18.01 (Fall 14) Solution to Problem Set 8

Part II

1. **solution**

   a) The integration of $\sec x$ is as following.

   $$\int \sec x dx = \int \frac{\cos x}{1-\sin^2 x} dx.$$

   Let $u = \sin x$, we have $du = \cos x dx$, and therefore

   $$\begin{aligned}\int \frac{\cos x}{1-\sin^2 x} dx &= \int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u} du \\ &= \frac{1}{2}\Big[\ln|1+u| - \ln|1-u|\Big] + C \\ &= \frac{1}{2}\Big[\ln(1+\sin x) - \ln(1-\sin x)\Big] + C \\ &= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C \\ &= \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + C\end{aligned}$$

   b) We have

   $$\begin{aligned}\ln\sqrt{\frac{1+\sin x}{1-\sin x}} + C &= \ln\frac{1+\sin x}{\sqrt{(1+\sin x)(1-\sin x)}} + C \\ &= \ln\frac{1+\sin x}{\sqrt{\cos^2 x}} + C \\ &= \ln\frac{1+\sin x}{|\cos x|} + C \\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

2. **solution**

   For the first hump, solve for $y = e^x \cos x = 0$, we have $x = \frac{\pi}{2}$. Use the shell method, the volume is given by

   $$\int_0^{\frac{\pi}{2}} 2\pi x e^x \cos x dx.$$

Now, compute the antiderivative,

$$\begin{aligned}
I &= \int xe^x \cos x dx = \int xe^x d\sin x \\
&= x\sin xe^x - \int \sin x d(xe^x) = x\sin xe^x - \int \sin x(e^x + xe^x)dx \\
&= x\sin xe^x - \int \sin xe^x dx - \int x\sin xe^x dx \\
&= x\sin xe^x - \int \sin xe^x dx + \int xe^x d\cos x \\
&= x(\sin x + \cos x)e^x - \int \sin xe^x dx - \int \cos x d(xe^x) \\
&= x(\sin x + \cos x)e^x - \int \sin xe^x dx - \int \cos x(e^x + xe^x)dx \\
&= x(\sin x + \cos x)e^x - \int \sin xe^x dx - \int \cos xe^x dx - \int x\cos xe^x dx \\
&= x(\sin x + \cos x)e^x - \int \sin xe^x dx - \int \cos xe^x dx - I.
\end{aligned}$$

Therefore,

$$\begin{aligned}
2I &= x(\sin x + \cos x)e^x - \int \sin xe^x dx - \int \cos xe^x dx, \\
I &= \frac{1}{2}\Big[x(\sin x + \cos x)e^x - \int \sin xe^x dx - \int \cos xe^x dx\Big].
\end{aligned}$$

Now, we need to compute $I_1 = \int \sin xe^x dx$ and $I_2 = \int \cos xe^x dx$.

$$\begin{aligned}
I_1 &= \int \sin xe^x dx = -\int e^x d\cos x = -e^x \cos x + \int \cos xe^x dx \\
&= -e^x \cos x + \int e^x d\sin x = -e^x \cos x + e^x \sin x - \int \sin xe^x dx \\
&= -e^x \cos x + e^x \sin x + C - I_1.
\end{aligned}$$

Therefore, we have

$$I_1 = \frac{1}{2}\Big(-e^x \cos x + e^x \sin x\Big) + C.$$

Also, we have

$$I_2 = I_1 + e^x \cos x = \frac{1}{2}\Big(e^x \cos x + e^x \sin x\Big) + C.$$

We can get

$$I = \frac{1}{2}\Big[x(\sin x + \cos x)e^x - I_1 - I_2\Big] = \frac{1}{2}\Big[x(\sin x + \cos x)e^x - \sin xe^x\Big] + C.$$

The volume is given by

$$2\pi(I(\frac{\pi}{2}) - I(0)) = 2\pi \cdot \frac{1}{2}(\frac{\pi}{2} - 1)e^{\frac{\pi}{2}} = \pi(\frac{\pi}{2} - 1)e^{\frac{\pi}{2}}.$$

3. **solution**

After canceling $\int e^x \sinh x dx$ from each side, we should get $e^x \cosh x - e^x \sinh x = C$ for some constant C instead of $e^x \cosh x - e^x \sinh x = 0$. Use the original definition of $\sinh x$ and $\cosh x$, we can compute explicitly that $C = 1$.

4. **solution**

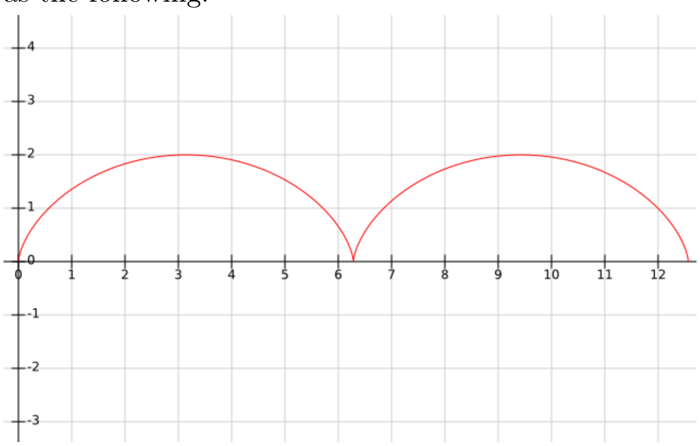
a) We have

$$\frac{dx}{dt} = 1 - \cos t,$$
$$\frac{dy}{dt} = \sin t.$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}.$$

It will be infinite when $\cos t = 1$, i.e. $t = 2\pi k$ for some integer k. The graph is as the following:



b) The arclength is given by

$$\begin{aligned}
\int_0^{2\pi} \sqrt{1 + (\frac{dy}{dx})^2} dx &= \int_0^{2\pi} \sqrt{1 + (\frac{\frac{dy}{dt}}{\frac{dx}{dt}})^2 \frac{dx}{dt}} dt \\
&= \int_0^{2\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} dt \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos t} dt = \int_0^{2\pi} \sqrt{4 \sin^2 \frac{t}{2}} dt \\
&= \int_0^{2\pi} 2|\sin \frac{t}{2}| dt = 2 \int_0^{2\pi} \sin \frac{t}{2} dt \\
&= -4 \cos \frac{t}{2} \Big|_0^{2\pi} = 8.
\end{aligned}$$

c) The surface area is given by

$$\begin{aligned}\int_0^{2\pi} 2\pi y\sqrt{1+(\frac{dy}{dx})^2}dx &= 2\pi\int_0^{2\pi}(1-\cos t)\sqrt{(1-\cos t)^2+\sin^2 t}dt \\ &= 4\pi\int_0^{2\pi}(1-\cos t)\sin\frac{t}{2}dt = 4\pi\int_0^{2\pi}2\sin^2\frac{t}{2}\sin\frac{t}{2}dt \\ &= 8\pi\int_0^{2\pi}\sin^3\frac{t}{2}dt = -16\pi\int_0^{2\pi}\sin^2\frac{t}{2}d\cos\frac{t}{2} \\ &= -16\pi\int_0^{2\pi}(1-\cos^2\frac{t}{2})d\cos\frac{t}{2} \\ &= -16\pi[\cos\frac{t}{2}-\frac{1}{3}\cos^3\frac{t}{2}]_0^{2\pi} = \frac{64}{3}\pi\end{aligned}$$

5. **solution**

When the curve is vertical, we have $\frac{dx}{dt}=0$. Since

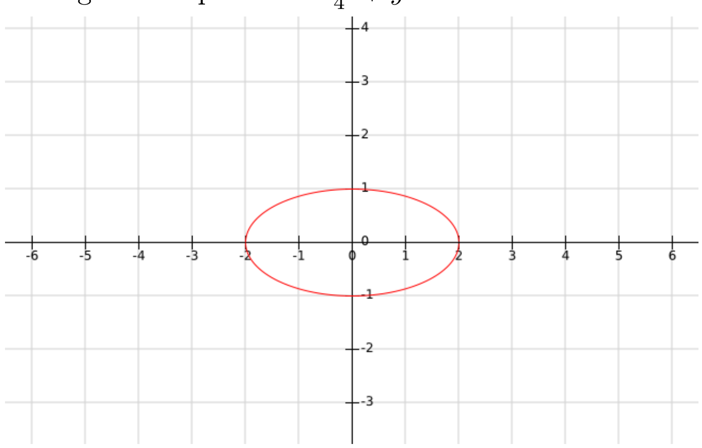
$$\frac{dx}{dt} = (\cos t)(\ln t),$$

The first point where $\frac{dx}{dt}=0$ is $t=\frac{\pi}{2}$. The length is given by

$$\int_1^{\frac{\pi}{2}}\sqrt{x'(t)^2+y'(t)^2}dt = \int_1^{\frac{\pi}{2}}\sqrt{(\ln t)^2}dt = \int_1^{\frac{\pi}{2}}\ln t dt = [t\ln t - t]_1^{\frac{\pi}{2}} = \frac{\pi}{2}\ln\frac{\pi}{2}-\frac{\pi}{2}+1.$$

6. **solution**

a) The graph is as below. The curve is traced counterclockwise as t increases. And the algebraic equation is $\frac{x^2}{4}+y^2=1$.



b) Let (x,y) be a point on the new curve, we know that $R_{-\theta}(x,y)$ is a point in the previous curve. Therefore,

$$\frac{1}{4}[(x+\sqrt{3})\cos\theta+y\sin\theta-\sqrt{3}]^2+[-(x+\sqrt{3})\cos\theta+y\cos\theta]^2=1.$$

We have

$$\begin{aligned}
4 =&[(x+\sqrt{3})\cos\theta + y\sin\theta - \sqrt{3}]^2 + 4[-(x+\sqrt{3})\sin\theta + y\cos\theta]^2 \\
=&(x+\sqrt{3})^2\cos^2\theta + y^2\sin^2\theta + 3 + 2(x+\sqrt{3})y\cos\theta\sin\theta - 2\sqrt{3}(x+\sqrt{3})\cos\theta \\
&- 2\sqrt{3}y\sin\theta + 4[(x+\sqrt{3})^2\sin^2\theta + y^2\cos^2\theta - 2(x+\sqrt{3})y\sin\theta\cos\theta] \\
=&(x+\sqrt{3})^2(\cos^2\theta + 4\sin^2\theta) + y^2(\sin^2\theta + 4\cos^2\theta) - 6(x+\sqrt{3})y\cos\theta\sin\theta \\
&+ 3 - 2\sqrt{3}(x+\sqrt{3})\cos\theta - 2\sqrt{3}y\sin\theta.
\end{aligned}$$

When $\theta = \frac{\pi}{4}$, we have

$$4 = \frac{5}{2}(x+\sqrt{3})^2 + \frac{5}{2}y^2 - 3(x+\sqrt{3})y + 3 - \sqrt{6}(x+\sqrt{3}) - \sqrt{6}y.$$

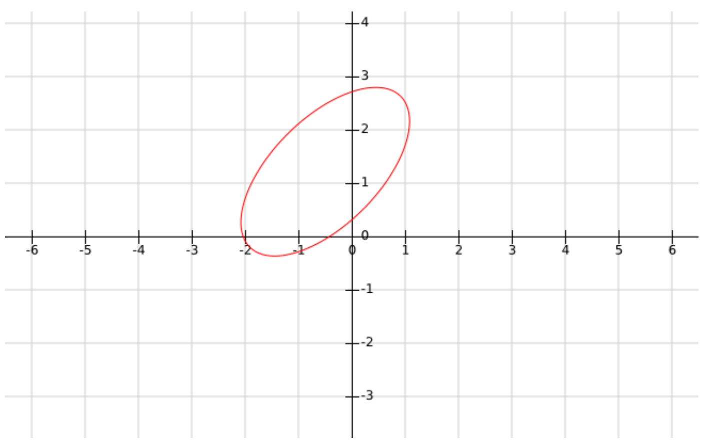
When $\theta = \frac{\pi}{2}$, we have

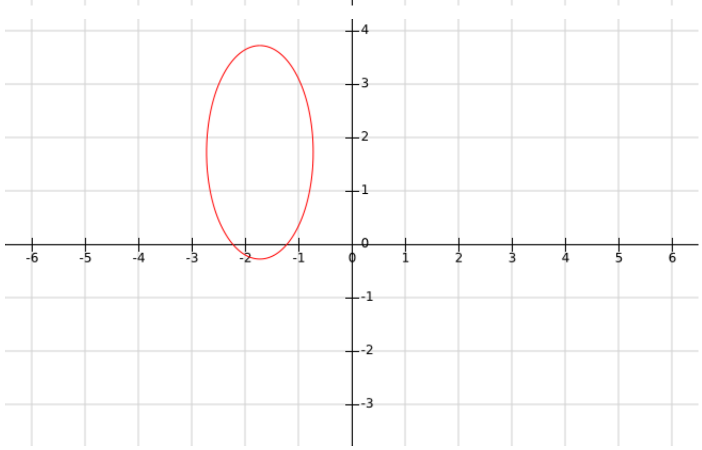
$$4 = 4(x+\sqrt{3})^2 + y^2 + 3 - 2\sqrt{3}y.$$

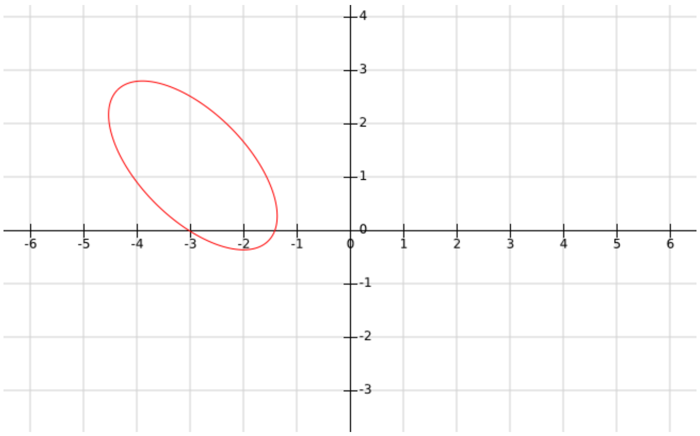
When $\theta = \frac{3\pi}{4}$, we have

$$4 = \frac{5}{2}(x+\sqrt{3})^2 + \frac{5}{2}y^2 + 3(x+\sqrt{3})y + 3 + 2\sqrt{6}(x+\sqrt{3}) - \sqrt{6}y.$$

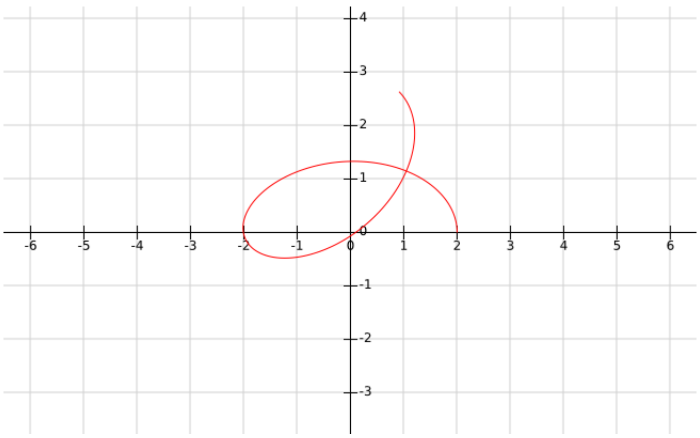
The graph is as the following.







c) The curve for $0 \leq t \leq 2\pi$ is the following. The curve does not end at the starting point.



The curve for $0 \leq t \leq 16\pi$ is the following.