MIDTERM 3 - 18.01 - FALL 2014.

Name:

Email:

Please put a check by your recitation section.

	Instructor	Time
☐	B.Yang	MW 10
☐	M. Hoyois	MW 11
☐	M. Hoyois	MW 12
☐	X. Sun	MW 1
☐	R. Chang	MW 2

Problem #	Max points possible	Actual score
1	15	
2	20	
3	10	
4	15	
5	20	
6	20	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**

Good luck!

Problem 1. (15 points)

Find the solution $y = f(t)$ to the differential equation

$$y' = t(1+y)$$

with $y(0) = 1$ (and $y' = dy/dt$).

Solution:

We first separate variables:

$$\frac{dy}{1+y} = tdt.$$

We then integrate each side to deduce that there exists a constant c such that

$$\ln|1+y| = \frac{1}{2}t^2 + c.$$

Since $y(0) = 1$, we have $\ln 2 = c$, and thus

$$\ln(1+y) = \frac{1}{2}t^2 + \ln 2.$$

We now solve for y :

$$1 + y = e^{\frac{1}{2}t^2 + \ln 2} = 2e^{\frac{t^2}{2}},$$

$$y = 2e^{\frac{t^2}{2}} - 1.$$

Problem 2. (5 + 5 + 5 + 5 = 20 points)

Let R be the finite region in the *first quadrant* of the (x, y) plane (that is, the quadrant with $x \geq 0$ and $y \geq 0$) that is trapped between the x axis, the line $x = \pi/4$, and the curve $y = \tan x$. Let S_1 be the solid that is obtained upon revolving R about the y axis and let S_2 be the solid that is obtained upon revolving R about the x axis.

Remark: In this problem, you do not have to compute any of the integrals.

a) Using the disc method, write down an integral representing the volume of S_1.

b) Using the cylindrical shell method, write down an integral representing the volume of S_1.

c) Using the disc method, write down an integral representing the volume of S_2.

d) Using the cylindrical shell method, write down an integral representing the volume of S_2.

Solution:

a) To find the volume V of S_1, we first note that the y axis is the axis of symmetry of the discs (which have holes) and that y varies from 0 to $\arctan(\pi/4) = 1$. Each disc has outer radius $\pi/4$, inner radius $\arctan y$, and thickness dy. The volume dV of the disc is therefore $dV = \pi \times$ (outer radius^2 $-$ inner radius^2) $\times$ $d$thickness $= \pi\left((\pi/4)^2 - \arctan^2 y\right)\, dy$. The total volume is therefore

$$V = \int dV = \int_0^1 \pi\left((\pi/4)^2 - \arctan^2 y\right)\, dy.$$

b) In this case x varies from 0 to $\pi/4$ and each shell has radius x, height $\tan x$, and thickness dx. The volume dV of the shell is therefore $dV = 2\pi \times$ radius $\times$ height $\times$ thickness $= 2\pi x \tan x\, dx$. The total volume is therefore

$$V = \int dV = \int_0^{\pi/4} 2\pi x \tan x\, dx.$$

c) To find the volume V of S_2, we first note that the x axis is the axis of symmetry of the discs and that x varies from 0 to $\pi/4$. Each disc has radius $\tan x$ and thickness dx. The volume dV of the disc is therefore $dV = \pi \times$ radius^2 $\times$ thickness $= \pi \tan^2 x\, dx$. The total volume is therefore

$$V = \int dV = \int_0^{\pi/4} \pi \tan^2 x\, dx.$$

d) In this case, y varies from 0 to $\arctan(\pi/4) = 1$ and each shell has radius y, height $\pi/4 - \arctan y$, and thickness dy. The volume dV of the shell is therefore $dV = 2\pi \times$ radius $\times$ height $\times$ thickness $= 2\pi y(\pi/4 - \arctan y)\, dy$. The total volume is therefore

$$V = \int dV = \int_0^1 2\pi y(\pi/4 - \arctan y)\, dy.$$

Problem 3. (5 + 5 = 10 points)
a) State the first fundamental theorem of calculus.
b) State the second fundamental theorem of calculus.

Solution:
a) If f is continuous and $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
b) If f is continuous and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

Problem 4. (15 points) Compute the following limit:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\cos(\pi k\Delta x)\Delta x,$$

where $\Delta x = \dfrac{1}{2n}$.

Solution:

$\displaystyle\sum_{k=1}^{n}\cos(\pi k\Delta x)\Delta x$ is a Riemann sum for the function $f(x) = \cos(\pi x)$ over the interval $[0, 1/2]$. As $n \to \infty$, $\Delta x \to 0$, and thus the Riemann sum converges to the definite integral

$$\int_0^{1/2}\cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x)|_0^{1/2} = \frac{1}{\pi}.$$

Problem 5. (10 + 10 = 20 points) Evaluate the following two definite integrals.

a)

$$\int_0^1 x\sin(\pi x^2)\,dx$$

b)

$$\int_{1/2}^0 \frac{x^2}{\sqrt{1-x^3}}\,dx$$

Solution:

a) We make the substitution $u=\pi x^2$, $du = 2\pi x\,dx$. Then the integral becomes

$$\frac{1}{2\pi}\int_0^\pi \sin u\,du = -\frac{1}{2\pi}\cos u\,|_0^\pi = \frac{1}{\pi}.$$

b) We make the substitution $u=x^3$, $du=3x^2\,dx$. Then the integral becomes

$$\frac{1}{3}\int_{u=1/8}^{u=0}(1-u)^{-1/2}\,du = -\frac{2}{3}(1-u)^{1/2}\Big|_{u=1/8}^{u=0} = \frac{2}{3}\left(\sqrt{\frac{7}{8}}-1\right).$$

Problem 6. (10 + 10 = 20 points)

The following limit can be viewed as the derivative of a function F at a point x_0 :

$$\lim_{x\to 1}\left(\frac{1}{x-1}\int_1^{x^2} e^{-t^2}\,dt\right).$$

a) Find F and x_0. Be sure to explain your answer to receive credit.
b) Compute the limit.

Solution:
a) Let

$$F(x) = \int_1^{x^2} e^{-t^2}\,dt.$$

Note that $F(1) = 0$. Hence, the the limit of interest is

$$\lim_{x\to 1}\frac{F(x)-F(1)}{x-1} = F'(1).$$

That is, $x_0 = 1$.

b) By FTC2 and the chain rule, we have that

$$F'(x) = e^{-(x^2)^2}\times\frac{d}{dx}x^2 = 2xe^{-x^4}.$$

Thus, the limit is $F'(1) = 2e^{-1}$.