MIDTERM 1 - 18.01 - FALL 2017.

Name:

Email:

Please put a check by your recitation section.

	Instructor	**Time**
☐	Miles Couchman	MW 1
☐	Kristin Kurianski	MW 1
☐	Yu Pan	MW 10
☐	Yu Pan	MW 11
☐	Jiewon Park	MW 12
☐	Jake Wellens	MW 12
☐	Siddharth Venkatesh	MW 2

Problem #	Max points possible	Actual score
1	20	
2	15	
3	20	
4	15	
5	15	
6	15	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**

Good luck!

Problem 1. (10 + 10 = 20 points) Compute the derivatives of the following two functions of x.

a) $\dfrac{\tan x}{\sqrt{x^2-1}}$

b) x^{x^2}

Solution: a)

$$y' = \frac{(\sec x)^2}{\sqrt{x^2-1}} - x\frac{\tan x}{(x^2-1)^{3/2}}$$

b) We set $y = x^{x^2}$. Then $\ln y = x^2 \ln x$. Hence, differentiating, we obtain

$$\frac{y'}{y} = 2x \ln x + x.$$

Thus,

$$y' = x^{x^2}(2x \ln x + x).$$

Problem 2. (5 + 10 = 15 points)

a) State the analytic definition of the derivative of a function.

b) Using only your definition from part a, decide whether or not the function $f(x) = |x|^{3/2}$ is differentiable at the point $x = 0$. If your answer is "yes," then what is $f'(0)$?

Solution: a)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

b) We compute

$$f'(0) := \lim_{\Delta x \to 0} \frac{|\Delta x|^{3/2}}{\Delta x} = \lim_{\Delta x \to 0} \left\{ \sqrt{|\Delta x|}\, \underbrace{\frac{|\Delta x|}{\Delta x}}_{\leq 1} \right\} = 0.$$

Thus, f is differentiable at $x = 0$ with $f'(0) = 0$.

Problem 3. (20 points) Find the equation of the tangent line to the following curve at the point $(x,y)=(1,0)$:

$$e^{xy}+x^2y+\sqrt{x}=2.$$

Solution: Using implicit differentiation, we compute that

$$(xe^{xy}+x^2)y'+ye^{xy}+2xy+(1/2)x^{-1/2}=0.$$

Setting $(x,y)=(1,0)$, we obtain $2y'+1/2=0$, that is,

$$y'=-\frac{1}{4}.$$

Hence, the equation of the tangent line through $(1,0)$ is

$$y=-\frac{1}{4}(x-1).$$

Problem 4. (7 + 8 = 15 points) Compute the following limits. You may *not* use L'Hôpital's rule, if you know what that is.

Hint: Relate the limits below to the derivative of a function.

a) $\lim\limits_{\Delta x \to 0} \dfrac{(8+\Delta x)^{1/3} - 2}{\Delta x}$

b) $\lim\limits_{h \to 0} \dfrac{2^{h^2} - 1}{h^2}$

Solution:

a) This limit is the derivative $f(x) = x^{1/3}$ at $x = 8$. Since $f'(x) = (1/3)x^{-2/3}$ and $8^{-2/3} = 1/4$, the limit is $1/12$.

b) This limit is the derivative $f(x) = 2^x$ at $x = 0$. Since $f'(x) = 2^x \ln 2$, the limit is $\ln 2$.

Problem 5. (15 points) For $x \geq 0$, the function $y = f(x) = e^{2x+x^3-3}$ has an inverse function $g(y)$. Compute $g'(1)$, or equivalently, $\frac{d}{dy}g(y)$ at the point $y = 1$.

Hint: Note that $f(1) = 1$.

Solution: The definition of an inverse function yields that $g(1) = 1$. Differentiating the relation $g(f(x)) = x$ with respect to x and using the chain rule, we deduce $g'(f(x)) = \dfrac{1}{f'(x)}$. Hence, $g'(1) = \dfrac{1}{f'(1)}$. Since $f'(x) = (2 + 3x^2)e^{2x+x^3-3}$, we have $f'(1) = 5$. In total, we have

$$g'(1) = \frac{1}{5}.$$

Problem 6. (15 points) Let $f(x)$ be a function with the following graph:

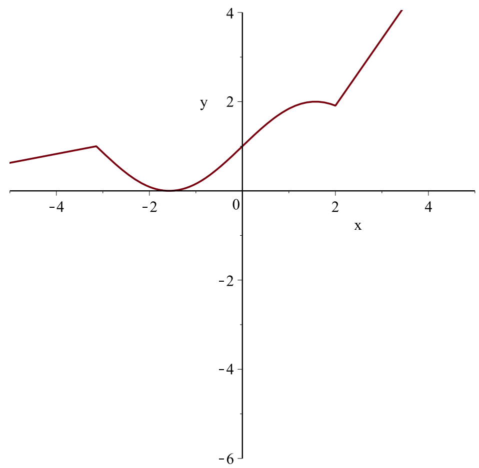

FIGURE 1. Graph of $f(x)$

Sketch the graph of $f'(x)$ on the blank graph below. Your picture should be qualitatively accurate, but it doesn't have to be quantitatively perfect.

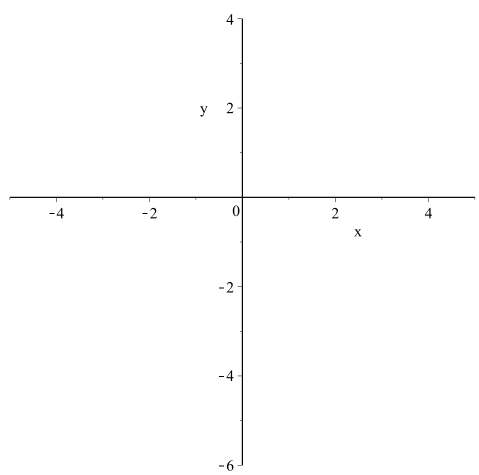

FIGURE 2. Draw your graph of $f'(x)$ here

Solution:

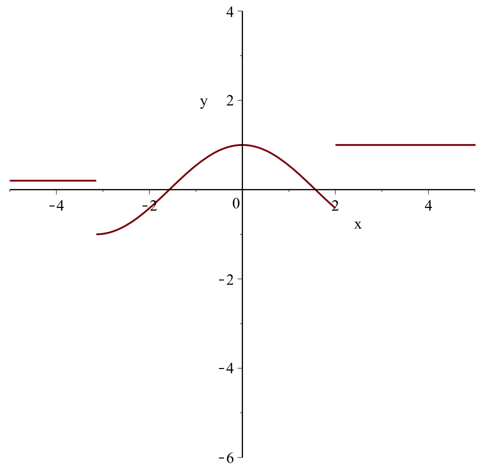

FIGURE 3. Graph of $f'(x)$