Cálculo III
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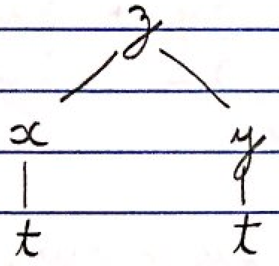
Aluno: Abrantes

Exercícios:
Regra da Cadeia

Outubro/2019

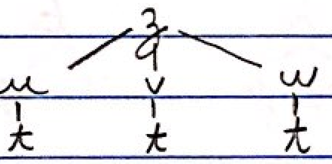
I - Desenhe um diagrama de árvore e escreva uma fórmula da regra da cadeia para cada derivada (Exercícios 13-24)

1 - $\frac{dz}{dt}$ para $z = f(x,y)$, $x = g(t)$, $y = h(t)$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

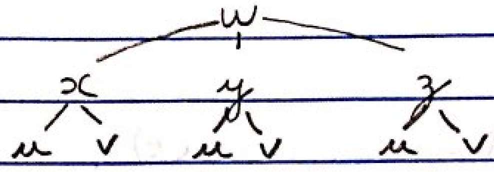
2 - $\frac{dz}{dt}$ para $z = f(u,v,w)$, $u = g(t)$, $v = h(t)$, $w = k(t)$



$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dt}$$

②

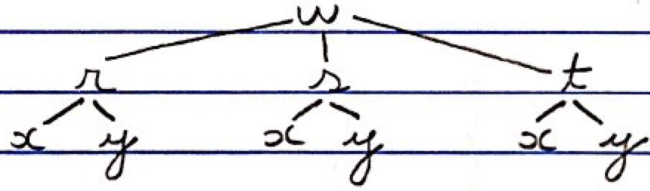
③ - $\dfrac{\partial w}{\partial u}$ e $\dfrac{\partial w}{\partial v}$, para $w = h(x,y,z)$, $x = f(u,v)$, $y = g(u,v)$, $z = k(u,v)$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

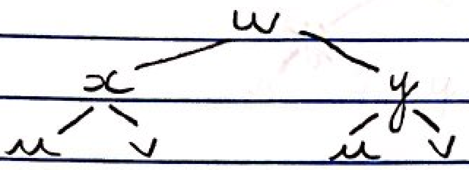
④ - $\dfrac{\partial w}{\partial x}$ e $\dfrac{\partial w}{\partial y}$, para $w = f(r,s,t)$, $r = g(x,y)$, $s = h(x,y)$, $t = k(x,y)$



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial y}$$

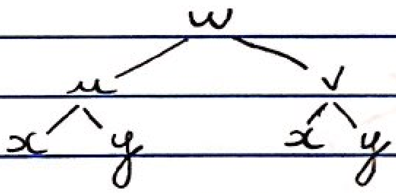
5 - $\frac{\partial w}{\partial u}$ e $\frac{\partial w}{\partial v}$, para $w = g(x, y)$, $x = h(u, v)$, $y = k(u, v)$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}$$

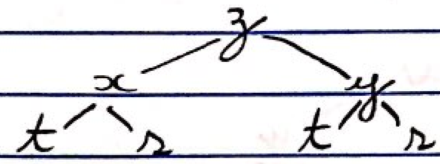
6 - $\frac{\partial w}{\partial x}$ e $\frac{\partial w}{\partial y}$, para $w = g(u, v)$, $u = h(x, y)$, $v = k(x, y)$



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y}$$

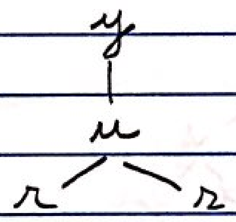
7 - $\frac{\partial z}{\partial t}$ e $\frac{\partial z}{\partial s}$, para $z = f(x,y)$, $x = g(t,s)$, $y = h(t,s)$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

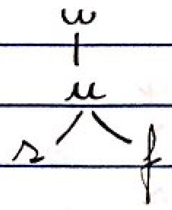
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

8 - $\frac{\partial y}{\partial r}$, para $y = f(u)$, $u = g(r,s)$



$$\frac{\partial y}{\partial r} = \frac{dy}{du} \cdot \frac{\partial u}{\partial r}$$

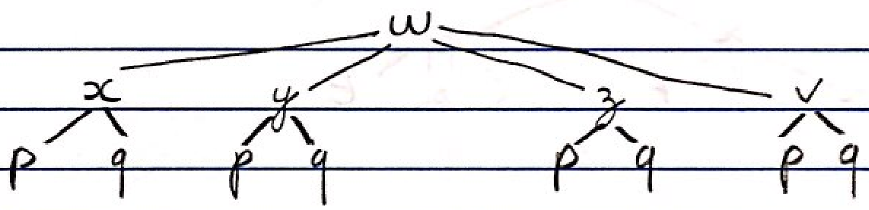
9 - $\frac{\partial w}{\partial r}$ e $\frac{\partial w}{\partial f}$, para $w = g(u)$, $u = h(r, f)$.



$$\frac{\partial w}{\partial r} = \frac{dw}{du} \cdot \frac{\partial u}{\partial r}$$

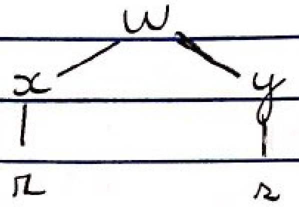
$$\frac{\partial w}{\partial f} = \frac{dw}{du} \cdot \frac{\partial u}{\partial f}$$

10 - $\frac{\partial w}{\partial p}$, para $w = f(x, y, z, v)$, $x = g(p, q)$, $y = h(p, q)$, $z = j(p, q)$, $v = k(p, q)$



$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial p} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial p} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial p}$$

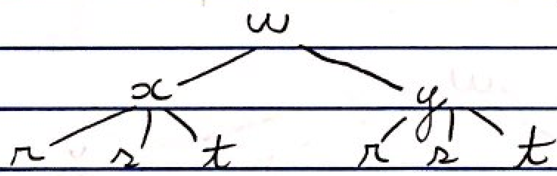
11- $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$, $w = f(x, y)$; $x = g(r)$, $y = h(s)$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dr}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial y} \cdot \frac{dy}{ds}$$

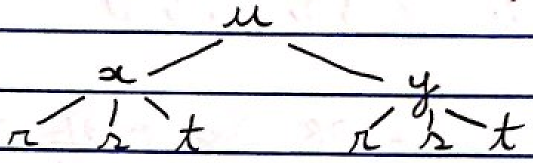
12- $\frac{\partial w}{\partial s}$, $w = g(x, y)$; $x = h(r, s, t)$, $y = k(r, s, t)$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

II - Use um diagrama em árvore para escrever a regra da cadeia (17-20)

17 - $u = f(x,y)$, $x = x(r,s,t)$, $y = y(r,s,t)$

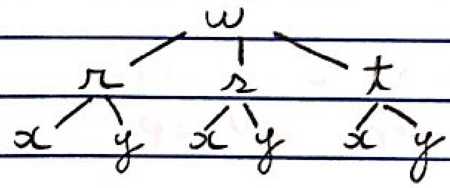


$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

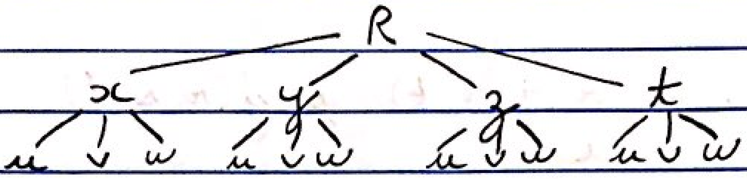
19 - $w = f(r,s,t)$, $r = r(x,y)$, $s = s(x,y)$, $t = t(x,y)$



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial y}$$

18) $R = f(x, y, z, t)$, $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, $t = t(u, v, w)$

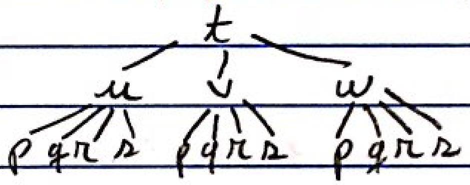


$$\frac{\partial R}{\partial u} = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial R}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial u} + \frac{\partial R}{\partial t}\cdot\frac{\partial t}{\partial u}$$

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial R}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial v} + \frac{\partial R}{\partial t}\cdot\frac{\partial t}{\partial v}$$

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial R}{\partial z}\cdot\frac{\partial z}{\partial w} + \frac{\partial R}{\partial t}\cdot\frac{\partial t}{\partial w}$$

20) $t = f(u, v, w)$, $u = u(p, q, r, s)$, $v = v(p, q, r, s)$, $w = w(p, q, r, s)$



$$\frac{\partial t}{\partial p} = \frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial p} + \frac{\partial t}{\partial v}\frac{\partial v}{\partial p} + \frac{\partial t}{\partial w}\cdot\frac{\partial w}{\partial p}$$

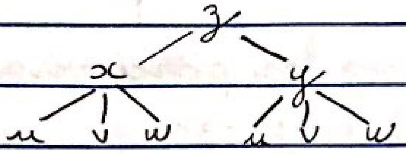
$$\frac{\partial t}{\partial q} = \frac{\partial t}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial q} + \frac{\partial t}{\partial w}\cdot\frac{\partial w}{\partial q}$$

$$\frac{\partial t}{\partial r} = \frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial r} + \frac{\partial t}{\partial w}\frac{\partial w}{\partial r}$$

$$\frac{\partial t}{\partial s} = \frac{\partial t}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial t}{\partial v}\cdot\frac{\partial v}{\partial s} + \frac{\partial t}{\partial w}\cdot\frac{\partial w}{\partial s}$$

III - Utilize a regra da cadeia para determinar as derivadas parciais indicadas (21-26)

21 - $z = x^2 + xy^3$; $x = uv^2 + w^3$, $y = u + ve^w$



$\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$

$u = 2$, $w = 0$, $v = 1$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial}{\partial x}(x^2+xy^3)\cdot\frac{\partial}{\partial u}(uv^2+w^3) + \frac{\partial}{\partial y}(x^2+xy^3)\cdot\frac{\partial}{\partial u}(u+ve^w)$$

$$= (2x+y^3)(v^2) + (3xy^2)(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= (2x+y^3)\cdot\frac{\partial}{\partial v}(uv^2+w^3) + (3xy^2)\cdot\frac{\partial}{\partial v}(u+ve^w)$$

$$= (2x+y^3)\cdot(2uv) + (3xy^2)(e^w)$$

Para $(u, v, w) = (2, 1, 0)$; temos:

$x = uv^2 + w^3$	$y = u + ve^w$
$= 2(1)^2 + 0^3$	$= 2 + 1e^0$
$= 2$	$= 3$

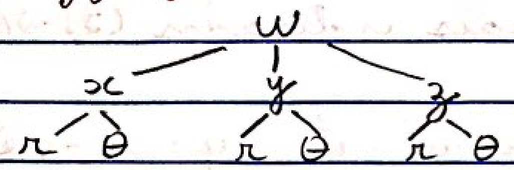
Então:

$$\frac{\partial z}{\partial u} = (2x+y^3)(v^2) + (3xy^2)$$
$$= (2\cdot 2 + 3^3)(1) + (3\cdot 2\cdot 3^2)$$
$$= (4+27) + 54 = 85$$

Então:

$$\frac{\partial z}{\partial v} = (2x+y^3)(2uv) + (3xy^2)\,e^w$$
$$= (31)(4) + (54)(1)$$
$$= 178$$

23. $w = xy + yz - zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$



$\dfrac{\partial w}{\partial r}$, $\dfrac{\partial w}{\partial \theta}$

$r = 2$, $\theta = \pi/2$

Encontrando as derivadas parciais:

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy - yz - zx) = y - z$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xy - yz - zx) = x - z$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(xy - yz - zx) = -y - x$$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos(\theta)) = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(r\theta) = \theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}(r\theta) = r$$

Pela regra da cadeia:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= (y-z)(\cos\theta) + (x-z)(\sin\theta) + (-y-x)\theta$$

$$\frac{\partial w}{\partial \theta} = (y-z)(-r\sin\theta) + (x-z)(r\cos\theta) + (-y-x)r$$

Quando $r=2$ e $\theta = \pi/2$, temos:

$$\begin{array}{l|l|l} x = r\cos\theta & y = r\sin\theta & z = r\theta \\ \quad = 2\cos(\frac{\pi}{2}) & \quad = 2(1) & \quad = \cancel{2}\pi/\cancel{2} \\ \quad = 0 & \quad = 2 & \quad = \pi \end{array}$$

Avaliando $\frac{\partial w}{\partial r}$ e $\frac{\partial w}{\partial \theta}$ no ponto $(r,\theta) = (2, \pi/2)$:

$$\frac{\partial w}{\partial r} = (y-z)(\cos\theta) + (x-z)(\sin\theta) + (-y-x)\theta$$
$$= (2-\pi)(\cos \pi/2) + (0-\pi)(\sin \pi/2) + (-2-0)\pi/2$$
$$= \quad 0 \quad - \pi \quad - \pi$$
$$= -2\pi$$

$$\frac{\partial w}{\partial \theta} = (y-z)(-r\sin\theta) + (x-z)(r\cos\theta) + (-y-x)r$$
$$= (2-\pi)(-2\sin \pi/2) + (0-\pi)(2\cos \pi/2) + (-2-0)2$$
$$= (-4+2\pi) + 0 + (-4)$$
$$= 2\pi - 8$$

IV. Use a regra da cadeia para achar $\frac{dz}{dt}$ ou $\frac{dw}{dt}$ (Exercícios 1-6):

1. $z = x^2 + y^2 + xy$, $x = \operatorname{sen} t$, $y = e^t$

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & & y \arrow[d, dash] \\ t & & t \end{tikzcd}$$

Encontrando as derivadas necessárias:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + xy) = 2x + y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + xy) = 2y + x$$

$$\frac{dx}{dt} = \frac{d}{dt}\sin t = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}e^t = e^t$$

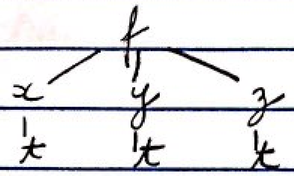
Assim temos que:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x + y)(\cos t) + (2y + x)(e^t)$$

V - Problema da hélice

$f(x, y, z)$, $x = \cos t$, $y = \text{sen}\, t$, $z = t$



$$\left\{\begin{array}{l} f_x = \cos t \\ f_y = \text{sen}\, t \\ f_z = t^2 + t - 2 \end{array}\right.$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$= (\cos t)(-\sin t) + (\sin t)(\cos t) + (t^2 + t - 2)(1)$$

$$= \cancel{-\cos t \sin t} + \cancel{\cos t \sin t} + t^2 + t - 2$$

$$= t^2 + t - 2$$

Para ocorrer um valor extremo, $\frac{df}{dt} = 0$.
Temos então:

$$\frac{df}{dt} = 0$$

$$t^2 + t - 2 = 0$$

$$\Delta = b^2 - 4ac$$
$$= 1^2 - 4(1)(-2)$$
$$= 1 + 8$$
$$= 9$$

$$t' = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-1 + \sqrt{9}}{2(1)} = 1$$

$$t'' = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-1 - \sqrt{9}}{2(1)} = -2$$

Portanto os valores extremos ocorrem quanto $t = 1$ ou $t = -2$. Os pontos da função onde ocorrem os extremos são então:

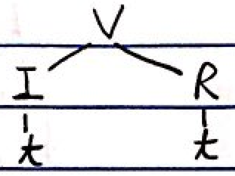
Se $t = 1$

$f[\cos(1), \sin(1), 1]$ //

Se $t = -2$

$f[\cos(-2), \sin(-2), -2]$ //

VI - Problema da bateria



Assim:

$$\frac{dV}{dt} = \frac{\partial V}{\partial I} \cdot \frac{dI}{dt} + \frac{\partial V}{\partial R} \cdot \frac{dR}{dt}$$

Avaliar como a corrente varia (dI/dt) quando:

$R = 600$ ohms	Note que:
$I = 0{,}04$ A	$V = IR \Rightarrow I = \frac{V}{R}$
$dR/dt = 0{,}5$ ohms/s	
$dV/dt = -0{,}01$ V/s	$\Rightarrow R = \frac{V}{I}$

Portanto:

$$\frac{dV}{dt} = \frac{\partial V}{\partial I} \cdot \frac{dI}{dt} + \frac{\partial V}{\partial R} \cdot \frac{dR}{dt}$$

$$\frac{dV}{dt} = R \cdot \frac{dI}{dt} + I \cdot \frac{dR}{dt}$$

$$-0{,}01 = 600 \frac{dI}{dt} + 0{,}04(0{,}5)$$

$$-0{,}01 - 0{,}02 = 600 \frac{dI}{dt}$$

$$-0{,}03 = 600 \frac{dI}{dt}$$

$$\frac{dI}{dt} = -5 \times 10^{-5} \text{ A/s}$$

Ferramentas utilizadas:

- Calculadora HP-50g
- Mathematica 12.0
- Wolfram Alpla