MATH 18.01, FALL 2014 - PROBLEM SET # 8

Professor: Jared Speck

Due: by Monday 5:00pm on 11-24-14
(in the boxes outside of Room E17-131; write your name, recitation instructor, and recitation meeting days/time on your homework)

18.01 Supplementary Notes (including Exercises and Solutions) are available on the course web page: `http://math.mit.edu/~jspeck/18.01_Fall%202014/1801_CourseWebsite.html`. This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

Part I consists of exercises given and solved in the Supplementary Notes. It will be graded quickly, checking that all is there and the solutions not copied.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises given here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

You are encouraged to use graphing calculators, software, etc. to <u>check</u> your answers and to explore calculus. However, (unless otherwise indicated) we strongly discourage you from using these tools to <u>solve</u> problems, perform computations, graph functions, etc. An extremely important aspect of learning calculus is developing these skills. You will not be allowed to use any such tools on the exams.

Part I (20 points)

Notation: The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Lecture 24. (Thurs., Nov. 13) Integrating rational functions, partial fractions.
Read: Read 10.6, Notes F.
Homework: Notes 5E: 2, 3, 5, 6, 10h (complete the square).

Lecture 25. (Fri., Nov. 14) Integration by parts. Reduction formulas.
Read: 10.7.
Homework: Notes 5F: 1a, 2d, followed by 2b, 3.

<u>**Material beyond Lecture 25 will not appear on Midterm # 4**</u>

Lecture 26. (Tues., Nov. 18) Parametric equations; arc length. Surface area.

Read: 17.1, 7.5, 7.6.

Homework: Notes 4E: 2, 3, 8; Notes 4F: 1d, 4, 5, 8; Notes 4G: 2, 5.

If a curve is given by $x = x(t), y = y(t)$ then to find its arc length, use ds in the form

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

and integrate from start to finish: from $t = t_0$ to $t = t_1$.

Part II (50 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. *You must show your work; "bare" solutions will receive very little credit.*

iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem **0**" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (Thurs., Nov. 13; partial fractions; 3 + 2 = 5 points)

a) Derive a formula for $\int \sec x\,dx$ by first writing $\sec x = \frac{\cos x}{1-\sin^2 x}$ (verify this), and then making a substitution for $\sin x$ and using partial fractions. (Your final answer must be expressed in terms of x.)

b) Convert the formula into the one we derived in class by multiplying the fraction in the answer on both top and bottom by $1+\sin x$. (Note that $(1/2)\ln v = \ln\sqrt{v}$.)

2. (Fri., Nov. 14; integration by parts; 10 points) Let R be the region in the first quadrant of the (x, y) plane $(x \geq 0, y \geq 0)$ under the first hump of the curve $y = e^x \cos x$ and above the $x-$ axis. The region R is rotated about the $y-$axis to generate a solid of revolution S. Use the shell method to find the volume of S.

3. (Fri., Nov. 14; integration by parts; 5 points) Recall that $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$. Consider the following sequence of integration by parts:

$$\begin{aligned}\int e^x \sinh x\,dx &= e^x \cosh x - \int e^x \cosh x\,dx \\ &= e^x \cosh x - e^x \sinh x + \int e^x \sinh x\,dx.\end{aligned}$$

By canceling $\int e^x \sinh x\,dx$ from each side, one would conclude that $e^x \cosh x = e^x \sinh x$ and hence $\cosh x = \sinh x$. However, these last two equalities are false. Explain the mistake in reasoning.

4. (Tues., Nov. 18; parametric equations, arc length, surface area; 2 + 4 + 4 = 10 points) In this problem, you will investigate a parametric curve in the (x, y) plane called a *cycloid*. You already encountered essentially the same curve in problem 4 of pset 3b. The curve is defined by

$$x = t - \sin t,$$
$$y = 1 - \cos t.$$

The curve describes the motion of a point $P = (x, y)$ on the outer rim of a wheel of radius 1 as the wheel rolls to the right at a constant angular velocity (in problem 4 of pset 3b the wheel was rolling to the left). Note that the point P is initially at the bottom of wheel (the center of the wheel starts at $(0, 1)$).

a) Express dy/dx in terms of t and find the values of t and (x, y) where dy/dx is infinite. Then sketch the cycloid (in the (x, y) plane) for $0 \leq t \leq 4\pi$.

b) Compute the arc length of one full period of the cycloid ($0 \leq t \leq 2\pi$). This will tell you the total distance traveled by the point P in the plane after one full period.

c) The region under one full period of the cycloid ($0 \leq t \leq 2\pi$) and above the $x-$axis is revolved about the $x-$axis to generate a solid of revolution S. Find the *surface area* of S.

5. (Tues., Nov. 18; parametric equations; 5 points) Consider the curve defined by the parametric equations

$$x = \int_1^t (\cos v)(\ln v)\, dv,$$
$$y = \int_1^t (\sin v)(\ln v)\, dv$$

for $t \geq 1$. Find the arc length of the portion of the curve that starts from the origin in the (x, y) plane and ends at the first non-origin point where the curve is vertical.

6. (Tues., Nov. 18; parametric equations; $1 + 7 + 4 = 12$ points) A standard ellipse is a "closed curve." That is, when you trace it out, it "ends up where it started after one full orbit." In the simplest models of our solar system, a detailed analysis of Newton's laws of motion shows the following fact (which was first shown by Newton and which is now known as his solution to his classic two-body problem): a planet traces out a standard ellipse as it revolves around the sun. However, more accurate models take into account an effect called "apsidal precession," which distorts the ellipses. Precession is caused by many factors including the gravitational force exerted by other planets, the fact that planets are not perfect "point particles," and the fact that Newton's theory of gravity is not as accurate as Einstein's theory of gravity (general relativity). In part c), you will investigate a simple model of precession. In this problem, you will not investigate the cause of the precession (investigating the cause is something you would do in an advanced course on celestial mechanics); your goal will only be to understand the shape of the curve.

In parts b) and c), you are strongly encouraged to use computer software such as Maple or Mathematica to assist you with plotting and computations.

a) Consider the following ellipse in parametric form for $0 \leq t \leq 2\pi$:

$$x = 2\cos t,$$
$$y = \sin t.$$

This ellipse describes[1] a planet that is orbiting around the sun, which is located at the focal point[2] $F = (-\sqrt{3}, 0)$. Find an algebraic equation in x and y for the curve. Then plot the portion of the curve corresponding to the parameter range $0 \leq t \leq 2\pi$. Is the curve traced out clockwise or counterclockwise as t increases?

b) Let θ be a *constant.* Given any point (x, y) in the plane, define $R_\theta(x, y)$ to be the "rotated point" in the plane with coordinates

$$R_\theta(x, y) = \Big((x + \sqrt{3})\cos\theta - y\sin\theta - \sqrt{3}, (x + \sqrt{3})\sin\theta + y\cos\theta\Big).$$

In linear algebra, you will learn that $R_\theta(x, y)$ is simply the rotation of the point (x, y) by θ radians counterclockwise[3] about the focus $F = (-\sqrt{3}, 0)$. More precisely, both (x, y) and $R_\theta(x, y)$ lie on the same circle of radius $r = \sqrt{(x + \sqrt{3})^2 + y^2}$ centered at F, but $R_\theta(x, y)$ has been rotated counterclockwise by θ about F (notice that F itself doesn't move at all, that is, $R_\theta(-\sqrt{3}, 0) = (-\sqrt{3}, 0)$). Using these facts and your insight from part a), plot the following rotated ellipse in parametric form for $0 \leq t \leq 2\pi$ when $\theta = \pi/4$:

$$x = (2\cos t + \sqrt{3})\cos\theta - \sin t \sin\theta - \sqrt{3},$$
$$y = (2\cos t + \sqrt{3})\sin\theta + \sin t\cos\theta.$$

Then find an algebraic relation in x and y for the curve. *Hint: Given any point (x, y) on the rotated ellipse, define $(\widetilde{x}, \widetilde{y}) = R_{-\theta}(x, y)$. Then $(\widetilde{x}, \widetilde{y})$ is a point on the original non-rotated ellipse from part a).*

Then do the same exercises when $\theta = \pi/2$ and $\theta = 3\pi/4$.

c) Now consider the case when the angle depends on $t : \theta(t) = t/8$. You can think of this curve as an ellipse that is "continuously being rotated about its focal point F as it is traced out, where the angular velocity of the rotation is constantly 1/8." Some possible physical causes of such "precession" were mentioned above. Sketch the following curve in parametric form for $0 \leq t \leq 2\pi$ (with the help of a computer if you want):

$$x = (2\cos t + \sqrt{3})\cos(\theta(t)) - \sin t\sin(\theta(t)) - \sqrt{3},$$
$$y = (2\cos t + \sqrt{3})\sin(\theta(t)) + \sin t\cos(\theta(t)).$$

Carefully plot the starting and end points of the curve. Does the curve end up where it started as t varies from 0 to 2π? Finally, extend your plot to the range $0 \leq t \leq 16\pi$.

[1] The actual orbits of the planets in our solar system are much more circular than the ellipse in this problem.

[2] A completely precise solution of Newton's two-body problem shows that both bodies revolve about their common center of mass. However, if one body is much more massive than the other (in the present problem, the sun is much more massive than the planet), then the common center of mass approximately coincides with the location of the more massive body. That is, the more massive body approximately stays in place.

[3] Some people in the UK call this "widdershins."