November 10, 2014

1

(a)
In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. Thus we have $\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$

(b)
Area $= \int_0^{2\pi} \int_0^{\sqrt{\frac{1}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}} r dr d\theta$

(c)

$$
\begin{aligned}
&\int_0^{2\pi} \int_0^{\sqrt{\frac{1}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}} r dr d\theta \\
=&4 \cdot \int_0^{\frac{\pi}{2}} \frac{1}{2(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2})} d\theta \\
=&4 \cdot \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{2(\frac{1}{a^2} + \frac{\tan^2\theta}{b^2})} d\theta \\
=&4 \cdot \int_0^{\frac{\pi}{2}} \frac{1}{2(\frac{1}{a^2} + \frac{\tan^2\theta}{b^2})} d\tan\theta \\
=&2ab \cdot \int_0^{\frac{\pi}{2}} \frac{1}{(1 + \frac{a^2\tan^2\theta}{b^2})} d(\frac{a\tan\theta}{b}) \\
=&2ab \int_0^{\pi/2} d(\arctan(\frac{a\tan\theta}{b})) \\
=&2ab \cdot (\pi/2 - 0) \\
=&\pi ab
\end{aligned}
$$

We can evaluate the integral from 0 to $\pi/2$ and then multiply the result by 4 because the ellipse is symmetric with respect to both x-axis and y-axis.

2

We only have to compute $\lim\limits_{E\to 1^-} f(E)(1-E^2)$. Set $x = 1-E$. Then

$$\begin{aligned}
&\lim_{E\to 1^-} f(E)(1-E^2)\\
=&\lim_{x\to 0^+} f(1-x)x(2-x)\\
=&\lim_{x\to 0^+} \frac{\tanh^{-1}(1-x)}{1-x}x(2-x)\\
=&\lim_{x\to 0^+} \frac{1}{2}\ln(\frac{2-x}{x})\frac{x(2-x)}{1-x}\\
=&\lim_{x\to 0^+} x\ln(\frac{2-x}{x})\\
=&0
\end{aligned}$$

Thus $\lim\limits_{E\to 1^-} S(E) = 2\pi$. This makes sense because when $E \to 1^-$, the ellipsoid will go to the two-sided unit disk whose surface area is 2π

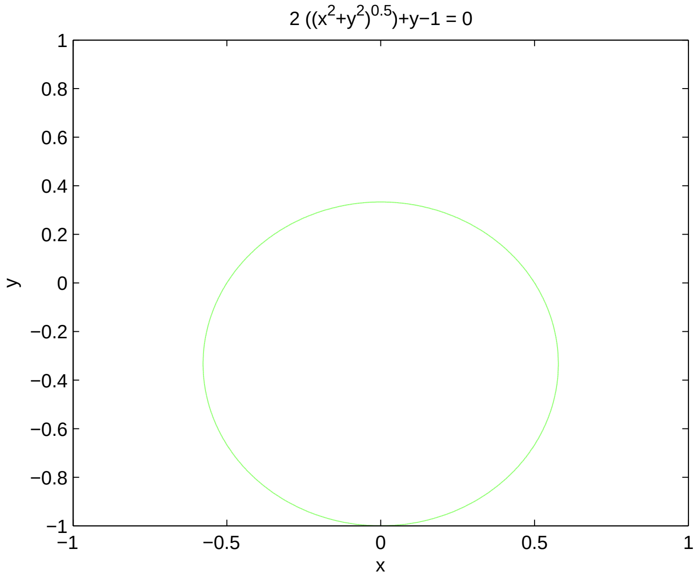
3

(a)
Area $= \int_0^{\arctan\frac{h}{a}} \int_0^{\frac{a}{\cos\theta}} r dr d\theta = \frac{1}{2}\int_0^{\arctan\frac{h}{a}} \frac{a^2}{\cos^2\theta} d\theta = \frac{1}{2}a^2 \cdot \frac{h}{a} = \frac{1}{2}ah$

(b)
$2\sqrt{x^2+y^2} + y = 1$
The graph is on the next page.

(c)
Area $= \int_{-1/2}^{1/2} (\sqrt{\frac{4}{9} - \frac{4}{3}x^2} - \frac{1}{3})dx = \frac{2\pi}{9\sqrt{3}} - \frac{1}{6}$

(d)
Area $= \int_0^\pi \int_0^{\frac{1}{2+\sin\theta}} r dr d\theta = \frac{1}{2}\int_0^\pi \frac{1}{(2+\sin\theta)^2} d\theta$.
Set $\phi = \theta - \pi/2$. Then the integral is equal to $\frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{1}{(2+\cos\phi)^2} d\phi = \int_{-\pi/2}^{\pi/2} \frac{1}{(1+2\cos^2\phi/2)^2} d(\phi/2)$.
Set $v = \tan\phi/2$. The integral is equal to $\int_{-1}^{1} \frac{1+v^2}{(3+v^2)^2} dv = \frac{2\pi}{9\sqrt{3}} - \frac{1}{6}$



4

(a)

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} e^{-x^2} dx \\ &= 2\int_{0}^{+\infty} e^{-x^2} dx \\ &< 2\sum_{i=0}^{+\infty} 2^{-i} \\ &= 4 \end{aligned}$$

Thus it is a finite positive number.

(b)

$$\begin{aligned} V &= \int_{0}^{+\infty} 2\pi x e^{-x^2} dx \\ &= -\pi \int_{0}^{+\infty} d(e^{-x^2}) \\ &= \pi \end{aligned}$$

(c)

Suppose the point (x, y, z) is on the surface and its corresponding point on the $xz-$plane is $(x_0, 0, z_0)$. Then we have $z = z_0$ and $x^2 + y^2 = x_0^2$. Since $(x_0, 0, z_0)$ is on the curve $z = e^{-x^2}$, we see that $z_0 = e^{-x_0^2}$. Thus $z = e^{-(x^2+y^2)}$.

(d)
$A(x) = \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dy = e^{-x^2} I$

(e)
This is the method of slices.

(f)
$V = \int_{-\infty}^{+\infty} e^{-x^2} I dx = I \cdot I = I^2$. Thus $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$.

5

(a)
Choose an integer n that is bigger than r. Then

$$0 \leq \lim_{x \to +\infty} x^r e^{-x} \leq \lim_{x \to +\infty} \frac{x^n}{e^x} \leq n! \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

(b)
From (a) we see that $\lim\limits_{x \to +\infty} x^{r+2} e^{-x} = 0$. Thus there exists a integer N such that $x^{r+2} e^{-x} \leq 1$ when $x \geq N$. Then

$$\int_0^{+\infty} x^r e^{-x} dx = \int_0^N x^r e^{-x} dx + \int_N^{+\infty} x^r e^{-x} dx \leq N \cdot N^r + \int_N^{+\infty} \frac{1}{x^2} dx \leq N^{r+1} + \frac{1}{N}$$

Thus the improper integral converges.

(c)

$$\begin{aligned}
A(n+1) &= \int_0^{+\infty} x^{n+1} e^{-x} dx \\
&= -\int_0^{+\infty} x^{n+1} d(e^{-x}) \\
&= \int_0^{+\infty} e^{-x} d(x^{n+1}) \\
&= (n+1) \int_0^{+\infty} x^n e^{-x} dx \\
&= (n+1) A(n)
\end{aligned}$$

(d)

$$A(0) = \int_0^{+\infty} e^{-x} dx = 1$$

Thus $A(n) = n!$.

(e)
Set $x = u^2$. Then $dx = 2udu = 2\sqrt{x}du$.

$$A(-\frac{1}{2}) = 2\int_0^{+\infty} e^{-u^2} du = \sqrt{\pi}$$

6

The Taylor Series for $F'(x) = 3x^2e^{-x^6}$ at 0 is $\sum\limits_{i=0}^{+\infty} \frac{3\cdot(-1)^i}{i!}x^{6i+2}$. Thus the Taylor Series for $F(x)$ is $\sum\limits_{i=0}^{+\infty} \frac{3\cdot(-1)^i}{i!(6i+3)}x^{6i+3}$