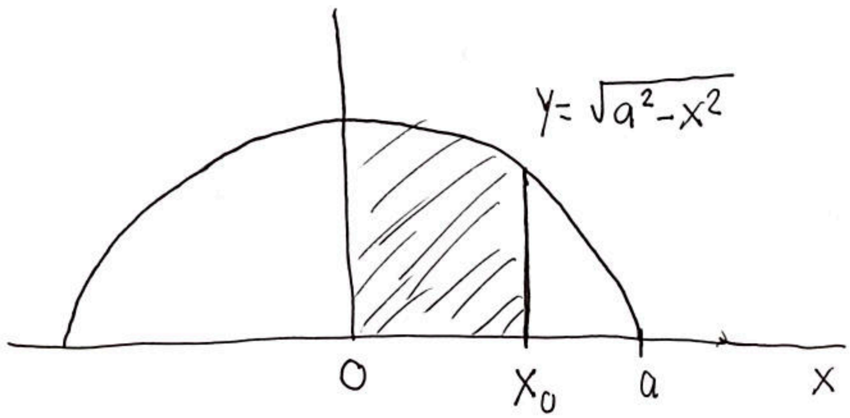
• Integration by Inverse Substitution; Completing the Square

Trigonometric Substitutions Continued



• Find the area of the shaded portion:

$$\int_0^{x_0} \sqrt{a^2 - x^2}\, dx$$

• Set $x = a\sin u$, $dx = a\cos u\, du$

• $a^2 - x^2 = a^2 - a^2\sin^2 u = a^2\cos^2 u \Rightarrow \sqrt{a^2 - x^2} = \underbrace{a\cos u}_{\text{No more square root!}}$

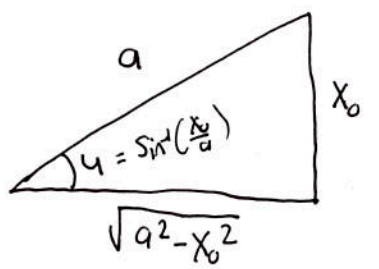
• $\int \sqrt{a^2 - x^2}\, dx = \int a^2\cos^2 u\, du = a^2 \int \frac{1 + \cos(2u)}{2}\, du$

(recall $\cos^2 u = \frac{1 + \cos 2u}{2}$)

$$= a^2\left[\frac{u}{2} + \frac{\sin 2u}{4}\right] + C$$

• We want to express our answer in terms of x_0

- When $x=0$, $a \sin u = 0$, and therefore $u=0$
- When $x = x_0$, $a \sin u = x_0$, and therefore $u = \sin^{-1}\left(\frac{x_0}{a}\right)$
- $\frac{\sin(2u)}{4} = \frac{2 \sin u \cos u}{4} = \frac{1}{2} \sin u \cos u$
- $\sin u = \sin\left(\sin^{-1}\left(\frac{x_0}{a}\right)\right) = \frac{x_0}{a}$
- To compute $\cos\left(\sin^{-1}\left(\frac{x_0}{a}\right)\right)$, we draw a right triangle:



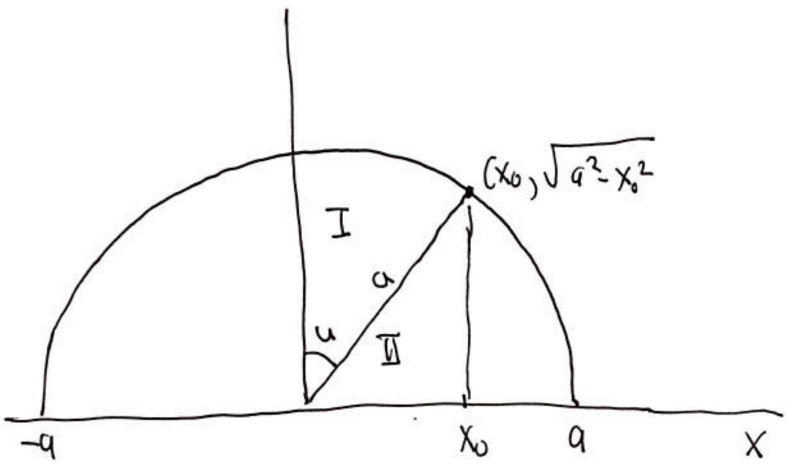
We see that $\cos u = \frac{\sqrt{a^2 - x_0^2}}{a}$

- In total: $\displaystyle\int_0^{x_0} \sqrt{a^2 - x^2}\, dx = a^2 \left[\frac{u}{4} + \frac{1}{2} \sin u \cos u \right] - 0$

$$= a^2 \left[\frac{\sin^{-1}\left(\frac{x_0}{a}\right)}{2} + \frac{1}{2} \left(\frac{x_0}{a}\right) \frac{\sqrt{a^2 - x_0^2}}{a} \right]$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{x_0}{a}\right) + \frac{1}{2} x_0 \sqrt{a^2 - x_0^2}$$

- Can check our answer:



- Area of I = area of a sector with angle u = $\frac{1}{2} a^2 u$
- Area of II = $\frac{1}{2}$ × base × height = $\frac{1}{2} x_0 \sqrt{a^2 - x_0^2}$

- Here is a list of integrals that can be computed using a trig substitution and a trig identity:

Integral	Substitution	Trig Identity
• $\int \frac{dx}{\sqrt{x^2+1}}$	• $x = \tan u$	• $\tan^2 u + 1 = \sec^2 u$
• $\int \frac{dx}{\sqrt{x^2-1}}$	• $x = \sec u$	• $\sec^2 u - 1 = \tan^2 u$
• $\int \frac{dx}{\sqrt{1-x^2}}$	• $x = \sin u$	• $1 - \sin^2 u = \cos^2 u$

- How about evaluating an integral such as

$$\int \frac{dx}{\sqrt{x^2 + 4x}}$$

- When you have a linear + quadratic term under the square root, complete the square:
  - $x^2 + 4x = (\text{Something})^2 \pm \text{Constant}$
  - $(x+2)^2 = x^2 + 4x + 4 \Rightarrow x^2 + 4x = (x+2)^2 - 4$
  - Now make a substitution: $v = x + 2$, $dv = dx$:

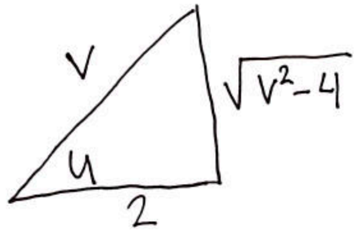
$$\int \frac{dx}{\sqrt{(x+2)^2 - 4}} = \int \frac{dv}{\sqrt{v^2 - 4}}$$

- Now let $v = 2\sec u$ $\quad dv = 2\sec u \tan u$

$$v^2 - 4 = 4(\sec^2 u - 1) = 4\tan^2 u \Rightarrow$$

$$\int \frac{dv}{\sqrt{v^2 - 4}} = \int \frac{2\sec u \tan u}{2\tan u} = \int \sec u = \ln(\sec u + \tan u) + C_1.$$

- We now use a right triangle to express our answer in terms of v: (recall $v = 2\sec u$)



- It follows that $\tan u = \dfrac{\sqrt{v^2-4}}{2}$, $\sec u = \dfrac{v}{2}$

- $$\int \sec u \, du = \ln\left(\frac{v}{2} + \frac{\sqrt{v^2-4}}{2}\right) + C_1$$
$$= \ln\left(v + \sqrt{v^2-4}\right) + \tilde{C}_1, \quad \tilde{C}_1 = C_1 - \ln 2.$$

- In total:

$$\int \frac{dx}{\sqrt{x^2+4x}} = \ln\left(x+2 + \sqrt{x^2+4x}\right) + \tilde{C}_1$$