# Solution to PSet 3

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## Part II 1

### a)

$v'(9) \approx \frac{v(9)-v(8)}{9-8} = 10.813,\ v'(10) \approx \frac{v(10)-v(9)}{10-9} = 11.239.$

### b)

$v''(10) \approx \frac{v'(10)-v'(9)}{10-9} = 0.426.$

### c)

By quartic approximation, for $d$ near 10

$$v(d) \approx v(10) + v'(10)(d-10) + \frac{1}{2}v''(10)(d-10)^2.$$

Plug in $d = 11$,

$$v(11) \approx v(10) + v'(10)(11-10) + \frac{1}{2}v''(10)(11-10)^2 \approx 113.435.$$

## Part II 2

### a)

Let $y = \tanh^{-1}(E)$. Then $E = \tanh y$. Since $\cosh(0) = 1$ and $\sinh(0) = 0$. As $E$ tends to 0, $y$ tends to 0. Therefore

$$\lim_{E\to 0} \frac{\tanh^{-1}(E)}{E} = \lim_{E\to 0} \frac{y}{\tanh y} = (\lim_{E\to 0} \frac{\sinh y}{y})^{-1} = 1 = f(0).$$

## b)

$$\frac{dE}{dy} = \frac{d\tanh y}{dy} = \frac{\sinh'(y)\cosh y - \cosh' y \sinh y}{\cosh^2 y} = \frac{1}{\cosh^2 y}.$$

Therefore

$$\frac{d\tanh^{-1}(E)}{dE} = \frac{dy}{dE} = \cosh^2 y = \frac{1}{1-\tanh^2 y} = \frac{1}{1-E^2}.$$

## c)

$$\left(\frac{1}{1-E^2}\right)'\Big|_{E=0} = 0, \ \left(\frac{1}{1-E^2}\right)''\Big|_{E=0} = 2.$$

So the quartic approximation of $1/(1-E^2)$ is $1+E^2$, which means

$$A_0 = 1, A_1 = 0, A_2 = 1.$$

## d)

By the assumption, the quartic approximation of $1/(1-E^2)$ obtained using the cubic approximation of $\tanh^{-1}(E)$ is

$$B_1 + 2B_2E + 3B_3E^2.$$

Compared with c), we have

$$B_1 = A_0 = 1, B_2 = A_1/2 = 0, B_3 = A_2/3 = 1/3.$$

Finally, $B_0 = \tanh^{-1}(0) = 0$.

## e)

For $E \neq 0$ but small,

$$f(E) = \frac{\tanh^{-1}(E)}{E} \approx \frac{B_0 + B_1E + B_2E^2 + B_3E^3 + O(E^4)}{E} = 1 + \frac{1}{3}E^2 + O(E^3).$$

Since $f(0) = 1$, the quadratic approximation of $f$ near 0 is $1 + \frac{1}{3}E^2 + O(E^3)$.

## f)

near $E = 0$,

$$S = 2\pi(1 + f(E)(1 - E^2) \approx 2\pi(1 + (1 + \frac{1}{3}E^2 + O(E^3))(1 - E^2))$$
$$= 4\pi - \frac{4\pi}{3}E^2 + O(E^3).$$

The last quantity is the quadratic approximation of $S$ near 0.

## g)

From the quadratic approximation of $S$ near $E = 0$, when $E$ is very small, $S$ is smaller than $S(0) = 4\pi$, which means it decreases the surface area.

# Section 4.1:18

$$\frac{dy}{dx} = mx^{m-1}(1-x)^n + x^m(-n)(1-x)^{n-1}.$$

## (a)

If $m$ is even, $x^m \geq 0$.

For $x \leq 1$, $(1-x)^n \geq 0$. So for $x \in (-\infty, 1]$, $x^m(1-x)^m \geq x^m(1-x)^n|_{x=0} = 0$, which means $y$ has a minimum at $x = 0$.

## (b)

If $n$ is even, $(1-x)^n \geq 0$.

For $x \in [0, \infty)$, $x^m \geq 0$. So for $x \in [0, \infty)$, $x^m(1-x)^n \geq x^m(1-x)^m|_{x=1} = 0$, which means $y$ has a minimum at $x = 1$.

## (c)

Near $x = m/(m+n)$, $y > 0$. So $y$ achieves its maximum at $m/(m+n)$ is equivalent to $\log y$ achieves its maximum.

$$(\log y)' = \frac{m}{x} - \frac{n}{1-x}.$$

$(\log y)'|_{x=m/(m+n)} = 0$. Moreover, for $x > m/(m+n)$, $(\log y)' < 0$, while for $x < m/(m+n)$, $(\log y)' > 0$. So $\log y$ achieves its maximum at $x = m/(m+n)$.

## Section 4.1: 22

If

$$f' = (x-1)(x+2) = x^2 + x - 2.$$

Then $f$ has a maximum at $-2$ and minimum at $x = 1$. So we can take

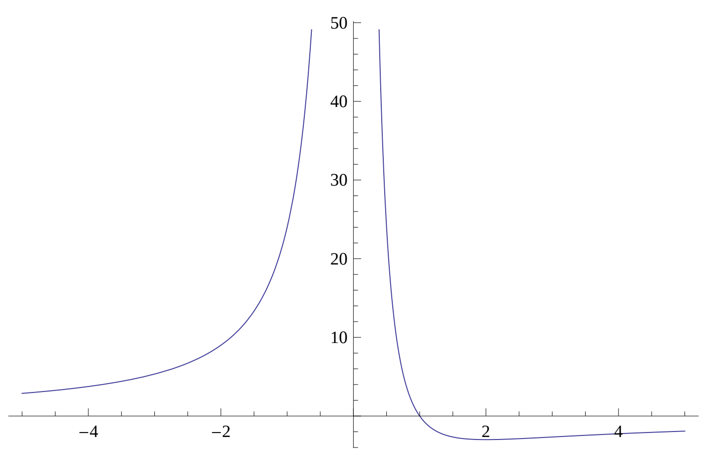
$$f = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x.$$

## Section 4.2: 12

$$y'' = \frac{72 - 24x}{x^4}.$$

For $x < 3$, $y'' > 0$ and the curve is concave up. For $x > 3$, $y'' < 0$ and the curve is concave down. For $x = 3$, $y'' = 0$ and $(3, -8/3)$ is a point of inflection.

Graph for Section 4.2:12



## Section 4.2: 16

It is not possible.

Pick a point $a$. Since $f'(a) < 0$ and $f''(a) < 0$, for all $x > a$, $f(x)$ will stay below its tangent line at $(a, f(a))$, which is $y - f(a) = f'(a)(x - a)$.

Since the tangent line goes to $-\infty$ as $x$ goes to $\infty$, so does $f(x)$.

## Section 4.2: 18

$$x^2 + y^2 = a^2, \quad 2x + 2yy' = 0 \quad 2 + 2y'^2 + 2yy'' = 0.$$

Therefore

$$y' = \frac{-x}{y}, \quad y'' = \frac{-1 - y'^2}{y} = \frac{-y^2 - x^2}{y^3} = \frac{-a^2}{y^3}.$$

Therefore

$$\text{if } y > 0,\ y'' = \frac{-a^2}{(a^2 - x^2)^{3/2}};$$
$$\text{if } y < 0,\ y'' = \frac{a^2}{(a^2 - x^2)^{3/2}};$$

Since $yy'' = a^2/y^2 > 0$ when $y \neq 0.y$ and $y''$ always have the opposite sign.

## Section 4.2: 24

The second derivative of a cubic curve is a linear function $y = 6ax + 2b$. It has a single zero point at $x_0 = -b/3a$. The sign of the linear function changes at this point. So this point is the unique point of inflection of the cubic curve.

In both these three cases, If $a > 0$, the curve is concave down at $(-\infty, a)$ and concave up at $(a, \infty)$. If $a < 0$, the concavity takes the opposite direction.

The derive of the general cubic curve is

$$y' = 3ax^2 + 2bx + c.$$

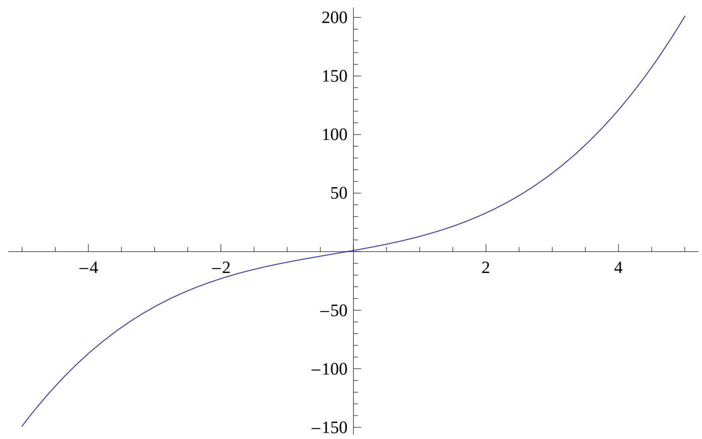
If $b^2 < 3ac$, $y' > 0$ or $y' < 0$ for all $x$ depending on $a > 0$ or $a < 0$. Therefore the cubic curve is an increasing or decreasing function according to whether $a > 0$ or $a < 0$.

If $b^2 = 3ac$, $y' \geq 0$ or $y' \leq 0$ for all $x$ depending on $a > 0$ or $a < 0$. Moreover $y'$ achieve 0 at $x_0 = -b/3a$. Therefore the cubic curve is an monotone function with an point of inflection at $x_0 = -3a/b$. And at this point, the derivative of $x$ is also 0. In fact $y$ is a translation $y = ax^3$.

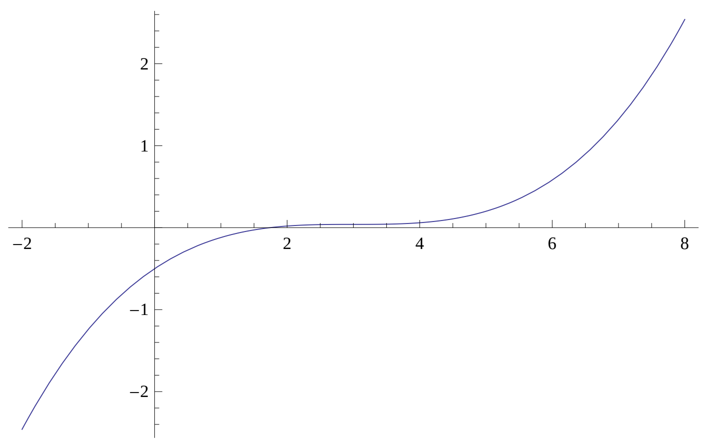
If $b^2 > 3ac$, $y' = 0$ has two zero points, which we denote by $x_1 < x_2$. At point $x_1$ and $x_2$ $f'$ change its sign. We also know that $x_1 < x_0 < x_2$ since $2x_0 = x_1 + x_2$. If $a > 0$, the function is increasing/decreasing/increasing in $(\infty, x_1)/(x_1, x_2)/(x_2, \infty)$ respectively. The curve reaches its maximum and minimum at $x_1$ and $x_2$ respectively. For $a < 0$, we change the direction of the monotonicity.

The following are the graphs of examples in these three categories.

$b^2 < 3ac$




$b^2 = 3ac$




$b^2 > 3ac$