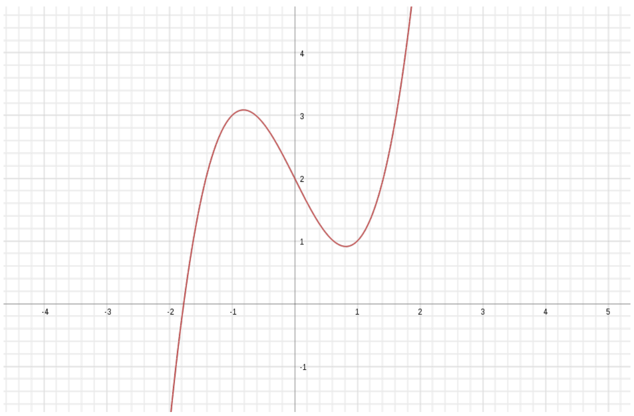
18.01 (Fall 14) Solution to Problem Set 4

Part II

1. **solution**

   a) The graph of $f(x) = x^3 - 2x + 2$ is as the following:



   b) Build the table as following,

Base point	$f(x)$	$f'(x)$
$x_0 = 0$	2	-2
$x_1 = 1$	1	1
$x_2 = 0$	2	-2
$x_3 = 1$	1	1
$x_4 = 0$	2	-2

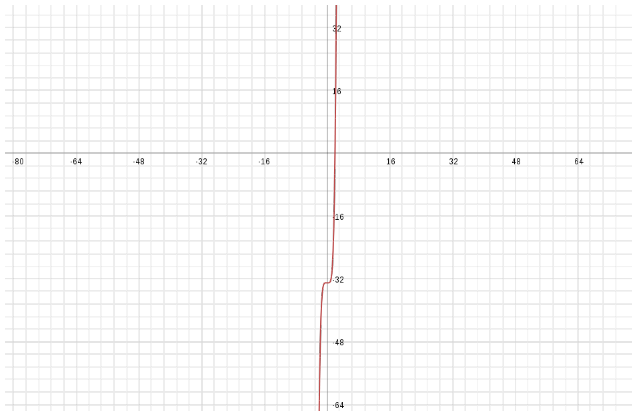
   c) Let the base point be $x_0 = 0.1$

Base point	$f(x)$	$f'(x)$
$x_0 = 0.1$	1.801	-1.97
$x_1 = 1.014213198$	1.014822114	1.085885233
$x_2 = 0.079655767$	1.841193885	-1.980964876
$x_3 = 1.009098741$	1.009347855	1.056349277
$x_4 = 0.053592947$	1.892968036	-1.991383388

   We can observe that the solution is getting closed to the solution in part b) and alternating between 0 and 1. It can't converge to the solution of $f(x) = 0$.

2. **solution**

   a) The graph of $f(x) = x^5 - 33$ is the following:



b) We can construct the table as following:

Base point	$f(x)$	$f'(x)$
$x_0 = 2$	-1	5
$x_1 = 2.2$	18.53632	117.128
$x_2 = 2.0417430503380917$	2.4817825573292355729	86.8909105664736227199
$x_3 = 2.0131810000482560$	$6.8471009117051644486 \times 10^{-2}$	82.1299004121447587233
$x_4 = 2.0123473084356588$	$5.6686479349152928256 \times 10^{-5}$	81.9939394858550766023
$x_5 = 2.0123466170860334$	$3.8949700815000230813 \times 10^{-11}$	81.9938268085853046356
$x_6 = 2.0123466170855583$	$1.83888387100106410423 \times 10^{-23}$	81.9938268085078831805

c) We can observe that for all $k \geq 1$, $x_k > 33^{\frac{1}{5}}$. Also, for all $k \geq 1$, we have $x_k > x_{k+1}$. The second part comes from the first part because when $f(x) > 0$, $f'(x) > 0$ the sequences is decreasing. The first part is because the function is convex here, therefore, the function will be above the tangent line and the next base point will be greater than $33^{\frac{1}{5}}$.

d) Use the quadratic approximation of $f = \sqrt[5]{x}$ at $x_0 = 32$. we have $f(x_0) = 2$, $f'(x_0) = \frac{1}{80}$, $f''(x_0) = -\frac{1}{3200}$. Therefore,

$$f(x) \approx 2 + \frac{1}{80}(x-32) - \frac{1}{6400}(x-32)^2.$$

Therefore, we can get $\sqrt[5]{33} \approx 2 + \frac{1}{80} - \frac{1}{6400} = 2.01234375$ Note that $\sqrt[5]{33} = 2.0123466170855583$. The quadratic approximation have only 6 digit of accuracy.

3. **solution**

a) No, the function cannot exist. Proof: Since $f(0) = -1$, $f(2) = 4$, By mean value theorem, there is $c \in (0,2)$ such that $f'(c) = (f(2) - f(0))/(2-0) = 2.5 > 2$. This contradicts $f'(x) \leq 2$.

b) We assume the contrary: there is a function f which has more than 1 fixed point and $f'(x) \neq 1$. Let a, b be different fixed points of f and $a < b$. Define $g(x) = f(x) - x$, We have $g(a) = f(a) - a = 0$, $g(b) = f(b) - b = 0$. By mean value theorem, there is some $c \in (a,b)$ such that $g'(c) = (g(b)-g(a))/(b-a) = 0$. On the other hand, $g'(x) = f'(x) - 1 \neq 0$ and therefore $g'(c) \neq 0$. We get a contradiction.

4. **solution**

a) For all $x > 0$, from the mean value theorem, there is some $y \in [0, x]$ such that $f(x) - f(0) = f'(y)(x - 0) = f'(y)x$. Since $f'(y) \geq 0$, $x > 0$, we have $f(x) \geq f(0) = 0$.

b) Let $f(x) = x - \ln(1+x)$. We have $f(0) = 0 - \ln 1 = 0$, $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$. Therefore, from part a), for all $x > 0$, $f(x) \geq 0$, i.e. for all $x \geq 0$, $\ln(1+x) \leq x$.

c) Let $f_2(x) = x - x^2/2 - \ln(1+x)$. We have $f_2(0) = 0$, $f_2'(x) = 1 - x - \frac{1}{1+x} = \frac{-x^2}{1+x} < 0$. We have $(-f_2)' > 0$. Therefore, from part a), for all $x > 0$, $-f_2(x) \geq 0$, i.e. for all $x \geq 0$, $\ln(1+x) \geq x - \frac{x^2}{2}$.

Let $f_3(x) = x - x^2/2 + x^3/3 - \ln(1+x)$. We have $f_2(0) = 0$, $f_2'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{x^3}{1+x} > 0$. We have $f_3' > 0$. Therefore, from part a), for all $x > 0$, $f_3(x) \geq 0$, i.e. for all $x \geq 0$, $\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$.

d) Let $p_k(x) = \sum_{l=1}^{k} (-1)^{l-1} x^l / l$, we can make a conjecture that for $x \geq 0$, $\ln x \geq p_k(x)$ for even k, $\ln x \leq p_k(x)$ for odd k.

Proof. Let $f_k(x) = p_k(x) - \ln(1+x)$, we have $f_k(0) = 0$,

$$\begin{aligned} f_k(x)' &= \sum_{l=1}^{k} (-1)^{l-1} x^{l-1} - \frac{1}{x+1} = \sum_{l=0}^{k-1} (-1)^l x^l - \frac{1}{x+1} \\ &= \frac{(x+1)\sum_{l=0}^{k-1}(-1)^l x^l - 1}{x+1} \\ &= \frac{1 + (-1)^{k-1} x^k - 1}{x+1} = \frac{(-1)^{k-1} x^k}{x+1}. \end{aligned}$$

Therefore, $f_k(x) > 0$ for odd k and $f_k(x) < 0$ for even k. From part a), we can get the desired result. □

e) Set $u = -x$, we want to show that $\ln(1-u) \leq -u$ for $0 \leq u < 1$. Let $g(u) = -u - \ln(1-u)$. We have $g(0) = 0$, $g'(u) = -1 + \frac{1}{1-u} = \frac{u}{1-u} > 0$. Therefore, from part a), we have $g(u) \geq 0$ for $0 \leq u < 1$, i.e. $\ln(1+x) \leq x$ for $-1 < x \leq 0$.