MATH 18.01 - MIDTERM 3 - SOME REVIEW PROBLEMS WITH SOLUTIONS

18.01 Calculus, Fall 2014	Professor: Jared Speck

Problem 1.
a) Find all solutions $y = f(x)$ to the differential equation $y' = x^2 y \ln y$.
b) Find the solution to the initial value problem $y' = x^2 y \ln y$, $y(0) = e$.

Problem 2. Find the area in between the curves $y = x^2$, $y = 1 - x^2$ and $x = 1$.

Problem 3. Consider the region R bounded by the lines $x = 4$, $x = 9$, $y = 0$ and the curve $y = x^{3/2}$. The region R is revolved around the line $x = 1$ to generate a solid S. Find the volume of S.

Problem 4. Let $f(x)$ be a function defined for all $x \geq 0$. Suppose that $\int_0^\infty f(x)\,dx$ is a finite number, where $\int_0^\infty f(x)\,dx = \lim_{N\to\infty} \int_0^N f(x)\,dx$. For each $N > 0$, consider the function $g_N(x) = f(Nx)$. Show that the average value of $g_N(x)$ on the interval $[0, 2]$ converges to 0 as $N \to \infty$.

Problem 5. Let

$$F(x) = \int_0^1 e^{-t^2 x}\,dt.$$

Show that when $x > 0$, we have

$$F'(x) = -\frac{1}{2}x^{-1}F(x) + \frac{1}{2}x^{-1}e^{-x}.$$

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Problem 7. Let $N \geq 0$ be an integer. Use a Riemann sum argument to show that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n i^N}{n^{N+1}} = \frac{1}{N+1}.$$

Hint: Relate the above sums to Riemann sums that converge to the integral as $n \to \infty$.

Problem 1. a) Find all solutions $y = f(x)$ to the differential equation $y' = x^2 y \ln y$.

Solution: We first write the differential equation as

$$\frac{dy}{y \ln y} = x^2 dx.$$

We now integrate both sides and use the substitution $u = \ln y$, $du = \frac{1}{y} dy$ to deduce that

$$\ln |\ln y| = \frac{1}{3}x^3 + C.$$

Exponentiating twice, we deduce that

$$y = e^{ce^{(1/3)x^3}},$$
$$c = \pm e^C.$$

b) Find the solution to the initial value problem $y' = x^2 y \ln y$, $y(0) = e$.

Solution: We have to solve for c. Setting $x = 0, y = e$ in the above formula for y in terms of x, we deduce that

$$e = e^{ce^{(1/3)0^3}} = e^c,$$

which implies that $c = 1$. The solution is therefore $y = e^{e^{(1/3)x^3}}$.

Problem 2. Find the area in between the curves $y = x^2$, $y = 1 - x^2$ and $x = 1$.

Solution: The graphs of $y = x^2, y = 1 - x^2$ intersect when $x^2 = 1 - x^2$, or equivalently, when $2x^2 = 1$. This equation has the solutions

$$x = \pm\sqrt{\frac{1}{2}}.$$

The region of interest can be split into two regions: $-\sqrt{\frac{1}{2}} \leq x \leq \sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{2}} \leq x \leq 1$. In the first region, the top curve is $y = 1 - x^2$ and the bottom one is $y = x^2$. The area of this region is therefore

$$\begin{aligned}\int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} (1 - x^2) - x^2 \, dx &= \left[x - \frac{2}{3}x^3\right]_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} \\ &= \frac{4}{3\sqrt{2}}.\end{aligned}$$

In the second region, the top curve is $y = x^2$ and the bottom one is $y = 1 - x^2$. The area of this region is therefore

$$\begin{aligned}\int_{\sqrt{\frac{1}{2}}}^{1} x^2 - (1 - x^2) \, dx &= \left[\frac{2}{3}x^3 - x\right]_{\sqrt{\frac{1}{2}}}^{1} \\ &= -\frac{1}{3} + \frac{2}{3\sqrt{2}}.\end{aligned}$$

Therefore, the total area of the two regions is

$$\frac{4}{3\sqrt{2}} - \frac{1}{3} + \frac{2}{3\sqrt{2}} = \sqrt{2} - \frac{1}{3}.$$

Problem 3. Consider the region R bounded by the lines $x = 4$, $x = 9$, $y = 0$ and the curve $y = x^{3/2}$. The region R is revolved around the line $x = 1$ to generate a solid S. Find the volume of S.

Solution: We divide S into cylindrical shells whose axes are parallel to the $y-$axis. For $4 \leq x \leq 9$, each shell has a height $x^{3/2}$, a radius $x - 1$, and a width dx. Thus, the volume of the shell is $dV = 2\pi(x-1)x^{3/2}\,dx$. To find the volume V of S, we integrate dV :

$$\begin{aligned}
V = \int dV &= \int_{x=4}^{x=9} 2\pi(x-1)x^{3/2}\,dx \\
&= 2\pi \int_{x=4}^{x=9} x^{5/2}\,dx - 2\pi \int_{x=4}^{x=9} x^{3/2}\,dx \\
&= 2\pi \left[\frac{2}{7}x^{7/2} - \frac{2}{5}x^{5/2}\right]_{x=4}^{x=9} \\
&= 2\pi \left(\frac{2}{7}3^7 - \frac{2}{7}2^7 - \frac{2}{5}3^5 + \frac{2}{5}2^5\right).
\end{aligned}$$

Problem 4. Let $f(x)$ be a function defined for all $x \geq 0$. Suppose that $\int_0^\infty f(x)\,dx$ is a finite number, where $\int_0^\infty f(x)\,dx = \lim_{N\to\infty} \int_0^N f(x)\,dx$. For each $N > 0$, consider the function $g_N(x) = f(Nx)$. Show that the average value of $g_N(x)$ on the interval $[0, 2]$ converges to 0 as $N \to \infty$.

Solution: By definition, the average value of $g_N(x)$ on $[0, 2]$ is $\frac{1}{2}\int_0^2 g_N(x)\,dx = \frac{1}{2}\int_0^2 f(Nx)\,dx$. By making the substitution $u = Nx$, $du = Ndx$, we see that the average value of g_N is

$$\frac{1}{2N}\int_{u=0}^{u=2N} f(u)\,du.$$

By assumption, we have that $\lim_{N\to\infty} \int_{u=0}^{u=2N} f(u)\,du = I$, where I is some finite number. Therefore, it follows that

$$\begin{aligned}
&\lim_{N\to\infty} \left\{\frac{1}{2N}\int_{u=0}^{u=2N} f(u)\,du\right\} \\
&= \lim_{N\to\infty} \frac{1}{2N} \times \lim_{N\to\infty} \int_{u=0}^{u=2N} f(u)\,du \\
&= 0 \times I = 0
\end{aligned}$$

as desired.

Problem 5. Let

$$F(x) = \int_0^1 e^{-t^2 x}\,dt.$$

Show that when $x > 0$, we have

$$F'(x) = -\frac{1}{2}x^{-1}F(x) + \frac{1}{2}x^{-1}e^{-x}.$$

Solution: We make the change of variables $u = t\sqrt{x}$, $du = dt\sqrt{x}$. Then we have the identity

$$F(x) = \int_0^1 e^{-t^2 x}\, dt = x^{-1/2}\int_0^{\sqrt{x}} e^{-u^2}\, du.$$

We now use the product rule, the second fundamental theorem of calculus, and the chain rule to deduce that

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\left(x^{-1/2}\int_0^{\sqrt{x}} e^{-u^2}\, du\right) \\
&= \left(\frac{d}{dx}x^{-1/2}\right)\int_0^{\sqrt{x}} e^{-u^2}\, du + x^{-1/2}\frac{d}{dx}\int_0^{\sqrt{x}} e^{-u^2}\, du \\
&= -\frac{1}{2}x^{-3/2}\int_0^{\sqrt{x}} e^{-u^2}\, du + x^{-1/2}e^{-(\sqrt{x})^2}\frac{d}{dx}\sqrt{x} \\
&= -\frac{1}{2}x^{-3/2}\int_0^{\sqrt{x}} e^{-u^2}\, du + \frac{1}{2}x^{-1}e^{-(\sqrt{x})^2} \\
&= -\frac{1}{2}x^{-3/2}\int_0^{\sqrt{x}} e^{-u^2}\, du + \frac{1}{2}x^{-1}e^{-x} \\
&= -\frac{1}{2}x^{-1}F(x) + \frac{1}{2}x^{-1}e^{-x}
\end{aligned}$$

as desired.

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Solution: We first calculate the area $A(r)$ of the square cross section of the solid that is located a perpendicular distance r from the center of the wooden sphere (where $0 \le r \le a$ accounts for half of the solid). By the pythagorean theorem, the diagonal length of the square cross section is $d = 2\sqrt{a^2 - r^2}$. Since the area A of a square with a diagonal of length d is $A = d^2/2$, it follows that $A(r) = 2(a^2 - r^2)$. Therefore, the volume of a thin slice of width dr is $A(r)\, dr$.

By symmetry, to find the total volume, we can double the volume of one half of the solid (which is obtained by integrating $A(r)\, dr$ from $r = 0$ to $r = a$):

$$\begin{aligned}
V &= 2\int_{r=0}^{a} 2(a^2 - r^2)\, dr \\
&= 4\left[a^2 r - \frac{1}{3}r^3\right]_{r=0}^{r=a} \\
&= \frac{8}{3}a^3.
\end{aligned}$$

Problem 7. Let $N \geq 0$ be an integer. Use a Riemann sum argument to show that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^N}{n^{N+1}} = \frac{1}{N+1}.$$

Hint: Relate the above sums to Riemann sums that converge to the integral as $n \to \infty$.

We first note that

$$\int_0^1 x^N\,dx = \frac{1}{N+1}.$$

We then approximate the integral using right Riemann sums with n equal intervals. More precisely, let $f(x) = x^N$. Then with $\Delta x = 1/n$, the right Riemann sum is

$$\begin{aligned}\sum_{i=1}^{n} f(i/n)\Delta x &= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^N \frac{1}{n} \\ &= \frac{\sum_{i=1}^{n} i^N}{n^{N+1}}.\end{aligned}$$

As $n \to \infty$, the Riemann rectangles become infinitely thin and the Riemann sums converge to the integral $\frac{1}{N+1}$. That is,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^N}{n^{N+1}} = \frac{1}{N+1}$$

as desired.