MATH 18.01 - MIDTERM 1 - SOME REVIEW PROBLEMS WITH SOLUTIONS

18.01 Calculus, Fall 2017   Professor: Jared Speck

Problem 1. Compute the second derivative of the function $f(x) = \arctan x$.

Problem 2. Compute the derivative of the function $f(x) = \sin(x)x^{x^x}$.

Problem 3. Compute $\lim_{x \to \pi/2} \frac{\cos(3x)}{\cos(x)}$. At this point in the course, you are forbidden from using L'Hôpital's rule.

Problem 4. Compute the derivative $\frac{dy}{dx}$ for the curve $x^2 + y^3 + xy = 7$ at the point $(x, y) = (2, 1)$.

Problem 5. Find the equation for the tangent line to $y = \ln(x/3)$ at $x = 3e$.

Problem 6. Consider the function

$$
f(x) = \begin{cases} -\frac{\sin x}{2x} & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \frac{\cos x - 1}{x^2} & \text{if } x > 0. \end{cases}
$$

Is $f(x)$ continuous at $x = 0$? If not, then is the discontinuity removable?

Problem 7. Prove the quotient rule $(u/v)' = (u'v - uv')/v^2$ using only the definition of a derivative.

Problem 8. Let r be a real number. Compute $\lim_{h \to 0} \frac{(1+2h)^r - 1}{h}$ by interpreting this limit as a derivative.

Solutions

Problem 1. Compute the second derivative of the function $f(x) = \arctan x$.

Solution:

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}, \qquad \text{(you should know how to prove this),}$$
$$\frac{d^2}{dx^2} \arctan x = \frac{d}{dx} \frac{1}{1 + x^2} = \frac{-2x}{(1 + x^2)^2}.$$

Problem 2. Compute the derivative of the function $f(x) = \sin(x) x^{x^x}$.

Solution:
We first compute the derivative of x^x using logarithmic differentiation:

$$y = x^x,$$
$$\ln y = \ln(x^x) = x \ln x,$$
$$\frac{y'}{y} = 1 + \ln x,$$
$$y' = (1 + \ln x) x^x.$$

We now compute the derivative of x^{x^x} :

$$z = x^{x^x},$$
$$\ln z = \ln(x^{x^x}) = x^x \ln x,$$
$$\frac{z'}{z} = \ln x \frac{d}{dx} x^x + x^x \frac{d}{dx} \ln x = \ln x(1 + \ln x) x^x + x^x \frac{1}{x},$$
$$z' = x^{x^x} \left\{ \ln x(1 + \ln x) x^x + x^x \frac{1}{x} \right\}.$$

Finally, we compute the derivative of $\sin(x) x^{x^x}$ using the product rule:

$$\frac{d}{dx} \left(\sin(x) x^{x^x} \right) = x^{x^x} \frac{d}{dx} \sin(x) + \sin(x) \frac{d}{dx} x^{x^x}$$
$$= x^{x^x} \cos x + \sin(x) x^{x^x} \left\{ \ln x(1 + \ln x) x^x + x^x \frac{1}{x} \right\}.$$

Problem 3. Compute $\lim_{x \to \pi/2} \frac{\cos(3x)}{\cos(x)}$. At this point in the course, you are forbidden from using L'Hôpital's rule.

Solution:

$$\lim_{x \to \pi/2} \frac{\cos(3x)}{\cos(x)} = \lim_{x \to \pi/2} \frac{\cos(3x) - \cos(3\pi/2)}{\cos(x) - \cos(\pi/2)}$$

$$= 3 \lim_{x \to \pi/2} \frac{\cos(3x) - \cos(3\pi/2)}{3x - 3\pi/2} \frac{x - \pi/2}{\cos(x) - \cos(\pi/2)}$$

$$= 3 \lim_{h \to 3\pi/2} \frac{\cos(h) - \cos(3\pi/2)}{h - 3\pi/2} \times \frac{1}{\lim_{h \to \pi/2} \frac{\cos(h) - \cos(\pi/2)}{h - \pi/2}}$$

$$= 3 \frac{d}{du} \cos(u)|_{u = 3\pi/2} \times \frac{1}{\frac{d}{du} \cos(u)|_{u = \pi/2}},$$

$$= 3(-\sin(3\pi/2)) \times \frac{1}{-\sin(\pi/2)}$$

$$= 3(1)(-1) = -3.$$

Problem 4. Compute the derivative $\frac{dy}{dx}$ for the curve $x^2 + y^3 + xy = 7$ at the point $(x, y) = (2, 1)$.

Solution:

$$2x + 3y^2 y' + xy' + y = 0,$$

$$y' = -\frac{2x + y}{3y^2 + x} = -\frac{5}{5} = -1.$$

Problem 5. Find the equation for the tangent line to $y = f(x) = \ln(x/3)$ at $x = 3e$.

Solution:

$$f(3e) = \ln e = 1,$$

$$f'(x) = \frac{1}{x/3} \times \frac{1}{3} = \frac{1}{x},$$

$$f'(3e) = \frac{1}{3e}.$$

Tangent line (with $x_0 = 3e$):

$$y - f(x_0) = f'(x_0)(x - x_0),$$

$$y - 1 = \frac{1}{3e}(x - 3e).$$

Problem 6. Consider the function

$$f(x) = \begin{cases} -\frac{\sin x}{2x} & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \frac{\cos x - 1}{x^2} & \text{if } x > 0. \end{cases}$$

Is $f(x)$ continuous at $x = 0$? If not, then is the discontinuity removable?

Solution:
We recall the following important limits, which have been previously investigated in this course:

$$\lim_{x \to 0^-} -\frac{\sin x}{2x} = -\frac{1}{2},$$
$$\lim_{x \to 0^+} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

Therefore, since $\lim_{x \to 0} f(x) = -\frac{1}{2} \neq f(0)$, $f(x)$ is not continuous at $x = 0$.

However, if we redefine f by setting $f(0) = -\frac{1}{2}$, then for the new f, we have $\lim_{x \to 0} f(x) = f(0)$. Thus, the redefined f is continuous (and hence the original f has a removable discontinuity at $x = 0$).

Problem 7. Prove the quotient rule $(u/v)' = (u'v - uv')/v^2$ using only the definition of a derivative.

Solution:
Given a point x and a small number Δx, we define

$$\Delta u = u(x + \Delta x) - u(x), \qquad \qquad \Delta v = v(x + \Delta x) - v(x).$$

Since u, v are differentiable (and therefore continuous), we have

$$\lim_{\Delta x \to 0} \Delta u = 0,$$
$$\lim_{\Delta x \to 0} \Delta v = 0,$$
$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u',$$
$$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v'.$$

Using the above facts, we have that

$$
\begin{aligned}
\left(\frac{u}{v}\right)' &= \lim_{\Delta x \to 0} \frac{\frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{(u + \Delta u)v - (v + \Delta v)u}{v(v + \Delta v)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{v\Delta u - u\Delta v}{v(v + \Delta v)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \frac{v}{v(v + \Delta v)} - \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \frac{u}{v(v + \Delta v)} \\
&= \frac{u'v}{v^2} - \frac{v'u}{v^2} \\
&= \frac{u'v - uv'}{v^2}.
\end{aligned}
$$

Problem 8. Let r be a real number. Compute $\lim_{h\to 0} \frac{(1+2h)^r - 1}{h}$ by interpreting this limit as a derivative.

Solution:

$$\lim_{h\to 0} \frac{(1+2h)^r - 1}{h} = \frac{d}{dx}(1+2x)^r|_{x=0} = r(1+2x)^{r-1} \times 2|_{x=0} = 2r.$$