MIDTERM 3 - 18.01 - FALL 2017.

Name:

Email:

Please put a check by your recitation section.

	Instructor	Time
☐	Miles Couchman	MW 1
☐	Kristin Kurianski	MW 1
☐	Yu Pan	MW 10
☐	Yu Pan	MW 11
☐	Jiewon Park	MW 12
☐	Jake Wellens	MW 12
☐	Siddharth Venkatesh	MW 2

Problem #	Max points possible	Actual score
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**

Good luck!

Problem 1. (10 + 5 points)

a) Find all solutions $y = f(x)$ to the differential equation

$$y' = e^{x+y}$$

(where $y' = (dy/dx)$).

b) Find the particular solution with $y(0) = 1$.

Solution:

a) Note that $y' = e^x e^y$. Hence, we can separate variables:

$$\frac{dy}{e^y} = e^x dx.$$

We then integrate each side to deduce that there exists a constant C such that

$$e^{-y} = -e^x + C.$$

Solving for y, we find that

$$y = -\ln(C - e^x).$$

b) To find the particular solution with $y(0) = 1$, we set $x = 0$, $y = 1$, and we solve for C:

$$1 = -\ln(C - 1),$$
$$C = 1 + e^{-1}.$$

It follows that the solution of interest is

$$y = -\ln(1 + e^{-1} - e^x).$$

Problem 2. (15 points) Write down a definite integral representing the area of the region in the first quadrant of the (x, y) plane (that is, $x \geq 0$ and $y \geq 0$) that is trapped in between the graphs of $y = x^3$ and $x = y^3$. Show your work, including a picture, to receive credit.

Remark: You do not have to compute an integral; just write it down.

Solution: The curve $y = x^3$ intersects the curve $x = y^3$ when $x = x^9$, which is equivalent to $x(x-1)(x+1)(x^2+1)(x^4+1) = 0$. The two roots of this polynomial with $x \geq 0$ are $x = 0$ and $x = 1$, which correspond to the intersection points $(0, 0)$ and $(1, 1)$. From the figure below, we see that the curve $y = x^{1/3}$ (i.e., $x = y^3$) lies above the curve $y = x^3$ in the region of interest. Hence, the area of the region of interest is given by the following expression:

$$\int_0^1 (x^{1/3} - x^3)\, dx.$$

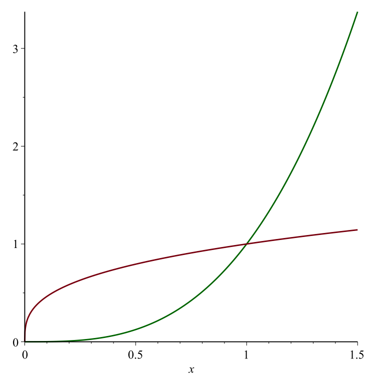

FIGURE 1. The region of interest

Problem 3. (10 + 10 = 20 points) Let R be the finite region in the (x, y) plane that is trapped in between the curve $y = \cos x$, the curve $y = \sin x$, the y-axis, and the line $x = x_{int}$, where x_{int} is the smallest positive x value at which the curves $y = \cos x$ and $y = \sin x$ intersect.

Remark: In this problem, you do not have to compute any integrals; just write them down.

a) Let S_1 be the solid that is obtained upon revolving R about the line $x = -1$. Write down a definite integral representing the volume of S_1. Show your work, including a picture, to receive credit. Be sure to clearly indicate which volume method you are using.

b) Let S_2 be the solid that is obtained upon revolving R about the line $y = -1$. Write down a definite integral representing the volume of S_2. Show your work, including a picture, to receive credit. Be sure to clearly indicate which volume method you are using.

Solution:

The region R is as follows, where the top curve is $y = \cos x$ and the bottom curve is $y = \sin x$:

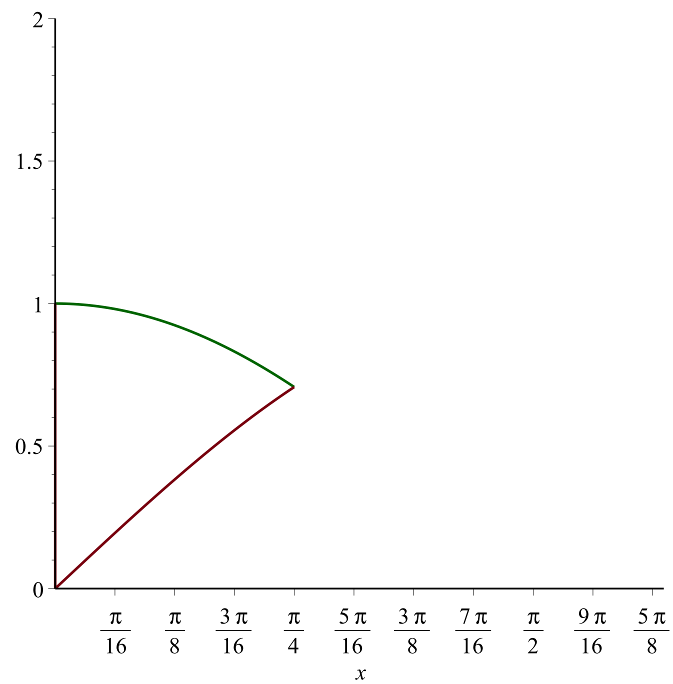

FIGURE 2. The region R

a) To find the volume V of S_1, we will use the cylindrical shell method. We first note that $x_{int} = \pi/4$. Next, we note that the axes through the center of the shells are parallel to the y axis. Each shell has radius $1 + x$, height $\cos x - \sin x$, and thickness dx. The volume of each shell is therefore $dV = 2\pi \times$ radius $\times$ height $\times$ thickness

$= 2\pi(1+x)(\cos x - \sin x)\,dx$. Moreover, we note that x varies from 0 to $\pi/4$. Thus, the total volume is

$$V = \int dV = \int_0^{\pi/4} 2\pi(1+x)(\cos x - \sin x)\,dx.$$

b) To find the volume V of S_2, we will use the disk/washer method. We first note that the axes of symmetry of the disks are parallel to the x axis. Each washer has an inner (i.e., smaller) radius equal to $\sin x$ and an outer (i.e., larger) radius equal to $\cos x$. The volume of a washer of thickness dx is therefore $dV = \pi\{(\text{outer radius})^2 - (\text{inner radius})^2\} \times \text{thickness} = \pi\left\{(\cos x + 1)^2 - (\sin x + 1)^2\right\}\,dx$. From the remarks we made in part a), we see that x varies from 0 to $\pi/4$. Thus, the total volume is

$$V = \int dV = \int_0^{\pi/4} \pi\left\{(\cos x + 1)^2 - (\sin x + 1)^2\right\}\,dx.$$

Problem 4. (15 points) Compute the following limit by interpreting it as the limit of Riemann sums for a function $f(x)$ on an interval $[a, b]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n\left(1+\frac{i}{n}\right)}$$

Hint: n is the number of (equal-length) sub-intervals of $[a, b]$ in the Riemann sum.

Solution: The function is $f(x) = \frac{1}{1+x}$ and the interval is $[a, b] = [0, 1]$. Thus, as $n \to \infty$, the Riemann sums $\sum_{i=1}^{n} \frac{1}{n\left(1+\frac{i}{n}\right)}$ converge to the integral

$$\int_0^1 \frac{1}{1+x}\, dx = \ln(1+x)\,\big|_0^1 = \ln 2.$$

Problem 5. (15 points) Let $N > 0$ be a number. Compute the average value of the function

$$f(x) = \sqrt{x}$$

on the interval $[0, N]$. Your answer will depend on N.

Solution: The average value of $f(x)$ is

$$\frac{1}{N}\int_0^N \sqrt{x}\,dx = \frac{1}{N}\frac{2}{3}x^{3/2}\Big|_0^N = \frac{2}{3}\sqrt{N}.$$

Problem 6. (20 points) Compute

$$\lim_{t\to 2} \frac{1}{t-2} \int_2^t y^y \, dy$$

by recognizing the limit as the derivative of a function $f(x)$ at some point x_0.

Hint: You will not be able to evaluate the integral. Instead, find $f(x)$ and the point x_0 and then use an important result that we discussed in class.

Solution: Let

$$f(x) = \int_2^x y^y \, dy.$$

Then since $f(2) = 0$, we see, in view of the analytic definition of the derivative of a function, that the limit in question is equal to $\lim_{t\to 2} \frac{f(t)-f(2)}{t-2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = f'(2)$. By FTC2,

$$f'(x) = x^x.$$

Thus, $f'(2) = 2^2 = 4$.