MATH 18.01 - MIDTERM 4 REVIEW: SUMMARY OF SOME KEY CONCEPTS

18.01 Calculus, Fall 2017	Professor: Jared Speck

a. Numerical integration

(a) Riemann sums $\int_a^b f(x)\,dx \approx \sum_{i=1}^n y_i \Delta x$
  (i) $y_i = f(x_i)$
  (ii) The x_i belong to the i^{th} subinterval (the precise definition of x_i depends on whether you are using right sums, left sums, upper sums, lower sums, etc.)
  (iii) $\Delta x = \frac{b-a}{n}$
  (iv) Upper sums: always use the tallest possible rectangle for each subinterval
  (v) Lower sums: always use the shortest possible rectangle for each subinterval
  (vi) Left sums: the rectangle height for each subinterval is determined by the point on the graph of $y = f(x)$ lying above the left endpoint
  (vii) Right sums: the rectangle height for each subinterval is determined by the point on the graph of $y = f(x)$ lying above the right endpoint

(b) Trapezoid rule $\displaystyle\int_a^b f(x)\,dx \approx \Delta x \left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2} \right)$
  (i) $\Delta x = \dfrac{b-a}{n}$
  (ii) $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\cdots$, $x_n = a + n\Delta x = b$

(c) Simpson's method (n must be even)
  (i) $\displaystyle\int_a^b f(x)\,dx \approx \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$
  (ii) $\Delta x = \dfrac{b-a}{n}$
  (iii) $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\cdots$, $x_n = a + n\Delta x = b$

b. Computing $\int (\sin x)^n (\cos x)^m\,dx$

(a) If m is odd, let $u = \sin x$, $du = \cos x\,dx$, and substitute $(\cos x)^2 = 1 - (\sin x)^2$ to transform the integral into a u integral.

(b) If n is odd, interchange the roles of $\sin x$ and $\cos x$ and proceed as above.

(c) If m, n both even, make repeated use of the trig identities $(\cos x)^2 = \frac{1}{2}[1 + \cos(2x)]$ and $(\sin x)^2 = \frac{1}{2}[1 - \cos(2x)]$.

c. Computing $\int (\sec x)^n (\tan x)^m\,dx$

(a) $\int \tan x\,dx = -\ln|\cos x| + C$.

(b) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.

(c) $\int (\sec x)^2\, dx = \tan x + C$.
(d) $\int \sec x \tan x\, dx = \sec x + C$.
(e) If m is odd, let $u = \sec x, du = \sec x \tan x$ and substitute $(\tan x)^2 = (\sec x)^2 - 1$ to transform the integral into a u integral.
(f) If n is even, let $u = \tan x, du = (\sec x)^2$ and substitute $(\sec x)^2 = 1 + (\tan x)^2$ to transform the integral into a u integral.
(g) If m is even and n is odd, then we haven't studied how to evaluate the integral.

d. Inverse trig substitution

(a) Is useful for evaluating $\int \sqrt{ax^2 + bx + c}\, dx$ because it gets rid of the square root (when a, b, c are constants).
(b) To evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$, let $x = \tan u$, $dx = (\sec u)^2\, du$, and substitute $(\tan u)^2 + 1 = (\sec u)^2$.
(c) To evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$, let $x = \sec u$, $dx = \sec u \tan u\, du$, and substitute $(\sec u)^2 - 1 = (\tan u)^2$.
(d) To evaluate $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$, let $x = \sin u$, $dx = \cos u\, dx$, and substitute $1 - (\sin u)^2 = (\cos u)^2$.
(e) To evaluate e.g. $\int \frac{dx}{\sqrt{x^2+2x+2}}$, first complete the square: $x^2 + 2x + 2 = (x+1)^2 + 1$. Then let $v = x + 1$, $dv = dx$, and proceed as above.
(f) You can draw a suitable right triangle to help you express the final answer in terms of x.

e. Partial fractions

(a) Is a strategy for evaluation $\int \frac{P(x)}{Q(x)}$, where P, Q are polynomials and the degree of P is $<$ the degree of Q.
(b) You have to factor $Q(x)$ to its fullest extent.
(c) If $Q(x) = (x+a)(x+b)$, guess $\frac{P(x)}{Q(x)} = \frac{A}{x+a} + \frac{B}{x+b}$ and solve for the constants A, B using e.g. the cover-up method. Then integrate the right-hand side using prior techniques.
(d) If $Q(x) = (x+a)(x+b)^2$, guess $\frac{P(x)}{Q(x)} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{(x+b)^2}$ and solve for the constants A, B, C (the cover-up method does not work for B.) Then integrate the right-hand side using prior techniques.
(e) If $Q(x) = (x+a)^2(x+b)^3$, guess $\frac{P(x)}{Q(x)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b} + \frac{D}{(x+b)^2} + \frac{E}{(x+b)^3}$ (hopefully the general pattern for a guess is clear).
(f) If $Q(x) = (x+a)(x^2+b)$, guess $\frac{P(x)}{Q(x)} = \frac{A}{x+a} + \frac{B_0+B_1x}{x^2+b}$ and solve for the constants A, B_0, B_1 (the cover-up method works only on A.) Then integrate the right-hand side using prior techniques. A similar idea would allow you to treat other quadratic factors (with no real roots) in place of $x^2 + b$ (you might have to complete the square first).
(g) If $Q(x) = (x+a)^2(x^2+b)^2$, guess $\frac{P(x)}{Q(x)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C_0+C_1x}{x^2+b} + \frac{D_0+D_1x}{(x^2+b)^2}$ (again, hopefully the general pattern for a guess is clear).

f. Integration by parts
  (a) Is simply the product rule combined with FTC 1:
  (b) $\int u\,dv = uv - \int v\,du$

g. Parametric curves
  (a) Are curves in the (x, y) plane expressed as

$$x = F(t),$$
$$y = G(t),$$

$a \le t \le b$, where t is called the *parameter*.

h. Arc length of a curve
  (a) Arc length is equal to $\int_a^b ds$.
  (b) a is the parameter starting point, b is the parameter end point.
  (c) For curves in parametric form, $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(F'(t))^2 + (G'(t))^2}\, dt$ (Pythagorean theorem).
  (d) For curves $y = f(x)$, the formula reduces to $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (f'(x))^2}\, dx$ (and x is the parameter).

i. Surface area of a solid formed by revolving a curve around the x-axis (for revolution around the y-axis, interchange the roles of x and y in everything that follows)
  (a) Divide the surface into small strips that are portions of cones (the cone strip radii are parallel to the y-axis, and the cone strip axes of symmetry are parallel to the x-axis).
  (b) Surface area is given by
  $\int$ conical strip circumference $\times$ slant edge length
  $= \int 2\pi$ conical strip radius $\times\, ds$
  $= \int_{t=a}^{t=b} 2\pi \overbrace{G(t)}^{y} \overbrace{\sqrt{(F'(t))^2 + (G'(t))^2}}^{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}\, dt.$
  (c) a is the parameter starting point, b is the parameter end point.
  (d) For curves $y = f(x)$, the formula reduces to $\int_{x=a}^{x=b} 2\pi \overbrace{f(x)}^{y} \overbrace{\sqrt{1 + (f'(x))^2}}^{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\, dx.$