MIDTERM 4 - 18.01 - FALL 2014.

Name:

Email:

Please put a check by your recitation section.

	Instructor	Time
☐	B.Yang	MW 10
☐	M. Hoyois	MW 11
☐	M. Hoyois	MW 12
☐	X. Sun	MW 1
☐	R. Chang	MW 2

Problem #	Max points possible	Actual score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Directions:

- Write your answers directly on the exam.
- No books, notes, or electronic devices can be used on the exam.
- Partial credit can be given if you show your work.
- **Don't forget to write your name and email and to indicate your recitation instructor above.**
- A formula sheet is attached.

Good luck!

Formula sheet

$$(\sin x)^2 + (\cos x)^2 = 1, \qquad (\sec x)^2 = (\tan x)^2 + 1$$

$$(\sin x)^2 = \frac{1}{2} - \frac{1}{2}\cos(2x), \qquad (\cos x)^2 = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\cos(2x) = (\cos x)^2 - (\sin x)^2, \qquad \sin(2x) = 2\sin x \cos x$$

$$\frac{d}{dx}\tan x = (\sec x)^2, \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \tan x\, dx = \ln|\sec x| + C, \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Problem 1. (10 + 10 = 20 points) Find the following two approximations to the definite integral

$$\int_0^{\pi/3} \ln(\sec(x))\,dx.$$

Also, in each case, decide whether your approximations are larger or smaller than the exact value.

a) Use lower Riemann sums with two equal-length subintervals. *Remark: Recall that a "lower Riemann sum" is such that on each subinterval, the approximating rectangle has a height that is equal to the minimum height of the function on that subinterval.*

b) Use the trapezoid rule with two equal-length subintervals.

Solution: a) We partition $[0,\pi/3] = [0,\pi/6] \cup [\pi/6,\pi/3]$. Let $f(x) = \ln(\sec(x))$. Note that for $0 \le x \le \pi/3$, we have

$$f(x) \geq 0, \tag{1}$$

$$f'(x) = \tan x \geq 0, \tag{2}$$

$$f''(x) = \sec^2(x) \geq 0. \tag{3}$$

In particular, f is increasing. Moreover, we compute that $f(0) = 0$, $f(\pi/6) = \ln(2/\sqrt{3})$, and $f(\pi/3) = \ln 2$. Hence, on the first subinterval, the minimum value of f is $f(0) = 0$, and on the second subinterval, the minimum value of f is $f(\pi/6) = \ln(2/\sqrt{3})$. Hence, the lower Riemann sum is

$$f(0) \times \frac{\pi}{6} + f(\pi/6) \times \frac{\pi}{6} = \frac{\pi}{6} \times \ln\left(\frac{2}{\sqrt{3}}\right).$$

Lower Riemann sums always yield an *underestimate* of the actual value of the integral.

b) Using the same partition, we compute the trapezoid approximation as follows:

$$\begin{aligned}\frac{1}{2}f(0) \times \frac{\pi}{6} + f(\pi/6) \times \frac{\pi}{6} + \frac{1}{2}f(\pi/3) \times \frac{\pi}{6} &= 0 + \ln(2/\sqrt{3}) \times \frac{\pi}{6} + \frac{1}{2}\ln 2 \times \frac{\pi}{6} \\ &= \frac{\pi}{6} \times \ln\left(\frac{2\sqrt{2}}{\sqrt{3}}\right).\end{aligned}$$

The trapezoid rule approximation yields an *overestimate* of the actual value of the integral because the graph of $y = f(x)$ is concave up.

Problem 2. (20 points) Evaluate the following integral:

$$\int \arcsin x \, dx$$

Solution: We use integration by parts with $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}}\, dx$, $dv = 1\, dx$, $v = x$ to compute that

$$\int u\, dv = uv - \int v\, du = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

The last integral can be evaluated with the substitution $u = x^2$, $du = 2x\, dx$, which yields

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = -(1-x^2)^{1/2} + C.$$

In total, we have

$$\int \arcsin x \, dx = x \arcsin x + (1-x^2)^{1/2} + C.$$

Problem 3. (20 points) Evaluate the following integral:

$$\int \frac{1}{x(x+1)^2}\,dx$$

Solution: The general form of the partial fraction decomposition relation for $\dfrac{1}{x(x+1)^2}$ is

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

The cover-up method yields $A = 1$, $C = -1$ and thus

$$\frac{1}{x(x+1)^2} = \frac{1}{x} + \frac{B}{x+1} - \frac{1}{(x+1)^2}.$$

Setting $x = 1$, we find that

$$\frac{1}{4} = 1 + \frac{B}{2} - \frac{1}{4}$$

and thus $B = -1$. We have therefore derived the following partial fraction decomposition:

$$\frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}.$$

Integrating both sides of the previous identity with respect to x, we conclude that

$$\int \frac{1}{x(x+1)^2}\,dx = \ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + const.$$

Problem 4. (20 points) Evaluate the following integral:

$$\int (x^2 + 2x + 2)^{-1/2}\, dx$$

Be sure to state your final answer in terms of x.

Solution: We first complete the square: $x^2+2x+2 = (x+1)^2+1$. We then make the substitution $x+1 = \tan\theta$, $dx = (\sec\theta)^2\, d\theta$, $(x+1)^2+1 = (\tan\theta)^2+1 = (\sec\theta)^2$ to conclude that

$$\begin{aligned}
\int (x^2 + 2x + 2)^{-1/2}\, dx &= \int (\sec\theta)^{-1} \times (\sec\theta)^2\, d\theta \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\sqrt{(x+1)^2+1} + x + 1\right| + C.
\end{aligned}$$

Problem 5. (20 points) Evaluate the following integral:

$$\int (\sin x)^3 (\cos x)^3 \, dx$$

Solution: Using the identity $(\sin x)^2 + (\cos x)^2 = 1$ and the substitution $u = \cos x$, $du = -\sin x \, dx$, we conclude that

$$\begin{aligned}
\int (\sin x)^3 (\cos x)^3 \, dx &= \int (\sin x)^2 (\cos x)^3 \sin x \, dx \\
&= \int \left\{ 1 - (\cos x)^2 \right\} (\cos x)^3 \sin x \, dx \\
&= -\int (1 - u^2) u^3 \, du \\
&= \frac{u^6}{6} - \frac{u^4}{4} + C \\
&= \frac{(\cos x)^6}{6} - \frac{(\cos x)^4}{4} + C.
\end{aligned}$$

Alternatively, we could use the substitution $u = \sin x$, $du = \cos x \, dx$ to derive the equivalent solution

$$\begin{aligned}
\int (\sin x)^3 (\cos x)^3 \, dx &= \int (\sin x)^3 (\cos x)^2 \cos x \, dx \\
&= \int (\sin x)^3 \left\{ 1 - (\sin x)^2 \right\} \cos x \, dx \\
&= \int u^3 (1 - u^2) \, du \\
&= \frac{u^4}{4} - \frac{u^6}{6} + C' \\
&= \frac{(\sin x)^4}{4} - \frac{(\sin x)^6}{6} + C',
\end{aligned}$$

where $C = C' + 1/12$.