MIDTERM 1 - 18.01 - FALL 2014.

Name:

Email:

Please put a check by your recitation section.

	Instructor	Time
☐	B.Yang	MW 10
☐	M. Hoyois	MW 11
☐	M. Hoyois	MW 12
☐	X. Sun	MW 1
☐	R. Chang	MW 2

Problem #	Max points possible	Actual score
1	20	
2	15	
3	15	
4	20	
5	15	
6	15	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**

Good luck!

Problem 1. (10 + 5 + 5 = 20 points)

a) Compute the derivative of the function $f(x) = x^{x^2}$.
b) Compute the first and second derivatives of the function $f(x) = xe^x$.
c) Let n be a positive integer. Based on your answer to part b), guess a formula for the n^{th} derivative of the function $f(x) = xe^x$ and use mathematical induction to show that your formula is correct.

Solution:
1)

$$\begin{aligned} y = x^{x^2} &\implies \ln y = x^2 \ln x \implies \frac{y'}{y} = 2x \ln x + x \\ &\implies y' = x^{x^2}(2x \ln x + x) \end{aligned}$$

b)

$$y = xe^x \implies y' = xe^x + e^x \implies y'' = xe^x + 2e^x.$$

c) We will use induction to show that $\frac{d^n}{dx^n}(xe^x) = xe^x + ne^x$. The base case $n = 1$ was verified in part b). Suppose that $\frac{d^n}{dx^n}(xe^x) = xe^x + ne^x$. Then

$$\begin{aligned} \frac{d^{n+1}}{dx^{n+1}}(xe^x) &= \frac{d}{dx}\left\{\frac{d^n}{dx^n}(xe^x)\right\} \\ &= \frac{d}{dx}(xe^x + ne^x) \\ &= xe^x + e^x + ne^x \\ &= xe^x + (n+1)e^x. \end{aligned}$$

We have thus proved that case $n + 1$ is a consequence of case n, which completes the induction.

Problem 2. (5 + 10 = 15 points)
a) Let $f(x)$ be a function. State the analytic definition of $f'(x)$ (in terms of a limit).

b) Consider the function $f(x) = x|x|$. Decide whether or not $f(x)$ is differentiable at the point $x_0 = 0$. To receive credit, your argument *must involve* the analytic definition of the derivative, and you must fully explain your reasoning.

Solution: a)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Solution: b) Since $f(0) = 0$,

$$\begin{aligned} f'(0) &= \lim_{\Delta x \to 0} \frac{f(\Delta x) - \overbrace{f(0)}^{0}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x |\Delta x|}{\Delta x} \\ &= \lim_{\Delta x \to 0} |\Delta x| = 0. \end{aligned}$$

Thus, f is differentiable at $x_0 = 0$ and $f'(0) = 0$.

Problem 3. (15 points) Find the equation of the tangent line at the point

$$(x_0, y_0) = (1, \frac{\pi}{2})$$

to the following curve in the xy plane:

$$\sin(xy) + x^2 y = 1 + \frac{\pi}{2}.$$

Solution: Implicitly differentiating the equation with respect to x, we find that

$$y' \left\{ x\cos(xy) + x^2 \right\} + y\cos(xy) + 2xy = 0.$$

We now set $(x, y) = (1, \frac{\pi}{2})$ in the above equation to deduce that

$$y'(1 \cdot 0 + 1) + \frac{\pi}{2} \cdot 0 + 2 \cdot 1 \cdot \frac{\pi}{2} = 0,$$
$$y' = -\pi.$$

Since the equation of the tangent line at (x_0, y_0) is $y - y_0 = f'(x_0)(x - x_0)$, when $(x_0, y_0) = (1, \frac{\pi}{2})$, the line can be expressed as

$$y - \frac{\pi}{2} = -\pi(x - 1).$$

Problem 4. (10 + 10 = 20 points) Compute the following limits by recognizing that they are equal to $f'(x_0)$, where you have to figure out what f and x_0 are. You may *not* use L'Hôpital's rule, if you know what that is.

$$\text{a) } \lim_{\Delta x \to 0} \frac{(10 + 2\Delta x)^{100} - 10^{100}}{\Delta x}$$
$$\text{b) } \lim_{\Delta x \to 0} \frac{(\pi/2 + \Delta x)^2 \cos(\pi/2 + \Delta x)}{\Delta x}$$

Solution: a)

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{(10 + 2\Delta x)^{100} - 10^{100}}{\Delta x} &= 2 \lim_{\Delta x \to 0} \frac{(10 + 2\Delta x)^{100} - 10^{100}}{2\Delta x} \\ &= 2 \frac{d}{dx} x^{100}|_{x=10} = 2 \cdot 100 x^{99}|_{x=10} = 2 \cdot 100 \cdot 10^{99} = 2 \cdot 10^{101}.\end{aligned}$$

b)

$$\lim_{\Delta x \to 0} \frac{(\pi/2 + \Delta x)^2 \cos(\pi/2 + \Delta x)}{\Delta x} = \frac{d}{dx}(x^2 \cos x)|_{x=\pi/2} = \left\{2x \cos x - x^2 \sin x\right\}|_{x=\pi/2} = -\left(\frac{\pi}{2}\right)^2.$$

Problem 5. (15 points) Let $y = f(x) = \sin x + x$. Let $g(y)$ be the inverse function of f, that is, $g(y) = x$ when $y = f(x)$. Find an expression for $\frac{d}{dy}g(y)$ when $y = f(x)$. You are allowed to express your answer in terms of x, that is, in the form $\frac{d}{dy}g(y) =$ expression involving x.

Hint: Do not try to find a formula for $g(y)$; you won't be able to do it.

Solution:

By the chain rule, we have

$$\frac{d}{dx}\overbrace{g(y)}^{x} = \frac{d}{dy}g(y)\frac{dy}{dx} = 1.$$

Since $\frac{dy}{dx} = \cos x + 1$, we have

$$\frac{d}{dy}g(y) = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x + 1}.$$

Problem 6. (15 points) Suppose that $f(x)$ is a continuous function and $f(0) = -1$. Suppose furthermore that $f(x)$ is differentiable at every x value except $x = -3$ and $x = 2$. Finally, suppose that the graph of $f'(x)$ is as follows:

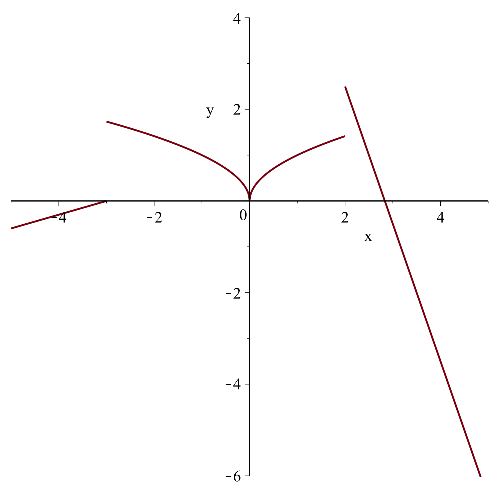

FIGURE 1. Graph of $f'(x)$

Sketch the graph of $f(x)$ on the blank graph below. Your picture should be qualitatively accurate, but it doesn't have to be quantitatively perfect.

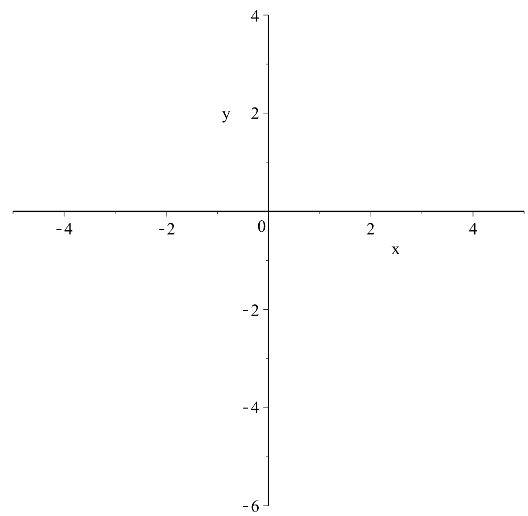

FIGURE 2. Draw your graph of $f(x)$ here

Solution:

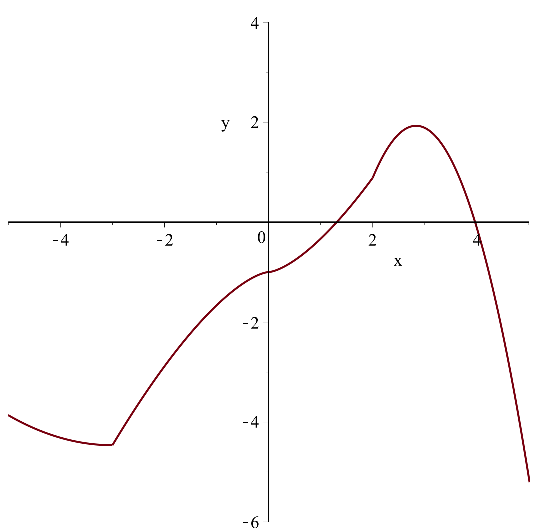

FIGURE 3. Graph of $f(x)$