MATH 18.01 - MIDTERM 2 - SOME REVIEW PROBLEMS WITH SOLUTIONS

18.01 Calculus, Fall 2014	Professor: Jared Speck

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Problem 2. Show that for any two numbers a and b, $|\sin a - \sin b| \leq |a - b|$.

Problem 3. Section 4.5: 8.

Problem 4. Use Newton's method to estimate the zero of $f(x) = x^3 + 5x - 7$. Start with the base point $x_0 = 1$ and compute x_1, x_2.

Problem 5. Graph the function $f(x) = |x|^{5/2} - 3|x|^{3/2} + |x|^{1/2}$. Indicate all zeros, critical points, inflection points, points of discontinuity, regions where $f(x)$ is increasing/decreasing, and regions where $f(x)$ is concave up/down.

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Problem 7. Compute the following antiderivatives:

$$\text{a)} \int \sin(x^x) x^x (1 + \ln x)\, dx$$

$$\text{b)} \int \frac{\arctan(3x)}{(1 + 9x^2)\sqrt{1 + [\arctan(3x)]^2}}\, dx$$

Problem 8. Consider the function $f(x) = (1 + x)^\alpha [1 + \ln(1 + \beta x)]$, where α and β are constants. Find the constants α and β that make the graph of $f(x)$ "as flat as possible" near $x = 0$. The choice $\beta = 0$ is forbidden.

Solutions

Problem 1. pg. 160 problem 85

Solution: If the woman runs the distance L along the $x-$axis, then she must swim the distance $\sqrt{b^2 + (L-a)^2}$. The total time she spends to reach the point (a, b) is

$$T = \frac{L}{r} + \frac{\sqrt{b^2 + (L-a)^2}}{s}.$$

The range of L values under consideration is $0 \leq L$.

To find the critical points of T, we first compute

$$\frac{dT}{dL} = \frac{1}{r} + [b^2 + (L-a)^2]^{-1/2} \frac{(L-a)}{s}.$$

Setting $\frac{dT}{dL} = 0$, we solve for the critical point $L_{critical}$ as follows:

$$L_{critical} = a - \frac{b}{\sqrt{\frac{r^2}{s^2} - 1}}.$$

As long as the above formula leads to $L_{critical} > 0$, it is straightforward to verify that $\frac{dT}{dL} > 0$ when $L > L_{critical}$ and $\frac{dT}{dL} < 0$ when $L < L_{critical}$. Thus, as long as $L_{critical} > 0$, $L_{critical}$ is in fact the minimum value.

Problem 2. Show that for any two numbers a and b, $|\sin a - \sin b| \leq |a - b|$.

Solution: Let $f(x) = \sin x$. By the mean value theorem, there exists a point c in between a and b such that $|\sin a - \sin b| = |f'(c)||b - a| = |\cos c||b - a| \leq |b - a|$.

Problem 3. Section 4.5: 8.

Solution: Assume that the boy is standing at the origin in the x, y plane and that the kite is at the location (x, y). Let D denote the length of the string. By the pythagorean theorem, we have

$$D^2 = x^2 + y^2.$$

Using the chain rule, we differentiate each side of the equation with respect to t to deduce that

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

We are told that

$$y = 80,$$
$$D = 100,$$

from which it follows that $x = 60$. We are also told that

$$\frac{dx}{dt} = 20,$$
$$\frac{dy}{dt} = 0.$$

Plugging these numbers into the above equation, we deduce that

$$\begin{aligned}\frac{dD}{dt} &= \frac{x}{D}\frac{dx}{dt} + \frac{y}{D}\frac{dy}{dt} \\ &= \frac{60}{100} \times 20 + 0 \\ &= 12.\end{aligned}$$

Problem 4. Use Newton's method to estimate the zero of $f(x) = x^3 + 5x - 7$. Start with the base point $x_0 = 1$ and compute x_1, x_2.

Solution: Newton's iterate formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Since $f'(x) = 3x^2 + 5$, we have

$$x_{k+1} = x_k - \frac{x_k^3 + 5x_k - 7}{3x_k^2 + 5}.$$

We then set $x_0 = 1$ and compute

$$\begin{aligned}x_1 &= 1 - \frac{-1}{8} = \frac{9}{8}, \\ x_2 &= \frac{9}{8} - \frac{\frac{9^3}{8^3} + 5\frac{9}{8} - 7}{3\frac{9^2}{8^2} + 5} \\ &= \frac{9}{8} - \frac{9^3 + 45 \times 8^2 - 7 \times 8^3}{3 \times 9^2 \times 8 + 5 \times 8^3} \\ &= \frac{9}{8} - \frac{25}{4504}.\end{aligned}$$

Problem 5. Graph the function $f(x) = |x|^{5/2} - 3|x|^{3/2} + |x|^{1/2}$. Indicate all zeros, critical points, inflection points, points of discontinuity, regions where $f(x)$ is increasing/decreasing, and regions where $f(x)$ is concave up/down.

Solution: The function is even, so we only need to consider $x \geq 0$. We first note that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to 0$ as $x \to 0^+$.

We then compute that for $x > 0$, we have

$$\begin{aligned}f'(x) &= \frac{1}{2}\frac{(5x^2 - 9x + 1)}{\sqrt{x}}, \\ f''(x) &= \frac{1}{4}\frac{(15x^2 - 9x - 1)}{x^{3/2}}.\end{aligned}$$

To find the critical points in the region $x > 0$, we set $f'(x) = 0$ and solve via the quadratic formula:

$$x_{critical\pm} = \frac{9 \pm \sqrt{61}}{10}.$$

Note that both of these numbers are positive. In between 0 and $x_{critical-}$, $f' > 0$ and so f is increasing. In between $x_{critical-}$ and $x_{critical+}$, $f' < 0$ and so f is decreasing. In between $x_{critical}$ and ∞, $f' > 0$ and so f is increasing. Also, $f'(x)$ becomes infinite as $x \to 0^+$.

To find the inflection points, we set $f''(x) = 0$ and solve via the quadratic formula:

$$x_{inflection} = \frac{9 + \sqrt{141}}{30}.$$

Note that we have discarded the other root since it is not positive. In between 0 and $x_{inflection}$, $f'' < 0$ and so f is concave down. In between $x_{inflection}$ and ∞, $f'' > 0$ and so f is concave up.

The full graph is given in the figure below.

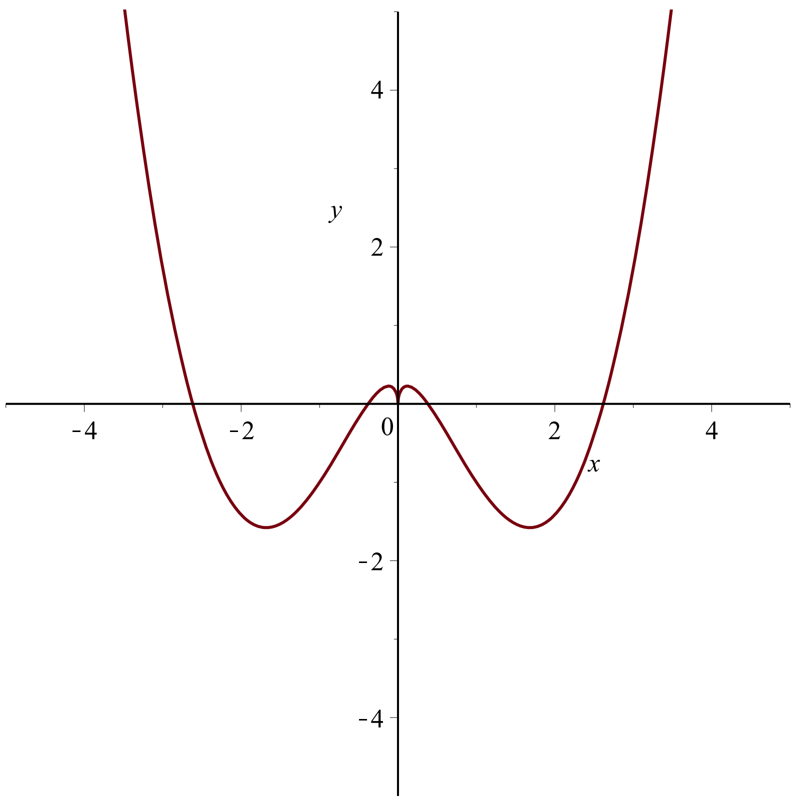

FIGURE 1. Graph of $f(x)$

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Solution: We will use the hint in the book. In particular, since the length of the base and the area are given, this implies that the height $h = 2\text{area}/(\text{length of base})$ is fixed. Suppose that the vertex has coordinates (x, h). Without loss of generality, we can assume that $x \geq 0$ (otherwise, we just flip the triangle about the y axis). Assume that the two vertices of the base are at $(-a, 0)$ and $(a, 0)$, where a is a constant. Then by the the pythagorean theorem, the lengths of the other two sides are

$$\ell_1 = \sqrt{(x+a)^2 + h^2},$$
$$\ell_2 = \sqrt{(x-a)^2 + h^2}.$$

We therefore want to minimize the function

$$f(x) = \ell_1 + \ell_2 = \sqrt{(x+a)^2 + h^2} + \sqrt{(x-a)^2 + h^2}$$

over the region $x \geq 0$. Clearly $f(x) \to \infty$ as $x \to \infty$, so the minimizer will be not "lie at $x = \infty$." To locate the critical points of $f(x)$, we first compute

$$f'(x) = \frac{x+a}{\sqrt{(x+a)^2 + h^2}} + \frac{x-a}{\sqrt{(x-a)^2 + h^2}}.$$

We then set $f'(x) = 0$ to deduce the equation

$$\frac{x+a}{\sqrt{(x+a)^2 + h^2}} = -\frac{x-a}{\sqrt{(x-a)^2 + h^2}}.$$

Squaring the equation to make life easier, we deduce

$$\frac{(x+a)^2}{(x+a)^2 + h^2} = \frac{(x-a)^2}{(x-a)^2 + h^2},$$

which is equivalent to

$$\frac{1}{1 + \frac{h^2}{(x+a)^2}} = \frac{1}{1 + \frac{h^2}{(x-a)^2}}.$$

We then see that

$$(x+a)^2 = (x-a)^2.$$

The above equation has only the solution $x = 0$. Thus, the only critical point is also an endpoint. Therefore, $x = 0$ must be the minimum. Since $x = 0$ implies that the triangle is isosceles, we have proved the desired result.

Problem 7. Compute the following antiderivatives:

$$\text{a)} \int \sin(x^x) x^x (1 + \ln x)\, dx$$
$$\text{b)} \int \frac{\arctan(3x)}{(1 + 9x^2)\sqrt{1 + [\arctan(3x)]^2}}\, dx$$

Solution: a) We set $u = x^x$. This implies (by logarithmic differentiation) that $du = x^x(1+\ln x)dx$. After these substitutions, the integral becomes

$$\int \sin u\, du = -\cos u + c = -\cos(x^x) + c.$$

b) We first make the substitution $u = \arctan(3x)$, $du = 3(1+9x^2)^{-1}dx$, which leads to the integral

$$\frac{1}{3} \int \frac{u}{\sqrt{1+u^2}}\, du.$$

We then make the second substitution $v = u^2$, $dv = 2udu$, and the integral becomes

$$\begin{aligned}\frac{1}{6}\int \frac{dv}{\sqrt{1+v}}\,dv &= \frac{1}{6}\int (1+v)^{-1/2} \\ &= \frac{1}{3}(1+v)^{1/2} + c = \frac{1}{3}(1+u^2)^{1/2} + c \\ &= \frac{1}{3}(1 + [\arctan(3x)]^2)^{1/2} + c.\end{aligned}$$

Problem 8. Consider the function $f(x) = (1+x)^{\alpha}\left[1 + \ln(1+\beta x)\right]$, where α and β are constants. Find the constants α and β that make the graph of $f(x)$ "as flat as possible" near $x = 0$. The choice $(\alpha, \beta) = (0, 0)$ is forbidden.

Solution: We first compute the quadratic approximation to $f(x)$:

$$\begin{aligned}f(x) &= \left(1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2} + O(x^3)\right)\left(1 + \beta x - \frac{\beta^2 x^2}{2} + O(x^3)\right) \\ &= 1 + (\alpha+\beta)x + \left(\alpha\beta + \frac{\alpha(\alpha-1)}{2} - \frac{\beta^2}{2}\right)x^2 + O(x^3).\end{aligned}$$

To make the graph of $f(x)$ as flat as possible, we set the coefficients of x and x^2 equal to 0 :

$$\alpha + \beta = 0,$$

$$\alpha\beta + \frac{\alpha(\alpha-1)}{2} - \frac{\beta^2}{2} = 0.$$

The first equation implies that $\alpha = -\beta$. Inserting this information into the second equation, we deduce

$$-\alpha^2 - \frac{1}{2}\alpha = 0.$$

This equation has the forbidden solution $\alpha = 0$ (forbidden because it leads to $\beta = 0$) and also the solution $\alpha = -1/2$. Thus,

$$(\alpha, \beta) = (-1/2, 1/2),$$

and $f(x) = (1+x)^{-1/2}\left[1 + \ln(1 + \frac{1}{2}x)\right]$.