- Ex (product rule)

$$\frac{d}{dx}(x\cos x) = \left(\frac{d}{dx}x\right)\cos x + x\frac{d}{dx}\cos x$$
$$= 1\cdot\cos x + x(-\sin x)$$
$$= \cos x - x\sin x$$

- Ex (quotient rule)

We have seen

$$\frac{d}{dx}x^n = nx^{n-1} \qquad n = 0, 1, 2, 3, \ldots$$

what about $n = -1, -2 \ldots$

$n = -1$:

$$\frac{d}{dx}\frac{1}{x} = \frac{x\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}x}{x^2} = \frac{-1}{x^2} = -x^{-2}$$

- In your HW: $\frac{d}{dx}x^n = nx^{n-1}$ when $n = -1, -2, -3, \ldots$ by induction

You will see an alternate proof later today.

- Chain rule and higher derivatives

- Today: ① How to compute the derivative of the composition of two functions (Chain rule).
  ② Higher derivatives

① Chain rule example

- $y = f(g(t))$ • $f(x) = \sin x$ • $x = g(t) = t^2$

To find $\frac{dy}{dt}$, write

$t_0 = t_0$	$t = t_0 + \Delta t$
$x_0 = g(t_0)$	$x = x_0 + \Delta x$
$y_0 = f(x_0)$	$y = y_0 + \Delta y$

- $$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta t}$$

- As $\Delta t \to 0$, $\Delta x \to 0$ since g is continuous.
- And $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$, $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$
- Thus, $\frac{\Delta y}{\Delta t} \to \boxed{\frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{dy}{dt}}$: Chain rule
- In the example, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dx} = \cos x$.
- Thus, $\frac{d}{dt}\sin(t^2) = \frac{dy}{dx}\cdot\frac{dx}{dt} = \cos x \cdot 2t = 2t\cos t^2$.

- Alternate Chain rule notation:

$$\frac{d}{dt} f(g(t)) = f'(g(t))\, g'(t)$$

$$\left(\text{or } \frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x) \right).$$

- Example continued: Composition of functions $f(x) = \sin x$ and $g(x) = x^2$
  - $(f \circ g)(x) = f(g(x)) = \sin(x^2)$
  - $g \circ f(x) = g(f(x)) = \sin^2(x)$
    - Note: $f \circ g \neq g \circ f$ in this example

$$\begin{tikzcd} {} \arrow[r, "x"] & \boxed{g} \arrow[r, "g(x)"] & \boxed{f} \arrow[r, "f(g(x))"] & {} \end{tikzcd}$$

- Example: $\frac{d}{dx} \overbrace{\cos(\frac{1}{x})}^{y} = \ ?$ • $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

  - let $u = \frac{1}{x}$
  - $\frac{dy}{du} = \frac{d}{du}(\cos u) = -\sin u$ • $\frac{du}{dx} = \frac{-1}{x^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin u}{x^2} = \frac{\sin(\frac{1}{x})}{x^2}$$

Example: $\frac{d}{dx} x^{-n} = ?$ $n = 1, 2, 3, \dots$

- Solution 1: $x^{-n} = \left(\frac{1}{x}\right)^n$

$$\Rightarrow \frac{d}{dx} x^{-n} = \frac{d}{dx}\left(\frac{1}{x}\right)^n = n\left(\frac{1}{x}\right)^{n-1} \cdot \left(-\frac{1}{x^2}\right)$$

$$= -n x^{-(n-1)} \cdot x^{-2} = -n x^{-n-1}.$$

- Solution 2: $x^{-n} = \frac{1}{x^n}$

$$\Rightarrow \frac{d}{dx} x^{-n} = \frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{1}{(x^n)^2} \cdot \frac{d}{dx} x^n$$

$$= -x^{-2n} \cdot n x^{n-1}$$

$$= -n x^{-n-1}.$$

Example: $\frac{d}{dx} \sin(\cos(x)) = \cos(\cos x) \cdot (-\sin x)$

$$= -\sin x \cos(\cos(x)).$$

- Higher derivatives
  - This is not a difficult concept; just keep differentiating.
  - If $h = f'$, we write $h' = (f')' = f''$!
    - Notation:

$f'(x)$	Df	$\frac{df}{dx}$	
$f''(x)$	$D^2 f$	$\frac{d^2 f}{dx^2}$	
$f'''(x)$	$D^3 f$	$\frac{d^3 f}{dx^3}$	
$f^{(n)}(x)$	$D^n f$	$\frac{d^n f}{dx^n}$	

Example: $D^n x^n = ?$

- $Dx = 1$
- $D^2 x^2 = D(2x) = 2 = 1 \cdot 2$
- $D^3 x^3 = D(3x^2) = 2 \cdot 3 \cdot Dx = 6 = 1 \cdot 2 \cdot 3$

⋮

- $D^n x^n = n!$ (educated guess)

"n factorial"; $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$

- Proof by induction: (base case $n=1$ is done). So suppose we know $\overbrace{D^n x^n = n!}^{\text{"case } n\text{"}}$. We then want to show that $D^{n+1} x^{n+1} = (n+1)!$ (i.e., that the "case $n+1$" holds)
- $D^{n+1} x^{n+1} = D^n (D x^{n+1}) = D^n ((n+1) x^n) = (n+1) D^n x^n = (n+1) \cdot n! = (n+1)!$ $\Rightarrow$ done!

case n