MATH 18.01 - FINAL EXAM REVIEW: SUMMARY OF SOME KEY CONCEPTS

18.01 Calculus, Fall 2014	Professor: Jared Speck

a. Parametric curves
  (a) Are curves in the (x, y) plane expressed as

$$x = F(t),$$
$$y = G(t),$$

  $a \leq t \leq b$, where t is called the *parameter*.

b. Arc length of a curve
  (a) Arc length is equal to $\int_a^b ds$.
  (b) a is the parameter starting point, b is the parameter end point.
  (c) For curves in parametric form, $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(F'(t))^2 + (G'(t))^2}\, dt$ (Pythagorean theorem).
  (d) For curves $y = f(x)$, the formula reduces to $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (f'(x))^2}\, dx$ (and x is the parameter).

c. Surface area of a solid formed by revolving a curve around the $x-$axis (for revolution around the $y-$axis, interchange the roles of x and y in everything that follows)
  (a) Divide the surface into small strips that are portions of cones (the cone strip radii are parallel to the $y-$axis, and the cone strip axes of symmetry are parallel to the $x-$axis).
  (b) Surface area is given by
  $\int$ conical strip circumference $\times$ slant edge length
  $= \int 2\pi$ conical strip radius $\times\, ds$
  $= \int_{t=a}^{t=b} 2\pi \overbrace{G(t)}^{y} \overbrace{\sqrt{(F'(t))^2 + (G'(t))^2}}^{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}\, dt.$
  (c) a is the parameter starting point, b is the parameter end point.
  (d) For curves $y = f(x)$, the formula reduces to $\int_{x=a}^{x=b} 2\pi \overbrace{f(x)}^{y} \overbrace{\sqrt{1 + (f'(x))^2}}^{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\, dx.$

d. Polar coordinates
  (a) $x = r\cos\theta$, $y = r\sin\theta$
  (b) In the standard formulation, $r = \sqrt{x^2 + y^2}$, θ is the polar angle, and $0 \leq \theta < 2\pi$

(c) Area in polar coordinates: Area under the curve $r = f(\theta)$ in between the angles θ_1 and θ_2 is given by Area $= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} [f(\theta)]^2\, d\theta$

e. L'Hôpital's rule

(a) Sometimes allows one to evaluate limits of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 0^0, 1^∞

(b) Many of the above limits can be massaged into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, where L'Hôpital's rule can sometimes directly be applied. For example, the 0^0 case can be massaged into the $\frac{0}{0}$ case with the help of $\ln$.

(c) In the "$\frac{0}{0}$" case: If f, g are differentiable functions, a is a finite number, $f(a) = g(a) = 0$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Furthermore, it is sometimes true that $L = \frac{f'(a)}{g'(a)}$ (for example, when $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $g'(a) \neq 0$).

(d) In the "$\frac{\infty}{\infty}$" case: If f, g are differentiable functions, a is a finite number, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

(e) Analogous statements hold if we replace $\lim_{x\to a}$ with $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.

f. Improper integrals

(a) If $f(x)$ is continuous for $0 \le x < \infty$, then by definition, $\int_0^\infty f(x)\, dx = \lim_{M\to\infty} \int_0^M f(x)\, dx$

(i) If the limit exists, we say the improper integral *converges*. Otherwise, we say it *diverges*.

(b) If $f(x)$ is continuous for $a < x \le b$ but is *not* continuous at $x = a$, then by definition, $\int_a^b f(x)\, dx = \lim_{x_0\to a^+} \int_{x_0}^b f(x)\, dx$

(i) If the limit exists, we say the improper integral *converges*. Otherwise, we say it *diverges*.

g. Infinite series

(a) Are series of the form $\sum_{k=0}^\infty a_k = a_0 + a_1 + a_2 + a_3 + \cdots$

(b) By definition, $\sum_{k=0}^\infty a_k = \lim_{M\to\infty} S_M$, where $S_M = \sum_{k=0}^M a_k = a_0 + a_1 + a_2 + \cdots + a_M$ is the M^{th} partial sum.

(i) If $\lim_{M\to\infty} S_M$ exists, we say the series *converges*. Otherwise, we say it *diverges*.

(c) Geometric series: $\sum_{k=0}^\infty x^k = \frac{1}{1-x}$ if $|x| < 1$. $\sum_{k=0}^\infty x^k$ diverges if $|x| \ge 1$.

(d) Comparison: If $0 \le a_k \le b_k$ for all large k, and if $\sum_{k=0}^\infty a_k$ diverges, then $\sum_{k=0}^\infty b_k$ diverges too (divergence of smaller $\implies$ divergence of bigger). If $0 \le a_k \le b_k$ for all large k, and if $\sum_{k=0}^\infty b_k$ converges, then $\sum_{k=0}^\infty a_k$ converges too (convergence of bigger $\implies$ convergence of smaller).

(e) Limit comparison test: If $a_k \ge 0, b_k \ge 0$ for all large k and $a_k \sim b_k$, then $\sum_{k=0}^\infty a_k$ converges if and only if $\sum_{k=0}^\infty b_k$ converges. Here, $a_k \sim b_k$ means that there exists a non-zero number L such that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.

(f) Integral comparison: If $f(x)$ is continuous, $f(x) \ge 0$ for all x, and $f(x)$ is decreasing for all large x, then $\sum_{k=0}^\infty f(k)$ converges if and only if the improper integral $\int_{x=0}^\infty f(x)\, dx$ converges.

h. Taylor's series with base point $b = 0$

(a) For x near $0 : f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
(b) $a_n = \frac{f^{(n)}(0)}{n!}$, where $f^{(n)}$ is the n^{th} derivative of f
(c) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
(d) $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \cdots$
(e) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \mp \cdots$
(f) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots$

i. Taylor's series with base point b

(a) For x near $b : f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + a_3(x-b)^3 + \cdots$
(b) $a_n = \frac{f^{(n)}(b)}{n!}$
(c) $x^A = b^A + Ab^{A-1}(x-b) + \frac{A(A-1)b^{A-2}}{2!}(x-b)^2 + \frac{A(A-1)(A-2)b^{A-3}}{3!}(x-b)^3 + \cdots$