INT.   IMPROPER INTEGRALS

In deciding whether an improper integral converges or diverges, it is often awkward or impossible to try to decide this by actually carrying out the integration, i.e., finding an antiderivative explicitly. For example both of these two improper integrals converge:

$$\int_1^\infty \frac{dx}{x^6 + 3x^3 + 2x^2 + 1} \quad \text{and} \quad \int_1^\infty \frac{dx}{\sqrt{x^3 + 1}},$$

but there is no explicit antiderivative for the second integral, and finding one for the first would be a hairy exercise in partial fractions, even if one were able to factor the denominator.

Instead of explicit integration, therefore, we show they converge by using estimation instead, comparing them with simpler integrals which are known to converge. Thus, for the first,

$$\frac{1}{x^6 + 3x^3 + 2x^2 + 1} \le \frac{1}{x^6}, \quad x > 0,$$

so that

$$\int_1^\infty \frac{dx}{x^6 + 3x^3 + 2x^2 + 1} \le \int_1^\infty \frac{dx}{x^6} = \frac{1}{5}.$$

Since the right hand integral converges, so does the left, which is smaller (but still positive). In a similar way, for the second integral, we estimate

$$\frac{1}{\sqrt{x^3 + 1}} \le \frac{1}{x^{3/2}}, \quad x > 0,$$

so that

$$\int_1^\infty \frac{dx}{\sqrt{x^3 + 1}} \le \int_1^\infty \frac{dx}{x^{3/2}} = 2.$$

In the same way we can show the divergence say of $\int_1^\infty \frac{x\,dx}{\sqrt{x^3 + 1}}$:

$$\frac{x}{\sqrt{x^3 + 1}} \ge \frac{1}{2\sqrt{x}}, \quad x > 1,$$

this last being equivalent to $2x^{3/2} \ge \sqrt{x^3 + 1}$, i.e., to $8x^3 \ge x^3 + 1$; thus we get

$$\int_1^\infty \frac{x\,dx}{\sqrt{x^3 + 1}} \ge \int_1^\infty \frac{dx}{2\sqrt{x}} = \infty.$$

We call the general principle we are using here the

Comparison Test for Improper Integrals. If $0 \leq f(x) \leq g(x)$ for $a \leq x < \infty$,

$$(1) \qquad \int_a^\infty g(x)\,dx \text{ converges} \Rightarrow \int_a^\infty f(x)\,dx \text{ converges};$$

$$(1') \qquad \int_a^\infty f(x)\,dx \text{ diverges} \Rightarrow \int_a^\infty g(x)\,dx \text{ diverges}.$$

In other words, if the area under $g(x)$ is finite and $f(x)$ lies below $g(x)$ (but still over the x-axis), then the area under $f(x)$ will be finite also. Or equivalently, if the area under $f(x)$ is infinite, so is the area under $g(x)$.
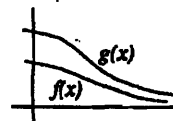
In using the test, the lower limit of integration is of no importance, since if $a < b$,

$$\int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx, \qquad \text{so that}$$

$$(3) \qquad \int_a^\infty f(x)\,dx \text{ converges} \quad \Leftrightarrow \quad \int_b^\infty f(x)\,dx \text{ converges}.$$

As another example of the use of the comparison test, we recall one used to illustrate the Second Fundamental Theorem (cf. p. FT.3):

$$e^{-x^2} \leq e^{-x} \quad \text{for } x \geq 1, \text{ and therefore}$$

$$\int_0^\infty e^{-x^2}\,dx \text{ converges, since } \int_0^\infty e^{-x}\,dx \text{ converges}.$$

(Here we use (3) above to shift the lower limit from 1 to 0; also the comparison test (1).)

In using the comparison test, we have to have some standard integrals that we know converge or diverge, to use for comparison purposes. The most useful are:

$$(4) \qquad \int_1^\infty \frac{dx}{x^p} \quad \text{converges if } p > 1, \text{ diverges if } p \leq 1;$$

$$(5) \qquad \int_0^\infty e^{-ax} \quad \text{converges if } a > 0.$$

It is important to notice that in using the test, you must make the inequality go in the right direction, which means you must *guess in advance* whether the integral will converge or diverge. For example, to test $\int_2^\infty \frac{x\,dx}{x^3 - 2}$, we see that as $x \to \infty$,

$$\frac{x}{x^3 - 2} \sim \frac{x}{x^3} = \frac{1}{x^2},$$

so from (4) we guess it will converge. Unfortunately, $\dfrac{x}{x^3 - 2} > \dfrac{1}{x^2}$, $x \geq 2$, so we can't use $\int_1^\infty \dfrac{dx}{x^2}$ as the comparison integral; however, only a slight change is needed:

$$\frac{x}{x^3 - 2} < \frac{2}{x^2}, \quad x \geq 2,$$

as we see by cross-multiplying: $x^3 < 2x^3 - 4$; thus our integral converges, using (1). $\quad \square$

Improper integrals of the second kind

The comparison test also works for improper integrals of the form

$$\int_{a^+}^{b} f(x)\,dx, \quad \text{where} \quad \lim_{x \to a^+} f(x) = \infty, \quad \text{and} \quad \int_{a}^{b^-} f(x)\,dx, \quad \text{where} \quad \lim_{x \to b^-} f(x) = \infty.$$

Sometimes these are called improper integrals of the **second kind** — the first kind being the previous type of improper integral, where one of the limits of integration is ∞ or $-\infty$.
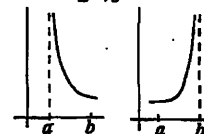
For improper integrals of the second kind, useful standard comparison integrals are

(6)
$$\int_{a}^{b^-} \frac{dx}{(b-x)^p} \quad \text{and} \quad \int_{a^+}^{b} \frac{dx}{(x-a)^p},$$

which *converge if* $p < 1$, *diverge if* $p \geq 1$. Note that this is just the opposite of (4). However, it's easy to remember which is which if you think of the picture at the right, which compares the graphs for $p < 1$ and $p > 1$.
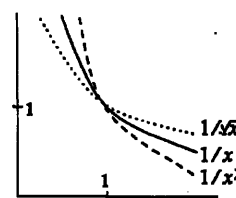
Example. Test $\int_0^1 \frac{dx}{\sqrt{2x - x^2}}$ for convergence.

Solution. The integrand becomes infinite when $x = 0$ and when $x = 2$, but 2 is of no importance, since it's not in the interval over which we are integrating. In making our guess as to convergence or divergence, we note that

$$\frac{1}{\sqrt{2x - x^2}} \approx \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x}}, \quad x \approx 0^+.$$

Thus, according to (6) above, we expect convergence, and make the comparison

(7)
$$\frac{1}{\sqrt{2x - x^2}} \leq \frac{1}{\sqrt{x}}.$$

Where is (7) valid (if at all)? Crossmultiplying, it claims that

$$\sqrt{x} \leq \sqrt{2x - x^2}, \quad \text{or squaring,} \quad x \leq 2x - x^2, \quad \text{or} \quad x^2 \leq x.$$

This last inequality is true if $0 \leq x \leq 1$, so we're safe, since this is the region of integration. So we see by the comparison test above that our integral converges because according to (6), the integral $\int_{0^+}^1 \frac{dx}{\sqrt{x}}$ converges.

Remark. It isn't customary to include the $+$ and $-$ symbols in a^+ and b^- when one writes integrals of the second kind. We only did it here to point out where they are improper, and on which side. It doesn't appear in the solutions to the exercises on improper integrals.

Exercises: Section 6B