• Trigonometric Integrals & Substitution

• Goal : Compute $\int \sin^n x \cos^m x \, dx \qquad n=0,1,2,\ldots \quad m=0,1,2,\ldots$

. Method A

Suppose that either m or n is odd

Ex $\int \sin^3 x \cos^2 x \, dx$

Strategy: Use identity $\sin^2 x + \cos^2 x = 1$

to rewrite the integral as $\int f(\cos x) \sin x \, dx$.

Then make the substitution $u = \cos x \quad du = -\sin x \, dx$.

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1-\cos^2 x)\cos^2 x \sin x \, dx$$

$$= \int (1-u^2)u^2 \cdot (-du) = \int -u^2 + u^4 \, du$$

$$= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Ex: $\int \cos^3 x\,dx = \int \cos^2 x \cdot \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$

This time, let $u = \sin x$, $du = \cos x\,dx$

$$\int (1-\sin^2 x)\cos x\,dx = \int (1-u^2)\,du = u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C.$$

Method B

- Requires both m and n to be even
- Requires a double angle formula such as

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

Ex: $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Ex $\int \sin^2 x \cos^2 x \, dx = \int \frac{(1-\cos 2x)(1+\cos 2x)}{2 \cdot 2} dx$

$= \int \frac{1}{4} - \frac{1}{4} \cos^2 2x \, dx = \int \frac{1}{4} - \frac{1}{8}(1+\cos 4x) \, dx$

$= \frac{1}{8} x - \frac{\sin 4x}{32} + C$

- A shortcut for this example: $\sin 2x = 2 \sin x \cos x$

$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1}{2} \sin 2x\right)^2 dx = \frac{1}{4} \int \frac{1 - \cos 4x}{2} dx$

$= \frac{1}{8} x - \frac{\sin 4x}{32} + C$ (as above)

- Next family of integrals (won't finish today):

- $\int \sec^n x \tan^m x \, dx \qquad m=0,1,2,\dots \quad n=0,1,2\dots$

Recall identity: $\sec^2 x = 1+\tan^2 x$

let's check it: $\frac{1}{\cos^2 x} \overset{?}{=} 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\overbrace{\cos^2 x+\sin^2 x}^{1}}{\cos^2 x}$ ✓

- Basic integrals:

$\int \sec^2 x = \tan x + C \quad$ (because $\frac{d}{dx} \tan x = \sec^2 x$)

$\int \sec x \tan x dx = \sec x + C \quad$ (because $\frac{d}{dx} \sec x = \sec x \tan x$)

- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

  $(u = \cos x \quad du = -\sin x \, dx)$

  $= \int -\frac{du}{u} = -\ln|u| + c = -\ln|\cos x| + c$

- How to handle even powers of secant:

  $$\int \sec^4 x \, dx = \int \sec^2 x \cdot \overbrace{\sec^2 x}^{\text{Save a copy of } \sec^2 x} \, dx = \int (1+\tan^2 x) \sec^2 x \, dx$$

  $\cdot\, u = \tan x \qquad \cdot\, du = \sec^2 x \, dx$

  $$\int (1+\tan^2 x) \sec^2 x \, dx = \int (1+u^2) \, du = u + \frac{u^3}{3} + C$$

  $$= \tan x + \frac{\tan^3 x}{3} + C$$

- How to handle an odd power of tangent?

- $$\int \tan^3 x \sec x\,dx = \int \tan^2 x \cdot \overbrace{\sec x \tan x\,dx}^{\text{save a copy of } \sec x \tan x = \frac{d}{dx}\sec x}$$

  - $u = \sec x$
  - $du = \sec x \tan x$

- $$\int \tan^2 x \sec x \tan x\,dx = \int (\sec^2 x - 1)\overbrace{\sec x \tan x\,dx}^{du}$$

$$= \int (u^2 - 1)\,du = \frac{u^3}{3} - u + C = \frac{\sec^3 x}{3} - \sec x + C$$

Ex (tricky) $$\int \sec x\,dx = \int \sec x \cdot \frac{(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

- $u = \sec x + \tan x$ $\quad du = (\sec x \tan x + \sec^2 x)\,dx$
$\quad = (\sec^2 x + \sec x \tan x)\,dx$

$$\int \sec x\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

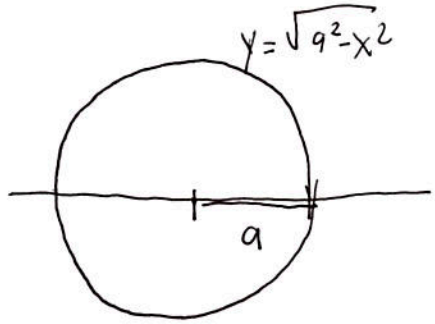
- $\int \sec^{odd} x \cdot \tan^{even} x\,dx$ is in general complicated.

  We will discuss these integrals later...

• Trigonometric Substitution

Big idea: Trig integrals are useful for evaluating some integrals involving square roots.

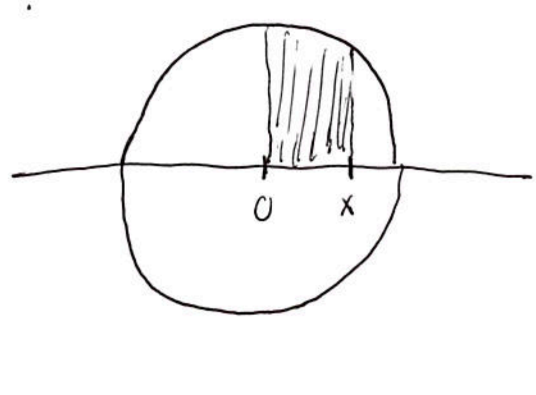
Ex:



We already know that the top half of the circle has area

$$\int_{-a}^{a} \sqrt{a^2-x^2}\,dx = \frac{\pi a^2}{2}$$

What about ?



- We need to evaluate the integral

$$\int_{t=0}^{t=x} \sqrt{a^2-t^2}\, dt$$

- Set $t = a\sin u \qquad dt = a\cos u\, du$

Changed integration endpoints

$$\int_0^x \sqrt{a^2-t^2}\, dt = \int_{u=0}^{u=\sin^{-1}(x/a)} (a\cos u)\, a\cos u\, du = a^2 \int_{u=0}^{u=\sin^{-1}(x/a)} \cos^2 u\, du$$

$$\left(t = a\sin u \implies u = \sin^{-1}\left(\tfrac{t}{a}\right) \right)$$

- $$a^2 \int_{u=0}^{u=\sin^{-1}(x/a)} \cos^2 u\, du = a^2 \left(\frac{u}{2} + \frac{\overbrace{\sin 2u}^{=2\sin u\cos u}}{4} \right) \Bigg|_{u=0}^{u=\sin^{-1}\left(\frac{x}{a}\right)}$$

$$= a^2 \left(\frac{u}{2} + \frac{\sin u \cos u}{2} \right) \Bigg|_{u=0}^{u=\sin^{-1}\left(\frac{x}{a}\right)}$$

$$= \frac{a^2 \sin^{-1}\left(\frac{x}{a}\right)}{2} + \frac{a^2}{2} \sin\left(\sin^{-1}\left(\tfrac{x}{a}\right)\right) \cdot \cos\left(\sin^{-1}\left(\tfrac{x}{a}\right)\right)$$