18.01 Solutions to Problem Set 1 Part II

Problem 1: Nonexistence of sin and cos cancellations

a) Since $E(x)$ is an even function, $E(-x) = E(x)$ for all real numbers x. Similarly, since O is odd, $O(-x) = -O(x)$ for all $x \in \mathbb{R}$. Suppose now that

$$E(x) + O(x) = 0$$

for all x. Since every real number x has a negative $-x$, we also have

$$E(-x) + O(-x) = 0$$

for all x. Using the equations we derived above, we get

$$E(x) - O(x) = 0$$

for all x. Adding the first and third equation together, we see that $2E(x) = 0$ for all x and hence E is just the 0 function. Similarly, subtracting the two equations tells us that $2O(x) = 0$ for all x and hence O is also the 0 function.

b) Suppose $A\sin(ax) + B\cos(bx) = 0$ for all x. If we let $E(x) = B\cos(bx)$ and $O(x) = A\sin(ax)$, then we see that E is an even function, O is an odd function and E and O satisfy the equation of part a). Hence, by part a), we must have $E(x) = 0$ for all x and $O(x) = 0$ for all x.

The only way we can have $B\cos(bx) = 0$ for all x is for $B = 0$, as plugging in $x = 0$ gives $0 = B\cos(0) = B$. The only way $A\sin(ax) = 0$ for all x is if either $A = 0$ or $a = 0$. If neither is true, then plugging in $x = \frac{\pi}{2a}$, we get $A\sin(ax) = A \neq 0$. So, B must be 0 and at least one of A or a must be 0.

Problem 2: Error analysis

a) Since $V = \frac{\pi}{9}h^3$, $h = \left(\frac{9V}{\pi}\right)^{\frac{1}{3}}$. Let us use the symbol a to denote $\left(\frac{9}{\pi}\right)^{\frac{1}{3}}$. Hence, $h = aV^{\frac{1}{3}}$. By the analytic definition of the derivative,

$$\begin{aligned}
h'(V) &= \lim_{\Delta V \to 0} a\frac{(V+\Delta V)^{\frac{1}{3}} - V^{\frac{1}{3}}}{\Delta V} \\
&= \lim_{\Delta V \to 0} a\frac{(V+\Delta V)^{\frac{1}{3}} - V^{\frac{1}{3}}}{\Delta V} \frac{(V+\Delta V)^{\frac{2}{3}} + V^{\frac{1}{3}}(V+\Delta V)^{\frac{1}{3}} + V^{\frac{2}{3}}}{(V+\Delta V)^{\frac{2}{3}} + V^{\frac{1}{3}}(V+\Delta V)^{\frac{1}{3}} + V^{\frac{2}{3}}} \\
&= \lim_{\Delta V \to 0} a\frac{\Delta V}{\Delta V} \lim_{\Delta V \to 0} \frac{1}{(V+\Delta V)^{\frac{2}{3}} + V^{\frac{1}{3}}(V+\Delta V)^{\frac{1}{3}} + V^{\frac{2}{3}}}
\end{aligned}$$

Cancelling the ΔV in the numerator and denominator in the first limit, we can then evaluate the second limit by simply setting ΔV equal to 0, as there are no factors of zero left in the denominator. Hence, we get

$$h'(V) = \frac{1}{3}\left(\frac{9}{\pi}\right)^{\frac{1}{3}} V^{\frac{-2}{3}}.$$

Alternative Solution: The analytic definition of the derivative tells us that

$$h'(V) = \lim_{\Delta V \to 0} \frac{h(V+\Delta V) - h(V)}{\Delta V}.$$

Instead of writing everything in terms of V, we can write everything in terms of h and note that sending ΔV to 0 is the same as sending Δh to 0 since V is a continuous function of h. Using b to denote $\frac{\pi}{9}$, we have

$$
\begin{aligned}
h'(V) &= \lim_{\Delta h \to 0} \frac{\Delta h}{V(h+\Delta h) - V(h)} \\
&= \lim_{\Delta h \to 0} \frac{1}{b} \frac{\Delta h}{(h+\Delta h)^3 - h^3} \\
&= \lim_{\Delta h \to 0} \frac{1}{b} \frac{\Delta h}{\Delta h((h+\Delta h)^2 + h(h+\Delta h) + h^2)} \\
&= \frac{1}{b} \frac{1}{3h^2}
\end{aligned}
$$

Now, plugging in $h = \left(\frac{9V}{\pi}\right)^{\frac{1}{3}}$ and $b = \frac{\pi}{9}$, we get

$$
h'(V) = \frac{1}{3} \left(\frac{9}{\pi}\right)^{\frac{1}{3}} V^{\frac{-2}{3}}
$$

the same answer as before.

b) If $h_0 = 1$, then $V_0 = \frac{\pi}{9} h^3 = \frac{\pi}{9}$. The approximate height error Δh is given by the formula

$$
\Delta h \approx h'(V_0)\Delta V = \frac{1}{3} \left(\frac{9}{\pi}\right)^{\frac{1}{3}} \left(\frac{9}{\pi}\right)^{\frac{2}{3}} \Delta V = \frac{1}{3}\frac{9}{\pi}\Delta V = \frac{3\Delta V}{\pi}.
$$

c) Since $3 \approx \pi$, $|\Delta h| \approx |\Delta V|$. As the scoops can only measure volumes with error $|\Delta V|$ around 0.1, it is impossible to fill the cone with $|\Delta h| \leq 0.2$.

d) Using $V(h) = \frac{\pi}{9} h^3$, we have $V'(h) = \frac{\pi}{3} h^2$ and hence $V'(h_0) = \frac{\pi}{3}$. Thus, we have

$$
\Delta V \approx \frac{\pi}{3} \Delta h \approx \Delta h
$$

and hence we cannot have $\Delta h \leq 0.2$ if ΔV is around 0.1.

Problem 3: Section 3.1

14. The first condition on a, b, c comes from requiring both curves to actually contain the point $(3, 3)$. Plugging in $x = 3, y = 3$, we get

$$
3 = 9 + 3a + b \text{ and } 3 = 3c - 9.
$$

Hence,

$$
3a + b = -6 \text{ and } c = 4.
$$

If they also have the same tangent at $(3, 3)$, the derivatives $\frac{dy}{dx}$ must be equal at $(3, 3)$.

$$
\frac{d}{dx}\left(x^2 + ax + b\right)|_{x=3} = (2x + a)|_{x=3} = 6 + a
$$

and

$$
\frac{d}{dx}\left(cx - x^2\right)|_{x=3} = (c - 2x)|_{x=3} = c - 6 = -2.
$$

Hence, $6 + a = -2$, i.e., $a = -8$ and $b = 18$.

18. Suppose the tangent line that passes through $(0,2)$ is tangent to the curve $y=x^3$ at a point $(x_0,y_0)=(x_0,x_0^3)$. Then, the slope of the tangent line is $\frac{y_0-2}{x_0}=\frac{x_0^3-2}{x_0}$. This slope is also equal to the derivative of the function $y=x^3$ at (x_0^3,x_0). Hence,

$$\frac{x_0^3-2}{x_0}=3x_0^2.$$

Thus, $3x_0^3=x_0^3-2$, which implies $x_0=-1$. Thus, the equation of the line is

$$y=3x_0^2x+2=3x+2.$$

21 (a) If the point (x_0,y_0) is on the parabola, $y_0=\frac{x_0^2}{4p}$. The slope of the tangent line at this point is

$$\frac{dy}{dx}|_{(x_0,y_0)}=\frac{x_0}{2p}.$$

Thus, if the y-intercept is c, the equation of the line is

$$y=\frac{x_0}{2p}x_0+c.$$

Plugging in $x=x_0, y=y_0=\frac{x_0^2}{4p}$, we have

$$c=\frac{x_0^2}{4p}-\frac{x_0^2}{2p}=-\frac{x_0^2}{4p}=-y_0.$$

21 (b) Let us use the distance formula to compute the side lengths of the triangle.

The side a with vertices $(0,p)$ and $(0,-y_0)$ has length $p+y_0=p+\frac{x_0^2}{4p}$.

The side b with vertices $(0,p)$ and (x_0,y_0) has length

$$\sqrt{x_0^2+(p-y_0)^2}=\sqrt{x_0^2+p^2+\frac{x_0^4}{16p^2}-\frac{x_0^2}{2}}.$$

Squaring both sides gives

$$a^2=p^2+\frac{x_0^4}{16p^2}+\frac{x_0^2}{2}=b^2.$$

Hence, $a=b$ and the corresponding triangle is isoceles.

21 (c) Suppose the light ray starts at the focus $F=(0,p)$ and hits the parabola at the point $A=(x_0,y_0)$. Let $B=(0,-y_0)$. By part (a), the line containing the points A and B is tangent to the parabola at A and by part (b), the segments FA and FB have the same length. Let C be a point on the vertical line passing through A above A. Then, since the line passing through AC is parallel to the axis, the angle FAC is the same as the angle FBA, which is the same as the angle FAB because the corresponding sides have the same length. Hence, the ray AC makes the same angle with the tangent line as the ray FA and hence must be the path of the light ray after reflection.

Problem 4: Section 2.5

19 (f)

$$\lim_{x\to 0}\frac{\sin^2 x}{3x^2}=\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=\frac{1}{3}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2=\frac{1}{3}.$$

19 (g)

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \frac{2}{3}\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}\frac{3x}{2x} = \frac{2}{3}\lim_{x\to 0}\frac{\sin 2x}{2x}\lim_{x\to 0}\frac{3x}{\sin 3x} = \frac{2}{3}.$$

20 (a)

$$\lim_{x\to 0}\frac{\sin x}{3\sqrt{x}} = \lim_{x\to 0}\frac{\sin x}{3\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0}\frac{1}{3}\frac{\sin x}{x}\sqrt{x} = \frac{1}{3}\cdot 1\cdot 0 = 0.$$

20 (b)

$$\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\sin x = 1\cdot 0 = 0.$$

20 (g)

$$\lim_{x\to 0}\frac{3x^2+4x}{\sin 2x} = \lim_{x\to 0}\frac{3x}{2}\frac{2x}{\sin 2x} + \lim_{x\to 0} 2\frac{2x}{\sin 2x} = 0+2 = 2.$$

22 (a) Conjecture:

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = \frac{1}{2}.$$

22 (b) Proof: Note that $1-\cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$. Hence,

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0}\frac{1}{2}\frac{\sin^2\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)^2} = \frac{1}{2}.$$

Problem 5: Induction and derivatives of powers of x.

a) Using the quotient rule

$$\frac{d}{dx}\frac{1}{x} = \frac{\frac{d1}{dx}\cdot x - 1\cdot\frac{dx}{dx}}{x^2} = -\frac{1}{x^2}.$$

b) We want to show, using induction on n, that $\frac{d}{dx}x^{-n} = -nx^{-n-1}$ for $n>0$. Whenever we need to do an induction proof, we need to prove a base case, when $n=1$, by hand and then show that if the statement is true for n, then it is true for $n+1$. We have shown the base case in part (a).

So, assume that $\frac{d}{dx}x^{-n} = -nx^{n-1}$. Then,

$$\begin{aligned}\frac{d}{dx}x^{-n-1} &= \frac{d}{dx}(x^{-1}x^{-n-1})\\ &= x^{-n}\frac{d}{dx}x^{-1} + x^{-1}\frac{d}{dx}x^{-n}\\ &= -x^{-n-2} - nx^{-1}x^{-n-1} \text{ (by the induction hypothesis)}\\ &= -(n+1)x^{-n-2}.\end{aligned}$$

c) Let us use induction to prove that $\frac{d}{dx}x^n = nx^{n-1}$ for $n\geq 1$. As mentioned in part b), we first need to prove a base case. For $n=1$,

$$\frac{d}{dx}x^n = \frac{d}{dx}x = 1 = 1x^0,$$

as desired. Let us now assume, as induction hypothesis, that the statement is true for n. Then,

$$\begin{aligned}\frac{d}{dx}x^{n+1} &= \frac{d}{dx}(xx^n) \\ &= x^n\frac{d}{dx}x + x\frac{d}{dx}x^n \\ &= x^n + xnx^{n-1} \\ &= (n+1)x^n\end{aligned}$$

as desired.