MATH 18.01 - FINAL EXAM - SOME REVIEW PROBLEMS WITH SOLUTIONS

18.01 Calculus, Fall 2014	Professor: Jared Speck

Problem 1. Consider the following curve in parametric form:

$$x = t + \ln t,$$
$$y = t - \ln t.$$

Find the arc length of the portion of the curve corresponding to $1 \leq t \leq 2$.

Hint: To evaluate the arc length integral, first make the inverse trig substitution $t = \tan u$. *Then multiply the top and bottom by* $\sin u$ *and make the substitution* $v = \cos u$. *Finally, use partial fractions to evaluate the* v *integral.*

Problem 2. Consider the following curve in the plane:

$$x^2 y = \frac{1}{2}.$$

Let $a \geq 1$ be a number. The portion of the curve with $1 \leq x \leq a$ is revolved around the $x-$axis to generate a solid of revolution. Show that the surface area of the solid *does not become infinite* when $a \to \infty$.

Problem 3. Compute the Taylor series (at the base point 0) for the function

$$\frac{1}{(1-x)^3}.$$

Problem 4. Compute the Taylor series (at the base point 0) for the function

$$f(x) = \arcsin x.$$

In particular, compute the first four non-zero terms in the series and illustrate the general pattern for the coefficients.

Problem 5. Find all numbers $p \geq 0$ such that the series

$$\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k^p}\right)$$

converges.

Problem 6. Find all numbers $p \geq 0$ such that the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p + 1}$$

converges.

Problem 7. Show that if x is *not* a number of the form $x = \frac{\pi}{2} + m\pi$ for some integer m, then

$$\sec^2 x = \sum_{k=0}^{\infty} (\sin x)^{2k}.$$

Problem 8. Let $f(x) = e^{x^2}$. Compute

$$f^{(2014)}(0),$$

where $f^{(2014)}(x)$ is the 2014^{th} derivative of $f(x)$.

Hint: Taylor (The good kind of Taylor, not the Swift kind $\cdots$*)*

Problem 9. Consider the following curve in the plane, described in polar form:

$$r = \ln \theta.$$

Find the area of the region in the second quadrant ($x \leq 0, y \geq 0$) that is trapped between the $x-$axis, the $y-$axis, and the curve.

Problem 10. Compute

$$\lim_{x \to \infty} \frac{e^x + \cos^2 x}{e^x}.$$

Solutions

Problem 1. Consider the following curve in parametric form:

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$$y = t - \ln t.$$

Find the arc length of the portion of the curve corresponding to $1 \le t \le 2$.

Hint: To evaluate the arc length integral, first make the inverse trig substitution $t = \tan u$*. Then multiply the top and bottom by* $\sin u$ *and make the substitution* $v = \cos u$*. Finally, use partial fractions to evaluate the* v *integral.*

Solution: We first compute that

$$\begin{aligned}
\frac{dx}{dt} &= 1 + \frac{1}{t}, \\
\frac{dy}{dt} &= 1 - \frac{1}{t}, \\
ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \sqrt{2}\sqrt{1 + \frac{1}{t^2}} \\
&= \sqrt{2}\frac{\sqrt{t^2+1}}{t}.
\end{aligned}$$

The arc length is therefore

$$\text{Arc Length} = \int ds = \sqrt{2}\int_1^2 \frac{\sqrt{t^2+1}}{t}\, dt.$$

To evaluate the integral, we first make the inverse trig substitution $t = \tan u$, $dt = \sec^2 u$, use $\tan^2 u + 1 = \sec^2 u$, multiply the top and bottom by $\sin u$, and use $\sin^2 = 1 - \cos^2 u$, which leads to

$$\begin{aligned}
\int \frac{\sqrt{t^2+1}}{t}\, dt &= \int \frac{\sqrt{\sec^2 u}}{\tan u} \sec^2 u\, du \\
&= \int \frac{1}{\sin u \cos^2 u}\, du \\
&= \int \frac{\sin u}{\sin^2 u \cos^2 u}\, du \\
&= \int \frac{\sin u}{(1 - \cos^2 u)\cos^2 u}\, du.
\end{aligned}$$

We then make the substitution $v = \cos u$, $dv = -\sin u\, du$, which leads to

$$\begin{aligned}
\int \frac{\sin u}{(1 - \cos^2 u)\cos^2 u}\, du &= -\int \frac{1}{(1 - v^2)v^2}\, dv \\
&= -\int \frac{1}{v^2(1 - v)(1 + v)}\, dv.
\end{aligned}$$

To evaluate the v integral, we first make a partial fraction decomposition:

$$\frac{1}{v^2(1-v)(1+v)} = \frac{A}{v} + \frac{B}{v^2} + \frac{C}{1-v} + \frac{D}{1+v}.$$

The cover up method yields $B = 1$, $C = \frac{1}{2}$, $D = \frac{1}{2}$. Then plugging the value $v = 2$ into both sides of the decomposition yields that $A = 0$. Hence,

$$\frac{1}{v^2(1-v)(1+v)} = \frac{1}{v^2} + \frac{1}{2}\left(\frac{1}{1-v}\right) + \frac{1}{2}\left(\frac{1}{1+v}\right),$$

and

$$\begin{aligned}
&-\int \frac{1}{v^2(1-v)(1+v)}\,dv = \frac{1}{v} + \frac{1}{2}\ln\left|\frac{1-v}{1+v}\right| + c \\
&= \frac{1}{\cos u} + \frac{1}{2}\ln\left|\frac{1-\cos u}{1+\cos u}\right| + c \\
&= \sqrt{1+t^2} + \frac{1}{2}\ln\left|\frac{1-(1+t^2)^{-1/2}}{1+(1+t^2)^{-1/2}}\right| + c \\
&= \sqrt{1+t^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}\right| + c.
\end{aligned}$$

In our calculations above, we used a right triangle to deduce that

$$\cos u = (1+t^2)^{-1/2}.$$

Finally, we reinsert the factor of $\sqrt{2}$ and the bounds of integration to conclude that

$$\begin{aligned}
\text{Arc Length} &= \sqrt{2}\int_1^2 \frac{\sqrt{t^2+1}}{t}\,dt \\
&= \sqrt{2}\left\{\sqrt{1+t^2} + \frac{1}{2}\ln\left|\frac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}\right|\right\}_{t=1}^{t=2} \\
&= \sqrt{2}\left\{\sqrt{5} - \sqrt{2} + \frac{1}{2}\ln\left|\frac{\sqrt{5}-1}{\sqrt{5}+1}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right\} \\
&= \sqrt{2}\left\{\sqrt{5} - \sqrt{2} + \frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{5}+1}{\sqrt{5}-1}\right|\right\}.
\end{aligned}$$

Problem 2. Consider the following curve in the plane:

$$x^2 y = \frac{1}{2}.$$

Let $a \geq 1$ be a number. The portion of the curve with $1 \leq x \leq a$ is revolved around the $x-$axis to generate a solid of revolution. Show that the surface area of the solid *does not become infinite* when $a \to \infty$.

Solution: We first compute that

$$\begin{aligned} y &= \frac{1}{2}x^{-2}, \\ \frac{dy}{dx} &= -x^{-3}, \\ ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + x^{-6}}\, dx. \end{aligned}$$

The surface area of the region of interest is

$$\int \text{thin conical strip circumference} \times \text{conical strip slant height} = \int 2\pi y\, ds.$$

Hence, the surface area is

$$\int_1^a 2\pi y\, ds = \int_1^a 2\pi \frac{1}{2} x^{-2} \sqrt{1 + x^{-6}}\, dx.$$

Now when $x \geq 1$, we have $\sqrt{1 + x^{-6}} \leq \sqrt{1 + 1^{-6}} = \sqrt{2}$. Hence, the above integral is

$$\begin{aligned} &\leq \pi \times \sqrt{2} \times \int_1^a x^{-2}\, dx \\ &= \sqrt{2}\pi \left[-x^{-1}\right]_1^a \\ &= \sqrt{2}\pi \left(1 - \frac{1}{a}\right). \end{aligned}$$

Now as $a \to \infty$, the above term converges to the *finite number* $\sqrt{2}\pi$.

Problem 3. Compute the Taylor series (at the base point 0) for the function

$$\frac{1}{(1-x)^3}.$$

Solution: We first note that

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\frac{1}{1-x}.$$

The standard formula for the geometric series yields that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

Twice differentiating the above series term-by-term and multiplying by $1/2$, we deduce that

$$\begin{aligned} \frac{1}{(1-x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\sum_{k=0}^{\infty}\frac{d^2}{dx^2}x^k &= \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-2} \\ &= 1 + 3x + 6x^2 + \cdots . \end{aligned}$$

The above series is the desired Taylor series.

Problem 4. Compute the Taylor series (at the base point 0) for the function

$$f(x) = \arcsin x.$$

In particular, compute the first four non-zero terms in the series and illustrate the general pattern for the coefficients.

Solution: Our strategy is to first compute the Taylor series for $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ and to then integrate that series term-by-term to recover the Taylor Series for $\arcsin x$. More precisely, we first use the fundamental theorem of calculus to deduce that

$$\arcsin x = \int_0^x \frac{dy}{\sqrt{1-y^2}}.$$

We will compute the Taylor series for the integrand $\frac{1}{\sqrt{1-y^2}}$ in two steps. In the first step, we compute the Taylor series for the function

$$\frac{1}{\sqrt{1-u}} = (1-u)^{-1/2}.$$

Differentiating with respect to u, we deduce that

$$\begin{aligned}
\frac{d}{du}(1-u)^{-1/2} &= \frac{1}{2}(1-u)^{-3/2},\\
\frac{d^2}{du^2}(1-u)^{-1/2} &= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)(1-u)^{-5/2},\\
\frac{d^3}{du^3}(1-u)^{-1/2} &= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)(1-u)^{-7/2},\\
&\cdots
\end{aligned}$$

Setting $u = 0$ in the above formulas, we deduce that

$$\begin{aligned}
\frac{d}{du}(1-u)^{-1/2}|_{u=0} &= \frac{1}{2},\\
\frac{d^2}{du^2}(1-u)^{-1/2}|_{u=0} &= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right),\\
\frac{d^3}{du^3}(1-u)^{-1/2}|_{u=0} &= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right),\\
&\cdots.
\end{aligned}$$

From the Taylor formula

$$\begin{aligned}
f(u) &= a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots,\\
a_n &= \frac{f^{(n)}(0)}{n!},
\end{aligned}$$

and the above computations, it follows that

$$\frac{1}{\sqrt{1-u}} = 1 + \frac{1}{2}\times\frac{1}{1!}u + \frac{1\times 3}{2\times 2}\times\frac{1}{2!}u^2 + \frac{1\times 3\times 5}{2\times 2\times 2}\times\frac{1}{3!}u^3 + \cdots.$$

We now substitute $u = y^2$ to deduce the Taylor series for the integrand $\frac{1}{\sqrt{1-y^2}}$:

$$\frac{1}{\sqrt{1-y^2}} = 1 + \frac{1}{2} \times \frac{1}{1!}y^2 + \frac{1 \times 3}{2 \times 2} \times \frac{1}{2!}y^4 + \frac{1 \times 3 \times 5}{2 \times 2 \times 2} \times \frac{1}{3!}y^6 + \cdots .$$

Integrating this series term-by-term, we conclude that

$$\begin{aligned}
\arcsin x &= \int_0^x \frac{dy}{\sqrt{1-y^2}} \\
&= \int_0^x \left(1 + \frac{1}{2} \times \frac{1}{1!}y^2 + \frac{1 \times 3}{2 \times 2} \times \frac{1}{2!}y^4 + \frac{1 \times 3 \times 5}{2 \times 2 \times 2} \times \frac{1}{3!}y^6 + \cdots \right) dy \\
&= \left(y + \frac{1}{2} \times \frac{1}{1!}\frac{1}{3}y^3 + \frac{1 \times 3}{2 \times 2} \times \frac{1}{2!}\frac{1}{5}y^5 + \frac{1 \times 3 \times 5}{2 \times 2 \times 2} \times \frac{1}{3!}\frac{1}{7}y^7 + \cdots \right) \Big|_{y=0}^{y=x} \\
&= x + \frac{1}{2} \times \frac{1}{1!}\frac{1}{3}x^3 + \frac{1 \times 3}{2 \times 2} \times \frac{1}{2!}\frac{1}{5}x^5 + \frac{1 \times 3 \times 5}{2 \times 2 \times 2} \times \frac{1}{3!}\frac{1}{7}x^7 + \cdots .
\end{aligned}$$

Problem 5. Find all numbers $p \geq 0$ such that the series

$$\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k^p}\right)$$

converges.

Solution: We first claim that for large k, $\ln\left(1 + \frac{1}{k^p}\right) \sim \frac{1}{k^p}$. To verify this claim, we use L'Hôpital's rule in the $\frac{\infty}{\infty}$ form to deduce that

$$\begin{aligned}
\lim_{k \to \infty} \frac{\ln\left(1 + \frac{1}{k^p}\right)}{\frac{1}{k^p}} &= \lim_{k \to \infty} \frac{\frac{1}{1+\frac{1}{k^p}} \times (-p)k^{-p-1}}{(-p)k^{-p-1}} \\
&= \lim_{k \to \infty} \frac{1}{1 + \frac{1}{k^p}} \\
&= 1.
\end{aligned}$$

Therefore, by limit comparison,

$$\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k^p}\right)$$

converges if and only if

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges. By the integral comparison test, this latter series converges if and only if $p > 1$.

Problem 6. Find all numbers $p \geq 0$ such that the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p + 1}$$

converges.

Solution: We first note that

$$\frac{1}{k(\ln k)^p + 1} \sim \frac{1}{k(\ln k)^p},$$

as $k \to \infty$, so by limit comparison, it suffices to investigate the convergence of

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}.$$

By the integral comparison test, it suffices to investigate the convergence of

$$\int_{x=2}^{\infty} \frac{dx}{x(\ln x)^p} = \lim_{M \to \infty} \int_{x=2}^{M} \frac{dx}{x(\ln x)^p}.$$

To this end, we make the substitution $u = \ln x$, $du = \frac{dx}{x}$, which leads to

$$\int \frac{dx}{x(\ln x)^p} = \int \frac{du}{u^p} = \begin{cases} \frac{1}{1-p} u^{1-p}, & \text{if } p \neq 1, \\ \ln u, & \text{if } p = 1. \end{cases}$$

$$= \begin{cases} \frac{1}{1-p} (\ln x)^{1-p}, & \text{if } p \neq 1, \\ \ln(\ln x), & \text{if } p = 1. \end{cases}$$

Hence, we compute that

$$\lim_{M \to \infty} \int_{x=2}^{M} \frac{dx}{x(\ln x)^p} = \begin{cases} \frac{1}{p-1} (\ln 2)^{1-p}, & \text{if } p > 1, \\ \text{limit does not exist}, & \text{if } p \leq 1. \end{cases}$$

Thus,

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p + 1}$$

converges if $p > 1$ and diverges if $0 \leq p \leq 1$.

Problem 7. Show that if x is *not* a number of the form $x = \frac{\pi}{2} + m\pi$ for some integer m, then

$$\sec^2 x = \sum_{k=0}^{\infty} (\sin x)^{2k}.$$

Solution: As long as x is not of the form $x = \frac{\pi}{2} + m\pi$ for some integer m, we have that $|\sin x| < 1$. Thus, under this assumption, it follows that the geometric series

$$\sum_{k=0}^{\infty} (\sin x)^{2k} = \sum_{k=0}^{\infty} [\sin^2 x]^k$$

converges to

$$\frac{1}{1 - \sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Problem 8. Let $f(x) = e^{x^2}$. Compute

$$f^{(2014)}(0),$$

where $f^{(2014)}(x)$ is the 2014^{th} derivative of $f(x)$.

Hint: Taylor (The good kind of Taylor, not the Swift kind $\cdots$)

Solution: Using Taylor's formula, we simply have to expand

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

and use the relation

$$a_n = \frac{f^{(n)}(0)}{n!}$$

(with $n = 2014$).

To this end, we first recall the expansion for e^u :

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots$$

We then substitute $u = x^2$ to deduce that

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

From this series expansion, it follows that $a_{2014} = \frac{1}{1007!}$ and hence

$$\begin{aligned} f^{(2014)}(0) = 2014! \times a_{2014} &= \frac{2014!}{1007!} \\ &= 2014 \times 2013 \times 2012 \times \cdots \times 1009 \times 1008. \end{aligned}$$

Problem 9. Consider the following curve in the plane, described in polar form:

$$r = \ln \theta.$$

Find the area of the region in the second quadrant ($x \leq 0, y \geq 0$) that is trapped between the $x-$axis, the $y-$axis, and the curve.

Solution: We first note that the second quadrant corresponds to $\frac{\pi}{2} \leq \theta \leq \pi$. We then recall the standard formula for the area under a curve in polar coordinates:

$$\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} (\ln \theta)^2 \, d\theta.$$

To evaluate the integral, we first use the integration by parts relations $u = \ln \theta$, $du = \frac{d\theta}{\theta}$, $dv = \ln \theta \, d\theta$, $v = \theta \ln \theta - \theta$ to deduce that

$$\begin{aligned}\int (\ln \theta)^2 \, d\theta = \int u \, dv = uv - \int v \, du \\ &= \theta(\ln \theta)^2 - \theta \ln \theta - \int (\ln \theta - 1) \, d\theta \\ &= \theta(\ln \theta)^2 - 2\theta \ln \theta + 2\theta.\end{aligned}$$

Inserting the factor of $\frac{1}{2}$ and the bounds of integration, we conclude that the area of the region of interest is

$$\begin{aligned}\frac{1}{2}\int_{\pi/2}^{\pi} (\ln \theta)^2 \, d\theta = \int u \, dv = \left[\frac{1}{2}\theta(\ln \theta)^2 - \theta \ln \theta + \theta\right]_{\theta=\pi/2}^{\theta=\pi} \\ &= \frac{\pi}{2}(\ln \pi)^2 - \pi \ln \pi - \frac{\pi}{4}\left[\ln\left(\frac{\pi}{2}\right)\right]^2 + \frac{\pi}{2}\ln\left(\frac{\pi}{2}\right) + \frac{\pi}{2}.\end{aligned}$$

Problem 10. Compute

$$\lim_{x\to\infty} \frac{e^x + \cos^2 x}{e^x}.$$

Solution: This is a "$\frac{\infty}{\infty}$" type limit. If you try to use L'Hôpital's rule repeatedly, then you deduce that

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x + \cos^2 x}{e^x} &= \lim_{x\to\infty} \frac{e^x - 2\sin x\cos x}{e^x} \\ &= \lim_{x\to\infty} \frac{e^x - \sin(2x)}{e^x} \\ &= \lim_{x\to\infty} \frac{e^x - 2\cos(2x)}{e^x} \\ &= \lim_{x\to\infty} \frac{e^x + 4\sin(2x)}{e^x} \\ &= \cdots .\end{aligned}$$

This will go on indefinitely and hence you will not solve the problem in this fashion.

To solve the problem, we avoid L'Hôpital's rule altogether and instead multiply the top and bottom of the original fraction by e^{-x}. After performing some algebra, we see that we have to compute

$$\lim_{x\to\infty} 1 + e^{-x}\cos^2 x.$$

This limit is 1 because e^{-x} goes to 0 as $x \to \infty$, while $\cos^2 x$ is never larger than one.