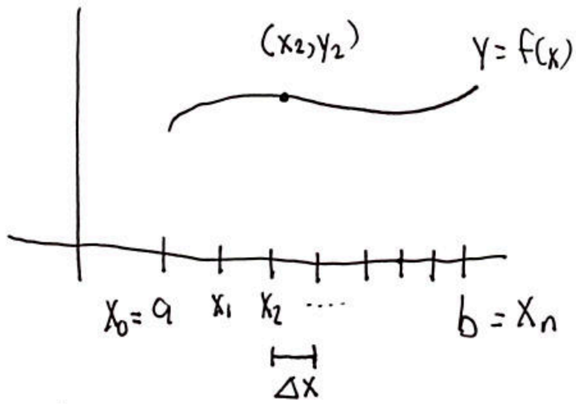
- Work, Average Value, Probability

- How to compute the average value of a function



- Average value of f over $[a,b]$

$$\approx \frac{y_1 + \cdots + y_n}{n} \quad \text{, where } y_1 = f(x_1),\ y_2 = f(x_2), \ldots$$

- The length Δx of each subinterval is $\Delta x = \frac{b-a}{n}$ (equal lengths)

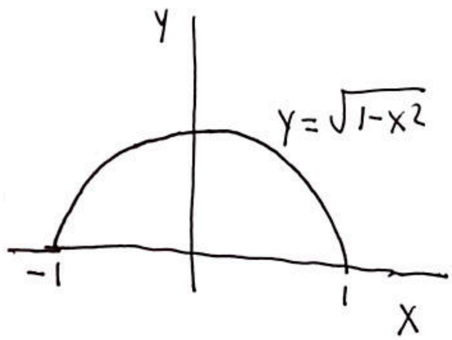
- $$\int_a^b f(x)\,dx = \text{limit of Riemann Sums} = \lim_{n\to\infty} (y_1 + \cdots + y_n)\underbrace{\left(\frac{b-a}{n}\right)}_{\Delta x}$$

- Thus: $$\underbrace{\lim_{n\to\infty} \frac{y_1 + \cdots + y_n}{n}}_{n\text{th average}} = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{"continuous average"}}$$

Ex: The average of a constant c is equal to c:

$$\frac{1}{b-a}\int_a^b 31\,dx = 31$$

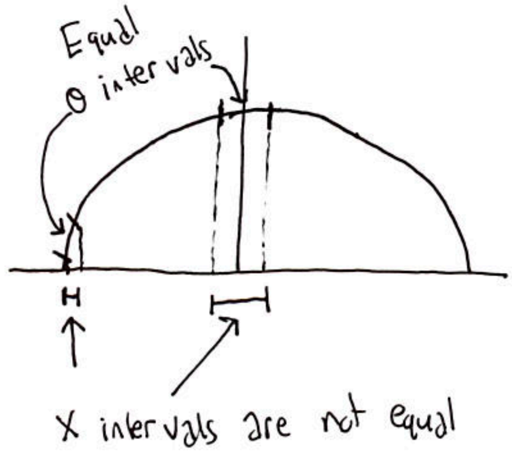
Ex:



The average height of $y=\sqrt{1-x^2}$ on the interval $-1\le x\le 1$

is $$\frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\times \text{Area of Semicircle} = \frac{\pi}{4}$$

Ex: Find the average height y on a semicircle with respect to arc length (use $d\theta$, not dx)



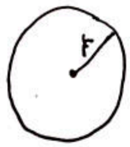
This is an average computed using a different weight than in the previous example

- Along the circle, $y = \sin\theta$, $0 \leq \theta \leq \pi$

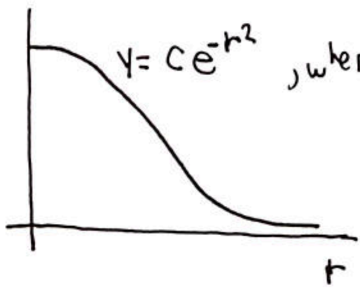
Thus, Average $= \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \sin\theta \, d\theta = \dfrac{1}{\pi} \left[-\cos\theta\right]_0^{\pi} = \dfrac{1}{\pi}\left[-\cos(\pi) + \cos(0)\right]$

$= \dfrac{2}{\pi}$

<u>Ex</u>: Dart board:

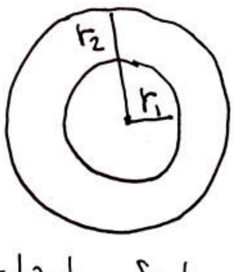


- You aim for the center, but your aim is not perfect.
- Let r denote the distance from the center
- Let's assume that your accuracy is "normally distributed":



- The number of hits within a given ring with $r_1 < r < r_2$

is $\quad c \displaystyle\int_{r_1}^{r_2} e^{-r^2} (2\pi r)\, dr$



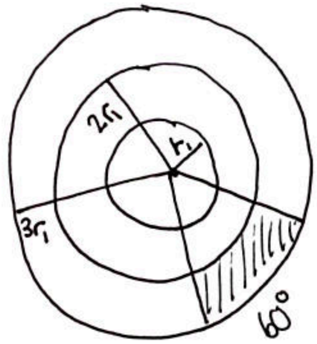
- c is a constant such that $c \displaystyle\int_0^{\infty} e^{-r^2} (2\pi r) dr = 1$

"The total probability is 1"

Above: • ce^{-r^2} = probability density of hitting some fixed point at a distance r from the center

- $2\pi r\, dr$ = area of a ring of width dr and radius r
- $ce^{-r^2}(2\pi r) dr$ = probability of hitting somewhere within the ring of width dr and radius r

Ex



shaded region makes up $\frac{1}{6}$ of the area in between $2r_1$ and $3r_1$

Let's find the probability of hitting the shaded region

- Probability $= \dfrac{\text{part}}{\text{whole}}$

$$= \frac{\frac{1}{6}\displaystyle\int_{2r_1}^{3r_1} c e^{-r^2}(2\pi r)\,dr}{\displaystyle\int_0^{\infty} c e^{-r^2}(2\pi r)\,dr}$$

- $\displaystyle\int_a^b r e^{-r^2}\,dr = -\frac{1}{2}e^{-r^2}\Big]_a^b = \frac{1}{2}\left(e^{-a^2} - e^{-b^2}\right)$

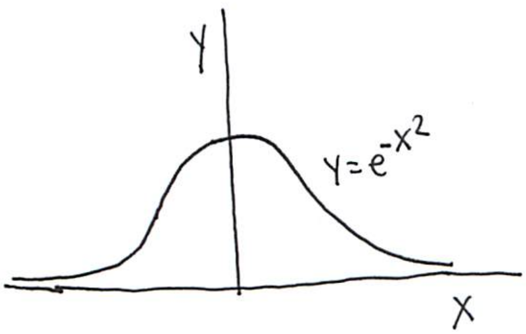
- $\displaystyle\int_0^{\infty} r e^{-r^2}\,dr = -\frac{1}{2}e^{-r^2}\Big]_0^{R\to\infty} = \underbrace{-\frac{1}{2}e^{-R^2}}_{\to 0 \text{ as } R\to\infty}\ {}^{R\to\infty} + \frac{1}{2}e^{-0^2} = \frac{1}{2}$

- Thus, Probability $= \dfrac{\frac{1}{6}\displaystyle\int_{2r_1}^{3r_1} e^{-r^2} r\,dr}{\displaystyle\int_0^{\infty} e^{-r^2} r\,dr} = \dfrac{1}{3}\displaystyle\int_{2r_1}^{3r_1} e^{-r^2} r\,dr = \dfrac{1}{6}\left(e^{-(2r_1)^2} - e^{-(3r_1)^2}\right)$

$$= \frac{1}{6}\left(e^{-4r_1^2} - e^{-9r_1^2}\right)$$

<u>Ex</u>: Compute $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$

- This integral represents the area under the curve $y = e^{-x^2}$ for $-\infty < x < \infty$:

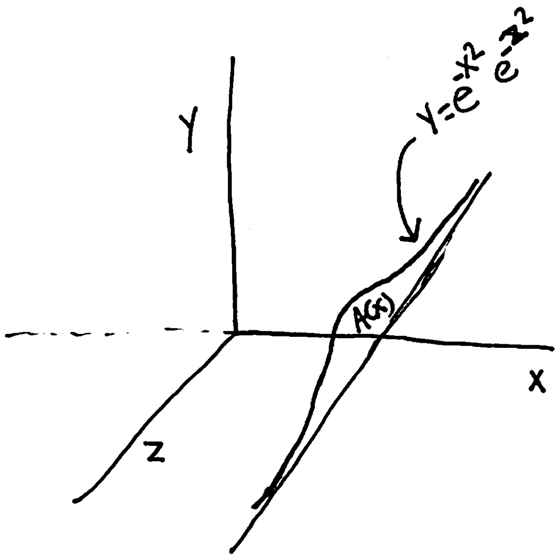


- This is one of the most important integrals in calculus

- Let's first revolve this graph about the y-axis and find the volume using the shell method:

$$V = \int_0^{\infty} \underbrace{e^{-x^2}}_{\text{shell height}} \underbrace{2\pi x}_{x = \text{shell radius}} \overbrace{dx}^{\text{shell thickness}} = -\pi\, e^{-x^2} \Big]_0^{R \to \infty}$$

$$= \pi \left[e^0 - e^{-R^2} \right]_{R \to \infty} = \pi$$



- Let's now find the volume of the same solid by using slices of constant x values.
- Let $A(x)$ = area of the slice
- dx = thickness of slice

- Use polar coordinates $r^2 = x^2 + z^2$ in the (x,z) plane
- $y = e^{-r^2} = e^{-(x^2+z^2)} = e^{-x^2} e^{-z^2}$

- Note that $A(x) = \displaystyle\int_{z=-\infty}^{\infty} e^{-r^2}\,dz = e^{-x^2} \int_{z=-\infty}^{\infty} e^{-z^2}\,dz = e^{-x^2} I$

It follows that

$$\pi \overset{\text{from before}}{=} V = \int_{-\infty}^{\infty} A(x)\,dx = \int_{-\infty}^{\infty} e^{-x^2} I\,dx = I \int_{-\infty}^{\infty} e^{-x^2}\,dx = I^2$$

- Thus, $I = \sqrt{\pi} = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx.$

- Equivalently:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\,dx = 1.$$

- Here is a rescaled version of the above formula (it can be derived by u substitution):

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\, dx = 1$$

- $\frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-x^2}{2\sigma^2}}$ is known as a "normal distribution" with standard deviation σ