MIDTERM 4 - 18.01 - FALL 2017

Name:

Email:

Please put a check by your recitation section.

	Instructor	Time
☐	Miles Couchman	MW 1
☐	Kristin Kurianski	MW 1
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☐	Yu Pan	MW 11
☐	Jiewon Park	MW 12
☐	Jake Wellens	MW 12
☐	Siddharth Venkatesh	MW 2

Problem #	Max points possible	Actual score
1	15	
2	15	
3	15	
4	20	
5	15	
6	20	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**
- A formula sheet is attached.

Good luck!

Formula sheet

$$(\sin x)^2 + (\cos x)^2 = 1, \qquad (\sec x)^2 = (\tan x)^2 + 1$$

$$(\sin x)^2 = \frac{1}{2} - \frac{1}{2}\cos(2x), \qquad (\cos x)^2 = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\cos(2x) = (\cos x)^2 - (\sin x)^2, \qquad \sin(2x) = 2\sin x\cos x$$

$$\frac{d}{dx}\tan x = (\sec x)^2, \qquad \frac{d}{dx}\sec x = \sec x\tan x$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \tan x\,dx = \ln|\sec x| + C, \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Problem 1. (15 points) Use the trapezoid rule with two equal-length subintervals to approximate the integral

$$\int_e^{3e} \ln x \, dx.$$

Then decide whether your approximation is larger or smaller than the exact value and justify your answer geometrically.

Solution: a) We partition $[e, 3e] = [e, 2e] \cup [2e, 3e]$. Let $f(x) = \ln x$. Note that for $e \leq x \leq 3e$, we have

$$f(x) \geq 1, \tag{1}$$

$$f'(x) = 1/x > 0, \tag{2}$$

$$f''(x) = -1/x^2 < 0. \tag{3}$$

In particular, the graph of $y = f(x)$ is concave down on the domain of interest. Moreover, we compute that $f(e) = 1$, $f(2e) = \ln 2 + 1$, and $f(3e) = \ln 3 + 1$. Thus, the trapezoid approximation is as follows:

$$\begin{aligned} \frac{1}{2}f(e) \times e + f(2e) \times e + \frac{1}{2}f(3e) \times e &= \frac{1}{2} \times e + (\ln 2 + 1) \times e + \frac{1}{2} \times (\ln 3 + 1) \times e \\ &= e \times \left(2 + \ln 2 + \frac{1}{2}\ln 3\right) = e \times \left(2 + \ln \sqrt{12}\right). \end{aligned}$$

The trapezoid rule approximation yields an *underestimate* of the actual value of the integral because the graph of $y = f(x)$ is concave down on the domain of interest.

Problem 2. (15 points) Evaluate the following integral:

$$\int \arcsin x \, dx$$

Solution: We use integration by parts with $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}}\,dx$, $dv = dx$, $v = x$ to compute that

$$\begin{aligned}\int u\,dv = uv - \int v\,du &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx \\ &= x\arcsin x + \sqrt{1-x^2} + C,\end{aligned}$$

where to evaluate $-\int \frac{x}{\sqrt{1-x^2}}\,dx$, we used the substitution $w = 1-x^2$, $dw = -2x\,dx$ to obtain $-\int \frac{x}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw = \sqrt{w} + C = \sqrt{1-x^2} + C$.

Problem 3. (15 points) Evaluate the following integral:

$$\int (\tan x)^5 \sec x \, dx$$

Solution: Using the trig identity $(\sec x)^2 = (\tan x)^2 + 1$, we rewrite the integral as

$$\int \left\{(\sec x)^2 - 1\right\}^2 \times \sec x \tan x \, dx.$$

To evaluate the above integral, we make the substitution $u = \sec x$, $du = \sec x \tan x \, dx$ and rewrite it as

$$\int (u^2 - 1)^2 \, du = \int u^4 - 2u^2 + 1 \, du.$$

Finally, we note that the above integral evaluates to

$$\frac{u^5}{5} - \frac{2u^3}{3} + u + C = \frac{(\sec x)^5}{5} - \frac{2(\sec x)^3}{3} + \sec x + C.$$

Problem 4. (20 points) Evaluate the following integral:

$$\int \frac{1}{x^2(x^2-1)}\,dx$$

Solution: The general form of the partial fraction decomposition relation for $\dfrac{1}{x^2(x^2-1)} = \dfrac{1}{x^2(x-1)(x+1)}$ is

$$\frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}.$$

The cover-up method yields $B = -1$, $C = 1/2$, and $D = -1/2$. Hence,

$$\frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} - \frac{1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

Finally, setting $x = 2$, we find that $1/12 = A/2 - 1/4 + 1/2 - 1/6$ and thus $A = 0$. We have therefore derived the following partial fraction decomposition:

$$\frac{1}{x^2(x^2-1)} = -\frac{1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

Integrating both sides of the previous identity with respect to x, we conclude that

$$\begin{aligned}\int \frac{1}{x^2(x^2-1)}\,dx &= \frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\ &= \frac{1}{x} + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.\end{aligned}$$

Problem 5. (15 points) Evaluate the following integral:

$$\int \frac{1}{\sqrt{x^2 - 2x}}\, dx$$

Be sure to state your final answer in terms of x.

Solution: We first complete the square: $x^2 - 2x = (x-1)^2 - 1$. We then make the substitution $x - 1 = \sec\theta$, $dx = (\sec\theta)\tan\theta\, d\theta$, $(x-1)^2 - 1 = (\sec\theta)^2 - 1 = (\tan\theta)^2$ and compute that

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2 - 2x}}\, dx &= \int \frac{(\sec\theta)\tan\theta}{\tan\theta}\, d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C = \ln\left|x - 1 + \sqrt{x^2 - 2x}\right| + C.
\end{aligned}$$

Problem 6. (10 + 10 = 20 points) Consider the parametric curve defined by

$$x = t^4 + t + 1,$$
$$y = t^3 + t$$

for $t \geq 0$.

a) Write down an integral (do not evaluate the integral!) representing the arc length of the portion of the curve that connects the points $(x, y) = (1, 0)$ and $(x, y) = (3, 2)$.

b) The portion of the curve from part a) is revolved around the line $x = -1$ to generate a solid of revolution. Write down an integral (do not evaluate the integral!) representing the surface area of this solid. *To receive credit, be sure to explain the geometric meaning of the various factors in your integral.*

Solution:

a) The arc length is the integral of ds, where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. We now compute that

$$\frac{dx}{dt} = 4t^3 + 1, \qquad \frac{dy}{dt} = 3t^2 + 1.$$

Thus,

$$ds = \sqrt{16t^6 + 9t^4 + 8t^3 + 6t^2 + 2}\, dt.$$

Next, we note that the curve portion of interest corresponds to $0 \leq t \leq 1$. Thus, the arc length of the curve portion is equal to the following integral:

$$\int_{t=0}^{1} \sqrt{16t^6 + 9t^4 + 8t^3 + 6t^2 + 2}\, dt.$$

b) The surface area is the integral of $2\pi(x+1)\, ds$, where $2\pi(x+1)$ represents the circumference of the approximating cone pieces and ds is the slant edge length of the approximating cone pieces. That is, the surface area is

$$2\pi \int_{t=0}^{1} (t^4 + t + 2)\sqrt{16t^6 + 9t^4 + 8t^3 + 6t^2 + 2}\, dt.$$