MATH 18.01, FALL 2017 - PROBLEM SET # 9

Professor: Jared Speck

Due: by <u>Wednesday</u> 4:00pm on 12-6-17
(in the boxes outside of Room 4-174; write your name, recitation instructor, and recitation meeting days/time on your homework)

18.01 Supplementary Notes (including Exercises and Solutions) are available on the course web page: `http://math.mit.edu/~jspeck/18.01_Fall%202017/1801_CourseWebsite.html`. This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

Part I consists of exercises given and solved in the Supplementary Notes. It will be graded quickly, checking that all is there and the solutions not copied.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises given here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

You are encouraged to use graphing calculators, software, etc. to <u>check</u> your answers and to explore calculus. However, (unless otherwise indicated) we strongly discourage you from using these tools to <u>solve</u> problems, perform computations, graph functions, etc. An extremely important aspect of learning calculus is developing these skills. You will not be allowed to use any such tools on the exams.

Part I (15 points)

Notation: The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Lecture 27. (Thurs., Nov. 30) Polar coordinates. Area in polar coordinates.
 Read: 16.1; 16.2 (lightly, for the pictures); 16.3 to the top of p. 570; 16.5 to the middle of p. 581.
 Homework: Notes 4H: 1bfg, 2a, 3f; Notes 4I: 2, 3.

Lecture 28. (Fri., Dec. 1) Indeterminate forms. L'Hôpital's rule. Growth rate of functions.
 Read: 12.2; 12.3 (Examples 1-3, Remark 1).
 Homework: Notes 6A: 1befgj, 5, 6c.

You do not have to turn in the homework for Lectures 29-31, but you are expected to know the material for the final exam.

Lecture 29. (Tues., Dec. 5) Improper integrals.
   Read: 12.4; Notes INT.
   Homework: Notes 6B: 1, 2, 7afkm, 8c.

Lecture 30. (Thurs., Dec. 7) Infinite series and convergence tests. Infinite series. Simple convergence tests. Geometric series. Harmonic series.
   Read: pp. 439-442 (top). Comparison tests. pp. 451-453 (skip proof in Example 3). Integral test. pp. 455-457 (top).
   Homework: 7A: 1abc; 7B: 1abf, 2acde.

Lecture 31. (Tues., Dec. 12) Taylor series.
   Read: 14.4 through p. 498 (bottom); skip everything involving the remainder term $R_n(x)$. Differentiation and integration of series. Read: 14.3-p.490 (top); Examples 1-5.
   Homework: Notes 7C: 1ac; Notes 7D: 1adg, 2bde (skip radius of convergence).

Part II (27 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. *You must show your work; "bare" solutions will receive very little credit.*

iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem **0**" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (Thurs., Nov. 30 for parts a and b, Tues., Dec. 5 for part c; Area in polar coordinates, Improper integrals; $2 + 2 = 4$ points; turn in parts a and b only) Let $a > 0, b > 0$ be constants, and consider the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

a) Find the equation for the ellipse in polar coordinates.

b) Write down the integral that expresses the area of the ellipse in polar coordinates.

c) Calculate the area of the ellipse using polar integration. Explain why you can first consider only the domain $0 \leq \theta \leq \pi/2$ (where θ is the standard angular polar coordinate) and then multiply this answer by a well-chosen constant. Note that your answer will depend on a and b. *Hint: To carry out the integration, try multiplying the top and bottom of the integrand by $(\sec \theta)^2$. Then after making a substitution, you should be able to integrate the resulting expression without too much difficulty. The resulting integral will be an improper integral.*

Solution: a) $r^2(\dfrac{\cos^2 \theta}{a^2} + \dfrac{\sin^2 \theta}{b^2}) = 1.$

b) $A = \displaystyle\int_0^{2\pi} \dfrac{1}{2(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2})} \, d\theta.$

c) Observe that the ellipse equation is symmetrical along x axis and y axis. So the area

$$A = 4\int_0^{\frac{\pi}{2}} \dfrac{1}{2(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2})} \, d\theta = 4\int_0^{\frac{\pi}{2}} \dfrac{a}{2b \cos^2 \theta} \dfrac{ab}{1 + (\frac{a \tan \theta}{b})^2} \, d\theta$$

Let $u = \dfrac{a \tan \theta}{b}$, then $du = \dfrac{a}{b \cos^2 \theta}$. By u substitution,

$$A = 4 \int_0^\infty \frac{ab}{2} \frac{1}{1 + u^2} \, d\theta.$$

Using improper intergral,

$$A = \lim_{M \to \infty} 4 \int_0^M \frac{ab}{2} \frac{1}{1 + u^2} \, d\theta = \lim_{M \to \infty} 2ab \arctan \theta \Big|_0^M = \pi ab.$$

2. (Fri., Dec. 1; L'Hôpital's rule; 5 points) In this problem, you will revisit the ellipse you encountered in problem set 3A:

$$x^2 + \left(\frac{y}{a}\right)^2 = 1.$$

Recall that the eccentricity of the ellipse is

$$E = \sqrt{1 - a^2}.$$

As in problem set 3A, we imagine that the ellipse is rotated about the y axis to generate an oblate ellipsoid. The surface area S of the ellipsoid (viewed as a function of the eccentricity) is

$$S(E) = 2\pi \left(1 + f(E) \times (1 - E^2)\right),$$

where

$$f(E) = \begin{cases} \frac{\tanh^{-1}(E)}{E} & \text{if } E \neq 0, \\ 1 & \text{if } E = 0. \end{cases}$$

Also, in problem set 5, you showed that

$$\tanh^{-1}(E) = \ln \sqrt{\left|\frac{1 + E}{1 - E}\right|} = \frac{1}{2} \ln \left|\frac{1 + E}{1 - E}\right|.$$

Compute

$$\lim_{E \to 1^-} S(E).$$

Give a geometric interpretation of your answer.

Solution:

$$\lim_{E \to 1^-} S(E) = 2\pi + \pi \lim_{E \to 1^-} \frac{(1 - E^2) \ln \left|\frac{1+E}{1-E}\right|}{E} = 2\pi + \pi \lim_{E \to 1^-} \frac{\ln \left|\frac{1+E}{1-E}\right|}{\frac{E}{(1-E^2)}}.$$

It is a $\dfrac{\infty}{\infty}$ style, by L'Hôpital's rule,

$$\lim_{E \to 1^-} \frac{\ln \left|\frac{1+E}{1-E}\right|}{\frac{E}{(1-E^2)}} = \lim_{E \to 1^-} \frac{\frac{1-E}{1+E} \frac{2}{(1-E)^2}}{\frac{1+E^2}{(1-E^2)^2}} = \lim_{E \to 1^-} \frac{2(1 - E^2)}{1 + E^2} = 0.$$

So $\lim_{E \to 1^-} S(E) = 2\pi$.

Geometrically, when E approaches 1^-, $a \to 0$. The equation of the ellipse can be write as $(ax)^2 + y^2 = a^2$. When a goes to 0, the ellipse approaches to two points $(-1, 0)$ and $(1, 0)$ on the x-axis. So the involution along y-axis gives a circle with $S(E) = 2\pi$.

3. (Thurs., Nov. 1; Area in polar coordinates; $2 + 2 + 3 + 3 = 10$ points)

a) Use polar coordinates to find the area of the right triangle with vertices at $(x, y) = (0, 0)$, $(x, y) = (a, 0)$, and $(x, y) = (a, h)$. (This is one of the least convenient ways to find the area of a triangle.)

b) Find the equation in (x, y) coordinates for the curve

$$r = \frac{1}{2 + \sin \theta}.$$

Then sketch the curve.

c) Find the area of the region $0 \le r \le \dfrac{1}{2 + \sin \theta}$, $0 \le \theta \le \pi$ using the rectangular coordinate formula you found in part b).

d) Find the area of the region in part c) using polar coordinates. One way to evaluate the integral is to change variables to $\phi = \theta - \pi/2$ and then use the half angle formula $1 + \cos \phi = 2 \cos^2(\phi/2)$ followed by the inverse trig substitution $v = \tan(\phi/2)$. This area was already computed in part c), but the polar coordinate formula is still valuable because it gives the area of any sector, not just the one that is bounded by a horizontal line. The area swept out from the viewpoint of the focus of an ellipse, parabola, or hyperbola is related by Kepler's law to the speed of planets and comets.

Solution: a) We have the equation $r \cos \theta = a$, where θ ranges from 0 to $\arctan(\frac{h}{a})$. So the area

$$A = \int_0^{\arctan \frac{h}{a}} \frac{a^2}{2 \cos^2 \theta} \, d\theta = \frac{a^2}{2} \arctan \theta \Big|_0^{\arctan \frac{h}{a}} = \frac{ah}{2}.$$

b) $2r + r \sin \theta = 1$. By polar coordinates, we have $2\sqrt{x^2 + y^2} + y = 1$, which means $x^2 + y^2 = (\frac{y-1}{2})^2$. We have

$$x^2 + \frac{3}{4} \left(y + \frac{1}{3} \right)^2 = \frac{1}{3}.$$
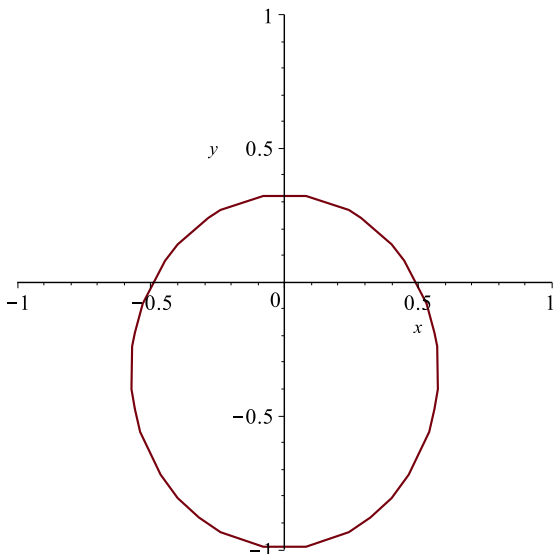
c) The region is the part where $y \geq 0$. We $y = 0$, $x = \pm 1/2$. So the area is

$$A = \int_{-1/2}^{1/2} \sqrt{\frac{4}{3}\left(\frac{1}{3} - x^2\right)} - \frac{1}{3} \, dx = \int_{-1/2}^{1/2} \frac{2}{3}\sqrt{1 - (\sqrt{3}x)^2} - \frac{1}{3} \, dx = \frac{2}{3}\int_{-1/2}^{1/2} \sqrt{1 - (\sqrt{3}x)^2} \, dx - \frac{1}{3}.$$

Let $x = \frac{1}{\sqrt{3}}\sin\theta$, then $dx = \frac{1}{\sqrt{3}}\cos\theta d\theta$.

$$\int_{-1/2}^{1/2} \sqrt{1 - (\sqrt{3}x)^2} \, dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{\sqrt{3}}\cos^2\theta \, d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2\sqrt{3}}(\cos 2\theta + 1) \, d\theta = \frac{1}{2\sqrt{3}}\left(\frac{1}{2}\sin 2\theta + \theta\right)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{1}{4} + \frac{\pi}{3\sqrt{3}}.$$

So $A = \frac{2}{3}\left(\frac{1}{4} + \frac{\pi}{3\sqrt{3}}\right) - \frac{1}{3} = \frac{\pi}{9\sqrt{2}} - \frac{1}{6}$.

d) By polar coordinates, we have

$$A = \int_0^\pi \frac{1}{2(2 + \sin\theta)^2} \, d\theta.$$

Let $\phi = \theta - \frac{\pi}{2}$, we have

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2(2 + \cos\phi)^2} \, d\phi.$$

By the double angle formula $\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}$, we have

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2(3\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2})^2} \, d\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2\cos^4\frac{\phi}{2}} \frac{1}{(3 + \tan^2\frac{\phi}{2})^2} \, d\phi.$$

Let $u = \tan\frac{\phi}{2}$, then $du = \frac{1}{2\cos^2\frac{\phi}{2}}$, we have that

$$A = \int_{-1}^1 \frac{u^2 + 1}{(3 + u^2)^2} \, du = \int_{-1}^1 \frac{1}{3 + u^2} - \frac{2}{(3 + u^2)^2} \, du.$$

$$\int_{-1}^1 \frac{1}{3 + u^2} \, du = \frac{1}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}}\Big|_{-1}^1 = \frac{\pi}{3\sqrt{3}}.$$

$$\int_{-1}^1 \frac{2}{(3 + u^2)^2} \, du = \frac{2}{9}\int_{-1}^1 \frac{1}{(1 + (\frac{u}{\sqrt{3}})^2)^2} \, du.$$

Let $\frac{u}{\sqrt{3}} = \tan\psi$, then $du = \sqrt{3}\sec^2\psi$. We have that

$$\int_{-1}^1 \frac{2}{(3 + u^2)^2} \, du = \frac{2}{9}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{3}\cos^2\psi \, d\psi = \frac{2}{9}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{3}\frac{\cos 2\psi + 1}{2} \, d\psi = \frac{\sqrt{3}}{9}\left(\frac{\sin 2\psi}{2} + \psi\right)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{1}{6} + \frac{\pi}{9\sqrt{3}}.$$

So $A = \frac{\pi}{3\sqrt{3}} - \left(\frac{1}{6} + \frac{\pi}{9\sqrt{3}}\right) = \frac{2\pi}{9\sqrt{3}} - \frac{1}{6}$.

4. (Tues., Dec. 5; Improper integrals; 0 points; you do not have to turn in this problem) In this problem, you will calculate a famous improper integral that arises in many distinct mathematical settings. Specifically, you will calculate the improper integral

$$I = \int_{-\infty}^\infty e^{-x^2} \, dx.$$

The interesting point is that even though we don't know how to exactly evaluate $\int_a^b e^{-x^2} \, dx$ when a and b are finite numbers, we can nonetheless exactly evaluate this integral when $a = -\infty$ and $b = \infty$.

a) Use a comparison argument to show that the integral converges to a finite positive number (that is, that I is a finite positive number).

b) Let $z = f(x) = e^{-x^2}$. Let R be the region in the (x, z) plane under the graph of $f(x)$ and above the x-axis for $0 \leq x < \infty$. The region R is revolved around the (vertical) z-axis to generate a solid of revolution S. Use the shell method (with vertical shells whose axes of symmetry are parallel to the z-axis) to show that the volume V of S is

$$V = \pi.$$

c) You will soon compute V using an alternate method, namely the method of slices. To this end, first consider a third axis y going "into" the page (so that the set of points of the form (x, y, z) is all of three-dimensional space and the z-axis is vertical). Let S be the solid from part b). Show that the top surface of S is the set of points in three-dimensional space with $z = e^{-(x^2+y^2)}$.

d) For each x value with $-\infty < x < \infty$, consider the infinite plane P_x that is perpendicular to the x-axis. This plane intersects the solid S (from parts b and c) in an infinite two-dimensional region that we will call S_x. Let $A(x)$ be the area of this infinite region. Show that

$$A(x) = e^{-x^2} I,$$

where I is the number from part a).

e) Explain why

$$V = \int_{-\infty}^{\infty} A(x) \, dx.$$

f) Using part e), explain why $V = I^2$ and use this fact to exactly determine the value of

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx.$$

Solution: a) First, observe that e^{-x^2} is a positive function, so $I > 0$. Notice that e^{-x^2} is an even function, we have that

$$I = 2 \int_0^\infty e^{-x^2} \, dx = 2 \left(\int_0^1 e^{-x^2} \, dx + \int_1^\infty e^{-x^2} \, dx \right).$$

Since $\int_0^1 e^{-x^2} \, dx$ is finite, we only need to consider $\int_1^\infty e^{-x^2} \, dx$. Consider the function xe^{-x^2} and notice that $x < e^{-x^2} < xe^{(-x^2)}$ for $x > 1$. Moreover we have

$$\int_1^\infty xe^{-x^2} \, dx = \lim_{M \to \infty} -\frac{1}{2} e^{-x^2} \Big|_1^M = \frac{1}{2} e^{-1},$$

which means $\int_1^\infty xe^{-x^2} \, dx$ converges. By the comparison test, $\int_1^\infty e^{-x^2} \, dx$ converges. So I is a positive finite number.

b)

$$V = \int_0^\infty 2\pi xe^{-x^2} \, dx = \lim_{M \to \infty} -\pi e^{-x^2} \Big|_0^M = \pi.$$

c) The top surface S is given by $z = e^{-r^2}$, where r is the radius on of the revolution circle on the xy–plane, i.e., $r^2 = x^2 + y^2$. Thus S is given by $z = e^{-(x^2+y^2)}$.

d) Notice that S_x has fixed x and may vary in y and z direction.

$$A(x) = \int_{-\infty}^{\infty} z \, dy = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dy = e^{-x^2} \int_{-\infty}^{\infty} e^{-y^2} \, dy = e^{-x^2} I.$$

e) The volume is an integration of the area $A(x)$ for all $x \in (-\infty, \infty)$, so $V = \int_{-\infty}^{\infty} A(x) \, dx$.

f)

$$V = \int_{-\infty}^{\infty} A(x) \, dx = \int_{-\infty}^{\infty} e^{-x^2} I \, dx = I \int_{-\infty}^{\infty} e^{-x^2} \, dx = I^2.$$

So $I = \sqrt{\pi}$.

5. (Fri., Dec. 1 for part a, Tues., Dec. 5 for parts b-e; L'Hôpital's rule, Improper integrals; 5 points; turn in part a only)

a) Let $r \geq 0$ be a number. Use a comparison argument and L'Hôpital's rule to evaluate

$$\lim_{x \to \infty} x^r e^{-x}.$$

Hint: Try comparing the limit to $\lim_{x \to \infty} x^n e^{-x}$, where $n \geq 0$ is a well-chosen integer that is related to r.

b) Let $r \geq 0$ be a number. Use part a and a comparison argument to show that the improper integral

$$\int_0^{\infty} x^r e^{-x} \, dx$$

converges. (Do not integrate by parts or try to evaluate the integral.)

c) Let $n \geq 0$ be an integer and denote $A(n) = \int_0^{\infty} x^n e^{-x} \, dx$. Use integration by parts to find the constant c_n for which

$$A(n + 1) = c_n A(n).$$

(Use part a to help you evaluate the infinite limit that comes up in the analysis.)

d) Find $A(0)$ and deduce from part c the formula for $A(n)$, $n = 0, 1, 2, \cdots$.

e) Show that the improper integral corresponding to $A(-1/2)$ converges and evaluate it using a change of variables. *Hint: Your answer to problem 4 is relevant.*

Solution: a)
Let n be the smallest integer that is bigger than r and thus we have that $0 < x^r e^{-x} < x^n e^{-x}$ for $x > 1$. Consider $\lim_{x \to \infty} x^n e^{-x} = \lim_{x \to \infty} \dfrac{x^n}{e^x}$, it is of $\frac{\infty}{\infty}$ type. Using L'Hôpital's rule for n times, we have that

$$\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} n \frac{x^{n-1}}{e^x} = \cdots = \lim_{x \to \infty} n! \frac{1}{e^x} = 0.$$

By the comparison argument, we have that $0 = \lim_{x \to \infty} 0 < \lim_{x \to \infty} x^r e^{-x} < \lim_{x \to \infty} x^n e^{-x} = 0$. Thus $\lim_{x \to \infty} x^r e^{-x} = 0$.

b) By part a), we have $\lim\limits_{x\to\infty} \left(\dfrac{x}{2}\right)^r e^{-\frac{x}{2}} = 0$ and thus $\lim\limits_{x\to\infty} x^r e^{-\frac{x}{2}} = 0$, which means after some finite number A, we have $x^r < e^{\frac{x}{2}}$. Thus we consider $\displaystyle\int_0^\infty x^r e^{-x}\, dx = \int_0^A x^r e^{-x}\, dx + \int_A^\infty x^r e^{-x}\, dx$. The first part is finite and the second part

$$\int_A^\infty x^r e^{-x}\, dx < \int_A^\infty e^{\frac{x}{2}} e^{-x}\, dx = \int_A^\infty e^{-\frac{x}{2}}\, dx = \lim_{M\to\infty} -2e^{-\frac{x}{2}}\Big|_A^M = 2e^{-A} < \infty.$$

Therefore the limit $\displaystyle\int_0^\infty x^r e^{-x}\, dx$ converges.

c) By integration by parts, we have

$$A(n+1) = \int_0^\infty x^{n+1} e^{-x}\, dx = \left(\lim_{M\to\infty} -x^{n+1} e^{-x}\Big|_0^M\right) + \int_0^\infty (n+1) x^n e^{-x}\, dx.$$

By part a), we have that $\lim\limits_{M\to\infty} -M^{n+1} e^{-M} = 0$, and thus $A(n+1) = (n+1)A(n)$, i.e., $c_n = n+1$.

d) $A(0) = \displaystyle\int_0^\infty e^{-x}\, dx = \lim_{M\to\infty} -e^{-x}\Big|_0^M = 1$ By part c), we have that

$$A(n) = nA(n-1) = \cdots = n!A(1) = n!.$$

e) $A(-1/2) = \displaystyle\int_0^\infty x^{-1/2} e^{-x}\, dx = \int_0^1 x^{-1/2} e^{-x}\, dx + \int_1^\infty x^{-1/2} e^{-x}\, dx$. For the first part, notice that

$0 < x^{-1/2} e^{-x} < x^{-1/2}$ for $0 < x < 1$ and $\displaystyle\int_0^1 x^{-1/2}\, dx = 2x^{1/2}\Big|_0^1 = 2$. For the second part, notice that

$0 < x^{-1/2} e^{-x} < e^{-x}$ for $x > 1$ and $\displaystyle\int_1^\infty e^{-x}\, dx$ converges as we did in part d). By the comparison test, the second part converges. So $A(-1/2)$ converges.

To compute $A(-1/2)$, we let $u = \sqrt{x}$, and $du = \frac{1}{2\sqrt{x}}dx$. So

$$A(-1/2) = \int_0^\infty 2e^{-u^2}\, du = \int_{-\infty}^\infty e^{-u^2}\, du = I.$$

By question 4, we know that $A(-1/2) = \sqrt{\pi}$

6. (Tues. Dec. 12; Taylor series; 0 points; you do not have to turn in this problem) Let

$$F(x) = \int_0^{x^3} e^{-t^2}\, dt.$$

Find the Taylor series at 0 for $F(x)$.

Solution: We know that $e^x = \displaystyle\sum_{n=0}^\infty \dfrac{x^n}{n!}$. So

$$e^{-x^2} = \sum_{n=0}^\infty \frac{(-x^2)^n}{n!} = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{n!}.$$

Let $G(x) = \int_0^x e^{-t^2} \, dt$, which is the antiderivative of e^{-x^2}. So $G(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$. Notice that $F(x) = G(x^3)$, so we have that

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{n!(2n+1)}.$$