Improper integrals

- An improper integral is defined by

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty} \int_a^M f(x)\,dx$$

is said to <u>converge</u> if the limit exists

(<u>diverge</u> if the limit does not exist).

<u>Ex:</u> Let's compute $\int_0^\infty e^{-Kx}dx$ when $K>0$.

- $\int_0^M e^{-Kx}dx = -\frac{1}{K}e^{-Kx}\Big|_0^M = \frac{1}{K}(1-e^{-KM})$

- Taking the limit as $M\to\infty$, we find that $e^{-KM}\to 0$ and therefore $\int_0^\infty e^{-Kx}dx = \frac{1}{K}$.

Ex Replace x by t= time in seconds in the previous example.

- R= rate of decay = # of atoms that decay per second at time 0
- At later times $t>0$ the decay rate is Re^{-Kt} (smaller by an exponential factor e^{-Kt})
- Eventually (over time $0 \le t < \infty$) every atom decays. So the total # of atoms N is calculated using the formula we found in the previous example:

$$N = \int_0^\infty R e^{-Kt}\,dt = \frac{R}{K}.$$

- The half **life** H of a radioactive element is the time H at which the decay rate is half of what it was at the start. Thus

$$e^{-KH} = \frac{1}{2} \Rightarrow -KH = \ln(1/2) \Rightarrow K = \frac{\ln 2}{H}$$

- Hence, $R = NK = \frac{N \ln 2}{H}$

Polonium 210: $H = (138)(24)(60)$ seconds

- One gram of Polonium 210 emits $(1\text{gram})\left(\frac{6 \times 10^{23}}{210} \frac{\text{atoms}}{\text{gram}}\right)\frac{(\ln 2)}{H} \approx 1.66\,10^{14} \frac{\text{decays}}{\text{sec}} \approx 4500$ Curies

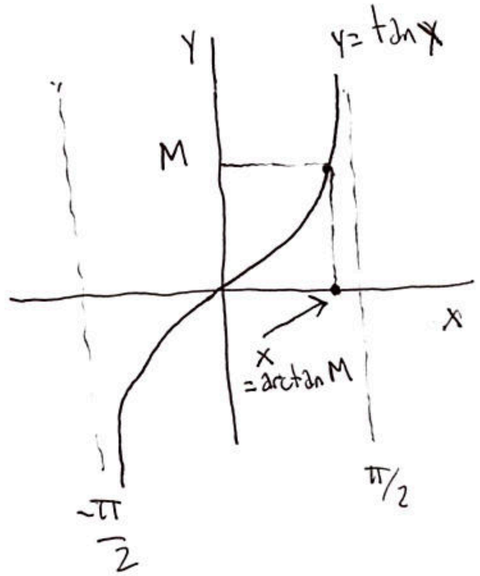
- $\frac{5.3 \text{ MeV}}{\text{decay}} \Rightarrow$ 1 gram of Polonium gives off 140 watts of radioactive energy.

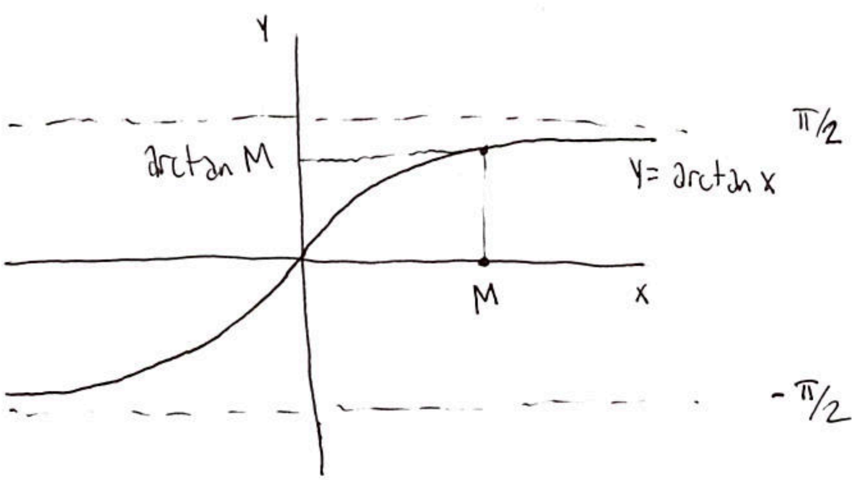
Ex. Let's calculate $\int_0^{\infty} \frac{dx}{1+x^2}$

- $\int_0^M \frac{dx}{1+x^2} = \arctan x \Big|_0^M = \arctan M$

- $\lim_{M \to \infty} \arctan M = \frac{\pi}{2}$.

- Hence, $\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$.





Ex: $\displaystyle\int_1^\infty \frac{dx}{x}$

- $\displaystyle\int_1^M \frac{dx}{x} = \ln x \Big|_1^M = \ln M - \ln 1 = \ln M \to \infty$ as $M \to \infty$

- Hence $\displaystyle\int_1^\infty \frac{dx}{x}$ is divergent

Ex: $\displaystyle\int_1^\infty \frac{dx}{x^p}$ $(p>1)$

$$\int_1^M \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p} \Big|_1^M = \frac{1}{1-p}\left(M^{1-p} - 1\right) \to \frac{1}{p-1} \text{ as } M \to \infty$$

because $1-p<0$.

- Hence $\displaystyle\int_1^\infty \frac{dx}{x^p}$ is convergent

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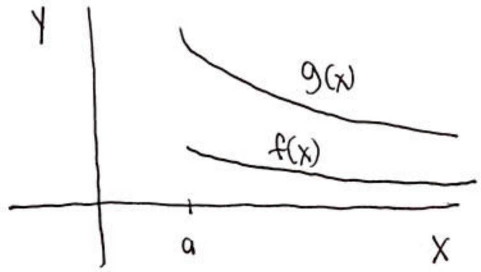
$$\int_1^M \frac{dx}{x^p} = \frac{1}{1-p}\left(M^{1-p} - 1\right) \to \infty \text{ as } M \to \infty$$

because $1-p>0$.

- Hence $\displaystyle\int_1^\infty \frac{dx}{x^p}$ is divergent.

- <u>Comparison Theorem.</u>

- Suppose $0 \le f(x) \le g(x)$ holds for $x \ge a$:



- Then if $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$.

  That is, if the area under g is finite, then the area under f, being smaller, must also be finite

- If $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$

  That is, if the area under f is infinite, then the area under g, being larger, must also be infinite.

- <u>Big idea</u>: Use the comparison theorem to decide whether $\int_a^\infty f(x)\,dx$ diverges or converges by comparing $f(x)$ to a simpler function.

<u>Ex</u>: Decide whether $\int_0^\infty \frac{dx}{\sqrt{x^3+1}}$ converges or diverges.

- First attempt: $\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{x^{3/2}}$

But $\int_0^\infty \frac{dx}{x^{3/2}}$ diverges because of the infinite behavior as $x \to 0^+$ (we will investigate this in more detail momentarily).

- Second attempt: $\int_0^\infty \frac{dx}{\sqrt{x^3+1}} = \int_0^1 \frac{dx}{\sqrt{x^3+1}} + \int_1^\infty \frac{dx}{\sqrt{x^3+1}}$

- The first integral yields some finite number.
- The second integral can now be shown to converge by the Comparison Theorem:

$$\int_1^\infty \frac{dx}{\sqrt{x^3+1}} < \int_1^\infty \frac{dx}{x^{3/2}} < \infty \quad \text{because } 3/2 > 1.$$

- Hence, $\int_0^\infty \frac{dx}{\sqrt{x^3+1}}$ converges.

Ex $\displaystyle\int_0^\infty e^{-x^3}\,dx$

- For $x \geq 1$, $x^3 \geq 1$ and $e^{-x^3} \leq e^{-x}$.

  Hence, $\displaystyle\int_1^\infty e^{-x^3}\,dx \leq \int_1^\infty e^{-x}\,dx = \frac{1}{e} < \infty$.

- Thus, $\displaystyle\int_0^\infty e^{-x^3}\,dx$ Converges as well by the Comparison theorem.

- The interval $\displaystyle\int_0^1 e^{-x^3}\,dx$ does not affect the Convergence or divergence of $\displaystyle\int_0^\infty e^{-x^3}\,dx$ Since the interval $0 \leq x \leq 1$ has finite length and Since e^{-x^3} remains finite on this interval.

- <u>Limit Comparison</u>:

  - Suppose that $0 \le f(x)$ and that $\lim_{x\to\infty} \frac{f(x)}{g(x)} \le 1$.
  - Then $f(x) \le 2g(x)$ for $x \ge a$, where a is some large number.
  - Hence, $\int_a^\infty f(x)dx \le 2\int_a^\infty g(x)\,dx$, and

    $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)dx$ converges.
  - Similarly, $\int_a^\infty f(x)dx$ diverges if $\int_a^\infty g(x)\,dx$ diverges and $\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$

Ex: $\displaystyle\int_0^\infty \frac{(x+10)dx}{x^2+1}$

- The limiting behavior as $x \to \infty$ is

$$\frac{x+10}{x^2+1} \simeq \frac{x}{x^2} = \frac{1}{x}$$

- Since $\displaystyle\int_1^\infty \frac{dx}{x} = \infty$, the integral $\displaystyle\int_0^\infty \frac{(x+10)}{x^2+1}dx$

  also diverges

- Improper integrals of the Second type

  - Let's examine $\int_0^1 \frac{dx}{\sqrt{x}}$
  - We know that $\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0^+$
  - However, $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{x^{1/2}}$

$= \lim_{a \to 0^+} 2x^{1/2} \Big|_a^1 = \lim_{a \to 0^+} 2 - 2a^{1/2} = 2.$

  - Hence, $\int_0^1 \frac{dx}{\sqrt{x}}$ converges to 2.
  - Similarly,

$$\int_0^1 x^{-p}\,dx = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$