18.01 - Problem Set #8, Part II Solutions

Problem 1

The area of the segment is,

$$A = 2\int_b^a \sqrt{a^2 - y^2}dy$$

Using the substitution $y = a\sin(u)\,,\ dy = a\cos(u)\,du$ gives,

$$\begin{aligned}
A &= 2\int_{\sin^{-1}\left(\frac{b}{a}\right)}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2(u)}a\cos(u)\,du \\
&= 2a^2\int_{\sin^{-1}\left(\frac{b}{a}\right)}^{\frac{\pi}{2}} \cos^2(u)\,du \\
&= a^2\int_{\sin^{-1}\left(\frac{b}{a}\right)}^{\frac{\pi}{2}} (1 + \cos(2u))\,du \\
&= a^2\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{a}\right) + \left.\frac{\sin(2u)}{2}\right|_{\sin^{-1}\left(\frac{b}{a}\right)}^{\frac{\pi}{2}}\right) \\
&= a^2\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{a}\right) - \frac{\sin\left(2\sin^{-1}\left(\frac{b}{a}\right)\right)}{2}\right) \\
&= a^2\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{a}\right) - \sin\left(\sin^{-1}\left(\frac{b}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{b}{a}\right)\right)\right) \\
&= a^2\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{a}\right) - \frac{b}{a}\sqrt{1 - \frac{b^2}{a^2}}\right) \\
&= a^2\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{b}{a}\right) - \frac{b}{a^2}\sqrt{a^2 - b^2}\right)
\end{aligned}$$

When $b = 0$, $A = \frac{\pi a^2}{2}$ which is half the area of a circle as expected. When $b = a$, $A = 0$ as expected.

Problem 2

a) First,

$$\sec(x) = \frac{1}{\cos(x)} = \frac{\cos(x)}{\cos^2(x)} = \frac{\cos(x)}{1 - \sin^2(x)}$$

Then,

$$\begin{aligned}
\int \sec(x)\,dx &= \int \frac{\cos(x)}{1 - \sin^2(x)}dx \\
&= \int \frac{\cos(x)}{(1 + \sin(x))(1 - \sin(x))}dx \\
&= \int \left(\frac{\cos(x)}{2(1 + \sin(x))} + \frac{\cos(x)}{2(1 - \sin(x))}\right)dx
\end{aligned}$$

Making the substitution $u = \sin(x)$ gives,

$$\begin{aligned}
\int \sec(x)\, dx &= \frac{1}{2} \int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du \\
&= \frac{1}{2} \left(\ln|1+u| - \ln|1-u| \right) + C \\
&= \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C \\
&= \ln \left(\sqrt{\left| \frac{1+\sin(x)}{1-\sin(x)} \right|} \right) + C
\end{aligned}$$

b)

$$\begin{aligned}
\ln \left(\sqrt{\left| \frac{1+\sin(x)}{1-\sin(x)} \right|} \right) &= \ln \left(\sqrt{\left| \frac{(1+\sin(x))^2}{\cos^2(x)} \right|} \right) \\
&= \ln \left(\sqrt{\left| \frac{(1+\sin(x))^2}{\cos^2(x)} \right|} \right) \\
&= \ln \left(\left| \frac{1+\sin(x)}{\cos(x)} \right| \right)
\end{aligned}$$

So,

$$\int \sec(x)\, dx = \ln\left(|\sec(x) + \tan(x)| \right) + C$$

Problem 3

The first positive root of $y = e^x \cos(x)$ is $x = \frac{\pi}{2}$. The volume of $\mathcal{S}$ is,

$$V = \int_0^{\frac{\pi}{2}} 2\pi x e^x \cos(x)\, dx$$

Before proceeding, we compute two integrals which we will need later. First, we compute $I_1 = \int e^x \cos(x)\, dx$ by using $u = \cos(x)$, $du = -\sin(x)\, dx$, $v = e^x$, $dv = e^x dx$ to integrate by parts which gives,

$$I_1 = \cos(x)\, e^x + \int e^x \sin(x)\, dx$$

Using $u = \sin(x)$, $du = \cos(x)\, dx$, $v = e^x$, $dv = e^x dx$ to integrate by parts again gives,

$$\begin{aligned}
I_1 &= \cos(x)\, e^x + \sin(x)\, e^x - I_1 + C \\
I_1 &= \frac{e^x}{2} \left(\sin(x) + \cos(x) \right) + C
\end{aligned}$$

In exactly the same manner, it can be shown that,

$$I_2 = \int e^x \sin(x)\, dx = \frac{e^x}{2} \left(\sin(x) - \cos(x) \right) + C$$

To compute V, we use $u = x$, $du = dx$, $v = \frac{e^x}{2} \left(\sin(x) + \cos(x) \right)$, $dv = e^x \cos(x)\, dx$ to integrate by parts,

$$
\begin{aligned}
V &= 2\pi\left[\frac{xe^x}{2}\left(\sin(x)+\cos(x)\right)-\int\frac{e^x}{2}\left(\sin(x)+\cos(x)\right)\right]_0^{\frac{\pi}{2}}\\
&= 2\pi\left[\frac{xe^x}{2}\left(\sin(x)+\cos(x)\right)-\frac{1}{2}\left(\frac{e^x}{2}\left(\sin(x)-\cos(x)\right)+\frac{e^x}{2}\left(\sin(x)+\cos(x)\right)\right)\right]_0^{\frac{\pi}{2}}\\
&= \pi e^{\frac{\pi}{2}}\left(\frac{\pi}{2}-1\right)
\end{aligned}
$$

Problem 4

It is true that we can use integration by parts to obtain,

$$\int e^x \sinh(x)\,dx = e^x\left(\cosh(x)-\sinh(x)\right)+\int e^x\sinh(x)\,dx$$

However, anti-derivatives are only unique up to an additive constant. Therefore, if $F(x)$ is an anti-derivative of $e^x\sinh(x)$, we have,

$$
\begin{aligned}
F(x)+C_1 &= e^x\left(\cosh(x)-\sinh(x)\right)+F(x)+C_2\\
C_1-C_2 &= e^x\left(\cosh(x)-\sinh(x)\right)
\end{aligned}
$$

We cannot assume that C_1 is equal to C_2 and so we cannot conclude that $e^x\left(\cosh(x)-\sinh(x)\right)=0$.

For interest, observe that,

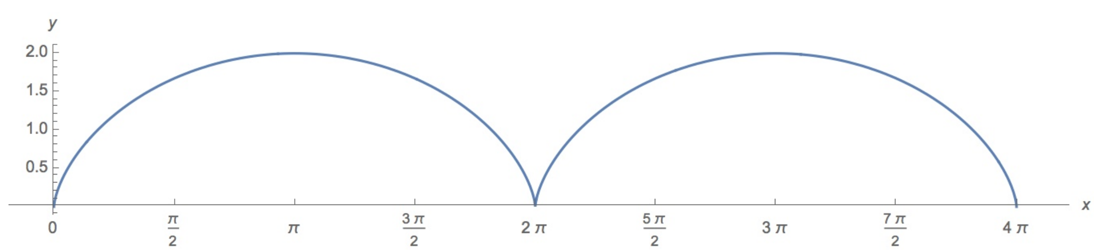
$$
\begin{aligned}
e^x\left(\cosh(x)-\sinh(x)\right) &= e^x\left[\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}\right]\\
&= e^xe^{-x}\\
&= 1
\end{aligned}
$$

Problem 5

a) We have,

$$
\begin{aligned}
dx &= \left(1-\cos(t)\right)dt\\
dy &= \sin(t)\,dt\\
\frac{dy}{dx} &= \frac{\sin(t)}{1-\cos(t)}
\end{aligned}
$$

We see that $\frac{dy}{dx}$ will be infinite when $\cos(t)=1$ which occurs at $t=2\pi n$, where n is an integer. This corresponds to the points $(x,y)=(2\pi n,0)$. A sketch of the cycloid is shown below:



b) The desired arc-length is,

$$
\begin{aligned}
s &= \int_0^{2\pi} \sqrt{dx^2 + dy^2} \\
&= \int_0^{2\pi} \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos(t)}dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}dt \\
&= 2\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt \\
&= -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} \\
&= 8
\end{aligned}
$$

c) The desired surface area is,

$$
\begin{aligned}
A &= \int_0^{2\pi} 2\pi y ds \\
&= \int_0^{2\pi} 2\pi\left(1 - \cos(t)\right)\sqrt{2}\sqrt{1 - \cos(t)}dt \\
&= 2\sqrt{2}\pi\int_0^{2\pi} \left(1 - \cos(t)\right)^{\frac{3}{2}} dt \\
&= 2\sqrt{2}\pi\int_0^{2\pi} \left(2\sin^2\left(\frac{t}{2}\right)\right)^{\frac{3}{2}} dt \\
&= 8\pi\int_0^{2\pi} \sin^3\left(\frac{t}{2}\right) dt \\
&= 8\pi\int_0^{2\pi} \left(1 - \cos^2\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right) dt
\end{aligned}
$$

Using the substitution $u = \cos\left(\frac{t}{2}\right)$, $du = -\frac{1}{2}\sin\left(\frac{t}{2}\right) dt$ gives,

$$
\begin{aligned}
A &= 16\pi\int_{-1}^{1} \left(1 - u^2\right) du \\
&= 32\pi\int_0^{1} \left(1 - u^2\right) du \\
&= \frac{64\pi}{3}
\end{aligned}
$$

Problem 6

Using FTC2, we have,

$$
\begin{aligned}
dx &= \cos(t)\ln(t)\,dt \\
dy &= \sin(t)\ln(t)\,dt \\
\frac{dy}{dx} &= \tan(t), \quad t > 1
\end{aligned}
$$

and so the first non-origin point where the curve is vertical occurs at $t = \frac{\pi}{2}$. The desired arc-length is then,

$$
\begin{aligned}
s &= \int_1^{\frac{\pi}{2}} \sqrt{dx^2+dy^2} \\
&= \int_1^{\frac{\pi}{2}} \sqrt{\ln^2(t)}dt \\
&= \int_1^{\frac{\pi}{2}} \ln(t)\,dt
\end{aligned}
$$

Using $u=\ln(t)$, $du=\frac{dt}{t}$, $v=t$, $dv=dt$ to integrate by parts gives,

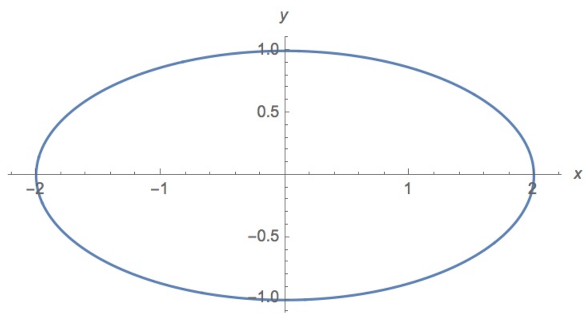
$$
\begin{aligned}
s &= t\ln(t)|_1^{\frac{\pi}{2}} - \int_1^{\frac{\pi}{2}} dt \\
&= \frac{\pi}{2}\ln\left(\frac{\pi}{2}\right) - \frac{\pi}{2} + 1 \\
&= \frac{\pi}{2}\left(\ln\left(\frac{\pi}{2}\right) - 1\right) + 1
\end{aligned}
$$

Problem 7

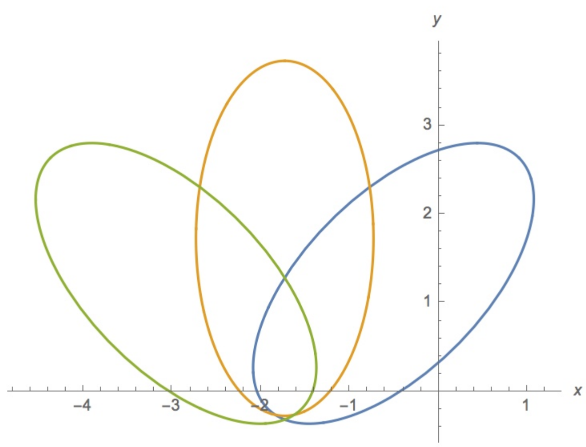
a) First, we find an algebraic equation for the curve in terms of x and y:

$$
\begin{aligned}
\frac{x^2}{4} + y^2 &= \cos^2(t) + \sin^2(t) \\
\frac{x^2}{4} + y^2 &= 1
\end{aligned}
$$

A plot of the curve is shown below. The curve is traced out in a counter-clockwise direction.



b) The rotated ellipses are shown below for $\theta=\frac{\pi}{4}$ (blue), $\theta=\frac{\pi}{2}$ (orange), and $\theta=\frac{3\pi}{4}$ (green).



Following the hint, $R_\theta\left(\tilde{x},\tilde{y}\right)=(x,y)$:

$$
\begin{aligned}
x &= \left(\tilde{x}+\sqrt{3}\right)\cos\left(\theta\right)-\tilde{y}\sin\left(\theta\right)-\sqrt{3} &(1)\\
y &= \left(\tilde{x}+\sqrt{3}\right)\sin\left(\theta\right)+\tilde{y}\cos\left(\theta\right) &(2)
\end{aligned}
$$

Re-writing equations (1) and (2) as,

$$
\begin{aligned}
\tilde{y} &= \frac{1}{\sin\left(\theta\right)}\left[\left(\tilde{x}+\sqrt{3}\right)\cos\left(\theta\right)-\sqrt{3}-x\right]\\
\tilde{y} &= \frac{1}{\cos\left(\theta\right)}\left[y-\left(\tilde{x}+\sqrt{3}\right)\sin\left(\theta\right)\right] &(3)
\end{aligned}
$$

and eliminating $\tilde{y}$ gives,

$$
\begin{aligned}
\frac{1}{\sin\left(\theta\right)}\left[\left(\tilde{x}+\sqrt{3}\right)\cos\left(\theta\right)-\sqrt{3}-x\right] &= \frac{1}{\cos\left(\theta\right)}\left[y-\left(\tilde{x}+\sqrt{3}\right)\sin\left(\theta\right)\right]\\
\left(\tilde{x}+\sqrt{3}\right)\cos\left(\theta\right)-\sqrt{3}-x &= \tan\left(\theta\right)y-\left(\tilde{x}+\sqrt{3}\right)\tan\left(\theta\right)\sin\left(\theta\right)\\
\left(\tilde{x}+\sqrt{3}\right)\left(\cos\left(\theta\right)+\tan\left(\theta\right)\sin\left(\theta\right)\right) &= \tan\left(\theta\right)y+\sqrt{3}+x\\
\tilde{x} &= \frac{\tan\left(\theta\right)y+\sqrt{3}+x}{\cos\left(\theta\right)+\tan\left(\theta\right)\sin\left(\theta\right)}-\sqrt{3}\\
&= \frac{\tan\left(\theta\right)y+\sqrt{3}+x}{\left(\frac{\cos^2(\theta)}{\cos(\theta)}+\frac{\sin^2(\theta)}{\cos(\theta)}\right)}-\sqrt{3}\\
&= \cos\left(\theta\right)\left(\tan\left(\theta\right)y+\sqrt{3}+x\right)-\sqrt{3}\\
\tilde{x} &= \sin\left(\theta\right)y+\cos\left(\theta\right)\left(\sqrt{3}+x\right)-\sqrt{3} &(4)
\end{aligned}
$$

Substituting equation (4) into equation (3) gives,

$$
\tilde{y}=\frac{y}{\cos\left(\theta\right)}-\left(\sin\left(\theta\right)y+\cos\left(\theta\right)\left(\sqrt{3}+x\right)\right)\tan\left(\theta\right) \qquad (5)
$$

Finally, we know that,

$$
\frac{\tilde{x}^2}{4}+\tilde{y}^2=1 \qquad (6)
$$

Substituting equations (4) and (5) into (6) gives the equation of the ellipse rotated counterclockwise by an angle θ (except for when $\cos\left(\theta\right)=0$):

$$
\frac{\left[\sin\left(\theta\right)y+\cos\left(\theta\right)\left(\sqrt{3}+x\right)-\sqrt{3}\right]^2}{4}+\left[\frac{y}{\cos\left(\theta\right)}-\left(\sin\left(\theta\right)y+\cos\left(\theta\right)\left(\sqrt{3}+x\right)\right)\tan\left(\theta\right)\right]^2=1 \qquad (7)
$$

For a rotation angle of $\theta=\frac{\pi}{4}$, the equation of the ellipse is,

$$
\boxed{\frac{\left[\frac{1}{\sqrt{2}}\left(y+x+\sqrt{3}\right)-\sqrt{3}\right]^2}{4}+\left[\sqrt{2}y-\frac{1}{\sqrt{2}}\left(y+x+\sqrt{3}\right)\right]^2=1}
$$

For a rotation angle of $\theta=\frac{3\pi}{4}$, the equation of the ellipse is,

$$
\boxed{\frac{\left[\frac{1}{\sqrt{2}}\left(y-x-\sqrt{3}\right)-\sqrt{3}\right]^2}{4}+\left[-\sqrt{2}y+\frac{1}{\sqrt{2}}\left(y-x-\sqrt{3}\right)\right]^2=1}
$$

For a rotation angle of $\theta = \frac{\pi}{2}$, equation (7) becomes singular. For $\theta = \frac{\pi}{2}$, equations (1) and (2) become,

$$\begin{aligned} x &= -\tilde{y} - \sqrt{3} \\ y &= \tilde{x} + \sqrt{3} \end{aligned}$$

These equations give,

$$\begin{aligned} \tilde{y} &= -\left(x + \sqrt{3}\right) \\ \tilde{x} &= y - \sqrt{3} \end{aligned}$$

Substituting these expressions into,

$$\frac{\tilde{x}^2}{4} + \tilde{y}^2 = 1$$

gives the equation of the ellipse rotated by an angle of $\theta = \frac{\pi}{2}$,

$$\boxed{\frac{\left(y - \sqrt{3}\right)^2}{4} + \left(x + \sqrt{3}\right)^3 = 1}$$

c) A plot of the precessing ellipse is shown below for $0 \le t \le 16\pi$. The curve does not close on itself at $t = 2\pi$ due to this precession.