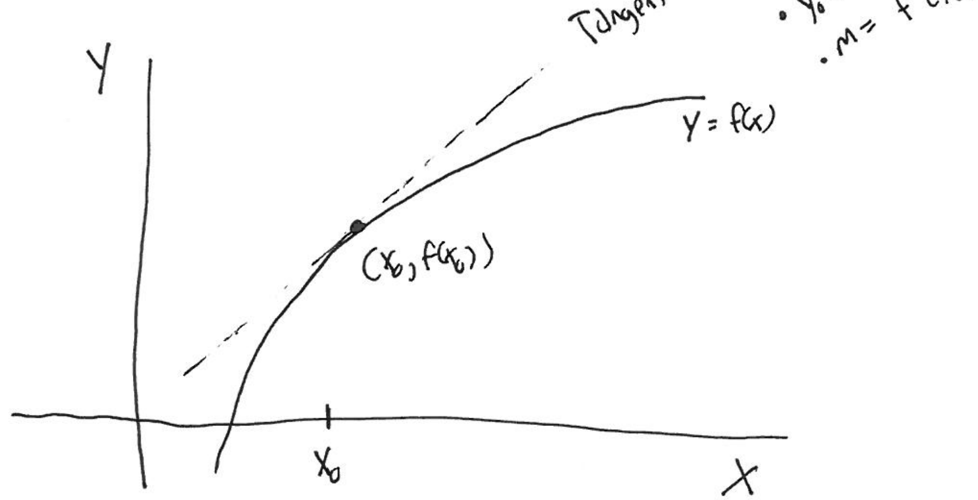
Linear + Quadratic Approximations

- Linear approximation



- The tangent line approximates $f(x)$.

  It gives a good approximation near x_0:

  $$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)} \quad (\text{when } x \approx x_0)$$

- The approximation might be very bad when x is not near x_0.
- Alternate notation: $f(x) = f(x_0) + f'(x_0)\Delta x + \underbrace{O((\Delta x)^2)}_{\text{quadratic error terms}}$, $\Delta x = x - x_0$
- Ex: $f(x) = \ln x$, $x_0 = 1$ (basepoint)

  $f'(x) = \frac{1}{x}$ $\qquad f(1) = 0$, $f'(1) = 1$.

  $\ln x \approx f(1) + f'(1)(x-1) = 0 + 1\cdot(x-1) = x - 1$ when x is near 1.

- Building block list of <u>linear</u> approximations:
  (we assume $x_0 = 0$ is the basepoint and $|x| \ll 1$).

① $\sin(x) \approx x$ when $x \approx 0$

② $\cos(x) \approx 1$ when $x \approx 0$

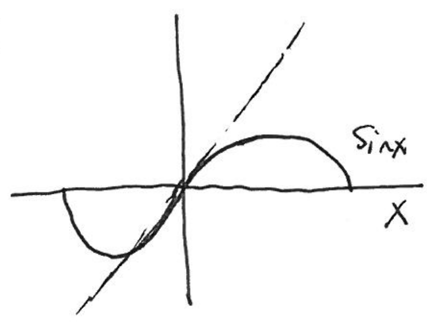
③ $e^x \approx 1 + x$ when $x \approx 0$

④ $\ln(1+x) \approx x$ when $x \approx 0$

⑤ $(1+x)^r \approx 1 + rx$ when $x \approx 0$.

- You should learn how to quickly derive these approximations.

Proof of ①:



If $f(x) = \sin x$, then $f'(x) = \cos x$.

$f(0) = 0$, $f'(0) = 1$

Therefore, $\sin x \approx 0 + 1 \cdot (x - 0) = x$ when $x \approx 0$.

The proofs of ②-⑤ are similar. We already proved ④.

Proof of ⑤: $f(x) = (1+x)^r$ $\quad f'(x) = r(1+x)^{r-1}$

$f(0) = 1$ $\quad f'(0) = r$

Therefore, $f(x) = (1+x)^r \approx 1 + r(x-0) = 1 + rx$

when $x \approx 0$.

Ex: Find the linear approximation of $f(x) = \frac{e^{-2x}}{\sqrt{1+x}}$ near $x_0 = 0$

- We can use the building blocks to give a short solution (without calculating $f'(x)$):

- $e^{-2x} \approx 1 + (-2x) = 1-2x$
- $\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \approx 1 - \frac{1}{2}x.$
- $\frac{e^{-2x}}{\sqrt{1+x}} \approx (1-2x)(1-\frac{1}{2}x) \approx 1 - \frac{5}{2}x$ when $x \approx 0$.
- Note that we have ignored all x^2, x^3, etc. terms. When $x \approx 0$, these terms are very small compared to $1-\frac{5}{2}x$.
- Note that $f(x) \approx 1 - \frac{5}{2}x$ means that $f'(0) = -\frac{5}{2}$ (we didn't even have to compute a formula for $f'(x)$!)

Ex: Compute $\lim_{x \to 0} \frac{(1+2x)^{10}-1}{x}$. Use $(1+2x)^{10} \approx 1 + (10)(2x)$
$= 1 + 20x$

- $\lim_{x \to 0} \frac{(1+2x)^{10}-1}{x} = \lim_{x \to 0} \frac{\cancel{1}+20x-\cancel{1}}{x} = 20.$

Quadratic Approximations

- Often times linear approximations are not accurate enough.
- Here is the basic formula for quadratic approximations:

$$\boxed{\begin{aligned} &f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \\ &\text{when } x \approx x_0 \end{aligned}}$$

- Note: If $f(x) = Ax^2 + Bx + C_1$, then

$$f'(x) = 2Ax + B$$
$$f''(x) = 2A.$$

Thus, for $x_0 = 0$, $f(0) = C_1$, $f'(0) = B$, $f''(0) = 2A$, and the quadratic approximation to $f(x)$ is $f(x) \approx C_1 + B \cdot x + \frac{1}{2} \cdot 2A \cdot x^2$

$$= Ax^2 + Bx + C_1.$$

- This explains the "$\frac{1}{2}$" in the formula and shows that the quadratic approximation is <u>exact</u> when f is a degree 2 polynomial.

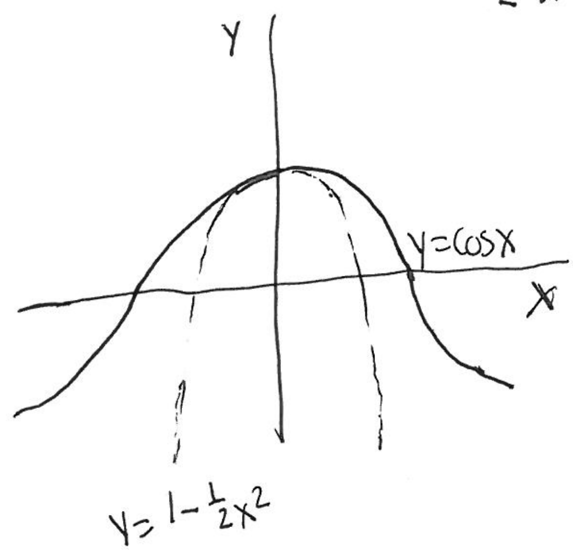
L7.5

$x_0 = 0$

Ex: $f(x) = \cos x \quad f(0) = 1$

$f'(x) = -\sin x \quad f'(0) = 0$

$f''(x) = -\cos x \quad f''(0) = -1$



$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$= 1 + 0\cdot x + \tfrac{1}{2}\cdot(-1)\cdot x^2$$

$$= 1 - \tfrac{1}{2}x^2 \quad \text{when } x \approx 0.$$

Building block quadratic approximations

for $x_0 = 0$, $x \approx 0$

1. $\sin(x) \approx x$ for $x \approx 0$
2. $\cos(x) \approx 1 - \frac{1}{2}x^2$ for $x \approx 0$
3. $e^x \approx 1 + x + \frac{1}{2}x^2$ for $x \approx 0$
4. $\ln(1+x) \approx x - \frac{1}{2}x^2$ for $x \approx 0$
5. $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$ for $x \approx 0$

Proofs: Are not that interesting
Just compute $f(0)$, $f'(0)$, $f''(0)$.

Ex: Find the quadratic approximation to $f(x) = \dfrac{e^{-2x}}{\sqrt{1+x}}$ near $x_0 = 0$

- We can use the quadratic approximation building blocks to give a relatively short answer:

- $e^{-2x} \approx 1 + (-2x) + \frac{1}{2}(-2x)^2 = 1 - 2x + 2x^2$
- $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \approx 1 + (-\frac{1}{2})x + \dfrac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2} x^2$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2.$$

- $f(x) \approx (1 - 2x + 2x^2)(1 - \frac{1}{2}x + \frac{3}{8}x^2)$

$$\approx 1 - \frac{5}{2}x + \frac{27}{8}x^2 \quad \text{when } x \approx 0.$$

- We have ignored all cubic & higher order terms since we are "expanding only to quadratic order".