MIDTERM 2 - 18.01 - FALL 2017.

Name:

Email:

Please put a check by your recitation section.

	Instructor	**Time**
☐	Miles Couchman	MW 1
☐	Kristin Kurianski	MW 1
☐	Yu Pan	MW 10
☐	Yu Pan	MW 11
☐	Jiewon Park	MW 12
☐	Jake Wellens	MW 12
☐	Siddharth Venkatesh	MW 2

Problem #	Max points possible	Actual score
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**

Good luck!

Problem 1. (15 points). Let

$$f(x) = (1+x)^{1/2}\sin x$$

Find the linear approximation of $f(x)$ near the base point $x_0 = 0$.

Solution: The linear approximation of $\sin x$ (near the base point $x_0 = 0$) is x. The linear approximation of $(1+x)^{1/2}$ is $1 + (1/2)x$. Thus, multiplying, we find that the linear approximation to $(1+x)^{1/2}\sin x$ is x.

Problem 2. (5 + 10 = 15 points).

a) Give a precise statement of the mean value theorem.

b) Use the mean value theorem to show that $\sqrt{1+x} < 1 + \frac{x}{2}$ whenever $x > 0$. To receive credit, your answer must explicitly invoke the mean value theorem.

Hint: Study the function $f(x) = 1 + \frac{x}{2} - \sqrt{1+x}$ *and its derivative.*

Solution: a) If $f(x)$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$, then there exists a number c with $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

b) The desired result is equivalent to showing that $f(x) > 0$ for $x > 0$. To proceed, we compute that $f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} = \frac{\sqrt{1+x}-1}{\sqrt{1+x}}$. Note that $f'(x) > 0$ for $x > 0$. Since $f(0) = 0$, we can use the mean value theorem to deduce that when $x > 0$, $f(x) = f'(c)(x - 0) = xf'(c)$ for some c in between 0 and x. The above discussion implies that $f'(c) > 0$. Thus, $f(x) > 0$ when $x > 0$, which is the desired result.

Problem 3. (10 + 10 = 20 points). Compute the following two antiderivatives:

a) $\int e^{\sin x} \cos x \, dx$

b) $\int \frac{x^2}{(1+x^3)^{3/2}} \, dx$

Solution: a) We set $u = \sin x$, $du = \cos x \, dx$, and compute that

$$\begin{aligned}
&\int e^{\sin x} \cos x \, dx \\
&= \int e^u \, du \\
&= e^u + c \\
&= e^{\sin x} + c.
\end{aligned}$$

b) We set $u = x^3$, $du = 3x^2 \, dx$, and compute that

$$\begin{aligned}
&\int \frac{x^2}{(1+x^3)^{3/2}} \, dx \\
&= \frac{1}{3} \int \frac{1}{(1+u)^{3/2}} \, du \\
&= -\frac{2}{3}(1+u)^{-1/2} + c \\
&= -\frac{2}{3}(1+x^3)^{-1/2} + c.
\end{aligned}$$

Problem 4. (15 points). Find numbers x and y such that *all* of the following three conditions hold:

(1) $x \geq 0$ and $y \geq 0$
(2) $x + y = 100$
(3) $x^3 + y^3$ is as large as possible, given (1) and (2)

Solution: We aim to maximize $x^3 + y^3 = x^3 + (100 - x)^3$. Hence, we set $f(x) = x^3 + (100 - x)^3$. Clearly the relevant domain of x values is $0 \leq x \leq 100$. We next compute that $f'(x) = 3x^2 - 3(100 - x)^2$. Setting $f'(x) = 0$ and solving for x, we deduce that the only critical point is $x = \frac{100^2}{200} = 50$. We now compute that $f(50) = 2 \cdot 50^3$. We now test the endpoints $x = 0$ and $x = 100$: $f(0) = f(100) = 100^3 = (2 \cdot 50)^3 = 8 \cdot 50^3$. Hence, the maximum occurs at the endpoints. That is, $(x, y) = (0, 100)$ (or $(100, 0)$).

Problem 5. (15 points). A cube-shaped balloon is being filled with air at the rate of 1 cubic meter per minute. Find the rate of increase of the balloon's outer surface area (in units of square meters per minute) when its volume is 2 cubic meters. You can assume that the balloon is cube-shaped for all time.

Solution: The volume of the cube is $V = S^3$, where S is the cube's side length. The surface area is $A = 6S^2 = 6V^{2/3}$. We let $V' := \frac{d}{dt}$, and similarly for A'. Then by the chain rule, we have $A' = 4V^{-1/3}V'$. We are told that $V' = 1$ (for all time), and we want to solve for A' at the moment that $V = 2$. Inserting these numbers into the above equation for A', we conclude that at the moment of interest, $A' = 4 \cdot (2^{-1/3}) \cdot 1 = \frac{4}{2^{1/3}} = 2^{5/3}$.

Problem 6. (20 points). Sketch the graph of the function

$$f(x) = \arctan(x^2)$$

Label any zeros of $f(x)$ by "Z", any critical points by "C", and any inflection points by "I". To receive full credit, you must clearly indicate the following (directly on your graph, or, if you prefer, in a separate table):

i) Any discontinuities or points of non-differentiability $f(x)$ might have.
ii) The limiting behavior of $f(x)$ as $x \to \pm\infty$.
iii) The regions on which $f(x)$ is positive and the regions on which $f(x)$ is negative.
iv) The regions on which $f(x)$ is increasing and the regions on which $f(x)$ is decreasing.
v) The regions on which $f(x)$ is concave up and the regions on which $f(x)$ is concave down.

Please be sure that you graph the correct function. If you accidentally graph a function other than the function $f(x)$ written above, then we can only award a small amount of credit at most.

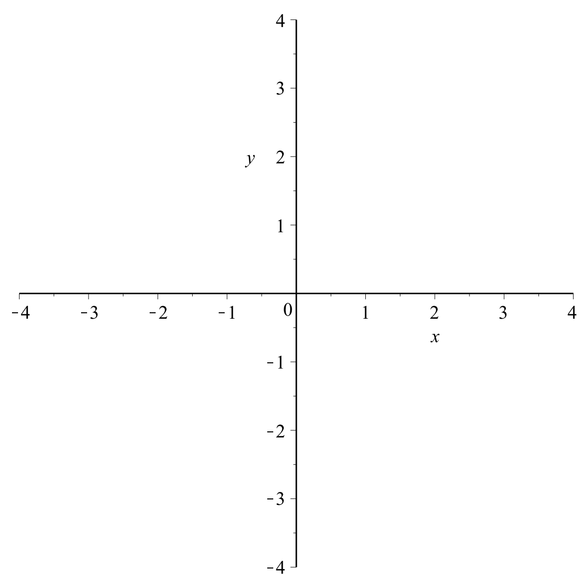
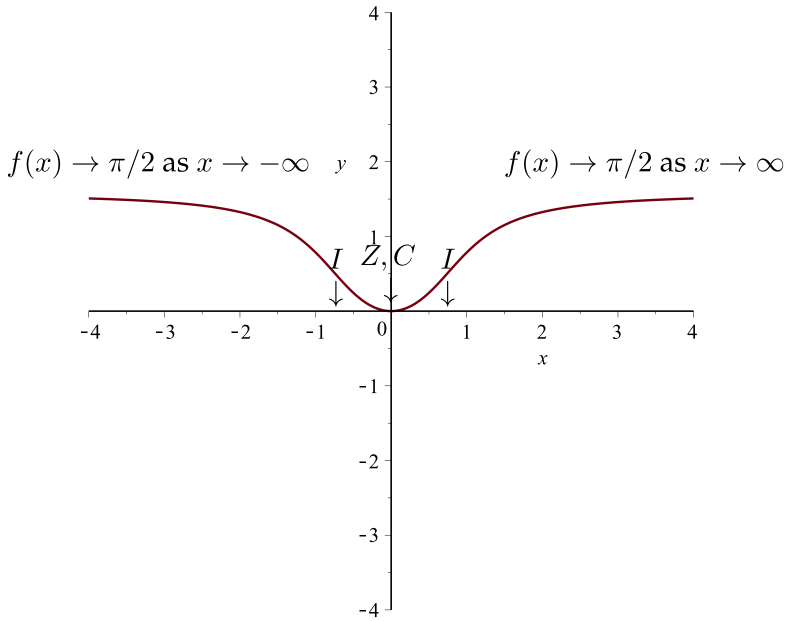


FIGURE 1. Draw your graph of $f(x)$ here

Solution: The graph of $f(x)$ is given in the figure below.



To justify the graph shown, we first note f has no discontinuities or points of non-differentiability. Next, we note that $\lim_{x\to\pm\infty} f(x) = \frac{\pi}{2}$, since $\lim_{x\to\pi/2^-} \tan x = \infty$.

We now compute the first and second derivatives of $f(x)$:

$$f'(x) = \frac{2x}{1+x^4},$$
$$f''(x) = \frac{2}{1+x^4} - \frac{8x^4}{(1+x^4)^2} = 2\frac{1-3x^4}{(1+x^4)^2}.$$

The above formulas imply that $x = 0$ is the only critical point of f (i.e., point where $f' = 0$) and $x = \pm\sqrt[4]{\frac{1}{3}}$ are the two inflection points (i.e., points where $f'' = 0$). Note that $x = 0$ is also the only zero of f.

To indicate some of the other features of $f(x)$, we make the following table:

x	$f(x)$	$f'(x)$	$f''(x)$
$x < -\sqrt[4]{\frac{1}{3}}$	$+$	$-$ (decreasing)	$-$ (concave down)
$-\sqrt[4]{\frac{1}{3}} < x < 0$	$+$	$-$ (decreasing)	$+$ (concave up)
$0 < x < \sqrt[4]{\frac{1}{3}}$	$+$	$+$ (increasing)	$+$ (concave up)
$\sqrt[4]{\frac{1}{3}} < x$	$+$	$+$ (increasing)	$-$ (concave down)