MATH 18.01 - FINAL EXAM REVIEW: SUMMARY OF SOME KEY CONCEPTS

18.01 Calculus, Fall 2017	Professor: Jared Speck

a. Polar coordinates
- (a) $x = r\cos\theta$, $y = r\sin\theta$
- (b) In the standard formulation, $r = \sqrt{x^2+y^2}$, θ is the polar angle, and $0 \leq \theta < 2\pi$
- (c) Area in polar coordinates: Area under the curve $r = f(\theta)$ in between the angles θ_1 and θ_2 is given by Area $= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} [f(\theta)]^2\, d\theta$

b. L'Hôpital's rule
- (a) Sometimes allows one to evaluate limits of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 0^0, 1^∞
- (b) Many of the above limits can be massaged into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, where L'Hôpital's rule can sometimes directly be applied. For example, the 0^0 case can be massaged into the $\frac{0}{0}$ case with the help of $\ln$.
- (c) In the "$\frac{0}{0}$" case: If f, g are differentiable functions, a is a finite number, $f(a) = g(a) = 0$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Furthermore, it is sometimes true that $L = \frac{f'(a)}{g'(a)}$ (for example, when $f'(x)$ and $g'(x)$ are continuous at $x = a$ and $g'(a) \neq 0$).
- (d) In the "$\frac{\infty}{\infty}$" case: If f, g are differentiable functions, a is a finite number, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.
- (e) Analogous statements hold if we replace $\lim_{x\to a}$ with $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.

c. Improper integrals
- (a) If $f(x)$ is continuous for $0 \leq x < \infty$, then by definition, $\int_0^\infty f(x)\, dx = \lim_{M\to\infty} \int_0^M f(x)\, dx$
  - (i) If the limit exists, we say the improper integral *converges*. Otherwise, we say it *diverges*.
- (b) If $f(x)$ is continuous for $a < x \leq b$ but is *not* continuous at $x = a$, then by definition, $\int_a^b f(x)\, dx = \lim_{x_0\to a^+} \int_{x_0}^b f(x)\, dx$
  - (i) If the limit exists, we say the improper integral *converges*. Otherwise, we say it *diverges*.

d. Infinite series
- (a) Are series of the form $\sum_{k=0}^\infty a_k = a_0 + a_1 + a_2 + a_3 + \cdots$
- (b) By definition, $\sum_{k=0}^\infty a_k = \lim_{M\to\infty} S_M$, where $S_M = \sum_{k=0}^M a_k = a_0 + a_1 + a_2 + \cdots + a_M$ is the M^{th} partial sum.
  - (i) If $\lim_{M\to\infty} S_M$ exists, we say the series *converges*. Otherwise, we say it *diverges*.

(c) Geometric series: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $|x| < 1$. $\sum_{k=0}^{\infty} x^k$ diverges if $|x| \geq 1$.

(d) Comparison: If $0 \leq a_k \leq b_k$ for all large k, and if $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges too (divergence of smaller $\Longrightarrow$ divergence of bigger). If $0 \leq a_k \leq b_k$ for all large k, and if $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges too (convergence of bigger $\Longrightarrow$ convergence of smaller).

(e) Limit comparison test: If $a_k \geq 0, b_k \geq 0$ for all large k and $a_k \sim b_k$, then $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges. Here, $a_k \sim b_k$ means that there exists a non-zero number L such that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.

(f) Integral comparison: If $f(x)$ is continuous, $f(x) \geq 0$ for all x, and $f(x)$ is decreasing for all large x, then $\sum_{k=0}^{\infty} f(k)$ converges if and only if the improper integral $\int_{x=0}^{\infty} f(x)\,dx$ converges.

e. Taylor's series with base point $b = 0$

(a) For x near $0 : f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

(b) $a_n = \frac{f^{(n)}(0)}{n!}$, where $f^{(n)}$ is the n^{th} derivative of f

(c) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(d) $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \cdots$

(e) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \mp \cdots$

(f) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots$

f. Taylor's series with base point b

(a) For x near $b : f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + a_3(x-b)^3 + \cdots$

(b) $a_n = \frac{f^{(n)}(b)}{n!}$

(c) $x^A = b^A + Ab^{A-1}(x-b) + \frac{A(A-1)b^{A-2}}{2!}(x-b)^2 + \frac{A(A-1)(A-2)b^{A-3}}{3!}(x-b)^3 + \cdots$