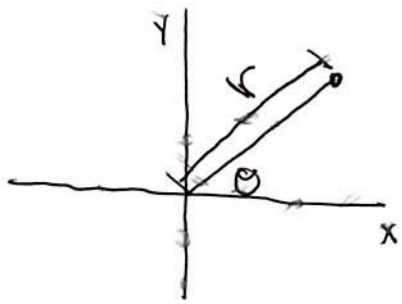
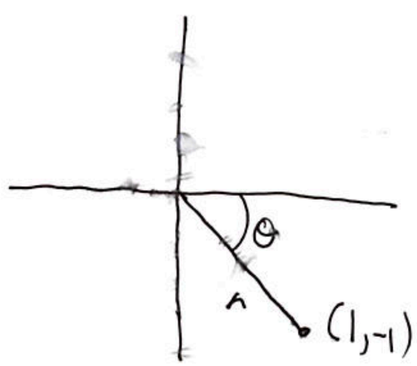
Polar Coordinates, Area in Polar Coordinates

- Polar Coordinates



- In polar coordinates, we specify an object's position in terms of its distance r from the origin and the angle θ that the ray from the origin to the point makes with respect to the x axis

Ex: What are the polar coordinates for the point with rectangular coordinates $(1,-1)$?



- $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
- $\theta = -\pi/4$

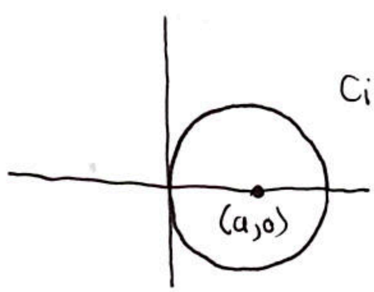
- The most common convention is $r \geq 0$ and $0 \leq \theta < 2\pi$
- Another common convention is $r \geq 0$ and $-\pi \leq \theta < \pi$
- Some conventions use

- No matter what the convention, the following formulas are always true:

  - $X = r\cos\theta$, • $Y = r\sin\theta$

<u>Ex</u>: $(1,-1)$ can be represented by $r = -\sqrt{2}$, $\theta = \frac{3\pi}{4}$:
$1 = x = -\sqrt{2}\cos\left(\frac{3\pi}{4}\right)$, $-1 = y = -\sqrt{2}\sin\left(\frac{3\pi}{4}\right)$

<u>Ex</u>: Consider a circle of radius a with its center at $x=a$, $y=0$. Let's find an equation that relates r to θ.



- In rectangular coordinates, the equation for the circle is

$$(x-a)^2 + y^2 = a^2$$

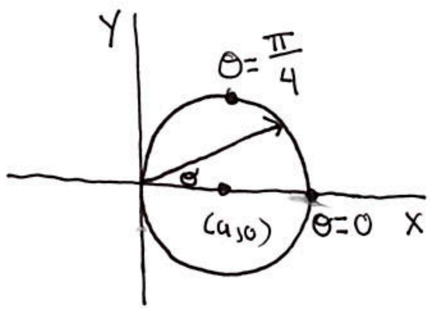
- We plug in $x = r\cos\theta$, $y = r\sin\theta$: $(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$

$\Rightarrow$ $r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta = a^2$

$\Rightarrow$ $r^2 - 2ar\cos\theta = 0$

$\Rightarrow$ $\boxed{r = 2a\cos\theta}$

- The range of $0 \leq \theta \leq \frac{\pi}{2}$ traces out the top half of the circle, while $-\frac{\pi}{2} \leq \theta \leq 0$ traces out the bottom half. Let's graph this.

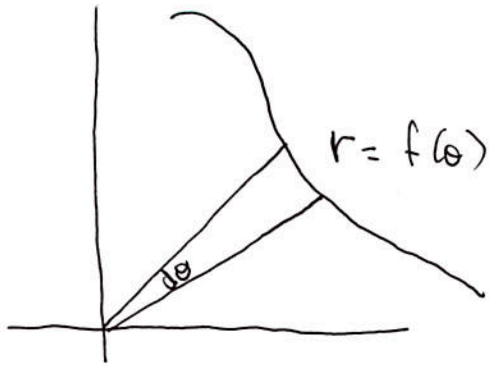


Graph of $r = 2a\cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

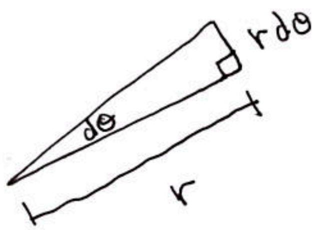
- At $\theta = 0$, $r = 2a \Rightarrow x = 2a,\ y = 0$
- At $\theta = \frac{\pi}{4}$, $r = 2a\cos\frac{\pi}{4} = a\sqrt{2}$

- The main issue is finding a range of θ values that traces the circle once. In this example $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ works.
  - $\theta = -\frac{\pi}{2}$ (down)
  - $\theta = \frac{\pi}{2}$ (up)

• Area in Polar Coordinates



• Let's find the area of a small slice



• The small slice is approximately a right triangle.

Area of Slice $\approx$ Area of right triangle

$$= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\, r\,(r\,d\theta)$$

• Total Area $= \displaystyle\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$

Ex: $r = 2a\cos\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (the circle from a previous example).

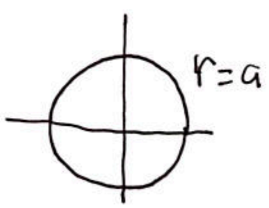
$$\text{Area} = \int_{\theta=-\pi/2}^{\pi/2} \frac{1}{2}(2a\cos\theta)^2 \, d\theta = 2a^2 \int_{\theta=-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

used trig id: $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$$= a^2 \int_{-\pi/2}^{\pi/2} (1+\cos(2\theta))\, d\theta = a^2 \int_{-\pi/2}^{\pi/2} d\theta + a^2 \int_{-\pi/2}^{\pi/2} \cos(2\theta)\, d\theta$$

$$= a^2 \left[\theta + \frac{\sin(2\theta)}{2}\right]_{\theta=-\pi/2}^{\theta=\pi/2} = \pi a^2 .$$

Ex: Circle centered at the Origin (radius $= a$)



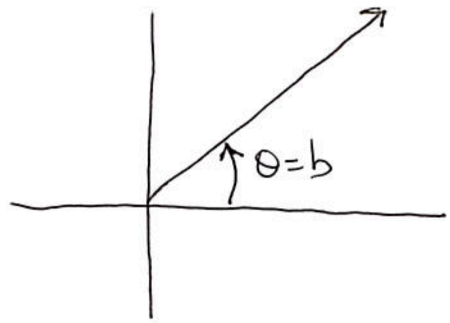
- $x = r\cos\theta \quad y = r\sin\theta$
- $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

- The equation of the circle is $x^2 + y^2 = a^2$, So $r = a$

$\Rightarrow x = a\cos\theta$, $y = a\sin\theta$

$$\text{Area} = \int_{\theta=0}^{2\pi} \frac{1}{2} a^2 \, d\theta = \frac{1}{2}\cdot a^2 \cdot 2\pi = \pi a^2$$

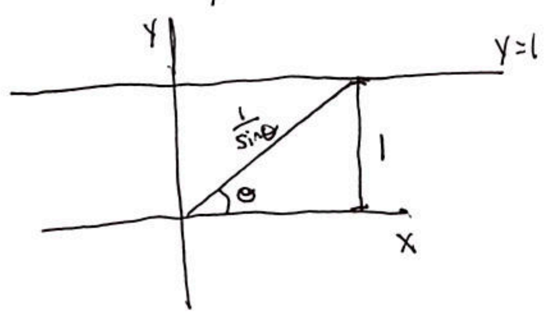
Ex A ray.



In this case $\theta = b$, and the range of r is $0 \leq r < \infty$

- $x = r\cos b$ • $y = r\sin b$

Ex The line $y = 1$



- To find the polar coordinate equation, plug in $y = r\sin\theta$, $x = r\cos\theta$ and solve for r:
  - $y = 1$
  - $r\sin\theta = 1$

  $\Rightarrow r = \frac{1}{\sin\theta}$ with $0 < \theta < \pi$

- $\underline{\text{Ex}}$. Finding the (x,y) coordinates from $r = f(\theta)$

As an example, let's consider $r = \dfrac{1}{1 + \frac{1}{2}\sin\theta}$

- $r + \frac{r}{2}\sin\theta = 1$

- Plug in $r = \sqrt{x^2+y^2}$, $\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$

$$\Rightarrow \quad \sqrt{x^2+y^2} + \frac{y}{2} = 1$$

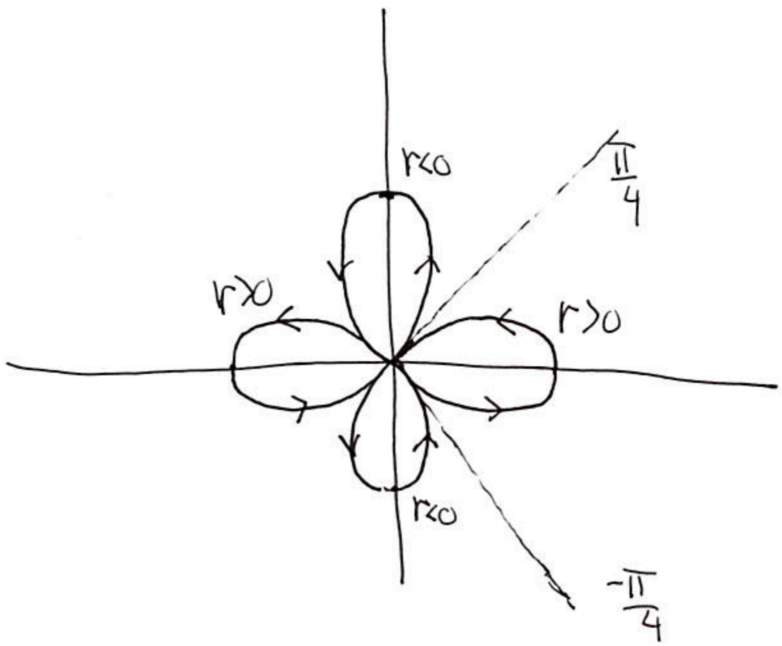
$$\Rightarrow \sqrt{x^2+y^2} = 1 - \frac{y}{2} \quad \Rightarrow \quad x^2+y^2 = \left(1-\frac{y}{2}\right)^2 = 1 - y + \frac{y^2}{4}$$

Finally, $x^2 + \frac{3y^2}{4} + y = 1$.

This is the equation of an ellipse with one focus at the origin

Ex A rose $r = \cos(2\theta)$

· The graph looks a bit like a flower



· For the first "petal," $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$