Cálculo III
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Exercícios:
Plano Tangente

Determine uma equação do plano tangente à superfície no ponto especificado.

1- $z = 3y^2 - 2x^2 + x$, $(2, -1, -3)$

Sendo a equação do plano tangente à superfície $z = f(x, y)$ no ponto $P(x_0, y_0, z_0)$ dada por

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

para encontrar a equação é necessário calcular f_x, f_y e resolver em $P(x_0, y_0, z_0)$. Assim:

$$f_x = \frac{\partial}{\partial x}(3y^2 - 2x^2 + x) = \cancel{3\frac{\partial}{\partial x}y^2} - 2\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial x}x =$$

$$\boxed{f_x = -4x + 1}$$

$$f_y = \frac{\partial}{\partial y}(3y^2 - 2x^2 + x) = 3\frac{\partial}{\partial y}y^2 \cancel{-2\frac{\partial}{\partial y}x^2} + \cancel{\frac{\partial}{\partial y}x} =$$

$$\boxed{f_y = 6y}$$

Encontrando os pontos $f_x(x_0, y_0)$ e $f_y(x_0, y_0)$:

$$\left.\frac{\partial}{\partial x}\right|_{(2,-1)} = -4(2) + 1 = -8 + 1 = \boxed{-7}$$

$$\left.\frac{\partial}{\partial y}\right|_{(2,-1)} = 6(-1) = \boxed{-6}$$

②

Como o valor de z_0 já é conhecido (-3), a equação do plano é então:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - (-3) = (-7)(x - 2) + (-6)(y - (-1))$$

$$z + 3 = -7x + 14 - 6y - 6$$

$$\boxed{z = -7x - 6y + 5}$$

②- $z = 3(x-1)^2 + 2(y+3)^2 + 7$; $(2, -2, 12)$

Cálculo de f_x e f_y:

$$f_x = \frac{\partial}{\partial x}\left[3(x-1)^2 + 2(y+3)^2 + 7\right]$$

$$= 3\frac{\partial}{\partial x}(x+1)^2 + \cancel{2\frac{\partial}{\partial x}(y+3)^2}^{\,0} + \cancel{\frac{\partial}{\partial x}7}^{\,0}$$

(sendo $u = x+1$:)

$$= 3\left(\frac{\partial}{\partial u}u^2 \, \cancel{\frac{\partial}{\partial x}(x+1)}^{\,1}\right) = 3(2u) = 6u = 6(x-1)$$

$$\boxed{f_x = 6x - 6}$$

$$f_y = \frac{\partial}{\partial y}\left[3(x-1)^2 + 2(y+3)^2 + 7\right]$$

$$= \cancel{3\frac{\partial}{\partial y}(x-1)^2}^{\,0} + 2\frac{\partial}{\partial y}(y+3)^2 + \cancel{\frac{\partial}{\partial y}7}^{\,0}$$

(sendo $v = y+3$):

$$= 2\left(\frac{\partial}{\partial v}v^2 \, \cancel{\frac{\partial}{\partial y}(y+3)}^{\,1}\right) = 2(2v) = 4v = 4(y+3)$$

$$\boxed{f_y = 4y + 12}$$

Resolvendo f_x e f_y no ponto $(2, -2, 12)$:

$$\left.\frac{\partial}{\partial x}\right|_{(2,-2)} = 6(2)-6 = 12-6 = \boxed{6}$$

$$\left.\frac{\partial}{\partial y}\right|_{(2,-2)} = 4(-2)+12 = -8+12 = \boxed{4}$$

A equação do plano tangente é então:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$z - 12 = 6(x-2) + 4(y-(-2))$$
$$z - \cancel{12} = 6x - \cancel{12} + 4y + 8$$
$$\boxed{z = 6x + 4y + 8}$$

3. $z = \sqrt{xy}$, $(1, 1, 1)$.

Cálculo de f_x e f_y:

$$f_x = \frac{\partial}{\partial x}\sqrt{xy} = \frac{\partial}{\partial x}(xy)^{1/2} \qquad \text{(sendo } u = xy\text{):}$$

$$= \frac{\partial}{\partial u} u^{1/2} \frac{\partial}{\partial x}(xy) = \frac{1}{2\sqrt{u}} \cdot y = \frac{y}{2\sqrt{xy}}$$

$$\boxed{f_x = \frac{y}{2\sqrt{xy}}}$$

$$\boxed{f_y = \frac{x}{2\sqrt{xy}}}$$, por simetria

4

Resolvendo f_x e f_y no ponto $(1, 1, 1)$:

$$\left.\frac{\partial}{\partial x}\right|_{(1,1)} = \frac{1}{2\sqrt{1 \times 1}} = \frac{1}{2}$$

$$\left.\frac{\partial}{\partial y}\right|_{(1,1)} = \frac{1}{2}$$, por simetria.

A equação do plano tangente é então:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

$$z - 1 = \frac{x-1}{2} + \frac{y-1}{2}$$

$$z - 1 = \frac{x+y-2}{2}$$

$$2z - \cancel{2} = x + y - \cancel{2} \quad \therefore \boxed{z = \frac{x+y}{2}}$$

4 - $z = x e^{xy}$, $(2, 0, 2)$

Cálculo de f_x e f_y:

$$f_x = \frac{\partial}{\partial x} x e^{xy} = e^{xy} \cancel{\frac{\partial x}{\partial x}}^{1} + x \frac{\partial}{\partial x} e^{xy}$$

(sendo $u = xy$)

$$= e^{xy} + x\left(\frac{\partial}{\partial u} e^{u} \cancel{\frac{\partial}{\partial x} xy}^{y}\right)$$

$$= e^{xy} + x(e^{u} y)$$

$$= e^{xy} + e^{xy} xy$$

$$\boxed{f_x = e^{xy}(xy+1)}$$

$$f_y = \frac{\partial}{\partial y} x e^{xy} = e^{xy} \cancel{\frac{\partial}{\partial y} x}^{0} + x \frac{\partial}{\partial y} e^{xy}$$

(sendo $u = xy$):

$$= x\left(\frac{\partial}{\partial u} e^{u} \cancel{\frac{\partial}{\partial y} xy}^{x}\right)$$

$$= x\left(e^{xy} x\right)$$

$$\boxed{f_y = x^2 e^{xy}}$$

Resolvendo f_x e f_y no ponto $(2,0,2)$:

$$\left.\frac{\partial}{\partial x}\right|_{(2,0)} = e^{2(0)}\left(2(0)+1\right)$$

$$= 1(1) = \boxed{1}$$

$$\left.\frac{\partial}{\partial y}\right|_{(2,0)} = 2^2 e^{2(0)} = 4(1) = \boxed{4}$$

A equação do plano tangente é então:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 2 = 1(x-2) + 4(y-0)$$

$$z \cancel{-2} = x \cancel{-2} + 4y$$

$$\boxed{z = x + 4y}$$

6.

5. $z = x \sin(x+y)$, $(-1, 1, 0)$

Cálculo de f_x e f_y:

$$f_x = \frac{\partial}{\partial x} x\sin(x+y) = \sin(x+y)\cancel{\frac{\partial}{\partial x} x}^{1} + x\frac{\partial}{\partial x}\sin(x+y)$$

$(u = x+y)$:

$$= \sin(x+y) + x\left(\frac{\partial}{\partial u}\sin(u)\cancel{\frac{\partial}{\partial x}(x+y)}^{1}\right)$$

$$= \sin(x+y) + x\left(\cos(x+y)\right)$$

$$\boxed{f_x = x\cos(x+y) + \sin(x+y)}$$

$$f_y = \frac{\partial}{\partial y} x\sin(x+y) = \cancel{\sin(x+y)\frac{\partial}{\partial y} x}^{0} + x\frac{\partial}{\partial y}\sin(x+y)$$

$$= x\left(\frac{\partial}{\partial u}\sin(u)\cancel{\frac{\partial}{\partial y}(x+y)}^{1}\right) = x\cos(x+y)$$

$$\boxed{f_y = x\cos(x+y)}$$

Resolvendo f_x e f_y no ponto $(-1, 1, 0)$:

$$\frac{\partial}{\partial x}\Big|_{(-1,1)} = (-1)\cos(\cancel{-1+1}^{0}) + \sin(\cancel{-1+1}^{0}) = \boxed{-1}$$

$$\frac{\partial}{\partial y}\Big|_{(-1,1)} = (-1)\cos(-1+1) = \boxed{-1}$$

A equação do plano tangente é então:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 0 = (-1)(x - (-1)) + (-1)(y - 1)$$

$$z = -x \cancel{-1} - y \cancel{+1}$$

$$\boxed{z = -x - y}$$

6- $z = \ln(x - 2y)$, $(3, 1, 0)$

Cálculo de f_x e f_y:

$$f_x = \frac{\partial}{\partial x} \ln(x - 2y) = \frac{\partial}{\partial u} \ln(u) \frac{\partial}{\partial x}(x - 2y) \qquad (u = x - 2y)$$

$$= \frac{1}{u}\left(\frac{\partial}{\partial x} x - 2 \cancel{\frac{\partial}{\partial x} y}^{0}\right) = \frac{1}{x - 2y}$$

$$\boxed{f_x = \frac{1}{x - 2y}}$$

$$f_y = \frac{\partial}{\partial y} \ln(x - 2y) = \frac{\partial}{\partial u} \ln(u) \frac{\partial}{\partial y}(x - 2y) \qquad (u = x - 2y)$$

$$= \frac{1}{u}\left(\cancel{\frac{\partial}{\partial y} x}^{0} - 2 \frac{\partial}{\partial y} y\right) = \frac{1}{x - 2y}(-2)$$

$$\boxed{f_y = \frac{-2}{x - 2y}}$$

⑧

Resolvendo fx e fy no ponto (3, 1, 0):

$$\left.\frac{\partial}{\partial x}\right|_{(3,1)} = \frac{1}{3-2(1)} = \boxed{1}$$

$$\left.\frac{\partial}{\partial y}\right|_{(3,1)} = \frac{-2}{3-2(1)} = \frac{-2}{1} = \boxed{-2}$$

A equação do plano tangente é então:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 0 = 1(x-3) + (-2)(y-1)$$

$$z = x - 3 - 2y + 2$$

$$\boxed{z = x - 2y - 1}$$

Recursos utilizados:
- Calculadora HP-50G
- GeoGebra