MATH 18.01 - MIDTERM 4 - SOME REVIEW PROBLEMS WITH SOLUTIONS

18.01 Calculus, Fall 2014	Professor: Jared Speck

Problem 1. Approximate the integral

$$\int_0^2 x^4\,dx$$

using first Simpson's rule with two equal intervals and then the trapezoid rule with two equal intervals. Compare your approximations to the exact value.

Problem 2. Evaluate the integral

$$\int e^x \sin x\,dx.$$

Problem 3. Evaluate the integral

$$\int \frac{dx}{(1-x^2)^{5/2}}.$$

Problem 4. Let $F(x) = \int_1^x \frac{\cos y}{y}\,dy$ and $G(x) = \int_1^x \frac{\sin y}{y^2}\,dy$. Show that there is a constant C such that $F(x) - G(x) = \frac{\sin x}{x} + C$. What is the value of C?

Problem 5. Evaluate the integral

$$\int \frac{dx}{x\sqrt{(\ln x)^2 - 4}}.$$

Solutions

Problem 1. Approximate the integral

$$\int_0^2 x^4\,dx$$

using first Simpson's rule with two equal intervals and then the trapezoid rule with two equal intervals. Compare your approximations to the exact value.

Solution: We set $f(x) = x^4$ and $\Delta x = \frac{2-0}{2} = 1$. Then the Simpson approximation is

$$\begin{aligned}\int_0^2 x^4\,dx &\approx \frac{\Delta x}{3}\left[f(0) + 4f(1) + f(2)\right]\\ &= \frac{1}{3}\left(0 + 4\times 1 + 16\right) = 20/3.\end{aligned}$$

The trapezoid approximation is

$$\begin{aligned}\int_0^2 x^4\,dx &\approx \Delta x\left[\frac{1}{2}f(0) + f(1) + \frac{1}{2}f(2)\right]\\ &= 1\times(0+1+8) = 9.\end{aligned}$$

The exact value of the integral is $\int_0^2 x^4\,dx = \frac{x^5}{5}\,\big|_{x=0}^{x=2} = 32/5$.

Problem 2. Evaluate the integral

$$\int e^x \sin x\,dx.$$

Solution: We integrate by parts with $u = e^x$, $du = e^x$, $dv = \sin x\,dx$, $v = -\cos x$, which leads to

$$\begin{aligned}\int e^x \sin x\,dx &= \int u\,dv = uv - \int v\,du\\ &= -e^x\cos x + \int e^x \cos x\,dx.\end{aligned}$$

Similarly, we compute that

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x \sin x\,dx.$$

Inserting this identity for $\int e^x\cos x\,dx$ back into the previous formula, we deduce that

$$\begin{aligned}\int e^x\sin x\,dx &= -e^x\cos x + \int e^x\cos x\,dx\\ &= -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx.\end{aligned}$$

We then add $\int e^x\sin x\,dx$ to both sides of the previous equation, divide by two, and add the integration constant C, which leads to

$$\int e^x\sin x\,dx = \frac{1}{2}\left\{e^x\sin x - e^x\cos x\right\} + C.$$

Problem 3. Evaluate the integral

$$\int \frac{dx}{(1-x^2)^{5/2}}.$$

Solution: We use the inverse trig substitution $x = \sin u$, $dx = \cos u\, du$, the identity $(\cos u)^2 = 1 - (\sin u)^2$, and the identity $(\sec u)^2 = 1 + (\tan u)^2$, which leads to

$$\begin{aligned}\int \frac{\cos u}{(\cos u)^5}\, du &= \int (\sec u)^4\, du \\ &= \int [1 + (\tan u)^2](\sec u)^2\, du.\end{aligned}$$

We then make the substitution $v = \tan u$, $dv = (\sec u)^2\, du$, which leads to

$$\begin{aligned}\int [1 + (\tan u)^2](\sec u)^2\, du &= \int 1 + v^2\, dv \\ &= v + \frac{v^3}{3} + C \\ &= \tan u + \frac{\tan^3 u}{3} + C \\ &= \frac{x}{(1-x^2)^{1/2}} + \frac{1}{3}\frac{x^3}{(1-x^2)^{3/2}} + C.\end{aligned}$$

In the last step above, we used a right triangle to deduce that $\tan u = \frac{x}{(1-x^2)^{1/2}}$.

Problem 4. Let $F(x) = \int_1^x \frac{\cos y}{y}\, dy$ and $G(x) = \int_1^x \frac{\sin y}{y^2}\, dy$. Show that there is a constant C such that $F(x) - G(x) = \frac{\sin x}{x} + C$. What is the value of C? **Solution:** We use integration by parts in the form $u = \frac{1}{y}$, $du = -\frac{1}{y^2}\, dy$, $dv = \cos y\, dy$, $v = \sin y$, $\int u\, dv = uv - \int v\, du$ to deduce that

$$\begin{aligned}\underbrace{\int_1^x \frac{\cos y}{y}\, dy}_{F(x)} &= \frac{\sin y}{y}\Big|_{y=1}^{y=x} + \int_1^x \frac{\sin y}{y^2}\, dy \\ &= \frac{\sin x}{x} - \sin 1 + \underbrace{\int_1^x \frac{\sin y}{y^2}\, dy}_{G(x)}.\end{aligned}$$

Hence,

$$F(x) - G(x) = \frac{\sin x}{x} - \sin 1,$$

and $C = \sin 1$.

Problem 5. Evaluate the integral

$$\int \frac{dx}{x\sqrt{(\ln x)^2 - 4}}.$$

Solution: We first make the substitution $u = \ln x$, $du = \frac{dx}{x}$, which leads to the integral

$$\int \frac{du}{(u^2 - 4)^{1/2}}.$$

We then make the inverse trigonometric substitution $u = 2\sec\theta$, $du = 2\sec\theta\tan\theta\, d\theta$ and use the identity $(\sec\theta)^2 - 1 = (\tan\theta)^2$, which leads to

$$\begin{aligned}
\int \frac{du}{(u^2 - 4)^{1/2}} &= \int \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C, \\
&= \ln\left|\frac{u}{2} + \sqrt{\frac{u^2}{4} - 1}\right| + C \\
&= \ln\left|\frac{\ln x}{2} + \sqrt{\frac{(\ln x)^2}{4} - 1}\right| + C.
\end{aligned}$$

In the next to last line above, we used a right triangle to deduce that $\tan\theta = \frac{\sqrt{u^2-4}}{2} = \sqrt{\frac{u^2}{4} - 1}$.