## 18.01 PROBLEM SET 1

**Due date:** Tuesday, September 16 **before 1pm.** Late work will be accepted only with a medical note or for another Institute-approved reason. You are encouraged to work with others, but the final write-up should be entirely your own and based on your understanding.

**Problem 1**(10 points) Water is poured into the trough shown in Figure **??** so that the height of the water increases at a constant rate $R = 0.05$ meters per second. In other words, after $t$ seconds the height $h(t)$ in meters of the water is given by

$$h(t) = 0.05t.$$

a)(5 points) Compute the area $A(t)$ (in units of $\mathrm{m}^2$) of the surface of the water after $t$ seconds, and compute the volume $V(t)$ (in units of $\mathrm{m}^3$) of the volume of the water after $t$ seconds.

b)(5 points) Compute the derivative $V'(t)$ of $V(t)$ with respect to $t$ and verify the formula

$$V'(t) = A(t) \times R.$$

Does the formula above make sense physically?

**Problem 2**(10 points) Under a "linearly increasing gravity" model, a projectile fired straight up in the air with an initial velocity of $v_0$ meters per second has height $s(t)$ meters after $t$ seconds, where

$$s(t) = -\frac{c_0}{6}t^3 - \frac{g}{2}t^2 + v_0 t,$$

where $g = 9.8$, and where $c_0$ is some positive constant which is unspecified.

a)(2 points) Find the velocity $v(t) = s'(t)$, the acceleration $a(t) = v'(t)$, and the rate of change of the acceleration $a'(t)$ at time $t > 0$. Use this to give a physical meaning to the constant $c_0$.

b)(4 points) At what time $t > 0$ is the velocity $v(t)$ of the projectile equal to zero?

c)(4 points) At what time $t > 0$ is the height $s(t)$ of the projectile equal to zero? To solve this, express $s(t)$ as a product of $t$ and a quadratic factor and use the quadratic formula to find the zeros of the quadratic factor.

**Remark**: Linearly increasing gravity actually does occur on the earth, although the constant $c_0$ is very small. At the equator the constant $c_0$ is larger than expected, leading some scientists to predict that the equator is "bulging out".

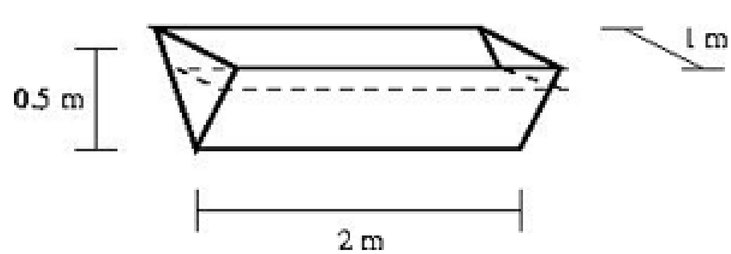


FIGURE 1. Water in a trough

**Problem 3**(10 points) The following theorem, sometimes called the *squeezing theorem*, is useful for computing some limits.

Let $f(x), g(x), h(x)$ be functions defined on an interval containing $a$. Assume that $f(x) \leq g(x) \leq h(x)$ for all $x$, except possibly $a$ itself. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and equals $L$.

a)(3 points) Let $g(x)$ be the function given by

$$g(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Find functions $f(x)$ and $h(x)$ as in the squeezing theorem to prove that $\lim_{x\to 0} g(x)$ exists and to compute what the limit equals. (Hint: For all $x \neq 0$, $-1 \leq \sin(\frac{1}{x}) \leq 1$).

b)(7 points) Let $u(x)$ be the function given by

$$u(x) = \begin{cases} x^2\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Use a) and the definition of derivative as a limit to prove that $u(x)$ is differentiable at 0 and to compute $u'(0)$. Do *not* try to compute the derivative using the product rule, chain rule, or the derivative of $\sin(x)$.

**Problem 4**(10 points) A circle $C$ in the plane is centered at the origin $O = (0, 0)$ with radius 5; in other words it is given by the equation

$$x^2 + y^2 = 25.$$

There are two lines $L_1$ and $L_2$ passing through the point $P = (13, 0)$ which are tangent to $C$. Determine the equations of $L_1$ and $L_2$.

**Geometric hint**: Let $Q$ be the point (to be determined) which is the intersection of $L_1$ and $C$. What can you say about the triangle $OPQ$? If you can *guess* the length of the segment $PQ$, then you can compute the slope of the line $L$ passing through $P$ and $Q$. After you have the slope, you can determine the equation of $L$ and use this to solve for the coordinates of $Q$. Once you have the coordinates, you can compute the slope of the tangent line to $C$ at $Q$ using implicit differentiation, thereby *proving* that $L$ is a tangent line (it is not enough to guess the right answer, you must work through this last step to prove your guess is correct). Don't forget that there are *two* tangent lines, and you must write the equations of both.

**Problem 5**(10 points)

a)(2 points) p. 91, Section 3.2, Exercise 14
b)(2 points) p. 91, Section 3.2, Exercise 40
c)(2 points) p. 107, Section 3.6, Exercise 8
d)(2 points) p. 107, Section 3.6, Exercise 26
e)(2 points) p. 264, Section 8.2, Exercise 10