Implicit differentiation & inverses

- Implicit differentiation

Ex • Previously, we have seen that $\frac{d}{dx} x^a = a x^{a-1}$ when a is an integer.

• Let's now show that this formula holds when a is rational, i.e, $a = \frac{m}{n}$, where m, n are integers and $n \neq 0$

• So Assume $\underbrace{y = x^{\frac{m}{n}}}_{y \text{ is clearly a function of } x}$. Then $y^n = x^m$.

We can differentiate both sides of the latter eqn. with respect to x and use the chain rule to conclude:

$$n y^{n-1} \frac{dy}{dx} = m x^{m-1}.$$

• Solving for $\frac{dy}{dx}$, we get $\frac{dy}{dx} = \frac{m}{n} \frac{x^{m-1}}{y^{n-1}}$

• Now plug in the fact that $y = x^{\frac{m}{n}}$ and simplify:

plug in

$$\frac{dy}{dx} = \frac{m}{n} \frac{x^{m-1}}{(x^{m/n})^{n-1}} = \frac{m}{n} \frac{x^{m-1}}{x^{\frac{mn-m}{n}}}$$

$$= \frac{m}{n} x^{m-1-\left(\frac{mn-m}{n}\right)} = \frac{m}{n} x^{\frac{\cancel{mn}-n-(\cancel{mn}-m)}{n}}$$

$$= \frac{m}{n} x^{\frac{m-n}{n}} = \boxed{\frac{m}{n} x^{\frac{m}{n}-1}}$$

as desired.

Ex: • The equation of a circle of radius 1 centered at the origin is $x^2 + y^2 = 1$.

• We can solve for y: $y = \pm\sqrt{1 - x^2}$

• Let's look at the + case: $y = \sqrt{1-x^2} = (1-x^2)^{1/2}$.

• By the chain rule: $\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

$$= \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{y}$$

• We can derive the same formula for $\frac{dy}{dx}$ using implicit differentiation:

• $x^2 + y^2 = 1$

• $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) = 0.$

• $\underbrace{\frac{d}{dx}x^2}_{2x} + \frac{d}{dx}y^2 = 0.$

By the chain rule, $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$. Therefore,

• $2x + 2y\frac{dy}{dx} = 0.$ We now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-x}{y},$$ as above.

<u>Ex</u>: $y^3 + xy^2 + 1 = 0$.

- It is not easy to explicitly solve for y as a function of x.
- However, we can still use implicit differentiation to find $\frac{dy}{dx}$: Just differentiate both sides of the eqn. with respect to x and use the chain rule to deduce that
  - $3y^2 \frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = \frac{d}{dx}(0) = 0$.
- Then solve for $\frac{dy}{dx}$ in terms of x, y:
  - $\frac{dy}{dx}(3y^2 + 2xy) = -y^2$
  - $$\boxed{\frac{dy}{dx} = \frac{-y^2}{3y^2 + 2xy}}$$

- Inverse functions

If $y = f(x)$ and $g(y) = x$, we call g the <u>inverse function</u> of f.

We sometimes denote it by f^{-1}:

$$x = g(y) = f^{-1}(y).$$

- Let's use implicit differentiation to find the derivative of f^{-1}

  - $y = f(x)$
  - $f^{-1}(y) = x$
  - Differentiate both sides w.r.t. x: $\frac{d}{dx} f^{-1}(y) = \frac{d}{dx} x = 1.$

By the chain rule:

$$\frac{d}{dx} f^{-1}(y) = \left\{ \frac{d}{dy} f^{-1}(y) \right\} \cdot \frac{dy}{dx} = 1.$$

Therefore: $\frac{d}{dy} f^{-1}(y) = \dfrac{1}{\frac{dy}{dx}}$

Ex: • $y = \arctan x$.

• $\tan y = x$

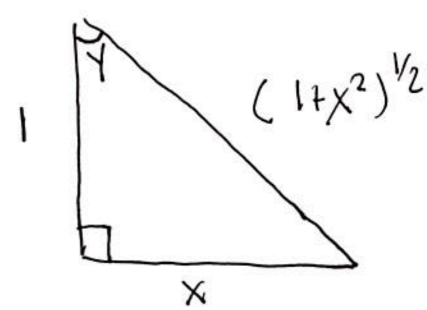
• $\frac{d}{dx}(\tan y) = \frac{d}{dx} x = 1$

• $\frac{d}{dy} \tan y \cdot \frac{dy}{dx} = 1$

• $\frac{1}{\cos^2 y} \frac{dy}{dx} = 1$

• $\frac{dy}{dx} = \cos^2 y = \cos^2 \arctan x$.

• We can simplify the right hand side using the following triangle:



• $\tan y = x$ in this picture.

• Thus, $\cos y = \frac{1}{\sqrt{1+x^2}}$

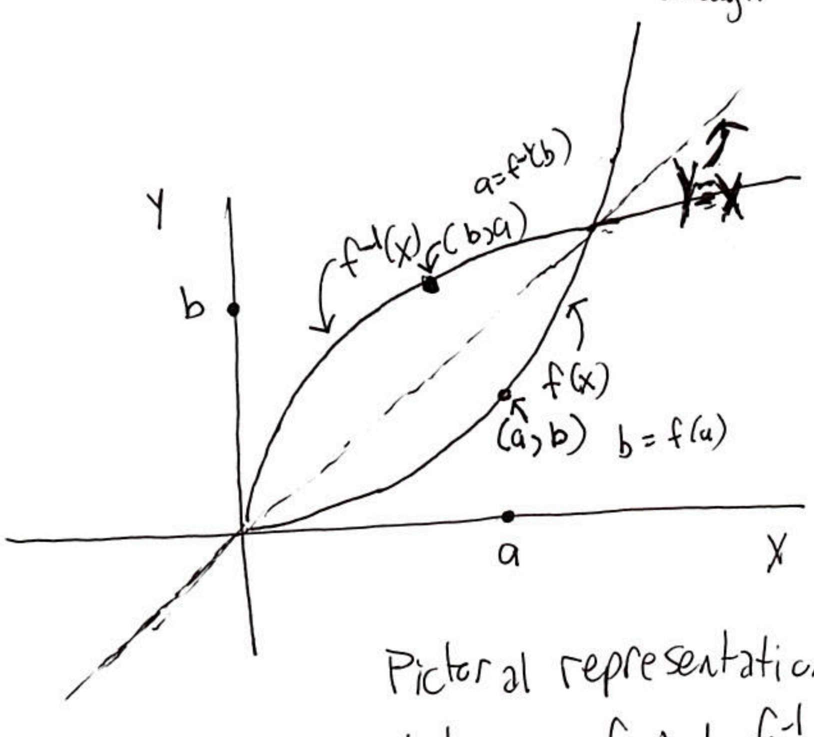
$\Rightarrow \cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$.

In total: if $y = \arctan x$, $\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$

Graphing an inverse function.

- Suppose that $y = f(x)$ and $g(y) = f^{-1}(y) = x$.
- Note that (a,b) lies on the graph of f if and only if $b = f(a)$.

Furthermore, $b = f(a)$ if and only if $a = f^{-1}(b)$.
Finally, $a = f^{-1}(b)$ if and only if (b, a) lies on the graph of f^{-1}.
We conclude: (a,b) lies on the graph of f if and only if (b,a) lies on the graph of f^{-1}. And (b,a) is the reflection of (a,b) through the line $y = x$:



Pictoral representation of the relationship between f and f^{-1}.