MATH 18.01, FALL 2017 - PROBLEM SET # 6 SOLUTIONS

Part II (50 points)

1. (Thurs., Oct. 26; Second Fundamental Theorem; 3 + 2 + 2 + 3 + 3 + 3 = 16 points) Let $\operatorname{sinc}(x)$ denote the "sinc" function

$$\operatorname{sinc}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{\sin x}{x} & \text{if } x \neq 0. \end{cases}$$

Now consider the "sine integral" function

$$\operatorname{Si}(x) = \int_0^x \operatorname{sinc}(t)\, dt.$$

Both of these functions frequently come up in Fourier analysis and signal processing and hence have been given their own names. *Remark: Si(x) cannot be expressed in terms of standard elementary functions.*

a) Compute $\operatorname{Si}'(x)$ and $\operatorname{Si}''(x)$. You will have to compute $\operatorname{Si}''(0)$ by using the definition of the derivative. *Hint: In computing Si$''(0)$, you can make use of the fact that* $\sin(\Delta x) = \Delta x + O((\Delta x)^3)$.

Solution: By the second fundamental theorem of calculus, $\operatorname{Si}'(x) = \operatorname{sinc}(x)$. Hence, $\operatorname{Si}''(x) = \operatorname{sinc}'(x)$. For $x \neq 0$, we can use the quotient rule to get

$$\operatorname{sinc}'(x) = \frac{x\cos x - \sin x}{x^2}.$$

For $x = 0$, we need to use the analytic definition of the derivative, which says that

$$\operatorname{sinc}'(0) = \lim_{\Delta x \to \infty} \frac{\frac{\sin(\Delta x)}{\Delta x)} - 1}{\Delta x} = \lim_{\Delta x \to \infty} \frac{1 - 1 + O((\Delta x)^2)}{\Delta x} = \lim_{\Delta x \to \infty} O(\Delta x) = 0.$$

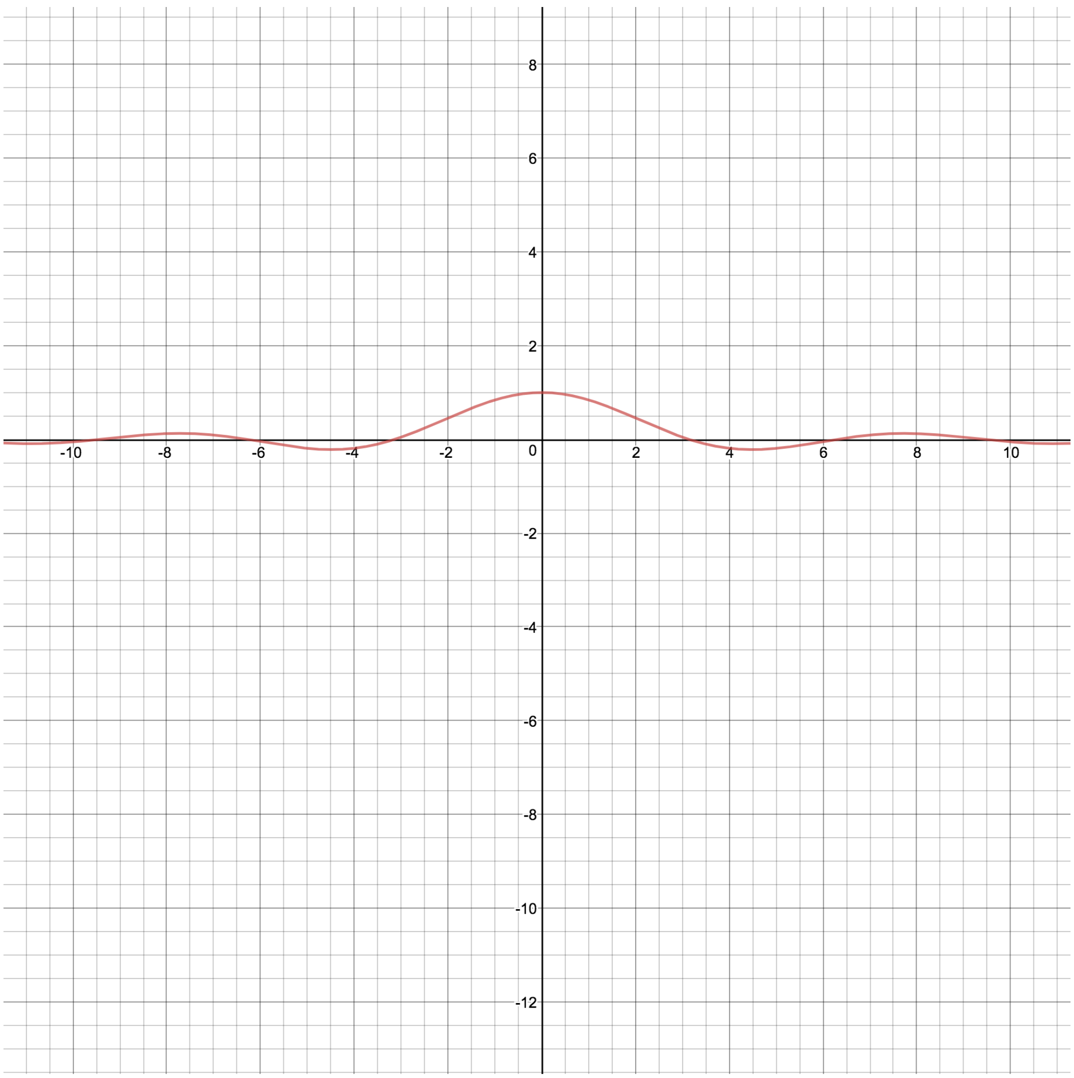
b) List the critical points of $\operatorname{Si}(x)$ in the entire range $-\infty < x < \infty$. Which critical points are local maxima and which ones are local minima?

Solution: The critical points are where $\operatorname{sinc}(x) = \operatorname{Si}'(x) = 0$. This happens when $\sin(x) = 0$, i.e., at $x = n\pi$ for $n \in \mathbb{Z}$ not equal to 0, since $\operatorname{sinc}(0) = 1$. To figure out which ones are local maxima/minima, we look at the sign of $\operatorname{sinc}(x)$ near $n\pi$. First note that since the function is even, if $n\pi$ is a local max/min, so is $-n\pi$. So we can assume n is positive, i.e., x is positive. For n even, and x slightly less than $n\pi$, $\sin(x) > 0$, and hence so is $\operatorname{sinc}(x)$, and for x slightly greater than $n\pi$, $\operatorname{sinc}(x) < 0$. The situation is reversed for n odd. Hence, the local maxima occur at $x = n\pi$ for n even and nonzero and the local minima occur at $n\pi$ for n odd.

c) Draw a rough sketch of $\mathrm{Si}'(x)$ and $\mathrm{Si}''(x)$. The drawings only have to be qualitatively correct, but make sure that the zeros of $\mathrm{Si}'(x)$ are accurately displayed.

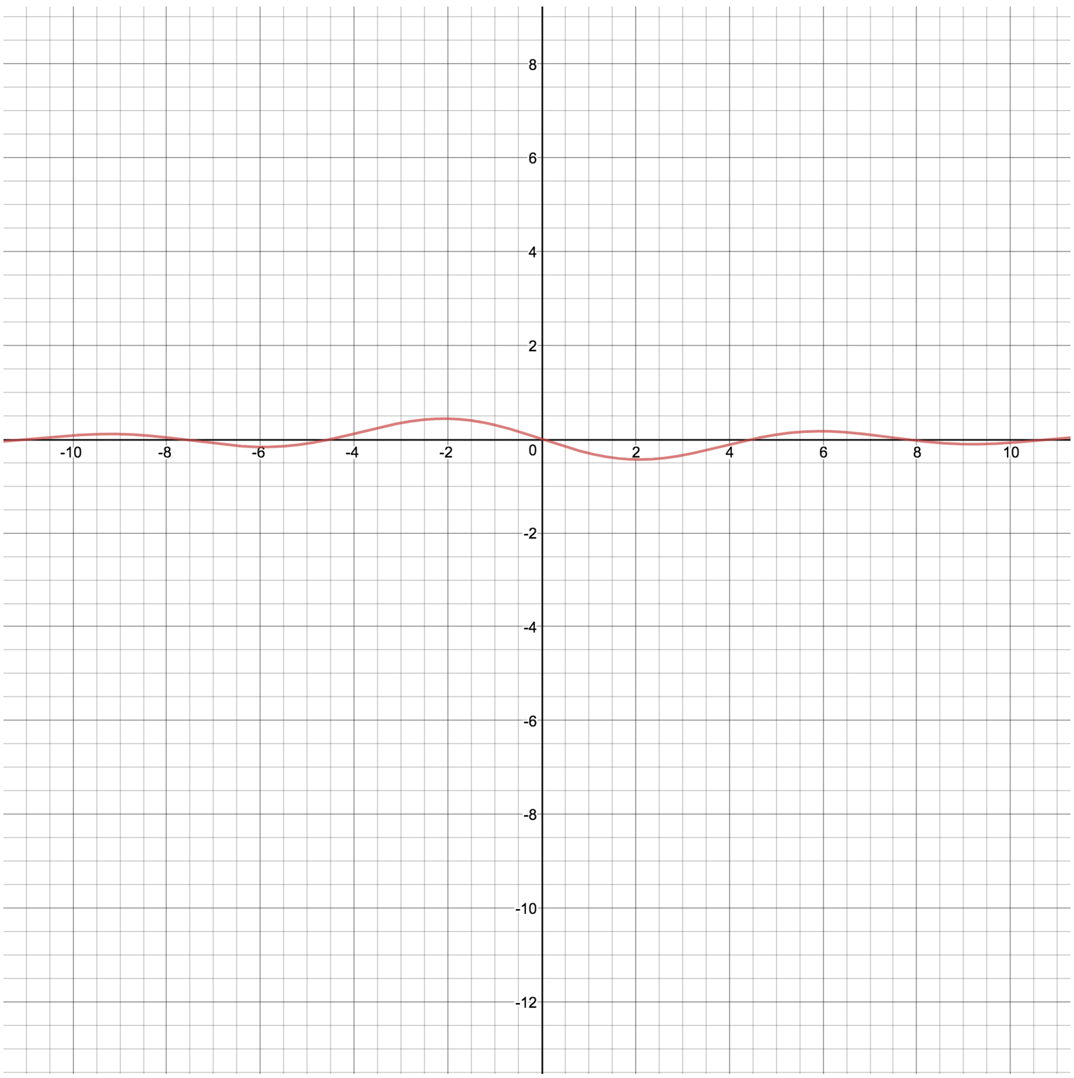
Solution:

Graph of $\mathrm{Si}'(x)$:



The zeros of $\mathrm{Si}'(x)$ are at $x = n\pi$ for $n \in \mathbb{Z}$.

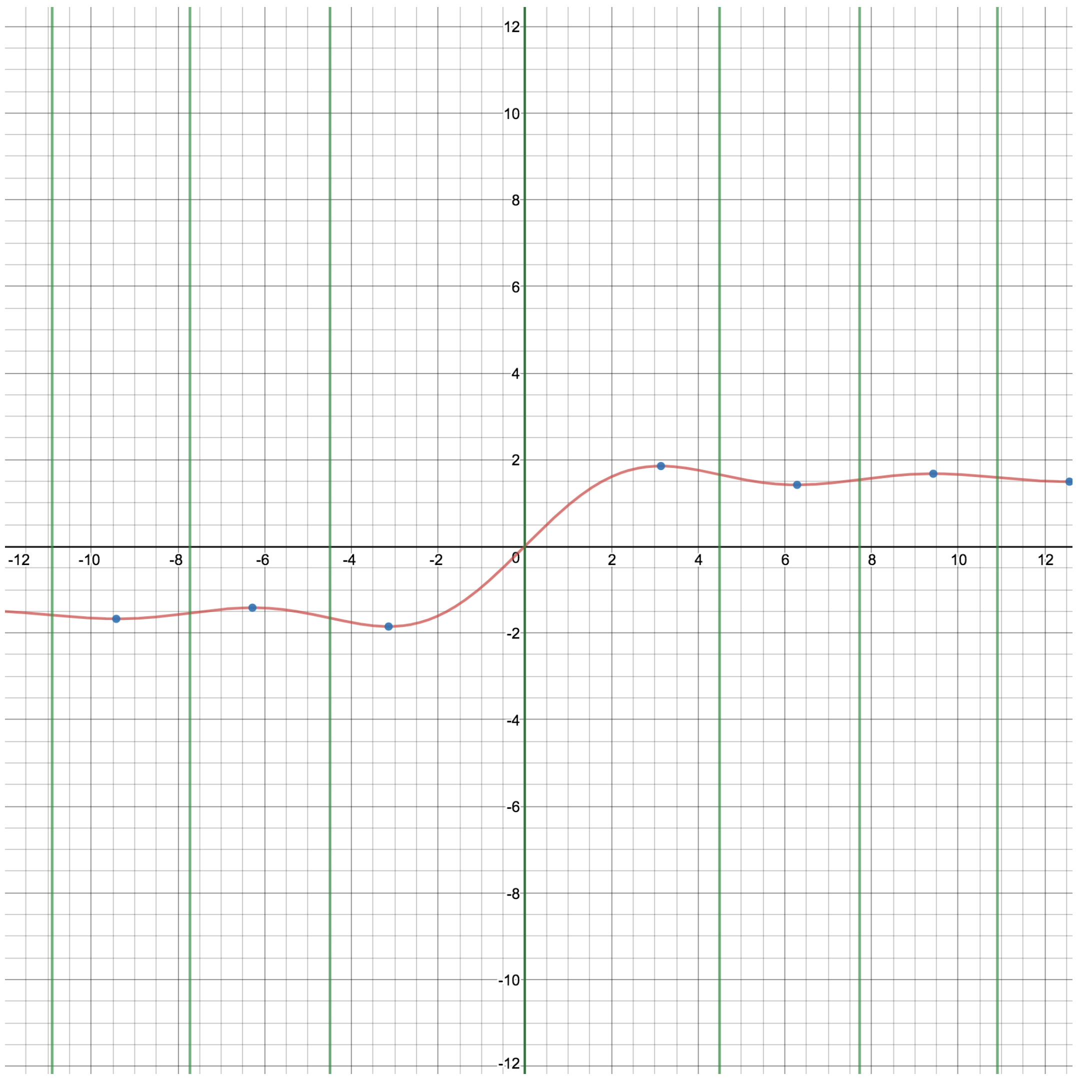
Graph of $\mathrm{Si}''(x)$:



The zeros of $\mathrm{Si}''(x)$ are implicitly given by $x = \tan(x)$.

d) Sketch the graph of $\mathrm{Si}(x)$ on the interval $-10\pi \leq x \leq 10\pi$ with labels for the critical points and inflection points. The drawing should be qualitatively correct and should reflect the shape of the graphs you sketched in part c).

Solution:
Zoomed in graph for $-4\pi < x < 4\pi$:



The graph has only one zero at $x = 0$ and is the graph of an odd function. The blue points are the critical points at $x = n\pi$. The green lines are the approximate solutions to $x = \tan x$ and mark the inflection points. The graph continues in this patter all the way out to 10π. It looks like a wave whose amplitude is getting smaller.

e) Let $r > 1$ be a real number, and define

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\sin(x^r)}{x} & \text{if } x \neq 0. \end{cases}$$

Remark: It is not too hard to show that $f(x)$ is continuous, even at $x = 0$. Consider the function

$$h(x) = \int_0^x f(t)\,dt.$$

Show that $h(x)$ can be expressed in terms of composition of Si with another function.

Solution: Consider the function $F(x) = \frac{\mathrm{Si}(x^r)}{r}$. By the chain rule,

$$F'(x) = \frac{\mathrm{Si}'(x^r)}{r}\frac{d}{dx}(x^r) = \mathrm{sinc}(x^r)x^{r-1} = f(x).$$

Hence, by the first fundamental theorem of calculus

$$h(x) = F(x) - F(0) = F(x).$$

Thus, $h(x)$ is, up to a scaling factor, the composition of $\mathrm{Si}(x)$ with x^r.

f) Compute

$$\lim_{x\to 3} \frac{x^2}{x-3}\int_3^x \mathrm{sinc}(t)\,dt.$$

Solution: By the fundamental theorem of calculus, the limit is equal to

$$\lim_{x\to 3} \frac{x^2(\mathrm{Si(x)} - \mathrm{Si}(3))}{x-3} = 9\lim_{x\to 3} \frac{\mathrm{Si}(x) - \mathrm{Si}(3)}{x-3} = 9\lim_{\Delta x\to 0} \frac{\mathrm{Si}(3+\Delta x) - \mathrm{Si}(3)}{\Delta x} = 9\mathrm{Si}'(3) = 3\sin 3.$$

2. (Fri., Oct. 27; volumes by slicing; 4 + 1 = 5 points) 7.3: 22

Solution:

(a) If V is the volume in the bowl and $A(h)$ is the surface area with the liquid having total height h, then

$$\frac{dV}{dt} = -cA(h)$$

for some constant $c > 0$.By the disks method,

$$V = \int_0^h A(x)\,dx.$$

Hence, by the fundamental theorem of calculus.

$$\frac{dV}{dh} = A(h).$$

Using the chain rule, we have

$$-cA(h) = \frac{dV}{dh}\frac{dh}{dt} = A(h)\frac{dh}{dt}.$$

Hence, $\frac{dh}{dt} = -c$.

(b) By integrating both sides of the equation obtained from part (a), we get

$$\int_0^t dh = \int_0^t -c\,dt$$

and hence

$$h(t) - h(0) = -ct.$$

Thus, if $h(t) = 0$, then $t = \frac{h_0}{c}$.

3. (Fri., Oct. 27; volumes by slicing; 10 points) Find the volume of the three-dimensional solid with $x > 0, y > 0, z > 0$ and

$$z^4 < x + y < z.$$

Hint: First find the area of the horizontal cross sections, which are perpendicular to the z axis.

Solution: First note that the z values go from 0 to 1, as $z^4 < z$. Now, the horizontal cross sections at a fixed height $0 < z < 1$ is the parallelogram bounded by the x-axis, the y-axis and the lines $x + y = z^4, x + y = z$. Hence, the area of the cross section is the difference between the area of the big triangle formed by the two axes and the line $x + y = z$ and the area of the smaller triangle formed by the axes and the line $x + y = z^4$. The big triangle has base and height z and hence area $\frac{z^2}{2}$, while the small triangle has area $\frac{z^8}{2}$. Hence, the area of the cross section is $A(z) = \frac{z^2 - z^8}{2}$.

The volume of the solid is

$$\int_0^1 A(z)\,dz = \int_0^1 \frac{z^2 - z^8}{2}\,dz = \frac{1}{6} - \frac{1}{18} = \frac{1}{9}.$$

4. (Tues., Oct. 31; shell and disk method; 2 + 1 + 2 + 1 = 6 points) (Donut with triangular cross sections)

a) An equilateral triangle in the (x, y) plane of side length ℓ has a base that runs along the x axis. The center of the triangle is a distance R from the y axis, where $R > \frac{1}{2}\ell$ (and thus the triangle does not intersect the y axis). The triangle is revolved around the y axis to create a solid. Use the cylindrical shell method to express the volume of the solid in terms of an integral. Your answer should depend on ℓ and R.

Solution: The triangle is built up of two parts. Between $x = R - \frac{l}{2}$ and $x = R$, the triangle is the region under the graph

$$y = \tan\left(\frac{\pi}{3}\right)\left(x - R + \frac{l}{2}\right) = \sqrt{3}\left(x - R + \frac{l}{2}\right).$$

From $x = R$ to $x = R + \frac{l}{2}$, the triangle is the region under the graph

$$y = -\sqrt{3}\left(x - R - \frac{l}{2}\right).$$

You can obtain these formulas by computing the slope and the x-intercept of the corresponding lines and noting that equilateral triangles have $\frac{\pi}{3}$ as all their angles. Hence, using the shells method on each piece separately, we get that

$$V = 2\pi\sqrt{3}\int_{R-\frac{l}{2}}^{R} x\left(x - R + \frac{l}{2}\right)\,dx + 2\pi\sqrt{3}\int_{R}^{R+\frac{l}{2}} x\left(R + \frac{l}{2} - x\right)\,dx.$$

b) Compute the integral from part a) to find a formula for the volume.

Solution: One possible way to solve the problem is just to do a straightforward integration but this leads to messy algebra. Let us simplify the problem a little. First, we split the integral into two parts:

$$V_1 = 2\pi\sqrt{3}\left(\int_{R-\frac{l}{2}}^{R} \frac{xl}{2}\,dx + \int_{R}^{R+\frac{l}{2}} \frac{xl}{2}\,dx\right)$$

and

$$V_2 = 2\pi\sqrt{3}\left(\int_{R-\frac{l}{2}}^{R} x(x - R)\,dx + \int_{R}^{R+\frac{l}{2}} x(R - x)\,dx\right).$$

The sums in the first integral can be combined since we are integrating the same function and the end points much up. Hence,

$$V_1 = 2\pi\sqrt{3}\int_{R-\frac{l}{2}}^{R+\frac{l}{2}} \frac{xl}{2}, dx = \pi\sqrt{3}l\frac{\left(R+\frac{l}{2}\right)^2 - \left(R-\frac{l}{2}\right)^2}{2} = \pi\sqrt{3}Rl^2.$$

To solve for V_2, we make the substitution $u = x - R$. Then,

$$\begin{aligned} V_2 &= 2\pi\sqrt{3}\left(\int_{-\frac{l}{2}}^{0} u^2 + Ru\,du + \int_0^{\frac{l}{2}} -u^2 - Ru\right) du \\ &= 2\pi\sqrt{3}\left(0 + \frac{l^3}{24} - R\frac{l^2}{8} - \frac{l^3}{24} - R\frac{l^2}{8}\right) \\ &= -\frac{\sqrt{3}\pi Rl^2}{2} \end{aligned}$$

Adding everything together, we get

$$V = \frac{\sqrt{3}\pi Rl^2}{2}.$$

c) Repeat parts a) and b), but this time using the disk method.

Solution: To use the disks method, we need to switch the roles of x and y in the formula given in the book, because we are revolving around the y-axis. Since the solid is separated from the y-axis, the volume is given by the formula

$$V = \int \pi(x_1^2 - x_2^2), dy.$$

where $x_1(y)$ is the function that describes the line further away from the y-axis, and $x_2(y)$ is the function describing the line closer to the y-axis. By the same method as in part (a), we can find the equation of the lines

$$x_1 = R + \frac{l}{2} - \frac{y}{\sqrt{3}} \text{ and } x_2 = \frac{y}{\sqrt{3}} + R - \frac{l}{2}.$$

The bounds of the integral are given by the range of the y-values. The minimum value is 0 and the maximum is the height of the triangle, which is $\frac{\sqrt{3}l}{2}$. Hence, the volume is

$$\begin{aligned} V &= \pi\int_0^{\frac{\sqrt{3}l}{2}} \frac{y^2}{3} - 2\frac{y}{\sqrt{3}}\left(R+\frac{l}{2}\right) + \left(R+\frac{l}{2}\right)^2 - \frac{y^2}{3} - 2\frac{y}{\sqrt{3}}\left(R-\frac{l}{2}\right) - \left(R-\frac{l}{2}\right)^2 dy \\ &= \pi\int_0^{\frac{\sqrt{3}l}{2}} -4\frac{y}{\sqrt{3}}R + 2Rl\,dy \\ &= \pi R\left(l^2\sqrt{3} - \frac{\sqrt{3}l^2}{2}\right) = \frac{\sqrt{3}\pi Rl^2}{2}. \end{aligned}$$

5. (Tues., Oct. 31; shell method; 3 + 7 = 10 points) 7.4: 12, 13.

Remark: Think of the "spherical ring" as a sphere that has been gored by a cylinder whose radius is smaller than the radius of the sphere, but whose length is infinite.

Solution:

12. The solid is a cone with vertex at $(0, h)$ and bottom face at $y = 0$, with radius equal to r. So the formula we expect to get is $V = \frac{\pi r^2 h}{3}$. Let us derive this using the cylindrical shell method.

$$V = 2\pi \int_0^r xy\,dx = 2\pi h \int_0^r x\left(1 - \frac{x}{r}\right)\,dx = 2\pi h\left(\left[\frac{x^2}{2}\right]_0^r - \left[\frac{x^3}{3r}\right]_0^r\right) = 2\pi h\left(\frac{r^2}{2} - \frac{r^2}{3}\right) = \frac{\pi r^2 h}{3}$$

as expected.

13. We can obtain the spherical ring by taking a segment of a circle of radius a and length of base $2h$ (with the base parallel to the y-axis) and revolving it around the y-axis. Since the base is parallel to the y-axis, it's equation is just $x = r$ for some constant r. To figure out this constant, note that $r^2 + h^2 = a^2$. Hence, $r = \sqrt{a^2 - h^2}$. By symmetry the surface above and below the x-axis will contribute the same total volume. Hence, we simply compute the volume of the solid obtained by rotating the piece of the ring above the x-axis and then multiply by 2.

The bounds on the x-value of the region that is revolved are r and a and the equation for the graph bounds the region is $y = \sqrt{a^2 - x^2}$ Hence, by the shells method,

$$V = 4\pi \int_r^a x\sqrt{a^2 - x^2}\,dx.$$

We use the substitution $u = a^2 - x^2$. Then, $du = -2x\,dx$. Hence,

$$V = -2\pi \int_{a^2-r^2}^{0} \sqrt{u}\,du = 2\pi \int_0^{a^2-r^2} \sqrt{u}\,du = 2\pi \int_0^{h^2} \sqrt{u}\,du = 2\pi \left[\frac{2u^{\frac{3}{2}}}{3}\right]_0^{h^2} = \frac{4\pi h^3}{3}.$$