MATH 18.01, FALL 2017 - PROBLEM SET # 3B

Professor: Jared Speck

Due: by Friday 1:45pm on 10-06-17 (turn in along with Problem Set #3*A*)
(in the boxes outside of Room 4-174 during the day; write your name, recitation instructor, and recitation meeting days/time on your homework)

18.01 Supplementary Notes (including Exercises and Solutions) are available on the course web page: `http://math.mit.edu/~jspeck/18.01_Fall%202017/1801_CourseWebsite.html`. This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

Part I consists of exercises given and solved in the Supplementary Notes. It will be graded quickly, checking that all is there and the solutions not copied.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises given here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

You are encouraged to use graphing calculators, software, etc. to check your answers and to explore calculus. However, (unless otherwise indicated) we strongly discourage you from using these tools to solve problems, perform computations, graph functions, etc. An extremely important aspect of learning calculus is developing these skills. You will not be allowed to use any such tools on the exams.

Part I (10 points)

Notation: The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Lecture 9. (Tues., Sept. 26) Maximum-minimum problems.
Read: 4.3, 4.4.
Homework: Notes 2C: 1, 2, 5, 11, 13.

Lecture 10. (Tues., Oct. 3) Related rate problems.
Read: 4.5.
Homework: Notes 2E: 2, 3, 5, 7.

Part II (50 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. *You must show your work; "bare" solutions will receive very little credit.*

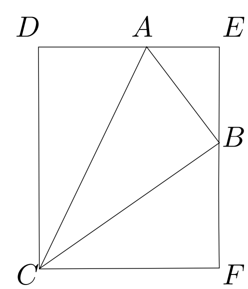
iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people whom you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem 0" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (Sept. 26; max-min problems; $2 + 10 + 5 = 17$ points) 4.3: 30 extra+ab. You may assume that the paper has a vertical length L, where $L \geq 2a$. Before doing the two problems stated in the text, first do the following "extra" problem: find the set of x values in terms of L and a such that the problem makes physical sense (i.e, such that $\triangle ABC$ is a triangle that fits on the sheet of paper with C on the left edge of the paper).



Solution: Extra: We need $|AB| > |AE|$, which means $x > a - x$. So we have $x > \frac{a}{2}$. Let $y = |CD|$, then $|BC| = y$ and $|AD| = |AB| = x$. So we have $|BE| = \sqrt{x^2 - (a-x)^2}$ and $|BF| = y - \sqrt{x^2 - (a-x)^2}$. Consider in the rect-triangle ΔBCF, we have that

$$a^2 + (y - \sqrt{x^2 - (a-x)^2})^2 = y^2.$$

$$y = \frac{a^2 + x^2 - (a-x)^2}{2\sqrt{x^2 - (a-x)^2}} = \frac{ax}{\sqrt{2ax - a^2}}$$

We also need $y \leq L$. Solving the inequality, we have $x \geq \frac{L^2 - L\sqrt{L^2 - a^2}}{a}$. In conclusion, we have $\frac{a}{2} < \frac{L^2 - L\sqrt{L^2 - a^2}}{a} \leq x \leq a$

a) The area of ΔABS is $A = \frac{1}{2}xy$. To minimize the area, we only need to minimize $S = A^2 = \frac{1}{4}x^2y^2$. $S = \frac{1}{4}x^2 \frac{(ax)^2}{2ax - a^2} = \frac{ax^4}{4(2x - a)}$. Take the derivative of S, we have

$$\frac{dS}{dx} = \frac{a}{4}\frac{4x^3(2x-a) - 2x^4}{(2x-a)^2} = \frac{a(6x^4 - 4ax^3)}{4(2x-a)^2}.$$

Note that we need $|AB| > |AE|$, which means $x > a - x$, i.e., $x > \frac{a}{2}$. So the bottom is never 0. To get the critical point, we make $\frac{dS}{dx} = 0$ and get $x = 0$ or $x = \frac{2a}{3}$. Again by $x > \frac{a}{2}$, the only critical point is $x = \frac{2a}{3}$. When $x < \frac{2a}{3}$, $\frac{dS}{dx} < 0$ and when $x > \frac{2a}{3}$, $\frac{dS}{dx} > 0$. Thus $x = \frac{2a}{3}$ is a local minimal. Note that it is the only critical point, thus it is a global minimum. When $x = \frac{2a}{3}$, $y = \frac{2\sqrt{3}a}{3} < 2a$ and the area $A = \frac{2\sqrt{3}a^2}{9}$.

b) To minimize the Length $|AC| = \sqrt{x^2 + y^2}$, we only need to minimize

$$F = |AC|^2 = x^2 + y^2 = x^2 + \frac{a^2x^2}{2ax - a^2} = \frac{2x^3}{2x - a}.$$

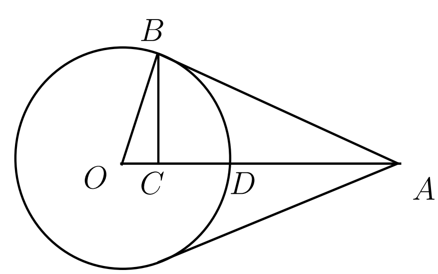
Take the derivative,

$$\frac{dF}{dx} = \frac{6x^2(2x - a) - 4x^3}{(2x - a)^2} = \frac{8x^3 - 6ax^2}{(2x - a)^2}.$$

Let $\frac{dF}{dx} = 0$, we get the only critical point $x = \frac{3a}{4}$. When $x < \frac{3a}{4}$, $\frac{dF}{dx} < 0$, and when $x > \frac{3a}{4}$, $\frac{dF}{dx} > 0$. So $x = \frac{3a}{4}$ is a local minimal. Since it is the only critical point, it is a global minimum. When $x = \frac{3a}{4}$, $y = \frac{3\sqrt{2}a}{4} < 2a$ and $|AC| = \frac{3\sqrt{3}a}{4}$.

2. (Sept. 26; max-min problems; 10 points) 4.4: 26

Solution:
First consider the ball with radius 1 and suppose the distance between the person A and center O is x. Draw a tangent line from AB to the sphere and let BC perpendicular to OA. Then the part one can see at A is a part of the sphere with height $|CD|$. So we need to compute $|CD|$. Note that ΔBOA is similar to ΔCOB. So $\frac{|OC|}{|OB|} = \frac{|OB|}{|OA|}$, which implies $|OC| = \frac{1}{x}$ and thus $|CD| = 1 - \frac{1}{x}$. The area one can see on this sphere is $2\pi(1 - \frac{1}{x})$.



Similarly, the area one can sea on the sphere with radius 2 is $2\pi \cdot 2(2 - \frac{4}{6-x})$. Therefore the total area

$$A = 2\pi(1 - \frac{1}{x}) + 4\pi(2 - \frac{4}{6-x}) = 2\pi(5 - \frac{1}{x} - \frac{8}{6-x}).$$

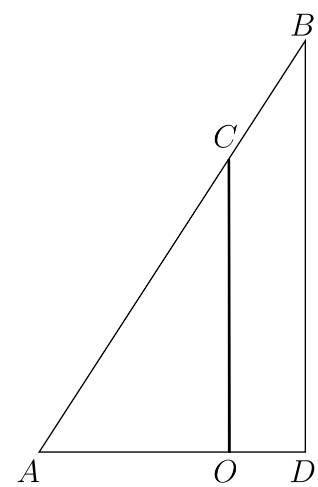
Take the derivative, we have

$$\frac{dA}{dx} = 2\pi\left(\frac{1}{x^2} - \frac{8}{(6-x)^2}\right) = 2\pi\frac{(6-x)^2 - 8x^2}{x^2(6-x)^2} = 2\pi\frac{-7x^2 - 12x + 36}{x^2(6-x)^2}.$$

Let $\frac{dA}{dx}=0$, we have $x=\frac{-6+12\sqrt{2}}{7}$. Note that $\frac{dA}{dx}$ is positive on the left of $\frac{-6+12\sqrt{2}}{7}$ and negative on the right of $\frac{-6+12\sqrt{2}}{7}$. So $x=\frac{-6+12\sqrt{2}}{7}$ is a local maximal. Since it is the only critical point, it is a global maximal.

3. (Oct. 3; related rates; 7 + 3 = 10 points) 4.5: 16ab

Solution:
In the figure on the right, the ladder AB leans on the wall OC. Let x be the distance of the bottom of the ladder away from the wall $|AO|$ and y be the distance of the top of the ladder away from the ground $|BD|$. Note that two triangles ΔAOC and ΔADB are similar. So we have $\frac{|OC|}{|AC|}=\frac{|DB|}{|AB|}$. Since $|OC|=12$ and $|AB|=20$, $|BD|=y$, we have $|AC|=\frac{12\cdot 20}{y}$. Consider in the rect-triangle ΔAOC, we have that



$$x^2+12^2=\left(\frac{12\cdot 20}{y}\right)^2.$$

Differentiate on both sides in terms of t, we have that

$$2x\cdot x'(t)=\frac{-2(12\cdot 20)^2}{y^3}y'(t).$$

a) When $|BC|=5$ and $|AC|=15$, so $y=|DB|=16$ and $x=|AO|=9$. We also know that $x'(t)=5$. So $y'(t)=-\frac{16}{5}$. The top of the ladder approach the ground at a rate of $\frac{16}{5}=3.2$ ft/min.

b) When $|BC|=0$, $|AC|=20$, $y=12$, $x=|AO|=16$. So $y'(t)=-\frac{12}{5}=-2.4$. The top of the ladder approach the ground at a rate of 2.4 ft/min.

4. (Oct. 3; related rates; motion of a wheel; 2 + 2 + 2 + 2 + 2 = 10 points) A circular wheel of radius 1 rolls (without slipping) to the *left* along the x axis in the (x,y) plane. The wheel starts out at rest with its center at the coordinate $(0,1)$. Let p be the point on the edge of the wheel that is initially at the location $(1,1)$. Let L denote the line passing through the center of the wheel and p, and let H denote the horizontal line through the center of the wheel. The angle θ (in radians) that L makes with H is initially 0. As the wheel rolls to the left, the point p will move in the (x,y) plane. The motion of p can be viewed as the superposition of two kinds of motion: **i)** the motion induced by the motion of the center of the wheel, which is traveling to the left; **ii)** the rotation of the point p about the center of the wheel.

Assume that the wheel starts out at $t=0$. Suppose that after t units of time, the wheel has rolled into a new position. In the new position, L has rotated and now makes an angle θ with the fixed horizontal line H. That is, p has been rotated counterclockwise by $\theta(t)$ radians and has also been translated to the left by some yet unknown amount. The new position of p is denoted by $(x_p(t),y_p(t))$. The new position of the center of the wheel is denoted by $(x_{center}(t),1)$.

Remark: in this problem, the angle θ is not restricted to the range $0 \leq \theta \leq 2\pi$. For example, after the wheel makes two complete rotations, θ will have increased from 0 to 4π. Also, throughout this problem, we will assume that θ is a differentiable function of t with $\theta' \geq 0$.

a) Express x_{center} in terms of θ.

Solution: The length that the center moved is the same as the arc length p moved. So $x_{center} = -\theta$.

b) Express x_p and y_p in terms of θ.

Solution: First think of the rotation, we have that $(x_p, y_p) - (x_{center}, 1) = (\cos\theta, \sin\theta)$. So $x_p = \cos\theta - \theta$ and $y_p = \sin\theta + 1$.

c) Express x'_{center}, x'_p, and y'_p in terms of θ'. Here, $x'_{center} = \frac{d}{dt}x_{center}$, etc.

Solution: $x'_{center} = -\theta'$, $x'_p = (-\sin\theta - 1)\theta'$, and $y'_p = \cos\theta \cdot \theta'$.

d) Suppose that at time t, the point p lies directly above the center of the wheel. How are $x'_p(t)$ and $y'_p(t)$ related to $x'_{center}(t)$?

Solution: When the point p lies directly above the center, $\theta = (2k + \frac{1}{2})\pi$. $x'_p(t) = -2\theta'(t) = 2x'_{center}(t)$ and $y'_p = 0$.

e) Suppose that at time t, the point p lies directly below the center of the wheel. How are $x'_p(t)$ and $y'_p(t)$ related to $x'_{center}(t)$?

Solution: When the point p lies directly below the center, $\theta = (2k + \frac{3}{2})\pi$. $x'_p(t) = (1 - 1)\theta'(t) = 0$ and $y'_p = 0$.