18.01 - Problem Set #3A, Part II Solutions

Problem 1.

a) We find,

$$\begin{aligned} v'(9) &\approx \frac{v(9)-v(8)}{1} = \frac{90.744-79.931}{1} = 10.813 \\ v'(10) &\approx \frac{v(10)-v(9)}{1} = \frac{101.983-90.744}{1} = 11.239 \end{aligned}$$

b) We find,

$$v''(10) \approx \frac{v'(10)-v'(9)}{1} = \frac{11.239-10.813}{1} = 0.426$$

c) A quadratic approximation of $v(d)$ near $d=10$ is,

$$v(d) \approx v(10) + v'(10)(d-10) + \frac{v''(10)}{2}(d-10)^2$$

Using our approximations of $v'(10)$ and $v''(10)$ from parts a) and b) gives,

$$v(d) \approx 101.983 + 11.239(d-10) + \frac{0.426}{2}(d-10)^2 \tag{1}$$

Using equation (1), we can now estimate the value $v(11)$:

$$v(11) \approx 101.983 + 11.239 + \frac{0.426}{2} = 113.435$$

Problem 2.

a) To show that $f(E)$ is continuous at $E=0$ we must show that,

$$\lim_{E\to 0^+} \frac{\tanh^{-1}(E)}{E} = f(0) = 1 \tag{2}$$

Note that we are only considering the right-handed limit as $E \geq 0$. We proceed by evaluating the limit:

$$\begin{aligned} \lim_{E\to 0^+} \frac{\tanh^{-1}(E)}{E} &= \lim_{E\to 0^+} \frac{\tanh^{-1}(E)-\tanh^{-1}(0)}{E-0} \\ &= \left.\frac{d}{dE}\tanh^{-1}(E)\right|_{E=0} \\ &= \left.\frac{1}{1-E^2}\right|_{E=0} \\ &= 1 \end{aligned}$$

We have shown that equation (2) is true and so $f(E)$ is continuous at $E=0$.

b)
Method 1: Implicit Differentiation

$$\begin{aligned} y &= \tanh^{-1}(E) \\ \tanh(y) &= E \end{aligned}$$

Taking the derivative of both sides gives,

$$\begin{aligned} \frac{d}{dE}\tanh(y) &= \frac{d}{dE}(E) \\ \operatorname{sech}^2(y)\frac{dy}{dE} &= 1 \\ \frac{dy}{dE} &= \frac{1}{\operatorname{sech}^2(y)} \end{aligned}$$

Using the identity $\operatorname{sech}^2(y) = 1 - \tanh^2(y)$ we find,

$$\begin{aligned} \frac{dy}{dE} &= \frac{1}{1-\tanh^2(y)} \\ &= \frac{1}{1-E^2} \end{aligned}$$

Method 2: Explicit Differentiation
First, we find an explicit expression for $y = \tanh^{-1}(x)$:

$$\begin{aligned} \tanh(y) &= E \\ \frac{e^y - e^{-y}}{e^y + e^{-y}} &= E \\ (1-E)\,e^y &= (1+E)\,e^{-y} \\ e^{2y} &= \frac{1+E}{1-E} \\ e^y &= \sqrt{\frac{1+E}{1-E}} \\ y &= \frac{1}{2}\ln\left(\frac{1+E}{1-E}\right) \qquad (3) \end{aligned}$$

We now differentiate equation (3):

$$\begin{aligned} \frac{dy}{dE} &= \frac{1}{2\left(\frac{1+E}{1-E}\right)}\left(\frac{1-E+1+E}{(1-E)^2}\right) \\ &= \frac{1-E}{1+E}\left(\frac{1}{(1-E)^2}\right) \\ &= \frac{1}{1-E^2} \end{aligned}$$

c) We know that $(1+x)^r \approx 1 + rx$. In this case, we have $x = -E^2$ and $r = -1$ and so,

$$\frac{1}{1-E^2} \approx 1 + E^2$$

d) We are told to assume that,

$$\begin{aligned}\frac{d}{dE}\left(B_0 + B_1E + B_2E^2 + B_3E^3\right) &= 1 + E^2\\ B_1 + 2B_2E + 3B_3E^2 &= 1 + E^2\end{aligned}$$

From this, we find that,

$$B_1 = 1, \quad B_2 = 0, \quad B_3 = \frac{1}{3}$$

We also know that $B_0 = \tanh^{-1}(0) = 0$. So, our cubic approximation is,

$$\tanh^{-1}(E) \approx E + \frac{E^3}{3} \tag{4}$$

e) Using equation (4), we find that,

$$f(E) = \frac{\tanh^{-1}(E)}{E} \approx \frac{E + \frac{E^3}{3}}{E} = 1 + \frac{E^2}{3} \tag{5}$$

f) Using equation (5), we find that,

$$\begin{aligned} S(E) &= 2\pi\left(1 + f(E)\left(1 - E^2\right)\right)\\ &\approx 2\pi\left(1 + \left(1 + \frac{E^2}{3}\right)\left(1 - E^2\right)\right)\\ &\approx 2\pi\left(1 + 1 - E^2 + \frac{E^2}{3}\right)\\ &= 4\pi\left(1 - \frac{E^2}{3}\right) \end{aligned} \tag{6}$$

g) From equation (6), we see that for $E > 0$, $S(E) < S(0)$. Therefore, slightly squashing the sphere causes the surface area to decrease.

Problem 3.

18.

First, we determine the critical points of $y(x)$ by examining the derivative,

$$\begin{aligned} y'(x) &= mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1}\\ &= x^{m-1}(1-x)^{n-1}\left[m(1-x) - nx\right]\\ &= (m+n)\,x^{m-1}(1-x)^{n-1}\left[\frac{m}{m+n} - x\right] \end{aligned}$$

We see that there are three critical points at,

$$x = 0, \frac{m}{m+n}, 1$$

a) Consider the sign of $y'(x)$ around $x = 0$:

	$x < 0$	$0 < x < \frac{m}{m+n}$
x^{m-1} (odd function)	-	$+$
$(1-x)^{n-1}$	$+$	$+$
$\frac{m}{m+n} - x$	$+$	$+$
$y'(x)$	-	$+$

We see that $x = 0$ is a minimum if m is even.

b) Consider the sign of $y'(x)$ around $x = 1$:

	$\frac{m}{m+n} < x < 1$	$1 < x$
x^{m-1}	+	+
$(1-x)^{n-1}$ (odd function)	+	-
$\frac{m}{m+n} - x$	-	-
$y'(x)$	-	+

We see that $x = 1$ is a minimum if n is even.

c) Consider the sign of $y'(x)$ around $x = \frac{m}{m+n}$:

	$0 < x < \frac{m}{m+n}$	$\frac{m}{m+n} < x < 1$
x^{m-1}	+	+
$(1-x)^{n-1}$	+	+
$\frac{m}{m+n} - x$	+	-
$y'(x)$	+	-

We see that $x = \frac{m}{m+n}$ is always a maximum.

22.

In order for $f(x)$ to have critical points at $x = -2$ and $x = 1$, $f'(x)$ must have zeros at $x = -2$ and $x = 1$. Further, in order for $x = -2$ to be a maximum and $x = 1$ to be a minimum, we need $f'(x) > 0$ for $x < -2$, $f'(x) < 0$ for $-2 < x < 1$, and $f'(x) > 0$ for $x > 1$. We can satisfy these requirements by choosing,

$$\begin{aligned} f'(x) &= (x+2)(x-1) \\ &= x^2 + x - 2 \end{aligned}$$

By inspection, we find that the function $f(x)$ that corresponds to this derivative is,

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x$$

Problem 4.

12.

To identify points of inflection, we need to consider the second derivative of $y = \frac{12}{x^2} - \frac{12}{x}$.

$$\begin{aligned} y'(x) &= 12\left(-\frac{2}{x^3} + \frac{1}{x^2}\right) \\ y''(x) &= 12\left(\frac{6}{x^4} - \frac{2}{x^3}\right) \\ &= \frac{24(3-x)}{x^4} \end{aligned}$$

We see that $y''(x)$ has a single zero at $x = 3$ and that $y''(x) > 0$ for $x < 3$ and $y''(x) < 0$ for $x > 3$. Therefore, there is a single point of inflection at $x = 3$ and the graph is concave up for $x < 3$ and concave down for $x > 3$.

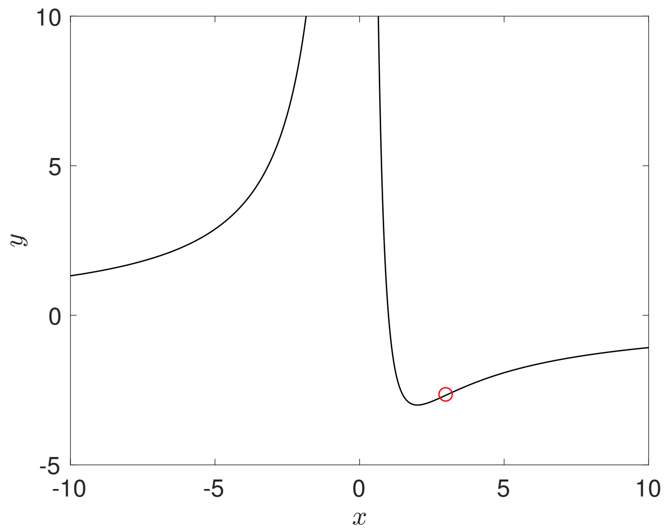


Figure 1: A plot of $y = 12/x^2 - 12/x$. There is a vertical asymptote at $x = 0$ and the single inflection point at $x = 3$ is marked.

16.

It is not possible. Assume that $f'(x) < 0$ and $f''(x) < 0$ for all x. This means that for two points $x_2 > x_1$, $f'(x_2) < f'(x_1) < 0$ (i.e. the slope of the graph gets more and more negative as x increases). Therefore, since the function f decreases at a faster and faster rate as x increases, if $f(x_0) > 0$ for some x_0, the graph $y = f(x)$ must eventually cross the x-axis at some $x > x_0$.

18.

Using implicit differentiation we find,

$$\begin{aligned}
\frac{d^2}{dx^2}\left(x^2+y^2\right) &= \frac{d^2}{dx^2}\left(a^2\right) \\
\frac{d}{dx}\left(2x+2y\frac{dy}{dx}\right) &= 0 \qquad (7) \\
1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 &= 0 \\
\frac{d^2y}{dx^2} &= -\frac{1+\left(\frac{dy}{dx}\right)^2}{y} \qquad (8)
\end{aligned}$$

As the numerator of the RHS of equation (8) is always positive, the sign of y'' is always opposite to the sign of y.

Finally, we can re-express equation (8) in terms of x. From equation (7), we see that $\frac{dy}{dx} = -\frac{x}{y}$. Substituting this into equation (8) gives,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{1+\frac{x^2}{y^2}}{y} \\
&= -\frac{x^2+y^2}{y^3}
\end{aligned}$$

As $x^2 + y^2 = a^2$ is a multi-valued function, we have $y = \pm\sqrt{a^2 - x^2}$, and so,

$$\frac{d^2y}{dx^2} = \begin{cases} -\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}} & \text{for top half of circle } (y>0) \\ \frac{a^2}{(a^2-x^2)^{\frac{3}{2}}} & \text{for bottom half of circle } (y<0) \end{cases}$$

24.

To identify points of inflection, we need to consider the second derivative of y:

$$\begin{aligned} y'(x) &= 3ax^2 + 2bx + c \\ y''(x) &= 6ax + 2b \end{aligned}$$

We see that there is a single inflection point at,

$$\begin{aligned} 0 &= 6ax + 2b \\ x &= -\frac{b}{3a} \end{aligned}$$

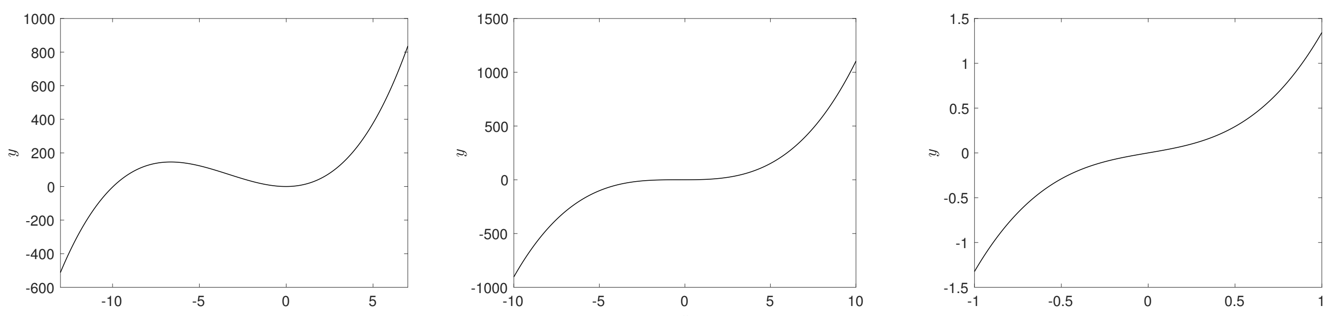
Next, we consider the critical points where $y'(x) = 0$:

$$\begin{aligned} 3ax^2 + 2bx + c &= 0 \\ x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\ &= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} \end{aligned}$$

There are three cases to consider:

1. $b^2 > 3ac$: $y(x)$ has one critical point to the left of the inflection point and one critical point to the right of the inflection point.
2. $b^2 = 3ac$: $y(x)$ has a single critical point at the point of inflection.
3. $b^2 < 3ac$: $y(x)$ no critical points.

An example of each case is shown below.



(a) Case 1: $y = x^3 + 10x^2 + x/3$ (b) Case 2: $y = x^3 + x^2 + x/3$ (c) Case 3: $y = x^3 + x^2/100 + x/3$

Figure 2