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Disciplina: Cálculo III

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Exercício 3: Derivadas parciais de 1ª e 2ª ordem

Derivadas parciais de outras ordens

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54) $f(x,y) = \sin^2(mx+ny)$

$$\frac{\partial^2}{\partial x^2}\sin^2(mx+ny) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\sin(mx+ny))^2\right\}$$

Sendo $u = \sin(mx+ny)$; pela regra da cadeia:

$$= \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial u}u^2 \frac{\partial}{\partial x}\sin(mx+ny)\right\} = \frac{\partial}{\partial x}\left\{2u \frac{\partial}{\partial x}\sin(mx+ny)\right\}$$

Sendo $v = mx+ny$; pela regra da cadeia:

$$= \frac{\partial}{\partial x}\left\{2u\left[\frac{\partial}{\partial v}\sin(v)\frac{\partial}{\partial x}(\cancelto{m}{mx} + \cancelto{0}{ny})\right]\right\}$$

$$= \frac{\partial}{\partial x}\left\{2u\left(m\cos(v)\right)\right\}$$

$$= \frac{\partial}{\partial x}\left\{2m\sin(mx+ny)\cos(mx+ny)\right\}$$

Nesse ponto já sabemos que $\frac{\partial}{\partial x}\sin^2(mx+ny)$ é igual à $2m\sin(mx+ny)\cos(mx+ny)$. Continuando:

$$= 2m\frac{\partial}{\partial x}\sin(mx+ny)\cos(mx+ny)$$

$$= 2m\left\{\cos(mx+ny)\frac{\partial}{\partial x}\sin(mx+ny) + \sin(mx+ny)\frac{\partial}{\partial x}\cos(mx+ny)\right\}$$

Sendo $u = mx+ny$, pela regra da cadeia:

$$= 2m\left\{\cos(u)\left[\frac{\partial}{\partial u}\sin(u)\,\frac{\partial}{\partial x}\left(\cancelto{m}{mx}+\cancelto{0}{ny}\right)\right]+\sin(u)\left[\frac{\partial}{\partial u}\cos(u)\,\frac{\partial}{\partial x}\left(\cancelto{m}{mx}+\cancelto{0}{ny}\right)\right]\right\}$$

$$= 2m\left\{\cos(u)\left(m\cos(u)\right)+\sin(u)\left(-\sin(u)m\right)\right\}$$

$$= 2m\left\{m\cos^2(u) - m\sin^2(u)\right\}$$

$$= 2m^2\left(\cos^2(mx+ny) - \sin^2(mx+ny)\right)$$

Portanto, já sabemos que:

$$\boxed{\frac{\partial}{\partial x}\sin^2(mx+ny) = 2m\sin(mx+ny)\cos(mx+ny)}$$

$$\boxed{\frac{\partial^2}{\partial x^2}\sin^2(mx+ny) = 2m^2\left(\cos^2(mx+ny) - \sin^2(mx+ny)\right)}$$

Agora o cálculo de:

$$\frac{\partial^2}{\partial y\partial x}\sin^2(mx+ny) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\sin^2(mx+ny)\right\}$$

$$= \frac{\partial}{\partial y}\left\{2m\sin(mx+ny)\cos(mx+ny)\right\}$$

$$= 2m\left\{\frac{\partial}{\partial y}\sin(mx+ny)\cos(mx+ny)\right\}$$

Pela regra do produto:

$$= 2m\left\{\cos(mx+ny)\frac{\partial}{\partial y}\sin(mx+ny) + \sin(mx+ny)\frac{\partial}{\partial y}\cos(mx+ny)\right\}$$

Sendo $u = mx+ny$, pela regra da cadeia:

$$= 2m\left\{\cos(u)\left[\frac{\partial}{\partial u}\sin(u)\frac{\partial}{\partial y}(\overset{0}{\cancel{mx}}+\overset{n}{\cancel{ny}})\right]+\sin(u)\left[\frac{\partial}{\partial u}\cos(u)\frac{\partial}{\partial y}(\overset{0}{\cancel{mx}}+\overset{n}{\cancel{ny}})\right]\right\}$$

$$= 2m\left\{\cos(u)[n\cos(u)] + \sin(u)[-n\sin(u)]\right\}$$

$$= 2m\left\{n\cos^2(u) - n\sin^2(u)\right\}$$

$$= 2mn\left(\cos^2(mx+ny) - \sin^2(mx+ny)\right)$$

Portanto:

$$\boxed{\frac{\partial^2}{\partial y\partial x}\sin^2(mx+ny) = 2mn\left(\cos^2(mx+ny)-\sin^2(mx+ny)\right)}$$

Agora o cálculo de:

$$\frac{\partial^2}{\partial y^2}\sin^2(mx+ny) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}(\sin(mx+ny))^2\right\}$$

Sendo $u = \sin(mx+ny)$, pela regra da cadeia:

$$= \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial u}u^2\frac{\partial}{\partial y}\sin(mx+ny)\right\}$$ Sendo $v = mx+ny$:

$$= \frac{\partial}{\partial y}\left\{2u\left[\frac{\partial}{\partial v}\sin(v)\frac{\partial}{\partial y}(\overset{0}{\cancel{mx}}+\overset{n}{\cancel{ny}})\right]\right\}$$

$$= \frac{\partial}{\partial y}\left\{ 2u\,(n\cos(v)) \right\}$$

$$= \frac{\partial}{\partial y}\left\{ 2n\sin(mx+ny)\cos(mx+ny) \right\}$$

Portanto, já sabemos que:

$$\boxed{\frac{\partial}{\partial y}\sin^2(mx+ny) = 2n\sin(mx+ny)\cos(mx+ny)}$$

Continuando o cálculo:

$$= \frac{\partial}{\partial y}\, 2n\sin(mx+ny)\cos(mx+ny)$$

$$= 2n\frac{\partial}{\partial y}\sin(mx+ny)\cos(mx+ny)$$

Pela regra do produto:

$$= 2n\left\{ \cos(mx+ny)\frac{\partial}{\partial y}\sin(mx+ny) + \sin(mx+ny)\frac{\partial}{\partial y}\cos(mx+ny) \right\}$$

Sendo $u = mx+ny$, pela regra da cadeia:

$$= 2n\left\{ \cos(u)\left[\frac{\partial}{\partial u}\sin(u)\,\frac{\partial}{\partial y}(\overset{0}{\cancel{mx}}+\overset{n}{\cancel{ny}})\right] + \sin(u)\left[\frac{\partial}{\partial u}\cos(u)\,\frac{\partial}{\partial y}(\overset{0}{\cancel{mx}}+\overset{n}{\cancel{ny}})\right] \right\}$$

$$= 2n\left\{ \cos(u)\left[n\cos(u)\right] + \sin(u)\left[-n\sin(u)\right] \right\}$$

$$= 2n\left(m\cos^2(u) - n\sin^2(u)\right)$$

$$= 2n^2\left[\cos^2(mx+ny) - \sin^2(mx+ny)\right]$$

Portanto, temos que:

$$\boxed{\frac{\partial^2}{\partial y^2}\sin^2(mx+ny) = 2n^2\left[\cos^2(mx+ny) - \sin^2(mx+ny)\right]}$$

A última derivada a ser calculada é:

$$\frac{\partial^2}{\partial x\,\partial y}\sin^2(mx+ny) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\sin^2(mx+ny)\right\}$$

$$= \frac{\partial}{\partial x}\left\{2n\sin(mx+ny)\cos(mx+ny)\right\}$$

$$= 2n\frac{\partial}{\partial x}\left\{\sin(mx+ny)\cos(mx+ny)\right\}$$

Pela regra do produto:

$$= 2n\left\{\cos(mx+ny)\frac{\partial}{\partial x}\sin(mx+ny) + \sin(mx+y)\frac{\partial}{\partial x}\cos(mx+ny)\right\}$$

Sendo $u = mx+ny$, pela regra da cadeia:

$$= 2n\left\{\cos(u)\left[\frac{\partial}{\partial u}\sin(u)\frac{\partial}{\partial x}(\overset{m}{\cancel{mx}}+\overset{0}{\cancel{ny}})\right] + \sin(u)\left[\frac{\partial}{\partial u}\cos(u)\frac{\partial}{\partial x}(\overset{m}{\cancel{mx}}+\overset{0}{\cancel{ny}})\right]\right\}$$

$$= 2n\left\{\cos(u)\left[m\cos(u)\right] + \sin(u)\left[-m\sin(u)\right]\right\}$$

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$$= 2n\{m\cos^2(u) - m\sin^2(u)\}$$

$$= 2mn\left[\cos^2(mx+ny) - \sin^2(mx+ny)\right]$$

Portanto:

$$\boxed{\frac{\partial}{\partial x\partial y}\sin^2(mx+ny) = 2mn\left[\cos^2(mx+ny) - \sin^2(mx+ny)\right]}$$

Fim da questão 54.

Obs: talvez seja melhor calcular primeiro as derivadas de primeira ordem e, só depois, as de segunda ordem, para organizar melhor as contas.

Obs. 2: a notação no padrão da Wolfram Language (Mathematica e Wolfram Alpha) me ajudou a compreender e subdividir os cálculos, ao custo de grandes contas no papel. Custo/benefício?

55) $w = \sqrt{u^2 + v^2}$

Cálculo das derivadas de primeira ordem:

$$\frac{\partial}{\partial u}\sqrt{u^2+v^2} = \frac{\partial}{\partial u}(u^2+v^2)^{1/2}$$

Sendo $h = u^2+v^2$, pela regra da cadeia:

$$\frac{\partial}{\partial h} h^{1/2} \frac{\partial}{\partial u}(u^2+\cancel{v^2}^{0}) = \frac{1}{\cancel{2}\sqrt{h}}(\cancel{2}u) = \boxed{\frac{u}{\sqrt{u^2+v^2}}}$$

$\frac{\partial}{\partial v}\sqrt{u^2+v^2}$ = por semelhança com $\frac{\partial}{\partial u}$, sabemos que o resultado será $\boxed{\frac{v}{\sqrt{u^2+v^2}}}$

Cálculo das derivadas de segunda ordem:

$$\frac{\partial^2}{\partial u^2}\sqrt{u^2+v^2} = \frac{\partial}{\partial u}\left\{\frac{\partial}{\partial u}\sqrt{u^2+v^2}\right\} = \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2}}\right)$$

$$= \frac{\partial}{\partial u} u(u^2+v^2)^{-1/2}$$

Pela regra do produto:

$$= (u^2+v^2)^{-1/2}\cancel{\frac{\partial}{\partial u}}^{1} + u\frac{\partial}{\partial u}(u^2+v^2)^{-1/2}$$

Sendo $h = u^2+v^2$, pela regra da cadeia:

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$$= \frac{1}{\sqrt{u^2+v^2}} + u\left[\frac{\partial h^{-1/2}}{\partial h}\frac{\partial(u^2+\cancel{v^2}^{\,0})}{\partial u}\right]$$

$$= \frac{1}{\sqrt{u^2+v^2}} + u\left[-\frac{1}{\cancel{2}} h^{-3/2} \cdot \cancel{2}u\right]$$

$$= \frac{1}{\sqrt{u^2+v^2}} + u\left[\frac{-u}{\sqrt{h^3}}\right]$$

$$= \boxed{\frac{1}{\sqrt{u^2+v^2}} - \frac{u^2}{\sqrt{(u^2+v^2)^3}}}$$

$\frac{\partial^2 \sqrt{u^2+v^2}}{\partial v^2}$ = por semelhança com $\frac{\partial^2}{\partial u^2}$, sabemos

que o resultado é: $\boxed{\frac{1}{\sqrt{u^2+v^2}} - \frac{v^2}{\sqrt{(u^2+v^2)^3}}}$

Cálculo de:

$$\frac{\partial^2 \sqrt{u^2+v^2}}{\partial u \partial v} = \frac{\partial}{\partial u}\left\{\frac{\partial \sqrt{u^2+v^2}}{\partial v}\right\} = \frac{\partial}{\partial u}\left\{\frac{v}{\sqrt{u^2+v^2}}\right\}$$

$$= \frac{\partial}{\partial u}\left\{v(u^2+v^2)^{-1/2}\right\}$$

Pela regra do produto:

$$= (u^2+v^2)^{-1/2}\cancel{\frac{\partial(v)}{\partial u}}^{\,0} + v\frac{\partial}{\partial u}(u^2+v^2)^{-1/2}$$

$$= v\frac{\partial}{\partial u}(u^2+v^2)^{-1/2}$$

Sendo $h = u^2 - v^2$, pela regra da cadeia:

$$= v \left[\frac{\partial h^{-3/2}}{\partial h} \frac{\partial}{\partial u}(u^2 + \cancel{v^2}^{\,0}) \right]$$

$$= v \left[-\frac{1}{\cancel{2}} \cdot h^{-3/2} \cdot \cancel{2}u \right] = v \left[\frac{-u}{\sqrt{(u^2+v^2)^3}} \right]$$

$$= \boxed{\frac{-uv}{\sqrt{(u^2+v^2)^3}}}$$

$\frac{\partial^2}{\partial v \partial u}\sqrt{u^2+v^2} =$ por semelhança com $\frac{\partial^2}{\partial u \partial v}$, sabemos que o resultado será $\boxed{\frac{-uv}{\sqrt{(u^2+v^2)^3}}}$

Fim da questão 55.

Obs.: até agora foi possível deduzir o resultado de $\frac{\partial^2}{\partial a \partial b}$ a partir de $\frac{\partial^2}{\partial b \partial a}$, pois as funções eram contínuas e a comutatividade da adição permitia. De qualquer forma, tomar cuidado com isso!

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$62)\ u = \ln(x+2y)$

Cálculo das derivadas de primeira ordem:

$$\frac{\partial}{\partial x}\ln(x+2y) = \frac{\partial}{\partial u}\ln(u)\,\frac{\partial}{\partial x}(x+\cancel{2y}^{\,0})$$

$u = x+2y$
e regra cadeia

$$= \frac{1}{u} = \boxed{\frac{1}{x+2y}}$$

$$\frac{\partial}{\partial y}\ln(x+2y) = \frac{\partial}{\partial u}\ln(u)\,\frac{\partial}{\partial y}(\cancel{x}^{\,0}+2y)$$

$u = x+2y$
e regra da cadeia

$$= \boxed{\frac{2}{x+2y}}$$

Agora a verificação do Teorema de Clairaut:

$$\frac{\partial^2}{\partial x\partial y}\ln(x+2y) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\ln(x+2y)\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{2}{x+2y}\right) = 2\left(\frac{\partial}{\partial x}(x+2y)^{-1}\right)$$

Sendo $u = x+2y$, pela regra da cadeia:

$$= 2\left(\frac{\partial}{\partial u}u^{-1}\,\frac{\partial}{\partial x}(\cancel{x}^{\,1}+\cancel{2y}^{\,0})\right) = 2\left(\frac{-1}{u^2}\right) =$$

$$= \boxed{\frac{-2}{(x+2y)^2}}$$

$$\frac{\partial^2}{\partial y \partial x} \ln(x+2y) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x} \ln(x+2y)\right\}$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{x+2y}\right) = \frac{\partial}{\partial y}(x+2y)^{-1}$$

Sendo $u = x+2y$, pela regra da cadeia:

$$= \frac{\partial u^{-1}}{\partial u} \cdot \frac{\partial}{\partial y}(\cancel{x}^{0} + 2\cancel{y}^{2})$$

$$= \frac{-1}{u^2}(2) = \boxed{\frac{-2}{(x+2y)^2}}$$

Portanto, $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$.

Fim da questão 62.
Sem outras observações.

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60) $u = e^{xy}\sin(y)$

Cálculo das derivadas de primeira ordem:

$$\frac{\partial}{\partial x} e^{xy}\sin(y) =$$

Pela regra do produto:

$$= \sin(y)\frac{\partial}{\partial x} e^{xy} + e^{xy}\cancel{\frac{\partial}{\partial x}\sin(y)}^{\,0}$$

Sendo $u = xy$, pela regra da cadeia:

$$= \sin(y)\left[\frac{\partial}{\partial u} e^{u} \frac{\partial}{\partial x} xy\right]$$

$$= \sin(y)\left[e^{u} \cdot y\right] = \boxed{e^{xy}\sin(y)y}$$

$$\frac{\partial}{\partial y} e^{xy}\sin(y) =$$

Pela regra do produto:

$$= \sin(y)\frac{\partial}{\partial y} e^{xy} + e^{xy}\cancel{\frac{\partial}{\partial y}\sin(y)}^{\,\cos(y)}$$

$$= e^{xy}\cos(y) + \sin(y)\left[\frac{\partial}{\partial u} e^{u} \cancel{\frac{\partial}{\partial y} xy}^{\,x}\right]$$ $u = xy$ regra da cadeia

$$= e^{xy}\cos(y) + e^{xy}\sin(y)x = \boxed{e^{xy}(\cos(y) + x\sin(y))}$$

Agora a verificação do teorema de Clairaut:

$$\frac{\partial^2}{\partial x \partial y} e^{xy}\sin(y) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y} e^{xy}\sin(y)\right\} = \frac{\partial}{\partial x}\left\{e^{xy}(\cos(y)+x\sin(y))\right\}$$

Pela regra do produto:

$$= [\cos(y)+x\sin(y)]\frac{\partial}{\partial x}e^{xy} + e^{xy}\frac{\partial}{\partial x}[\overset{0}{\cancel{\cos(y)}} + \overset{\sin(y)}{\cancel{x\sin(y)}}]$$

Sendo $u = xy$, pela regra da cadeia:

$$= [\cos(y)+x\sin(y)]\left(\overset{e^u}{\cancel{\frac{\partial}{\partial u}e^u}}\ \overset{y}{\cancel{\frac{\partial}{\partial x}xy}}\right) + e^{xy}\sin(y)$$

$$= [\cos(y)+x\sin(y)](e^{xy}y) + e^{xy}\sin(y)$$

$$= e^{xy}\cos(y)y + e^{xy}\sin(y)xy + e^{xy}\sin(y)$$

$$= \boxed{e^{xy}(y\cos(y) + xy\sin(y) + \sin(y))}$$

$$\frac{\partial^2}{\partial y \partial x} e^{xy}\sin(y) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x} e^{xy}\sin(y)\right\} = \frac{\partial}{\partial y}\left\{e^{xy}\sin(y)y\right\}$$

Pela regra do produto:

$$= [\sin(y)y]\frac{\partial}{\partial y}e^{xy} + e^{xy}\frac{\partial}{\partial y}(\sin(y)y)$$

Pela regra da cadeia, $u = xy$:

$$= [\sin(y)y]\left(\overset{e^u}{\cancel{\frac{\partial}{\partial y}e^u}}\ \overset{x}{\cancel{\frac{\partial}{\partial y}xy}}\right) + e^{xy}\frac{\partial}{\partial y}(\sin(y)y)$$

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$$= e^{xy}xy\sin(y) + e^{xy}\frac{\partial(\sin(y)y)}{\partial y}$$

Pela regra do produto:

$$= e^{xy}xy\sin(y) + e^{xy}\left[y\frac{\partial}{\partial y}\sin(y) + \sin(y)\frac{\partial}{\partial y}y\right]$$

$$= e^{xy}xy\sin(y) + e^{xy}\left[y\cos(y) + \sin(y)\right]$$

$$= e^{xy}xy\sin(y) + e^{xy}y\cos(y) + e^{xy}\sin(y)$$

$$= \boxed{e^{xy}(y\cos(y) + xy\sin(y) + \sin(y))}$$

Portanto: $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$

Fim da questão 60.
Sem outras observações.

64) $f(x,y) = \sin(2x+5y)$. $f_{yxy} = ?$

Cálculo de f_y:

$$\frac{\partial}{\partial y}\sin(2x+5y) =$$

Sendo $u = 2x+5y$, pela regra da cadeia:

$$= \frac{\partial}{\partial u}\cancel{\sin(u)}^{\cos(u)} \cdot \frac{\partial}{\partial y}(\cancel{2x}^{0} + \cancel{5y}^{5})$$

$$= \boxed{5\cos(2x+5y)}$$

Cálculo de f_{yx}:

$$\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\sin(2x+5y)\right\} = \frac{\partial}{\partial x}\left\{5\cos(2x+5y)\right\}$$

$$= 5\frac{\partial}{\partial x}\cos(2x+5y)$$

Sendo $u = 2x+5y$, pela regra da cadeia:

$$= 5\left[\frac{\partial}{\partial u}\cancel{\cos(u)}^{-\sin(u)} \frac{\partial}{\partial x}(\cancel{2x}^{2} + \cancel{5y}^{0})\right]$$

$$= \boxed{-10\sin(2x+5y)}$$

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Cálculo de f_{yxy}:

$$\frac{\partial}{\partial y}\left\{\frac{\partial^2}{\partial x \partial y}\sin(2x+5y)\right\} = \frac{\partial}{\partial y}\left\{-10\sin(2x+5y)\right\}$$

$$= -10\frac{\partial}{\partial y}\sin(2x+5y)$$

Sendo $u = 2x+5y$, pela regra da cadeia:

$$= -10\left[\frac{\partial}{\partial u}\overset{\cos(u)}{\cancel{\sin(u)}}\frac{\partial}{\partial y}(\overset{0}{\cancel{2x}}+\overset{5}{\cancel{5y}})\right]$$

$$= \boxed{-50\cos(2x+5y)}$$

Fim da questão 64.
Sem outras observações.

65) $f(x,y,z) = e^{xyz^2}$; $f_{xyz} = ?$

Cálculo de f_x:

$$\frac{\partial}{\partial x} \cdot e^{xyz^2} =$$

Pela regra da cadeia, e considerando $u = xyz^2$:

$$= \frac{\partial}{\partial u} e^u \frac{\partial}{\partial x} xyz^2 = e^u \left[yz^2 \frac{\partial x}{\partial x} + x \cancel{\frac{\partial}{\partial x} yz^2}^{0} \right]$$

$$= \boxed{e^{xyz^2} yz^2}$$

Cálculo de f_{xy}:

$$\frac{\partial^2}{\partial y \partial x} = \frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial x} e^{xyz^2} \right\} = \frac{\partial}{\partial y} e^{xyz^2} yz^2 =$$

Pela regra do produto:

$$= yz^2 \frac{\partial}{\partial y} e^{xyz^2} + e^{xyz^2} \cancel{\frac{\partial}{\partial y}} y z^2 \, z^2$$

Sendo $u = xyz^2$, pela regra da cadeia:

$$= yz^2 \left(\frac{\partial}{\partial u} e^u \cancel{\frac{\partial}{\partial y}} \overset{xz^2}{\cancel{y}} xz^2 \right) + e^{xyz^2} z^2$$

$$= yz^2 \left(e^{xyz^2} xz^2 \right) + e^{xyz^2} z^2 = e^{xyz^2} xyz^4 + e^{xyz^2} z^2$$

$$= \boxed{z^2 e^{xyz^2} \left(xyz^2 + 1 \right)}$$

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Agora o cálculo de f_{xyz}:

$$\frac{\partial^3}{\partial z \partial y \partial x} = \frac{\partial}{\partial z}\left\{\frac{\partial^2}{\partial y \partial x} e^{xyz^2}\right\} = \frac{\partial}{\partial z}\left\{z^2 e^{xyz^2}(xyz^2+1)\right\} =$$

Pela regra do produto:

$$= (xyz^2+1)\frac{\partial}{\partial z} z^2 e^{xyz^2} + z^2 e^{xyz^2}\frac{\partial}{\partial z}(xyz^2+1)$$

$$= (xyz^2+1)\left[e^{xyz^2}\frac{\partial}{\partial z}z^2 + z^2\frac{\partial}{\partial z}e^{xyz^2}\right] + z^2 e^{xyz^2}\left[xy\frac{\partial}{\partial z}z^2 + \cancelto{0}{\frac{\partial}{\partial z}1}\right]$$

$$= (xyz^2+1)\left[2z e^{xyz^2} + z^2\frac{\partial}{\partial z}e^{xyz^2}\right] + z^2 e^{xyz^2}(2xyz)$$

$$= (xyz^2+1)\left[2z e^{xyz^2} + z^2\left(\frac{\partial}{\partial u}e^{u}\frac{\partial}{\partial z}xyz^2\right)\right] + 2xyz^3 e^{xyz^2}$$

$$= (xyz^2+1)\left[2z e^{xyz^2} + z^2\left(e^{xyz^2}2xyz\right)\right] + 2xyz^3 e^{xyz^2}$$

$$= (xyz^2+1)\left[2z e^{xyz^2} + 2xyz^3 e^{xyz^2}\right] + 2xyz^3 e^{xyz^2}$$

$$= 2xyz^3 e^{xyz^2} + 2x^2y^2z^5 e^{xyz^2} + 2z e^{xyz^2} + 2xyz^3 e^{xyz^2} + 2xyz^3 e^{xyz^2}$$

$$= 6xyz^3 e^{xyz^2} + 2x^2y^2z^5 e^{xyz^2} + 2z e^{xyz^2}$$

$$= \boxed{2z e^{xyz^2}\left(3xyz^2 + x^2y^2z^4 + 1\right)}$$

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69) $w = \dfrac{x}{y+2z}$; $\dfrac{\partial^3 w}{\partial z \partial y \partial x}$, $\dfrac{\partial^3 w}{\partial x^2 \partial y}$

Primeira Parte: cálculo de f_x:

$$\frac{\partial}{\partial x}\left(\frac{x}{y+2z}\right) = \frac{\partial}{\partial x} x(y+2z)^{-1}$$

Pela regra do produto:

$$= (y+2z)^{-1} \frac{\partial \cancel{x}^{1}}{\cancel{\partial x}} + x \cancel{\frac{\partial (y+2z)^{-1}}{\partial x}}^{0}$$

$$= \boxed{\frac{1}{y+2z} = \frac{\partial}{\partial x}}$$

Cálculo de f_{xy}:

$$\frac{\partial^2}{\partial y \partial x}\left\{\frac{x}{y+2x}\right\} = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\left(\frac{x}{y+2x}\right)\right\} = \frac{\partial}{\partial y}\left(\frac{1}{y+2z}\right) =$$

$$= \frac{\partial}{\partial y}(y+2z)^{-1}$$

Sendo $u = y+2z$, pela regra da cadeia:

$$= \frac{\partial}{\partial u} u^{-1} \frac{\partial}{\partial y}(\cancel{y}^{1} + \cancel{2z}^{0}) = \frac{-1}{u^2} = \boxed{-\frac{1}{(y+2z)^2} = \frac{\partial^2}{\partial y \partial x}}$$

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Cálculo de f_{xyz}:

$$\frac{\partial^3}{\partial z \partial y \partial x}\left(\frac{x}{y+2z}\right) = \frac{\partial}{\partial z}\left\{\frac{\partial^2}{\partial y \partial x}\left(\frac{x}{y+2z}\right)\right\}$$

$$= \frac{\partial}{\partial z}\left(-\frac{1}{(y+2z)^2}\right) = \frac{\partial}{\partial z} -(y+2z)^{-2} = -\frac{\partial}{\partial z}(y+2z)^{-2}$$

Sendo $u = y+2z$, pela regra da cadeia:

$$= -\left(\frac{\partial}{\partial u} u^{-2} \frac{\partial}{\partial z}(\cancel{y}^{0} + \cancel{2z}^{2})\right) = -\left(\frac{-2}{u^3} \times 2\right) = \boxed{\frac{4}{(y+2z)^3} = \frac{\partial^3}{\partial z \partial y \partial x}}$$

Segunda Parte:

Dúvida: será que eu poderia supor continuidade e evitar o cálculo de $\frac{\partial^2}{\partial x \partial y}$, já que o cálculo de $\frac{\partial^2}{\partial y \partial x}$ já está feito? Pelo Teorema de Clairaut isso é válido. Como a função não é definida no ponto $(x,0,0)$ fico em dúvida. Mas considerando que o cálculo da derivada leva em conta somente o limite quando $(x, y \to 0, z \to 0)$, sem incluir o ponto $(x,0,0)$, acho que posso usar Clairaut. Assim:

$$\frac{\partial^3}{\partial x^2 \partial y}\left(\frac{x}{y+2z}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial^2}{\partial x \partial y}\left(\frac{x}{y+2z}\right)\right\}$$

$$= \frac{\partial}{\partial x} -(y+2z)^{-2} = \boxed{0 = \frac{\partial^3}{\partial x^2 \partial y}}$$

70) $u = x^a y^b z^c$; $\dfrac{\partial^6 u}{\partial x \partial y^2 \partial z^3}$

Cálculo de $\dfrac{\partial}{\partial z} x^a y^b z^c$:

Pela regra do produto:

$$= z^c \cancel{\frac{\partial}{\partial z} x^a y^b}^{\,0} + x^a b^b \frac{\partial}{\partial z} z^c = \boxed{c\, x^a y^b z^{c-1} = \frac{\partial}{\partial z}}$$

Cálculo de $\dfrac{\partial^2}{\partial z^2} x^a y^b z^c =$

$$\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial z} x^a y^b z^c\right\} = \frac{\partial}{\partial z} c\, x^a y^b z^{c-1}$$

Pela regra do produto:

$$= c\left(z^{c-1} \cancel{\frac{\partial}{\partial z} x^a y^b}^{\,0} + x^a y^b \frac{\partial}{\partial z} z^{c-1}\right) = \boxed{(c-1)(c)\, x^a y^b z^{c-2} = \frac{\partial}{\partial z^2}}$$

Todas as outras derivadas, pela regra do produto, seguirão o mesmo padrão. Assim podemos deduzir que:

$$\boxed{\frac{\partial^6}{\partial x \partial y^2 \partial z^3} x^a y^b z^c = (a)(b)(b-1)(c)(c-1)(c-2)\, x^{a-1} y^{b-2} z^{c-3}}$$

Fim da questão 70.
Sem outras observações.

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Ferramentas utilizadas para estudo, aprendizagem e conferência de respostas:

- James Stewart, Cálculo, 8ª ed.
- Wolfram Mathematica
- Wolfram Alpha
- Calculadora HP-50G

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