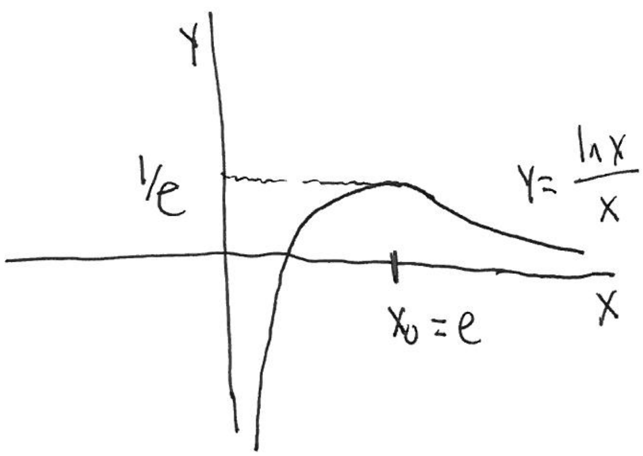
Min-Max Problems

Ex: $y = \frac{\ln x}{x}$ (Last time)

$x > 0$



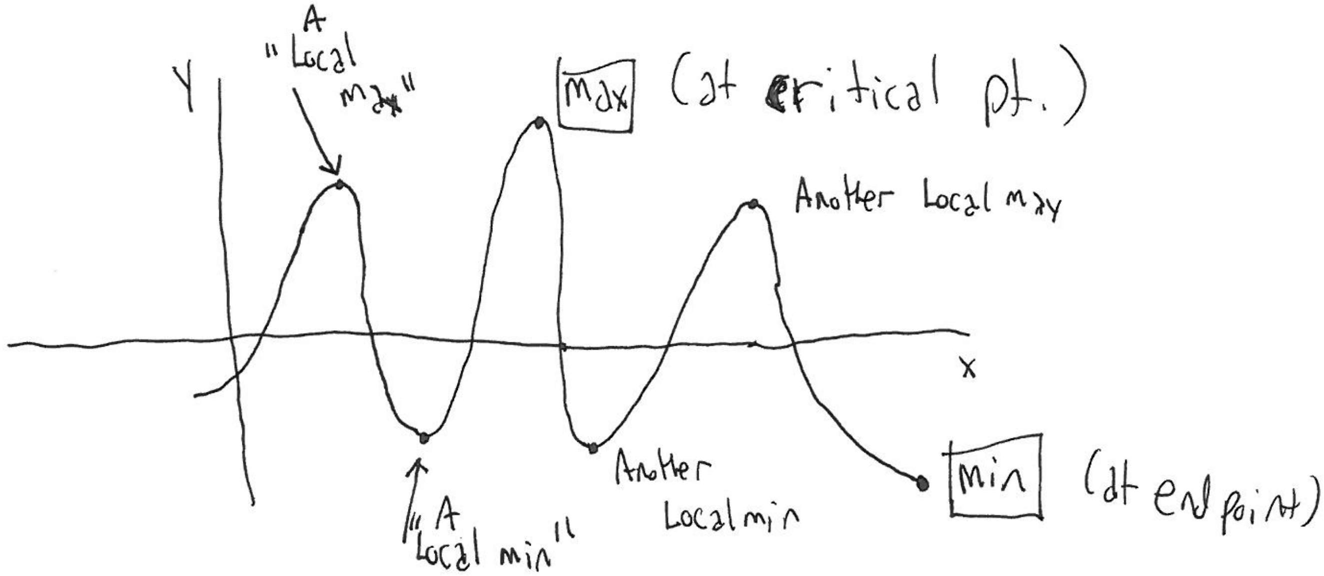
- Q1: At which point is the max achieved?

  A1: $x = e$

- Q2: What is the maximum value?

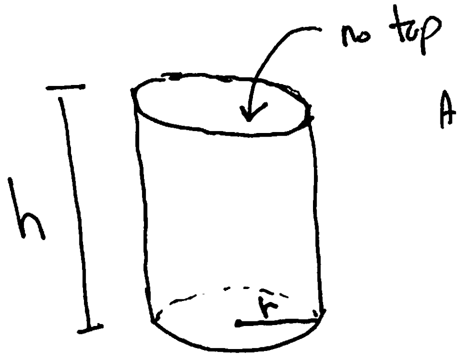
  A2: $\frac{1}{e}$

Ex:



- Big idea: Search for the max among critical points and end points (and also points where the function is not differentiable, if there are any

Ex.:



A can of radius r and height h

Q: Suppose the volume of the can is known to be V. Find the open-topped can with the least surface area.

Steps

① Picture ✓

② Decide on variables: r, h, V, S = surface area

③ ⓐ Identify constraints: $V = \pi r^2 h$ = constant

ⓑ Find a formula for the variable you want to solve for:

$$S = \pi r^2 + 2\pi r h = \text{area of bottom} + \text{area of side}$$

· The goal is to minimize S assuming that V is constant.

④ Use the constraint to express everything in terms of the variable r and the constant V:

$$h = \frac{V}{\pi r^2} \; ; \; S = \pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = \pi r^2 + \frac{2V}{r}$$

5. Find the critical points (at which $\frac{d\$}{dr} = 0$) and the endpoints. $\$$ will achieve its max & min at one of these places.

$$\frac{d\$}{dr} = 2\pi r - \frac{2V}{r^2} = 0$$

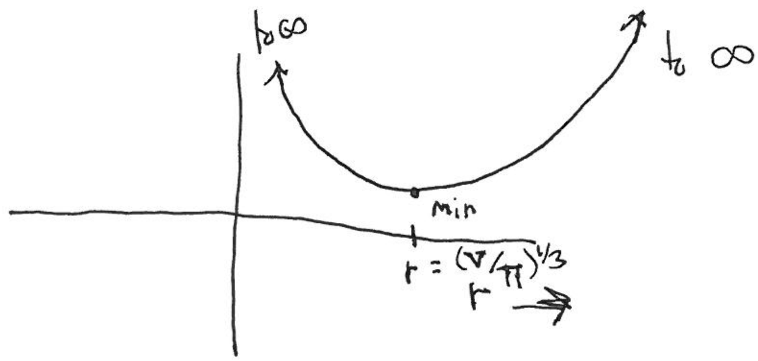
$$\Rightarrow \pi r^3 - V = 0$$

$$\Rightarrow r^3 = \frac{V}{\pi} \Rightarrow r = \left(\frac{V}{\pi}\right)^{1/3}.$$

- The endpoints are $r = 0$, "$r = \infty$"

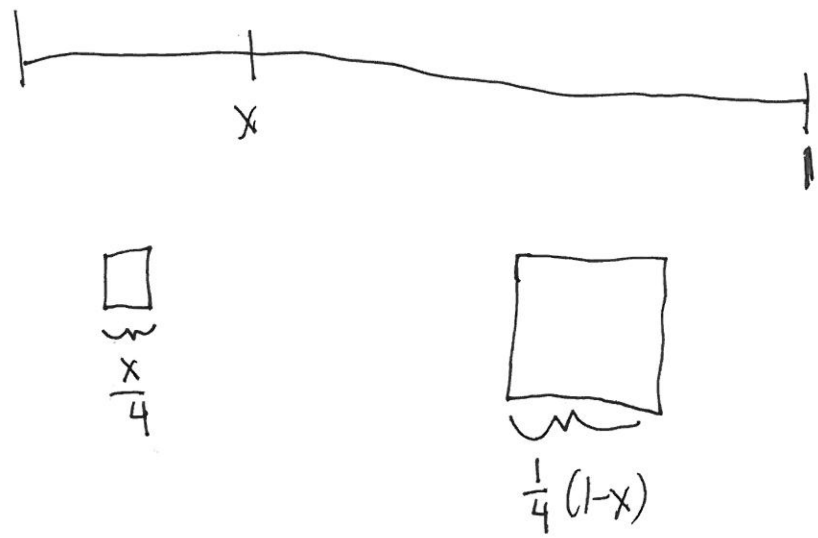
$$\$ = \pi r^2 + \frac{2V}{r}, \quad 0 < r < \infty$$

- As $r \to 0^+$, the second term goes to ∞, so $\$ \to \infty$ too
- As $r \to \infty$, the first term goes to ∞, so $\$ \to \infty$ too
- Thus, the minimum is achieved at the critical pt. $r = \left(\frac{V}{\pi}\right)^{1/3}$, and not at an endpoint.



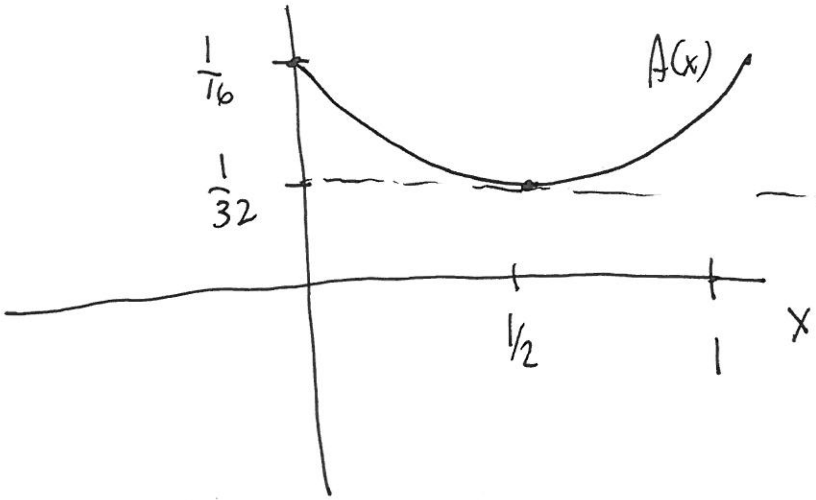
- We still need to find the minimum value of S at the critical point, and also the values of r and h:
- $r = \left(\frac{V}{\pi}\right)^{1/3}$; • $h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\frac{V}{\pi}\right)^{2/3}} = \left(\frac{V}{\pi}\right)^{1/3}$
- $S = \pi r^2 + \frac{2V}{r} = \pi\left(\frac{V}{\pi}\right)^{2/3} + 2V\left(\frac{V}{\pi}\right)^{1/3} = 3\pi^{-1/3}V^{2/3}$.

Ex: Consider a ~~[illegible]~~ wire of length 1, cut into two pieces. Bend each piece into a square. Let's figure out where to cut the wire in order to enclose as much area in the two squares as possible.



- The first square has sides of length $\frac{x}{4}$. Its area is $\frac{x^2}{16}$. The second square has length $\frac{1-x}{4}$. Its area is $\frac{(1-x)^2}{16}$.
- The total area is $A(x) = \frac{x^2}{16} + \frac{(1-x)^2}{16}$
- $A'(x) = \frac{1}{16}\left(2x + 2(1-x)(-1)\right) = \frac{1}{16}(4x-2) \overset{?}{=} 0$ to find critical pts.
  - Thus, $x = \frac{1}{2}$ is a critical pt.
  - And $A(\frac{1}{2}) = \frac{(\frac{1}{2})^2}{16} + \frac{(\frac{1}{2})^2}{16} = \frac{1}{32}$ is the corresponding critical value.
- Now we have to check the end points $x=0$ and $x=1$:

$A(0) = \frac{0}{16} + \frac{1}{16} = \frac{1}{16}$. $A(1) = \frac{1}{16} + \frac{0}{16} = \frac{1}{16}$



- We see that the minimum area was achieved when $x = 1/2$, and the maximum when $x=0$ or $x=1$.
  - The max corresponds to using the whole length to make one square.

⊛ Don't forget to check the endpoints.