Differentials & Antiderivatives

Differentials

New notation: $y = f(x)$

$$\boxed{dy = f'(x)\,dx}$$

- dy, dx are called differentials
- You can think of $\frac{dy}{dx} = f'(x)$ as a quotient of differentials
- One way this is used is for linear approximations:

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

Ex: Approximate $65^{1/3}$

Method 1 (review of linear approximation)

- $f(x) = x^{1/3}$
- $f'(x) = \frac{1}{3} x^{-2/3}$
- $f(x) \approx f(a) + f'(a)(x-a)$ = Eqn. for tangent line at a.
- $x^{1/3} \approx a^{1/3} + \frac{1}{3} a^{-2/3}(x-a)$ for x near a
- A good base point is $a = 64$ because $64^{1/3} = 4$
- Let $x = 65$
- $65^{1/3} \approx 64^{1/3} + \frac{1}{3} 64^{-2/3}(65-64) = 4 + \frac{1}{3}\left(\frac{1}{16}\right)(1) = 4 + \frac{1}{48} \approx 4.02$
- Similarly $(64.1)^{1/3} \approx 4 + \frac{1}{\ldots}$

- Method 2 (review)

  - $65^{1/3} = (64+1)^{1/3} = \left[64\left(1+\frac{1}{64}\right)\right]^{1/3} = 64^{1/3}\left(1+\frac{1}{64}\right)^{1/3}$
  $= 4\left(1+\frac{1}{64}\right)^{1/3}$

  - Next, use the approximation

$$\underset{\uparrow \text{ for } x \text{ near } 0}{(1+x)^r \approx 1+rx} \quad \text{with } r=\frac{1}{3} \text{ and } x=\frac{1}{64}$$

$$\Rightarrow 65^{1/3} \approx 4\left\{1+\frac{1}{3}\left(\frac{1}{64}\right)\right\} = \underbrace{4+\frac{1}{48}}_{\text{Same answer as method 1}}$$

- Method 3 (using differential notation)
  - $y = x^{1/3}\big|_{x=64} = 4$
  - $dy = \frac{1}{3}x^{-2/3}\big|_{x=64}\cdot dx = \frac{1}{3}\cdot\frac{1}{16}\,dx = \frac{1}{48}\,dx$
  - We set $dx=1$, since $x+dx=65$.
  - $dy=\frac{1}{48}$ when $dx=1$
  - $y+\Delta y \approx y+dy = 4+\frac{1}{48}$ = Same as in methods 1 & 2
  - The common theme behind all of these methods is:
    - $y = x^{1/3}$ $\quad \frac{dy}{dx}\Big| = \frac{1}{3}x^{-2/3}\Big|$ ...

• Antiderivatives

• $F(x) = \int f(x)\,dx$ means that F is an antiderivative of f.

• Other ways of saying this are:

$$F'(x) = f(x) \quad \text{, or} \quad dF = f(x)\,dx.$$

Basic examples

① $\int \sin dx = -\cos x + c$, where c is any constant

② $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ for $n \neq -1$

③ $\int \dfrac{dx}{x} = \ln|x| + c$ (this takes care of $n=-1$ in ②)

④ $\int \sec^2 x\,dx = \tan x + c$

⑤ $\int \dfrac{dx}{\sqrt{1-x^2}} = \arcsin x + c$

⑥ $\int \dfrac{dx}{1+x^2} = \arctan x + c$

- Let's prove ③ by taking the derivative of $\ln|x|$

Case i) $x > 0$ then $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$ ✓

Case ii) $x < 0$ then $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot\frac{d}{dx}(-x)$

$= \frac{1}{x}$

chain rule

- <u>Thm</u>: Antiderivatives are unique up to an additive constant. That is, if $F'(x) = f(x)$ and $G'(x) = f(x)$, then $G(x) = F(x) + C$ for some constant factor c.

Proof: $(G-F)' = G' - F' = f - f = 0$.

By the mean value theorem corollary,

$G(x) - F(x) = C$ for some constant c.

That is, $G(x) = F(x) + C$.

Ex: "Substitution"

- $\int x^3(x^4+2)^5\,dx$.

Substitution $u = x^4+2$, $du = 4x^3dx$, $(x^4+2)^5 = u^2$,
$x^3dx = \frac{1}{4}\,du$

- Hence $\int x^3(x^4+2)^5dx = \frac{1}{4}\int u^5\,du = \frac{u^6}{4\cdot 6} + 3 = \frac{1}{24}(x^4+2)^6 + C$

Ex: "Guess"

- We guess that $\int \frac{x}{\sqrt{1+x^2}}\,dx = (1+x^2)^{1/2} + C$.

To test our guess, we differentiate:

$$\frac{d}{dx}(1+x^2)^{1/2} = \frac{1}{2}(1+x^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{1+x^2}}.$$

So we were right

Ex: "Guess" . We guess that $\int e^{6x}dx = e^{6x} + C$.

We test by differentiation: $\frac{d}{dx}e^{6x} = 6e^{6x}$.

So we were off by a factor of 6.

The correct answer is therefore $\frac{1}{6}e^{6x} + C$.

Ex: $\int x e^{-x^2} dx$

We guess $e^{-x^2} + C$.

we test by differentiating: $\frac{d}{dx} e^{-x^2} = -2x e^{-x^2}$.

So we were off by a factor of -2.

The correct answer is $-\frac{1}{2} e^{-x^2}$.

Ex:

- $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C$
- Another answer is $-\frac{1}{2} \cos^2 x + C$. Both are correct.

We will test by differentiating:

- $\frac{d}{dx} \left(\frac{1}{2} \sin^2 x \right) = \frac{1}{2} \cdot 2 \cdot \sin x \cdot \cos x = \sin x \cos x$ ✓
- $\frac{d}{dx} \left(-\frac{1}{2} \cos^2 x \right) = -\frac{1}{2} \cdot 2 \cdot \cos x \cdot (-\sin x) = \sin x \cos x$ ✓
- The two answers differ only by a constant (because $\sin^2 x + \cos^2 x = 1$).

- Ex $\int \frac{dx}{x \ln x}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln(x)| + C.$$