• Parametric Equations

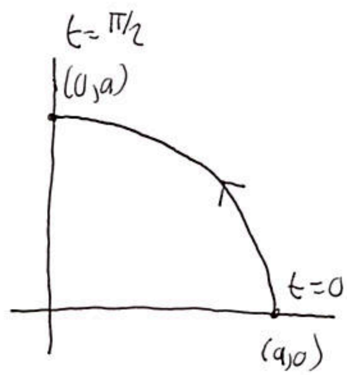
<u>Ex</u>: Alternative way to describe a circle of radius a:

- $x = a\cos t$
- $y = a\sin t$

- a is a constant and t is a variable

- There is a relationship between x and y:

$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$$



- when $t=0$, $x = a\cos 0 = a$, $y = a\sin 0 = 0$
- when $t = \pi/2$, $x = a\cos\pi/2 = 0$, $y = a\sin\pi/2 = a$
- For $0 \le t \le \pi/2$, a quarter circle is traced counter-clockwise

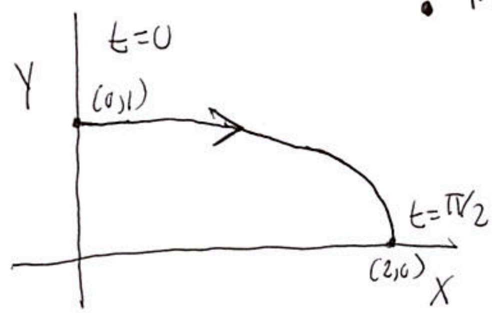
<u>Ex</u>: Arclength ds for the previous example:

- $dx = -a\sin t\,dt$ • $dy = a\cos t\,dt$

- $$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(-a\sin t\,dt)^2 + (a\cos t\,dt)^2}$$
$$= \sqrt{(a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$$

Ex: An ellipse in parametric form

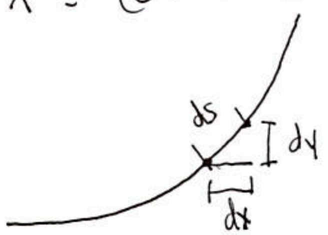
- $x = 2\sin t$ • $y = \cos t$
- $\frac{x^2}{4} + y^2 = \sin^2 t + \cos^2 t = 1$



- The ellipse is traced out clockwise as t increases from 0 to π/2

Parametric Equations, Arclength, Surface Area

- Ex: We will compute the arc length of a curve in parametric form.
  - $x = t^2$ for $0 \le t \le 1$
  - $y = t^3$
  - $x^3 = (t^2)^3 = t^6 = (t^3)^2 = y^2 \Rightarrow y = x^{2/3}$, $0 \le x \le 1$



- $dx = 2t\,dt$
- $dy = 3t^2\,dt$

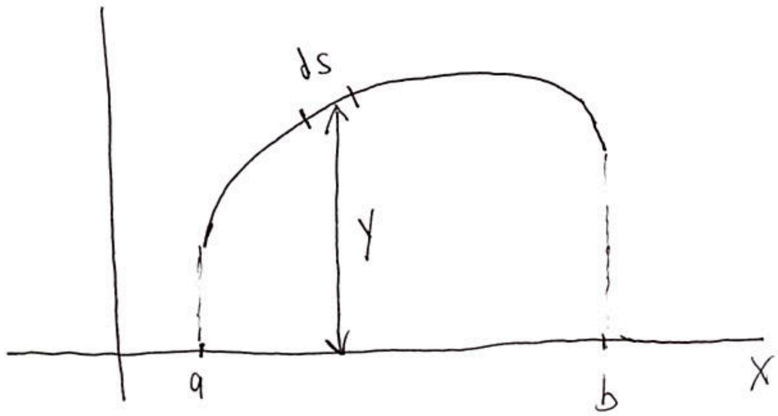
- $ds^2 = dx^2 + dy^2 = (2t\,dt)^2 + (3t^2 dt)^2 = (4t^2 + 9t^4)\,dt^2$

- $$\text{Length} = \int_{t=0}^{t=1} ds = \int_0^1 \sqrt{4t^2 + 9t^4}\,dt$$

$$= \int_0^1 t\sqrt{4 + 9t^2}\,dt = \left.\frac{(4+9t^2)^{3/2}}{27}\right]_0^1$$

$$= \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)$$

- Surface Area (Surfaces of revolution)

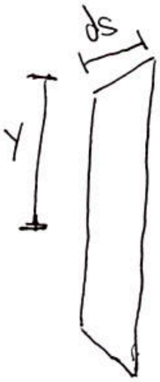


- Suppose the strip of width ds is revolved around the x axis

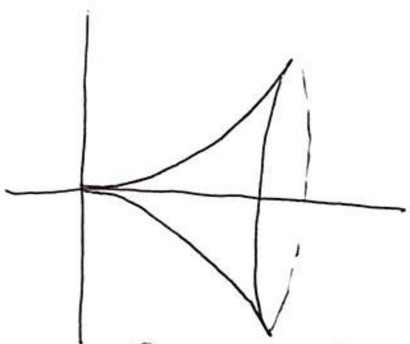
- The surface area of the thin strip is

$$\underbrace{2\pi y}_{\text{Circumference}}\ \underbrace{ds}_{\text{slant width}}$$

Side View:



Ex: Revolve $\begin{array}{l} x = t^2 \\ y = t^3 \end{array}$ $0 \le t \le 1$ around the x-axis



Curved Surface of a trumpet

- $ds = t\sqrt{4+9t^2}\,dt$

- $\text{Area} = \displaystyle\int 2\pi y\,ds = \int_{t=0}^{1} 2\pi \cdot t^3 \cdot t\sqrt{4+9t^2}\,dt$

$$= 2\pi \int_0^1 t^4\sqrt{4+9t^2}\,dt$$

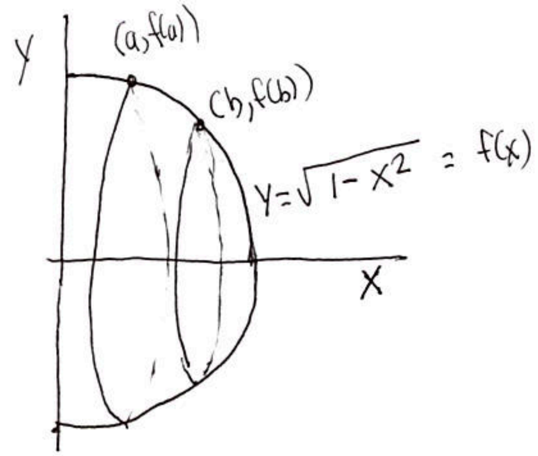
- To evaluate the integral, use the trig substitution
  - $t = \frac{2}{3}\tan u$ • $dt = \frac{2}{3}\sec^2 u\,du$ • $\tan^2 u + 1 = \sec^2 u$

- $\displaystyle\int t^4 (4+9t^2)^{1/2}\,dt = \int \left(\tfrac{2}{3}\tan u\right)^4 \left[4 + 9\left(\tfrac{4}{9}\tan^2 u\right)\right]^{1/2} \tfrac{2}{3}\sec^2 u\,du$

$$= \left(\frac{2}{3}\right)^5 \int \tan^4 u\,(2\sec u)\sec^2 u\,du$$

- This is a tan-sec integral. In principle, you could compute the integral, but it would take a long time

Ex: Let's compute the surface area of a portion of the unit sphere



- $y = \sqrt{1-x^2}$
- $ds = \dfrac{dx}{\sqrt{1-x^2}}$ (previously calculated)
- $\text{Area} = \displaystyle\int_{x=a}^{x=b} 2\pi y \, ds = \int_a^b 2\pi \sqrt{1-x^2} \frac{dx}{\sqrt{1-x^2}} = \int_a^b 2\pi \, dx$

$$= 2\pi (b-a).$$

- For the whole sphere: $a = -1,\ b = 1$

$$\text{Area} = 2\pi(1-(-1)) = 4\pi$$