Applications of FTC 2 to logarithms

- We will now use FTC 2 to provide an alternate approach to studying the function $\ln(x)$.

- We introduce the "new" function
$$L(x) = \int_1^x \frac{dt}{t}. \quad \text{Note: } L(1) = 0$$

- Then FTC 2 implies
$$L'(x) = \frac{1}{x}.$$

- Recall that $\frac{d}{dx} \ln x = \frac{1}{x}$ and $\ln(1) = 0$.

- Thus, by the Mean Value Theorem argument:
$\frac{d}{dx}\left(L(x) - \ln(x)\right) = 0$, and hence
$L(x) = \ln(x) + C$. The constant c must be 0 since $L(1) = \ln(1) = 0$. Hence, $\boxed{L(x) = \ln(x)}$

- We can derive some important properties of $\ln(x)$ by using the representation

$$L(x) = \int_1^x \frac{dt}{t}$$

Claim 1: $L(ab) = L(a) + L(b)$.

Proof: By the definition of $L(ab)$ and $L(a)$, we have $L(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{dt}$

$$= L(a) + \int_a^{ab} \frac{dt}{t}.$$

- We now make the substitution $t = au$.

Then $dt = a\,du$ and $a < t < ab \Rightarrow 1 < u < b$.

Thus, $\int_a^{ab} \frac{dt}{t} = \int_{u=1}^{u=b} \frac{a\,du}{au} = \int_1^{b} \frac{du}{u} = L(b)$.

In total: $L(ab) = L(a) + L(b)$ as desired.

<u>Claim 2</u>: $L(x) \to \infty$ as $x \to \infty$.

Proof: We will show that $L(2^n) \to \infty$ as the integer $n \to \infty$. Then, since $L'(x) = \frac{1}{x} > 0$ (when $x > 0$), L is increasing. This fills in the gaps in between the powers of 2.

- We use claim 1 to compute:

$$L(2^n) = L(\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}}) = \underbrace{L(2) + L(2) + \cdots + L(2)}_{n \text{ times}} = n\,L(2).$$

Hence, since $nL(2) \to \infty$ as $n \to \infty$, so does $L(2^n)$.

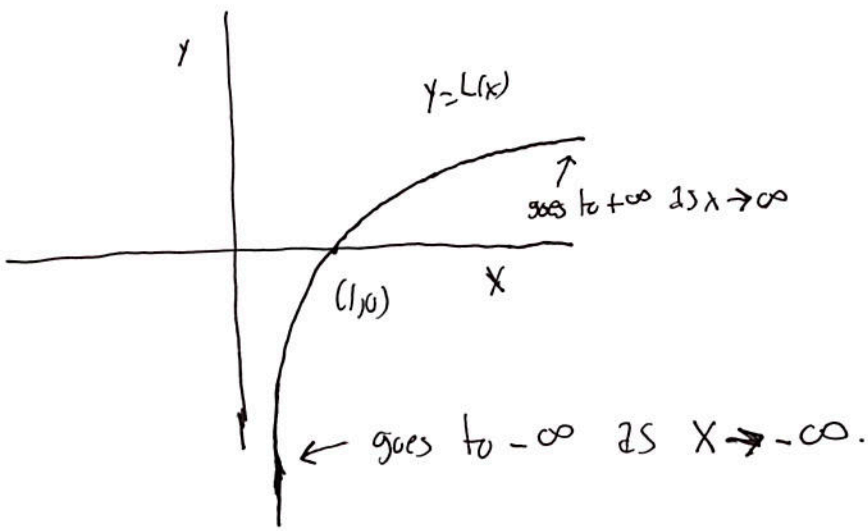
<u>Claim 3</u>: $L(x) \to -\infty$ as $x \to 0^+$.

Proof: $0 = L(1) = L(x \cdot \frac{1}{x}) = L(x) + L(\frac{1}{x})$ (by claim 1).

Now as $x \to 0^+$, $\frac{1}{x} \to \infty$, and hence claim 2 implies that $L(\frac{1}{x}) \to \infty$. Thus, $L(x) = -L(\frac{1}{x}) \to -\infty$ as $x \to 0^+$.

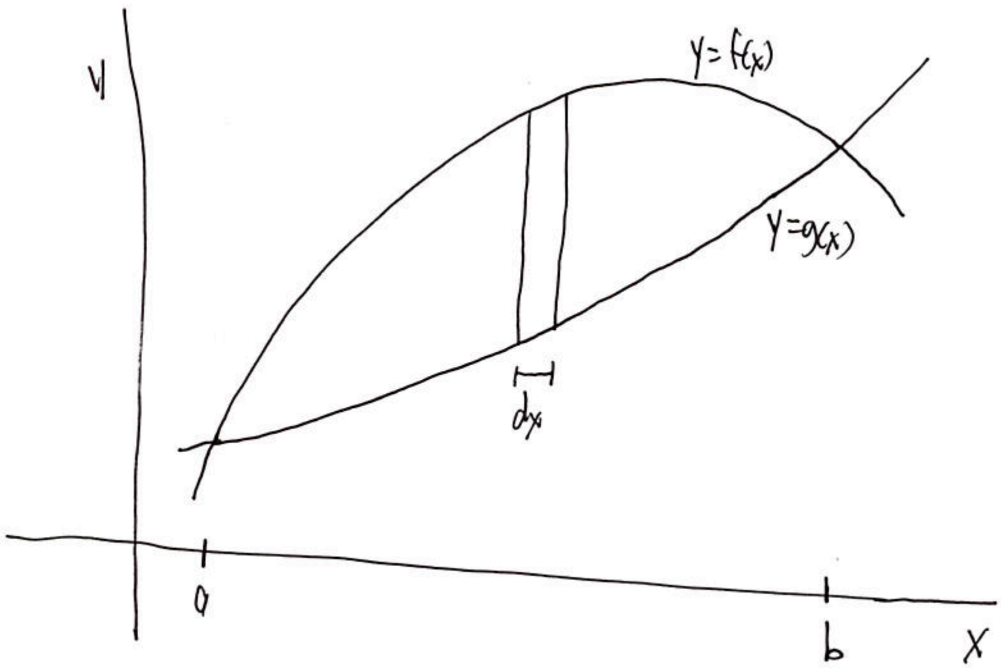
- We also compute: $L''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$.
  Thus, the graph of $L(x)$ is concave down when $x > 0$.

- In total, we have shown that the graph of $L(x)$ is as follows:



- We can <u>define</u> $\ln(x) = L(x)$, <u>define</u> e to be the number such that $L(e) = 1$, <u>define</u> e^x to be the inverse of $L(x)$, and <u>define</u> $a^x = e^{xL(a)}$.

- Applications of FTC to Geometry (Volumes + Areas)
- Area between two curves



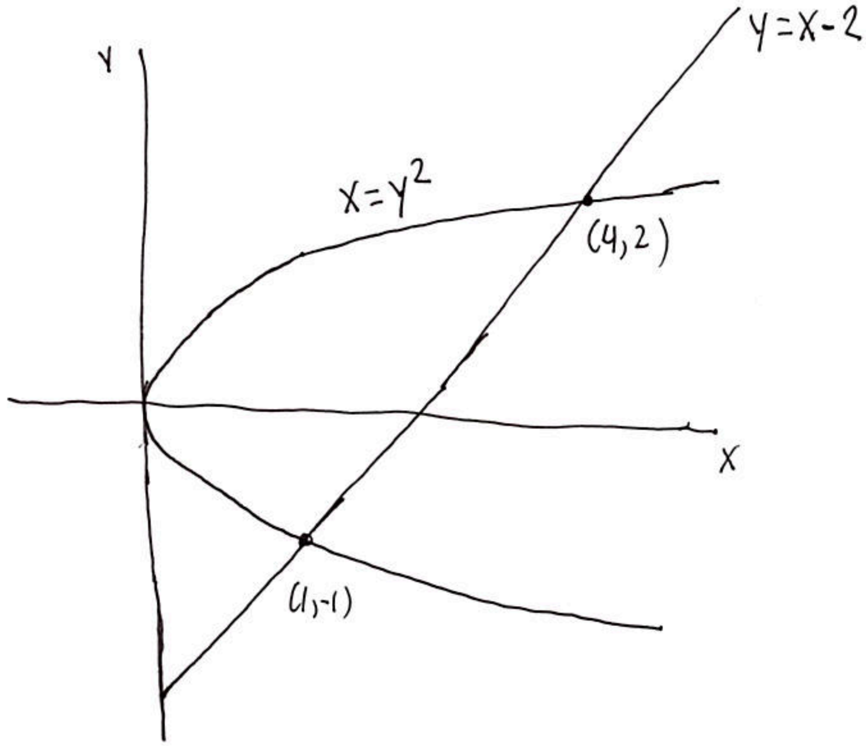
- The area A between the curves is

$$A = \int_a^b \left(f(x) - g(x) \right) dx .$$

  - a and b are the "crossing points"

Ex: Find the area in between the region

$x = y^2$ and $y = x - 2$



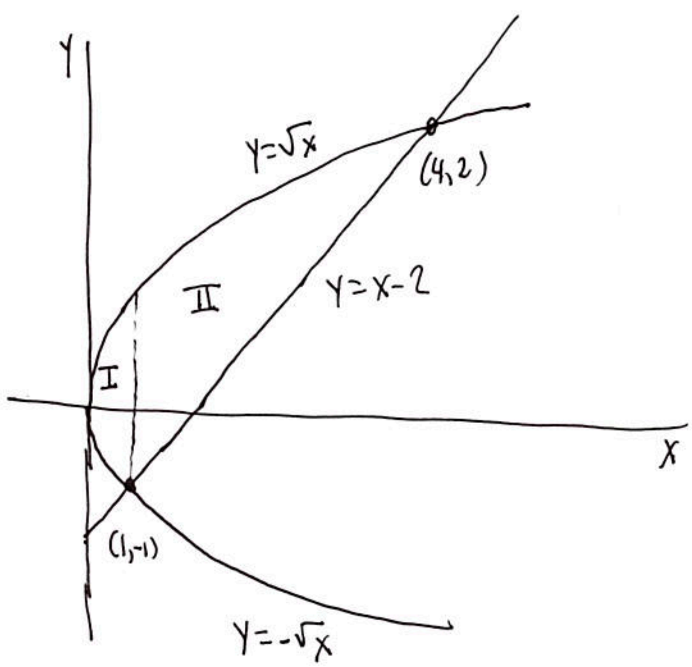
To find the crossing points, we solve the equation: $y + 2 = x = y^2$ for y.

- $y^2 - y - 2 = 0$
- $(y-2)(y+1) = 0$. $y = -1, 2$.
- We then solve for the x-values corresponding to each y value. These are $x = 1, 4$ respectively
- In total, the two crossing points are

$(1,-1)$ and $(4,2)$.

- There are two ways to find the area between the curves.

<u>Hard way:</u> Vertical Slices: If we use vertical slices, we need to consider two different regions.



- The area of region I is $\int_{x=0}^{1} \sqrt{x} - (-\sqrt{x})\, dx = 2\int_{x=0}^{1} \sqrt{x}\, dx = \frac{4}{3}x^{3/2}\Big]_0^1 = \frac{4}{3}.$
- The area of region II is $\int_{x=1}^{x=4} \sqrt{x} - (x-2)\, dx = \frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\Big]_1^4$

$$= \frac{2}{3}\cdot 4^{3/2} - \frac{1}{2}\cdot 4^2 + 2\cdot 4 - \left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

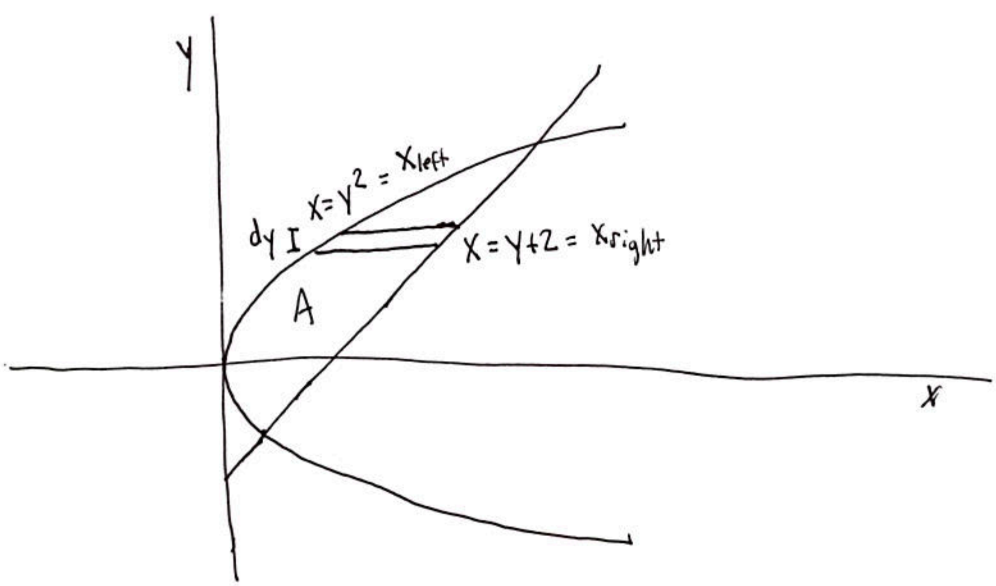
$$= \frac{19}{6}$$

Then
$A = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}.$

• <u>Easy Way</u>: Horizontal Slices. Using this method, we subtract the left curve from the right one:

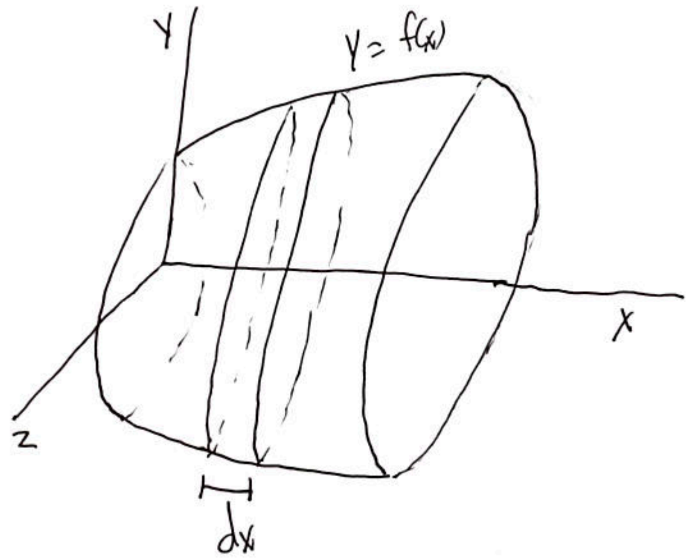
$$A = \int x_{right} - x_{left}\, dy = \int_{y=-1}^{y=2} (y+2) - y^2\, dy$$

$$= \frac{y^2}{2} + 2y - \frac{1}{3}y^3 \Big]_{-1}^{2} = \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}.$$



- Volumes of Solids of revolution

- Consider the solid of revolution formed by rotating the curve $y = f(x)$ about the x-axis (coming out of the page)



- We want to figure out the volume of the solid by first figuring out the volume of a slice of width dx and then adding up the volumes of the slices (i.e., integrating dx).

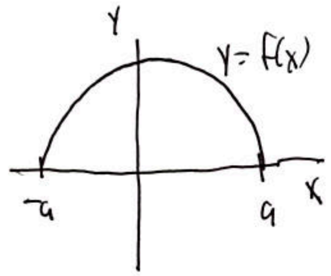
- Each slice is approximately a disk of width dx, radius $y = f(x)$, and cross sectional area πy^2.
- The volume of one slice is therefore

$$dV = \pi y^2 \, dx = \pi [f(x)]^2 dx$$

(for a solid of revolution around the x-axis).

- We can then integrate dx to find the total volume.

- <u>Ex</u>: Find the volume of a sphere of radius a.



The equation for the upper half circle is $y = f(x) = \sqrt{a^2 - x^2}$

- If we rotate this half circle about the x-axis, we get a sphere of radius a.
- In total: $V = \int \pi y^2 dx = \int_{x=-a}^{x=a} \pi (a^2 - x^2) dx = \pi a^2 x - \frac{\pi}{3} x^3 \Big]_{-a}^{a} = \frac{4}{3} \pi a^3.$

- By symmetry, we could have integrated from 0 to a and then doubled the result

$V = 2 \int_0^a \pi (a^2 - x^2) dx = 2\pi \left[a^2 x - \frac{1}{3} x^3\right]_0^a = \frac{4\pi}{3} a^3.$ This saves some work.