MATH 18.01 - MIDTERM 4 - SOME REVIEW PROBLEMS WITH SOLUTIONS

18.01 Calculus, Fall 2017	Professor: Jared Speck

Problem 1. Approximate the integral

$$\int_0^2 x^4\,dx$$

using first Simpson's rule with two equal intervals and then the trapezoid rule with two equal intervals. Compare your approximations to the exact value.

Problem 2. Evaluate the integral

$$\int e^x \sin x\,dx.$$

Problem 3. Evaluate the integral

$$\int \frac{dx}{(1-x^2)^{5/2}}.$$

Problem 4. Let $F(x) = \int_1^x \frac{\cos y}{y}\,dy$ and $G(x) = \int_1^x \frac{\sin y}{y^2}\,dy$. Show that there is a constant C such that $F(x) - G(x) = \frac{\sin x}{x} + C$. What is the value of C?

Problem 5. Evaluate the integral

$$\int \frac{dx}{x\sqrt{(\ln x)^2 - 4}}.$$

Problem 6. Consider the following curve in parametric form:

$$x = t + \ln t,$$
$$y = t - \ln t.$$

Find the arc length of the portion of the curve corresponding to $1 \le t \le 2$.

Solutions

Problem 1. Approximate the integral

$$\int_0^2 x^4\,dx$$

using first Simpson's rule with two equal intervals and then the trapezoid rule with two equal intervals. Compare your approximations to the exact value.

Solution: We set $f(x) = x^4$ and $\Delta x = \frac{2-0}{2} = 1$. Then the Simpson approximation is

$$\begin{aligned}\int_0^2 x^4\,dx &\approx \frac{\Delta x}{3}\left[f(0) + 4f(1) + f(2)\right]\\ &= \frac{1}{3}\left(0 + 4\times 1 + 16\right) = 20/3.\end{aligned}$$

The trapezoid approximation is

$$\begin{aligned}\int_0^2 x^4\,dx &\approx \Delta x\left[\frac{1}{2}f(0) + f(1) + \frac{1}{2}f(2)\right]\\ &= 1\times(0+1+8) = 9.\end{aligned}$$

The exact value of the integral is $\int_0^2 x^4\,dx = \frac{x^5}{5}\,\Big|_{x=0}^{x=2} = 32/5.$

Problem 2. Evaluate the integral

$$\int e^x \sin x\,dx.$$

Solution: We integrate by parts with $u = e^x, du = e^x$, $dv = \sin x\,dx$, $v = -\cos x$, which leads to

$$\begin{aligned}\int e^x \sin x\,dx &= \int u\,dv = uv - \int v\,du\\ &= -e^x\cos x + \int e^x \cos x\,dx.\end{aligned}$$

Similarly, we compute that

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Inserting this identity for $\int e^x \cos x\,dx$ back into the previous formula, we deduce that

$$\begin{aligned}\int e^x \sin x\,dx &= -e^x\cos x + \int e^x\cos x\,dx\\ &= -e^x\cos x + e^x\sin x - \int e^x \sin x\,dx.\end{aligned}$$

We then add $\int e^x \sin x\,dx$ to both sides of the previous equation, divide by two, and add the integration constant C, which leads to

$$\int e^x \sin x\,dx = \frac{1}{2}\left\{e^x\sin x - e^x\cos x\right\} + C.$$

Problem 3. Evaluate the integral

$$\int \frac{dx}{(1-x^2)^{5/2}}.$$

Solution: We use the inverse trig substitution $x = \sin u$, $dx = \cos u\, du$, the identity $(\cos u)^2 = 1 - (\sin u)^2$, and the identity $(\sec u)^2 = 1 + (\tan u)^2$, which leads to

$$\begin{aligned}\int \frac{\cos u}{(\cos u)^5}\, du &= \int (\sec u)^4\, du \\ &= \int [1 + (\tan u)^2](\sec u)^2\, du.\end{aligned}$$

We then make the substitution $v = \tan u$, $dv = (\sec u)^2\, du$, which leads to

$$\begin{aligned}\int [1 + (\tan u)^2](\sec u)^2\, du &= \int 1 + v^2\, dv \\ &= v + \frac{v^3}{3} + C \\ &= \tan u + \frac{\tan^3 u}{3} + C \\ &= \frac{x}{(1-x^2)^{1/2}} + \frac{1}{3}\frac{x^3}{(1-x^2)^{3/2}} + C.\end{aligned}$$

In the last step above, we used a right triangle to deduce that $\tan u = \frac{x}{(1-x^2)^{1/2}}$.

Problem 4. Let $F(x) = \int_1^x \frac{\cos y}{y}\, dy$ and $G(x) = \int_1^x \frac{\sin y}{y^2}\, dy$. Show that there is a constant C such that $F(x) - G(x) = \frac{\sin x}{x} + C$. What is the value of C? **Solution:** We use integration by parts in the form $u = \frac{1}{y}$, $du = -\frac{1}{y^2}\, dy$, $dv = \cos y\, dy$, $v = \sin y$, $\int u\, dv = uv - \int v\, du$ to deduce that

$$\begin{aligned}\underbrace{\int_1^x \frac{\cos y}{y}\, dy}_{F(x)} &= \frac{\sin y}{y}\, \Big|_{y=1}^{y=x} + \int_1^x \frac{\sin y}{y^2}\, dy \\ &= \frac{\sin x}{x} - \sin 1 + \underbrace{\int_1^x \frac{\sin y}{y^2}\, dy}_{G(x)}.\end{aligned}$$

Hence,

$$F(x) - G(x) = \frac{\sin x}{x} - \sin 1,$$

and $C = \sin 1$.

Problem 5. Evaluate the integral

$$\int \frac{dx}{x\sqrt{(\ln x)^2 - 4}}.$$

Solution: We first make the substitution $u = \ln x$, $du = \frac{dx}{x}$, which leads to the integral

$$\int \frac{du}{(u^2-4)^{1/2}}.$$

We then make the inverse trigonometric substitution $u = 2\sec\theta$, $du = 2\sec\theta\tan\theta\,d\theta$ and use the identity $(\sec\theta)^2 - 1 = (\tan\theta)^2$, which leads to

$$\begin{aligned}
\int \frac{du}{(u^2-4)^{1/2}} &= \int \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta + \tan\theta| + C, \\
&= \ln\left|\frac{u}{2} + \sqrt{\frac{u^2}{4} - 1}\right| + C \\
&= \ln\left|\frac{\ln x}{2} + \sqrt{\frac{(\ln x)^2}{4} - 1}\right| + C.
\end{aligned}$$

In the next-to-last line above, we used a right triangle to deduce that $\tan\theta = \frac{\sqrt{u^2-4}}{2} = \sqrt{\frac{u^2}{4} - 1}$.

Problem 6. Consider the following curve in parametric form:

$$\begin{aligned}
x &= t + \ln t, \\
y &= t - \ln t.
\end{aligned}$$

Find the arc length of the portion of the curve corresponding to $1 \leq t \leq 2$.

Solution: We first compute that

$$\begin{aligned}
\frac{dx}{dt} &= 1 + \frac{1}{t}, \\
\frac{dy}{dt} &= 1 - \frac{1}{t}, \\
ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \sqrt{2}\sqrt{1 + \frac{1}{t^2}} \\
&= \sqrt{2}\frac{\sqrt{t^2+1}}{t}.
\end{aligned}$$

The arc length is therefore

$$\text{Arc Length} = \int ds = \sqrt{2}\int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt.$$

To evaluate the integral, we first make the inverse trig substitution $t = \tan u$, $dt = \sec^2 u$, use $\tan^2 u + 1 = \sec^2 u$, multiply the top and bottom by $\sin u$, and use $\sin^2 = 1 - \cos^2 u$, which leads to

$$\begin{aligned}
\int \frac{\sqrt{t^2+1}}{t}\, dt &= \int \frac{\sqrt{\sec^2 u}}{\tan u} \sec^2 u \, du \\
&= \int \frac{1}{\sin u \cos^2 u}\, du \\
&= \int \frac{\sin u}{\sin^2 u \cos^2 u}\, du \\
&= \int \frac{\sin u}{(1-\cos^2 u)\cos^2 u}\, du.
\end{aligned}$$

We then make the substitution $v = \cos u$, $dv = -\sin u \, du$, which leads to

$$\begin{aligned}
\int \frac{\sin u}{(1-\cos^2 u)\cos^2 u}\, du &= -\int \frac{1}{(1-v^2)v^2}\, dv \\
&= -\int \frac{1}{v^2(1-v)(1+v)}\, dv.
\end{aligned}$$

To evaluate the v integral, we first make a partial fraction decomposition:

$$\frac{1}{v^2(1-v)(1+v)} = \frac{A}{v} + \frac{B}{v^2} + \frac{C}{1-v} + \frac{D}{1+v}.$$

The cover up method yields $B = 1$, $C = \frac{1}{2}$, $D = \frac{1}{2}$. Then plugging the value $v = 2$ into both sides of the decomposition yields that $A = 0$. Hence,

$$\frac{1}{v^2(1-v)(1+v)} = \frac{1}{v^2} + \frac{1}{2}\left(\frac{1}{1-v}\right) + \frac{1}{2}\left(\frac{1}{1+v}\right),$$

and

$$\begin{aligned}
-\int \frac{1}{v^2(1-v)(1+v)}\, dv &= \frac{1}{v} + \frac{1}{2}\ln\left|\frac{1-v}{1+v}\right| + c \\
&= \frac{1}{\cos u} + \frac{1}{2}\ln\left|\frac{1-\cos u}{1+\cos u}\right| + c \\
&= \sqrt{1+t^2} + \frac{1}{2}\ln\left|\frac{1-(1+t^2)^{-1/2}}{1+(1+t^2)^{-1/2}}\right| + c \\
&= \sqrt{1+t^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}\right| + c.
\end{aligned}$$

In our calculations above, we used a right triangle to deduce that

$$\cos u = (1+t^2)^{-1/2}.$$

Finally, we reinsert the factor of $\sqrt{2}$ and the bounds of integration to conclude that

$$\begin{aligned}
\text{Arc Length} &= \sqrt{2}\int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt \\
&= \sqrt{2}\left\{\sqrt{1+t^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}\right|\right\}_{t=1}^{t=2} \\
&= \sqrt{2}\left\{\sqrt{5}-\sqrt{2} + \frac{1}{2}\ln\left|\frac{\sqrt{5}-1}{\sqrt{5}+1}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right\} \\
&= \sqrt{2}\left\{\sqrt{5}-\sqrt{2} + \frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{5}+1}{\sqrt{5}-1}\right|\right\}.
\end{aligned}$$