18.01 PROBLEM SET 2

Due date: Tuesday, September 30 **before 1pm.** Write your name, the date, the name of your recitation instructor, and the hour of your recitation on the top of your paper. Late work will be accepted only with a medical note or for another Institute-approved reason. You are encouraged to work with others, but the final write-up should be entirely your own and based on your understanding.

Problem 1(15 points) You are probably aware that when you throw a ball, you should throw the ball at an angle of $\pi/4$ (i.e., 45 degrees) to the ground in order to make the ball go farthest. In this problem you will determine at what angle to throw the ball if you are *throwing against a horizontal wind.*

The setup is as in Figure 1. Here y is the vertical component of displacement, and x is the horizontal component of displacement. The equations of motion are:

$$\begin{cases} x(t) &= (v_0\cos(\theta) - w)t, \\ y(t) &= -\frac{g}{2}t^2 + v_0\sin(\theta)t \end{cases} \tag{1}$$

Here θ is an angle ranging between 0 and π, and v_0, w and g are positive constants.

(a)(5 points) Define $t_f > 0$ to be the moment when the ball hits the ground, i.e., $y(t_f) = 0$, and define $X(\theta)$ to be the x-component of displacement at the time t_f. Solve for t_f and substitute this into $x(t)$ to find an equation for $X(\theta)$ involving $\sin(\theta)$ and $\cos(\theta)$.

(b)(5 points) Find the derivative $\frac{dX}{d\theta}$, and express as a quadratic polynomial in $\cos(\theta)$ using the identity $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$.

(c)(5 points) Set $\frac{dX}{d\theta} = 0$ and solve the resulting quadratic polynomial in $\cos(\theta)$ to determine the value of $\cos(\theta)$ that maximizes $X(\theta)$. If w is small and positive, is this angle greater than or less than $\pi/4$?

Problem 2(10 points) Two services offered by a bank are *savings accounts* and *loans*. If a customer makes an initial deposit of A dollars in a savings account with continuously compounded interest at an annual rate of a, then after t years the bank owes the customer $A(1+a)^t$ dollars (neglecting fees). If a customer takes a loan of B dollars with continuously compounded interest at an annual rate of b, then after t years the customer owes the bank $B(1+b)^t$ dollars (neglecting fees). In order to make a profit, the interest rate for loans, b, is always higher than the interest rate for savings accounts, a. To simplify the computations, define $\alpha = \ln(1+a)$ and $\beta = \ln(1+b)$.

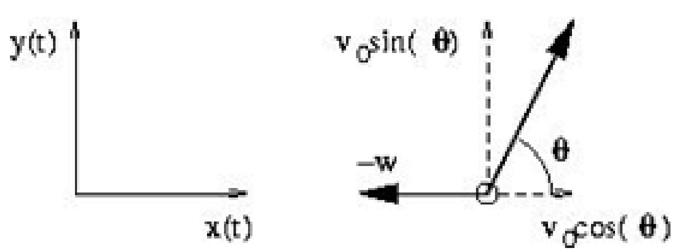

FIGURE 1. Ball thrown against the wind

A customer places A dollars in a savings account at an annual rate of a. The bank in turn loans a smaller amount B dollars to a second customer at a higher interest rate b (some portion of the first customer's deposit will be deposited in the Federal Reserve Bank, used for paying bank employees, etc.). The bank's net assets from these two transactions (measured in dollars) after t years is

$$A(t) = Be^{\beta t} - Ae^{\alpha t}. \tag{2}$$

In the long run, obviously the bank will have positive net assets. But in the short run, the bank has a net liability

$$L(t) = Ae^{\alpha t} - Be^{\beta t}. \tag{3}$$

Note that the liability for just the savings account is

$$M(t) = Ae^{\alpha t}. \tag{4}$$

(a)(5 points) Determine the time t at which the net liability $L(t)$ is maximum. Assume that $A\alpha > B\beta$. Leave your answer in the form

$$e^{(\beta-\alpha)t} = \text{something}. \tag{5}$$

(b)(5 points) Consider the ratio $L(t)/M(t)$. Using your answer to (a), determine the value of this ratio at the time when the net liability $L(t)$ is maximum. Simplify your answer as much as possible. How does this ratio depend on the amounts A and B?

Problem 3(10 points) In Problem 2 (b) from Problem Set 1, you computed that a projectile fired under a linearly increasing gravity model reaches its maximum at the time

$$t_{\max} = \frac{g}{c_0}\left(\sqrt{1+\frac{2c_0 v_0}{g^2}} - 1\right), \tag{6}$$

and in Problem 2 (c), you computed that it will strike the ground at the time

$$t_f = \frac{3g}{2c_0}\left(\sqrt{1+\frac{8c_0 v_0}{3g^2}} - 1\right). \tag{7}$$

(a)(5 points) Treating c_0 as the variable, determine the linearization of the square root in $t_{\max}$ for $c_0 = 0$ and substitute this in to get the approximate value of $t_{\max}$.

(b)(5 points) Same as above for t_f.

Problem 4(15 points) Consider the function $f(x) = x\cos(x) - \sin(x)$.

(a)(5 points) Determine whether $f(x)$ is an even function, an odd function or neither. What, if anything, does this tell you about the value of $f(0)$? What, if anything, does this tell you about whether f has a maximum or a minimum at $x = 0$?

(b)(5 points) Determine all critical points of $f(x)$ and determine the value of $f(x)$ at each critical point. Which of these are maxima and which of these are minima?

(c)(5 points) Sketch the graph of $f(x)$ on the interval $[-4\pi, 4\pi]$. Label all maxima and minima. *Do not* try to determine the zeros of $f(x)$ (this is just a rough sketch).