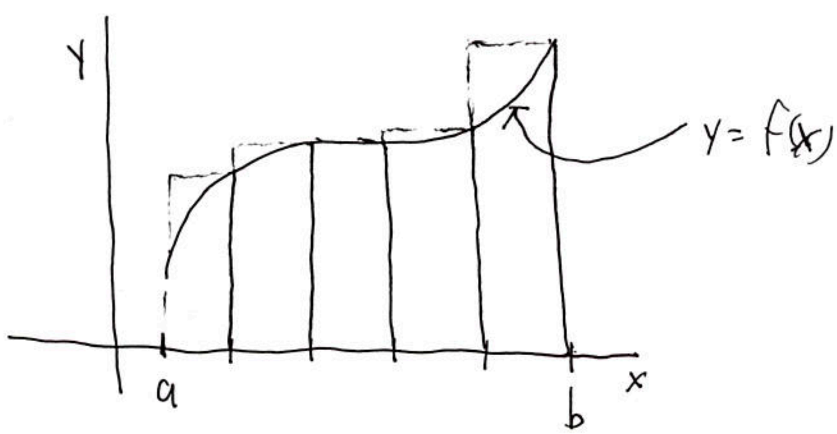
• Definite integrals

- Integrals are used to calculate cumulative totals, averages, & areas.

- Area under a curve
  1. Divide region into rectangles
  2. Add up area of rectangles
  3. Take limit as rectangles become thin



Ex $f(x) = x^2$ $\quad a=0,\ b=$ arbitrary

① Divide $[0,b]$ into n intervals of length $\frac{b}{n}$

② Heights of rectangles: • 1st : $x = \frac{b}{n}$, height $= \left(\frac{b}{n}\right)^2$

• 2nd $x = \frac{2b}{n}$, height $= \left(\frac{2b}{n}\right)^2$

Add up the areas of the rectangles:

$$\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \cdots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2 = \frac{b^3}{n^3}\left(1^2+2^2+\cdots+n^2\right)$$

In a separate step below, we will show that

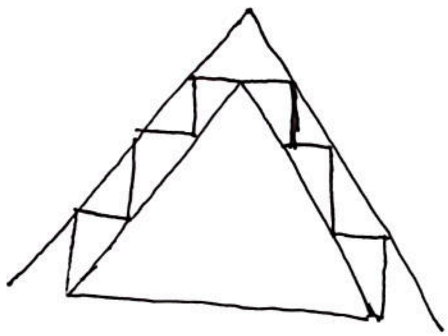
$$(*) \quad \frac{1}{3} < \frac{1^2+2^2+\cdots+n^2}{n^3} < \frac{1}{3}\,\frac{(n+1)^3}{n^3}$$

- Thus as $n \to \infty$ (i.e, as the rectangles become infinitely thin), we have

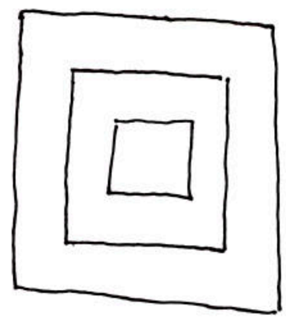
$$\frac{b^3}{n^3}\left(1^2+2^2+\cdots+n^2\right) \xrightarrow{n\to\infty} \frac{1}{3}b^3.$$

Hence, the area under the curve $y = x^2$ from 0 to b is $\frac{b^3}{3}$.

- To establish the inequalities in (*), we consider the following 3-d staircase pyramid:



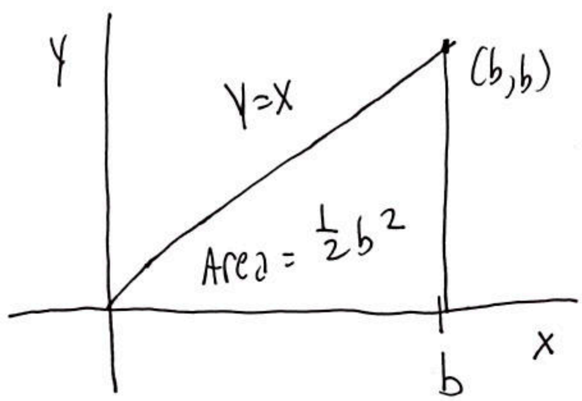
$n = 3$



- 1st level is an $n \times n \times 1$ layer of volume n^2
- 2nd level is an $(n-1) \times (n-1) \times 1$ layer of volume $(n-1)^2$, etc.
- The total volume of the staircase pyramid is
  $n^2 + (n-1)^2 + \cdots + 1$
- The volume of the staircase pyramid is $>$ the volume of the inner prism:
  $$1^2 + 2^2 + \cdots + n^2 > \frac{1}{3} \cdot \text{base} \cdot \text{height} = \frac{1}{3} n^2 \cdot n = \frac{1}{3} n^3$$
- Similarly, it is $<$ the volume of the outer prism:
  $$1^2 + 2^2 + \cdots + n^2 < \frac{1}{3} (n+1)^2 (n+1) = \frac{1}{3} (n+1)^3$$
- These two inequalities together imply (*) as desired.

Ex: $f(x) = x$. Reasoning similar to the previous example gives that $\frac{b^2}{n^2}(1 + 2 + \cdots + n) \to \frac{1}{2} b^2$ as $n \to \infty$.

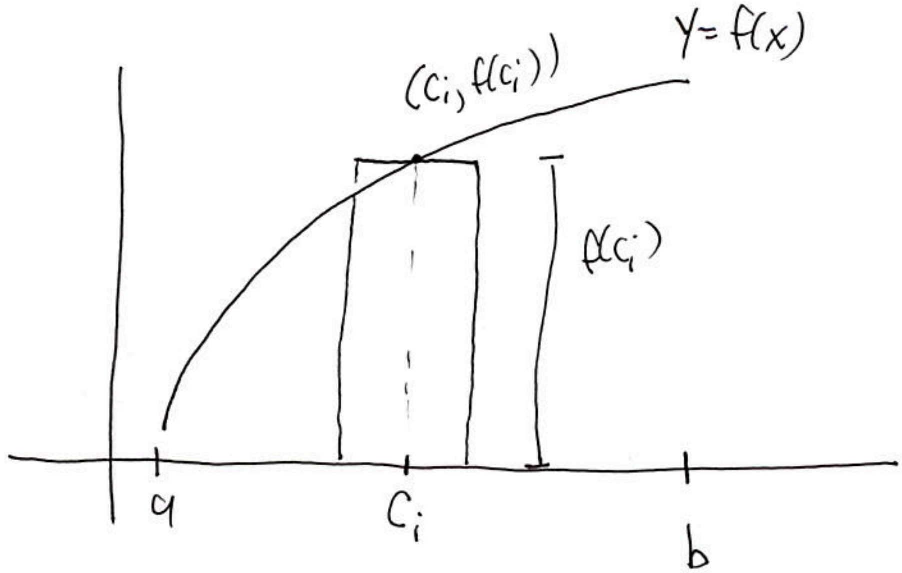
- Thus, the area under the curve from 0 to b is $\frac{1}{2} b^2$. This is the area of the triangle below:



- Important pattern:
  - $\frac{d}{db}\left(\frac{b^3}{3}\right) = b^2$
  - $\frac{d}{db}\left(\frac{b^2}{2}\right) = b$

- The area $A(b)$ under the curve $y = f(x)$ from 0 to b should satisfy $A'(b) = f(b)$.

- General Picture



- Divide $[a,b]$ into n equal pieces of length $\Delta x = \frac{b-a}{n}$
- Pick <u>any</u> c_i in the ith interval and use $f(c_i)$ as the height of the rectangle
- Sum of areas: $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$
- Using summation notation, we can write the sum as

$$\underbrace{\sum_{i=1}^{n} f(c_i)\Delta x}_{\text{Riemann Sum}}$$

- <u>Definition</u>

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \underbrace{\int_a^b f(x)\,dx}_{\text{Definite integral}}$$

- $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ above the interval $[a,b]$.