Indeterminate Forms: L'Hôpital's Rule

- Sometimes when we are computing limits, we encounter indeterminate expressions such as

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Ex: $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \frac{0}{0} = ??$

Ex: One way to deal with these difficulties is by using algebra to simplify the expressions.

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}\,x+1} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$$

- Alternate approach:
  - L'Hôpital's rule, easy version: If $f(a)=g(a)=0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \quad \text{as long as } g'(a) \neq 0.$$

Proof: $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-\overbrace{f(a)}^{0}}{x-a}}{\frac{g(x)-\overbrace{g(a)}^{0}}{x-a}} = \frac{f'(a)}{g'(a)}.$$

- In the previous example,

$$a=1, \quad f(x) = x^3-1, \quad g(x) = x^2-1, \quad f(a) = g(a) = 0$$

$$f'(x) = 3x^2, \quad g'(x) = 2x,$$

$$f'(a) = 3, \quad g'(a) = 2, \quad \frac{f'(a)}{g'(a)} = \frac{3}{2}.$$

<u>Ex</u>: Let's apply L'Hôpital's rule to

$$\lim_{x\to1} \frac{x^{15}-1}{x^3-1}:$$

$$\lim_{x\to1} \frac{x^{15}-1}{x^3-1} = \lim_{x\to1} \frac{15x^{14}}{3x^2} = \frac{15}{3} = 5.$$

- Alternate approach - linear approximation

$f(x) = x^{15}-1$, $a=1$, $f(a)=0$, $m = f'(1) = 15$,

- $f(x) \approx m(x-a) + f(a) = 15(x-1)$.
- Similarly, $g(x) = x^3-1 \approx 3(x-1)$.

Hence, $\dfrac{x^{15}-1}{x^3-1} \approx \dfrac{15(x-1)}{3(x-1)} = 5$

Ex: Let's apply L'Hôpital's rule to compute

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{3\cos(3x)}{1} = 3.$$

- This is the same as

$$\frac{d}{dx} \sin(3x)\Big|_{x=0} = 3\cos(3x)\Big|_{x=0} = 3$$

Ex: $$\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x\to \frac{\pi}{4}} \frac{\cos x + \sin x}{1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

Remark: Derivatives $\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$ are always a $\frac{0}{0}$ type limit

Ex $$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{-\sin x}{1} = 0$$

Ex: $$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = \frac{-1}{2}$$

- Alternate approach: quadratic approximation:

$$\cos x \approx 1 - \frac{1}{2}x^2 \quad \text{when } x \approx 0$$

$$\Rightarrow \quad \frac{\cos x - 1}{x^2} \approx \frac{\left(1 - \frac{1}{2}x^2\right) - 1}{x^2} = -\frac{1}{2}$$

Ex $$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} = \text{this limit does not exist!}$$

- Since $\lim_{x\to 0} \frac{\cos x}{2x}$ is not of the form $\frac{0}{0}$,

You cannot and should not apply L'Hôpital's rule!

- L'Hôpital's rule can also be used on limits of the form $\frac{\infty}{\infty}$ or if $x \to \pm\infty$

- Let's figure out which function goes to ∞ faster as $x \to \infty$: x, e^{ax}, or $\ln x$.

<u>Ex</u>: For $a > 0$ $\lim_{x \to \infty} \frac{e^{ax}}{x} \overset{\text{L'Hôpital}}{=} \lim_{x \to \infty} \frac{ae^{ax}}{1} = +\infty$

Hence, when $a > 0$, e^{ax} grows faster than x

<u>Ex</u> $\lim_{x \to \infty} \frac{e^{ax}}{x^{10}} \overset{\text{L'Hôpital}}{=} \lim_{x \to \infty} \frac{ae^{ax}}{10x^9} = \lim_{x \to \infty} \frac{a^2 e^{ax}}{10 \cdot 9 x^8}$

$= \cdots = \lim_{x \to \infty} \frac{a^{10} e^{ax}}{10!} = +\infty$

- Alternate solution: $\frac{e^{ax}}{x^{10}} = \left(\frac{e^{\frac{a}{10}x}}{x} \right)^{10}$

We have already shown that $\frac{e^{\frac{a}{10}x}}{x} \to \infty$ as $x \to \infty$

Thus $\lim_{x \to \infty} \frac{e^{ax}}{x^{10}} = \lim_{x \to \infty} \left(\frac{e^{\frac{ax}{10}}}{x} \right)^{10} = \infty^{10} = \infty.$

Ex $\lim_{x\to\infty} \frac{\ln x}{x^{1/3}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} 3x^{-1/3} = 0$

- Combining the previous examples, we see that when $a > 0$ and $x \to \infty$,

$$\ln x \ll x^{1/3} \ll x \ll x^{10} \ll e^{ax}$$

- L'Hôpital's rule directly applies to $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
- However, we sometimes encounter other indeterminate limits such as $0 \cdot \infty$, 1^∞, and 0^0
  - Using algebra, exponentials, and logs, we can put these other indeterminate limits into standard L'Hôpital form.

Ex. Let's calculate $\lim_{x \to 0} x^x$ (0^0 form)

- First note: $x^x = e^{\ln x^x} = e^{x \ln x}$

- Let's calculate $\lim_{x \to 0} x \ln x$ ($0 \cdot (-\infty)$ form)

- We try putting it into $\frac{0}{0}$ form:

$$\frac{x}{1/\ln x}$$

- However, since we don't know how to find $\lim_{x \to 0} 1/\ln x$, this approach is not helpful.

- So we instead try to put it into $\frac{\infty}{\infty}$ form:

$$\frac{\ln x}{1/x}$$

- By L'Hôpital, we find that

$$\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{1/x} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} -x = 0$$

- Thus, $\lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x} = e^{\lim_{x \to 0} x \ln x} = e^0 = 1.$

since e^u is a continuous function of u