MATH 18.01, FALL 2017 - PROBLEM SET # 4

Professor: Jared Speck

Due: by Friday 1:45pm on 10-13-17
(in the boxes outside of Room 4-174; write your name, recitation instructor, and recitation meeting days/time on your homework)

18.01 Supplementary Notes (including Exercises and Solutions) are available on the course web page: `http://math.mit.edu/~jspeck/18.01_Fall%202017/1801_CourseWebsite.html`. This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

Part I consists of exercises given and solved in the Supplementary Notes. It will be graded quickly, checking that all is there and the solutions not copied.

Part II consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises given here because they do not fit conveniently into an exam or short-answer format. See the guidelines below for what collaboration is acceptable, and follow them.

To encourage you to keep up with the lectures, both Part I and Part II tell you for each problem on which day you will have the needed background for it.

You are encouraged to use graphing calculators, software, etc. to <u>check</u> your answers and to explore calculus. However, (unless otherwise indicated) we strongly discourage you from using these tools to <u>solve</u> problems, perform computations, graph functions, etc. An extremely important aspect of learning calculus is developing these skills. You will not be allowed to use any such tools on the exams.

Part I (5 points)

Notation: The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Lecture 11. (Thurs., Oct. 5) Newton's method.
Read: 4.6, (4.7 is optional).

Lecture 12. (Fr., Oct. 6) Mean-value theorem. Inequalities.
Read: 2.6 to middle of p. 77; Notes MVT.
Homework: Notes 2G: 1b, 2b, 5, 6.

Part II (30 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don't understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. *You must show your work; "bare" solutions will receive very little credit.*

iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

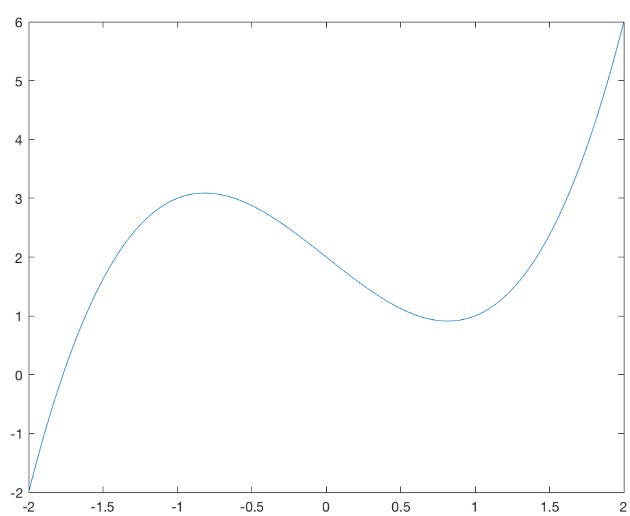
iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation." This includes visits outside recitation to your recitation instructor. If you don't know a name, you must nevertheless identify the person, as in, "tutor in Room 2-106," or "the student next to me in recitation." Optional: note which of these people or resources, if any, were particularly helpful to you.

This "Problem **0**" will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (Oct. 5; Newton's method; $1 + 2 + 2 = 5$ points) This problem will show you why it is important to choose a good starting point x_0 when you are applying Newton's method. Consider the function $f(x) = x^3 - 2x + 2$.

a) Plot the graph of $f(x)$. The graph will help you make sense of the rest of the problem.



b) Use Newton's method with the base point $x_0 = 0$ in an effort to locate a zero of $f(x)$. Compute the first 4 iterates an explain what is happening.

Newton's method gives $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 2x_k + 2}{3x_k^2 - 2}$. With $x_0 = 0$ we compute that $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 0$. Inductively, $x_{2k-1} = 1$ and $x_{2k} = 0$ where $k = 0, 1, 2, \cdots$.

c) Using a calculator, compute the first 4 Newton iterates starting with a base point x_0 near 0, but not equal to 0. The repeat your calculations for a slightly different choice of x_0 (but still near 0). Explain how the behavior of the iterates compares to the behavior you observed in part b). In particular, do the iterates converge to the solution x to $f(x) = 0$?

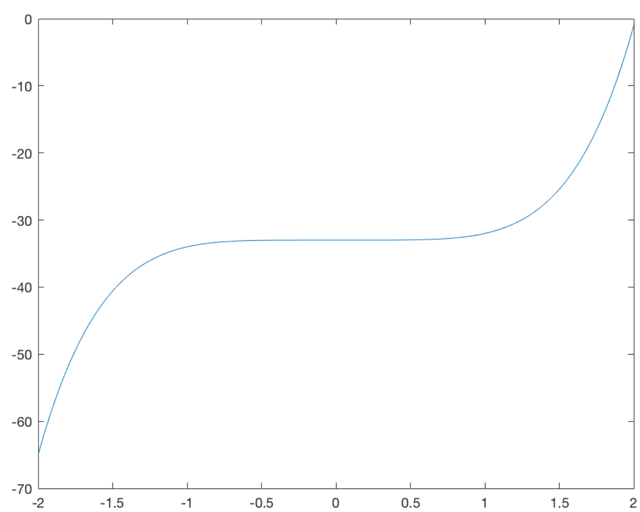
x_0	0.1
x_1	1.01421
x_2	0.079639
x_3	1.00909
x_4	0.0521785

x_0	-0.1
x_1	1.01624
x_2	0.0901732
x_3	1.01161
x_4	0.0658576

In above iterates, x_{2k-1} is converging to 1 and x_{2k} is converging to 0. As neither $x = 0$ nor $x = 1$ yields $f(x) = 0$, the iterates do not converge to the solution x to $f(x) = 0$.

2. (Oct. 5; Newton's method; 1 + 1 + 2 + 2 = 6 points)

a) Sketch the graph of the function $f(x) = x^5 - 33$. The graph will help you make sense of the rest of the problem.



b) With the help of a computer, compute $33^{1/5}$ to 15 significant figures by using Newton's method starting with $x_0 = 2$. How many steps does it take to achieve this degree of accuracy?

Newton's method gives $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^5 - 33}{5x_k^4}$.

x_k	15 significant figures
x_0	2.00000000000000
x_1	2.01250000000000
x_2	2.01234664046397
x_3	2.01234661708556
x_4	2.01234661708556

It takes 3 iterates to compute $33^{1/5}$ to 15 significant figures.

c) For each of x_0, x_1, x_2, x_3, indicate whether x_k is larger or smaller than $33^{1/5}$. Then indicate whether x_k is larger or smaller than x_{k-1}. For the points x_k that are bigger than $33^{1/5}$, explain how your answers are connected to graph you sketched in part a).

- $x_0 < 33^{1/5}$, $x_1 > 33^{1/5}$, $x_2 > 33^{1/5}$, $x_3 > 33^{1/5}$. These can be checked by computing x_k^5 using the approximate values of x_k from (b); for instance, $x_2^5 \approx (2.01234664046397)^5 = 33.0000019\cdots > 33$.
- $x_1 > x_0$, $x_2 < x_1$, $x_3 < x_2$.

Since $x_0^5 = 32 < 33$, x_0 lies on the left of $33^{1/5}$. Noting that the graph of the function $f(x) = x^5 - 33$ is concave up when $x > 0$, the tangent line to the graph of $f(x)$ at $(x_0, f(x_0))$ has x-intercept greater than $33^{1/5}$. This is why $x_1 > 33^{1/5}$. When $x > 33^{1/5}$, observe that $f(x) > 0$, f is increasing, and the graph of $f(x)$ is concave up. Therefore the tangent line to the graph of $f(x)$ at $(x_1, f(x_1))$ has x-intercept between $33^{1/5}$ and x_1, which is x_2; similar for the tangent line at $(x_2, f(x_2))$.

d) Find a quadratic approximation to $33^{1/5}$ and estimate the error between your quadratic approximation and the precise value of $33^{1/5}$. How many digits of accuracy does your quadratic approximation achieve? Remark: when constructing the quadratic approximation, it is helpful to note that $33^{1/5} = \{32(1 + 1/32)\}^{1/5}$.

Recall the quadratic approximation $(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3)$ when x is small. Letting $p = 1/5$ and $x = 1/32$, we see that $(1+1/32)^{1/5} \approx 1 + \frac{1}{5\cdot 32} + \frac{1}{2} \cdot \frac{1}{5} \cdot (-\frac{4}{5}) \cdot \frac{1}{32^2} = 1.006171875$; error is on the scale of $\frac{1}{32^3} = 0.000030517578125$. Now note that $33^{1/5} = \{32(1+1/32)\}^{1/5} = 2 \cdot (1+1/32)^{1/5} \approx 2 \times 1.006171875 = 2.01234375$; error is on the scale of $2 \cdot \frac{1}{32^3} = 0.00006\cdots$. We can say the first 4 digits are correct, and $33^{1/5} \approx 2.012$.

3. (Oct. 6; Mean value theorem; 3 + 3 = 6 points)

a) Decide whether or not there is a differentiable function $f(x)$ with the following properties: $f(0) = -1$, $f(2) = 4$, and $f'(x) \le 2$ for all x. Explain your answer.

Suppose that such a function $f(x)$ exists. By the mean value theorem, there exists x_0 such that $f'(x_0) = \frac{f(2)-f(0)}{2-0} = \frac{5}{2}$. This contradicts the assumption that $f'(x) \le 2$ for all x. Therefore there cannot exist such a function $f(x)$.

b) Let $f(x)$ be a differentiable function. A point a is said to be a *fixed point* if $f(a) = a$. Suppose that there is no point x such that $f'(x) = 1$. Show that there exists at most one fixed point.

Suppose there are at least two fixed points $x_1 \neq x_2$. We may assume $x_1 < x_2$ without loss of generality. By the mean value theorem, there exists x_3 such that $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{x_2-x_1}{x_2-x_1} = 1$.

This contradicts the assumption that there is no point x such that $f'(x) = 1$. Therefore there can be at most one fixed point.

4. (Oct. 6; Mean value theorem; 2 + 2 + 2 + 2 + 2 = 10 points)

a) Let $f(x)$ be a differentiable function. Use the mean value theorem to show that if $f(0) = 0$ and $f'(x) \geq 0$, then $f(x) \geq 0$ for all $x \geq 0$.

Suppose to the contrary that $f(x_0) < 0$ for some $x_0 > 0$. By the mean value theorem, there exists x_1 such that $0 \leq x_1 \leq x_0$ and $f'(x_1) = \frac{f(x_0)-f(0)}{x_0-0} = \frac{f(x_0)}{x_0} < 0$. This contradicts the assumption that $f'(x) \geq 0$ for all x. Therefore, $f(x) \geq 0$ for all $x \geq 0$.

b) Using part a), show that $\ln(1 + x) \leq x$ for $x \geq 0$. *Hint: Use* $f(x) = x - \ln(1 + x)$.

Let $f(x) = x - \ln(1+x)$. Then $f(0) = 0$ and $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \geq 0$ when $x \geq 0$. By part a), it follows that $f(x) \geq 0$ for all $x \geq 0$, or equivalently, $\ln(1 + x) \leq x$.

c) Use the same method as in b) to show that $\ln(1+x) \geq x - x^2/2$ and $\ln(1+x) \leq x - x^2/2 + x^3/3$ for $x \geq 0$.

Let $g(x) = \ln(1 + x) - x + x^2/2$. Then $g(0) = 0$ and $g'(x) = 1/(1 + x) - 1 + x = x^2/(1 + x) \geq 0$ when $x \geq 0$. By part a), $g(x) \geq 0$ for all $x \geq 0$, or equivalently, $\ln(1 + x) \geq x - x^2/2$.

Let $h(x) = x - x^2/2 + x^3/3 - \ln(1+x)$. Then $h(0) = 0$ and $h'(x) = 1 - x + x^2 - 1/(1+x) = x^3/(1+x) \geq 0$ when $x \geq 0$. By part a), $h(x) \geq 0$ for all $x \geq 0$, or equivalently, $\ln(1 + x) \leq x - x^2/2 + x^3/3$.

d) Find the pattern in b) and c) and make a general conjecture; you do not have to prove your conjecture.

Conjecture: for any natural number k, $\ln(1 + x) \geq x - x^2/2 + x^3/3 - x^4/4 + \cdots - x^{2k}/2k$ and $\ln(1 + x) \leq x - x^2/2 + x^3/3 - x^4/4 + \cdots + x^{2k-1}/(2k - 1)$ for all $x \geq 0$.

e) Show that $\ln(1 + x) \leq x$ for $-1 < x \leq 0$. (Use the change of variable $u = -x$.)

Let $u = -x$. We need to show that $\ln(1 - u) \leq -u$ for $0 \leq u < 1$. Let $f(u) = -u - \ln(1 - u)$. Then $f(0) = 0$ and $f'(u) = -1 + 1/(1 - u) = u/(1 - u) \geq 0$ when $0 \leq u < 1$. By the same argument as in part a), $f(u) \geq 0$ for all $0 \leq u < 1$. Equivalently, $\ln(1 + x) \leq x$ for $-1 < x \leq 0$.