Exponential & Log, Logarithmic Differentiation, Hyperbolic functions

- $a > 1$ (a is the base)
- $a^0 = 1$, $a^1 = a$, $a^2 = a \cdot a$
- $a^{x_1 + x_2} = a^{x_1} a^{x_2}$
- $(a^{x_1})^{x_2} = a^{x_1 x_2}$
- $a^{p/q} = \sqrt[q]{a^p}$ when p, q are integers
- To define a^r for all real #s r, just "fill in by continuity"

Main goal: Compute $\frac{d}{dx} a^x$:

- $$\frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x}$$

$$= \boxed{M(a) \cdot a^x},$$ where we have named

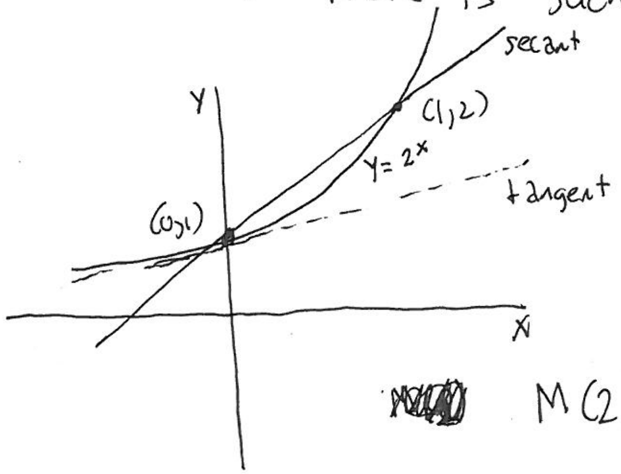
$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

Two ways of thinking about $M(a)$:

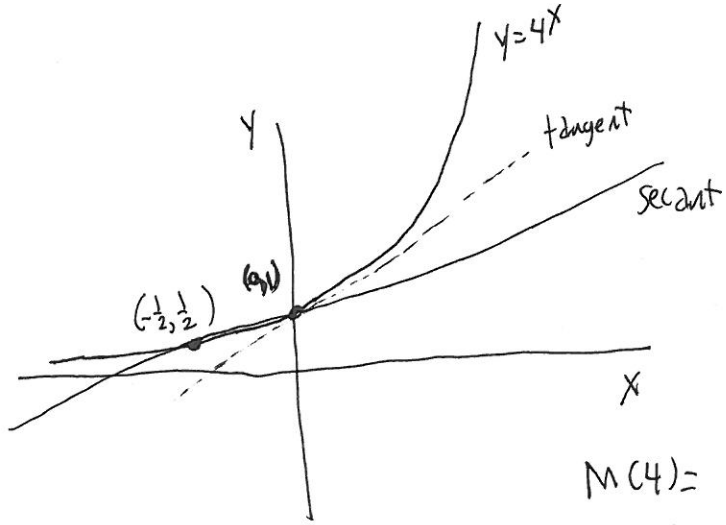
① Analytically : $M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = \frac{d}{dx} a^x \Big|_{x=0}$

② Geometrically: $M(a)$ is the slope of the graph $y = a^x$ at $x = 0$.

- We want to define e to be the number such that $M(e)=1$. Let's see why there is such a number.



$$
\begin{aligned}
M(2) &= \text{slope of tangent at } x=0 \\
&< \text{slope of secant} \\
&= 1.
\end{aligned}
$$



$$
\begin{aligned}
M(4) &= \text{slope of tangent at } x=0 \\
&> \text{slope of secant} \\
&= 1
\end{aligned}
$$

- Somewhere in between 2 and 4, there is a number e with $M(e)=1$.

- e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

- $\frac{d}{dx} e^x = M(e) = 1$ when $x = 0$.

- A closely related function is $\ln(x)$, which is sometimes denoted by $\log(x)$.

- $\ln(x)$ is defined to be the inverse function of e^x:

$$\boxed{\text{if } \; y = e^x, \; \text{ then } \; \ln(y) = x}$$

$$\boxed{\text{if } \; w = \ln(x), \; \text{ then } \; e^w = x}$$

- Basic properties:
  - $\ln(1) = 0$, $\ln(x) < 0$ for $0 < x < 1$, $\ln(x) > 0$ for $x > 1$.
  - $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$

- We can use implicit differentiation to compute $\frac{d}{dx} \ln(x)$:

  - $w = \ln(x)$

$\Rightarrow$ - $e^w = x$

- $\frac{d}{dx} e^w = \frac{d}{dx} x = 1$

- $\frac{d}{dx} e^w = \underbrace{\frac{d}{dw} e^w}_{e^w} \cdot \frac{dw}{dx} = 1$

  - $e^w \frac{dw}{dx} = 1$

  - $\frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x}$

  - $\boxed{\frac{dw}{dx} = \frac{1}{x} \quad \text{when } w = \ln(x)}$

- How to compute $\frac{d}{dx} a^x$

Method 1): • $a = e^{\ln a}$

• $a^x = (e^{\ln a})^x = e^{x \ln a}$

• $\frac{d}{dx} a^x = \frac{d}{dx} e^{x\ln a} \overset{\text{Chain rule}}{=} e^{x \ln a} \cdot \underbrace{\frac{d}{dx}(x\ln a)}_{\ln a}$

$= a^x \cdot \ln a$

$\Rightarrow \boxed{\frac{d}{dx} a^x = \ln a \cdot a^x}$

• Recall: $\frac{d}{dx} a^x = M(a)\, a^x$.

• Thus, $M(a) = \ln(a)$.

Method 2): Logarithmic differentiation:

• The basic idea is to compute $\frac{d}{dx} f(x)$ by first computing $\frac{d}{dx} \ln(f(x))$ (which might be easier to compute).

• Set $u = f(x)$. Then

• $\frac{d}{dx} \ln u = \frac{d}{du} \ln u \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$. Since $\frac{du}{dx} = f'$, we have

$$\boxed{(\ln f)' = \frac{f'}{f} \quad \text{or} \quad f' = f\,(\ln(f))'}$$

- Let's apply this to $f(x) = a^x$
- $\ln f(x) = x \ln a$

$$\Rightarrow \frac{d}{dx} \ln f = \frac{d}{dx} \ln a^x = \frac{d}{dx}(x \ln a) = \ln a$$

$$\Rightarrow \frac{f'}{f} = \ln a \Rightarrow f' = \ln a \, f$$

$$\Rightarrow \boxed{\frac{d}{dx} a^x = \ln a \, a^x}$$

- Ex: . $f(x) = (\sin x + 2)^{\tan x}$

. $\ln f(x) = \tan(x) \ln(\sin x + 2)$

. $\frac{d}{dx} \ln f(x) = \frac{\tan x}{\sin x + 2} \cdot \cos x + \sec^2 x \ln(\sin x + 2)$

$$\Rightarrow \frac{f'}{f} = \left(\frac{\tan x}{\sin x + 2} \cdot \cos x + \sec^2 x \ln(\sin x + 2) \right)$$

$$\Rightarrow f' = (\sin x + 2)^{\tan x} \left(\frac{\tan x}{\sin x + 2} \cos x + \sec^2 x \ln(\sin x + 2) \right)$$

<u>Ex:</u> We can use ln to evaluate:

$$\lim_{K\to\infty} \left(1+\frac{1}{K}\right)^K$$

- Set $f(K) = \left(1+\frac{1}{K}\right)^K$
- $g(K) = \ln f(K) = K \ln\left(1+\frac{1}{K}\right) = \dfrac{\ln\left(1+\frac{1}{K}\right)}{\frac{1}{K}}$
- We are interested in the behavior of $g(K)$ and $f(K)$ as $K\to\infty$.
- We can set $h = \frac{1}{K}$ and let $h\to 0$ instead:

$$\lim_{K\to\infty} g(K) = \lim_{h\to 0} \frac{\ln(1+h)}{h} = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h}$$

$$\stackrel{\text{def}}{=} \frac{d}{dx} \ln x \Big|_{x=1} = 1.$$

- Since $g(K)\to 1$ as $K\to\infty$,

$$\left(1+\frac{1}{K}\right)^K = f(K) = e^{g(K)} \to e^1 = e.$$

- Thus, $\displaystyle\lim_{K\to\infty} \left(1+\frac{1}{K}\right)^K = e.$

Hyperbolic Sine + Cosine

- $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$
- $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$
- $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$
- $\dfrac{d}{dx}\sinh(x) = \dfrac{d}{dx}\left(\dfrac{e^{x}-e^{-x}}{2}\right) = \dfrac{e^{x} - (-e^{-x})}{2} = \cosh x$
- Can check: $\dfrac{d}{dx}\cosh(x) = \sinh(x)$
- Basic Identity: $\cosh^2(x) - \sinh^2(x) = 1$.

Proof: $\cosh^2(x) - \sinh^2(x) = \left(\dfrac{e^{x}+e^{-x}}{2}\right)^2 - \left(\dfrac{e^{x}-e^{-x}}{2}\right)^2$

$$= \frac{1}{4}\left(\cancel{e^{2x}} + 2e^{x}\cdot e^{x} + \cancel{e^{-2x}}\right) - \frac{1}{4}\left(\cancel{e^{2x}} - 2e^{x}\cdot e^{-x} + \cancel{e^{-2x}}\right)$$

$$= \frac{1}{4}(2+2) = 1.$$

- Remark: If $u = \cosh(x)$ and $v = \sinh(x)$, then $u^2 - v^2 = 1$. Since this equation describes a hyperbola, sinh and cosh are called the "hyperbolic functions"