- Derivatives of Products, quotients, sine, and cosine

  - Derivative Formulas
  - Specific examples: $\frac{d}{dx} x^n$, $\frac{d}{dx}\left(\frac{1}{x}\right)$, etc.
  - General examples: $(u+v)' = u' + v'$, $(cu)' = cu'$
    (c is a const.)

- Notation: $(u+v)(x) = u(x) + v(x)$; $uv(x) = u(x)v(x)$

- Theorem: $(u+v)' = u' + v'$ (General)

  Proof: $$(u+v)'(x) = \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x}$$

  $$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x}$$

  $$= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x} \right\}$$

  $$= u'(x) + v'(x).$$

- Similarly, we can prove $(cu)' = cu'$ when c is a constant.

- Derivatives of $\sin x$ and $\cos x$ (Specific)

- Last time: $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = 0$.

Therefore:

- $\frac{d}{dx}(\sin x)\Big|_{x=0} = \lim\limits_{\Delta x\to 0} \frac{\sin(0+\Delta x) - \sin(0)}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{\sin(\Delta x)}{\Delta x} = 1$
- $\frac{d}{dx}(\cos x)\Big|_{x=0} = \lim\limits_{\Delta x\to 0} \frac{\cos(0+\Delta x) - \cos(0)}{\Delta x} = \lim\limits_{\Delta x\to 0} \frac{\cos(\Delta x)-1}{\Delta x} = 0$

- Let's now compute the derivatives of $\sin x$ and $\cos x$ for all values of x.

- We will need the following trig formulas:

$\sin(a+b) = \sin a \cos b + \sin b \cos a.$

- $$\frac{d}{dx}\sin x = \lim_{\Delta x\to 0} \frac{\overbrace{\sin x \cos\Delta x + \cos x \sin\Delta x}^{\sin(x+\Delta x)} - \sin x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \left\{ \frac{\sin x(\cos\Delta x - 1)}{\Delta x} + \frac{\cos x \sin\Delta x}{\Delta x} \right\}$$

$$= \lim_{\Delta x\to 0} \sin x \left(\frac{(\cos\Delta x - 1)}{\Delta x} \right) + \lim_{\Delta x\to 0} \cos x \left(\frac{\sin\Delta x}{\Delta x} \right)$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

- Similarly, we can show that $\frac{d}{dx}\cos x = -\sin x$

- Product formula (General):

<u>Thm</u>: $(uv)' = u'v + uv'$

<u>Proof</u>: $$(uv)' = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)\,v(x+\Delta x) - u(x)\,v(x)}{\Delta x}$$

- Key idea: insert $0 = u(x+\Delta x)\,v(x) - u(x+\Delta x)\,v(x)$ into the numerator:

$$(uv)' = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)\,v(x) - u(x)v(x) + u(x+\Delta x)\,v(x+\Delta x) - u(x+\Delta x)\,v(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{ \left(\frac{u(x+\Delta x) - u(x)}{\Delta x} \right) v(x) + u(x+\Delta x) \left(\frac{v(x+\Delta x) - v(x)}{\Delta x} \right) \right\}$$

$$= \left(\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \right) \cdot v(x) + \overbrace{\left(\lim_{\Delta x \to 0} u(x+\Delta x) \right)}^{(= u(x),\ \text{since } u \text{ is continuous}} \cdot \left(\lim_{\Delta x \to 0} \frac{v(x+\Delta x) - v(x)}{\Delta x} \right)$$

$$= u'(x) \cdot v(x) + u(x) \cdot v'(x).$$

- We have assumed that u and v are differentiable, which implies they are

- Quotient Rule (General)
- Theorem: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
- Proof: Set $\Delta u = u(x+\Delta x) - u(x)$,
  $\Delta v = v(x+\Delta x) - v(x)$

Then $\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)} = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}$

common denominator

$$= \frac{(u+\Delta u)v - u(v+\Delta v)}{(v+\Delta v)v}$$

Cancel $uv - vu$

$$= \frac{(\Delta u)v - u(\Delta v)}{(v+\Delta v)v}.$$

. Thus, $\frac{1}{\Delta x}\left\{\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right\} = \frac{\left(\frac{\Delta u}{\Delta x}\right)v - u\left(\frac{\Delta v}{\Delta x}\right)}{(v+\Delta v)v} \longrightarrow \frac{u'v - uv'}{v^2}.$

as $\Delta x \to 0$

We used the fact that $\Delta v \to 0$ as $\Delta x \to 0$, since v is continuous.