

18.01 (Fall 2014) Solutions to Problem Set 1

Part II

1. solution

- a) For any x , we have $E(x) + O(x) = 0$. By the condition that $E(x)$ is even and $O(x)$ is odd, we have $0 = E(-x) + O(-x) = E(x) - O(x)$. Using these 2 equations, we can solve for $E(x)$ and $O(x)$ and get $E(x) = O(x) = 0$ for any x .
- b) From part a), since $A \sin(ax)$ is odd and $B \cos(bx)$ is even, we deduce that $A \sin(ax) = 0$ and $B \cos(bx) = 0$. Putting $x = 0$ into the second equation, we have $B = 0$. For the equation $A \sin(ax) = 0$, either $A = 0$ or $\sin(ax) = 0$ for any x . Therefore, we can get either $A = 0$ or $a = 0$.

2. solution

- a) We have $V = \frac{\pi}{9}h^3$. Thus, $h(V) = \sqrt[3]{\frac{9}{\pi}}V^{\frac{1}{3}}$. From the definition of derivative, we have

$$\begin{aligned} h'(V) &= \lim_{\Delta V \rightarrow 0} \frac{h(V + \Delta V) - h(V)}{\Delta V} \\ &= \sqrt[3]{\frac{9}{\pi}} \lim_{\Delta V \rightarrow 0} \frac{(V + \Delta V)^{\frac{1}{3}} - V^{\frac{1}{3}}}{\Delta V} \\ &= \sqrt[3]{\frac{9}{\pi}} \lim_{\Delta V \rightarrow 0} \frac{((V + \Delta V)^{\frac{1}{3}} - V^{\frac{1}{3}})((V + \Delta V)^{\frac{2}{3}} + (V + \Delta V)^{\frac{1}{3}}V^{\frac{1}{3}} + V^{\frac{2}{3}})}{\Delta V((V + \Delta V)^{\frac{2}{3}} + (V + \Delta V)^{\frac{1}{3}}V^{\frac{1}{3}} + V^{\frac{2}{3}})} \\ &= \sqrt[3]{\frac{9}{\pi}} \lim_{\Delta V \rightarrow 0} \frac{(V + \Delta V) - V}{\Delta V((V + \Delta V)^{\frac{2}{3}} + (V + \Delta V)^{\frac{1}{3}}V^{\frac{1}{3}} + V^{\frac{2}{3}})} \\ &= \sqrt[3]{\frac{9}{\pi}} \lim_{\Delta V \rightarrow 0} \frac{1}{((V + \Delta V)^{\frac{2}{3}} + (V + \Delta V)^{\frac{1}{3}}V^{\frac{1}{3}} + V^{\frac{2}{3}})} \\ &= \sqrt[3]{\frac{9}{\pi}} \frac{1}{3V^{\frac{2}{3}}} = \sqrt[3]{\frac{1}{3\pi}}V^{-\frac{2}{3}} \end{aligned}$$

- b) We first note that $V_0 = \frac{\pi}{9}h_0^3 = \frac{\pi}{9}$. Using $\Delta h = h'(V_0)\Delta V$, we have

$$\Delta h = h'(V_0)\Delta V = \frac{3}{\pi}\Delta V$$

- c) We substitute $\Delta V = 0.1$ into the previous equation. We find that $\Delta h = \frac{0.3}{\pi} > 0.02$. Therefore, the tool is not good enough.
- d) We have $V'(h) = \frac{\pi}{3}h^2$. Therefore, using the corresponding approximate error formula, we have $\Delta V = V'(h_0)\Delta h = \frac{\pi}{3}\Delta h$. This is exactly the same equation as in b). Therefore, it will not change the answer in c).

3. solution

Section 3.1: 14 Both curves must pass through $(3, 3)$. We can get the equations

$$3 = 9 + 3a + b$$

$$3 = 3c - 9$$

Thus, $c = 4$. Now, both curves must have the same slope when $x = 3$. Differentiating both equations with respect to x yields

$$y' = 2x + a$$

$$y' = c - 2x$$

Plugging $c = 4$, $x = 3$ into the above equations and setting the equations equal to each other, we find that $a = -8$. Solving for b , we find that $b = 18$.

Section 3.1: 18 The slope of the curve at (x_0, x_0^3) is given by $y'(x_0) = 3x_0^2$. Therefore, the tangent line at (x_0, x_0^3) is given by $y - x_0^3 = 3x_0^2(x - x_0)$. Plugging in $(x, y) = (0, 2)$ yields $2 - x_0^3 = -3x_0^3$. We have $2 = -2x_0^3$, $x_0 = -1$. The desired tangent is given by $y + 1 = 3(x + 1)$ or $y = 3x + 2$.

Section 3.1: 21

- a) The equation of the parabola is $y = \frac{1}{4p}x^2$. Differentiation with respect to x yields $y' = \frac{x}{2p}$. The tangent at (x_0, y_0) is given by $y - y_0 = \frac{x_0}{2p}(x - x_0)$. Plugging in $x = 0$, we find that the y -intercept is $y = -\frac{x_0^2}{2p} + y_0 = -2y_0 + y_0 = -y_0$.
- b) The distance from $(0, p)$ to $(0, -y_0)$ is $p + y_0$. The distance from $(0, p)$ to (x_0, y_0) is

$$\begin{aligned}\sqrt{x_0^2 + (y_0 - p)^2} &= \sqrt{4py_0 + y_0^2 - 2py_0 + p^2} \\ &= \sqrt{y_0^2 + 2py_0 + p^2} = y_0 + p\end{aligned}$$

Therefore, the triangle with these vertices is isosceles.

- c) We name following points: $X = (x_0, y_0)$, $F = (0, p)$, $Z = (0, -y_0)$. Any isosceles triangle has the same base angles. The angle of incidence $\angle FXZ = \angle FZX$ is equal to the angle between the tangent and the vertical line. Since the angle of incidence is equal to the angle of reflection, after reflection, the ray should be vertical, that is, point upward.

4. solution

Section 2.5: 19f)

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{3x^2} = \frac{1}{3} \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) = \frac{1}{3} \cdot 1 \cdot 1 = \frac{1}{3}$$

Section 2.5: 19g)

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \frac{2}{3} \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \right) \left(\lim_{x \rightarrow 0} \frac{3x}{\sin 3x} \right) = \frac{2}{3} \cdot 1 \cdot 1 = \frac{2}{3}$$

Section 2.5: 20a)

$$\lim_{x \rightarrow 0} \frac{\sin x}{3\sqrt{x}} = \frac{1}{3} \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left(\lim_{x \rightarrow 0} \sqrt{x} \right) = \frac{1}{3} \cdot 1 \cdot 0 = 0$$

Section 2.5: 20b)

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left(\lim_{x \rightarrow 0} \sin x \right) = 1 \cdot 0 = 0$$

Section 2.5: 20g)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3x^2 + 4x}{\sin 2x} &= \lim_{x \rightarrow 0} \frac{3x^2}{\sin 2x} + \lim_{x \rightarrow 0} \frac{4x}{\sin 2x} \\ &= \frac{3}{2} \left(\lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \right) \left(\lim_{x \rightarrow 0} x \right) + 2 \lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \\ &= \frac{3}{2} \cdot 1 \cdot 0 + 2 \cdot 1 = 2 \end{aligned}$$

Section 2.5: 22a) Let $f(\theta) = \frac{1 - \cos \theta}{\theta^2}$

θ	$f(\theta)$
1	0.45969769
0.1	0.49958347
0.01	0.49999583
0.001	0.49999996

We can make a conjecture that $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2}$.

Section 2.5: 22b) Using the trig identity $\cos \theta = 1 - 2\sin^2(\frac{\theta}{2})$, we find that

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} &= \lim_{\theta \rightarrow 0} \frac{2\sin^2(\frac{\theta}{2})}{\theta^2} \\ &= \frac{1}{2} \left(\lim_{\theta \rightarrow 0} \frac{\sin(\frac{\theta}{2})}{\frac{\theta}{2}} \right) \left(\lim_{\theta \rightarrow 0} \frac{\sin(\frac{\theta}{2})}{\frac{\theta}{2}} \right) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}. \end{aligned}$$

5. solution

a)

$$\frac{d}{dx} x^{-1} = \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{\frac{d}{dx} 1 \cdot x - \frac{d}{dx} x \cdot 1}{x^2} = \frac{-1}{x^2} = -x^{-2}$$

- b) From a), we have $\frac{d}{dx}x^{-1} = -x^{-2}$. Now, suppose we know that $\frac{d}{dx}x^{-n} = -nx^{-(n+1)}$. For the case $n + 1$, we have $x^{-(n+1)} = x^{-n}x^{-1}$. Using the product rule, we have

$$\begin{aligned}\frac{d}{dx}x^{-(n+1)} &= \frac{d}{dx}(x^{-n}x^{-1}) = \frac{d}{dx}x^{-n} \cdot x^{-1} + x^{-n} \cdot \frac{d}{dx}x^{-1} \\ &= -nx^{-(n+1)} \cdot x^{-1} + x^{-n} \cdot (-x^{-2}) = -nx^{-(n+2)} - x^{-(n+2)} \\ &= -(n+1)x^{-(n+2)}\end{aligned}$$

Therefore, by the method of induction, the formula $\frac{d}{dx}x^{-n} = -nx^{-(n+1)}$ holds for all positive integers n .

- c) When $n = 1$, we have $\frac{d}{dx}x = 1$. Now, suppose we know that $\frac{d}{dx}x^n = nx^{n-1}$. For the case $n + 1$, we have $x^{n+1} = x^n \cdot x$. Using the product rule, we have

$$\begin{aligned}\frac{d}{dx}x^{n+1} &= \frac{d}{dx}(x^n \cdot x) = \frac{d}{dx}x^n \cdot x + x^n \cdot \frac{d}{dx}x \\ &= nx^{n-1} \cdot x + x^n \cdot 1 = nx^n + x^n = (n+1)x^n.\end{aligned}$$

Therefore, by the method of induction, the formula $\frac{d}{dx}x^n = nx^{n-1}$ holds for all positive integers n .