

## MATH 18.01, FALL 2017 - PROBLEM SET # 3A

**Professor:** Jared Speck

**Due: by 1:45pm on 10-6-17**

(in the boxes outside of Room 4-174 during the day; write your name and recitation instructor on your homework)

**18.01 Supplementary Notes (including Exercises and Solutions)** are available on the course web page: [http://math.mit.edu/~jspeck/18.01\\_Fall%202017/1801\\_CourseWebsite.html](http://math.mit.edu/~jspeck/18.01_Fall%202017/1801_CourseWebsite.html). This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

**Part I** consists of exercises given and solved in the Supplementary Notes. It will be graded quickly, checking that all is there and the solutions not copied.

**Part II** consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises given here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

**You are encouraged to use graphing calculators, software, etc. to check your answers and to explore calculus. However, (unless otherwise indicated) we strongly discourage you from using these tools to solve problems, perform computations, graph functions, etc. An extremely important aspect of learning calculus is developing these skills. You will not be allowed to use any such tools on the exams.**

### Part I (15 points)

**Notation:** The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

**Lecture 7.** (Thurs., Sept. 21) Linear and quadratic approximations.

*Read:* Notes A.

**Homework:** Notes 2A: 2, 3, 7, 11, 12ade.

**Lecture 8.** (Friday, Sept. 22) Curve-sketching.

*Read:* 4.1, 4.2.

**Homework:** Notes 2B: 1abcefhi, 2abcefhi, 4, 6ab, 7ab.

**Part II** (40 points)

**Directions and Rules:** Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. *You must show your work; "bare" solutions will receive very little credit.*

iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

**0.** (not until due date; 3 points) Write the names of all the people whom you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem 0" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

**1.** (Sept. 21; quadratic approximations;  $3 + 2 + 3 = 8$  points) Your electric guitar speaker has a continuous dial that varies from 0 to 11. Assume that the volume output  $v$  of the speaker is a twice differentiable function of the dial number  $d$ . Using your decibel meter, you have measured the volume output of your speaker at the following three settings:

	$d = 8$	$d = 9$	$d = 10$
$v$ (in decibels)	79.931	90.744	101.983

You are very curious to know the volume output when  $d = 11$ , but since you live in a dorm, you don't dare turn the dial to 11. Luckily, you are enrolled in 18.01 and are therefore able to estimate the output using the following procedure.

a) Estimate  $v'(9)$  and  $v'(10)$  by using difference quotients of the form  $\frac{f(x) - f(x - \Delta x)}{\Delta x}$  for  $\Delta x = 1$ .

b) Use part a) and a similar procedure to estimate  $v''(10)$ .

c) Derive a quadratic approximation of  $v(d)$  near  $d = 10$ . In your quadratic approximation, you can use the approximate values for  $v'$  and  $v''$  from parts a) and b). Then use your quadratic approximation to estimate  $v(11)$ .

**2.** (Sept. 21; limits; continuity; quadratic approximations;  $2 + 2 + 1 + 2 + 1 + 1 + 1 = 10$  points) In this problem, you will investigate what happens to the surface area of a sphere when you slightly squash it. A perfect unit sphere can be visualized in the following way: consider the set of points in the  $(x, y)$  plane that satisfy the equation  $x^2 + y^2 = 1$ . This curve describes a circle of radius 1 centered at the origin. Now imagine that this circle is continuously rotated in physical space about the  $y$  axis

until it has spun a full 360 degrees. The resulting 3D object traced out is a sphere of radius 1 in space centered at the origin.

Now instead of the circle, consider the curve

$$x^2 + \left(\frac{y}{a}\right)^2 = 1,$$

where  $a$  is some number with  $0 < a < 1$ . This curve describes an ellipse in the  $(x, y)$  plane centered at the origin. The horizontal axis of the ellipse has length 2 and the vertical axis has length  $2a$ . Now imagine that this ellipse is rotated about the  $y$  axis in the manner described above. The resulting 3D object that gets traced out is called an *oblate ellipsoid*, which means that it is a sphere that has been squashed<sup>1</sup> in one direction (in this case the  $y$  axis is the squashed direction). This geometric object comes up when one studies the shape of the earth: the earth's rotation gives it a slightly squashed character.<sup>2</sup>

There is an important quantity associated to the oblate ellipsoid called the *eccentricity*  $E$ . The eccentricity is equal to 0 for a perfect sphere, and for a oblate ellipsoid, it measures how squashed it is:

$$E = \sqrt{1 - a^2}.$$

In vector calculus, you will develop the tools necessary to calculate the surface area of the oblate ellipsoid. Here, I will only provide you with the formula. The surface area  $S$  can be expressed in terms of  $E$  as follows:

$$S = 2\pi (1 + f(E) \times (1 - E^2)),$$

where

$$f(E) = \begin{cases} \frac{\tanh^{-1}(E)}{E} & \text{if } E \neq 0, \\ 1 & \text{if } E = 0. \end{cases}$$

In the above formula,  $\tanh^{-1}(E)$  is the inverse of the hyperbolic tangent function. That is,  $y = \tanh^{-1}(E)$  if and only if  $\tanh(y) = E$ , where  $\tanh(E) = \sinh(E)/\cosh(E)$ .

a) Show that  $f(E)$  is continuous at  $E = 0$ .

b) Show that  $(d/dE) \tanh^{-1}(E) = 1/(1 - E^2)$ .

c) Find the quadratic approximation to  $1/(1 - E^2)$  near  $E = 0$ . That is, find the constants  $A_0$ ,  $A_1$  and  $A_2$  such that  $1/(1 - E^2) = A_0 + A_1E + A_2E^2 + O(E^3)$  near  $E = 0$ .

d) Imagine that you have made a *cubic* approximation to  $\tanh^{-1}(E)$  near  $E = 0$ . That is, assume that you have approximated  $\tanh^{-1}(E) = B_0 + B_1E + B_2E^2 + B_3E^3 + O(E^4)$  for constants  $B_0, B_1, B_2, B_3$ . Assume that  $B_0 + B_1E + B_2E^2$  is the usual quadratic approximation to  $\tanh^{-1}(E)$  near  $E = 0$ . Assume in addition that the derivative of your cubic approximation to  $\tanh^{-1}(E)$  is precisely the quadratic approximation to  $1/(1 - E^2) = (d/dE) \tanh^{-1}(E)$  that you found in part c). Under these assumptions, find the constants  $B_0, B_1, B_2, B_3$ .

<sup>1</sup>The word “squashed” may be a bit misleading in the present context. One can show using the methods of vector calculus that the volume of the oblate ellipsoid of interest is  $\frac{4}{3}\pi a^2$ . Since the “unsquashed” perfect unit sphere has volume  $\frac{4}{3}\pi$ , by my definition of “squashing,” some volume can be lost.

<sup>2</sup>In reality, the earth is quite bumpy, and therefore the oblate ellipsoid model of the earth may not be accurate enough for all applications.

e) Use part d) to find the *quadratic* approximation to  $f(E)$  near  $E = 0$ .

f) Use part e) to find the quadratic approximation to  $S$  near  $E = 0$ .

*Remark: The quadratic approximation procedure outlined in this problem can be rigorously justified. It turns out that the quadratic approximations you are deriving are in some sense the “best possible ones.” These ideas fall under the scope of Taylor’s series, which we will discuss at the end of the course.*

g) Does slightly squashing the sphere increase or decrease its surface area? Put differently, when  $E$  is positive but near 0, does your quadratic approximation predict that the surface area of the ellipsoid will be greater than or less than the surface area of the perfect sphere of radius 1? Note that the perfect sphere has  $E = 0$ .

**3.** (Sept. 26;  $2 + 2 + 2 + 4 = 10$  points) Section 4.1: 18abc, 22

**4.** (Sept. 26;  $2 + 2 + 2 + 3 = 9$  points) Section 4.2: 12, 16, 18, 24