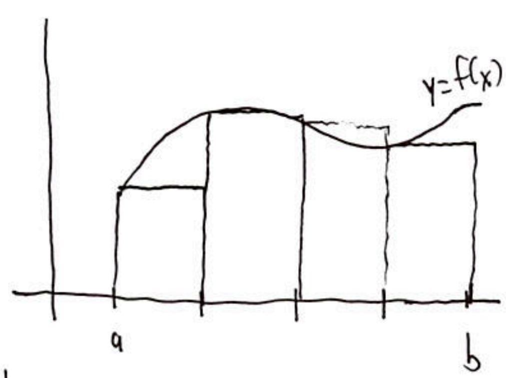
Numerical Integration

- Today's goal: How to approximate $\int_a^b \text{complicated}(x)\,dx$ when no elementary antiderivative formula exists.

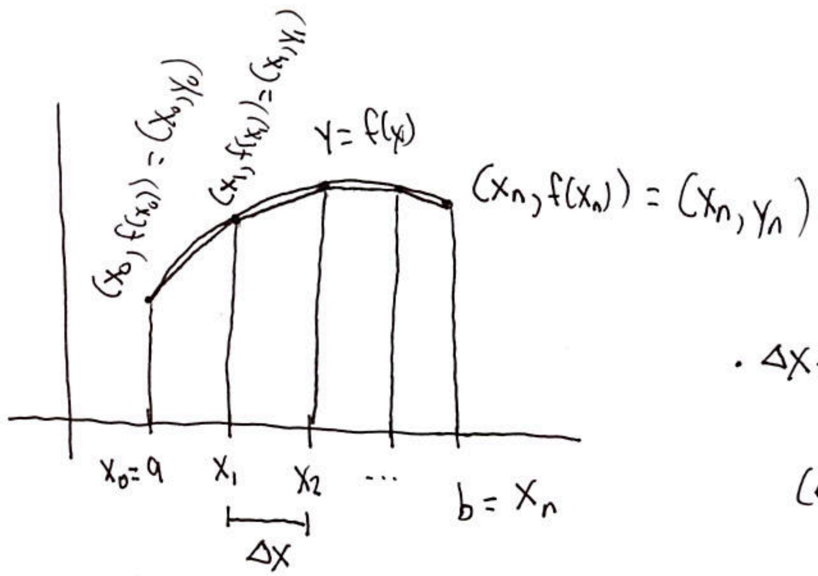
Ex: $\int_0^x \cos(t^2)\,dt$, $\int_0^3 e^{-x^2}\,dx$

Method 1: Riemann Sums



$$\int_a^b f(x)\,dx \approx \underbrace{\sum \text{areas of rectangles}}_{\text{Riemann Sum}}$$

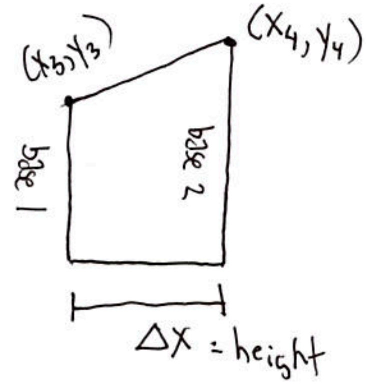
Method 2: Trapezoid Rule



- $\Delta x = \dfrac{b-a}{n}$

(equal length intervals)

- The trapezoid rule is the average of left and right Riemann sums
- It gives a more symmetric treatment of the two endpoints a and b



- Area of trapezoid

$$= \text{height} \times \frac{1}{2}(\text{base } 1 + \text{base } 2)$$

$$= \left(\frac{y_3 + y_4}{2}\right)\Delta x$$

- Total trapezoidal area $= \Delta x \left(\dfrac{y_0 + y_1}{2} + \dfrac{y_1 + y_2}{2} + \dfrac{y_2 + y_3}{2} + \dots + \dfrac{y_{n-1} + y_n}{2} \right)$

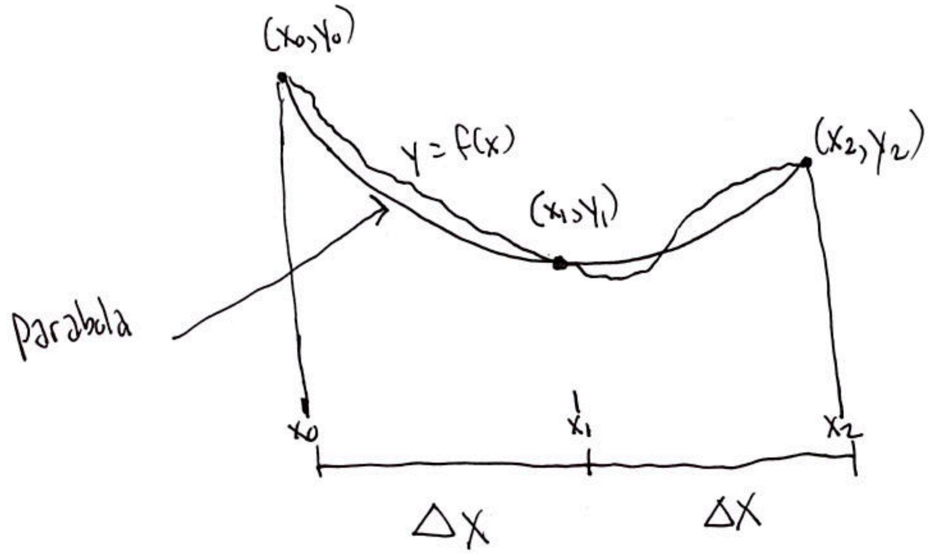
$$= \Delta x \left(\frac{y_0}{2} + y_1 + y_2 + \dots + y_{n-1} + \frac{y_n}{2} \right)$$

- Simpson's rule:

- Basic idea: replace one of the straight trapezoid edges with a parabola

- Requires an even number of intervals

- Often yields more accurate results than trapezoid rule



Can check: Area under parabola

$$= \text{base} \cdot (\text{weighted avg. height}) = 2\Delta x \cdot \left(\frac{y_0 + 4y_1 + y_2}{6} \right)$$

- Simpson's rule for n intervals (n even)

- Total parabolic area

$$= 2\Delta x \left(\frac{1}{6}\right)\left\{ (y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right\}$$

Notice the following coefficient pattern

$$\begin{array}{rccccccc}
 & 1 & 4 & 1 & & & & \\
 & & & 1 & 4 & 1 & & \\
+ & & & & & 1 & 4 & 1 \\
\hline
 & 1 & 4 & 2 & 4 & 2 & 4 & 1
\end{array}$$

Thus: Simpson's rule $\Rightarrow$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

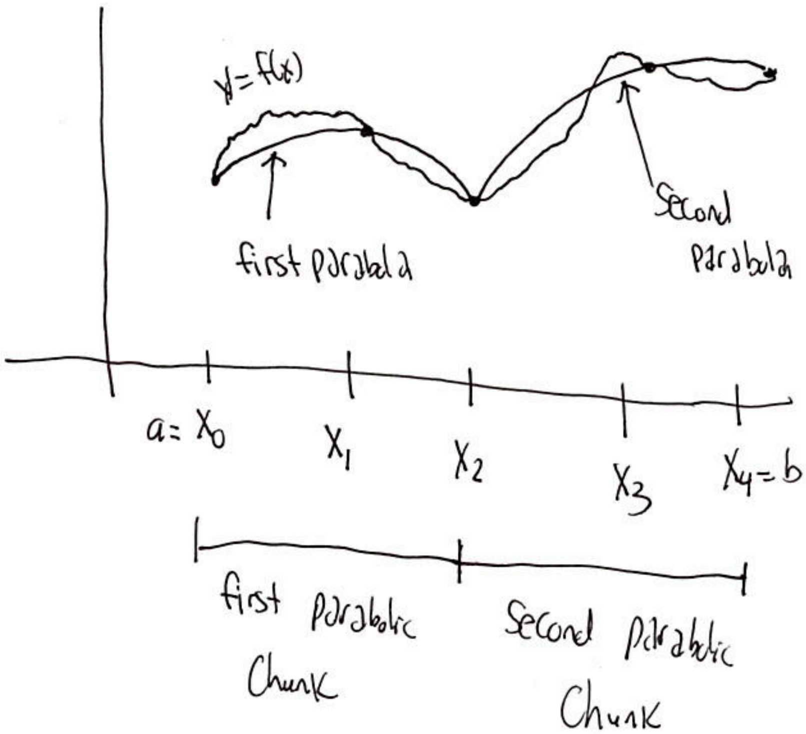
- Let's check this when $f(x) = 1$ for all x.

Then LHS $= n\Delta x$.

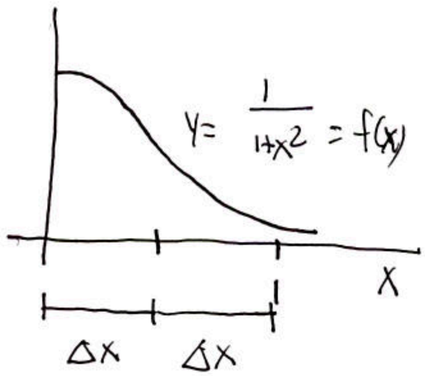
$$\text{RHS} = \frac{\Delta x}{3}\left(1 + 1 + 4\left(\frac{n}{2}\right) + 2\left(\frac{n}{2} - 1\right)\right) = n\Delta x$$

LHS $=$ RHS when $f(x) = 1$.

Picture when $n = 4$:



<u>Ex</u>: Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using the trapezoid rule and Simpson's method with two equal intervals



X	f(x)
0	1
$\frac{1}{2}$	$\frac{4}{5}$
1	$\frac{1}{2}$

- By trapezoid rule: $\Delta x \left(\frac{1}{2} y_0 + y_1 + \frac{1}{2} y_2 \right)$

  $= \frac{1}{2}\left(\frac{1}{2}\cdot 1 + \frac{4}{5} + \frac{1}{2}\cdot\frac{1}{2} \right) = .775$

- By Simpson's rule: $\frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2 \right) = \frac{\frac{1}{2}}{3}\left(1 + 4\cdot\frac{4}{5} + \frac{1}{2} \right) = .78333\ldots$

- Exact answer: $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x \Big]_0^1 = \tan^{-1}(1) = \frac{\pi}{4} \approx .785$

- Can show: $|\text{Simpson} - \text{Exact}| \sim (\Delta x)^4$