Calculus 1A: Differentiation

MITx 18.01.1x

2018/08/17 – 2018/11/20

Sumário

1 Getting started (2018/08/17)

1.1 Overview and logistics

1.1.1 Meet the course team

Professor David Jerison David Jerison received his Ph.D. from Princeton University in 1980, and joined the mathematics faculty at MIT in 1981. In 1985, he received an A.P. Sloan Foundation Fellowship and a Presidential Young Investigator Award. In 1999 he was elected to the American Academy of Arts and Sciences. In 2004, he was selected for a Margaret MacVicar Faculty Fellowship in recognition of his teaching. In 2012, the American Mathematical Society awarded him and his collaborator Jack Lee the Bergman Prize in Complex Analysis.

Figura 1: Professor David Jerison

Professor Jerison's research focuses on PDEs and Fourier Analysis. He has taught single variable calculus, multivariable calculus, and differential equations at MIT several times each.

Professor Gigliola Staffilan Gigliola Staffilani is the Abby Rockefeller Mauzé Professor of Mathematics since 2007. She received her Ph.D. from the University of Chicago in 1995. Following faculty appointments at Stanford, Princeton, and Brown, she joined the MIT mathematics faculty in 2002. She received both a teaching award and a research fellowship while at Stanford. She received a Sloan Foundation Fellowship in 2000. In 2014 she was elected to the American Academy of Arts and Sciences.

Figura 2: Professor Gigliola Staffilani

Professor Staffilani is an analyst, with a concentration on dispersive nonlinear PDEs. She has taught multivariable calculus several times at MIT, as well as differential equations.

Instructor Jen French Jen French is an MITx Digital Learning Scientist in the MIT math department. She earned her Ph.D. in mathematics from MIT in 2010, with specialization in Algebraic Topology. After teaching after school math for elementary aged students and working with the Teaching and Learning Lab at MIT developing interdisciplinary curricular videos tying foundational concepts in math and science to engineering design themes, she joined MITx in 2013. She has developed videos, visual interactives, and problems providing immediate feedback using the edX platform residentially in the MIT math department to aid student learning. She has developed the calculus series (3 courses) and differential equations series (5 courses) available here on edX.

Figura 3: Instructor Jen French

Instructor Stephen Wang Stephen Wang earned a Ph.D. in mathematics from the University of Chicago in 2006, where he specialized in geometry. He has earned teaching awards from both Chicago and Harvard University, and has also been a faculty member at Haverford College and Bucknell University before jumping on board the calculus team at MIT. In fall 2015 he joined the Rice University mathematics faculty.

Figura 4: Instructor Stephen Wang

Special thanks to ... Huge thanks to Prof. Arthur Mattuck for starting it all. Big thanks to Timothy Hall for asking David Jerison the question, how do ziplines behave mathematically. We also thank David Custer and Susan

Ruff who helped with real life ziplines and shared MIT student experiments on ziplines.

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This course was funded in part by:
Class of 1960 Alumni Funds
2014–2015 Alumni Class Funds Grant
Wertheimer Fund

1.1.2 Course description

Discover the derivative — what it is, how to compute it, and when to apply it in solving real world problems. Part 1 of 3.

How does the final velocity on a zip line change when the starting point is raised or lowered by a matter of centimeters? What is the accuracy of a GPS position measurement? How fast should an airplane travel to minimize fuel consumption? The answers to all of these questions involve the derivative.

But what is the derivative? You will learn its mathematical notation, physical meaning, geometric interpretation, and be able to move fluently between these representations of the derivative. You will discover how to differentiate any function you can think up, and develop a powerful intuition to be able to sketch the graph of many functions. You will make linear and quadratic approximations of functions to simplify computations and gain intuition for system behavior. You will learn to maximize and minimize functions to optimize properties like cost, efficiency, energy, and power.

This course, in combination with *18.01.2x Calculus 1B: Integration*, covers the AP Calculus AB curriculum.

This course, in combination with *18.01.2x Calculus 1B: Integration* and *18.01.3x Calculus 1C: Coordinate Systems and Infinite Series*, covers the AP Calculus BC curriculum.

If you intend to take an AP exam, we strongly suggest that you familiarize yourself with the AP exam to prepare for it.

1.1.3 The making of this course

This course was created using latex2edX, a free tool developed at MIT for creating content for edX written in LaTeX. LaTeX is a typesetting language that is fantastic for writing math! Occasionally, the equations you see in the webpage (which are rendered in mathjax) do not load appropriately. Our apologies. The easiest fix is to simply reload the page. Another solution is to change browsers. (Firefox seems to render mathjax less reliably than Chrome or Safari. However, frequent changes to edX will cause disruptions in our content.)

Note edX is not supported on tablet devices. That said, users report that 95% of the problems can be done on a tablet device, but if weird errors are creeping in (especially with formula input type problems) you may try switching to a laptop or desktop computer.

1.1.4 How to succeed

Prerequisites This course has a global audience with students from a wide variety of backgrounds. To succeed in this course, you must have a solid foundation in

1. Algebra
2. Geometry
3. Trigonometry
4. Exponents
5. Logarithms
6. Limits

We know that many students may not have solid foundation in limits, so we have included an optional Unit 0 that introduces Limits. Understanding limits is essential to understand the first lecture on the definition of the derivative in Unit 1, some Homework problems on Continuity and Differentiability in Unit 1, and the first lecture on Limiting behavior and sketching functions in Unit 4.

Because we know you come from different backgrounds, we want to help you to choose the best path through this content. To aid us in this, please take the "Choose your calculus adventure" diagnostics. This will help you to determine if you have the skills to succeed, what skills you may need to review, and which units you may be able to skip!

Reference materials The material we provide in the Courseware contains all of the content you need for this course. However, there are many good calculus texts that have a great deal of problems and alternate explanations that may help you. Most widely used calculus texts are adequate.

There is also the free web resource Khan Academy. Links to other web resources can be found on the Course Info page under the header "Related Links". Feel free to share other resources on the course wiki or through the discussion forum.

1.1.5 Grading

There are 4 categories of graded problems in 18.01.1x: in-lecture Exercises, Part A Homework, Part B Homework, and the Final Exam.

- **Exercises:** These are the problems that are interspersed between videos in each lecture. These problems count for 20% of your grade. These problems will be used to motivate theory, practice a concept you just learned, and review material from previous sequences that we are using. While you are graded on these problems, they are low-stakes: you have multiple attempts, and have the opportunity to look at the answer after you have submitted a correct answer or run out of attempts. This is where you will do the majority of your learning. We encourage you to make mistakes and learn from them!

- **Part A Homework:** Each unit has 1 Part A Homework assignment, which gives you an opportunity to practice what you learned. These problems count for 10% of your total grade. Wait until the end of the unit to attempt these problems. These problems help you identify the concepts that you have forgotten, and aid in long-term retention. These problems are mostly mechanical–asking you to practice methods, and techniques learned in each unit. Each problem typically tests knowledge from only one section in a unit. (We won't necessarily tell you which one though!)

- **Part B Homework:** Each unit has 1 Part B Homework assignment. The part B homework counts for 25% of your total grade. The problems on this homework combine ideas from all of the sequences in the unit. These problems are mostly in the form of word problems which ask you to apply the methods learned to new scenarios.

- **Final:** The final exam is the culmination of your learning, and will account for 45% of your grade. These problems cover all of the material in this course. Several of the problems follow the AP short-answer format. However, we cannot grade the justifications to your reasoning here. To prepare for the AP exam, you should take and review the solutions to sample AP exams from the AP website.

Note: Please notice that Unit 0 is optional and the exercises and homework are intended for self study only and do not count towards your grade.

Certification To earn an ID verified certificate, you must earn 60% of the points in this course. You can see your progress towards certification by clicking on the Progress link above.

1.2 Using the EdX platform

1.2.1 Navigating EdX

This course was developed at MIT and is made available to you by the edX platform.The edX platform is a platform for learning! It allows people from around the world to access content for free, based on their own interests and background.

If you have never taken a course on edX, please take the short 1 hour course DemoX to familiarize yourself with the platform and its capabilities.

In this course, we have the following top-level resources:

- **Course:** This is the graded content of this course, as well as all learning materials.
- **Calendar:** All of the due dates are in UTC, and are available in the google calendar, which you can download into your own calendar so that you can have these due dates available in your own time zone.

- **Discussion:** This is a link to the full discussion forum. For specific discussions related to a problem or video, link through the discussion forum link at the bottom of each page. (See the discussion at the bottom of this page for help with these problems.)
- **Progress:** Use this tab to see how your are progressing through the content!

Course is where you will spend most of your time. This is where we put the content and assessments for your learning. Everything else is a resource to support your learning.

1.2.2 Example problem types

Take a moment to familiarize yourself with the main problem types we use in this course.

Checking and submitting an answer: The edX platform is able to check your answers and give you immediate feedback. When you "check" a problem, it is automatically submitted for grading purposes. Depending on the type of the problem you may have access to the "show answer" button. In the lecture exercises as well as the part A and part B Homework assignments, this option to show the answer will appear only after the due date has passed, you have run out of problem attempts, or you have already submitted the correct answer. You will never get detailed solutions to the final exam. Example: *This problem has unlimited attempts. If you get an answer wrong, you can simply try again until you get it right. How many weeks will this course be?*

Resetting a Problem: Some problems involve randomized parameters, or other elements that you may wish to reset to the original configuration. Here is an example where the variables and are randomized. After one attempt, you can click reset to see the values change! Example: *Let* $x_1 = 5$ *and* $x_2 = 65$. *Enter the numeric value of in the answer box below.*

Limited Number of Attempts 1: Most of the time, you will have a limited number of times that you can attempt a problem. To save an answer and keep it there until you come back, use the save button. Example: *How much does it cost to take an edX course?*

Limited Number of Attempts 2: Multiple choice problems will almost always have between 1 and 3 attempts. Example: *Which choice is correct?*

Formula Entry Problems: This is a math class, which means we are going to be using formulas. And sometimes, we want you to find these formulas. There are some rules for entering formulas into the text entry box (which follows rules for ASCII math). Use:

- $+$ to denote addition; e.g. $2 + 3$
- $-$ to denote subtraction; e.g. $x - 1$
- $*$ to denote multiplication; e.g. $2 * x$
- $\wedge$ to denote exponentiation; e.g. $x \wedge 2$ for x^2
- / to denote division; e.g. $7/x$ for $\frac{7}{x}$
- "pi" for the mathematical constant π
- "e" for the mathematical constant e
- sqrt(x), sin(x), cos(x), ln(x), arccos(x), etc. for the known functions $\sqrt{x}$, $\sin x$, $\cos x$, $\ln x$, etc. Note that parentheses are required.
- Use parentheses ( ) to specify order of operations.

Each formula entry box will have a Formula Input Help button below the answer button, where you can find these facts about how to enter formulas. (See the button below.) Example: *enter the function* $2e^{x-1} + \sqrt{y}$ *using the rules above. (Type 2 * e ^(x-1) + sqrt(y) into the answer box.)*

Drag and Drop Problems: Example: *Drag and drop the elements to create the quadratic formula*. Use the arrows on the horizontal bar to see more options to drag into the formula.

Sketch Input Problems: We created this sketch input problem type because being able to sketch functions to reason through problems is a big part of applying calculus as a problem-solving tool. Example: *Try drawing a smiley face. The mouth should lie below the x-axis, and the place an eye at the points and* $(-1, 2)$ *and* $(1, 2)$

1.3 Using the forum

1.3.1 Discussion forum

The discussion forum is the tool for connecting with the community of online learners in this course. Use the forum to ask questions, seek clarifications, report bugs, start or respond to topical discussions.

On most pages, there is a link at the bottom, which says "show discussion". Clicking this link will show the discussion forum associated with the videos and problems on that page.

"Netiquette": What to do

- **Be polite.** Make sure that your posts are respectful of the other students and staff in the course.
- Use the search button. Search for similar forum posts **before you post** using the magnifying glass icon. Many of your classmates will have the same question that you do! If you perform a search first, you may find the question and answer without needing to post yourself. This helps us keep the forum organized and useful!

- Reply to existing discussions when you see someone with the same question. This helps to organize responses.
- Use a descriptive and specific title to your post. This will attract the attention of TAs and classmates who can answer your question.
- Be very specific about where you need help. Are you stuck on a particular part of a problem? Are you confused by a particular concept? What have you done so far?
- Actively up-vote other posts, and other students will up-vote yours! The more up-votes your post has, the more likely they are to be seen.

"Netiquette": What not to do Follow common writing practices for online communication:

- Avoid TYPING IN ALL CAPS. Some people read this as shouting, even if that is not your intention.
- Avoid **typing in bold**. Some people read this as shouting, even if that is not your intention.
- Avoid unnecessary symbols, abbreviated words, texting shorthand, and replacing words with numbers (e.g. Pls don't rplce wrds w/#s).
- Avoid repeating letters or reeeeepeeaattinggggg chaaracterrrss.
- Avoid excessive punctuation!!!!!!!!

Cheating! We encourage you to communicate in the forum about problems, and get hints and help understanding the material from your fellow classmates and the course TAs. However:

- Please do not post solutions to lecture problems, homework problems (part A or part B), or final exam problems. These will be removed, and the student who posted will be contacted and dealt with individually.
- Do not post or copy solutions posted to the forum for any exercises. This is cheating.
- Do not copy solutions from yourself. This is cheating.

1.4 Choose your calculus adventure

1.4.1 Choose your own calculus adventure

You are interested in learning calculus, but we don't know very much about you or what you already know. So to help you learn best, please take the following diagnostics. These diagnostics will help you choose a path through the content that makes the most sense for you.

We want you to succeed, so make sure that you have the basic precalculus skills so you won't be frustrated! The first 4 pages test your readiness for this calculus class. If you aren't ready yet, don't worry, you can take this course later.

Some of you may already know a lot of calculus. To help you get started in the right place, we have further diagnostics. On pages 5 and 6, you can take the limits and derivatives diagnostics. We encourage you to take a look at these even if you don't know any calculus yet. But don't worry; we designed this course for people with varying backgrounds, including those with no calculus experience.

1.4.2 Algebra Problems

A1: What is the slope of the line through the points $(3, -5)$ and $(1, -1)$?

A2: The lines $3x + 27 = 7$ and $x - 3y = 6$ intersect in a point with what coordinates?

A3: Which expression is equivalent to $\left(\frac{1}{x} + \frac{1}{y}\right)^{-1}$?

A4: List all possible solutions to the equation $x^3 - x^2 - 2x = 0$ (Use decimals only, not fractions, and separate answers with commas.)

A5: A 0.25 mL sample of water drawn from a 5 liter flask contains 1.25×10^8 bacteria. Give the approximate number of bacteria in the flask, expressing your answer in scientific notation. (Scientific Notation: find a real number a between 1 and 10, and an integer n, such that $x = a \times 10^n$.)

A6: For what value of the constant a will the system of linear equations have no solution?

$$\begin{aligned} 6x - 5y &= 3 \\ 3x + ay &= 1 \end{aligned} \tag{1}$$

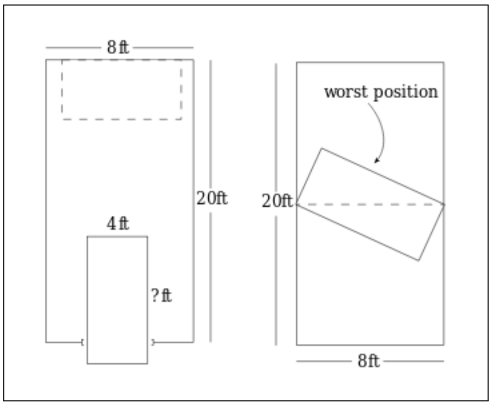
A7: Find the value of the constant a for which the polynomial $x^3 + ax^2 - 1$ will have -1 as a zero.

A8: If $a_n = \dfrac{x^n}{2^n n!}$, find $\dfrac{a_{n+1}}{a_n}$.

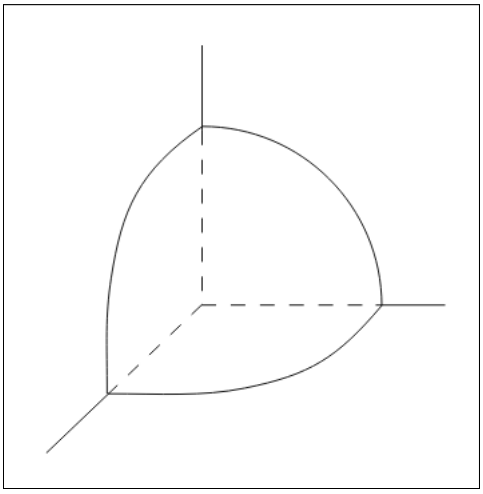
If you score 0.6 or above, you have a good grasp of algebraic manipulations, and can do them accurately enough to succeed in this class! Otherwise, this course will be very difficult for you. We recommend taking an algebra and/or trigonometry class to solidify your familiarity and accuracy before attempting this course.

1.4.3 Geometry

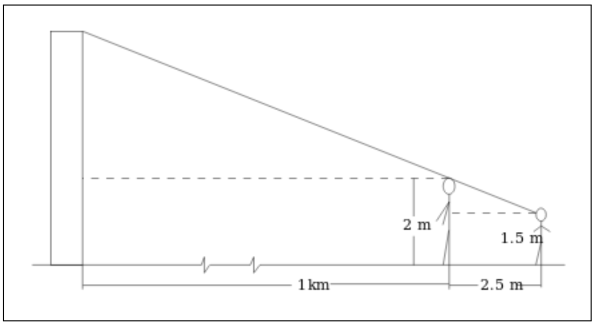
G1: A bed that is 4 feet wide must enter through a door along the 8 foot wall of a 8 by 20 foot room. What is the largest length of a bed that can be rotated to fit into the position shown by the dotted lines against the back wall?



G2: The four-sided solid shown is the part of the solid sphere (of radius 2, centered at the origin) in the first octant. Find its total surface area.

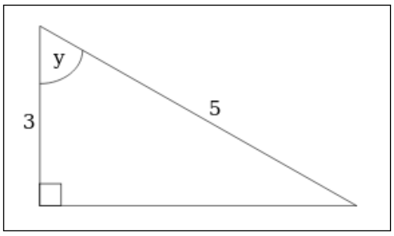


G3: To estimate the height of a skyscraper 1km in the distance, Jenny finds that if her friend Steve stands 2.5 meters away, the top of his head just lines up with the top of the building. Steve is 2 meters tall, and Jenny's eye is 1.5 meters from the ground. How high is the building? (The dotted lines may help you.)

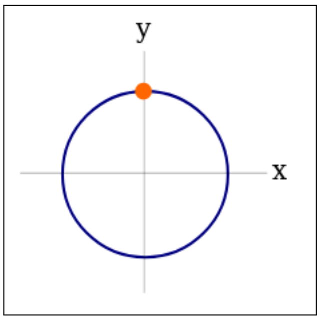


1.4.4 Trigonometry

T1: In the given right triangle, what is $\tan y$?



T2: A horse runs counterclockwise (anticlockwise) around the circular track of radius 400m at a constant speed, starting at the marked point. It completes one lap in three minutes. What is its coordinate after one minute? (If needed, you can use "pi" for π, and sqrt(5) for $\sqrt{5}$.)

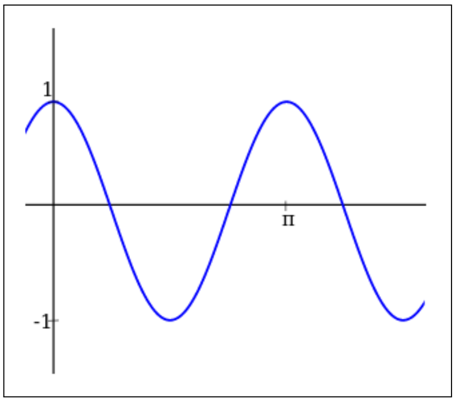


T3: Find the smallest positive solution to the equation $\sin 2x = \frac{1}{2}$; here x is in radians. (If needed, you can use "pi" for π, and sqrt(5) for $\sqrt{5}$. You can enter fractions using the forward slash / ; e.g. pi/2 for $\frac{\pi}{2}$.)

T4: A line with slope 1/2 makes an acute angle θ with the axis. What is $\sin\theta$? (If needed, you can use pi for π, and sqrt(5) for $\sqrt{5}$. You may enter your answer as a decimal number.)

T5: By using the trigonometric identity $\cos 2x = \cos^2 x - \sin^2 x$, and other identities, find the **positive** expression for $\sin\left(\frac{A}{2}\right)$ in terms of $\cos A$.

T6: The graph below represents which of these functions?



- ☐ $\sin x$
- ☐ $\cos x$
- ☐ $\sin(x/2)$
- ☐ $\cos(x/2)$
- ☐ $\sin 2x$
- ☐ $\cos 2x$

If you got a 0.6 or above, you have the foundational trigonometry understanding to succeed in this course. Otherwise, you will need to study trigonometry concurrent with this course in order to succeed!

1.4.5 Logarithms and exponentials

E1: If $\log_{10} a = 4.2$ and $\log_{10} b = 0.5$, what is $\log_{10} ab$?

E2: If $2^a = \frac{\sqrt{8}}{4^3}$, what is a?

E3: Which of the following is equal to $\sqrt{\frac{x^{16}(1+x^2)}{9}}$?

- ☐ $\frac{x^4(1+x)}{3}$
- ☐ $\frac{x^8(1+3)}{3}$
- ☐ $\frac{x^4(1+x^2)^{0.5}}{3}$
- ☐ $\frac{x^8(1+x^2)^{0.5}}{3}$
- ☐ $\frac{x^4(1+x^2)}{3}$

☐ $\frac{x^8(1+x^2)}{3}$

☐ None of the above

E4: Solve for x: $\log_{10}[(x+1)^2] = 2$. (Enter your answer as a list of x-values, separated by commas.)

E5: A pot of water (at sea level) is boiling; the heat is turned off at time $t = 0$, and 2 minutes later the water temperature has fallen to 80°C. If the temperature T (in ºC) is expressed in terms of time t (in minutes) by the law

$$T = Ae^{-kt} \tag{2}$$

find the values of the constants A and k.

If you got a 0.7 or higher, congratulations! You have an excellent understanding of logarithms and exponents! Good work. You can still succeed in this course if you got a 0.7 or lower, but we strongly recommend that you review logarithms and exponents before you get to the end of Unit 2: Differentiation where logarithms and exponents begin to take on a prominent role in the course.

1.4.6 Limits diagnostic

You will NOT see which problems you get correct and incorrect. Sorry if this is frustrating. We will be using these problems to assess whether or not our content teaches you about limits.

L1: What is $\lim_{x\to\infty} \dfrac{x^2-4}{2+x-4x^2}$?

☐ -2

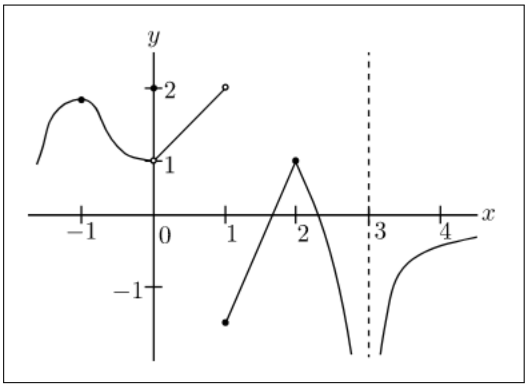
☐ $-\frac{1}{4}$

☐ $\frac{1}{2}$

☐ 1

☐ The limit does not exist.

L2: The graph of a function f is shown below. If the limit as $x \to \infty$ exists and f is not continuous at b, then $b =$?



☐ -1
☐ 0
☐ 1
☐ 2
☐ 3

L3: What is $\lim\limits_{x\to 3} \dfrac{6/x-2}{3-4x+x^2}$? (If the limit does not exist, enter DNE.)

L4: Which of the following functions have a removable discontinuity at $x = 2$?

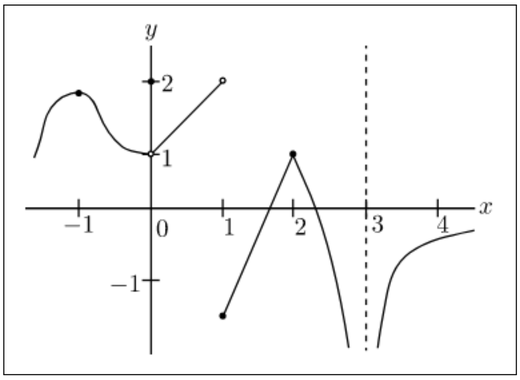
☐ $f(x) = \frac{x^2-x-2}{x-2}$

☐ $f(x) = \frac{1}{(x-2)^2}$

☐ $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$

☐ $f(x) = \begin{cases} x^3-1 & x > 2 \\ -x^2 & x \leq 2 \end{cases}$

L5: Identify the left-hand limit $\lim\limits_{x\to -1^-} f(x)$ based on the graph of shown below.



☐ 2
☐ 1
☐ 0
☐ −1
☐ −1.5
☐ Does not exist.

L6: Identify the right-handed limit $\lim_{x\to -1^+} \frac{x^2-1}{|x+1|}$. (Enter DNE if the limit does not exist.)

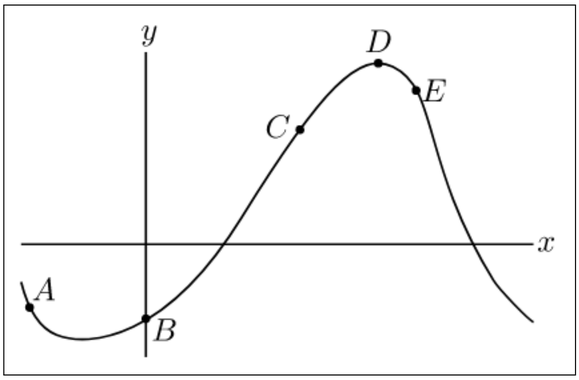
- If you got 0.65 or above, you have a good handle on limits. Move on to Unit 1. You can go back to Unit 0 at any time to fill any gaps in your understanding of limits.
- If you got between .35–.65 points, we recommend that you start by doing the in-lecture problems in Unit 0. You may be able to skip the video tutorials.
- Otherwise, start with Unit 0!

1.4.7 Derivatives diagnostic

You will not see which problems you get correct and incorrect. Sorry if this is frustrating. We will be using these problems to assess whether or not our content teaches you about derivatives.

C1: What is $\lim_{h\to 0} \frac{\cos(\pi/6+h)-\cos(\pi/6)}{h}$? (Enter the answer as a decimal. If the limit does not exist, enter DNE.)

C2: At which of the five points on the graph are $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ both negative?



C3: What is the average rate of change of the function $f(x) = x^4 - 5x$ between $x = 0$ and $x = 3$?

C4: The position of a particle moving along a line is $p(t) = 2t^3 - 24t^2 + 90t + 7$ for $t \geq 0$. For what values of t is the speed of the particle increasing?

- ☐ $3 < t < 4$ only
- ☐ $t > 4$ only
- ☐ $t > 5$ only
- ☐ $0 < t < 3$ and $t > 5$
- ☐ $3 < t < 4$ and $t > 5$

C5: Evaluate the limit $\lim_{x\to\infty} \frac{\ln x}{x^2}$:

- ☐ 0
- ☐ 1
- ☐ -1
- ☐ ∞
- ☐ $-\infty$

C6: If f is differentiable at $x = a$, which of the following must be true? Choose all of the following that must be true.

- ☐ f is continuous at $x = a$.
- ☐ $\lim_{x\to a} f(x)$ exists.
- ☐ $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.
- ☐ $f'(a)$ is defined.
- ☐ $f''(a)$ is defined.

C7: Let $f(x) = x^3 + 5x^2 - 7x - 1$. What is $f'(1)$?

C8: Let $g(x) = x^2 e^x$. What is $g'(1)$?

C9: Suppose that $f(x) = g(5x)$ for all x, and that both functions are differentiable. Which of the following is necessarily true?

- ☐ $f'(1) = g'(1)$
- ☐ $f'(5) = g'(1)$
- ☐ $f'(1) = g'(5)$
- ☐ $5f'(1) = g'(1)$
- ☐ $5f'(1) = g'(5)$
- ☐ $f'(1) = 5g'(1)$
- ☐ $f'(1) = 5g'(5)$
- ☐ None of the above.

C10: Let $f(x) = \dfrac{\ln(5t+1)}{\sqrt{t+1}}$. What is $f'(0)$?

If you got 0.8 or above, you have a good handle on the derivative. There is likely no material in Unit 0 or Unit 1 that you are not familiar with. You may choose to the the in-lecture exercises, but may wish to skip the video tutorials. If you got less than 0.8, that is to be expected! We assume you are here to learn about differentiation after all.

1.5 Syllabus and schedule

1.5.1 Syllabus and schedule

Getting started	Released: 17 August
Overview and logistics	Meet the course team
	Course description
	Prerequisites and resources
	Syllabus
	Grading and certification
	Discussion forum
Tutorial on using the edX platform	Navigating courseware
	Interactive problems
	Discussion forum
Choose your calculus adventure	How to succeed in this course
	Diagnostics
	Plan your path through course
Entrance survey	

Unit 0: Limits	Released: 17 August	(Optional)
Introduction to limits	Right- and left-hand limits	
	When limits do not exist	
	Graphical limits	
	Limit laws	
Continuity	Right- and left-continuity	
	Continuous functions	
	Jump and removable discontinuities	
	Intermediate Value Theorem	
Limits of quotients	When the denominator approaches zero	
	Infinite limits	
Homework 0 Part A		
		Hide

Unit 1: Introducing the derivative	Released: 22 August	Due: 15 September
What is the derivative?	Average rate of change	
	Instantaneous rate of change	
	Limit definition of derivative	
Geometric interpretation of the derivative	Slopes of tangent lines	
	Secant lines	
	Approximating derivatives graphically	
The derivative as a function	Where a function is increasing and decreasing	
	Concavity	
Calculating derivatives	Power rule	
Leibniz notation	Notation and interpretation of units	
Second derivatives and higher	Derivatives of derivatives	
	Acceleration	
Trigonometric functions:	Derivatives of sine and cosine	
	Modeling oscillations	
Homework 1 Part A		
Homework 1 Part B		

Unit 2: Differentiation	Released: 29 August	Due: 3 October

Linear approximation	Tangent line approximation
	Concavity and error
Product rule	Derivatives of products of functions
Quotient rule	Derivatives of quotients of functions
Chain rule	Derivatives of compositions of functions
Implicit differentiation	Derivatives of implicitly defined functions
Inverse functions	Graphing inverse functions
	Restricted domains
	Derivatives of inverses
Exponential functions	Formula
	Euler's constant
Logarithms	Natural logarithm
	Exponential and logarithmic differentiation
Homework 2 Part A	
Homework 2 Part B	

Unit 3: Approximations	Released: 29 August	Due: 24 October

Measurement error	Linear approximations
	Sensitivity to error
	Combining approximations
Quadratic approximations	Best fit parabola
	Big "O" notation
	Understanding error
Newton's method	Algorithm
	Why it works, how it can fail
Homework 3 Part A	
Homework 3 Part B	

Hide

Unit 4: Applications	Released: 29 August	Due: 14 November

Graphing and critical points	First derivative test
	Second derivative test
	Relative max and min
	Inflection points
Asymptotes and limiting behavior	L'hôpital's rule
	Asymptotic behavior
	Sketching functions
Optimization	Max min problems
	Checking end points
Related rates	Apply implicit differentiation
	Methods of problem solving
Homework 4 Part A	
Homework 4 Part B	

1.6 Entrance survey

1.6.1 Entrance survey

Welcome to this online course from MITx.

For us to offer the best course experience possible, we'd like to ask you to answer a few questions about yourself.

Whether you are just browsing or you are determined to complete the entire course, the more we know about you, the better we can serve all students in this course. As one of the first students in this new, free offering, your responses will be especially important to us.

There are no right or wrong answers or responses, and your honest feedback is very important to us. After reading the consent document below, you may click the right arrow below to proceed.

General Information About Survey Research in MITx. Please Read then Click Below to Continue.

Participation is voluntary. All survey responses are voluntary, students can skip any question at any time, and any responses have no effect on student assessments or participation.

What is the purpose of this research? We are interested in learning more about our participants' backgrounds, interests, and motivations, and encouraging engagement with the course, so we can do the best possible job designing, evaluating and refining this course. With this research we will understand how to best encourage engagement with online education .

How long will I take part in this research? Your participation will be the duration of the course.

What can I expect if I take part in this research? As a participant, you will be provided questions about yourself and other short prompts, which we will use to understand your participation in the course.

What are the risks and possible discomforts? If you choose to participate, we anticipate minimal risks and only the minor discomfort that might accompany online surveys.

Are there any benefits from being in this research study? We cannot promise any benefits to you or others from taking part in this research. However, possible benefits include your being more engaged with the course and better serving future students who participate in online courses.

If I take part in this research, how will my privacy be protected? What happens to the information you collect? Your instructor will not be able to identify your personal responses during the course and researchers will not attempt to identify individuals. Your data will not be made identifiable to anyone other than researchers and course staff, and it will be aggregated for analysis and publication purposes.

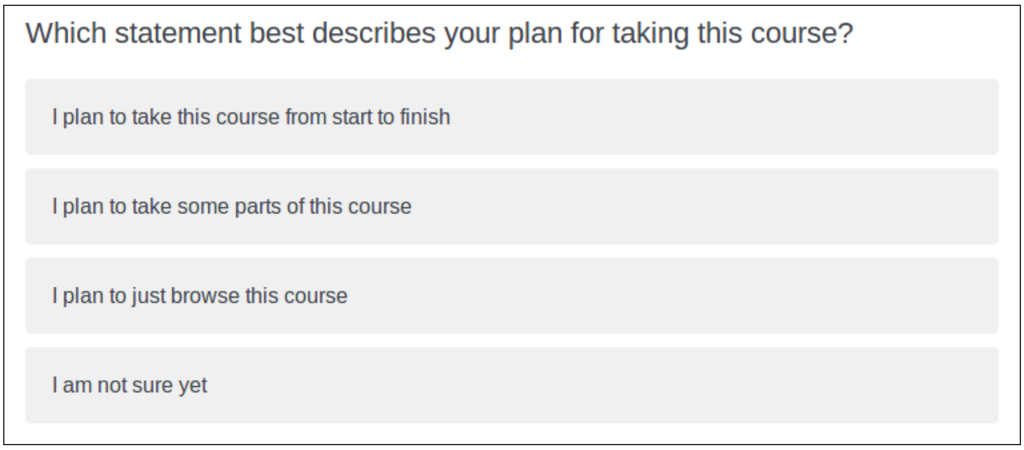
If I have any questions, concerns or complaints about this research study, who can I talk to? The researcher for this study is Justin Reich who can be reached at 617-715-2962, 600 Technology Square, NE49-2028, Cambridge, MA, 02139, jreich@mit.edu for any of the following:

- If you have questions, concerns, or complaints,
- If you would like to talk to the research team,
- If you think the research has harmed you, or
- If you wish to withdraw from the study.

This research has been reviewed by the Committee on the Use of Human Subjects in Research at Harvard University. They can be reached at 617-496-2847, 1414 Massachusetts Avenue, Second Floor, Cambridge, MA 02138, or cuhs@fas.harvard.edu for any of the following:

- If your questions, concerns, or complaints are not being answered by the research team,
- If you cannot reach the research team,
- If you want to talk to someone besides the research team, or
- If you have questions about your rights as a research participant.

Please print or save a copy of this form for your records. **If you agree to participate, please click "Next"to enter the survey.**


Which statement best describes your plan for taking this course?

I plan to take this course from start to finish

I plan to take some parts of this course

I plan to just browse this course

I am not sure yet

Do you intend to earn a verified certificate in this course?

- Yes
- No
- Unsure

How many course assessments (quizzes, tests, etc.) do you intend to complete?

- All assessments
- Most assessments
- A few assessments
- No assessments

Why did you enroll in this course?

	Does not apply to me	Applies to me
For enjoyment	○	○
To improve my English skills	○	○
To advance my education	○	○
To advance my career	○	○

How many hours do you intend to spend on this course each week? *Please enter a whole number.*

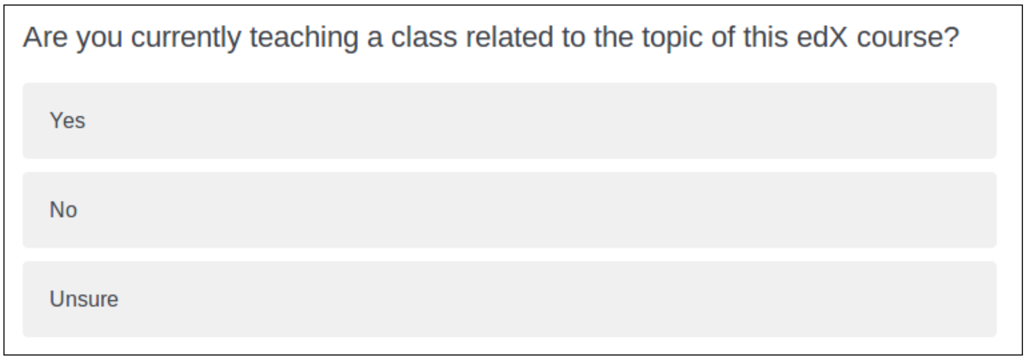
How familiar are you with the topics in this course?

- Extremely familiar
- Very familiar
- Somewhat familiar
- Slightly familiar
- Not at all familiar

How important is learning the materials in this course to you?

- Extremely important
- Very important
- Moderately important
- A little important
- Not at all important

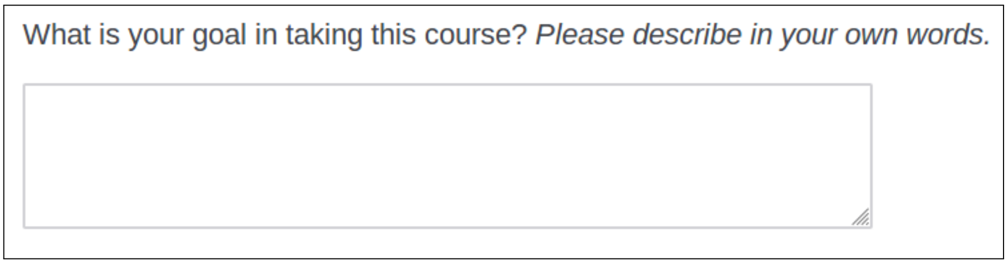
How many online courses have you completed in the past?
Please enter a whole number. If you have not completed any, enter "0".

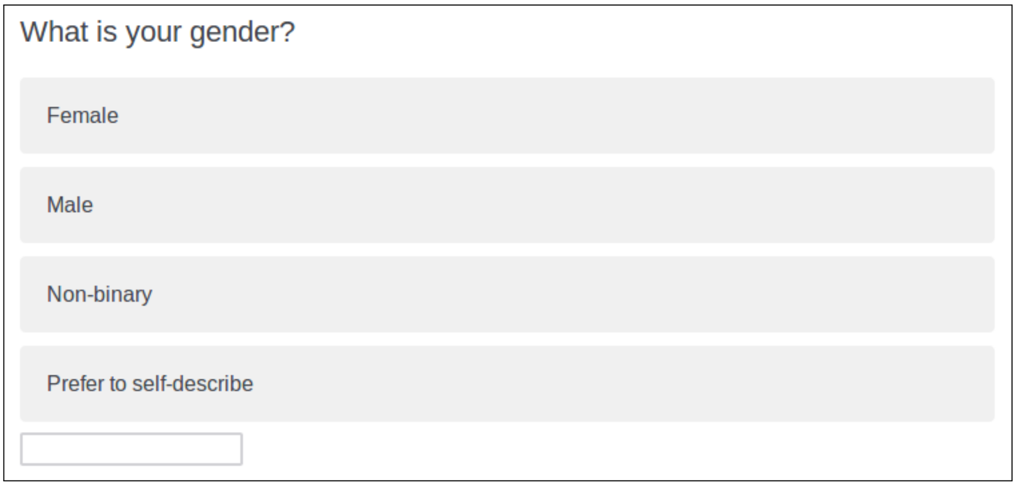

Are you currently teaching a class related to the topic of this edX course?

Yes

No

Unsure


What is your goal in taking this course? *Please describe in your own words.*

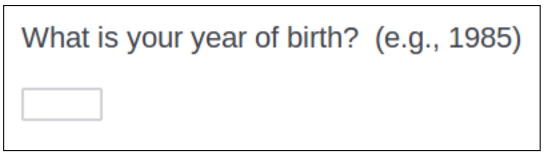

What is your gender?

Female

Male

Non-binary

Prefer to self-describe


What is your year of birth? (e.g., 1985)

In which country were you born?

-- Choose a country --

What is your current employment status?

- Employed
- Unemployed
- Full-time student
- Retired
- Other

What is the highest level of education **you** have completed?

- Doctorate/Ph.D.
- Masters
- Professional
- Bachelors
- Some College
- Associate
- Secondary/High School
- Middle school/Jr. High
- Elementary
- None

What is the highest level of education that **any of your parents** have completed?

- Doctorate/Ph.D.
- Masters
- Professional
- Bachelors
- Some College
- Associate
- Secondary/High School
- Middle school/Jr. High
- Elementary
- None

How would you describe your English language skills?

- Fluent
- Proficient
- Intermediate
- Basic
- Weak

Which of the following barriers to access education does this online course help you overcome? *Select all that apply, if any.*

- Confidence: I would not feel confident in an in-person class.
- Learning Difficulty: I have a harder time learning in in-person classes.
- Financial: I could not easily afford it otherwise.
- Scheduling: I could not make time for regular classes.
- Geographical: The nearest school is too far away.
- Other barrier:

What is the probability that you will complete enough of the coursework to earn a passing grade?

Use the scale below to forecast the probability that you will complete the course: "100" means that you **certainly will** complete the course (i.e. "100% chance"), while "0" means that you **certainly will not** complete the course (i.e. "0% chance").

The probability that I will complete this course is ...

0 10 20 30 40 50 60 70 80 90 100

Researchers from Harvard University and MIT are interested in learning more about the experiences of students taking this course, in order to better understand the edX experience and to improve courses for future students. At present, this study is limited to adults over 18.

Are you interested in participating in an interview to share your experience in this course? If you are interested, a researcher may contact you by email about participating in a study of edX students.

- Yes
- No

Do you have any affiliations with MIT? *Check all that apply.*

- Current Student
- Alumnus/Alumna
- Staff
- Faculty
- No Affiliation

We want everyone who signs up to meet their goals in this course. However, while many students who intend to finish the course will complete it, there are others who do not finish as much of the course as they had wanted. We'd like to know your thoughts about why some people do not follow through on their intentions.

Do you think there are some common reasons that explain why some students do not achieve the goals they set for themselves? Are there reasons you might not meet your own goals in this course?

Use the boxes below to describe some of these reasons.

(Note: You don't have to fill every box; just use the different boxes to separate distinct reasons).

Reason #1

Reason #2

Reason #3

Please write down a clear, concrete plan to follow through on your goals in the course. Plan-making can be a helpful tool in MOOCs! Successful students in previous courses have made detailed plans for how they will engage throughout the course. In the text boxes below, **write out your plans to complete your work for the course.**

Please be as specific as you can! Write clearly, in full sentences, so that someone else could understand what you mean.

When and where do you plan to engage with the course content?

What specific steps will you take to ensure you complete the required course work?

How will you overcome potential obstacles in the course?

Thank you for writing down your plans. Sticking to your plans can help you stay on track and achieve your goals in the course!

Take a moment now to read over your plans below, to make sure you remember them later. For example: write them down on paper, email them to yourself or a friend, add to a calendar with a reminder, or tell someone about them!

<u>YOUR PLANS FOR THIS COURSE</u>

OFFICE OF DIGITAL LEARNING

Thank you for your responses and we hope you enjoy the course.

Now you are ready to begin the class.

2 Unit 0: limits

2.1 Introduction to limits

2.1.1 Motivation

Video: Introduction to Limits

Calculus has two main concepts — the derivative and the integral. But in order to understand either of them, you first have to understand limits.

So let's talk limits. We'll start with a curve. Fix a point A on the curve. Choose a second point, B, which we're going to move. And draw a line through A and B. Let's look at what happens when B moves closer and closer to the point A.

This is an example of a limit. In the limit, the line becomes tangent to the curve at the point A. The slope of this line is the derivative at the point A. Now let's see how limits are related to integrals.

Integrals are used to measure areas of curvy regions like this. Measuring areas of curvy regions seems hard, but measuring areas of rectangles is easy, so we'll try to fill our region with rectangles.

Each rectangle has a certain width. As we make the width smaller, the total area of the rectangles gets closer and closer to the area of the curvy region.

The integral is the limit of the total area of the rectangles as the width tends to zero.

So that's why we start with limits. They're the foundation for everything else in calculus. At the beginning, limits may seem abstract, but very quickly you'll get used to them.

2.1.2 Introduction to limits

Objectives At the end of this sequence, and after some practice, you should be able to:

- Use a calculator to determine right and left hand limits.
- Identify right and left hand limits based on graphs.
- Determine if a limit exists based on values of right and left hand limits.
- Understand that the limit does not depend on the value of a function at the point of interest.

Contents: 14 pages, 6 videos (24 minutes 1x speed), 17 questions.

2.1.3 Moving closer and closer

Video: Moving closer and closer

Welcome.

Calculus is all about functions. You probably know that a function f takes an input x and gives an output f of x. But in calculus, we're not concerned with just one input and finding the output for that one input. We want to consider a whole range of inputs. So we would want to know what happens when the input "moves"or "varies."For instance, we could ask what happens as the input moves really close. Closer and closer to some point. Let's say 1.

And to be even more specific, let's say that x is moving towards 1 from the left. So if this is a number line, and we've got the point 1 right there, then x could start here, and just move closer and closer and closer towards 1, from the left. We'll use this arrow notation to denote that x is getting really, really close to 1. But a warning, this does not mean that x will ever actually equal 1. We're only concerned with values of x that are near one.

OK. Now that that's said, as x moves, we know that the output f of x is also going to move. And so the question that we can ask is as x moves closer and closer to 1 from the left, does f of x move closer and closer to some value of its own?

Let's be concrete here. And pick a particular function f. I'm going to choose f of x to be the square root of 3 minus 5 x plus x squared, plus x cubed, all over x minus 1. Kind of a complicated function, but you'll have to trust me that this is a good example. And what we can do in order to see what's happening to f as x approaches 1 from the left is just select certain values of x that are getting closer and closer to 1 from the left. So over here on the number line, we could start with x equals zero. And then they get closer, we could try x equals 0.5. Or even closer, maybe 0.9. Even 0.99. These sorts of values. And we want to know, what's happening to the output? So we can just plug these values into the function, and see whether the output gets closer and closer to anything.

Now there are technically infinitely many values of x that we could have chosen here. But let's just start with these four. Remember though that one value of x that we will definitely not consider is x equal to 1 itself. In fact, this function isn't even defined at x equals 1. We'd have a zero denominator. It is, however, defined when x is approaching one, and those are the values we're considering.

OK. Well let's make a table with our chosen inputs and the associated outputs, and let's just calculate those outputs. So when we plug in zero we'll get a square root of 3 on top divided by minus 1. So minus square root of 3, which is roughly minus 1.73. Next up is x equals 0.5. I'm going to have to bust out the calculator here. So we've got 3 minus 5 times 0.5 plus 0.5 squared plus 0.5 cubed, and then we need the square root, and then we need to divide by 0.5 minus 1. So 0.5 negative. So we get minus 1.87, roughly. So back to our table. We've got f of x moving from minus 1.73 to minus 1.87. Well that's not really enough data to tell if f is getting closer and closer to anything in particular. So let's take our next two values of x and plug those in. I'm going to fast forward through the calculations. You ready? x equals 0.9. All right? That's approximately minus 1.97, and finally 0.99, and we've got minus 1.997. So as we go down this table, f of x is getting really, really close to what looks like minus 2. So we can say that as x approaches 1 from the left, f of x approaches minus 2. Now f of x might never actually equal to minus 2, just as x never actually equals one, but it gets really, really close. And if it gets arbitrarily close, meaning as close as we could possibly want, then that's really all we'll care about. What I would like you to do now is to do this same exercise, except this time have x approach 1 from the right. You might be surprised at what you find. We'll talk afterwards.

Determine what happens to $f(x) = \dfrac{\sqrt{3 - 5x + x^2 + x^3}}{x - 1}$ as x approaches 1 from the right. Take values of x that are greater than 1, but getting closer and closer to 1. For instance, you could try $x = 1.1, 1.01, 1.001, 1.0001$, etc. What happens to $f(x)$ as x approaches 1 from the right?

- ☐ $f(x)$ gets closer and closer to a particular number
- ☐ $f(x)$ gets bigger and bigger in the positive direction without bound
- ☐ $f(x)$ gets bigger and bigger in the negative direction without bound
- ☐ None of the above

What value does $f(x)$ get closer to? Enter the number below; if there is no such value, enter capital (for "does not exist")

2.1.4 One-sided limits

Video: One-sided limits

2.1.5 Definitions of right-hand and left-hand limits

2.1.6 A few more limits

xx

2.1.7 Possible limits behaviors

xx

2.1.8 Quick limit questions

xx

2.1.9 The overall limit

xx

2.1.10 Limit definition

xx

2.1.11 Limits from graphs

xx

2.1.12 Review problems

xx

2.1.13 Limit laws

xx

2.1.14 Limit laws (2)

xx

2.1.15 Summary

xx

2.2 Continuity

2.2.1 Motivation

xx

2.2.2 How do we compute limits?

xx

2.2.3 Continuity

xx

2.2.4 Continuity questions

xx

2.2.5 More continuity questions

xx

2.2.6 Overall continuity

xx

2.2.7 Continuity continued

xx

2.2.8 Limit laws and continuity

xx

2.2.9 Review of continuity

xx

2.2.10 Catalog of continuous functions

xx

2.2.11 IVT intro

xx

2.2.12 Intermediate Value Theorem

xx

2.2.13 Basics

xx

2.2.14 Roots

xx

2.2.15 Summary

xx

2.3 Limits of quotients

2.3.1 Limits of quotients

xx

2.3.2 How do we compute limits of quotients?

xx

2.3.3 Limits and division

xx

2.3.4 Quotients of small numbers

xx

2.3.5 Small divided by small

xx

2.3.6 Limit law for division

xx

2.3.7 Limit laws

xx

2.3.8 Using the division limit law

xx

2.3.9 Division limit questions

xx

2.3.10 Review problems

xx

2.3.11 Limits that don't exist

xx

2.3.12 Another function

xx

2.3.13 Infinite limits 2

xx

2.3.14 What is the limit?

xx

2.3.15 Division involving infinite limits

xx

2.3.16 Summary

xx

2.4 Homework Unit 0: Part A

2.4.1 Part A Problems

xx

2.4.2 Part A Homework

xx

3 Unit 1: The Derivative (2018/08/29)

3.1 What is the derivative?

3.1.1 What is the derivative?

Video: What is the derivative?

I just got a speeding ticket in the mail. There were no police present on the turnpike, so my question for you is, how did they know I was speeding?

What I can tell you is that I went through the toll booth at the 50 mile marker at eight AM. I went through a second toll booth at the 220 mile marker at ten AM. And by the way, I was certainly going slower than the speed limit through the toll booths.

In this lesson, you'll figure out why I got this speeding ticket. We'll learn about average velocity, instantaneous velocity, and explore the relationship between them.

This relationship is crucial to the definition of the derivative; and the derivative is the subject of this entire course.

Note: In the United States, you cannot actually get a speeding ticket this way. A law was passed that prevents law enforcement from using toll booth data to give speeding tickets retroactively. Otherwise, no one would purchase automated toll booth passes!

3.1.2 Objectives

- Describe what is meant by an *average rate of change*, and compute them with appropriate *units*.
- Describe the difference and relationship between the average rate of change and an *instantaneous rate of change*.
- Use a *limit* to find the instantaneous rate of change, also known as the *derivative* at a point.
- Interpret the *sign of a derivative* — positive, negative, or zero — as having real-world meaning.

Contents: 13 pages; 6 videos (23 minutes 1x speed); 17 questions.

3.1.3 Rates of change

Derivatives are all about rates of change. What do we mean by a rate of change? Let's take an example.

Using the fact that you went through the toll booth at the 50 mile marker on the Turnpike at 8am, and you went through a second toll booth at the 220 mile marker at 10am.

Exercise 3.1.3-1: Average rate: On average, how fast were you traveling between 8 AM and 10 AM?

Exercise 3.1.3-2: Check units: OK, you entered a number there, and hopefully you got the right answer. But this is supposed to be measuring something real — a velocity, not just a number. In what units did we just measure our speed?

- ☐ kilometers
- ☐ miles
- ☐ hours
- ☐ kilometers per hour
- ☐ miles por hour
- ☐ hours per kilometer
- ☐ hours per mile
- ☐ something else

3.1.4 Average vs. Instantaneous

Exercise 3.1.4-1: Average: We calculated a velocity of 85 miles per hour, so we were definitely speeding. Does that mean that at the exact instant of 8 AM, the speed of the car was exactly 85 mph?

- ☐ yes
- ☐ no

Exercise 3.1.4-2: Instantaneous approximation: Which of the following would give us a better idea of the velocity at 8am?

- ☐ Do the same calculation, but between 8am and 12pm instead of between 8am and 10am.
- ☐ Do the same calculation, but between 8am and 8:01am instead of between 8am and 10am

Video: Average Velocity

OK, we've calculated the average velocity of our car between 8:00 and 10:00. But then we decided that that might not be such a good approximation if we wanted the velocity at 8:00 exactly. So we've decided that instead, we should do the same calculation just this time, between 8:00 and 8:01.

All right, we're going to have a lot of numbers floating around here. So let's get some notation to organize all this. We know that position is a function of time. So if we have t as representing time, then we can say that $f(t)$ is position.

So our initial data was that our position at 8:00 was 50 miles. So $f(8) = 50$. And our position at 10:00 was 220 miles.

Now previously, you had used this data to calculate that the average velocity between 8:00 and 10:00 was 85 miles per hour. And that 85 came from 220 minus 50. So we traveled 170 miles. And then you divide by 10 minus 8. That was the time, two hours, that the journey took.

So in our new notation, this is $f(10) - f(8)$ on top divided by $10 - 8$. So our numerator here is the change in position. And our denominator is the change in time. And when you divide those two, we get our average velocity of 85 miles per hour.

Now, calculus is all about variables changing, just all over the place. And so we have a special notation that we use to denote the change in a variable. So here, this numerator where we're saying the change in position, or the change in f, we often denote that by Δf. So Delta (Δ), this Greek letter here, this triangle, stands for difference.

And so we have the difference in f. This is not $\Delta \times f$. Delta isn't a thing in and of itself. It's just one quantity, Δf.

And similarly, on the denominator, we're going to have a quantity, Δt, the change in time.

So this $\frac{\Delta f}{\Delta t}$ is giving us our average velocity over the period of time from 8:00 to 10:00.

Now, we wanted to talk about 8:01. So in order to do that, I need to tell you the position of the car at 8:01. And in our notation, that's going to be $f(8+\frac{1}{60})$. So that's 8 and 1/60 hours. And let's say that the car was at mile marker 51 at that moment.

So given that information, what would you say is the average velocity of the car between 8:00 and 8:01? Why don't you think about that, and then we'll come back and discuss some more.

Exercise 3.1.4-3: Average velocity 2: Given that at 8:00 our position was 50 miles, and at 8:01 our position was 51 miles, what was our average velocity over the one minute from 8:00 to 8:01? Express your answer in miles per hour.

3.1.5 Instantaneous approximation continued

Video: Average Rate of Change

We're looking for the average velocity of our car between 8 o'clock and 8:01. And we were told the positions of the car at those two times. So f, which is our position function, of 8 is 50 miles. And $f(8:01)$, or 8 plus 1/60, is 51 miles. And from that, hopefully you are able to determine what the average velocity during this 1 minute was.

So you need Δf, the change in position, divided by Δt, the change in time. And if you did that, then on the top you would get 51 minus 50 miles. And on the bottom, we have the difference in time between 8:01 and 8 o'clock. And so we get 1 mile divided by 1 minute, or 1 mile per minute.

Now, we wanted this in miles per hour. So there are a couple ways to do that. One way is to just rewrite everything in terms of hours rather than minutes. So on the top we have $f(8+1/60)-f(8)$. That's the difference in position. And on the bottom we have the difference in the two times, so $(8+1/60)-8$. And that's in hours. And when you do that, you get 1 mile on top divided by 1/60 of an hour on the bottom, and that's 60 miles per hour.

A faster way might have been to just take this 1 mile per minute and multiply it by a conversion factor of 60 minutes per hour. And when we do that, the minutes cancel and we're just left with 60 miles per hour.

So this gave us our average velocity over this one minute, or our average rate of change of position with respect to time. Now, it's important to note that, in this course **when we say average rate of change, it doesn't have to be with respect to time: It can be any sort of thing**.

For instance, if you have some amount of gas, maybe some steam, then if you change the temperature of the gas, then the pressure changes. So pressure is a function of temperature, and we could talk about the average rate of change of pressure with respect to temperature, as the temperature goes from such and such to such and such.

In general, if you have any function f with some input variable x, then you can talk about the average rate of change of $f(x)$ with respect to x, as x goes from $x = a$ to $x = b$. And what this is, is exactly what we had above.

It's just the change in f, so Δf, divided by the change in x, Δx. And this is always going to be measured in units of the output divided by units of the input. And the formula is just as follows. So Δx — x is going from a to b, so Δx, the change in x, is going to be $b - a$. And Δf, well, how much does f change? f is going from $f(a)$ to $f(b)$, so its change is $f(b) - f(a)$.

And so this quotient right here is our average rate of change of f with respect to x as x goes from $x = a$ to $x = b$.

Exercise 3.1.5-1: Pressure changes: The pressure of a volume of gas changes as we change the temperature. If we measure the pressure in units of pascals, and temperature in degrees Celsius, in what units would we be measuring a rate of change of pressure with respect to temperature?

☐ pascals
☐ degrees Celsius
☐ degrees Celsius per pascal
☐ pascals per degree Celsius
☐ none of the above

Exercise 3.1.5-2: Pressure rate of change: If pressure is 210000 pascals at 30 degrees Celsius, and 220000 pascals at 32 degrees Celsius, what is the average rate of change of pressure over this range of temperature?

3.1.6 Getting closer to instantaneous

Exercise 3.1.6-1: Getting closer to instantaneous: We've calculated the average velocity of our car between 8:00 and 8:01. Great! But again we have the problem — this is merely the average velocity over a period of one minute. We still haven't found the velocity at the precise instant of 8am. Which of the following expressions would give us the precise velocity at exactly 8am?

☐ $\dfrac{f(10) - f(8)}{10 - 8}$

☐ $\dfrac{f(8+1/60)-f(8)}{(8+1/60)-8}$

☐ $\dfrac{f(8+1/3600)-f(8)}{(8+1/3600)-8}$

☐ none of the above

3.1.7 Finding a formula

Exercise 3.1.7-1: Finding a formula: All of the choices were of the form $\dfrac{f(b)-f(8)}{b-8}$. When b was 10, we know that gives us the average velocity over the span of time from 8:00 to 10:00, but that won't necessarily match the instantaneous velocity at 8:00. When we choose b to be one minute past 8, $8+1/60$, we probably get a more accurate measurement, but it's still an average velocity over a minute, not the instantaneous velocity. When b is only one second away, $8+1/3600$, even more accurate, but again, not exact. So is the answer simply to plug in 8 for b into the equation for the average velocity?

☐ Yes
☐ No

3.1.8 Derivative at a point

Video: The Derivative at a point

We've been thinking about average velocity over a period of time. We know that if we want the average velocity between eight o'clock and some other time b, then that's given by this formula. We have the change in position divided by the change in time.

But if we want the instantaneous velocity at eight o'clock exactly, then this formula is a little bit problematic. If we try plugging in b is 8:01, then that's giving it as an average velocity over a period of one minute. That's not the same thing as the instantaneous velocity at eight o'clock.

We could try b is eight o'clock and one second, but that's still an average velocity. Now, it's an OK approximation if we're talking about the velocity of a car. So the car's velocity is not going to change very much over that one second. But if we're talking about a dragonfly, then its velocity fluctuates all over the place.

The issue is this other variable b. If we want an instantaneous velocity at eight o'clock, there's no good way to choose one specific value for b in this formula that's going to work in every situation and that everyone can just agree on.

We know that the closer b is to 8, the better. But we can't plug in $b=8$, because we'd get $0/0$, and that's just ridiculous. So our solution is to take a *limit* as $b\to 8$ (as b approaches) eight o'clock.

So our formula for the instantaneous velocity at eight o'clock is a limit as b approaches eight of $f(b)-f(8)$ — that's the change in f — divided by $b-8$, the change in time: $\lim_{b \to 8} \frac{f(b) - f(8)}{b - 8}$. In other words, we're taking the limit of average velocities as b approaches 8, or as the time interval gets shorter and shorter.

This is a massive concept, the idea that you can take the limit of a bunch of average velocities and get an instantaneous velocity.

And we want to apply this not just to instantaneous velocity, but we want to talk about **instantaneous rates of change of any function**.

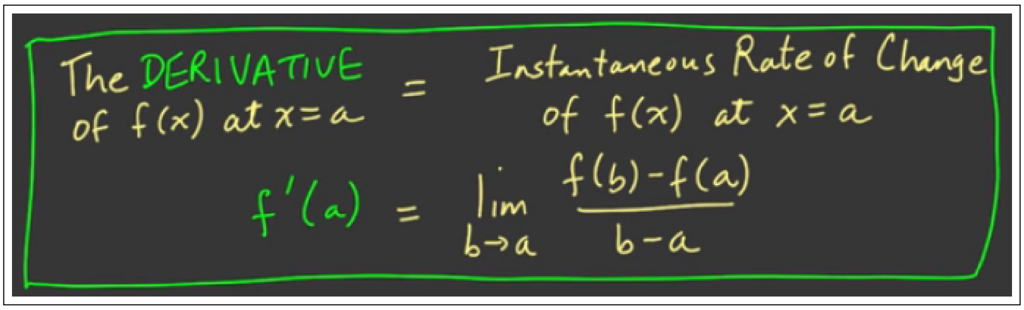
So let me erase this. If we want the instantaneous rate of change of a function $f(x)$, at some point, $x = a$. We're going to give this a special name. So we're going to call this the **derivative** of $f(x)$ at the point $x = a$: $\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$.

So this is our big idea. The **derivative at a point is measuring the instantaneous rate of change of the function at that point**. And the formula for it is exactly the same as what we had below.

We've got $\frac{f(b) - f(a)}{b - a}$. So that's an average rate of change. And then we take the limit as b approaches a. And that's the derivative.

So we give this a special notation. We're going to say that the derivative of f at a will be denoted by this f' "prime" of a: $f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$. So all of these things — this notation, this formula, this idea of an instantaneous rate of change — all of these are wrapped up in this word "derivative."

And that's what we're going to be learning about for the next several weeks. So let's start getting used to it.



3.1.9 Definition of the derivative

The *derivative* of a function $f(x)$ at a point $x = a$ is defined to be:

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{3}$$

This alternative definition is also common:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{4}$$

Note: The derivative of a function always measures an instantaneous rate of change of the function's output with respect to the function's input, so, just like an average rate of change, it would be measured in units of output per units of input.

Exercise 3.1.9-1: Review question: Which of the following $f'(a)$ measure? (Check as many as apply.)

☐ The size of f
☐ An average rate of change of f
☐ An instantaneous rate of change of f

Exercise 3.1.9-2: Units of the derivative: If x is temperature, measured in degrees, and $f(x)$ is pressure, measured in pascals, what units would be $f'(30)$ measured in?

☐ Pascals
☐ Degrees
☐ Degrees per Pascal
☐ Pascals per degree
☐ None of the above

3.1.10 Tossing a pumpkin

Video: Tossing a Pumpkin

Let's say we throw a pumpkin off of a building. The height of the pumpkin is going to be a function of time. So maybe it's given by this function: $f(t) = 100 + 20t - 5t^2$ meters where t is measured in seconds.

Just so you know a little bit about where this function came from, it's a quadratic polynomial, because we know that objects fall in parabolic arcs. And the 100, that's the height of the building. So that's where the pumpkin starts at time 0. The 20 is referring to an initial velocity with which we throw the pumpkin. And this $-5t^2$, that's the effect of gravity.

What we're going to do in this video is calculate the average velocity of the pumpkin between times $t = 0$ and $t = 1$. And then we're going to calculate the instantaneous velocity of the pumpkin at $t = 1$. And we're going to do this and see how those things differ.

So let's start with the average velocity. We know that average velocity is the change in height. So $\dfrac{f(1) - f(0)}{1 - 0}$. And we can just calculate these things. $f(1) = 115$. $f(0) = 100$. And that's in meters. And on the bottom, we have a time of one second. So we've got 15 meters per second for our average velocity. Wonderful.

Now, instantaneous velocity is asking for a derivative. So we want $f'(1)$. And we'll use our limit definition. So $f'(1) = \lim\limits_{b \to 1} \dfrac{f(b) - f(1)}{b - 1}$. So let's see what we can do to simplify this expression that we're taking the limit of.

So $\frac{f(b)-f(1)}{b-1}$, when we just plug these things in to our formula for f, we get $f(b) = 100 + 20b - 5b^2$ And then we're subtracting $f(1) = 115$. All this is over $b - 1$.

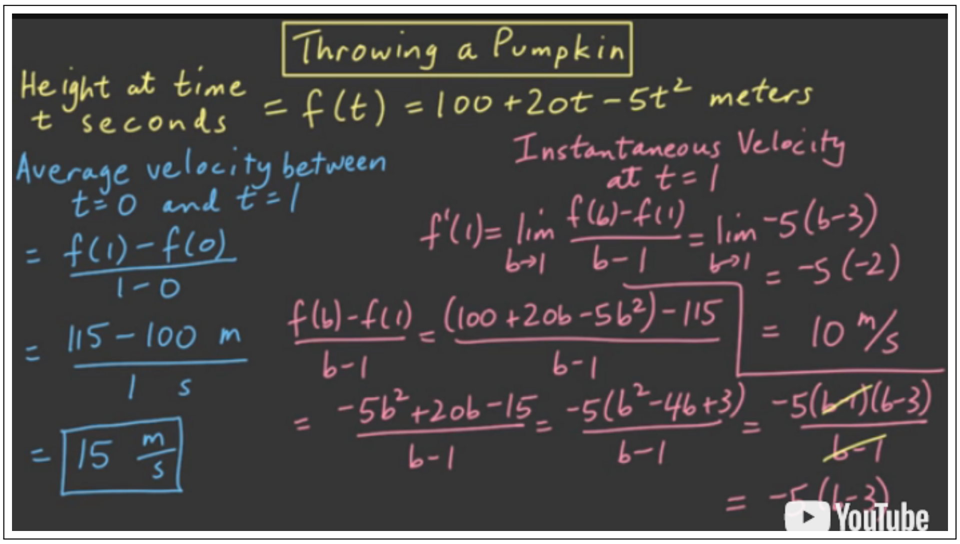
And simplifying a little bit, on the top, we're going to have minus 5b squared plus 20b minus 15 divided by b minus 1.

Now, remember that we want the limit as b approaches 1. But if we leave it in this form, then our numerator is going to be going to 0 as b approaches 1, and our denominator is also going to be going to 0. So that's not going to be good enough. We need to do some more work. And we're going to have to factor here.

So we're going to get on top minus 5 times b squared minus 4b plus 3. And on the bottom, we still have our b minus 1. And this polynomial we can factor further. We'll get minus 5 times b minus 1 times b minus 3 all over b minus 1. We get some cancellation here, and we're finally left with minus 5 times b minus 3.

So moving back up here, we say that this limit is the limit as b approaches 1 of minus 5 times b minus 3. Now this is continuous, so we can just plug in b equals 1. So we're going to get minus 5 times minus 2, which is 10.

And 10 what? Well, remember that the output of f is meters. The input that's going into f is measured in seconds, so the derivative of f should be measured in meters per second. So that's our instantaneous velocity at time t equals 1. It's 10 meters per second.



Let's take a moment and think about why this answer is different from the 15 meters per second that we got for the average velocity.

The average velocity is referring to a time interval from t equals 0 to t equals 1. And what happens during that time is — well, at t equals 0, that's when we throw the pumpkin. So it's starting with an initial upward velocity. And I'll use this arrow to denote that.

But then during the 1 second, gravity starts to take over. So the pumpkin is going to go slower and slower during the 1 second, until at the end of that period, which is at t equals 1 exactly, the pumpkin's velocity is slower than any velocity that came before.

So that's why the instantaneous velocity at t equals 1 is slower than the average velocity over the period from t equals 0 to t equals 1.

Let's do one other calculation really quickly. Let's find the instantaneous velocity at t equals 3.

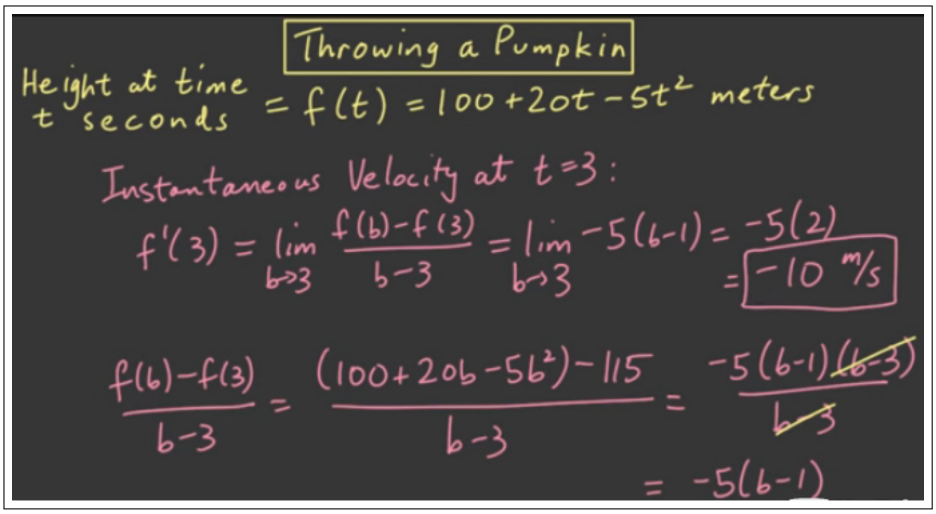
So I'll erase this. And we want f prime of 3. So by definition, that's just the limit as b approaches 3 of f of b minus f of 3 divided by b minus 3.

So to calculate the limit of this thing, let's try to simplify it first. We want f of b minus f of 3 divided by b minus 3. And that's equal to — well, f of b is just 100 plus 20b minus 5b squared. And then we're subtracting f of 3, which is 115. And then all of this is over b minus 3.

And notice that this numerator is exactly the same numerator as we had before in our f prime of 1 calculation. So it's going to factor in exactly the same way. So we're going to get minus 5 times b minus 1 times b minus 3. And all of this is divided by b minus 3, and we get cancellation.

So we end up with minus 5 times b minus 1. So putting that above, we want the limit as b approaches 3 of minus 5 times b minus 1. This is continuous now.

So we can just plug in b equals 3. And we're going to get minus 5 times 2, which is minus 10. And that's measured in meters per second.



So what's the deal with this minus sign? Why don't you take a moment and think about it.

3.1.11 A negative derivative?

Exercise 3.1.11-1: Understanding the answer: The height of the pumpkin at time t is $f(t) = 100 + 20t - 5t^2$. We got a negative number for $f'(3)$. Does this make sense?

- ☐ Yes, because at $t = 3$ the pumpkin is below the original height of the building.
- ☐ Yes, because at $t = 3$ the pumpkin has started to go down.
- ☐ No, because velocity cannot be negative.
- ☐ No, because height cannot be negative.

Video: Negative derivative

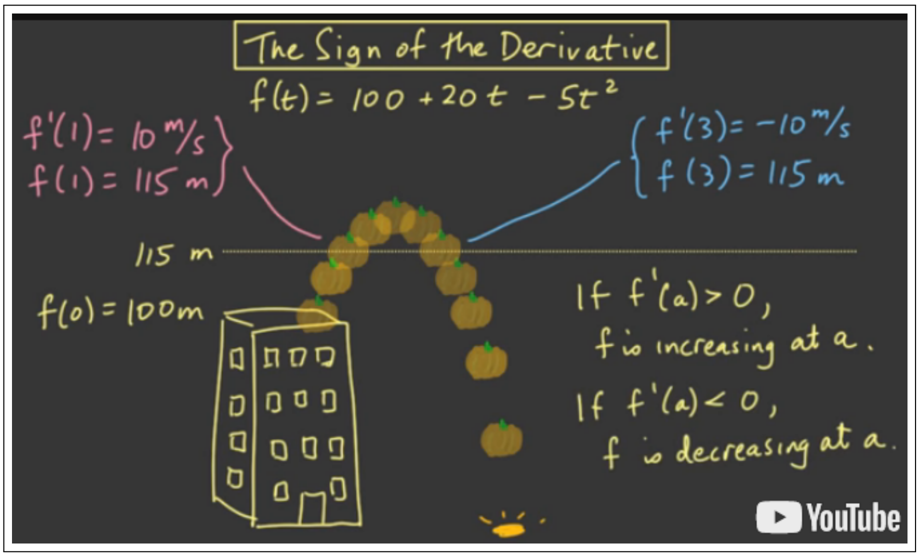
In our last video, we threw a pumpkin off a building. And its height was given by this function of time. We calculated these derivatives, f prime of 1 was 10 meters per second and f prime of 3 was minus 10 meters per second. So what does that mean?

Let's think about where this pumpkin has been, where its position has been. So at time t equals 0, we have that f of 0 is 100 meters. So that's the height of the building. That's where the pumpkin starts.

And we also calculated in the last video that f of 1 was 115 meters and f of 3 was also equal to 115 meters. So if we think about it, we can figure out what's going on. If 115 meters is here, then at t equals 1 second, the pumpkin is at that height on its way up. And at t equals 3 seconds, the pumpkin is, again, at that height, but this time it's on the way down. And eventually it goes splat.

So f prime of 1 is positive because it's measuring the instantaneous velocity when the pumpkin is moving upwards. Or other words, the height f is increasing. Whereas f prime of 3 is negative because it's measuring an instantaneous velocity at a time when f is decreasing, the height is going down.

So **the sign of a derivative tells us the direction in which the function is changing**. If f prime of a is positive, then f is increasing at that point. And if f prime of a is negative, then f is decreasing at that point.



We have one other thing to talk about. But first, we want you to get some practice calculating derivatives.

So why don't you calculate f prime of 2? And then we'll come back and discuss.

Exercise 3.1.11-2: Calculate derivative: Now you calculate $f'(2)$. Use the same limit definition as we did above!

Food for thought: What does an instantaneous velocity of 0 mean here?

$$\begin{aligned}
f'(2) &= \lim_{t\to 2}\frac{f(t)-f(2)}{t-2}\\
&= \lim_{t\to 2}\frac{(100+20t-5t^2)-120}{t-2}\\
&= \lim_{t\to 2}\frac{-20+20t-5t^2}{t-2}\\
&= \lim_{t\to 2}-5\frac{4-4t+t^2}{t-2}\\
&= \lim_{t\to 2}-5\frac{(t-2)^2}{t-2}\\
&= \lim_{t\to 2}-5(t-2) = 0m/s
\end{aligned}$$

3.1.12 Zero derivative

Video: Zero derivative

We've calculated f prime of 2 to be 0 meters per second. But what does that mean? Should we interpret it as saying that the pumpkin froze in midair? Well, no.

We know that pumpkins don't behave like that. Freezing in midair would require the velocity of the pumpkin to be 0 over a period of time. But what we have is a derivative. That's an instantaneous velocity. So it's a velocity only at the instant t equals 2 seconds.

To think about it a different way, we know that a positive derivative means that f is increasing. So the velocity would be upward. A negative derivative means that the velocity would be downward. There's an instant, however, in between. Right at the time the pumpkin reaches the top of its trajectory. The pumpkin has stopped going up. But it hasn't quite started to go down. And it's right at that moment that we say that the instantaneous velocity is 0.

3.1.13 Units of derivatives

Exercise 3.1.13-1: Units of derivatives question: A truck is going to make a trip of 500 km. If it is loaded with x pounds of cargo, its fuel efficiency on the trip will be $f(x)$ miles per gallon. What units is $f'(5000)$ measured in?

Note: The derivative of any function is always measured in units of the output of the function, divided by units of the input of the function. The output here is measured in miles per gallon (mi/g), and the input is measured in pounds (p), so the derivative is measured in miles per gallon per pound: $(mi/g)/p$, or $mi/(gp)$.

Exercise 3.1.13-2: Sign: Do you expect $f'(5000)$ to be positive, negative, or zero?

☐ positive
☐ negative
☐ zero

Note: If the weight of the cargo increases, the fuel efficiency of the truck will decrease. This means that $f'(5000)$ should be negative.

3.1.14 Summary

The Definition of the average rate of change of a function $f(x)$ over an interval $a \leq x \leq b$ is defined to be

$$\frac{f(b) - f(a)}{b - a} \tag{5}$$

The Definition of the derivative of a function $f(x)$ at a point $x = a$ is defined to be

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{6}$$