MATH 18.01, FALL 2017 - PROBLEM SET # 2

Professor: Jared Speck

Due: by 4:00pm on Friday 9-22-17
(in the boxes outside of Room 4-174 during the day; write your name and recitation instructor on your homework)

18.01 Supplementary Notes (including Exercises and Solutions) are available on the course web page: `http://math.mit.edu/~jspeck/18.01_Fall%202017/1801_CourseWebsite.html`. This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

Part I consists of exercises given and solved in the Supplementary Notes. It will be graded quickly, checking that all is there and the solutions not copied.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises given here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

You are encouraged to use graphing calculators, software, etc. to check your answers and to explore calculus. However, (unless otherwise indicated) we strongly discourage you from using these tools to solve problems, perform computations, graph functions, etc. An extremely important aspect of learning calculus is developing these skills. You will not be allowed to use any such tools on the exams.

Part I (30 points)

Notation: The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Lecture 4. (Thurs., Sept. 14) Chain rule; higher derivatives.
Read: 3.3, 3.6.
Homework: Notes 1F: 1ab, 2, 6, 7bd.
Homework: Notes 1J: 1abf.
Homework: Notes 1G: 1b, 5ab.

Lecture 5. (Fri., Sept. 15) Implicit differentiation; inverse functions.
Read: 3.5; Notes G: Section 5; 9.5 (bottom pp. 313-315).
Homework: Notes 1F: 3, 5, 8c.
Homework: Notes 1A: 5b.
Homework: Notes 5A: 1abc (just sin, cos, sec), 3fh.

Lecture 6. (Tues., Sept. 19) Exponentials and logs: definitions, algebra, applications; logarithmic differentiation; hyperbolic functions.

Read: Notes: Section X (8.2 has some of this); 8.3 to middle p. 267; 8.4 to top p. 271.

Read: 9.7 to p. 326.

Homework: Notes 1H: 1, 2, 3a, 5b.

Homework: Notes 1I: 1cdefm, 4a.

Homework: Notes 5A: 5abc (in 5a, you don't have to find the critical points or inflection points; just provide a rough sketch).

Part II (50 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, then ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises, you are expected to write the answer in your own words. *You must show your work; "bare" solutions will receive very little credit.*

iii) Write on your problem set whom you consulted and the sources you used. *This includes graphing calculators, software, etc.* (again, we strongly discourage you from relying on these tools to solve problems). If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people whom you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem **0**" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (now; slope, basic curve sketching; 1 + 2 = 3 points)

a) Find an even function $E(x)$ and an odd function $O(x)$ such that the function $f(x) = x^4/(x+1)$ can be decomposed as $f(x) = E(x) + O(x)$.

Solution: We can multiply $f(x)$ by $(x-1)/(x-1)$ to get

$$f(x) = \frac{x^4}{x+1} = \frac{x^4}{x+1} \cdot \frac{x-1}{x-1} = \frac{x^5 - x^4}{x^2-1} = \underbrace{\frac{x^5}{x^2-1}}_{I} + \underbrace{\frac{-x^4}{x^2-1}}_{II}.$$

Notice that x^5 is odd, x^4 is even, and $x^2 - 1$ is even. Since I is of the form *odd/even*, it is an odd function. Since II is of the form *even/even*, it is an even function. So

$$\boxed{O(x) = \frac{x^5}{x^2-1}} \quad \text{and} \quad \boxed{E(x) = -\frac{x^4}{x^2-1}}.$$

b) Graph $E(x)$. In the same picture, also provide a rough sketch of the graph of $E'(x)$ without performing any computations (except to check your answer if you want). Then do the same thing for $O(x)$ and $O'(x)$.

Solution:

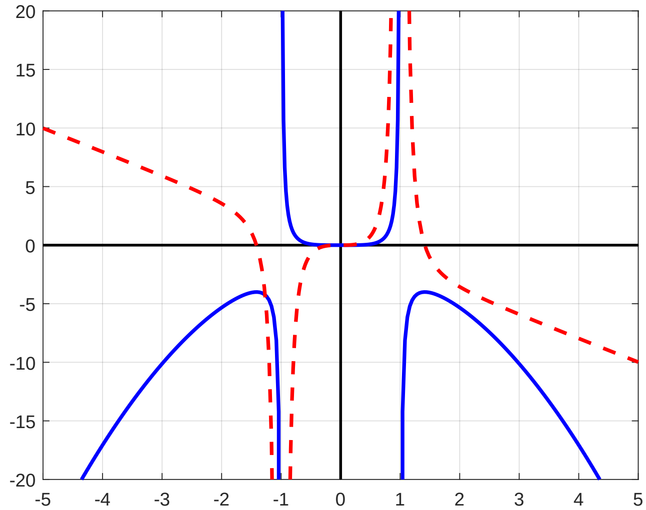

FIGURE 1. Graph of $E(x)$ in solid blue, $E'(x)$ in dashed red

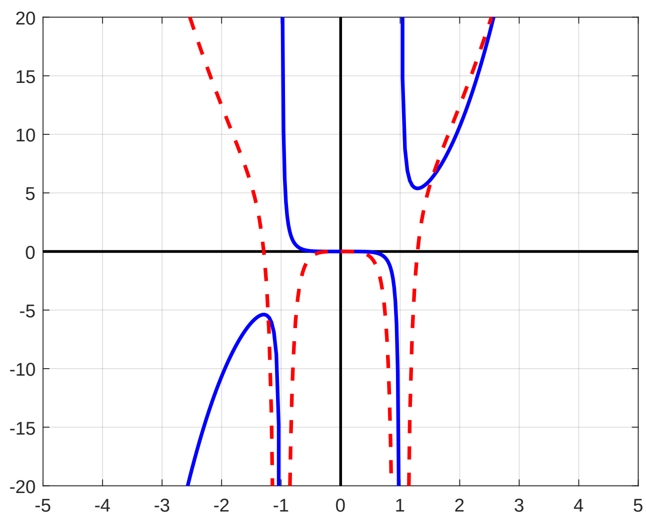

FIGURE 2. Graph of $O(x)$ in solid blue, $O'(x)$ in dashed red

2. (Sept. 14; chain rule, higher derivatives; 4 points) A function $f(x)$ is said to be *concave up* at x_0 if $f''(x_0) > 0$ (we will study this notion in detail later in the course). Suppose that $f(x)$ is a differentiable function with $f(x_0) > 0$ for all x_0. Suppose in addition that $f(x)$ is concave up at all x_0. Show that the function $g(x) = (f(x))^2$ is concave up at every x_0.

Solution: Using the chain rule, take the derivative of g twice to get

$$g'(x) = 2f'(x)f(x) \implies g''(x) = 2(f'(x))^2 + 2f''(x)f(x).$$

Since f is differentiable, $f'(x_0)$ exists and $(f'(x_0))^2 \geq 0$ for all x_0. We also know that f is concave up and positive for all x_0. Therefore, $f''(x_0) > 0$ and $f(x_0) > 0$ for all x_0. These facts imply that $g''(x_0) > 0$ for all x_0 and so g is concave up for every x_0.

3. (Sept. 15; chain rule and inverse trig differentiation; 3 + 3 = 6 points) Compute the following derivatives:

a) $(d/dx)[\arctan(x^2 + x + 1)]^{-3}$.

Solution: Apply chain rule once by letting

$$u = x^2 + x + 1 \quad \text{and} \quad f(u) = [\arctan(u)]^{-3}.$$

Then

$$\frac{du}{dx} = 2x + 1.$$

To compute df/du, we apply chain rule again. Let

$$w(u) = \arctan(u) \quad \text{and} \quad g(w) = w^{-3}.$$

Then

$$\frac{dw}{du} = \frac{1}{1+u^2} \quad \text{and} \quad \frac{dg}{dw} = -3w^{-4}.$$

So

$$\frac{df}{du} = \frac{dw}{du}\frac{dg}{dw} = \frac{1}{1+u^2} \cdot (-3w^{-4}) = \frac{-3[\arctan(u)]^{-4}}{1+u^2} = \frac{-3}{(1+u^2)[\arctan(u)]^4}.$$

Putting this all together gives us

$$(d/dx)[\arctan(x^2 + x + 1)]^{-3} = \frac{du}{dx}\frac{df}{du} = \boxed{\frac{-3(2x+1)}{(1+(x^2+x+1)^2)[\arctan(x^2+x+1)]^4}}$$

b) $(d/dy)\left(\sin y \cos y\right)$. Do this is in two different ways: i) First do it directly. ii) Then use a trig identity to write $\sin y \cos y = f(2y)$ for some trig function f (and then compute $(d/dy)f(2y)$). Show that your two answers agree.

Solution: (i) First, we use the product rule to compute the derivative directly to get

$$\frac{d}{dy}(\sin y \cos y) = \cos y \cos y + \sin y(-\sin y) = \boxed{\cos^2 y - \sin^2 y}.$$

(ii) Using the trig identity that $\sin y \cos y = \dfrac{1}{2}\sin(2y)$, we can use the chain rule to get

$$\frac{d}{dy}(\sin y \cos y) = \frac{d}{dy}\left(\frac{1}{2}\sin(2y)\right) = \boxed{\cos(2y)}.$$

Our two answers are equal by the trig identity

$$\cos^2 y - \sin^2 y = \cos(2y).$$

We can prove this as follows

$$\cos(2y) = \cos(y + y) = \cos(y)\cos(y) - \sin(y)\sin(y) = \cos^2(y) - \sin^2(y).$$

4. (Sept.15; implicit differentiation; slope; 5 + 7 = 12 points) Consider the following curve in the (x, y) plane: $(1/3)y^3 + xy^2 + x^2y + x^3 + x = 5/3$.

a) Find the number of points $P = (x_0, y_0)$ on the curve such that the tangent line to the curve at P is horizontal.

Solution: Implicit differentiation gives

$$\begin{aligned} & y^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} + 3x^2 + 1 = 0 \\ \implies & \frac{dy}{dx} = -\frac{3x^2 + 2xy + y^2 + 1}{y^2 + 2xy + x^2}. \end{aligned}$$

The point at which the curve has a horizontal tangent line would satisfy

$$\frac{dy}{dx} = 0$$

and so we would need

$$y^2 + 2xy + 3x^2 + 1 = 0.$$

Using the quadratic formula for y gives

$$y = \frac{-2x \pm \sqrt{4x^2 - 4(3x^2 + 1)}}{2} = -x \pm \sqrt{-2x^2 - 1}.$$

In order for this to have a real-valued solution, we require

$$-2x^2 - 1 \geq 0 \implies -2x^2 \geq 1 \implies x^2 \leq -\frac{1}{2},$$

which can never happen. $\boxed{\text{So no such points } P = (x_0, y_0) \text{ exist}}$.

b) Find the one point Q on the curve such that the tangent line to the curve at Q is vertical. *Remark: You only have to find the one point; you don't have to prove that there is only one Q. In order to prove that there is only one, you need some tools that we haven't yet developed.*

Solution: A vertical tangent line would require the denominator of dy/dx to be zero. This would happen for x and y satisfying

$$0 = y^2 + 2xy + x^2 = (x + y)^2.$$

This is satisfied when $y = -x$. Substituting this into the equation for the curve gives

$$\frac{1}{3}(-x)^3 + x(-x)^2 + x^2(-x) + x^3 + x = \frac{5}{3} \implies \frac{2}{3}x^3 + x - \frac{5}{3} = 0.$$

This occurs when $x = 1$ and so the curve has a vertical tangent at the point $\boxed{Q = (1, -1)}$.

5. (Sept. 15; inverse trig functions and differentiation; $2+2+2=6$ points) The function $\theta = \cos^{-1}(x)$ is the inverse of $\cos\theta$ for $0 \le \theta \le \pi$. Similarly, the function $\phi = \sin^{-1}(x)$ is the inverse of $\sin\phi$ for $-\pi/2 \le \phi \le \pi/2$.

a) Use implicit differentiation to find a formula for $(d/dx)\cos^{-1}(x)$. Pay particular attention to the sign of the square root.

Solution: We know that if $\theta = \cos^{-1}(x)$, then $x = \cos(\theta)$. Now we can use implicit differentiation to see that

$$1 = -\sin(\theta)\frac{d\theta}{dx} \implies \frac{d\theta}{dx} = -\frac{1}{\sin\theta}.$$

But we want our answer to be in terms of x, so we can use the trig identity

$$\sin^2\theta + \cos^2\theta = 1.$$

We are considering only values of θ for whichr $0 \le \theta \le \pi$. In this range of θ, we have that $\sin\theta \ge 0$. So we get

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

and hence

$$\frac{d}{dx}\theta = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1-\cos^2\theta}}.$$

Since we know that $\theta = \cos^{-1}(x)$ and $x = \cos(\theta)$, we can substitute these into our expression to get

$$\boxed{\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}}$$

b) Simplify the expression $\sin^{-1}(x) + \cos^{-1}(x)$ as much as possible.

Solution: If we have $\theta = \cos^{-1}(x)$, then $x = \cos(y)$. Recall that $\cos(\theta) = \sin(\pi/2 - \theta)$. So

$$x = \cos(\theta) = \sin(\pi/2 - \theta)$$

The range $0 \le \theta \le \pi$ means that $(\pi/2 - \theta)$ is in the range $-\pi/2 \le (\pi/2 - \theta) \le \pi/2$, which is the range we need for

$$\sin^{-1}(x) = \frac{\pi}{2} - \theta.$$

to make sense. So we have

$$\sin^{-1}(x) + \cos^{-1}(x) = \left(\frac{\pi}{2} - \theta\right) + \theta = \boxed{\frac{\pi}{2}}.$$

c) Explain how you can use parts a) and b) to quickly derive a formula for $(d/dx)\sin^{-1}(x)$ by doing *very little* additional work (you should be able to do the computation in your head).

Solution: From part (b), we have

$$\sin^{-1}(x) = \frac{\pi}{2} - \cos^{-1}(x).$$

Then we can use part (a) to see that

$$\frac{d}{dx}\sin^{-1}(x) = -\frac{d}{dx}\cos^{-1}(x) = \boxed{\frac{1}{\sqrt{1-x^2}}}.$$

6. (Sept. 19; logarithmic differentiation; 2 + 2 + 2 = 6 points)

a) Given two functions u and v, set $y = uv$. Note the identity $\log y = \log u + \log v$. Use this identity and logarithmic differentiation to give an alternative derivation (i.e., different than the derivation we discussed in class) of the product rule $(uv)' = u'v + uv'$.

Solution: For $y = uv$, we have $\log y = \log u + \log v$. Using logarithmic differentiation on both sides gives

$$\frac{y'}{y} = \frac{u'}{u} + \frac{v'}{v} = \frac{u'v + v'u}{uv} = \frac{u'v + v'u}{y}$$

and so

$$y' = \frac{u'v + v'u}{y} \cdot y = u'v + v'u.$$

b) Modify the approach from part a) in order to give an alternative derivation of the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

Solution: For $y = \frac{u}{v}$, we have $\log y = \log u - \log v$. Using logarithmic differentiation on both sides gives

$$\frac{y'}{y} = \frac{u'}{u} + \frac{v'}{v} = \frac{u'v + v'u}{uv}$$

and so

$$y' = \frac{u'v + v'u}{uv} \cdot y = \frac{u'v + v'u}{uv} \cdot \frac{u}{v} = \frac{u'v + v'u}{v^2}$$

c) Suppose now that you are given n functions $u_1(x), u_2(x), \cdots, u_n(x)$. Modify the approach from part a) in order to derive a formula for the derivative of the product $u_1 u_2 \cdots u_n$, i.e., a formula for $(d/dx)(u_1 u_2 \cdots u_n)$.

Solution: For $y = u_1 u_2 \cdots u_n$, we have $\log y = \log u_1 + \log u_2 + \cdots + \log u_n$. Differentiating both sides gives

$$\begin{aligned}\frac{y'}{y} &= \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n} \\ &= \frac{u_1'(u_2 \cdots u_n) + u_2'(u_1 u_3 \cdots u_n) + \cdots + u_n'(u_1 \cdots u_{n-1}')}{u_1 u_2 \cdots u_n} \\ &= \frac{u_1'(u_2 \cdots u_n) + u_2'(u_1 u_3 \cdots u_n) + \cdots + u_n'(u_1 \cdots u_{n-1}')}{y}\end{aligned}$$

and so

$$y' = \frac{u_1'(u_2 \cdots u_n) + u_2'(u_1 u_3 \cdots u_n) + \cdots + u_n'(u_1 \cdots u_{n-1}')}{y} \cdot y = u_1'(u_2 \cdots u_n) + u_2'(u_1 u_3 \cdots u_n) + \cdots + u_n'(u_1 \cdots u_{n-1}')$$

7. (Sept. 19; 2 + 2 + 2 + 2 + 2 = 10 points) Section 8.2: 8ac, 10, 11; Section 8.4: 18, 20 (only compute dy/dx).

Section 8.2: 8(a) Solution: Let E_1 and M_1 be the energy and magnitude of the first (and larger) earthquake, respectively. Let E_2 and M_2 be the energy and magnitude of the second (and smaller) earthquake, respectively. So

$$M_1 = \frac{2}{3}\log_{10}\frac{E_1}{E_0} \quad \text{and} \quad M_2 = \frac{2}{3}\log_{10}\frac{E_2}{E_0}.$$

Assuming $M_1 = M_2 + 1$, we have

$$\frac{2}{3}\log_{10}\frac{E_1}{E_0} = \frac{2}{3}\log_{10}\frac{E_2}{E_0} + 1 = \frac{2}{3}\log_{10}\frac{E_2}{E_0} + \log_{10} 10.$$

Log rules tell us that

$$\begin{aligned}
&\frac{2}{3}\log_{10} E_1 - \frac{2}{3}\log_{10} E_0 = \frac{2}{3}\log_{10} E_2 - \frac{2}{3}\log_{10} E_0 + \log_{10} 10 \\
\Longrightarrow\ &\frac{2}{3}\log_{10} E_1 = \frac{2}{3}\log_{10} E_2 + \log_{10} 10 \\
\Longrightarrow\ &\log_{10} E_1^{2/3} = \log_{10}(E_2^{2/3}\cdot 10) \\
\Longrightarrow\ &E_1^{2/3} = E_2^{2/3}\cdot 10 \\
\Longrightarrow\ &\left(\frac{E_1}{E_2}\right)^{2/3} = 10 \\
\Longrightarrow\ &\boxed{\frac{E_1}{E_2} = 10^{3/2} \approx 31.62 \text{ kW}\cdot\text{h}}
\end{aligned}$$

Section 8.2: 8(c) Solution: For an earthquake of magnitude 6, the energy released is given by

$$6 = \frac{2}{3}\log_{10}\frac{E}{E_0} \implies 9 = \log_{10}\frac{E}{E_0} \implies 10^9 = \frac{E}{E_0} \implies \boxed{E = 10^9 E_0 = 7\times 10^6 \text{ kW}\cdot\text{h}}$$

This would power the city for

$$\frac{7\times 10^6}{3\times 10^5} = \frac{70}{3} = \boxed{23.\overline{33}\text{ days}}.$$

Section 8.2: 10 Solution: Assume to the contrary that $\log_3 2$ is rational. In particular, $\log_3 2 = p/q$ where p and q are positive integers with greatest common divisor 1, i.e., p/q cannot be simplified further. The

$$\log_3 2 = \frac{p}{q} \implies 2 = 3^{p/q} \implies 2^q = 3^p$$

but this cannot happen for integar values of p and q. Namely, 2^q is even and 3^p is odd for integers p and q. Therefore $\log_3 2$ is irrational.

Section 8.2: 11 Solution: The flaw in the proof occurs at the first step, since $\log\frac{1}{2} < 0$ flips the inequality. The correct argument gives

$$\begin{aligned} 1 &< 2 \\ 1\cdot\log\frac{1}{2} &> 2\cdot\log\frac{1}{2} \\ \log\frac{1}{2} &> \log\left(\frac{1}{2}\right)^2 \\ \log\frac{1}{2} &> \log\frac{1}{4} \\ \frac{1}{2} &> \frac{1}{4}. \end{aligned}$$

Section 8.4: 18 Solution: We have

$$y = \sqrt[3]{(x+1)(x-2)(2x+7)}$$

and so

$$\log y = \frac{1}{3}\left[\ln(x+1) + \ln(x-2) + \ln(2x+7)\right].$$

Differentiating both sides gives

$$\begin{aligned} \frac{y'}{y} &= \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right) \\ \implies y' &= \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right) y \\ \implies & \boxed{y' = \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right)\sqrt[3]{(x+1)(x-2)(2x+7)}} \end{aligned}$$

Note: Other forms of this solution could be

$$y' = \frac{2x(3x+5)-11}{3((x-2)(x+1)(2x+7))^{2/3}}$$

or

$$\frac{6x^2+10x-11}{3(2x^3+5x^2-11x-14)^{2/3}}$$

etc.

Section 8.4: 20 Solution: We can compute dy/dx by using $\ln y = x\ln x$ to get

$$\frac{dy/dx}{y} = \ln x + x\cdot\frac{1}{x} = \ln x + 1 \implies \frac{dy}{dx} = y(\ln x + 1) \implies \boxed{\frac{dy}{dx} = x^x(\ln x + 1)}.$$

Or we can compute dy/dx by using $y = e^{x\ln x}$ to get

$$\frac{dy}{dx} = e^{x\ln x}\cdot\frac{d}{dx}(x\ln x) = e^{x\ln x}\left(\ln x + x\cdot\frac{1}{x}\right)$$

which gives

$$\boxed{\frac{dy}{dx} = x^x(\ln x + 1)}.$$