• First Fundamental Theorem of Calculus

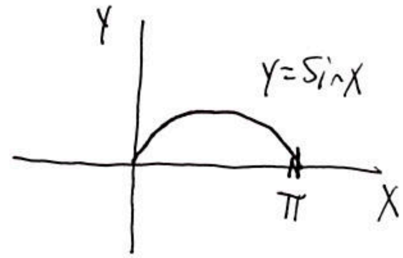
FTC I: If $f(x)$ is continuous and $F'(x) = f(x)$,

then $\int_a^b f(x)\,dx = F(b) - F(a)$

• Notation: $F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$

Ex $F(x) = \frac{x^3}{3}$ $F'(x) = x^2$ $\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$

Ex: Area under one hump of $\sin x$



$$\int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos\pi - (-\cos 0)$$
$$= -(-1) - (-1) = 2.$$

Ex $\displaystyle\int_0^1 x^5 dx = \frac{x^6}{6}\Big|_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$

- Intuitive interpretation of FTC I
  - $x(t)$ is a position
  - $v(t) = x'(t) = \frac{dx}{dt}$ = speed or rate of change of x

FTC I says:

$$\int_a^b v(t)\,dt = x(b) - x(a)$$

- The RHS is how far $x(t)$ went from time $t=a$ to time $t=b$ (difference between two odometer readings)
- The LHS is connected to speedometer readings.

$\displaystyle\sum_{i=1}^{n} v(t_i)\Delta t$ approximates the sum of distances traveled over times Δt

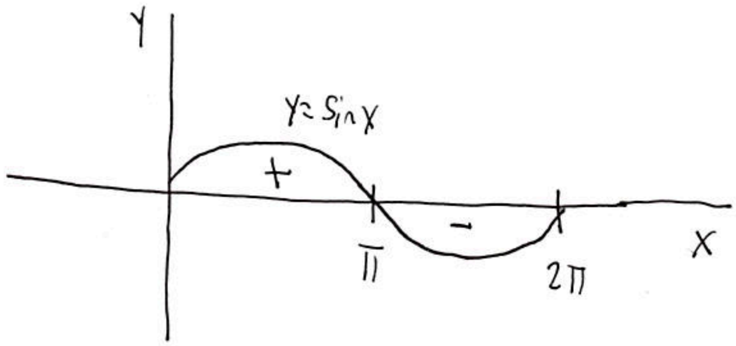
$\downarrow$ converges to

$\displaystyle\int_a^b v(t)dt$ as $\Delta t \to 0$

- If v changes sign, then the interpretation of $x(t)$ as an odometer reading is no longer valid. For example, imagine a round trip so that $x(b) - x(a) = 0$. Then the positive and negative velocities cancel each other, whereas an odometer would measure the total distance traveled.

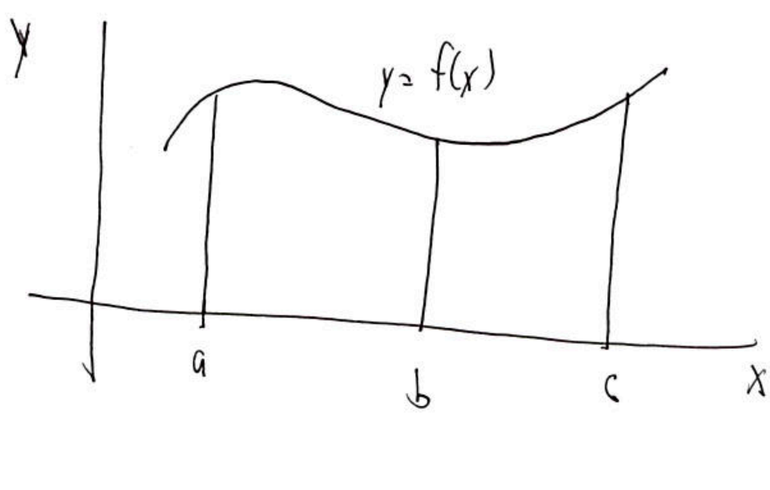
Ex: $\displaystyle\int_0^{2\pi} \sin x\,dx = -\cos x \Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = 0$

The integral represents the sum of the areas under the curve, above the x-axis, minus the areas below the x-axis



- Integrals have an important additive property:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



- New definition $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

- This definition renders the FTC1 valid no matter if $a<b$ or $b<a$.
- Also, the additive property works for a, b, c in any order

- Inequalities involving integrals:

If $a<b$ and $f(x) \leq g(x)$ when $a \leq x \leq b$, then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Ex: Estimate of e^x

- Since $1 \le e^x$ for $x \ge 0$, we have

$$1 = \int_0^1 1\,dx \le \int_0^1 e^x\,dx = e^x\Big|_0^1 = e-1.$$

Therefore, $e \ge 2$.

Ex. In the Mean Value Theorem lecture, we showed that

$$1+x \le e^x.$$ Therefore,

$$\int_0^1 1+x\,dx \le \int_0^1 e^x\,dx = e-1$$

- Now $\displaystyle\int_0^1 1+x\,dx = x + \frac{x^2}{2}\Big|_0^1 = \frac{3}{2}.$

Therefore, $e \ge \dfrac{5}{2}$

- <u>Change of Variable</u>:

- If $f(x) = g(u(x))$, then we write $du = u'(x)\,dx$

$$\int g(u)\,du = \int g(u(x))\,u'(x)\,dx = \int f(x)\,u'(x)\,dx \quad \text{(indefinite integrals)}$$

- For definite integrals:

$$\int_{x_1}^{x_2} f(x)\,u'(x)\,dx = \int_{u_1}^{u_2} g(u)\,du, \quad \text{where } u_1 = u(x_1),\ u_2 = u(x_2)$$

<u>Ex</u>: $\int_1^2 x^2(x^3+2)^4\,dx$

- Let $u = x^3+2$
- Then $du = 3x^2\,dx \Rightarrow x^2\,dx = \frac{du}{3}$
- $x_1 = 1 \Rightarrow u_1 = 1^3+2 = 3$
- $x_2 = 2 \Rightarrow u_2 = 2^3+2 = 10$

$$\int_1^2 \overbrace{x^2 \underbrace{(x^3+2)}_{u}{}^4\,dx}^{\frac{du}{3}} = \int_{u=3}^{u=10} u^4 \frac{du}{3} = \left.\frac{u^5}{15}\right|_3^{10}$$

$$= \frac{10^5 - 3^5}{15}$$