# • Integration by Parts, Reduction Formulas

## • Integration by parts

Recall the product rule: $(uv)' = u'v + uv'$

- We can rewrite this as $uv' = (uv)' - u'v$
- We can integrate this to get the formula for integration by parts:

$$\int uv' = uv - \int vu'$$

<u>Ex</u>: How to compute $\int \tan^{-1}x \, dx$ ?

- $\tan^{-1}x = \tan^{-1}x \cdot 1 = uv'$, where • $u = \tan^{-1}x$, $v' = 1$
  - $u' = \frac{1}{1+x^2}$ $v = x$

Use integration by parts:

$$\int \tan^{-1}x \, dx = \int uv' dx = uv - \int v du = \tan^{-1}x \cdot x - \int \frac{x \, dx}{1+x^2}$$

$$= x \tan^{-1}x - \frac{1}{2} \ln|1+x^2| + C$$

- Integration by parts for definite integrals
  - IBP formula: $uv' = (uv)' - u'v$
  - Let's take a definite integral of both sides

$$\int_a^b uv'\,dx = \int_a^b (uv)'\,dx - \int_a^b u'v\,dx$$

$$\overset{\text{by FTC I}}{=} uv\Big|_a^b - \int_a^b u'v\,dx$$

- Another notation for indefinite integration by parts:

$$\int u\,dv = uv - \int v\,du$$

  - This is the same because

$$dv = v'dx \Rightarrow uv'dx = u\,dv$$

and $du = u'dx \Rightarrow u'v\,dx = vu'dx = v\,du$

Ex: $\int \ln x \, dx$

- $u = \ln x \quad du = \frac{1}{x}\,dx \quad dv = dx \quad v = x$

$$\int \ln x \, dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C$$

- Alternate approach: "advanced guessing"

$$\frac{d}{dx} ? = \ln x$$

- Guess: $\frac{d}{dx}(x\ln x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1.$

- Our guess was not quite right, so we alter it:

$$\frac{d}{dx}\underbrace{(x\ln x - x)}_{\text{Correct answer}} = \ln x + 1 - 1 = \ln x \quad \checkmark$$

- Reduction Formulas (Recurrence Formulas)

- Ex: $\int (\ln x)^n dx$

- Let's try
  - $u = (\ln x)^n \Rightarrow du = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$
  - $v' = dx \quad v = x$

Using the IBP formula:

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1} \cdot \cancel{\frac{1}{x}} \cdot \cancel{x}\, dx$$

- Keep repeating integration by parts to get the full formula: $n \to n-1 \to n-2 \to n-3 \ldots$ etc.

Ex: $\int x^n e^x dx$ try:
- $u = x^n$ · $u' = nx^{n-1}$
- $v' = e^x$ · $v = e^x$

$$\int x^n e^x dx = x^n e^x - \int n x^{n-1} e^x dx$$

- Keep going: $n \to n-1 \to n-2 \ldots$ etc.

- Bad news: If you change the integrals just a little bit, they become impossible to evaluate: $\int (\tan^{-1}x)^2\,dx$ = impossible

$$\int \frac{e^x}{x}\,dx = \text{impossible}$$

- Many integrals can be evaluated numerically on a computer.

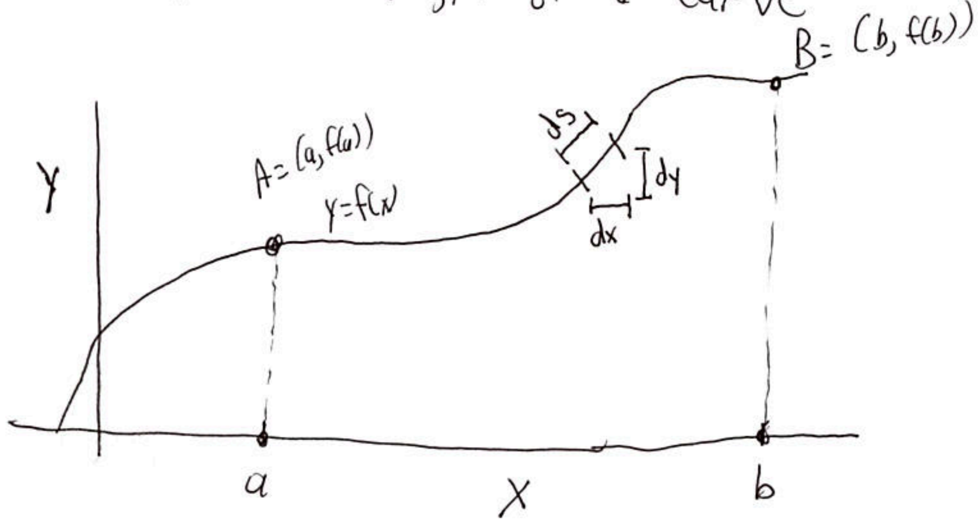
- Evaluating integrals by hand is still important. For example, sometimes you might need to understand how a whole family of integrals such as

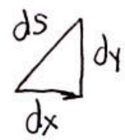
$$F(a) = \int_1^\infty \frac{e^x}{x^a}\,dx \quad \text{depends}$$

on the value of $a$.

# • Arc Length

Goal: Compute the length of a curve



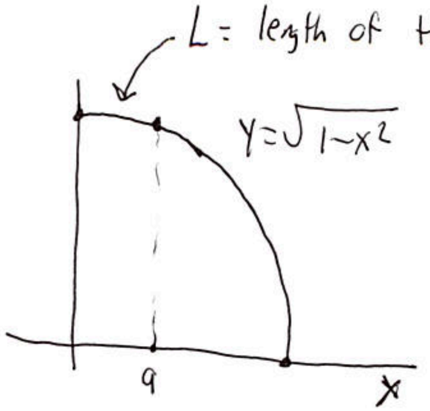




$ds$ = infinitesimal arc length

• $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

• To find the length of the portion of the curve in between $A$ and $B$, just integrate: $\text{Length} = \int_A^B ds = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Ex: Portion of a circle of radius 1

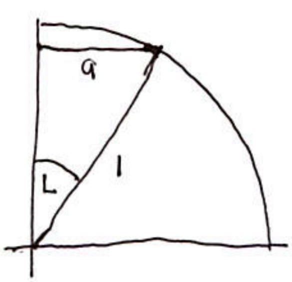



- $y = \sqrt{1-x^2}$
- $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}}$
- $ds = \sqrt{1 + \left(\dfrac{-x}{\sqrt{1-x^2}}\right)^2}\, dx = \sqrt{\dfrac{1-x^2 + x^2}{1-x^2}}\, dx$

$$= \sqrt{\frac{1}{1-x^2}}\, dx$$

- $L = \displaystyle\int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big]_0^a = \sin^{-1} a$

- Alternative derivation: $L = \angle$ in radians




- $\sin L = a$

$\Rightarrow L = \sin^{-1} a$