MIDTERM 2 - 18.01 - FALL 2014.

Name:

Email:

Please put a check by your recitation section.

	Instructor	**Time**
☐	B.Yang	MW 10
☐	M. Hoyois	MW 11
☐	M. Hoyois	MW 12
☐	X. Sun	MW 1
☐	R. Chang	MW 2

Problem #	Max points possible	Actual score
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**

Good luck!

Problem 1. (15 points)
Find the linear approximation of the function $f(x) = \ln(x^x)$ near $x = 1$.

Solution: We compute that

$$\begin{aligned} & f(x) = x \ln x, \qquad f'(x) = \ln x + 1 \\ \implies & f(1) = 0, \qquad f'(1) = 1. \end{aligned}$$

Hence,

$$\begin{aligned} f(x) = x \ln x &= f(1) + f'(1)(x-1) + O((x-1)^2) \\ &= x - 1 + O((x-1)^2). \end{aligned}$$

Problem 2. (5 + 5 + 5 = 15 points)

a) State the mean value theorem. Furthermore, draw a picture of a function $f(x)$ defined on the interval $0 \leq x \leq 1$ that illustrates the mean value theorem. Your picture must include the secant line through $(0, f(0))$ and $(1, f(1))$, and you must explain what the mean value theorem says about this secant line.

b) Show that

$$\sin x \leq x$$

for all real numbers $x \geq 0$.

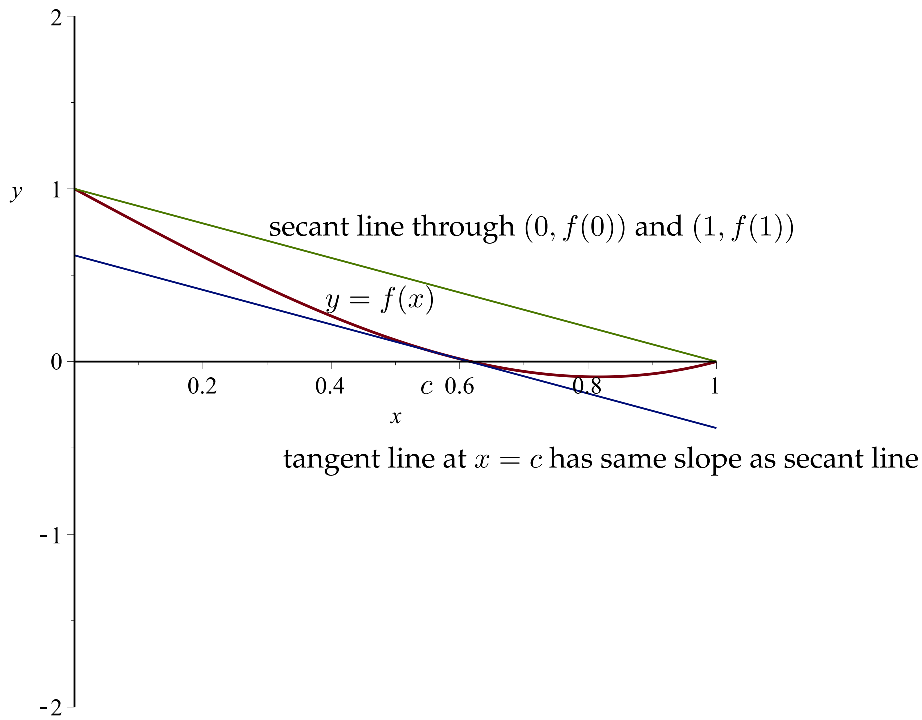
c) Show that

$$\cos x \geq 1 - \frac{1}{2}x^2$$

for all real numbers x.

Solution: **a)** The mean value theorem states that if f is differentiable for $a < x < b$ and if f is continuous for $a \leq x \leq b$, then there exists a point c with $a < c < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



b) We set $f_1(x) = x - \sin x$. Then $f_1(0) = 0$ and $f_1'(x) = 1 - \cos x \geq 0$. Hence, f_1 is a non-decreasing function for all $x \geq 0$. Thus, $f_1(x) \geq f_1(0) = 0$ for all $x \geq 0$, that is, $x \geq \sin x$ as desired.

c) Since both sides of the inequality are even, we only need to prove the inequality for $x \geq 0$. To this end, we set $f_2(x) = \cos x - (1 - x^2/2)$. We next note that $f_2(0) = 0$, and by part **b)**, $f_2'(x) = x - \sin x = f_1(x) \geq 0$ whenever $x \geq 0$. Thus, f_2 is a non-decreasing function for $x \geq 0$ and hence $f_2(x) \geq f_2(0) = 0$ whenever $x \geq 0$. Thus, $\cos x \geq 1 - x^2/2$ as desired.

Problem 3. (10 + 10 = 20 points) Compute the following two antiderivatives:

$$\text{a)} \int \frac{\ln x}{x}\, dx$$

$$\text{b)} \int (\cos x)^{1801} \sin x\, dx$$

Solution: a) We first set $u = \ln x, du = \frac{1}{x}\, dx$. We then compute that

$$\begin{aligned} \int \frac{\ln x}{x}\, dx &= \int u\, du \\ &= \frac{1}{2}u^2 + c \\ &= \frac{1}{2}(\ln(x))^2 + c. \end{aligned}$$

b) We set $u = \cos x$, $du = -\sin x\, dx$, and compute that

$$\begin{aligned} &\int (\cos x)^{1801} \sin x\, dx \\ &= -\int u^{1801}\, du \\ &= -\frac{1}{1802}u^{1802} + c \\ &= -\frac{1}{1802}(\cos x)^{1802} + c. \end{aligned}$$

Problem 4. (15 points) A rectangle with sides *parallel* to the x and y axes lies inside the curve $x^4 + y^4 = 1$ in the (x, y) plane. Note that the curve looks like a distorted circle. The bottom edge of the rectangle lies on the x axis and its upper two vertices lie on the curve. Find the dimensions of the rectangle that maximize its area. To receive full credit, explain your reasoning and provide a justification that you have found the dimensions that lead to the maximal area.

Solution:

All candidate rectangles have vertices of the form $(-x, 0)$, $(x, 0)$, $(-x, (1-x^4)^{1/4})$, $(x, (1-x^4)^{1/4})$, where $0 \leq x \leq 1$. The area $A(x)$ of the candidate rectangle is

$$A(x) = \text{base} \times \text{height} = 2x(1 - x^4)^{1/4}.$$

Since $A(0) = A(1) = 0$, the maximum does not occur at either of the endpoints. To find the critical points, we compute that

$$A'(x) = 2(1 - x^4)^{1/4} - 2x^4(1 - x^4)^{-3/4} = 2\frac{1 - 2x^4}{\sqrt[3]{1 - x^4}}.$$

Setting $A'(x) = 0$ to find the critical points x_{crit}, we find that

$$x_{crit} = \sqrt[4]{\frac{1}{2}}.$$

The corresponding height y_{crit} on the curve is obtained by solving for y when $x = x_{crit}$:

$$y_{crit} = \sqrt[4]{1 - x_{crit}^4} = \sqrt[4]{\frac{1}{2}}.$$

Thus, the rectangle of maximal area has base width $2x_{crit} = 2\sqrt[4]{1/2}$ and height $y_{crit} = \sqrt[4]{1/2}$.

Problem 5. (15 points) A solid cylinder of radius $r = 3$ meters and height $h = 4$ meters starts changing in size at time $t = 0$. Its radius begins shrinking at the rate of 1 meter per minute and its height begins growing at the rate of 2 meters per minute. At time $t = 0$, find the rate of change of the volume of the cylinder and decide whether the volume is increasing or decreasing.

Solution:
The volume of the cylinder is

$$V = \pi r^2 h.$$

Differentiating this relationship with respect to t and applying the chain and product rules, we find that

$$\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}.$$

Setting $r = 3$, $h = 4$, $\frac{dr}{dt} = -1$, and $\frac{dh}{dt} = 2$, we compute that

$$\frac{dV}{dt} = 24\pi \frac{dr}{dt} + 9\pi \frac{dh}{dt} = -24\pi + 18\pi = -6\pi \text{ meters}^3 \text{ per minute.}$$

Thus, the cylinder's volume is decreasing.

Problem 6. (20 points) Sketch the graph of the function

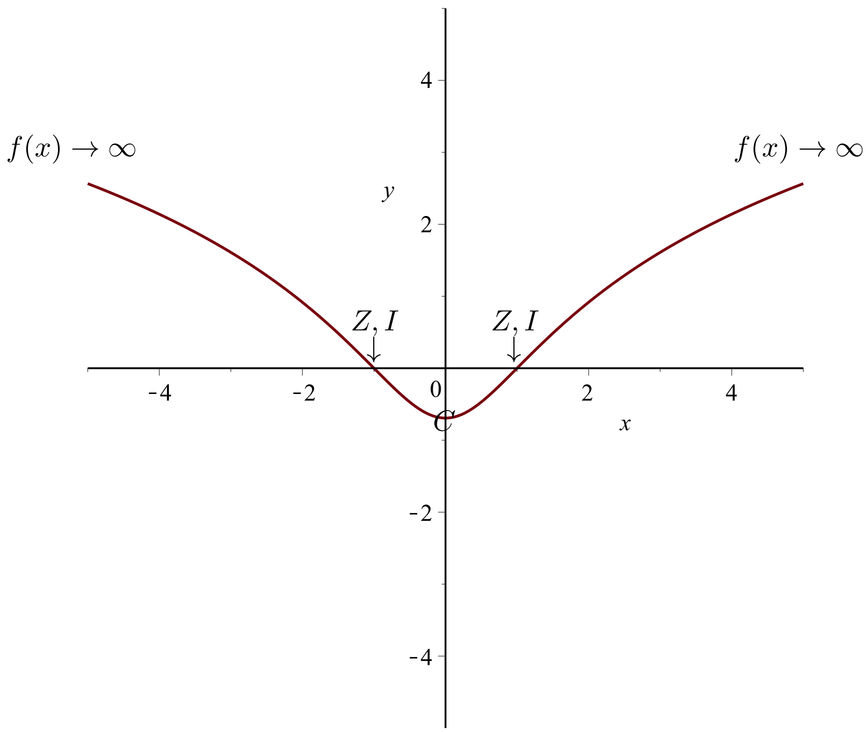
$$f(x) = \ln(x^2 + 1) - \ln 2.$$

Label the zeros of $f(x)$ by "Z," the critical points by "C," and the inflection points by "I." To receive full credit, you must clearly indicate:

- **i)** Any discontinuities or points of non-differentiability $f(x)$ may have.
- **ii)** The limiting behavior of $f(x)$ as $x \to \pm\infty$.
- **iii)** The regions on which $f(x)$ is positive and the regions on which $f(x)$ is negative.
- **iv)** The regions on which $f(x)$ is increasing and the regions on which $f(x)$ is decreasing.
- **v)** The regions on which $f(x)$ is concave up and the regions on which $f(x)$ is concave down.

Please be sure that you graph the correct function. If you accidentally graph a function other than the function $f(x)$ written above, then we can only award a small amount of credit at most.

Solution: The graph of $f(x)$ is given in the figure below.



To justify the graph shown, we first note f has no discontinuities and that $\lim_{x\to\pm\infty} f(x) = \infty$.

We next note that setting $f(x) = 0$ leads to the equation $x^2 + 1 = 2$, which has the two solutions $x = \pm 1$.

We now compute the first and second derivatives of $f(x)$:

$$f'(x) = \frac{2x}{x^2+1},$$
$$f''(x) = 2\frac{1-x^2}{(x^2+1)^2}.$$

The above formulas imply that $x = 0$ is the only critical point (i.e., point where $f' = 0$) and $x = \pm 1$ are the two inflection points (i.e., points where $f'' = 0$). To indicate some of the other features of $f(x)$, we make the following table:

x	$f(x)$	$f'(x)$	$f''(x)$
$x < -1$	$+$	$-$ (decreasing)	$-$ (concave down)
$-1 < x < 0$	$-$	$-$ (decreasing)	$+$ (concave up)
$0 < x < 1$	$-$	$+$ (increasing)	$+$ (concave up)
$1 < x$	$+$	$+$ (increasing)	$-$ (concave down)