Partial Fractions

- Definition of a rational function: $\dfrac{P(x)}{Q(x)}$

  $P(x), Q(x)$ both polynomials

- Goal: Compute $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

- Ex: $\dfrac{1}{x-1} + \dfrac{3}{x+2} = \dfrac{(x+2)+3(x-1)}{(x-1)(x+2)} = \dfrac{4x-1}{x^2+x-2}$

- Therefore: $\displaystyle\int \frac{4x-1}{x^2+x-2}\,dx = \int \frac{1}{x-1} + \frac{3}{x+2}\,dx$

  $= \ln|x-1| + 3\ln|x+2| + C$

- Big idea: in general, split $\dfrac{P(x)}{Q(x)}$ into simpler pieces

Ex: How to split $\frac{4x-1}{x^2+x-2}$ into simpler pieces:

First, factor the denominator: $x^2+x-2 = (x-1)(x+2)$

- Then guess: $$\frac{4x-1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

  and find A, B

- Slow way of finding A, B: Clear all denominators by multiplying by $(x-1)(x+2)$:

  $$4x-1 = A(x+2) + B(x-1)$$

  - Then set the coefficients of the various powers of x on each side equal to each other:

    $$4 = A + B$$
    $$-1 = 2A - B$$

    Then solve for A, B

- Faster way of solving for A, B: "Cover-up" method:

First Multiply both sides by $(x-1)$:

$$\frac{4x-1}{\cancel{(x-1)}(x+2)}\cancel{(x-1)} = A + \frac{B(x-1)}{x+2}$$

- Then set $x=1$ to make the B term on the right-hand side drop out:

$$\frac{4-1}{1+2} = A + B\cdot 0 \implies \boxed{A=1}$$

- Then multiply both sides by $x+2$ and set $x=-2$ to make the A term drop out:

- $$\frac{4x-1}{(x-1)\cancel{(x+2)}}\cancel{(x+2)} = \frac{A(x+2)}{x-1} + B$$

$$\frac{4(-2)-1}{-2-1} = B \implies \boxed{B=3}$$

- This method works when $Q(x)$ factors into <u>distinct</u> factors and the degree of P is <u>less</u> than the degree of Q

- If the factors of Q repeat, we slightly modify the approach:

Ex: $$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C_1}{x+2}$$

- Use the cover up method on the highest degree term in $x-1$: multiply both sides by $(x-1)^2$

$$\frac{x^2+2}{x+2} = B + \text{stuff } (x-1)$$

Set $x=1$: $\frac{1^2+2}{1+2} = B \Rightarrow B=1$

- C_1 can also be evaluated by the cover up method: multiply both sides by $x+2$ and set $x=-2$:

$$\frac{x^2+2}{(x-1)^2} = C_1 + \text{stuff} \cdot (x+2) \qquad \frac{(-2)^2+2}{(-3)^2} = C_1 \Rightarrow \boxed{C_1 = \frac{2}{3}}$$

- So far we have: $$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{2}{3}}{x+2}$$

- Cover up cannot be used to evaluate A. Instead, plug in a convenient x value into both sides, $x=0$

$$\frac{2}{(-1)^2 \cdot 2} = \frac{A}{-1} + \frac{1}{(-1)^2} + \frac{\frac{2}{3}}{2} \Rightarrow \boxed{A = \frac{1}{3}}$$

- In total: $$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{1}{3(x-1)} + \frac{1}{(x-1)^2} + \frac{2}{3(x+2)}$$

- $\Rightarrow \displaystyle\int \frac{x^2+2}{(x-1)^2(x+2)}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{(x-1)} + \frac{2}{3}\ln|x+2| + C$

- Not all polynomials factor completely without resorting to complex numbers.

- <u>Ex</u>: $$\frac{1}{(x^2+1)(x-1)} = \frac{A_1}{x-1} + \frac{B_1x+C_1}{x^2+1}$$

We find A_1 by using the cover up method (multiply both sides by $x-1$ and set $x=1$):

$$\frac{1}{1^2+1} = A_1 \Rightarrow \boxed{A_1 = \frac{1}{2}}$$

- We now have $$\frac{1}{(x^2+1)(x-1)} = \frac{\frac{1}{2}}{x-1} + \frac{B_1x+C_1}{x^2+1}$$

- To find C_1, plug in $x=0$:

$$\frac{1}{(1)(-1)} = \frac{\frac{1}{2}}{-1} + \frac{C_1}{1} \Rightarrow \boxed{C_1 = -\frac{1}{2}}$$

$$\Rightarrow \frac{1}{(x^2+1)(x-1)} = \frac{\frac{1}{2}}{x-1} + \frac{B_1x-\frac{1}{2}}{x^2+1}$$

- To find B_1, plug in any x value except 0 or 1:

$x=-1$: $$\frac{1}{2(-2)} = \frac{\frac{1}{2}}{-2} + \frac{B_1(-1)-\frac{1}{2}}{2} \Rightarrow \boxed{B_1 = -\frac{1}{2}}$$

$$\Rightarrow \int \frac{dx}{(x^2+1)(x-1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

- How to integrate $\int \frac{dx}{(x-1)^{10}}$?

$$\int \frac{dx}{(x-1)^{10}} = -\frac{1}{9}(x-1)^{-9} + C$$

- How to integrate $\int \frac{dx}{(x^2+1)^{10}}$?
  - Use inverse trig substitution
    - $x = \tan u$
    - $dx = \sec^2 u \, du$
    - $\tan^2 u + 1 = \sec^2 u$

$$\int \frac{dx}{(x^2+1)^{10}} = \int \frac{\sec^2 u \, du}{(\sec^2 u)^{10}} = \underbrace{\int \cos^{18} u \, du}_{\text{Could be evaluated using previously discussed methods}}$$