• Infinite Series and Convergence Tests

• Geometric Series

- A geometric series is an infinite sum of the form

$$1 + a + a^2 + a^3 + \cdots = S.$$

- There is a trick to evaluate this: multiply both sides by a:

$$a + a^2 + a^3 + \cdots = aS$$

- Then subtracting, we deduce

$$(1 + a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + \cdots) = S - aS.$$

That is, $1 = S - aS = S(1-a)$.

We conclude that $S = \dfrac{1}{1-a}$

- This only works when $|a| < 1$, i.e. $-1 < a < 1$
- $a = 1$ can't work : $1 + 1 + 1 + \cdots = \infty$
- $a = -1$ can't work : $1 - 1 + 1 - 1 + \cdots \neq \dfrac{1}{1-(-1)} = \dfrac{1}{2}$.

• Notation:

- An infinite sum is written as $\displaystyle\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots$
- The finite sum $S_n = \displaystyle\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$

is called the "n th partial sum" of the infinite series

- Definition: $$\sum_{k=0}^{\infty} a_k = S$$

is defined to be the same thing as

$$\lim_{n\to\infty} S_n = S, \quad \text{where} \quad S_n = \sum_{k=0}^{n} a_k$$

- We say the series <u>converges to S</u> if the limit exists and is finite.

- The importance of convergence is illustrated by the above example of the geometric series. If $a=1$, then $S = 1+1+1+\cdots = \infty$.

  But $S - aS = 1$ or $\infty - \infty = 1$

  does not make sense and is not usable.

- Another type of series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- We can use integrals to decide if this type of series converges. First, approximate the sum by an integral:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \approx \int_1^{\infty} \frac{dx}{x^p}.$$

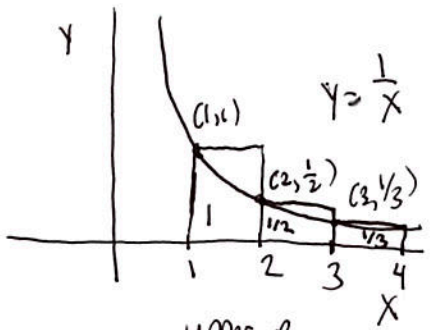
  - If the improper integral converges, then so does the infinite series.

- Remark: The integral comparison technique does <u>not</u> tell us the value of the infinite sum, only whether or not the infinite series converges.

  Ex: . Beyond this course: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

  - Recent result: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges to an irrational number

<u>Ex</u>: Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$



upper Riemann Sums

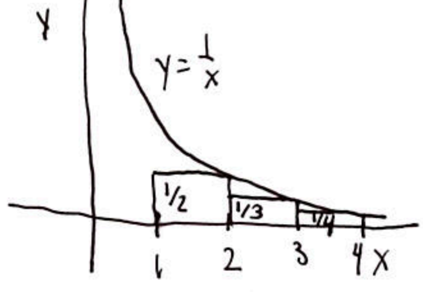
From the picture, we see that

$$\ln(N) = \int_1^N \frac{dx}{x} = \text{Area under curve from } x=1 \text{ to } x=N$$
$$\leq 1 + \frac{1}{2} + \cdots + \frac{1}{N-1} = S_{N-1} \leq S_N.$$

- As $N \to \infty$, $\ln(N) \to \infty$, so $S_N \to \infty$ as well.

  That is, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

- Actually, $S_N \to \infty$ rather slowly. To see this, let's examine lower Riemann Sums:



Lower Riemann Sum

- We see that: $S_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N} = 1 + \sum_{n=2}^{N} \frac{1}{n} \leq 1 + \int_1^N \frac{dx}{x} = 1 + \ln(N)$

- Hence, $\ln(N) < S_N < 1 + \ln(N)$

- Integral Comparison

Theorem: (Generalization of previous example):

Let $f(x)$ be a positive decreasing function.

Then $\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)\,dx \right| < f(1)$.

- Thus, either both terms converge or diverge
- This is what we mean when we write $\sum_{n=1}^{\infty} \frac{1}{n^p} \approx \int_1^{\infty} \frac{dx}{x^p}$

New notation: Suppose that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$ for some constant c with $0 < c < \infty$.

Then we write: $f(x) \sim g(x)$.

Theorem: Let $f(x), g(x)$ be positive functions and Suppose that $f(x) \sim g(x)$.

Then $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} g(n)$ either both converge or both diverge

Ex: Consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10}}$

- Note that $\frac{1}{\sqrt{n^2+10}} \sim \frac{1}{(n^2)^{1/2}} \sim \frac{1}{n}$.
- Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+10}}$