Singular-Perturbations-Based Analysis of Dynamic Consensus in Directed Networks of Heterogeneous Nonlinear Systems

Mohamed Maghenem Elena Panteley Antonio Loría

***Abstract*— We investigate conditions under which heterogeneous nonlinear systems, interconnected over a directed static network, may achieve synchrony. Due to the network's hetereogeneity, complete synchronization is impossible in general, but an emergent dynamics arises. This may be characterized by two dynamical systems evolving in two time-scales. The first, "slow", corresponds to the dynamics of the network on the synchronization manifold. The second, "fast", corresponds to that of the synchronization errors. We present a framework to analyze the emergent dynamics based on the behavior of the slow dynamics. Firstly, we give conditions under which if the slow dynamics admits a globally asymptotically stable (GAS) equilibrium, so do the networked systems. Secondly, we give conditions under which, if the slow dynamics admits an asymptotically stable orbit and a single unstable equilibrium point, there exists a unique periodic orbit that is almost-globally asymptotically stable. The emergent behavior is thus clear, the systems asymptotically synchronize in frequency and, in the limit, as the coupling strength grows unboundedly, the emergent dynamics approaches that of the slow system. Our analysis is established using singular-perturbations theory. In that regard, we also contribute with original statements on stability of disconnected invariant sets and limit cycles for systems in singular-perturbation form.**

***Index Terms*— Consensus, multi-agent systems, singular perturbations, network systems, synchronization.**

I. Introduction and Motivation

NETWORKS of nonlinear heterogeneous systems are both, ubiquitous and *complex*. Their ubiquity motivates their study across numerous research disciplines, as varied as Engineering Systems theory [1], Complexity theory [2] and even Philosophy of Science [3]. Their complexity is motor for two apparently antagonistic trains of thought that attempt to explain the collective behavior of networked systems in a broad sense: *reductionism* and *emergentism*. The first asserts that any whole can be reduced to its constituent parts—as in the case of networked linear systems [4], while the tenet of emergentism is that a new behavior appears as a consequence of the interaction of the said parts [3]—as in networks of heterogeneous nonlinear systems. What is more, one of the accepted definitions of Complexity is that it corresponds to the difference between the network as a whole and the sum of its parts and, in that regard, nonlinearity is a necessary condition for complexity to appear [5].

The authors are with the CNRS, France. M. Maghenem is with University of Grenoble Alpes, CNRS, Grenoble INP, GIPSA-Lab, France. E-mail: mohamed.maghenem@cnrs.fr; E. Panteley and A. Loría are with L2S, CNRS, 91192 Gif-sur-Yvette, France. E-mail: elena.panteley@cnrs.fr and antonio.loria@cnrs.fr

In this paper we show that, to some extent, emergentism and reductionism are not necessarily mutually exclusive, but their respective underlying postulates are both useful to assess the behavior of networks of heterogeneous nonlinear systems. We focus on systems with dynamics given by

$$\dot{x}_i = f_i(x_i) + u_i, \quad i \in \{1, 2, \ldots, N\},\ x_i \in \mathbb{R}^n, \tag{1}$$

where $i \in \{1, 2, \ldots, N\}$, $x_i \in \mathbb{R}^n$ is the state of the ith system and $u_i \in \mathbb{R}^n$ is the decentralized control input to each system, defined as the consensus control law

$$u_i := -\sigma\Big[l_{i1}(x_i - x_1) + \ldots + l_{iN}(x_i - x_N)\Big], \tag{2}$$

where l_{ij} are different non-negative real numbers denoting the individual interconnection weights and the scalar parameter $\sigma > 0$ is the common coupling strength.

The control law (2) is reminiscent of that commonly used in the literature on consensus control, in which the coupling strength $\sigma = 1$. This is specifically the case for networks of linear systems, in which case *complexity* hardly appears and the focus turns towards the nature of the interconnections. On one hand, these may be linear [4]; nonlinear [6], [7]; time-varying [8], [9]; switching [10], [11], state-dependent [12], dynamic [13]–[15], *etc.* On the other hand, the interconnections may be directed [16] or signed [17]. Yet another aspect that plays a crucial role in the behavior of networks of heterogeneous nonlinear systems is the coupling strength. Different kinds of emergent behavior may arise depending on whether σ is "weak" [18], [19] or "strong" [20], [21].

Now, in spite of the generality of the individual systems' dynamics, the control law (2) remains conservative relatively to other works in which output coupling is considered—see *e.g.* [7], [15], [21]–[23]. The case of output coupling is addressed also in [20], for heterogeneous nonlinear systems, but only practical asymptotic synchronization is established.

Our main interest in this paper is to assess the behavior of the corresponding closed-loop system, specifically, in the case that the coupling gain σ is larger than a certain threshold, but we restrict our analysis to networks with an underlying static directed graph. Akin to [20], we analyze the closed-loop networked system via a change of coordinates that exhibits an intrinsic dichotomous structure composed of two dynamics defined in orthogonal spaces. On one hand, one has a *reduced-order dynamics* with state $x_m \in \mathbb{R}^n$ and, on the other, the dynamics that corresponds to the synchronization errors, denoted by $e_i := x_i - x_m$. This characterization of the networked system is driven by the objective of characterizing synchronization phenomena that may appear (or not) as a result of the systems' interconnections.

Following [20], the systems reach *dynamic consensus* if, for all $i \leq N$, the synchronization errors e_i converge to zero asymptotically and the reduced-order dynamics[1] has an asymptotically stable invariant set. The dynamic consensus paradigm generalizes the more common *equilibrium* consensus, in which case the reduced-order dynamics is null, *i.e.* $\dot{x}_m = 0$, because the collective behavior is *static* and the state of the reduced-order dynamics satisfies $x_m(t) \equiv x_m(0)$, where $x_m(0)$ is a weighted average of the nodes' states' initial values. In the case of a network of oscillators, the reduced-order dynamics may admit an asymptotically stable equilibrium (an example is provided in [20]) or an asymptotically stable attractor. However, in general, asymptotic dynamic consensus is unreachable due to the heterogeneity [21]. An exception is that of systems that admit an internal model [24]–[26] but, in general, for nonlinear heterogeneous systems, dynamic consensus may be guaranteed only in a *practical* sense [20].

In this paper, we establish several original statements and address two generic cases: one in which the (slow) reduced-order dynamics admits a globally asymptotically stable equilibrium, and another one in which it admits a periodic solution. In the first case, we give sufficient conditions under which the origin for the networked system is globally asymptotically stable. In the second case, we prove that the synchronization errors converge to a unique attractive periodic orbit, so the systems achieve frequency synchronization [27], [28]. Moreover, for "large" values of the coupling strength σ, this orbit is "close" to that generated by periodic solutions of the reduced dynamics. Thus, the emergent dynamics approaches that of the reduced-order system, as the coupling gain grows.

The analysis is based on the recognized premise that in self-organized complex systems, emergence is *multi*-level, as in *occurring in multiple timescales* [2], [29]. Here, we only consider two, one that is *slow* and pertains to the reduced-order system and another that is *fast* and pertains to the synchronization errors. The analysis of multi-timescale systems may be carried out using the classical singular-perturbations approach [30]–[32] or the geometric one [33], [34]. Here, we rely on classical singular perturbation theory, but also on original refinements of some statements from [30] for systems admitting disconnected sets composed of equilibria and periodic orbits. The proofs of some technical statetements, however, are omitted due to space constraints; for a more complete manuscript see [35].

The model-reduction-and-multi-time-scale perspective is certainly not new, neither in systems theory [1] nor in other disciplines. In the seminal work [36], which follows up on [1], the authors consider a modular network composed of sparsely connected clusters of densely interconnected dynamical systems modeled by simple integrators—the paradigm is motivated by that of large electrical networks. Using classical singular-perturbation theory [30], [37], it is showed that such networks achieve synchronization at two levels, within and among the clusters. The analysis is based on relating the network's sparsity to a singular-perturbation parameter. These concepts have been revisited in many succeeding works, such as [38] and [16]. In the former, for networks of simple integrators through sector nonlinearities, and in the latter for linear homogeneous systems interconnected through time-varying persistently-exciting gains *a la* [8]. On the other hand, networks of linear homogeneous singularly-perturbed systems are considered in [39] and [40]. Thus, in all of the above, the setting is fundamentally different from the one studied here.

In [41], for a particular case-study of networked Andronov-Hopf oscillators, we use a coordinate transformation to exhibit the presence of the two-timescale emergent dynamics and singular-perturbation theory to analyze the collective behavior under the premise that the reduced-order system admits an asymptotically stable orbit. Based on the coordinate transformation introduced in [41], singular-perturbation theory is used in [21] on a wider class of nonlinear systems with rank-deficient coupling to establish synchronization in the practical sense. In [14] and [15], singular-perturbation theory is used to analyze networked systems over undirected graphs and restricted to the case in which the slow dynamics admits an asymptotically stable equilibrium.

Thus, there are several articles in the literature that explicitly use reduction and singular-perturbation theory, even in a multi-agent context. Yet, we are not aware of any work covering generic nonlinear heterogeneous systems interconnected over directed graphs and characterize the collective behavior with higher precision. Conceptually, in phase with the emergentism posit, we evince the appearance of a *complex* (in the sense of [5]) dynamic behavior, as a result of the systems' interactions. At the same time, we give a more precise characterization (well beyond practical asymptotic stability of the synchronization manifold) of the collective behavior of networked systems based on that of a reduced-order model.

The rest of the paper is organized as follows. In Section II we exhibit the network's reduced-order and synchronization dynamics, under an invertible coordinate transformation. In Sections III and IV we present our main results. In Section V, we present a case-study and in Section VI, we provide concluding remarks and comments on future research directions. The paper is completed with technical appendices.

Notation and definitions. Given a nonempty set $K \subset \mathbb{R}^n$, we define $|x|_K := \inf_{y \in K} |x - y|$, where $|s|$ denotes the Euclidean norm of s. For a set $O \subset \mathbb{R}^n$, $K \backslash O$ denotes the subset of elements of K that are not in O. For a matrix $A \in \mathbb{R}^{n \times n}$, $|A|$ denotes its norm, and $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the smallest and the largest eigenvalues of A, respectively. For a sequence $\{A_i\}_{i=1}^N \subset \Pi_{i=1}^N \mathbb{R}^{n_i \times n_i}$, $\underset{i\in\{1,2,\ldots,N\}}{\mathrm{blkdiag}}\{A_i\}$ is the block-diagonal matrix whose ith diagonal block corresponds

[1]In [20] we use the term *emergent* dynamics to refer to what we call here *reduced-order* dynamics. We correct here a misuse of terminology that we made in [20]. Indeed, *emergent* dynamics refers to the *complex* behavior that arises as an effect of the systems' interactions.

to A_i. By $\mathbf{1}_N \in \mathbb{R}^N$, we denote the vector whose entries are equal to 1. By $\otimes$, we denote the Kronecker product. For a complex number $\lambda \in \mathcal{C}$, $\Re e(\lambda)$ denotes the real part of λ and $\Im m(\lambda)$ denotes its imaginary part. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded.

According to [42], for $\dot{x} = f(x, \varepsilon)$, with $x \in \mathbb{R}^n$ and $\varepsilon \in [0, 1]$, the set $\mathcal{A} \subset \mathbb{R}^n$ is globally practically attractive if, for each $\beta > 0$, there exists $\varepsilon^\star > 0$ such that, for each $\varepsilon \leq \varepsilon^\star$ and for every solution $t \mapsto x(t)$, there exists $T > 0$ such that $|x(T)|_{\mathcal{A}} \leq \beta$. The set $\mathcal{A}$ is globally practically stable if there exists $\kappa \in \mathcal{K}$ such that, for each $\beta > 0$, there exists $\varepsilon^\star > 0$ such that, for each $\varepsilon \geq \varepsilon^\star$ and for every solution $t \mapsto x(t)$, we have $|x(t)|_{\mathcal{A}} \leq \kappa(|x(0)|_{\mathcal{A}}) + \beta$ for all $t \geq 0$. The set $\mathcal{A}$ is globally practically asymptotically stable (GPAS) if it is globally practically attractive and globally practically stable.

II. On strongly-coupled connected networks

A. The model and standing assumptions

Consider a group of N nonlinear systems as in (1) driven by the distributed control inputs, as defined in (2), where each $l_{ij} \geq 0$ is constant but not necessarily equal to l_{ji}. In particular, when there exists an interconnection from the jth node to the ith node, l_{ij} is strictly positive, but l_{ji} may be null, in which case, the interconnection is said to be unidirectional and the graph is said to be directed. We also assume that the graph is connected, which means that there exists a path (non-necessarily directed) between every pair of nodes [43]. More particularly, we pose the following hypothesis.

Assumption 1 (connected digraph): The network's digraph contains at least one directed spanning tree.

Remark 1: In some works, digraphs containing a directed spanning tree are called *quasi-strongly connected*—see *e.g.,* [43]. Under this property, the Laplacian L has exactly one eigenvalue (say, λ_1) that equals zero, while the others have positive real part, *i.e.*, $0 = \lambda_1 < \Re e\{\lambda_2\} \leq \ldots \leq \Re e\{\lambda_n\}$. Furthermore, the right eigenvectors corresponding to the simple eigenvalue $\lambda_1 = 0$ are spanned by $v_r = \mathbf{1}_N \in \mathbb{R}^N$, while the left eigenvectors are spanned by a vector that contains only non-negative elements [10], which we denote by v_l, and satisfies $1_N^\top v_l = 1$.

In addition, for each unit, we impose the following.

Assumption 2 (Regularity): The functions f_i are once continuously differentiable and, for each $i \in \{1, 2, \ldots, N\}$, there exists $x_i^* \in \mathbb{R}^n$ such that $f_i(x_i^*) = 0$. Without loss of generality[2], we assume that $x_i^* = 0$ for all $i \in \{1, 2, \ldots, N\}$.

Assumption 3 (Semi-passive units): Each agent is input-to-state strictly semi-passive—cf. [44], [45]. More precisely, for each $i \in \{1, 2, \ldots, N\}$, there exist a continuously differentiable storage function $V_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a class $\mathcal{K}_\infty$ function $\underline{\alpha}_i$, a positive constant ρ_i, a continuous function $H_i : \mathbb{R}^n \to \mathbb{R}$, and a continuous function $\psi_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i), \quad \dot{V}_i(x_i) \leq x_i^T u_i - H_i(x_i)$$

and $H_i(x_i) \geq \psi_i(|x_i|)$ for all $|x_i| \geq \rho_i$.

[2] If the system does not have an equilibrium at the origin, or at all for that matter, it is assumed that one may be imposed onto it via a preliminary feedback.

Assumption 3 is useful to assess the boundedness of solutions for system (1) in closed loop with (2) for a sufficiently large coupling strength σ. More precisely, we have the following result—see [46].

Lemma 1 (Global ultimate boundedness): Consider the systems in (1) in closed loop with the control inputs in (2) and let Assumptions 1–3 hold. Then, the closed-loop system is globally uniformly (in σ) ultimately bounded. That is, given $\sigma^* > 0$, there exists $r > 0$ such that, for any $R \geq 0$, there exists $\tau_R \geq 0$ such that, for each $\sigma \geq \sigma^*$ and for each solution $t \mapsto x(t)$, we have

$$|x(0)| \leq R \Longrightarrow |x(t)| \leq r \qquad \forall t \geq \tau_R,$$

where $x \in \mathbb{R}^{nN}$ denotes the network's state, *i.e.*, $x = [x_1^\top, \ \ldots, \ x_N^\top]^\top$. □

Remark 2: The proof follows along the same lines as that for [20, Proposition 2], which is constructive and provides an explicit formula for r. In that regard, we stress that, contrary to its formulation in [20], the statement therein holds for *directed* graphs.

Under the assumptions listed above, we investigate the problem of assessing the behavior of the networked closed-loop system (1)-(2). To this end, as it is customary, let us collect the individual interconnection coefficients l_{ij} into the Laplacian matrix $L := [\ell_{ij}] \in \mathbb{R}^{N \times N}$, where

$$\ell_{ij} = \begin{cases} \sum\limits_{k \in \mathcal{N}_i} a_{ik} & i = j \\ -a_{ij} & i \neq j. \end{cases} \tag{3}$$

Then, replacing (2) in (1) and using $x = [x_1^\top, \ \ldots, \ x_N^\top]^\top$, we see that the overall network dynamics takes the form

$$\dot{x} = F(x) - \sigma[L \otimes I_n]x, \tag{4}$$

where the function $F : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is given by

$$F(x) \quad := \quad [f_1(x_1) \ \cdots \ f_N(x_N)]^\top .$$

As in [20] and [47], to analyze the behavior of the network system (4), we acknowledge its dichotomous nature. In these references, as well as in many others—*e.g.*, [48]–[50], synchronization is defined as the property of the trajectories of each individual system following the trajectories of an "averaged" unit with state

$$x_m := [v_l^\top \otimes I_n]x. \tag{5}$$

The quotes in "averaged" are superfluous in the case of undirected networks, in which case $v_l = \mathbf{1}_N$, so $x_m = \frac{1}{N}\sum_{i=1}^N x_i$, but for directed connected networks the state x_m is more generically defined as a weighted average of the respective systems' states since $v_{li} \geq 0$ for all $i \in \{1, 2, ..., N\}$ and $v_l^\top \mathbf{1}_N = 1$.

In either case, a sensible way to define the synchronization errors e is as the difference between the units' states and x_m, that is,

$$e := x - [\mathbf{1}_N \otimes I_n]x_m. \tag{6}$$

Thus, in [20] and [47] the collective behavior of network systems is studied in function of the dynamics of the "averaged" unit x_m and that of the synchronization errors e.

In the next section, we introduce another change of coordinates to rewrite system (4) in an equivalent form that exhibits two motions: one that is generated by the averaged dynamics and another by a *projection* of the synchronization errors e on a certain subspace. This coordinate transformation is not a simple artifice for analysis, it exhibits two time-scales that are inherent to networked systems satisfying Assumption 1 and subject to a sufficiently large coupling σ. Other changes of variable that naturally lead to a time-scale separation have been proposed, specially for undirected graphs, *e.g.*, in [14], [38]. The one we adopt here, and describe below, is taken from [20], [47].

B. Characterization of the collective behavior

After Assumption 1 and Remark 1, because $\lambda_1 = 0$ has multiplicity one, the Laplacian admits the Jordan-block decomposition

$$L = U \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix} U^{-1}, \tag{7}$$

where $\Lambda \in \mathbb{R}^{(N-1)\times(N-1)}$ is composed by the Jordan blocks corresponding to the $N-1$ non-zero eigenvalues.

Remark 3: Note that even though a Jordan decomposition does not necessarily exist with a real matrix U, it is always possible to use the spectral-invariant-subspace decomposition as in [51, Theorem 1.5., p. 224]—see also [35, Lemma 13]—to generate a real matrix U.

The invertible matrix U is constituted, column-wise, of the right eigenvector of the Laplacian, $\mathbf{1}_N$, and a left-invertible matrix $V \in \mathbb{R}^{N\times(N-1)}$, which consists of the eigenvectors corresponding to the nonzero eigenvalues of L. That is,

$$U = [\mathbf{1}_N \quad V], \qquad U^{-1} = \begin{bmatrix} v_l^\top \\ V^\dagger \end{bmatrix}, \tag{8}$$

where $V^\dagger \in \mathbb{R}^{(N-1)\times N}$, and

$$v_l^\top V = 0, \quad V^\dagger V = I_{N-1}. \tag{9}$$

So, using (8) and (9), we also have the useful identity

$$VV^\dagger = I_N - \mathbf{1}_N v_l^\top.$$

Now, using U^{-1}, we define the new coordinates

$$\bar{x} := [U^{-1} \otimes I_n]x \tag{10}$$

and the inverse transformation

$$x := [U \otimes I_n]\bar{x}. \tag{11}$$

The interest of the coordinate $\bar{x}$ is that it consists in the familiar "averaged" states x_m and a projection of the synchronization errors e defined in (6) onto the subspace that is generated by $V^\dagger$, which is orthogonal to the right eigenvector $\mathbf{1}_N$. To better see this, note that such projection yields

$$[V^\dagger \otimes I_n]e = [V^\dagger \otimes I_n]\big[x - [\mathbf{1}_N \otimes I_n]x_m\big].$$

In the sequel, we refer to the left-hand side of the latter equation as the projected synchronization errors,

$$e_v := [V^\dagger \otimes I_n]e. \tag{12}$$

Hence, in view of (5), (8), (10), and (12), we have

$$\bar{x} = \begin{bmatrix} x_m \\ e_v \end{bmatrix} = \begin{bmatrix} [v_l^\top \otimes I_n]x \\ [V^\dagger \otimes I_n]x \end{bmatrix}. \tag{13}$$

In the new coordinates, the network system (4) is equivalently written as

$$\dot{\bar{x}} = [U^{-1} \otimes I_n]\big[F(x) - \sigma[L \otimes I_n]x\big],$$

which consists in two interconnected dynamics, that of the "averaged" state x_m and that of the projected synchronization errors e_v. Therefore, the behavior of the trajectories of (4) may be assessed via that of the latter dynamics. To this end, we use $\bar{x} = [x_m^\top \quad e_v^\top]^\top$ and $U = [\mathbf{1}_N \quad V]$ in (11) to write

$$x = [\mathbf{1}_N \otimes I_n]x_m + [V \otimes I_n]e_v. \tag{14}$$

Then, differentiating on both sides of (5), using (4), (14), and the fact that $v_l^\top L = 0$, we obtain

$$\dot{x}_m = F_m(x_m) + G_m(x_m, e_v), \tag{15}$$

where $F_m(x_m) := [v_l^\top \otimes I_n]F\big([\mathbf{1}_N \otimes I_n]x_m\big)$ and

$$G_m(x_m, e_v) := [v_l^\top \otimes I_n]\big[F\big([\mathbf{1}_N \otimes I_n]x_m + [V \otimes I_n]e_v\big) - F\big([\mathbf{1}_N \otimes I_n]x_m\big)\big].$$

Note that $F_m(x_m)$ effectively corresponds to an "averaged" drift of the systems in (1), *i.e.*,

$$F_m(x_m) = \sum_{i=1}^{N} v_{li} f_i(x_m),$$

$G_m(x_m, 0) = 0$ and, under Assumption 2, all these functions are smooth and there exists a continuous function $h : \mathbb{R}^{nN} \to \mathbb{R}_{\geq 0}$ such that

$$\big|G_m(x_m, e_v)\big| \leq h(x_m, e_v)|e_v| \qquad \forall (x_m, e_v) \in \mathbb{R}^{nN}.$$

On the other hand, by differentiating on both sides of $e_v = [V^\dagger \otimes I_n]x$ and using (7), (8), and (14), we obtain the *synchronization-errors dynamics*

$$\dot{e}_v = -\sigma[\Lambda \otimes I_n]e_v + G_e(x_m, e_v), \tag{16}$$

where

$$G_e(x_m, e_v) := [V^\dagger \otimes I_n]F\big([V \otimes I_n]e_v + [\mathbf{1}_N \otimes I_n]x_m\big).$$

The complete collective behavior of the networked control system (4), up to the globally invertible coordinate transformation in (10), may be assessed by analyzing that of the interconnected systems (15) and (16), which evolve in orthogonal spaces [20]. We see that the systems in (1) under the action of the control laws in (2) *synchronize*, in the sense that $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t) = \lim_{t\to\infty} x_m(t)$ for all $i,\ j \leq N$, if and only if the errors e_v tend asymptotically to zero. As we explain on p. 2, in this case we say that the systems achieve dynamic consensus. Therefore, because the solutions of (4) do not necessarily stabilize at a common equilibrium, the characterization of the networked systems' behavior would be incomplete unless one can ascertain what the individual systems do when they synchronize. Indeed, *a priori*, not even boundedness of solutions is guaranteed (whence Assumption 3, see [52]). To assess any kind of stable behavior, we analyze the network system (4) *on* the synchronization subspace corresponding to $e_v = 0$. On such a subspace, we have the *reduced-order dynamics*

$$\dot{x}_m = F_m(x_m). \tag{17}$$

Thus, the motion of the synchronized systems is fully determined by that of the reduced-order dynamics (17). In this regard, it is important to underline that (17), as well as the "averaged" dynamics (15) are independent of the coupling gain σ. This dynamics is *inherent* to the network and appears simply as a consequence of the graph's connectivity imposed by Assumption 1. The synchronization dynamics (16), on the other hand, clearly depends on the coupling strength σ. We are interested in investigating the synchronization behavior for 'large' values of the coupling strength. More precision about the meaning of 'large' is given farther below.

We consider two scenarii of interest. The first pertains to the case in which the reduced-order dynamics (17) admits the origin as a globally asymptotically stable equilibrium point. Our main statement in this case (Theorem 1 in Section III) is that, not only the networked system achieves dynamic consensus, but the origin $\{x = 0\}$ is GAS for (4). The second scenario pertains to the case in which the synchronization errors e_{vi} do not vanish (due to the systems' heterogeneity), but only are ultimately bounded—cf [20], and the reduced-order dynamics admits an unstable equilibrium and a stable periodic orbit generated by a limiting periodic solution $x_{mo}(t+\alpha_o)$ of period α_o. In this case, we prove that each error e_{vi} also becomes periodic for sufficiently large values of the coupling gain—see Theorem 2 in Section IV. This is significant because, since $e_{vi} := x_i - x_m$ and $x_m(t) \to x_{mo}(t + \alpha_o)$, we also have that, for each i, $x_i(t) \to \tilde{x}_i(t + \alpha_i)$ where $\tilde{x}_i(t + \alpha_i)$ denotes a periodic solution of period α_i. That is, roughly speaking, each system becomes periodic asymptotically. In addition, we show that as the coupling gain $\sigma \to \infty$, all the periods become equal to that of the reduced-order system, *i.e.,* $\alpha_i \to \alpha_o$ for all $i \leq N$. In other words, all the systems' periodic trajectories mutually synchronize both in amplitude and in frequency.

Remark 4: Part of the significance of our main result (Theorem 2 on p. 8) lies in characterizing the behavior of the networked system when the systems' solutions are ultimately bounded. More precisely, it is a well-known fact that if the solutions of a planar system tend asymptotically to the interior of a compact ball centered at the origin, they exhibit a periodic behavior. The counterpart of this statement for higher-dimensional systems is, in general, an open problem. Here, we identify a particular case in which it holds true.

C. Intrinsic two-time-scales decomposition

The analysis of (4), towards the statements described above, starts with the observation that Eqs. (15)-(16) may be written in the familiar singular-perturbation form [31], [32], *i.e.,*

$$\dot{x}_m = F_m(x_m) + G_m(x_m, e_v) \tag{18a}$$
$$\varepsilon\dot{e}_v = -(\Lambda \otimes I_n)\, e_v + \varepsilon G_e(x_m, e_v), \quad \varepsilon := 1/\sigma, \tag{18b}$$

in which, we recognize two time scales, "slow" and "fast", corresponding, respectively, to the dynamics of the averaged-unit states x_m and the projected synchronization errors e_v. Now, in accordance with singular-perturbation theory; see [32, p. 358], the behavior of (18) is ineluctably determined by that of the *slow* dynamics, obtained by setting $\varepsilon = 0$, which clearly corresponds to the reduced-order model (17). Thus, the rest of the paper is devoted to the analysis of (18) in the two cases evoked above, stability of the origin in Section III and orbital stability, in Section IV.

It is important to stress, however, that even though the analysis of singularly-perturbed systems as in (18) is not uncommon in recent literature, available methods hardly address the scenarii considered in this paper. For instance, Eqs. (18) are of the same form as [14, Eqs. (4)] or [15, Eqs. (13)], but these respective equations model different systems than described above. This is because the problem addressed therein is that of classical consensus (so $\dot{x}_m = 0$) over undirected graphs, but with dynamic interconnections. In the first part of [14], as in [15], the fast dynamics corresponds to that of the interconnections, and in the second part of [14], it stems from rapidly varying virtual systems, whose trajectories the actual systems are meant to follow. In a different context, in [38], large-scale networks of integrators are studied, such that certain groups of nodes that are densely connected among themselves synchronize fast, while the (average behaviors of the) groups synchronize at a slower pace—cf. [1]. To assess the stability of the consensus manifold, the statements in [14] rely mostly on Lyapunov theory, in [15] the geometric approach based on non-hyperbolic invariant manifolds [53] is used, and in [38] the authors rely on the theory of singular perturbations *a la* Tykhonov [54]. Apart from Theorem 1 in Section IV, such approaches appear unsuitable for our purposes since, our main interest is to assess the behavior of the networked systems even in the case that the synchronization errors do not vanish.

Now, even though in the sequel we focus on the analysis of the system in singularly-perturbed form, (18), we remark that our two main statements are formulated for system (4), which remains the main subject study in this paper. Therefore, we finish this section by re-expressing the properties of (4) in Assumptions 1–3 in terms of (18), in the form of the following, rather evident, statement that is extensively used in the sequel.

Lemma 2: Consider system (4) such that Assumptions 1–3 hold. Then, the resulting system (18), with states defined in (13), enjoys the following properties:

(i) the functions F_m, G_m, and G_e are continuously differentiable;
(ii) the origin $\{(x_m, e_v) = (0, 0)\}$ is an isolated equilibrium point;

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(iii) the solutions to (18) are globally uniformly (in σ) ultimately bounded;
(iv) the matrix $-\Lambda$ is Hurwitz. □

III. Case I: Global asymptotic stability of the origin

Our first original statement establishes global asymptotic stability for networked heterogeneous semi-passive systems, interconnected over a (not-necessarily strongly) connected digraph, provided that the coupling gain is sufficiently large.

Theorem 1 (GAS): Consider system (4) under Assumptions 1–3. Assume, also, that for system (17) the origin $\{x_m = 0\}$ is globally asymptotically stable and there exist $\rho > 0$, $c_\rho > 0$, a continuously differentiable Lyapunov function $V_m : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a class $\mathcal{K}_\infty$ function α, such that

$$\frac{\partial V_m(x_m)}{\partial x_m^\top} F_m(x_m) \leq -\alpha\left(|x_m|\right)^2 \tag{19}$$

for all $\bar{x} \in B_\rho$—recall that $\bar{x} := [x_m^\top \; e_v^\top]^\top$ and $B_\rho := \{x \in \mathbb{R}^q : |x| \leq \rho\}$—and

$$\max_{\bar{x}\in B_\rho} \left\{ \left|G_e(x_m, e_v) - G_e(0, e_v)\right|, \left|\frac{\partial V_m(x_m)}{\partial x_m^\top}\right| \right\} \leq c_\rho \alpha\left(|x_m|\right). \tag{20}$$

Then, there exists $\sigma^\star > 0$ such that, for all $\sigma \geq \sigma^\star$, the origin for (4) is globally asymptotically stable. □

The regularity conditions in (19)-(20) are required to ensure negativity of the time derivative of a Lyapunov function along the solutions to (18)—see the proof farther below. Similar conditions may be found in the literature; for instance in [14], where asymptotic consensus is established for nonlinear systems interconnected through (slowly-varying) dynamic undirected interconnections, but only within a certain domain of attraction, not globally.

Inequalities (19)-(20) are little conservative, as they are required to hold only on an arbitrary compact set containing the origin, but they are not necessary for global asymptotic stability of the origin for (17)—see [32, Exercise 9.24]. Then, again, if these conditions do not hold the origin may not be globally asymptotically stable either—in [35, Section 5.2], we provide an example that illustrates this claim. On the other hand, GAS for system (17) implies a relaxed form of (19) and of the second inequality in (20). This is used in Proposition 1 below, which is an original statement that generalizes the main results in [20] on practical asymptotic stability for networks over strongly connected digraphs. The proof of Proposition 1, and subsequently that of Theorem 1, builds upon GAS for (17) and the fact that under Assumptions 1–3 the system is globally ultimately bounded, uniformly in σ—see Lemma 1.

Proposition 1 (GPAS): Consider system (4) under Assumptions 1–3. In addition, assume that for system (17), the origin $\{x_m = 0\}$ is globally asymptotically stable. Then, the origin for (4) is globally practically asymptotically stable. □

Proof: We establish the equivalent statement that the origin $\{\bar{x} = 0\}$ for (18) is GPAS. For this, we remark first that after Assumptions 1-3 and Lemma 1, there exist $\sigma^* > 0$ and $r > 0$ such that, for any $R \geq 0$, there exists $\tau_R \geq 0$ such that $|\bar{x}_o| \leq R \Rightarrow |\bar{x}(t)| \leq r$ for all $t \geq \tau_R$, and all $\sigma \geq \sigma^*$. Now we show that, for every positive constant $r_f < r$, there exists $\sigma_f > 0$ such that, for each $\sigma \geq \sigma_f$, the compact set B_{r_f} is asymptotically stable for (18) on B_r [42].

To that end, let the assumption that the origin for (17) is globally asymptotically stable generate, by [32, Theorem 3.14], the converse Lyapunov function $W_m : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and positive constants α_{rr_f} and β_{rr_f} such that, for all $\bar{x} \in B_r \backslash B_{r_f}$,

$$\left|\frac{\partial W_m(x_m)}{\partial x_m}\right| \leq \beta_{rr_f} |x_m|, \tag{21}$$

$$\frac{\partial W_m(x_m)}{\partial x_m^\top} F_m(x_m) \leq -\alpha_{rr_f} |x_m|^2. \tag{22}$$

Furthermore, let Item (iv) in Lemma 2 generate a symmetric and positive definite matrix $P \in \mathbb{R}^{(N-1)\times(N-1)}$ such that $-P\Lambda - \Lambda^\top P \leq -I_N$. Then, the total derivative of the Lyapunov function candidate $V(\bar{x}) := W_m(x_m) + e_v^\top [P \otimes I_n] e_v$, along the trajectories of (18)—multiplying by σ on both sides of (18b)—yields

$$\begin{aligned}\dot{V}(\bar{x}) = {} & 2e_v^\top P \left[G_e(0, e_v) + (G_e(x_m, e_v) - G_e(0, e_v))\right] \\ & - \alpha_{rr_f} |x_m|^2 + \frac{\partial W_m}{\partial x_m^\top} G_m(x_m, e_v) - \sigma |e_v|^2. \end{aligned} \tag{23}$$

Now, after Assumption 2 and items (i)-(ii) of Lemma 2 it follows that there exists a constant $d_r > 0$ such that,

$$\max_{\bar{x}\in B_r} \left\{ |G_m(x_m, e_v)|, |G_e(0, e_v)| \right\} \leq d_r |e_v|, \tag{24}$$

$$|G_e(x_m, e_v) - G_e(0, e_v)| \leq d_r |x_m|, \quad \forall \bar{x} \in B_r. \tag{25}$$

From all the above it follows, in turn, that

$$\begin{aligned}\dot{V}(\bar{x}) \leq {} & -\alpha_{rr_f} |x_m|^2 + d_r \left[\lambda_{max}(P) + \beta_{rr_f}\right] |x_m| |e_v| \\ & - \left[\sigma - 2d_r \lambda_{max}(P)\right] |e_v|^2 \qquad \forall\, \bar{x} \in B_r \backslash B_{r_f}. \end{aligned}$$

Thus, for $\sigma_f := d_r^2 \left(\lambda_{max}(P) + \beta_{rr_f}\right)^2 + 4d_r \lambda_{max}(P)$, we conclude that, for each $\sigma \geq \sigma_f$, we have

$$\dot{V}(\bar{x}) \leq -\frac{1}{2}\alpha_{rr_f} |x_m|^2 - \frac{1}{2}\sigma |e_v|^2 \qquad \forall\, \bar{x} \in B_r \backslash B_{r_f}.$$

This, and uniform global ultimate boundedness, establishes the statement. ■

Remark 5: Inequalities (21), (22), and (25) are reminiscent of (19) and (20). However, we stress that (25) holds on a compact that is generated by the property of global uniform ultimate boundedness, *i.e.,* it is generated by Assumptions 1–3 via Lemma 1. In turn, Inequalities (21) and (22) hold for free, for a converse Lyapunov function W_m, but not around the origin, as required to establish global asymptotic stability, whence the conditions (19) and (20) in Theorem 1. Based on these and the global practical asymptotic stability property just established, the proof of Theorem 1 is constructed upon a similar argumentation as for Proposition 1.

Proof of Theorem 1: The conditions of Proposition 1 are met, so the set B_{r_f}, *for any* $r_f < r$ and r being generated by Lemma 1, is globally asymptotically stable. Pick r_f sufficiently small such that $r_f < \rho$, where ρ is defined in the theorem. It follows that there exists t_ρ such that the solutions generated from arbitrary initial conditions are contained in B_ρ for all $t \geq t_\rho$. Therefore, it is enough to prove that the origin is asymptotically stable for all solutions contained in B_ρ. To that end, akin to the proof of Proposition 1, consider the Lyapunov function $V(\bar{x}) := V_m(x_m) + e_v^\top [P \otimes I_n] e_v$, with V_m as defined in the theorem and P generated by Item (iv) of Lemma 2. Its time derivative along the solutions to (18) satisfies

$$\begin{aligned} \dot{V}(\bar{x}) = & \, 2e_v^\top P \left[G_e(0, e_v) + (G_e(x_m, e_v) - G_e(0, e_v))\right] \\ & - \alpha(|x_m|)^2 + \frac{\partial V_m}{\partial x_m^\top} G_m(x_m, e_v) - \sigma \left|e_v\right|^2. \end{aligned} \tag{26}$$

Now, a bound like (24) continues to hold with d_ρ, so after this and (20), we have

$$\left| \frac{\partial V(\bar{x})}{\partial x_m^\top} G_m(x_m, e_v) \right| \leq c_\rho d_\rho \alpha\left(|x_m|\right) |e_v| \quad \forall \bar{x} \in B_\rho. \tag{27}$$

Consequently, from all of the above it follows that

$$\begin{aligned} \dot{V}(\bar{x}) \leq & - \alpha(|x_m|)^2 + c_\rho [2\lambda_{max}(P) + d_\rho] \alpha(|x_m|) |e_v| \\ & - [\sigma - 2d_\rho \lambda_{max}(P)] |e_v|^2 \qquad \forall \bar{x} \in B_\rho. \end{aligned}$$

Thus, we conclude that, for each $\sigma \geq \sigma^\star := c_\rho^2[\lambda_{max}(P) + d_\rho]^2 + 4d_\rho\lambda_{max}(P)$,

$$\dot{V}(\bar{x}) \leq -\frac{1}{2}\alpha(|x_m|)^2 - \frac{1}{2}\sigma \left|e_v\right|^2 \qquad \forall \bar{x} \in B_\rho.$$

The statement follows. ■

IV. Case II: Almost global asymptotic orbital stability

In this section, we present our second and main statement, which pertains to the case when (17) admits a periodic orbit that is attractive from almost all initial conditions. Under this condition, Theorem 2 below establishes that for $\sigma > 0$ sufficiently large, the network system (4) also admits a unique periodic orbit, which is globally attractive from almost all initial conditions. In particular, frequency synchronization is achieved and the synchronization errors can be made arbitrary small by choosing σ sufficiently large. It is important to stress that our main statement establishes a precise periodic behavior for the network system (4) rather than just approaching the periodic solution to (17).

For completeness and clarity, we start by recalling some notions and tools related to the stability of periodic solutions to nonlinear systems of the form

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n, \tag{28}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is at least locally Lipschitz.

Definition 1 (Periodic solution and periodic orbit): A solution $t \mapsto \phi(t)$, or simply $\phi(t)$, to (28) is said to be α-periodic if there exists $\alpha > 0$ (the period) such that, for each $t \geq 0$,

$$\phi(t + \alpha) = \phi(t) \quad \text{and} \quad \phi(t + s) \neq \phi(t) \quad \forall s \in (0, \alpha).$$

Moreover, if the system (28) admits a periodic solution ϕ, we say that it admits a (closed) periodic orbit $\gamma \subset \mathbb{R}^n$ generated by the image of ϕ.

Then, according with Lyapunov theory, we may single out the following desired properties for periodic solutions.

Definition 2 (Orbital Stability): Let γ be a periodic orbit for (28).

- The orbit γ is orbitally stable if, for each $\varepsilon > 0$, there exist $\delta > 0$ and $T \geq 0$, such that, for each initial condition x_o satisfying $|x_o|_\gamma \leq \delta$, the solution ϕ starting from x_o satisfies $|\phi(t)|_\gamma \leq \varepsilon$ for all $t \geq T$.
- The orbit γ is orbitally asymptotically stable, if it is orbitally stable and attractive; *i.e.*, if there exists $R \in (0, +\infty]$ such that, for each x_o satisfying $|x_o|_\gamma \leq R$, the solution ϕ starting from x_o satisfies $\lim_{t\to\infty} |\phi(t)|_\gamma = 0$.
- The orbit γ is globally orbitally asymptotically stable if it is orbitally asymptotically stable with $R = +\infty$ and almost globally asymptotically stable if it is orbitally asymptotically stable for all $x_o \in \mathbb{R}^n \backslash \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^n$ has a null Lebesgue measure.

Finally, we recall some orbital stability criteria in terms of the so-called characteristics multipliers [55, Section III.7] which, for linear periodic systems, are the counterpart of eigenvalues for linear autonomous systems. To see this, we assume that f is continuously differentiable and we consider the α-periodic matrix $A(t) := \frac{\partial f}{\partial x^\top}(\phi(t))$, where $\phi(t)$ is the α-periodic solution to (28) generating the orbit γ. After Floquet theory—see *e.g.*, [56] and [57], there exist an α-periodic non-singular matrix $P : [t_o, +\infty] \to \mathbb{R}^{n\times n}$ and a constant matrix $B \in \mathbb{R}^{n\times n}$ such that the transition matrix associated to the linear time-varying system

$$\dot{x} = A(t)x \tag{29}$$

is given by $X(t) := P(t)e^{Bt}$ and the non-singular change of coordinates $y := P(t)^{-}x$ transforms the linear system (29) into $\dot{y} = By$.

Definition 3 (Characteristic multipliers): The characteristic multipliers of the α-periodic matrix $A(t)$ are the eigenvalues of the matrix $e^{B\alpha}$.

Definition 4 (Non-singular periodic orbit): The periodic orbit γ generated by the periodic solution $\phi(t)$ is non-singular if the matrix $A(t) := \frac{\partial f}{\partial x^\top}(\phi(t))$ admits a simple characteristic multiplier equal to 1.

Lemma 3 (Theorem 2.1, Section VI.2. [55]): Consider system (28) with f continuously differentiable and let ϕ be a non-trivial α-periodic solution generating the orbit γ. Assume that the matrix $A(t) := \frac{\partial f}{\partial x^\top}(\phi(t))$ is non-singular and all the characteristic multipliers, except one, have modulus strictly less than 1. Then, the resulting periodic orbit γ is asymptotically orbitally stable. □

Sufficient conditions for orbital stability

As mentioned above, generally speaking, the standing assumption in this section is that the reduced-order dynamics (17) admits an orbitally asymptotically stable periodic orbit. However, we remark that some nonlinear systems defined on compact and convex sets and that admit a limit cycle, also admit at least one equilibrium point [58]. This imposes

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particular richness to the network's collective behavior, and a considerable difficulty to analyze it since it translates into studying stability of a disconnected invariant set. In that light we pose the following hypothesis.

Assumption 4: The reduced-order dynamics (17) admits a unique compact invariant subset $\omega \subset \mathbb{R}^n$, which is globally attractive. More precisely, for each $x_{mo} \in \mathbb{R}^n$, the solution $x_m(t)$ starting from x_{mo} satisfies

$$\limsup_{t\to+\infty} |x_m(t)|_\omega = 0. \tag{30}$$

Furthermore, the set ω is composed of a non-singular periodic orbit γ_o (of period α_o and that is orbitally asymptotically stable) and, *either* the origin $\{x_m = 0\}$ if the latter is repulsive[3], *or* the homoclinic orbit $\gamma_1 := W_o^u(0) \cap W_o^s(0)$, if the origin is hyperbolic—$W_o^s(0)$ and $W_o^u(0)$ stand, respectively, for the global stable and unstable manifolds of the origin.[4].

Assumption 4 imposes, for the case in which the trajectories tend to diverge away from the origin, that the system admits a unique limit cycle. While difficult to verify in full generality, this assumption holds, *e.g.,* for a variety of planar oscillators admitting the origin as an unstable repulsive equilibrium, such as the Liénard equation or the Stuart-Landau oscillator [59]. Sufficient conditions for the existence of a unique limit cycle in more general cases are provided, *e.g.*, in [60]. In Section V we treat an example of Stuart-Landau oscillators.

Remark 6: Note that the global attractivity property in (30) plus the structure of the invariant set ω imply the existence of a Lyapunov function enjoying useful properties along the solutions to (17)—see [61, Theorem 1] or [35, Lemma 11].

Lemma 4: Under Assumption 4 and Item (i) in Lemma 2, the periodic orbit γ_o is almost globally orbitally asymptotically stable for (17). □

We are ready to present our main statement.

Theorem 2 (Almost global orbital asymptotic stability): Consider the network system (4) under the Assumptions 1–3 and such that the reduced-order dynamics (17) satisfies Assumption 4. Then, there exists $\sigma^f > 0$, such that, for all $\sigma \geq \sigma^f$,

(i) the networked system (4) admits a unique nontrivial periodic orbit $\mathcal{O}_{1/\sigma}$, of period $\alpha_{1/\sigma}$ and that is almost globally orbitally asymptotically stable.

(ii) as $\sigma \to \infty$, $\alpha_{1/\sigma} \to \alpha_o$ and $\mathcal{O}_{1/\sigma} \to \mathcal{O}_o$, where

$$\mathcal{O}_o := \{x \in \mathbb{R}^{nN} : x_m \in \gamma_o \text{ and } e_v = 0\}$$

—see (13). □

Item (i) of Theorem 2 states that if the reduced-order system (17) admits a limit cycle, then there exists a threshold coupling gain σ^f, such that, for any interconnection strength above it, *each* individual system in the network has an asymptotically stable orbit. In general, such limit cycles differ from one system to another, but all of them are generated by trajectories of the same period. This is sometimes referred to in the literature as frequency synchronization [27], [28]. Item (ii) states that as the coupling gain grows unboundedly, all the said limit cycles converge to the one generated by the periodic solution to the reduced-order dynamics (17), and the periods of the corresponding trajectories all converge to that of the reduced-order system's. In Section V, we illustrate these statements, notably via numerical simulations.

The proof of Theorem 2, which is provided farther below, follows a sequence of logical steps to assess the existence, uniqueness, and almost global orbital asymptotic stability of an orbit for (4). The analysis relies on studying the singularly-perturbed system (18), but we emphasize that the available literature on stability (of the origin or a compact set) for singularly-perturbed systems [54], [37], [32] does not apply to (18), when (17) admits a limit cycle and an isolated equilibrium point. Therefore, the proof of Theorem 2 relies on technical lemmata that are presented next, but due to space constraints, the proofs of the latter are omitted; they are available in [35].

- Lemma 5 establishes global asymptotic practical stability of the set
$$\{(x_m, e_v) \in \mathbb{R}^{nN} : x_m \in \gamma_o \cup \gamma_1 \text{ and } e_v = 0\};$$
- Lemma 6 establishes that, given a torus sufficiently tight around $\mathcal{O}_o$, for each coupling gain σ sufficiently large, system (17) admits a unique periodic orbit $\mathcal{O}_{1/\sigma}$ contained in the torus;
- Lemma 7 establishes that each such orbit $\mathcal{O}_{1/\sigma}$ is (locally) asymptotically stable and admits the aforementioned torus as a basin of attraction;
- Lemma 8 below and Lemma 11 in the Appendix provide a local analysis around the origin, to establish that it attracts only the solutions starting from a null-measure set.

Remark 7: Each one of the different Lemmata, used to prove Theorem 2, establishes the existence a coupling threshold, above which, a certain key property holds. From the proofs of those Lemmata, we can see that they do not allow to find the coupling gains explicitly, except, maybe, for Lemma 5, which uses a constructive Lyapunov argument. The proofs of the remaining Lemmata use some existence results that do not seem necessarily constructive. Providing an explicit value for σ^f is a challenging open problem.

Technical Lemmata

We start by introducing the following notations. Correspondingly to $\gamma_o \subset \mathbb{R}^n$ and $\gamma_1 \subset \mathbb{R}^n$, which denote, respectively, the closed periodic and homoclinic orbits for system (17)—see Assumption 4, we introduce their "lifting" $\Gamma_o \subset \mathbb{R}^{nN}$ and $\Gamma_1 \subset \mathbb{R}^{nN}$ onto the space of system (18), as

$$\Gamma_o := \left\{(x_m, e_v) \in \mathbb{R}^{nN} : x_m \in \gamma_o \text{ and } e_v = 0\right\}, \tag{31}$$
$$\Gamma_1 := \left\{(x_m, e_v) \in \mathbb{R}^{nN} : x_m \in \gamma_1 \text{ and } e_v = 0\right\}. \tag{32}$$

Furthermore, we denote by T_ρ the torus defined as

$$T_\rho := \left\{(e_v, x_m) \in \mathbb{R}^{N(n-1)} \times \mathbb{R}^n \;:\; |(x_m, e_v)|_{\Gamma_o} \leq \rho\right\}, \tag{33}$$

[3] *i.e.*, there exists $\sigma > 0$ such that, for each initial condition $|x(t_\circ)| \leq \sigma$, there exists $T > 0$ such that implies that $|x(t)| > \sigma$ for all $t \geq t_\circ + T$.

[4] *i.e.*, among the eigenvalues of $A := \frac{\partial f_m(x_m)}{\partial x_m}\big|_{x_m=0}$, $0 < k < n$ eigenvalues have positive real part and $n-k$ have negative real part.

and we use $\Gamma_\varepsilon \subset \mathbb{R}^{nN}$ to denote a closed orbit generated by a periodic solution to system (18), *if it exists*. That is, Γ_ε is a subset in the space of (x_m, e_v) that consists in the image points generated by the parameterized solutions of (18), $t \mapsto (x_m(t), e_v(t))$, that are periodic with period α_ε.

The first technical lemma provides a statement for system (18) on global practical asymptotic (GPAS) stability of the set

$$\Omega := \Gamma_o \cup \Gamma_1,$$

where Γ_o and Γ_1 are introduced in (31)-(32).

Lemma 5 (GPAS of Ω): Consider system (18) such that Items (i)–(iii) of Lemma 2 hold and let Assumption 4 be satisfied for the corresponding reduced-order system (17). Then, the set $\Omega := \Gamma_o \cup \Gamma_1$ is GPAS for (18). In particular, for any $\rho > 0$, there exists $\varepsilon_1(\rho) > 0$, such that, for each $\varepsilon \leq \varepsilon_1(\rho)$ and for each initial condition $x_o = (e_{vo}, z_{mo}) \in \mathbb{R}^{nN}$, the solution $(x_m(t), e_v(t))$ satisfies $\lim_{t\to\infty} |x_m(t), e_v(t)|_\Omega \leq \rho$. □

The next lemma establishes that, for all sufficiently small values of $\rho > 0$, there exist sufficiently small values of ε, such that there exists a unique periodic orbit $\Gamma_\varepsilon \subset T_\rho$ generated by a solution to (18) of period $\alpha_\varepsilon \approx \alpha_o$.

Lemma 6 (Existence of Γ_ε): Consider system (18) such that Items (i) and (iii) of Lemma 2 hold and let Assumption 4 hold for the reduced-order dynamics (17). Then, there exist $\rho_o > 0$ and a class $\mathcal{K}$ function ε_o such that, for each $\rho \in (0, \rho_o]$ and for each $\varepsilon \leq \varepsilon_o(\rho)$, system (18) has a unique nontrivial periodic orbit Γ_ε, which is strictly contained in T_ρ. Moreover, the period α_ε of the solution to (18) generating the orbit Γ_ε tends to α_o, which is the period of the solution to (17) generating the orbit Γ_o. □

Remark 8: The existence result in Lemma 6 follows from a direct application of Anosov Theorem; see Lemma 9 in Appendix I.

The next lemma establishes local asymptotic orbital stability of all periodic orbits Γ_ε lying inside the torus T_ρ for sufficiently small values of ε and ρ. Moreover, we show that the corresponding domain of attraction is uniform in ε.

Lemma 7 (Stability of Γ_ε): Let system (18) satisfy Items (i) and (iii) of Lemma 2 and let Assumption 4 be satisfied for the corresponding reduced-order dynamics (17). Then, there exist $\varepsilon^{**} > 0$ and $\rho^{**} > 0$ such that, for each $\varepsilon \leq \varepsilon^{**}$, each periodic orbit $\Gamma_\varepsilon \subset T_{\rho^{**}}$ generated by an α_ε-periodic solution to (18), with α_ε sufficiently close to α_o, is asymptotically orbitally stable with a domain of attraction that contains $T_{\rho^{**}}$. □

Remark 9: Lemma 7 is reminiscent of a statement established by Anosov— [30, Theorem 5]—that pertains to the case in which the periodic orbit γ_o for (17) is only non-singular (or hyperbolic). Although it is claimed in [30] that the proof therein translates directly to the case where γ_o is non-singular and asymptotically stable, in this paper, we provide an original proof for the latter case using the theory of perturbed matrices [51], [62].

The next statement links those from Lemmata 5–7. It establishes that if ε is sufficiently small, then the periodic behavior of the reduced-order system (17) is preserved for system (18), as well as its stability properties.

Proposition 2: Consider the dynamical system (18) under the assumption that Items (i)–(iv) of Lemma 2 hold and assume further that the reduced-order dynamics (17) satisfies Assumption 4. Then, there exists $\rho_o > 0$ such that, for each $\rho \in (0, \rho_o]$, there exists $\varepsilon_2(\rho) > 0$ such that, for each $\varepsilon \in (0, \varepsilon_2(\rho)]$,

(i) system (18) admits a unique orbit $\Gamma_\varepsilon \subset T_\rho$ generated by a non-trivial (α_ε)-periodic solution, with $\Gamma_\varepsilon \to \Gamma_o$ and $\alpha_\varepsilon \to \alpha_o$ as $\varepsilon \to 0$;

(ii) Γ_ε is (locally) asymptotically stable;

(iii) for any initial condition $\bar{x}_o \in \mathbb{R}^{nN}$ the corresponding solution $\bar{x}(t)$ to (18) either converges to Γ_ε or to a ρ-neighborhood of Γ_1, that is,

$$\limsup_{t\to\infty} |\bar{x}(t)|_{\Gamma_1} \leq \rho. \tag{34}$$

□

Proof: Items (i)-(iv) in Lemma 2 and Assumption 4 imply that the statements of Lemmata 5–7 hold. Then, let Lemma 7 generate $\varepsilon^{**} > 0$ and $\rho^{**} > 0$. Furthermore, let Lemma 6 generate $(\rho_o, \varepsilon_o(\cdot))$, for each $\rho \in (0, \rho_o]$, let Lemma 5 generate $(\varepsilon_1(\rho), \varepsilon_1(\rho^{**}))$, and let

$$\varepsilon \leq \varepsilon_2(\rho) := \min\{\varepsilon^{**}, \varepsilon_o(\rho), \varepsilon_o(\rho^{**}), \varepsilon_1(\rho), \varepsilon_1(\rho^{**})\}$$

be arbitrarily fixed.

After Lemma 6, there exists a unique periodic orbit $\Gamma_\varepsilon \subset T_\rho \cap T_{\rho^{**}} = T_{\min\{\rho,\rho^{**}\}}$ generated by a solution to (18). Now, given any sequence $\{\epsilon_i\}_{i=1}^\infty$ that converges to zero and such that $\epsilon_i \leq \varepsilon_2(\rho)$ for all $i \in \{1, 2, ...\}$, from the above, we know that, for i large enough, the unique orbit Γ_{ϵ_i} satisfies $\Gamma_{\epsilon_i} \subset T_{\rho_i}$, where $\rho_i := \varepsilon_o^{-1}(\epsilon_i)$—note that ε_o^{-1} exists and is of class $\mathcal{K}$ because so it ε_o. Item (i) of the proposition follows since T_{ρ_i} converges to Γ_o—see (33).

Next, after Lemma 7, we conclude that Γ_ε is orbitally asymptotically stable and $T_{\rho^{**}}$ is inside the domain of attraction of Γ_ε. This establishes Item (ii).

Finally, from Lemma 5 we conclude that each solution to (18) either converges to $T_{\min\{\rho,\rho^{**}\}} \subset T_{\rho^{**}}$, so it also converges to Γ_ε, or it converges to a $\min\{\rho, \rho^{**}\}-$neighborhood of Γ_1. This establishes Item (iii). ■

The last technical lemma provides a local stability analysis around the origin of (18). It states that the origin is a hyperbolic equilibrium point, for all sufficiently-small values of ε. Furthermore, inspired by the Stable Manifold Theorem [63, Theorem 13.4.1], we show that the stable and unstable manifolds around the origin are uniquely defined on a neighborhood whose size does not shrink with ε.

Lemma 8 (Local behavior around the origin): Consider system (18) and let Items (i)-(ii) of Lemma 2 hold. Assume further that the corresponding reduced-order dynamics (17) satisfies Assumption 4. Then, there exist $\rho^* > 0$, $\varepsilon^* > 0$, a neighborhood of the origin denoted $U \subset \mathbb{R}^n$, and $r > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*)$,

(i) system (18) admits a unique unstable and stable manifolds $(W_\varepsilon^u(0), W_\varepsilon^s(0))$ defined on U;

(ii) for each $\bar{x}(t)$ bounded solution to (18) starting from $\bar{x}_o \in U \backslash W^s_\varepsilon(0)$, there exists $T_1 > 0$ such that $|\bar{x}(t)| \geq r$ for all $t \geq T_1$;

(iii) for each $\bar{x}(t)$ solution to (18) such that $|\bar{x}(0)|_{\Gamma_1} \leq \rho^*$, there exists $T_2 > 0$ such that $|\bar{x}(T_2)| < r$. □

Proof of Theorem 2

Under Assumptions 1–3, Items (i)-(iv) of Lemma 2 hold for system (18). This and Assumption 4 imply that the statements of Proposition 2 and Lemmata 5–8 hold too. Then, let Lemma 8 generate $\rho^* > 0$ and $\varepsilon^* > 0$ and let Proposition 2 generate $(\rho_o, \varepsilon_2(\min\{\rho^*, \rho_o\}/4))$. We show that the statement of Theorem 2 holds with $\sigma^f := 1/\min\{\varepsilon^*, \varepsilon_2(\min\{\rho^*, \rho_o\}/4)\}$ in four ordered steps:

1) First, for any $\sigma \geq \sigma_f$ or, equivalently, any $\varepsilon = 1/\sigma$ satisfying $\varepsilon \leq \min\{\varepsilon^*, \varepsilon_2(\min\{\rho^*, \rho_o\}/4)\}$, we use Item (i) of Proposition 2 to conclude the existence of a unique nontrivial periodic orbit Γ_ε generated by a periodic solution to (18) of period α_ε. From Item (ii) of the same Proposition it follows that Γ_ε is locally asymptotically stable. In addition, from Item (iii) of Proposition 2 it follows that each solution $\bar{x}(t)$ to system (18) either converges to the orbit Γ_ε, otherwise, it converges to a $(\min\{\rho^*, \rho_o\}/4)$–neighborhood of Γ_1; that is, (34) holds with $\rho = \min\{\rho^*, \rho_o\}/4$ and, consequently, there exists $T < \infty$ such that
$$|\bar{x}(t)|_{\Gamma_1} = |(x_m(t), e_v(t))|_{\Gamma_1} \leq \min\{\rho^*, \rho_o\}/2 \quad \forall t \geq T. \tag{35}$$
2) Now, we introduce the backward propagation of the stable manifold $W^s_\varepsilon(0)$ introduced for (18) in Lemma 8. That is, we introduce set
$$R(W^s_\varepsilon(0)) := \{\bar{x}(t) : t \leq 0,\ \bar{x}(0) \in W^s_\varepsilon(0)\}$$
and prove by contradiction that the solution $\bar{x}$ to (18) satisfying (35) must start from the set $R(W^s_\varepsilon(0))$. Indeed, assume that the opposite holds. Then, using Item (iii) in Lemma 8, we conclude that the solution $\bar{x}$ must enter B_r at some $T^* \geq T$. In particular, $\bar{x}(T^*) \in U$ and $\bar{x}(T^*) \notin W^s_\varepsilon(0)$. So, using Item (ii) in Lemma 8, we conclude that there exists $T_1 > 0$ such that
$$|\bar{x}(T^* + t)| \geq r \qquad \forall t \geq T_1.$$
However, since $|\bar{x}(T^* + T_1)|_{\Gamma_1} \leq \min\{\rho^*, \rho_o\}/2$, it follows that $\bar{x}$ must enter B_r again under Item (iii) of Lemma 8, which contradicts Item (ii).
3) Next, we show that the set $R(W^s_\varepsilon(0))$ is a null measure set using contradiction. That is, let $S_o \subset R(W^s_\varepsilon(0))$ such that $\mu(S_o) \neq 0$. Assume without loss of generality that for some $T < 0$, we have
$$S_o \subset \{\bar{x}(t) : t \in [-T, 0],\ \bar{x}(0) \in W^s_\varepsilon(0),\ \bar{x} \text{ sol. to (18)}\}.$$
Note that
$$R^b(T, S_o) := \{\bar{x}(T) : \bar{x}(0) \in S_o\} \subset W^s_\varepsilon(0)$$
with $\mu(W^s_\varepsilon(0)) = 0$. However, using Lemma 11 from the Appendix, we conclude that $\mu(S_o) = 0$.
4) Finally, using the inverse transformation (11), it follows that the orbit
$$\mathcal{O}_\varepsilon := \{x \in \mathbb{R}^{nN} : (x_e, e_v) \in \Gamma_\varepsilon\}$$
is almost GAS for (4). The second statement follows from Lemma 6. ■

V. Case study: a network of Andronov-Hopf oscillators

To illustrate the use of our main theoretical findings, we address, as a case-study, the analysis of a network of Andronov-Hopf oscillators, also known as Stuart-Landau [59]. The equation of such oscillator represents a normal form of the bifurcation carrying the same name and is given by
$$\dot{z} = -\nu|z|^2 z + \mu z, \tag{36}$$
where $z \in \mathbb{C}$ denotes the state of the oscillator, while ν, $\mu \in \mathbb{C}$ are constant parameters: $\nu := \nu_R + i\nu_I$ and $\mu := \mu_R + i\mu_I$, with $i = \sqrt{-1}$ and $\nu_R > 0$. The analysis of (36) is well documented in the literature. For instance, via Lyapunov-exponents-based methods, as in [64] and [57], or using Lyapunov's direct method, as in [65] and [66].

The behavior of the system may also be explained using polar coordinates, which are useful to represent the system's trajectories on the complex plane—see Figure 1 below.

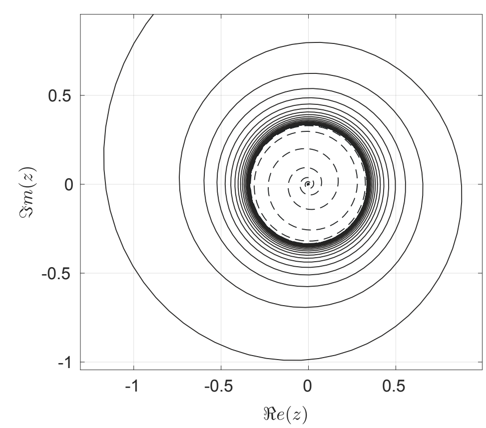

Fig. 1. Trajectories of the Andronov-Hopf oscillator on the complex plane with $\mu_R > 0$. In this case, the origin is an unstable equilibrium and all trajectories tend to the stable limit-cycle of radius $\sqrt{\frac{\mu_R}{\nu_R}}$ and period $\frac{2\pi\nu_R}{|\mu_I\nu_R - \nu_I\mu_R|}$.

Let $z = re^{i\varphi}$. Then, the equations for the radial amplitude r and the angular variable φ can be decoupled into:
$$\dot{r} = \mu_R r - \nu_R r^3 \tag{37a}$$
$$\dot{\varphi} = \mu_I - \nu_I r^2. \tag{37b}$$
If $\mu_R < 0$, Equation (37a) has only one stable fixed point at $r = 0$. Moreover, the latter is Lyapunov (globally exponentially) stable. However, if $\mu_R > 0$, this equation has a stable fixed point at $r = \sqrt{\frac{\mu_R}{\nu_R}}$, while $r = 0$ becomes unstable. This

implies that the trajectories of the system converge to a circle of radius $\sqrt{\frac{\mu_R}{\nu_R}}$, starting from initial conditions either inside or outside the circle. Thus, the latter is an attractor and the system (36) exhibits a periodic oscillation at its natural period $\frac{2\pi\nu_R}{|\mu_I\nu_R-\nu_I\mu_R|}$.

The bifurcation of the limit cycle from the origin that appears at the value $\mu_R = 0$ is known in the literature as the Andronov-Hopf bifurcation. The curves

$$\Gamma_\alpha := \left\{ \sqrt{\frac{\mu_R}{\nu_R}} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} : t \in \left[0, \frac{2\pi\nu_R}{|\mu_I\nu_R - \nu_I\mu_R|}\right] \right\}$$

define the limit cycle of the system.

Now, for the purpose of this paper, we consider N forced Andronov-Hopf oscillators

$$\dot{z}_k = -\nu_k|z_k|^2 z_k + \mu_k z_k + u_k \qquad k \in \{1,2,\dots,N\} \tag{38}$$

where $z_k \in \mathbb{C}$ denotes the kth oscillator's state, $u_k \in \mathbb{C}$ is its control input, and $\nu_k := \nu_{Rk} + i\mu_{Ik}$ and $\mu_k := \mu_{Rk} + i\mu_{Ik}$ are constant complex parameters. We assume that $\nu_{Rk} > 0$ for all $k \in \{1, 2, ..., N\}$.

We assign to the control inputs a distributed control law as in (2), modulo the obvious changes in the notation. It is assumed that the corresponding interconnection graph contains at least one directed spanning tree. Then, in a compact form, the overall network dynamics in closed loop takes the form

$$\dot{z} = F(z) - \sigma L z, \tag{39}$$

where $z := [z_1 \, \cdots \, z_N]^\top$, the function $F : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$F(z) := [f_1(z_1) \quad f_2(z_2) \quad \cdots \quad f_N(z_N)]^\top,$$
$$f_k(z_k) := -\nu_k|z_k|^2 z_k + \mu_k z_k,$$

the elements of L are defined in (3), and L has a unique null eigenvalue while others have positive real parts.

Note that the networked system (39) satisfies Assumptions 1–3. Indeed, Assumption 1 is explicitly imposed, and each single unit has a polynomial vector field, so Assumption 2 holds as well. We also remark that each unit in (38) is semi-passive (Assumption 3) since it is assumed that $\nu_{Rk} > 0$ for all $k \in \{1, 2, ..., N\}$. To better see this, consider the total derivative of the Lyapunov function candidate

$$V(z_k) := \frac{1}{2} z_k^* z_k, \tag{40}$$

where z_k^* is the complex conjugate of z_k; we obtain

$$\dot{V}(z_k) = -\nu_{Rk}|z_k|^4 + \mu_{Rk}|z_k|^2 + z_k^* u_k.$$

Now, after Remark 1, we may apply the coordinate transformation in (13) and rewrite the network dynamics in the singularly-perturbed form, where the reduced-order dynamics, defined by (17), has the form

$$\dot{z}_m = f_m(z_m) := -\nu_m|z_m|^2 z_m + \mu_m z_m, \tag{41}$$

where $z_m = v_l^\top z$, the parameter $\mu_m \in \mathbb{C}$ corresponds to the weighted average of the μ_ks, *i.e.*,

$$\mu_m := \mu_{mR} + i\mu_{mI},$$
$$\mu_{mR} := \sum_{j=1}^{N} v_{lj}\mu_{Rj}, \quad \mu_{mI} := \sum_{j=1}^{N} v_{lj}\mu_{Ij},$$

similarly for $\nu_m := \nu_{mR} + i\nu_{mI}$, with $\nu_{mR} > 0$.

Remark 10: With an abuse of terminology, we may refer to (41) as the average dynamics. However, we note that μ_{mR} and μ_{mI} do not correspond to simple averages of the individual parameters μ_{Rk} and μ_{Ik}; through the Laplacian's eigenvector v_l, they depend on the network's topology, through v_l.

The reduced-order dynamics (41) also corresponds to an Andronov-Hopf oscillator, which admits an invariant set composed of a compact invariant set that is composed of two disjoint invariant subsets, the origin and a limit cycle:

$$\omega = \left\{ z_m \in \mathbb{C} : |z_m| = \sqrt{\frac{\mu_{mR}}{\nu_{mR}}} \right\} \cup \{z_m = 0\}.$$

In particular, if $\mu_{mR} \leq 0$, the set ω reduces to the origin. If $\mu_{mR} > 0$, Assumption 4 holds.

Thus, we see that Andronov-Hopf oscillators enjoy several interesting properties, individually. Yet, as we show below, the behavior of a network of different such systems may vary considerably depending on the choice of the coupling gain, and on the individual systems' dynamics. The postulate of this paper, as that of [20], is that the collective behavior may be assessed by studying the evolution of the reduced-order system (41), using Theorems 1 and 2. In what follows, we analyze different scenarii that we regroup into two cases, depending on whether $\mu_{mR} \leq 0$ or $\mu_{mR} > 0$.

A. The case in which $\mu_{\mathrm{mR}} = 0$

Consider the Lyapunov function V defined in (40), *i.e.*, $V(z_m) := (1/2)z_m^* z_m$. Its total derivative, along the trajectories of (41) with $\mu_m = \mu_{mI}$, yields

$$\dot{V}(z_m) = -\nu_{mR}|z_m|^4, \quad \nu_{mR} := \sum_{j=1}^{N} v_{lj}\nu_{Rj}.$$

Since $\nu_{mR} > 0$, the origin $\{z_m = 0\}$ for the reduced-order system (41) is globally asymptotically stable.

Remark 11: Note that $\nu_{mR} > 0$ does not necessarily imply that $\nu_{Rk} > 0$ for each individual system. However, the latter is needed to verify Assumption 3.

Furthermore, after Item (i) of Theorem 1, we conclude that the origin for the overall networked system (39) is globally practically asymptotically stable. The same may also be concluded after the main results in [20], but neither of these theorems gives an assessment regarding the behavior of the trajectories of the networked system, within the compact set to which they converge. For instance, the origin may be unstable or exponentially stable, depending on the coupling gain σ and depending on the individual dynamics composing the network. To see clearer, consider the linearization of (39) around the origin, *i.e.*,

$$\dot{z} = A_\sigma z, \qquad \left\{ \begin{array}{lcl} A_\sigma & := & A_o - \sigma L, \\ A_o & := & \mathrm{diag}\{\mu_1, \mu_2, ..., \mu_N\}. \end{array} \right. \tag{42}$$

and the following examples.

Example 1 (unstable origin): Consider two Andronov-Hopf oscillators interconnected bidirectionally, hence with

$$L := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Let $\nu_1 = \nu_2 = 1$, $\mu_{R1} = -\mu_{R2} = \mu \in \mathbb{R}$, so $\mu_{I1} = \mu_{I2} = 0$ and $\mu_{mR} = 0$. After (42), we have $A_o := \text{diag}\{\mu, \ -\mu\}$ and

$$A_\sigma = \begin{bmatrix} -\sigma + \mu & \sigma \\ \sigma & -\sigma - \mu \end{bmatrix},$$

whose eigenvalues are $\lambda_{1,2}(A_\sigma) = -\sigma \mp \sqrt{\sigma^2 + \mu^2}$. That is, A_σ has two positive real eigenvalues for any $\sigma > 0$, so, after Lyapunov's first method, the origin is unstable for the network of Andronov-Hopf oscillators (39). □

Example 2 (exponentially stable origin): Consider two Andronov-Hopf oscillators interconnected bidirectionally as in Example 1 and with $\nu_1 = \nu_2 = 1$, $\mu_{R1} = -\mu_{R2} = 1$, and $\mu_{I1} = -\mu_{I2} = 2$, so $\mu_{mR} = 0$ and A_σ in (42) corresponds to

$$A_\sigma = \begin{bmatrix} -\sigma + 1 - 2i & \sigma \\ \sigma & -\sigma - 1 + 2i \end{bmatrix}.$$

The eigenvalues of A_σ above are

$$\lambda_{1,2}(A_\sigma) := -\sigma \pm \sqrt{\sigma^2 - 3 - 4i},$$

which have strictly negative real parts for all $\sigma > 0.8376$. We conclude that the origin for the networked system is (locally) exponentially stable for such values of sigma. □

More generally, we can state the following result, which follows directly from Lyapunov's direct method, but it is certainly difficult to apply to large networks; namely, it is increasingly difficult to find the threshold value of σ as the dimension of the network increases. Note, also, that the threshold depends on the network's topology since so does A_σ.

Proposition 3: Consider the network of Andronov-Hopf oscillators in (39), interconnected over a connected directed graph. Assume that there exists $\sigma^f > 0$ such that, for all $\sigma \geq \sigma^f$, the smallest (in norm) eigenvalue of A_σ, denoted by $\lambda_1(A_\sigma)$, has a strictly negative real part. Then, there exists $\sigma^f > 0$ such that, for all $\sigma \geq \sigma^f$, the origin $\{z = 0\}$ is locally exponentially stable. □

Thus, it may be inferred that while standard methods may be used to assess the local behavior of solutions, albeit with increasing difficulty in function of the network's size, such tools fail to address the more interesting problem of finding sufficient conditions for *global* exponential stability. To the best of our knowledge, this question remains open. Now, one may conjecture that since the origin for (39) is globally practically asymptotically stable, if the origin is also locally exponentially stable, the global property should follow if one could establish exponential stability in the large [67], [68] with a domain of attraction that does not shrink in function of the coupling gain. Such analysis, however, is tedious even for the simple examples above and is beyond this paper's scope.

B. The case in which $\mu_{mR} \neq 0$

From Examples 1 and 2, it is clear that the parameter μ_{mR} plays a crucial role in the collective behavior, which is hardly predictable from the sole inspection of the corresponding parameter for each individual system. Indeed, the network behavior varies significantly in function of the coupling gain. In what follows, we analyze it in the case that μ_{mR} is either positive or negative and we provide some illustrative numerical simulation results.

Proposition 4: Consider the network of Andronov-Hopf oscillators defined in (39) interconnected over a connected directed graph. For the corresponding reduced-order system (41). Then,

(i) if $\mu_{mR} < 0$, then there exists $\sigma^f > 0$ such that, for all $\sigma \geq \sigma^f$, the origin $\{z = 0\}$ is GAS;

(ii) if $\mu_{mR} > 0$, then there exists $\sigma^f > 0$ such that, for each $\sigma \geq \sigma^f$, system (39) has a unique nontrivial periodic orbit

$$\mathcal{O}_o := \{z \in \mathbb{C}^N : z_m \in \gamma_o \text{ and } e_v = 0\},$$
$$\gamma_o := \left\{ z_m \in \mathbb{C} : |z_m| = \sqrt{\frac{\mu_{mR}}{\nu_{mR}}} \right\}$$

which is almost globally asymptotically stable. □

Proposition 4 illustrates the interest of Theorems 1 and 2. Both statements rely on the analysis of the reduced-order dynamics (41) instead of recurring to a local analysis of the overall network. Not only the analysis is straightforward, as one can see from the proof below, but the statements guarantee global properties.

Proof of Proposition 4: The total derivative of $V(z_m) := (1/2)z_m^* z_m$ along the trajectories of (41), yields

$$\dot{V}(z_m) = -\nu_{mR}|z_m|^4 + \mu_{mR}|z_m|^2. \tag{43}$$

If $\mu_{mR} < 0$ global asymptotic stability for the reduced-order dynamics follows and, after Theorem 1, Item (i) of the proposition follows (*i.e.*, global asymptotic stability for the overall network). If $\mu_{mR} > 0$, the invariant orbit γ_o is almost globally asymptotically stable and the origin $\{z_m = 0\}$ is anti-stable—see [69]. So Item (ii) of Theorem 2 implies the second statement of the proposition. ■

C. Numerical results

For the system (39), using (43) and standard Lyapunov theory [32], we may conclude that if $\mu_{mR} > 0$, the solutions are ultimately bounded. However, nothing may be inferred about the behavior of the solutions within the ultimate bound. After Proposition 4, it follows that for sufficiently large values of the coupling gain, all oscillators describe periodic orbits and, moreover, that there exist a unique periodic orbit defined by the solutions of the reduced order system. To illustrate this fact, we used Matlab® to perform several numerical simulations for the system (39), for distinct values of the coupling gain σ.

The setting consists in five Andronov-Hopf oscillators (38), with $\nu_k = 1$ for all $k \in \{1, 2, \ldots, 5\}$, $\mu_k \in \{1 + i, 3 +$

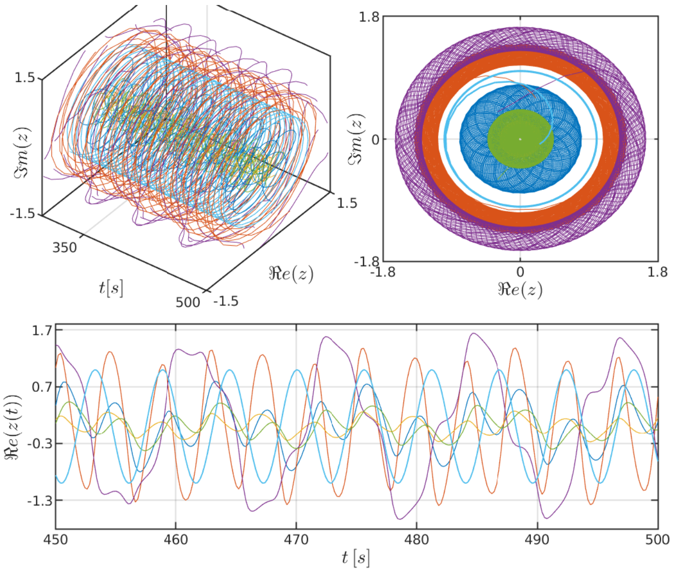

Fig. 2. Trajectories of (39) with $\boldsymbol{\sigma = 3}$, *i.e.*, below the threshold $\boldsymbol{\sigma^f}$ in Theorem 2. The interconnected oscillators loose their periodic behavior. NB: the time span for the NE plot is $\boldsymbol{t \in [0, 500]}$. The trajectories of the reduced-order system are showed in cyan.

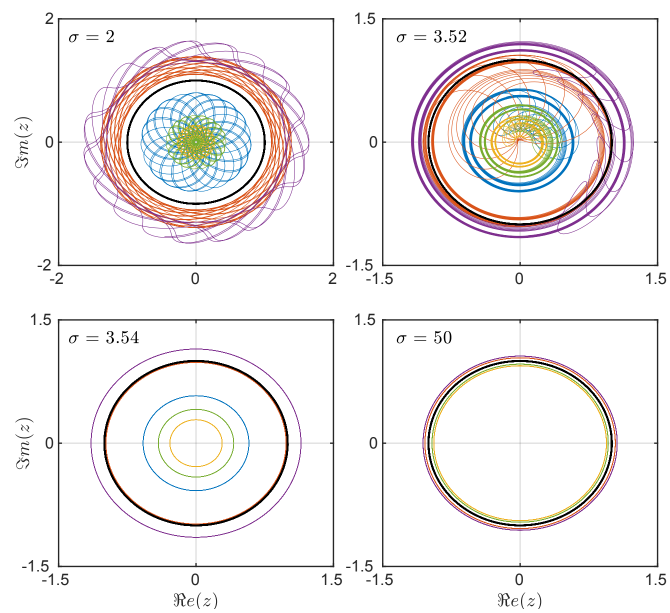

Fig. 3. Trajectories of (39) on the complex plane. The oscillators become periodic as $\boldsymbol{\sigma}$ is increased beyond the threshold $\boldsymbol{\sigma^f = 3.53}$, identified numerically. As $\boldsymbol{\sigma}$ increases further, the unique oscillators' orbits converge to that of the reduced-order system. The time span in all figures is $\boldsymbol{t \in [400, 500]}$. The orbit $\boldsymbol{\gamma_o}$ is showed in **black**.

$2i,\ -2+5i,\ 4-i,\ -1-i\}$, interconnected over a directed network with Laplacian matrix

$$L = \begin{bmatrix} 1 & -1/4 & -1/4 & -1/4 & -1/4 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The initial conditions were set to $z_{ok} \in \frac{1}{10}\{5-3i,\ 2+6i,\ 4+2i,\ -3+4i,\ -3-6i\}$. In Figure 2, we show the response of the interconnected systems with a relatively low interconnection gain. It is apparent that the Stuart-Landau oscillators do not exhibit periodic orbits, but a new dynamics emerges.

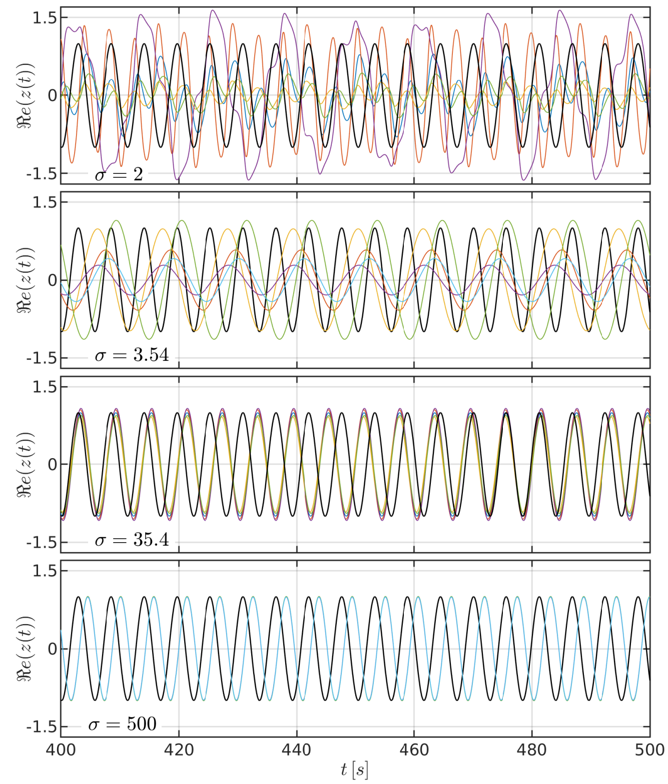

Fig. 4. Real part of the trajectories of (39) against time, in oscillatory steady state, *i.e.*, for $\boldsymbol{t \in [400, 500]}$. From the top: in the first plot one can appreciate that, for a relatively weak coupling gain, the trajectories are not periodic, except for that of the reduced-order system (in black). In the second plot, it is showed that for a coupling gain slightly above the threshold, the solutions become periodic, but of different amplitude. In the third plot, for a higher coupling gain, we see that all the oscillators synchronize in frequency, but not at the same frequency of the reduced order system. In the bottom plot, one sees that all oscillators, under very strong coupling, are in perfect synchrony, albeit out of phase relative to the reduced-order system.

In Figure 3 we present plots of $z(t)$ on the complex plane, for four different values of the coupling gain, for values below the threshold and above it. As Theorem 2 establishes, for gain values above the threshold not only the systems exhibit a periodic behavior, but as the gain increases, it approaches that of the reduced-order "averaged" system (41). Finally, in Figure 4, it is appreciated that the oscillators' periods tend to match with respect to each other and with that of the reduced-order system—Item (ii) in Theorem 2, as the coupling gain increases.

VI. Concluding Remarks

We presented a framework to assess the collective behavior of networks of heterogeneous nonlinear systems, in fair generality. Our approach allows to qualitatively characterize the collective behavior for "large" values of the coupling gains. For systems admitting periodic orbits of any dimension, we demonstrate that there exists a coupling gain threshold characterizing an inherent orbit to which each and all oscillators tend to asymptotically. On one hand, for any gain above this threshold each system admits an individual periodic motion. On the other hand, as the coupling gain increases, all oscillators' motions tend to coincide with that of an average dynamics. Although such behavior may be expected to hold locally, we demonstrate that it occurs for any initial conditions.

Our main results rely on original statements on singular-perturbation theory that might serve as starting point to establish interesting extensions. For instance, our analysis, so far, is restricted to networks of systems with state-coupling. Yet, many physical systems are interconnected through part or a function of the state. Statements for systems under output coupling pose a significant and well-motivated challenge.

Furthermore, our analysis does not provide explicit values of the coupling strength threshold, under which, the networked system exhibits the established behavior. Characterizing the emergent behavior, such as orbital asymptotic stability, both qualitatively and quantitatively (in terms of the coupling gain) is still another open and important problem, even in specific case-studies.

Finally, beyond the analysis problems solved in this paper, the control design problem is widely open. To find conditions under which a network of heterogeneous systems may be controlled so that it admits a *desired* reduced-order dynamics. Finally, we believe that extending the proposed framework for general classes of nonlinear systems such as hybrid systems is an interesting perspective as well.

Acknowledgments: The authors are indebted to Anes Lazri, PhD student at Univ Paris Saclay, for performing the illustrative numerical simulations and for valuable technical discussions on boundedness of solutions.

Appendix I: Background

The following result, which is a consequence of the main statements in [30], establishes the existence of periodic solutions for singularly-perturbed systems,

$$\begin{aligned} \dot z &= f(z,e,\varepsilon) \\ \varepsilon\dot e &= g(z,e,\varepsilon) \qquad (z,e)\in\mathbb{R}^{m_z}\times\mathbb{R}^{m_e}. \end{aligned} \tag{44}$$

Lemma 9: Consider the singularly perturbed system (44) such that the following properties hold:

1) the functions f and g are continuous with respect to (z,e,ε) and differentiable with respect to z and e. Moreover, the derivatives of f and g with respect to z and e depend continuously on (z,e,ε).
2) There is a unique function $h:\mathbb{R}^{m_z}\to\mathbb{R}^{m_e}$ such that $g(z,h(z),0)=0$.
3) The equilibrium state $y=0$ (with $y=e-h(z)$) of the boundary-layer system
$$y' = g(z, y+h(z), 0),$$
where $y' := dy/d(t/\varepsilon)$, is hyperbolic uniformly in z.
4) The unperturbed system
$$\dot{\bar z} = f(\bar z, h(\bar z), 0) \tag{45}$$
has a nontrivial nonsingular periodic orbit $\gamma_o\subset\mathbb{R}^{m_z}$.

Then, there exists $\rho_o>0$ and a class $\mathcal{K}$ function ε_o such that for each $\rho\in(0,\rho_o]$ and for each $\varepsilon\le\varepsilon_o(\rho)$, the system (44) has a unique nontrivial periodic orbit Γ_ε, which is strictly contained in the ρ-neighborhood of Γ_o, where

$$\Gamma_o := \{(z,e)\in\mathbb{R}^{m_z}\times\mathbb{R}^{m_e} : z\in\gamma_o \text{ and } e=h(z)\}.$$

Moreover, the period α_ε of the periodic solution to (44) generating the orbit Γ_ε tends to α_o the period of the solution to (45) generating the orbit Γ_o. □

Appendix II: Auxiliary Lemmata

Given a function of two scalar variables that is smooth in one and only continuous in the other, the following original lemma shows the existence of a smooth approximation to any given 'nonuniform' degree of precision.

Lemma 10: Consider a function $T:[0,1]\times[0,\alpha_o]\to\mathbb{R}^{n\times n}$ such that $\tau\mapsto T(\varepsilon,\tau)$ is continuous, $\varepsilon\mapsto T(\varepsilon,\tau)$ is continuously differentiable, and $\tau\mapsto T(0,\tau)$ is continuously differentiable. Then, for each $\rho>0$, there exists $\hat T:[0,1]\times[0,\alpha_o]\to\mathbb{R}^{n\times n}$ continuously differentiable such that

$$\hat T(0,\tau) = T(0,\tau) \qquad \forall\tau\in[0,\alpha_o], \tag{46}$$
$$|\hat T(\varepsilon,\tau) - T(\varepsilon,\tau)|_\infty \le \rho\varepsilon + o(\varepsilon) \qquad \forall\tau\in[0,\alpha_o], \tag{47}$$

and

$$\lim_{\varepsilon\to0}\dot{\hat T}(\varepsilon,\tau) = \lim_{\varepsilon\to0}\frac{\partial\hat T(\varepsilon,\tau)}{\partial\tau} = \dot T(0,\tau). \tag{48}$$

□

Proof: Since the matrix T is continuously differentiable in ε and continuous in τ, then it admits a first-order Taylor expansion of the form

$$T(\varepsilon,\tau) = T(0,\tau) + a(\tau)\varepsilon + g(\varepsilon,\tau),$$

where $a:[0,\alpha_o]\to\mathbb{R}^{n\times n}$ is continuous and $g:[0,1]\times[0,\alpha_o)\to\mathbb{R}^{n\times n}$ enjoys the same continuity and smoothness properties as T. Furthermore, there exists $M>0$ such that, for each $\tau\in[0,\alpha_o]$, we have

$$|g(\varepsilon,\tau)|\le M\varepsilon^2 \qquad \forall\varepsilon\in[0,1].$$

Now, we choose the matrix $\hat T$ as

$$\hat T(\varepsilon,\tau) = T(0,\tau) + \hat a(\tau)\varepsilon,$$

where $\hat a:[0,\alpha_o]\to\mathbb{R}^{n\times n}$ is a continuously differentiable approximation of a on $[0,\alpha_o]$ satisfying

$$\sup_{\tau\in[0,\alpha_o]}\{|a(\tau)-\hat a(\tau)|\}\le\rho.$$

To obtain the latter inequality we used StoneWeierstrass theorem stating that every continuous function defined on a closed interval $[0,\alpha_o]$ can be uniformly approximated as closely as desired by a polynomial function [70]. As a result, (46) holds. Furthermore, we note that

$$T(\varepsilon,\tau) - \hat T(\varepsilon,\tau) = (a(\tau)-\hat a(\tau))\varepsilon + g(\varepsilon,\tau),$$

which implies that (47) also holds. Finally, (48) holds under (46) and the continuous differentiability of $\hat T$. ■

Consider the dynamical system

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n, \tag{49}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and the origin $x = 0$ is a hyperbolic equilibrium point. The following lemma allows us to show that the propagation of the local stable manifold $W^s(0)$ using the backward solutions to (49) is a null-measure set.

Lemma 11: Consider the dynamical system (49) and let $S_o \subset \mathbb{R}^n$ and $T > 0$ such that, for each $x_o \in S_o$, the solution $x(t)$ is well defined on $[0, T]$. Furthermore, for each $t \in [0, T]$, we define the reachable set

$$R^b(t, S_o) := \{y \in \mathbb{R}^n : y = x(t),\ x(0) = x_o \in S_o\}.$$

Then, if there exists $\tau \in [0, T]$ such that $\mu(R^b(\tau, S_o)) = 0$ then $\mu(S_o) = 0$, where $\mu(\cdot)$ is the Lebesgue measure of $(\cdot)$. □

Proof: To find a contradiction, we assume that $\mu(S_o) > 0$ and $\mu(R^b(\tau, S_o)) = 0$ for some $\tau \in [0, T]$. Next, we introduce the mapping $\phi_\tau : S_o \to \mathbb{R}^n$ such that $\phi_\tau(x_o) := x(\tau)$, where $x(\tau)$ is the unique solution to (49) starting from x_o. Using [71, Theorem V.2.1], we conclude that the mapping ϕ_τ is continuous and clearly the reciprocal mapping satisfies $\phi_\tau^-(x_o) := x(-\tau, x_o)$. Hence, $\phi_\tau^-(\cdot)$ is also continuous and therefore ϕ_τ is a homeomorphism; thus, an open map. Let us now fix $x_o \in S_o$ arbitrary such that there exists $U(x_o)$ an open set containing x_o that is contained in S_o, the latter is possible to find since $\mu(S_o) \neq 0$. Let $\phi_\tau(U(x_o))$ be the image of $U(x_o)$ by the homeomorphism ϕ_τ. Since ϕ_τ is a homeomorphism, $\phi_\tau(U(x_o))$ is an open set containing $\phi_\tau(x_o)$. Hence, $\mu(\phi_\tau(U(x_o))) \neq 0$. However, $\phi_\tau(U(x_o)) \subset R^b(\tau, S_o)$ and we already assumed that $\mu(R^b(\tau, S_o)) = 0$, which yields to a contradiction. ■

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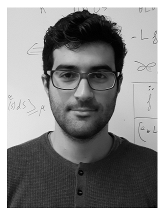

Mohamed Maghenem received his Control-Engineer degree from Polytechnical School of Algiers, Algeria, in 2013, his M.Sc. and PhD. degrees in Automatic Control from the University of Paris-Saclay, France, in 2014 and 2017, respectively. He was a Postdoctoral Fellow at the Electrical and Computer Engineering Department at the University of California at Santa Cruz from 2018 through 2021. M. Maghenem holds a research position at the French National Centre of Scientific Research (CNRS) since January 2021. His research interests include control systems theory (linear, non-linear, hybrid, and infinite dimensional) to ensure (stability, safety, reachability, synchronization, and robustness); with applications to power systems, mechanical systems, cyber-physical systems, and some partial differential equations.

Elena Panteley is a Senior Researcher at CNRS and a member of the Laboratory of Signals and Systems. She received her PhD. degree in Applied Mathematics from the State University of St. Petersburg, Russia, in 1997. From 1986 to 1998, she held a research position with the Institute for Problem of Mechanical Engineering, Russian Academy of Science, St. Petersburg. She is co-chair of the International Graduate School of Control of the European Embedded Control Institute (EECI-IGSC). Elena Panteley is the Book-reviews Editor for Automatica and Associate Editor for IEEE Control Systems Letters. Her research interests include stability and control of nonlinear dynamical systems and networked systems with applications to multi-agent systems.

Antonio Loría obtained his BSc degree on Electronics Engineering from ITESM, Monterrey, Mexico in 1991 and his MSc and PhD degrees in Control Engg. from UTC, France in 1993 and 1996, respectively. He has the honor of holding a researcher position at the French National Centre of Scientific Research (CNRS) since January 1999 (Senior Researcher since 2006). He has co-authored about 300 publications on control systems and stability theory and he served as Associate Editor, for a cumulated total in excess of 45 years, for the *IEEE Transactions on Automatic Control*, *Transactions on Control Systems Technology*, *Control Systems Letters*, and the *IEEE Conference Editorial Board*, as well as for *Automatica*, *Systems and Control Letters*, and *Revista Iberoamericana de Automática e Informática industrial*.