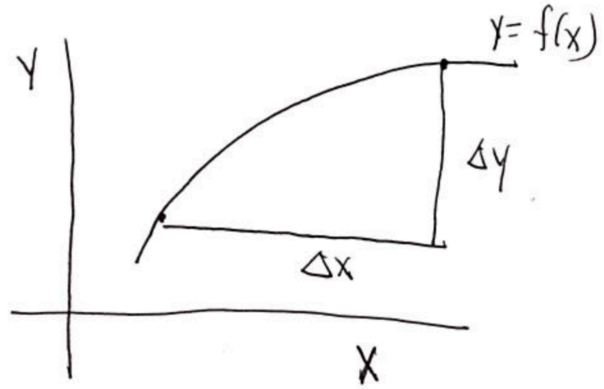
- Derivatives Cont.

Derivatives can be interpreted as a "rate of change"




$$\frac{\Delta y}{\Delta x} \rightarrow \frac{dy}{dx} \text{ as } \Delta x \to 0$$

"Average rate of change" "instantaneous rate of change"

## Examples

- $P$ = Population $\quad \frac{dP}{dt}$ = Population growth rate
- $\theta$ = temperature $\quad \frac{d\theta}{dx}$ = temperature gradient
- $S$ = distance $\quad \frac{ds}{dt}$ = speed
- Error analysis (HW)

Basic idea: $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$ when $\Delta x$ is small

$$\Delta y \approx \frac{dy}{dx} \Delta x$$

If you make an error of $\Delta x$ in measuring the input $x$, this will result in an output error $\Delta y \approx \frac{dy}{dx} \Delta x$

- Rate of Change example (D. Jerison).

A pumpkin is dropped from the top roof of a 400 ft. building. Newtonian Mechanics (we will just accept this for now) implies that the height above the ground $y$ of the pumpkin is

$$y = 400 - 16t^2. \quad (t \text{ in seconds, } y \text{ in feet})$$

- The pumpkin hits the ground when $y=0$:

$$400 - 16t^2 = 0.$$

Solve for $t$ => $\boxed{t = 5}$

- The average speed of the pumpkin during its journey:

$$\frac{\Delta y}{\Delta t} = \frac{400}{5}\frac{\text{ft.}}{\text{sec.}} = 80\,\frac{\text{ft.}}{\text{sec.}}$$

- The instantaneous velocity (velocity, unlike speed, can be negative) is:

$$\frac{dy}{dt} = -32t.$$

The instantaneous velocity when the pumpkin hits the ground: Set $t=5$:

$$\boxed{\frac{dy}{dt} = -32 \cdot 5 = -160\,\frac{\text{ft.}}{\text{sec.}}}$$

# Limits and Continuity

- Easy limits: $\lim_{x \to 3} \frac{x^2-1}{x-1} = \frac{3^2-1}{3-1} = \frac{8}{2} = 4$

With easy limits, you can just plug in the limiting value.

- Derivatives are never easy limits:

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The denominator goes to 0, so you have to be very careful. For differentiable functions, the numerator also goes to 0, and calculating the derivative is about figuring out how the ratio behaves.

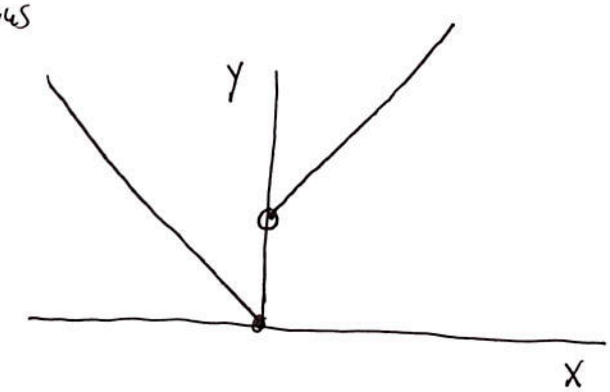
Key Remark: we never consider $\Delta x = 0$ when calculating a limit as $\Delta x \to 0$.

- Continuity

Definition: We say that $f(x)$ is continuous at $x_0$ if
$$\lim_{x \to x_0} f(x) = f(x_0).$$

- The definition can be broken into pieces:
  1. $\lim_{x \to x_0} f(x)$ exists.
  2. $f(x_0)$ exists
  3. The two numbers from ① and ② are the same.

EX: A discontinuous function:




$$f(x) = \begin{cases} x+1 & x > 0 \\ -x & x \leq 0 \end{cases}$$

- $\lim_{x \to 0} f(x)$ does not exist. So $f(x)$ is not continuous at $x=0$. (But it is left-continuous).

- Left and Right limits (definitions)

Right limit: $\lim\limits_{x \to x_0^+} f(x)$ means $\lim\limits_{x \to x_0} f(x)$ for $x > x_0$

Left limit $\lim\limits_{x \to x_0^-} f(x)$ means $\lim\limits_{x \to x_0} f(x)$ for $x < x_0$

- Note: $\lim\limits_{x \to x_0} f(x)$ exists if and only if iff

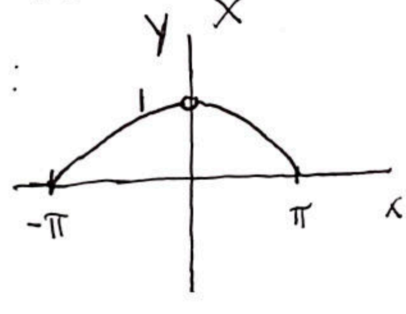
$\lim\limits_{x \to x_0^+} f(x)$ and $\lim\limits_{x \to x_0^-} f(x)$ both exist and are the same number.

- Def of <u>removable discontinuity</u>

We say $f$ has a removable discontinuity at $x_0$ If $\lim\limits_{x \to x_0} f(x)$ exists, but this limit does not equal $f(x_0)$.

<u>Ex</u>: $f(x) = \dfrac{\sin(x)}{x}$ is not even defined at $x=0$.

But here is the graph:

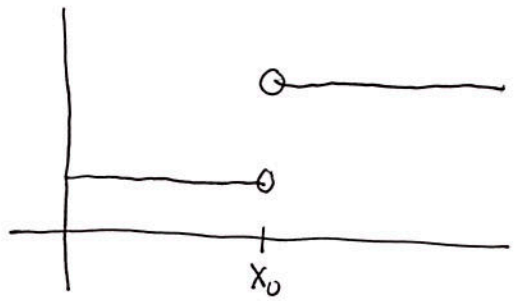



- We will soon see: $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

Thus, $f$ has a removable discontinuity at $x=0$.

- Jump discontinuites (e.g. shock waves)

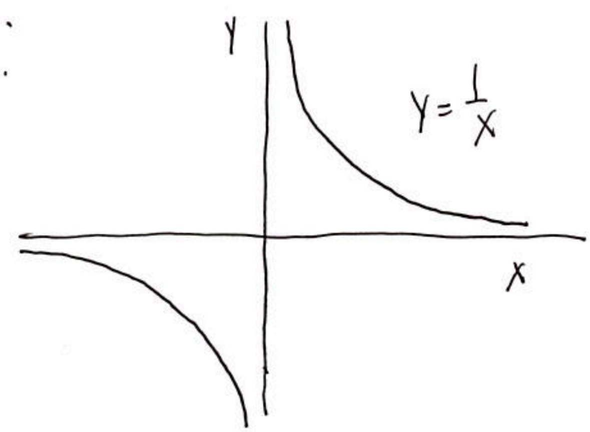
Example:




$\lim_{x \to x_0^+} f(x)$ exists, $\lim_{x \to x_0^-} f(x)$ exists, but the limits are not equal.

- Infinite discontinuities

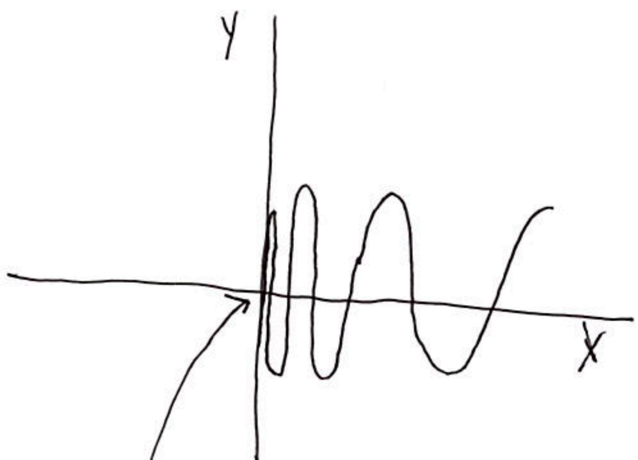
Example:




$\lim_{x \to 0^+} \frac{1}{x} = \infty$ $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

- Awful discontinuities

<u>Example</u>: $f(x) = \sin\left(\frac{1}{x}\right)$

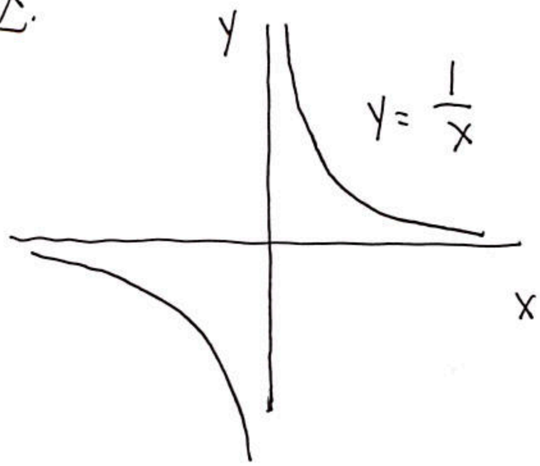


"Oscillates infinitely" as $x \to 0$

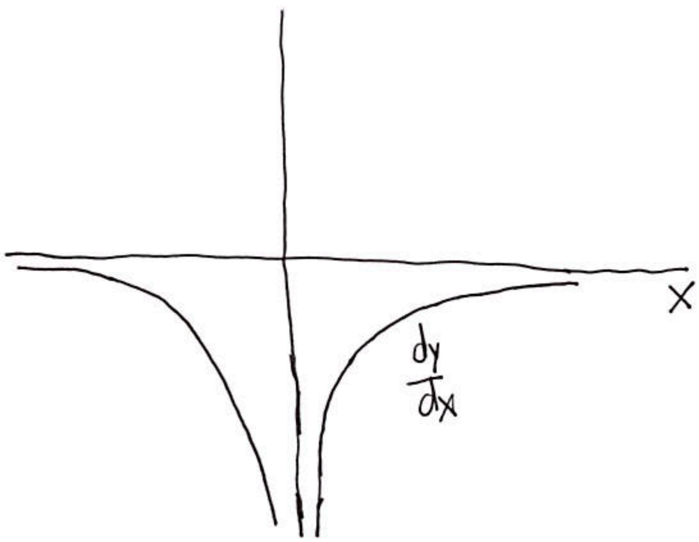
$\lim_{x \to 0} f(x)$ does not exist.

- Picturing the derivative

Ex:




A graphical representation of $\frac{dy}{dx}$




Note: We did not need to compute $\frac{dy}{dx}$ in order to provide a rough sketch of it.

- Big Thm: Differentiable $\Rightarrow$ Continuous.

  More precisely, if $f(x)$ is differentiable at $x_0$, then $f(x)$ is continuous at $x_0$.

Proof: $\lim\limits_{x\to x_0}\left[f(x)-f(x_0)\right]$

$$= \lim_{x\to x_0}\left\{\left[\frac{f(x)-f(x_0)}{x-x_0}\right]\cdot(x-x_0)\right\}$$

Remember, $x-x_0$ is <u>never</u> equal to $0$ when computing $\lim\limits_{x\to x_0}$.

$$= \left\{\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right\}\cdot\lim_{x\to x_0}(x-x_0)$$

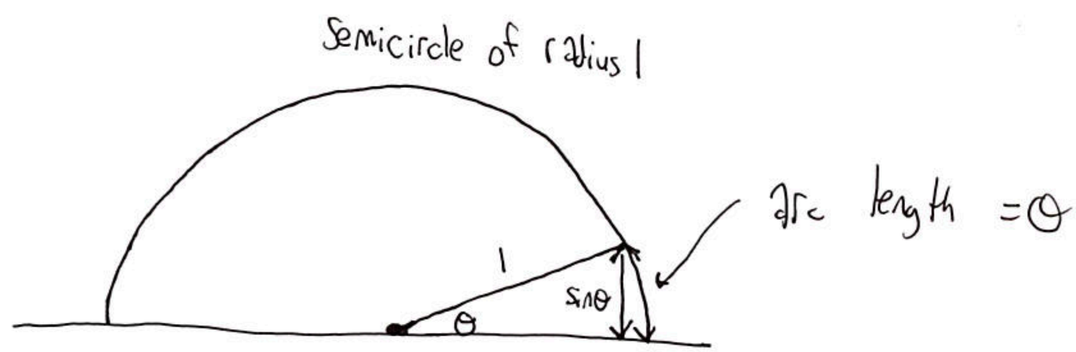
$$= f'(x_0)\cdot 0 \quad = 0.$$

- Two important trig limits

$$\boxed{\bullet \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad ; \quad \bullet \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0}$$

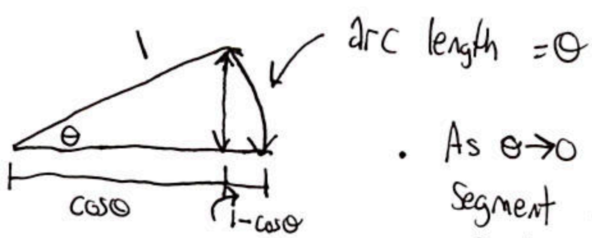
Remark: $\theta$ is in radians

- Geometric "proof" of first limit:




- As $\theta \to 0$, $\sin\theta$ and the arc length become very close to each other: $\frac{\sin\theta}{\theta} \to 1$

- Geometric "proof" of second limit: examine the bottom edge of the same triangle




- As $\theta \to 0$, the length $1-\cos\theta$ of the short segment becomes much smaller than the arc of length $\theta$: $\frac{1-\cos\theta}{\theta} \to 0$ as $\theta \to 0$.