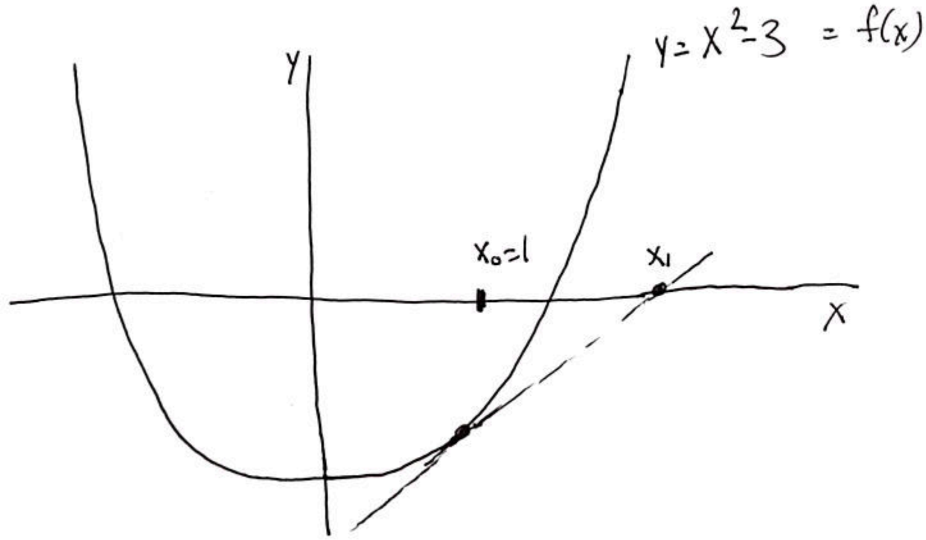
• Newton's method

- Newton's method is a powerful tool for solving equations of the form $f(x)=0$

- <u>Ex</u>: $f(x)=x^2-3$. In other words, solve $x^2-3=0$ for x.
  - We know the exact answer is $x=\sqrt{3}$.
  - Newton's method gives a numerical approximation to the exact answer with the help of <u>tangent lines</u>



- Our goal is to find where the graph crosses the x-axis. We start with the guess $x_0=1$. We plug x_0 into f and find that $y_0=f(x_0)=1-3=-2$, which isn't very close to 0.

- Our next guess is x_1, which is where the tangent line to the function at x_0 crosses the x axis.

- The eqn. for the tangent line is

$$y - y_0 = m(x - x_0), \text{ where } m = f'(x_0)$$

- When the tangent line intersects the x-axis, $y=0$.

Thus, $-y_0 = m(x_1 - x_0)$

$$x_1 = x_0 - \frac{y_0}{m}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

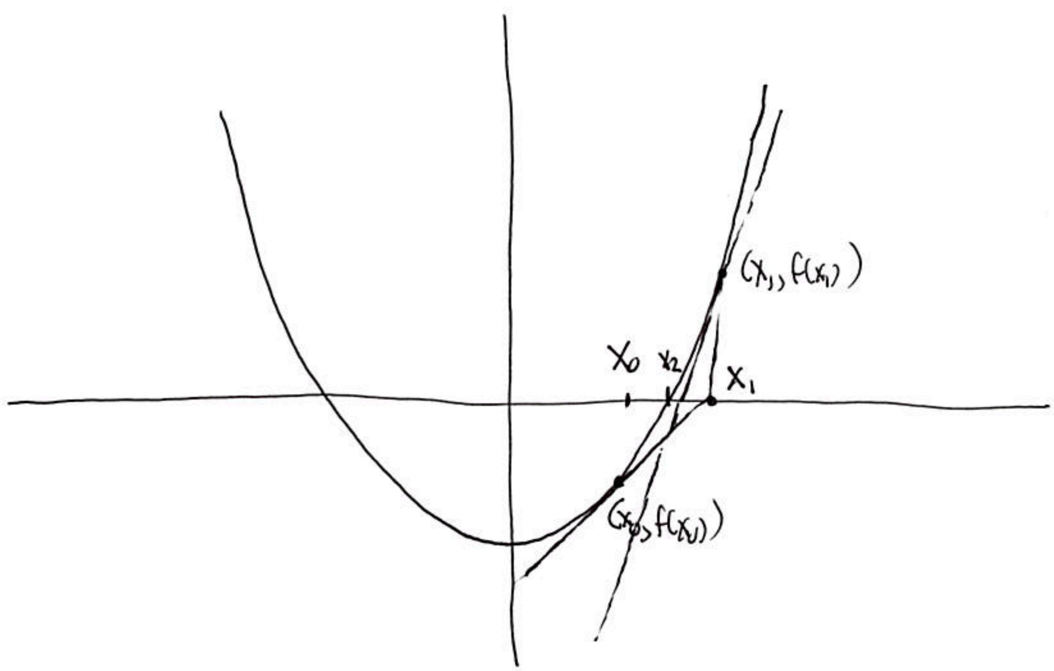


Illustration of Newton's method for $f(x) = x^2 - 3$

- In our example, $f(x) = x^2 - 3$, $f'(x) = 2x$

Thus, $X_1 = X_0 - \frac{(X_0^2 - 3)}{2X_0} = X_0 - \frac{1}{2}X_0 + \frac{3}{2X_0}$

$$X_1 = \frac{1}{2}X_0 + \frac{3}{2X_0}$$

- The main idea is to repeat this process:

$$X_2 = \frac{1}{2}X_1 + \frac{3}{2X_1}$$

$$X_3 = \frac{1}{2}X_2 + \frac{3}{2X_2}$$

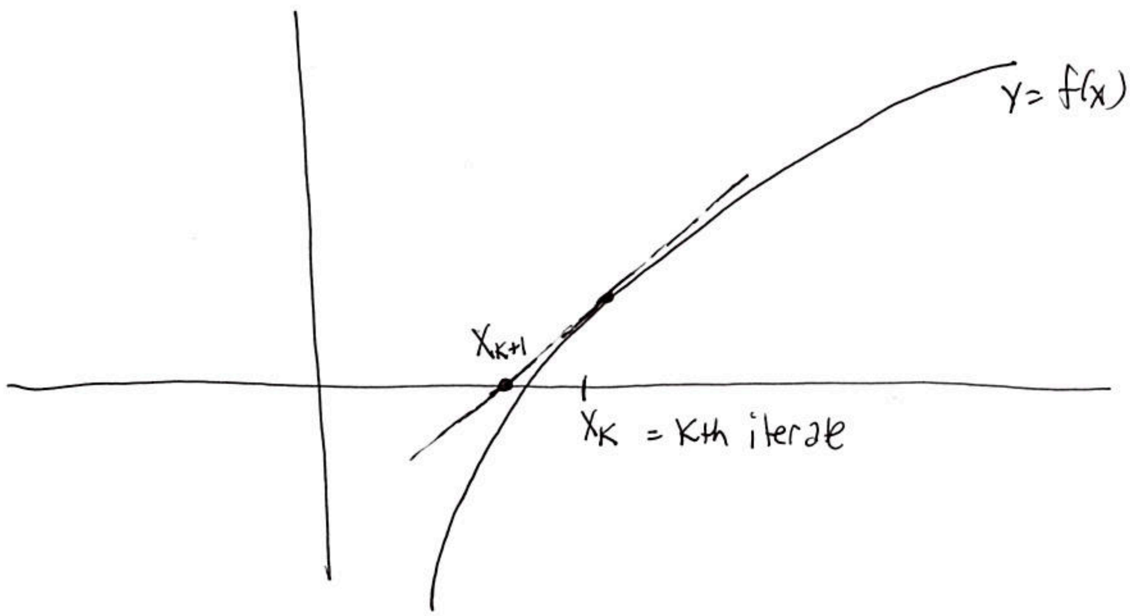
- This procedure approximates $\sqrt{3}$ very well!:

X	Y	accuracy: $\lvert y - \sqrt{3} \rvert$
X_0	1	
X_1	2	3×10^{-1}
X_2	$\frac{7}{4}$	2×10^{-2}
X_3	$\frac{7}{8} + \frac{6}{7}$	10^{-4}
X_4	$\frac{18817}{10864}$	3×10^{-9}

- Note that the # of digits of accuracy doubles with each iteration

Summary:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$



Example 1 considered the particular case of

$$f(x) = x^2 - 3$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \frac{1}{2} x_k + \frac{3}{2x_k}$$

- We now define $\bar{x} = \lim_{k \to \infty} x_k$

  ($x_k \to \bar{x}$ as $k \to \infty$)

- To evaluate $\bar{x}$, take the limit as $k \to \infty$ in the equation

$$x_{k+1} = \frac{1}{2} x_k + \frac{3}{2x_k}$$

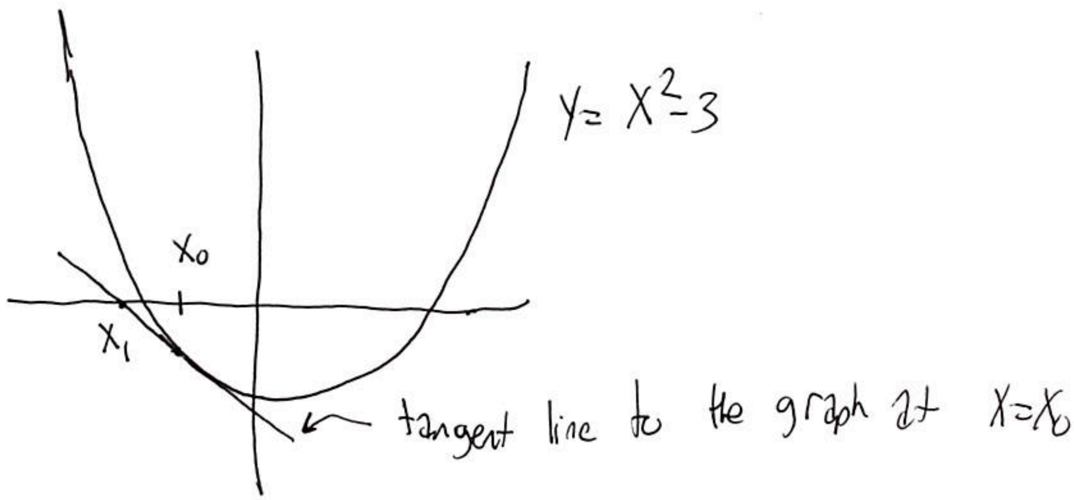
$$\Rightarrow \bar{x} = \frac{1}{2}\bar{x} + \frac{3}{2\bar{x}}$$

$$\Rightarrow \frac{1}{2}\bar{x} = \frac{3}{2\bar{x}}$$

$$\Rightarrow \boxed{\bar{x}^2 = 3},$$

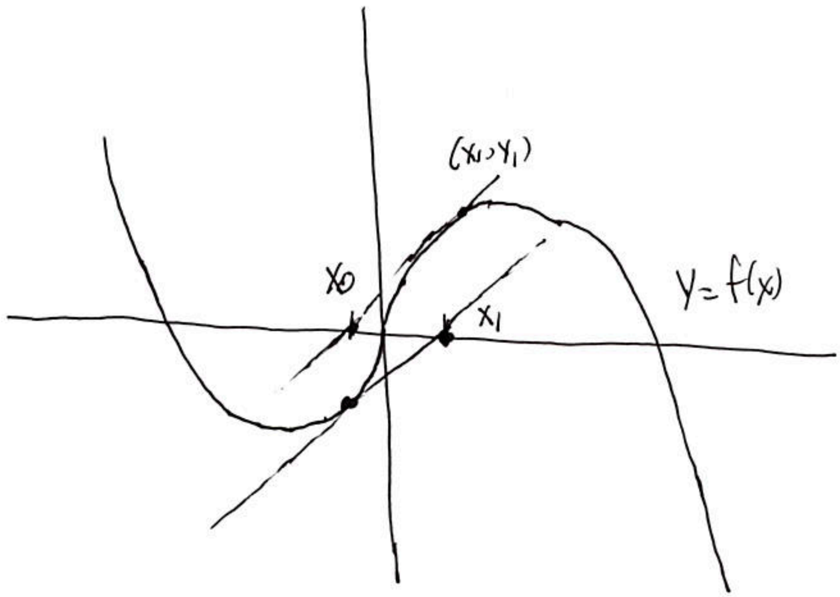
which is what we wanted!

- <u>Warning 1</u>: Newton's method can find an unexpected root:



If you take $x_0 = -1$, then $x_k \to -\sqrt{3}$ instead of $\sqrt{3}$

• Warning 2: Newton's method can fail completely:



In this example, $x_2 = x_0$, $x_3 = x_1$, etc.
It repeats in a cycle and never converges to a single value.