality that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. The number C is the intersection of the parabola with the y-axis, so $C = y_1$. The area beneath the parabola is

$$\int_{-h}^h (Ax^2 + Bx + C)dx = \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right)\Big|_{-h}^h = \frac{2A}{3}h^3 + 2hy_1.$$

We have $y_0 = Ah^2 - Bh + C$ and $y_2 = Ah^2 + Bh + C$, whence

$$2Ah^2 = y_0 + y_2 - 2y_1.$$

Thus, the area is

$$\frac{h}{3}(y_0 + y_2 - 2y_1) + 2hy_1 = \frac{h}{3}(y_0 + y_2 + 4y_1).$$

4.

x	0	1/8	1/4	3/8	1/2	5/8	3/4	7/8	1
sinc(x)	1	.997398	.989616	.976727	.958851	.936156	.908851	.877193	.841471

Recall that Simpson's rule with $2n$ intervals of length h is

$$\int_a^b f(x)dx \approx \frac{h}{3}\sum_{i=1}^{n}(y_{2i-2} + 4y_{2i-1} + y_{2i}).$$

Using 4 intervals of length 1/4, we get $\int_0^1 \mathrm{sinc}(x)dx \approx .946087$.
Using 8 intervals of length 1/8, we get $\int_0^1 \mathrm{sinc}(x)dx \approx .946083$.
The latter matches the actual value.

5. (a) Let $u = \tan x$, $du = \sec^2 x\, dx$. Then

$$\int \tan^{n+2} x\, dx = \int \tan^n x(\sec^2 x - 1)dx$$
$$= \int u^n du - \int \tan^n x\, dx = \frac{1}{n+1}\tan^{n+1} x - \int \tan^n x\, dx.$$

(b) Applying (a) for $n = 2$ and for $n = 0$, we get:

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\, dx = \frac{1}{3}\tan^3 x - \tan x + \int \tan^0 x\, dx$$
$$= \frac{1}{3}\tan^3 x - \tan x + x.$$

6. It is possible to find the area of the segment by elementary means, by taking the area of the whole sector and subtracting the area of the triangle. This gives:

$$\text{area(segment)} = \text{area(sector)} - \text{area(triangle)} = a^2 \arccos(b/a) - b\sqrt{a^2 - b^2}.$$

We can give a different proof using integration. If we place the x-axis on the chord with the origin at the closest point from the center of the circle, the segment becomes the region below the graph of $y = \sqrt{a^2 - x^2} - b$ between $\pm\sqrt{a^2 - b^2}$. Let's abbreviate $\sqrt{a^2 - b^2}$ to c and let

$$\alpha = \arccos(b/a) = \arcsin(c/a)$$

be the half-angle of the sector. Using the substituion $x = a\sin\theta$, we have

$$\sqrt{a^2 - x^2} = a\cos\theta \quad \text{and} \quad dx = a\cos\theta\, d\theta,$$

whence

$$\text{area(segment)} = \int_{-c}^{c}(\sqrt{a^2 - x^2} - b)dx = \int_{-\alpha}^{\alpha}(a^2\cos^2\theta - ba\cos\theta)d\theta$$
$$= \left[a^2\frac{\theta}{2} + a^2\frac{2\sin\theta\cos\theta}{4} - ba\sin\theta\right]_{-\alpha}^{\alpha} = a^2\alpha + bc - 2bc = a^2\alpha - bc.$$

Thus we recover the above formula.