MATH 18.01, FALL 2017 - PROBLEM SET #5 SOLUTIONS (PART II)

1. (Oct. 12; Antiderivatives; 2 + 2 + 3 = 7 points) Recall that in pset 3A, you showed that $(d/dx)\tanh^{-1} x = \dfrac{1}{1-x^2}$. Here, $\tanh^{-1}(x)$ denotes the inverse to the hyperbolic tangent function.

a) Find the two constants A and B such that

$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}.$$

This is called the *method of partial fractions*. We will discuss this method in much more detail later in the course.

Solution: Note,

$$\begin{aligned}\frac{A}{1+x} + \frac{B}{1-x} &= \frac{A(1-x)+B(1+x)}{(1+x)(1-x)} \\ &= \frac{(A+B)+(B-A)x}{1-x^2}.\end{aligned}$$

So we can choose $A = B = \frac{1}{2}$ and get

$$\frac{A}{1+x} + \frac{B}{1-x} = \frac{1}{1-x^2}.$$

b) Use part a) to compute

$$\int \frac{1}{1-x^2}\,dx.$$

Simplify your answer as much as possible.

Solution: Note,

$$\int \frac{1}{1+x}\,dx = \ln|1+x| + c.$$
$$\int \frac{1}{1-x}\,dx = -\ln|1-x| + c.$$

Therefore,

$$\begin{aligned}\int \frac{1}{1-x^2}\,dx &= \int \frac{1}{2}\frac{1}{1+x} + \frac{1}{2}\frac{1}{1-x}\,dx \\ &= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + c.\end{aligned}$$

c) Explain the connection between the function $\tanh^{-1}(x)$ and your answer to part b).

Solution: We know that

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}.$$

Therefore $\tanh^{-1}(x)$ and our answer in part b) are both the antiderivative to the function $\frac{1}{1-x^2}$. Antiderivatives differ by a constant term so we must have,

$$\tanh^{-1}(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) + c$$

for some constant c. Here we were allowed to remove the absolute value since $-1 < x < 1$. To find the constant we can substitute $x = 0$. We have $\tanh(0) = 0$, so $\tanh^{-1}(0) = 0$. Additionally $\ln(1) = 0$. Therefore we must have $c = 0$, and we conclude that

$$\tanh^{-1}(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x).$$

2. (Oct. 12; Antiderivatives; 6 points) Suppose that $E(x) = \int e^{-x^2}\,dx$. That is, suppose that $E(x)$ is a function such that $E'(x) = e^{-x^2}$. *Remark: $E(x)$ cannot be expressed in terms of simpler functions that you already know. In this way, you should think of $E(x)$ as a "new function."* Express the antiderivative

$$\int xe^{-x^4-2x^2-1}\,dx$$

in terms of the function E and polynomials.

Solution: Note,

$$e^{-x^4-2x^2-1} = e^{-(x^2+1)^2}.$$

We make the substitution $u = x^2 + 1$. We have,

$$du = 2xdx$$

so,

$$\begin{aligned}\int xe^{-x^4-2x^2-1}\,dx &= \frac{1}{2}\int (2x)e^{-(x^2+1)^2}\,dx \\ &= \frac{1}{2}\int e^{-u^2}\,du \\ &= \frac{1}{2}E(u) + c = \frac{1}{2}E(x^2+1) + c,\end{aligned}$$

for a constant c.

3. (Oct. 13; Differential equations and separation of variables; 2 + 1 + 4 = 7 points) Section 5.4: 20, Section 8.5: 8, pg. 291 problem 54ab; find the "blow-up" time t_0 in part b.

Solution to 5.4 - 20: We consider P as a function of time t measured in weeks since the curse was uttered. Then the problem tells us that $P(0) = 676$ and $\frac{dP}{dt} = -\sqrt{P}$. We are interested in know for what $t_1 > 0$ we have $P(t_1) = 0$ (i.e. when are all the people dead.) Separating variables gives us

$$P^{-1/2}\,dP = -dt.$$

Integrating we get

$$2P^{1/2} = -t + c$$

for some constant c. If we put $t = 0$ we get

$$c = 2(676)^{1/2} = 52.$$

We have $P = 0$ precisely when $2P^{1/2} = 0$, so this is precisely when $t = c$. In other words, $P(t) = 0$ when
$$t = 52.$$
So it takes 52 weeks (roughly a year) before all the people are dead.

Solution to 8.5 - 8: Let $M(t)$ be the fool's money in dollars at time t where t is the time since he started gambling measured in hours. We are told,
$$\frac{dM}{dt} = -\frac{1}{3}M(t).$$
Separating variables we get
$$\frac{1}{M}\,dM = -\frac{1}{3}dt.$$
Integrating we get
$$\ln(M) = -\frac{t}{3} + c.$$
Setting $t = 0$ we get
$$\ln(M(0)) = c.$$
Therefore,
$$M = e^{\frac{-t}{3}+c} = e^{\frac{-t}{3}}M(0).$$
We want to know for what t we have $M(t) = \frac{M(0)}{2}$. This is equivalent to finding the t such that
$$e^{\frac{-t}{3}} = \frac{1}{2}.$$
The solution to this is $t = -3\ln(1/2) = 3\ln(2)$. Therefore the fool will have lost half his money after $3\ln(2)$ hours.

Solution to pg. 291 problem 54ab ...: Separating variables in the doomsday equation we get
$$N^{-(1+\epsilon)}\,dN = k\,dt.$$
Integrating we get
$$\frac{N^{-\epsilon}}{-\epsilon} = kt + c.$$
Setting $t = 0$ we get
$$c = -\frac{N(0)^{-\epsilon}}{\epsilon} = \frac{-1}{\epsilon N(0)^{\epsilon}}.$$
So we get
$$N(t) = (-\epsilon(kt + c))^{\frac{1}{-\epsilon}} = \frac{1}{\left(\epsilon(\frac{1}{\epsilon N(0)^{\epsilon}} - kt)\right)^{1/\epsilon}}.$$
We notice that this is undefined at $t_0 = \frac{1}{k\epsilon N(0)^{\epsilon}}$, and as the denominator approaches 0 from the right as $t \to t_0^-$ we have,
$$\lim_{t\to t_0^-} N(t) = \infty.$$

4. (Oct. 13; Differential equations; 6 + 4 + 2 = 12 points) In this problem, you will learn about terminal velocity. Consider an object of mass m dropped (straight down) off of a very tall cliff. Its downward velocity is initially 0. Let $v(t)$ denote its *downward* velocity at time t (i.e. $v > 0$ means the object is falling towards the earth). In a very simple model of air resistance, the air provides a drag force that is proportional to v^2. That is, the air exerts an *upward* force of magnitude κv^2 on the object, where $\kappa > 0$ is a constant (which has dimensions of $\frac{\text{mass}}{\text{length}}$). Let $g > 0$ denote the downward acceleration of gravity. For the purposes of this problem, we can assume that g is a constant (approximately equal to 9.8 meter/sec^2). Under these assumptions, Newton's laws of motion imply that

$$\frac{d}{dt}v = -\frac{\kappa}{m}v^2 + g. \tag{1}$$

a) Use separation of variables to solve the differential equation with the initial condition $v(0) = 0$. *Hint: A modified version of the method of partial fractions from problem 1 might be useful...*

Solution: Separating the variables we get

$$\frac{\frac{-m}{\kappa}}{v^2 - \frac{mg}{\kappa}}dv = dt. \tag{2}$$

To be able to integrate the left hand side we follow the hint and try to find A and B such that

$$\frac{\frac{-m}{\kappa}}{v^2 - \frac{mg}{\kappa}} = \frac{A}{v - \sqrt{\frac{mg}{\kappa}}} + \frac{B}{v + \sqrt{\frac{mg}{\kappa}}}.$$

Since,

$$\begin{aligned}\frac{A}{v - \sqrt{\frac{mg}{\kappa}}} + \frac{B}{v + \sqrt{\frac{mg}{\kappa}}} &= \frac{A\left(v + \sqrt{\frac{mg}{\kappa}}\right) + B\left(v - \sqrt{\frac{mg}{\kappa}}\right)}{v^2 - \frac{mg}{\kappa}} \\ &= \frac{(A-B)\sqrt{\frac{mg}{\kappa}} + (A+B)v}{v^2 - \frac{mg}{\kappa}}\end{aligned}$$

we can just choose

$$B = -A, \qquad \text{and} \qquad A = \frac{-1}{2}\sqrt{\frac{m}{\kappa g}}$$

because then

$$(A-B)\sqrt{\frac{mg}{\kappa}} + (A+B)v = \frac{-2}{2}\sqrt{\frac{m}{\kappa g}}\sqrt{\frac{mg}{\kappa}} + 0v = -\frac{m}{\kappa}.$$

Thus we have,

$$\frac{\frac{-m}{\kappa}}{v^2 - \frac{mg}{\kappa}} = \frac{1}{2}\sqrt{\frac{m}{\kappa g}}\left(\frac{-1}{v - \sqrt{\frac{mg}{\kappa}}} + \frac{1}{v + \sqrt{\frac{mg}{\kappa}}}\right).$$

Thus integrating our separated ODE (2) we get

$$\begin{aligned}
t + c &= \frac{1}{2}\sqrt{\frac{m}{\kappa g}} \int \frac{-1}{v - \sqrt{\frac{mg}{\kappa}}} + \frac{1}{v + \sqrt{\frac{mg}{\kappa}}}\, dv \\
&= \frac{1}{2}\sqrt{\frac{m}{\kappa g}} \left(\ln\left|v + \sqrt{\frac{mg}{\kappa}}\right| - \ln\left|v - \sqrt{\frac{mg}{\kappa}}\right| \right) \\
&= \frac{1}{2}\sqrt{\frac{m}{\kappa g}} \left(\ln\left(\sqrt{\frac{mg}{\kappa}} + v\right) - \ln\left(\sqrt{\frac{mg}{\kappa}} - v\right) \right) \\
&= \frac{1}{2}\sqrt{\frac{m}{\kappa g}} \left(\ln\left(1 + v\frac{\sqrt{\kappa}}{\sqrt{mg}}\right) - \ln\left(1 - v\frac{\sqrt{\kappa}}{\sqrt{mg}}\right) \right) \\
&= \sqrt{\frac{m}{\kappa g}} \tanh^{-1}\left(v\frac{\sqrt{\kappa}}{\sqrt{mg}}\right).
\end{aligned}$$

where the last equality comes from $\tanh^{-1}(x) = \frac{1}{2}(\ln(1+x) - \ln(1-x))$. From this we get

$$\tanh^{-1}\left(v\frac{\sqrt{\kappa}}{\sqrt{mg}}\right) = \frac{\sqrt{\kappa g}}{\sqrt{m}}(t + c).$$

$$v = \frac{\sqrt{mg}}{\sqrt{\kappa}} \tanh\left(\frac{\sqrt{\kappa g}}{\sqrt{m}}(t + c)\right). \tag{3}$$

This is the general solution to the ODE. We need to impose the condition $v(0) = 0$. Substituting $t = 0$ in the solution we get

$$0 = \tanh\left(\frac{\sqrt{\kappa g}}{\sqrt{m}} c\right)$$

which implies $c = 0$. Therefore our solution to the differential equation (1) with initial condition $v(0) = 0$ is

$$v = \frac{\sqrt{mg}}{\sqrt{\kappa}} \tanh\left(\frac{\sqrt{\kappa g}}{\sqrt{m}} t\right).$$

b) Does $\lim_{t\to\infty} v(t)$ exist? If so, explain what happens when you plug the limiting velocity into the right-hand side of the differential equation (1) and why the result makes sense.

Solution: We have

$$\lim_{t\to\infty} \tanh(t) = 1$$

so

$$\begin{aligned}
\lim_{t\to\infty} v(t) &= \lim_{t\to\infty} \frac{\sqrt{mg}}{\sqrt{\kappa}} \tanh\left(\frac{\sqrt{\kappa g}}{\sqrt{m}} t\right) \\
&= \frac{\sqrt{mg}}{\sqrt{\kappa}}.
\end{aligned}$$

When we plug this limiting value into our ODE (1) we get

$$\frac{d}{dt}v = -\frac{\kappa}{m}\left(\frac{\sqrt{mg}}{\sqrt{\kappa}}\right)^2 + g$$

$$\frac{d}{dt}v = -g + g = 0.$$

In other words in the limit the derivative of velocity goes to 0 (in other words acceleration is 0 in the limit.) This is because the closer we get to terminal velocity the acceleration approaches 0. The velocity never actually reaches terminal velocity, but it will get arbitrarily close to it.

c) Explain how your answer to part b) changes if the object is thrown off the cliff (straight down) with initial velocity $v_0 > 0$.

Solution: Using the general solution (3) we still get the same limit in (b)

$$\begin{aligned}\lim_{t\to\infty} v(t) &= \lim_{t\to\infty} \frac{\sqrt{mg}}{\sqrt{\kappa}} \tanh\left(\frac{\sqrt{\kappa g}}{\sqrt{m}}(t+c)\right)\\ &= \frac{\sqrt{mg}}{\sqrt{\kappa}}.\end{aligned}$$

Therefore the answer doesn't change.

Our solution assumes that v_0 is less than the terminal velocity. If not, then our solution to (a) needs to be modified since we used

$$\ln\left|v - \sqrt{\frac{mg}{\kappa}}\right| = \ln\left(\sqrt{\frac{mg}{\kappa}} - v\right).$$

The answer to (b) would still come out the same though.

5. (Oct. 18; Definite integrals; 1 + 1 + 1 + 5 = 8 points) First do Section 6.3: 9. Then use this problem to help you calculate $\int_0^{\pi/2} \cos x\, dx$ by using right Riemann sums. By a "right Riemann sum," we mean that the right endpoint of the approximating rectangles lies on the graph of $y = \cos x$.

Solution to 9.3-9(a): Set $m = 1, \theta = x/2, n = 2k$ in the product formula to get

$$\sin\frac{x}{2}\cos kx = \frac{1}{2}\left[\sin\left(\frac{2k+1}{2}x\right) + \sin\left(\frac{1-2k}{2}x\right)\right]$$

which can be rewritten as

$$2\sin\frac{x}{2}\cos kx = \sin\left((k+\frac{1}{2})x\right) + \sin\left((k-\frac{1}{2})x\right)$$

Solution to 9.3-9(b): We have,

$$\begin{aligned}2\sin\left(\frac{1}{2}x\right)\sum_{k=1}^{n}\cos kx &= \sum_{k=1}^{n}\left[\sin\left(\left(k+\frac{1}{2}\right)x\right) - \sin\left(\left(k-\frac{1}{2}\right)x\right)\right]\\ &= \sum_{k=1}^{n}\sin\left(\left(k+\frac{1}{2}\right)x\right) - \sum_{k=1}^{n}\sin\left(\left(k-\frac{1}{2}\right)x\right)\\ &= \sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\frac{1}{2}x\right).\end{aligned}$$

So,

$$\sum_{k=1}^{n}\cos kx = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)}.$$

Solution to 9.3-9(c): The product formula with $\theta = x/2$ gives us

$$\sin\left(\frac{nx}{2}\right)\cos\left(\frac{(n+1)x}{2}\right) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\frac{1}{2}x\right)}{2}.$$

Thus,

$$\sum_{k=1}^{n} \cos kx = \frac{\sin\left(\frac{nx}{2}\right)\cos\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{1}{2}x\right)}.$$

Solution to main problem: If we split the interval $[0, \pi/2]$ into n equal intervals, then the associated right Riemann sum of $\cos x$ is

$$\sum_{k=1}^{n} \cos\left(\frac{k}{n}\frac{\pi}{2}\right)\frac{\pi}{2n}.$$

Setting $x = \frac{\pi}{2n}$ in our solution to 9.3-9(c) we get

$$\sum_{k=1}^{n} \cos\left(\frac{k}{n}\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{(n+1)\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}.$$

We can now calculate

$$\begin{aligned}
\int_0^{\pi/2} \cos(x)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \cos\left(\frac{k}{n}\frac{\pi}{2}\right)\frac{\pi}{2n} \\
&= \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{(n+1)\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}\frac{\pi}{2n} \\
&= \sin\left(\frac{\pi}{4}\right)\lim_{n\to\infty}\cos\left(\frac{(n+1)\pi}{4n}\right)\lim_{n\to\infty}\frac{\pi}{2n\sin\left(\frac{\pi}{4n}\right)} \\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\lim_{n\to\infty}\frac{(n+1)\pi}{4n}\right)\lim_{n\to\infty}\frac{2\frac{\pi}{4n}}{\sin\left(\frac{\pi}{4n}\right)} \\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)2 \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}2 = 1.
\end{aligned}$$

6. (Oct. 25; First fundamental theorem; properties of integrals; 4 + 3 = 7 points) Section 6.7: 12

Solution: We can draw a rectangle that goes from $(0,0)$ to (a, a^n). We can calculate the area of this rectangle in two ways. First we can use the basic formula which tells us that the rectangle must have area a^{n+1}. Secondly we can split the rectangle into two parts using the graph of $y = x^n$ and then calculate each part separately. The part below the curve is $\int_0^a x^n\,dx$. The area above the curve is equivalent to the area below the curve of $y = x^{1/n}$ from $x = 0$ to $x = a^n$ since we can go from one to the other by reflecting in the diagonal line $y = x$. Therefore the area of the second piece of our rectangle is $\int_0^{a^n} y^{1/n}\,dy$. Therefore we must have

$$\int_0^a x^n\,dx + \int_0^{a^n} y^{1/n}\,dy = a^{n+1}.$$

Let us now check that this equation actually holds. From the first fundamental theorem of calculus we know

$$\int_a^b x^k \, dx = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

So,

$$\int_0^a x^n \, dx = \frac{a^{n+1} - 0^{n+1}}{n+1} = \frac{a^{n+1}}{n+1}.$$

$$\int_0^{a^n} y^{1/n} = \frac{(a^n)^{1/n+1} - 0^{1/n+1}}{1/n+1} = \frac{na^{n+1}}{n+1}.$$

Thus,

$$\int_0^a x^n \, dx + \int_0^{a^n} y^{1/n} \, dy = \left(\frac{1}{n+1} + \frac{n}{n+1}\right) a^{n+1} = a^{n+1}$$

which proves our equation holds as we knew from geometric arguments.