• Differential Equations & Separation of Variables

• Goal: Given an equation of the form

(*) $\dfrac{dy}{dx} = F(x,y)$, Solve for y as a function of x. That is, find $y = f(x)$ so that eqn. (*) holds.

Ex: $\dfrac{dy}{dx} = g(x)$. Then $y = \int g(x)\,dx$.

We consider these types of equations as solved

Ex: $\left(\frac{d}{dx} + x\right)y = 0$ (equivalently: $\frac{dy}{dx} + xy = 0$).

- Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = -xy$.

The Key step is called Separation of variables:

- $$\frac{dy}{y} = -x\,dx$$

All y dependence on left

All x dependence on right

- Now take antiderivatives of both sides:
- $\int \frac{dy}{y} = -\int x\,dx$
- $\ln|y| = -\frac{x^2}{2} + c$
- $|y| = e^c \cdot e^{-x^2/2}$
- $y = a\,e^{-x^2/2}$ $\quad (a = \pm e^c)$

Remark: Even though $e^c \neq 0$, all possible choices of a (including 0) lead to a solution.

- In general:

  If $\dfrac{dy}{dx} = g(x)\,h(y)$, then

  - $\dfrac{dy}{h(y)} = g(x)\,dx$
  - Integrate both sides
  - $H(y) = G(x) + C$ (implicit formula for y)
    - $H(y) = \displaystyle\int \frac{dy}{h(y)}$ · $G(x) = \displaystyle\int g(x)\,dx$
  - $y = H^{-1}(G(x) + C)$, where H^{-1} is the inverse function of H

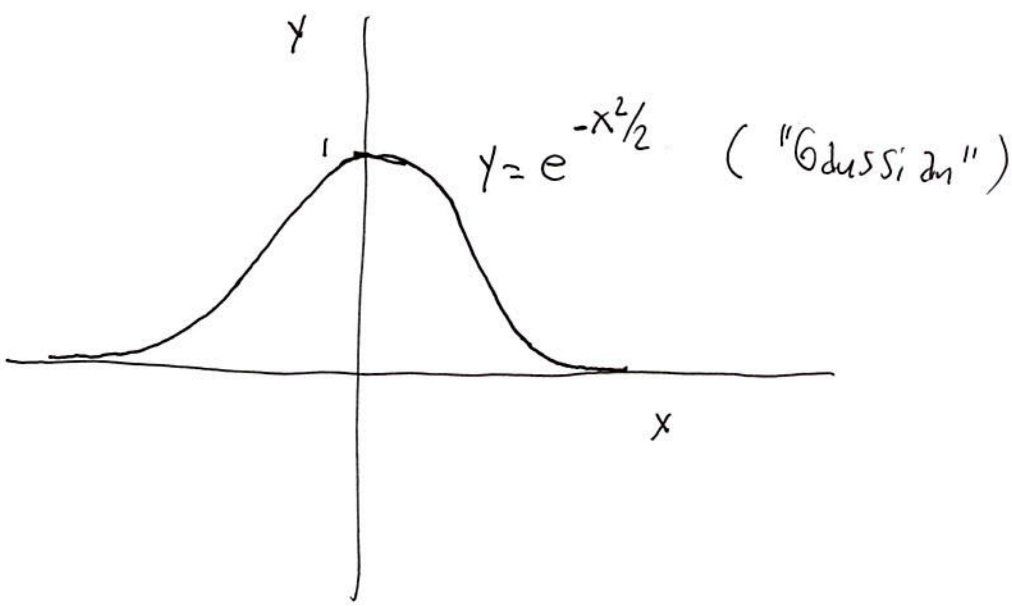
- In the previous example,
  - $g(x) = -x$ $\quad h(y) = y$
  - $G(x) = -\dfrac{x^2}{2}$ $\quad H(y) = \displaystyle\int \frac{dy}{y} = \ln|y|$

- The solution can be thought of as depending on its <u>initial condition</u>.
- For example, if $y(0) = 1$, then $y = e^{-x^2/2}$.

  Initial condition

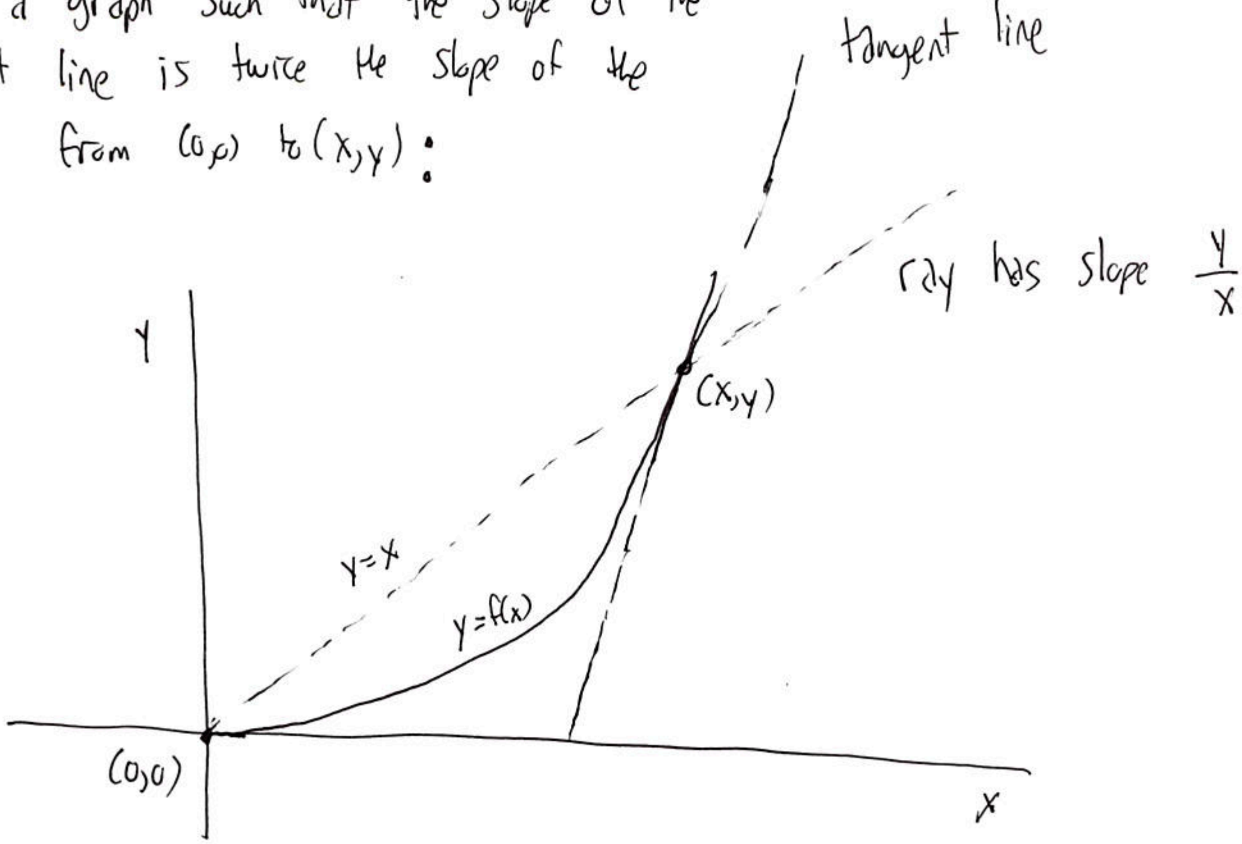
- If $y(0) = a$, then $y = a e^{-x^2/2}$

- Graph of $y = e^{-x^2/2}$:



Ex (Geometric)

- Find a graph such that the slope of the tangent line is twice the slope of the ray from $(0,0)$ to (x,y):



- Translate the problem into a differential equation: $\frac{dy}{dx} = 2\left(\frac{y}{x}\right)$
- Now solve:
- $\frac{dy}{y} = \frac{2dx}{x}$ (separation of variables)
- $\ln|y| = 2\ln|x| + C$ (antiderivatives)
- $|y| = e^c x^2$ (exponentiate and recall that $e^{2\ln|x|} = x^2$)
- $y = a x^2$

- $a<0$, $a>0$, $a=0$ are all possible solutions.

Examples:

- $y = x^2$ $(a=1)$
- $y = 2x^2$ $(a=2)$
- $y = -x^2$ $(a=-1)$
- $y = 0x^2 = 0$ $(a=0)$
- $y = -2x^2$ $(a=-2)$
- $y = 100x^2$ $(a=100)$

Ex: Find the curves that are perpendicular to the parablas from the previous example.

- Recall, if two lines respectively have slopes m_1 and m_2, then they are perpendicular if and only if

$$m_2 = -\frac{1}{m_1}$$

- Thus, $$\frac{dy}{dx} = -\frac{1}{\text{slope of parabola}} = -\frac{x}{2y}$$

- Solve:

- $y\,dy = -\frac{x}{2}\,dx$ (separation of variables)
- $\frac{y^2}{2} = -\frac{x^2}{4} + c$ (antiderivatives)
- $\frac{x^2}{4} + \frac{y^2}{2} = c$, a family of ellipses
  - $y = \pm\sqrt{2\left(c - \frac{x^2}{4}\right)}$
  - Ratio of x-semi-major axis to y-semi-minor axis is $\sqrt{2}$