Module 3

Relations and Functions:

- Cartesian Products and Relations
- Functions
- Plain and One-to-One
- Onto Functions
- Stirling Numbers of the Second Kind
- Special Functions
- The Pigeon-hole Principle
- Function Composition and
- Inverse Functions

- Properties of Relations
- Computer Recognition
- Zero-One Matrices
- Directed Graphs
- Partial Orders
- Hasse Diagrams
- Equivalence Relations and
- Partitions

Module-3 10 Hours

Relations

SYLLABUS

Relations and Functions: Cartesian Products and Relations, Functions – Plain and One-to-One, Onto Functions – Stirling Numbers of the Second Kind, Special Functions, The Pigeon-hole Principle, Function Composition and Inverse Functions

Introduction

Product set: If A and B are any 2 non-empty sets then the product set of A and B are the Cartesian product or product of A and B.

$$A \times B = \{(a, b) / (a \in A, b \in B)\}$$

$$A \times B \neq B \times A$$

Example: (a) Let, A = {1, 2, 3} B = {a, b}

Then, A X B = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

B X A = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}

A X B ≠ B x A

(b) Let, A = {1, 2} B = {a, b} C={x, y}

B X C = {(a, x), (a, y), (b, x), (b, y)}

A X (B X C) = {(1, (a, x)), (1, (a, y)), (1, (b, x)), (1, (b, y)),
(2, (a, x)) (2, (a, y)), (2, (b, x)), (2, (b, y))}

A X B = {(1, a), (1, b), (2, a), (2, b)}

(A X B) X C = {((1, a), x), ((1, a), y), ((1, b), x), ((1, b), y),
((2, a), x), ((2, a), y), ((2, b),x),((2,b),y),}

*Remarks:

a. A X (B X C) = (A X B) X C

b. A X A = A^2

c. If R is the set of all real numbers then R x R = R^2, set of all points in plane.

d. (a, b) = (c, d) if a = c and b = d

<u>Partition set:</u> Let 'A' be a non-empty set. A partition of 'A' or quotient set of 'A' is a collection P of subsets of

'A' such that.

(a) Every element of A belongs to some set in P

(b) If A_1 and A_2 are any two distinct members of P, then A_1 n A_2 = ϕ.

(c) The members of P are called 'blocks' or 'cells'.

<u>Example:</u>

Let,

A = {1, 2, 3, 4, 5} then,

P_1 = {{1, 2, 3}, {4}, {5}}

P_2 = {{1, 5}, {4, 3}, {2}}

P_3 = {{1}, {2}, {3}, {4}, {5}}

<u>Relations:</u> Let A and B be any two non-empty sets. A relation R from a set A to the set B is a subset of A x B.

If (a, b) Є R then we write a R b, otherwise we write a $\not R$ b (ie. a not related to b).

<u>Example:</u>

Let,

A = {1, 2, 3, 4, 5}, Let R be a relation on A defined as a R b if a<b. R = {(1, 2), (1, 3), (1, 4), (1, 5) (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

=> R $\subseteq$ A X A.

Domain of R: Dom (R) = {1, 2, 3, 4} $\subseteq$ A

Range of R: Ran (R) = {2, 3, 4, 5} $\subseteq$ B

Dom (R) = {x ЄA / x R y for some x Є A}

Ran (R) = {y Є B / x R y for some y Є B}

R - Relative set: If R is a relation from A to B and if x ЄA then the R relative set of x is defined as

$$R(x) = \{y \in B / x \, R \, y\}$$

If $A_1 \subseteq A$ then the R relative set of A_1 is defined as,

$$R(A_1) = \{y \in B / x \, R \, y \text{ for some } x \in A_1\}$$

$$= U \, R(x) \text{ for } x \in A_1$$

Example:

Let,

$A = \{a, b, c, d\}$

$R = \{(a, a), (a, b), (b, c), (c, a) \, (c, b) \, (d, a)\}$

$R(a) = \{a, b\}$

$R(b) = \{c\}$

$R(c) = \{a, b\}$

$R(d) = \{a\}$

Let,

$A_1 = \{a, c\}$ be a subset of A,

Then, $R(A_1) = R(a) \, U \, R(c)$

$= \{a, b\} \, U \, \{a, b\}$

$= \{a, b\}$

Matrix of a relation / Relation Matrix: Let $A = \{a_1, a_2, a_3 \ldots\ldots\ldots\ldots\ldots\ldots \, a_m\}$ and $B = \{b_1, b_2, b_3 \ldots b_n\}$ be any two finite sets.

Let R be relation from A to B then the matrix of the relation R is defined as the m x n matrix,

$$M_R = [M_{ij}]$$

Where $M_{ij} = 1$, if $(a_i, b_j) \in R$

$= 0$, if $(a_i, b_j) \notin R$

Example:

(a) Let,

A = {1, 2, 3} and B = {x, 4}

R = {(1, x) (1, 4), (2, 4) (3, x)}

Thus, $M_r = \begin{bmatrix} 10 \\ 01 \\ 10 \end{bmatrix}$

(b) Given $M_R = \begin{bmatrix} 1001 \\ 0110 \\ 1010 \end{bmatrix}$. Find Relation R.

Define set,

A = {a_1, a_2, a_3} and

B = {b_1, b_2, b_3, b_4}

R = {(a_1, b_2) (a_1, b_4) (a_2, b_2) (a_2, b_3) (a_3, b_1) (a_3, b_3)}

Digraph of a relation: Let A be a finite set and R be a relation on A. Then R can be represented pictorially as follows,

(a) Draw a small circle for each element of A and label the circles with the corresponding element of A. These circles are called "Vertices".

(b) Draw an arrow from a_i to a_j if a_i R a_j. These arrows are called "edges".

(c) The resulting picture representing the relation R is called the "directed graph of R" or "digraph of R".

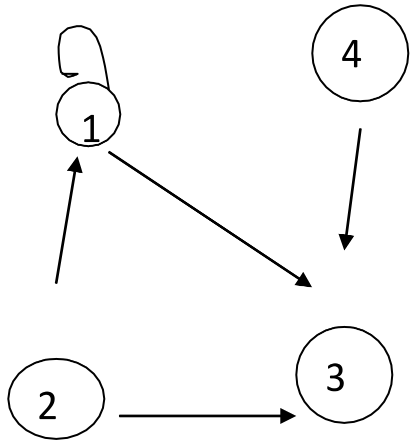
Example:

(a) Let, A be equal to the set

A = {1, 2, 3, 4}

R = {(1, 1), (1, 3), (2, 1), (2, 3), (3, 2), (4, 3)}

<u>Diagram:</u>



The "indegree" of a ЄA is the number of elements b Є A such that b R a.

The "outdegree" of a Є A is the number of elements b Є A such that a R b

Elements	Indegree	Outdegree
1	2	2
2	1	2
3	3	1
4	0	1

(b) If A = {1, 2, 3, 4} and B = {1, 4, 6, 8, 9} and R: A $\rightarrow$B defined by

a R b if $b = a^2$.Find the domain, Range, and M_R

A = {1, 2, 3, 4}   B = {1, 4, 6, 8, 9}

R = {(x, y)/x A, y B and $y = X^2$}

R = {(1, 1), (2, 4), (3, 9)}

Domain: Dom (R) = {1, 2, 3}

Range: Ran (R) = {1, 4, 9}

$$M_r = \begin{bmatrix} 10000 \\ 01000 \\ \\ 00001 \\ 00000 \end{bmatrix}$$

<u>Properties of a relation:</u>

1. **Reflexive:** Let R be a relation on a set A.

The "R is reflexive" if (a, a) Є R ∀ a ЄA or a R a, ∀ a Є A.

<u>Example:</u> A = {I, 2, 3}

R = {(1, 1), (2, 2) (1, 2), (3, 2) (3, 3)}

Therefore, R is reflexive.

A relation R on a set A is "non-reflexive" if ‗a‘ is not relation to ‗a‘ for some a ЄA or (a, a) ∉R for some a Є A

A = {1, 2, 3}

R = {(1, 1), (2, 1), (3, 2), (3, 3)} => (2, 2) ∉R

Therefore, R is *not-reflexive.*

2. **Irreflexive**: A relation R on a set A is irreflexive if a R a, ∀ a Є A.

<u>Example:</u> R = {(1, 2) (2, 1) (3, 2) (3, 1)}

(1, 1), (2, 2) (3, 3) ∉R hence R is irreflexive.

A relation R on a set A is –not irreflexive‖ if ‗a‘ is not Relation to ‗a‘ for some a Є A.

<u>Example:</u> R= {(1, 1) (1, 2) (2, 1) (3, 2) (3, 1)}

(1, 1) Є R hence R is ―not irreflexive‖.

3. **Symmetric Relation:** Let R be a relation on a set A, then R is –symmetric‖ if whenever a R b, then b R a; ∀ a Є A, b Є A.

<u>Example:</u> Let A = {1, 2, 3} and R = {(1, 1) (1, 2) (2, 1) (3, 2) (2, 3)}

Therefore, R is symmetric.

A relation R on a set A is said to be "not symmetric" if a R b and b ~~R~~ a for some a, b Є A.

Example: A = {1, 2, 3} and R = {(1, 2) (3, 2) (1, 3) (2, 1) (2, 3)}

Therefore, R is not symmetric.

4. **Asymmetric:** Let R be a relation on a set A then R is –Asymmetric‖, if whenever a Rb then b R a, ∀ a, b Є A.

R = {(1, 2), (1, 3) (3, 2)}

Therefore, R is asymmetric.

A relation R on a set A is said to be "not Asymmetric" if a R b and b R a for some a, b ЄA R = {(1, 1) (1, 2) (1, 3) (3, 2)}

R is not symmetric.

5. **Anti – symmetric:** Let R be a relation on a set A, then R is anti symmetric if whenever a R b and b R a then a = b (for some a, b Є A)

Example: Let, A = {1, 2, 3} and R = {(1, 1), (1, 2), (3, 2)}

R is anti-symmetric Є 1R1 and 1 = 1.

Example: R = {(1, 2) (2, 1)}

1R2, 2R1 but 2 = 1 hence R is not anti symmetric.

6. **Transitive Property:** Let R be a relation on a set A, then R is transitive if whenever a R b and b R c, then a R c ∀ a, b, c Є ¸ A.

Example: Let, A = {1, 2, 3} and R = {(1, 1), (1, 3), (2, 3), (3, 1) (2, 1), (3, 3)} (all should satisfy)

Equivalence relation: A Relation R is said to be an equivalence relation if it is,

(a) Reflexive

(b) Symmetric and

(c) Transitive.

Therefore, R is an equivalence Relation.

Symmetric: Let a R b

=> Є 1R2

2 is not Related to 1 and also b is not Related to a

Hence, R is not symmetric

Transitive: Let a R b and b R c

=> 1 R 2 and 2 R 3 but, 1 is not Related to 3 and also a is not Related to c

Hence, R is not transitive.

Therefore, R is not an equivalence Relation.

b. R = {(1, 2), (2, 1) (1, 3) (3, 1) (2, 3) (3, 2)}

Reflexive: a R a ∀ a ∈ A

=> 1 R1, 2 R 2, 3 R 3 not true,

Hence, R is not reflexive

Symmetric: Let a R b

=> 1 R 3

=> 3 R 1

=> b R a

Hence, R is symmetric.

Transitive: Let a R b and b R c

=> 1 R 2 and 2 R 3

=> 1 R 3

=> a R c

Hence, R is transitive

Therefore, R is not an equivalence Relation.

c. A = {1, 2, 3}

$R = A \times A = \{(1, 1)(1, 2)(1, 3)(2, 1)(2, 2)(2, 3)(3, 1)(3, 2)(3, 3)\}$

It is reflexive, symmetric and transitive and hence R is an equivalence Relation.

<u>Theorem</u>: "Let R be an equivalence relation on a set A, and P be the collection of all distinct R - relative set of A. Then P is a partition of A, and R is the equivalence relation determined by P"

OR

"Show that an equivalence relation R in a set S which is non-empty, determine a partition of S"

Proof: Given, $P = \{R(a) / \forall a \in A\}$

We know that $\forall a \in A$, we have, $a R a$

$\Rightarrow (a, a) \in R$

$\Rightarrow a \in R(a)$

Therefore, for every element of A belongs to one of the sets of P.

If R (a) and R (b) are 2 distinct relative sets $R(a) \cap R(b) = \Phi$

If possible, let $x \in R(a) \cap R(b)$

$\Rightarrow x \in R(a)$ and $x \in R(b)$

$\Rightarrow a R x$ and $b R x$

$\Rightarrow a R x$ and $x R b$ (since R is symmetric)

$\Rightarrow a R b$ (since R is transitive)

$\Rightarrow R(a) = R(b)$ (by theorem)

Therefore, If $R(a) = R(b)$, then $R(a) \cap R(b) = \Phi$.

Therefore, from the above, P is a partition of the set A.

This partition determines the relation R in the sense that a R b if a and b belong to the same block of the partition.

Hence proved…..

NOTE: The partition of a set A determined by an equivalence relation R is called the partition induced by R and is denoted by A/R.

Manipulation of relations:

1. **Complement:** Let R be a relation from A to B. The complement of R is a relation defined as a R b if a $R^{\sim}$ b, where $R^{\sim}$ is the complement of R.

=> (a, b) $R^{\sim}$ if (a, b) $R^{\sim}$

2. **Union:** Let R and S be 2 relations from A to B. The union R U S is a relation from A to B defined as,

a (R U S) b if either a R b or a S b

That is (a, b) Є R U S if either (a, b) Є R or (a, b) Є S.

3. **Intersection:** Let Rand S be relations from A to B. The intersection R n S is a relation from A to B defined as,

a (R n S) b if a R b and a S b

That is (a, b) Є R n S if (a, b) Є R and (a, b) Є S.

4. **Inverse:** Let R be a relation from A to B. The inverse R^{-1} is a relation from B to A defined as, a R b if b R^{-1} a

i.e., (a, b) Є R if (b, a) Є R^{-1}

Composition of relations: Let Rand S be relations from A to Band B to C respectively.

The composition of Rand S is the

relation S o R from A to C defined as,

a(S o R) c if there-exist b E B/a R b and b S c.

R^2 =R o R = {(a, a), (a, c) (a, b) (b, a) (b, c) (b, b) (c, a) (c, b) (c, c)}

S^2 = S o S = {(a, a) (b, b) (b, c) (b, a) (c, a) (c, c)}

Reflexive closure: Let R be a relation on a set' A'. Suppose R lacks a particular property, the smallest relation that contain R and which, processes the desired property is called the closure of R with respective a property in question.

Given a relation R on a set' A' the relation R1 =∆(A U R) is the "reflexive closure of R‖.

Example:

A = {1, 2, 3}

R = {(1, 1)(1,2)(2, 1)(1,3)(3, 2)} find the reflexive closure of R.

Solution: We know that, R is not reflexive because (2, 2) Є Rand (3, 3) Є R.

Now, A= {(1, 1) (2, 2) (3, 3)}

Therefore, R_1 = R U A= {(1, 1) (1, 2) (2, 1) (2, 2) (1, 3) (3, 2) (3, 3)}

R_1 is the reflexive closure of R.

Symmetric closure : If R is not symmetric then there exists (x, y) A such that (x, y) Є R, but (y, x) Є R. To make R symmetric we need to add the ordered pairs of R^{-1}.

R_1 = R U R^{-1} is the "symmetric closure of R".

A = {1, 2, 3}

R = {(1, 1) (1, 2) (2, 1) (1, 3) (3, 2)} find the symmetric closure of R.

Solution: We know that, R is not symmetric because (1, 3) Є R but (3, 1) Є R and (3, 2) Є R but (2, 3) Є R.

Example: R^{-1} = {(1, 1) (2, 1) (1, 2) (3, 1) (2, 3)}

Therefore, R_1 = R U R^{-l} = {(1, 1) (1, 2) (2, 1) (1, 3) (3, 1) (3, 2) (2, 3)}

R_1 is called the symmetric closure of R.

Transitive closure: Let R be a relation on a set A the smallest transition relation containing R is called the "Transitive closure of R".

Functions

SYLLABUS

Relations *contd.*: Properties of Relations, Computer Recognition – Zero-One Matrices and Directed Graphs, Partial Orders – Hasse Diagrams, Equivalence Relations and Partitions

Introduction

A person counting students present in a class assigns a number to each student under consideration. In this case a correspondence between two sets is established: between students understand whole numbers. Such correspondence is called functions. Functions are central to the study of physics and enumeration, but they occur in many other situations as well. For instance, the correspondence between the data stored in computer memory and the standard symbols a, b, c... z, 0, 1,...9,?,!, +... into strings of O's and I's for digital processing and the subsequent decoding of the strings obtained: these are functions. Thus, to understand the general use of functions, we must study their properties in the general terms of set theory, which is will be we do in this chapter.

Definition: Let A and B be two sets. A function f from A to B is a rule that assigned to each element x in A exactly one element y in B. It is denoted by

f: A $\rightarrow$B

Note:

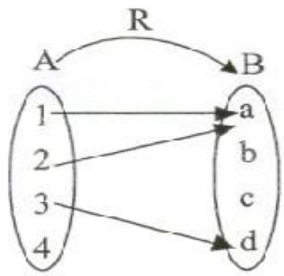
1. The set A is called domain of f.
2. The set B is called domain of f.

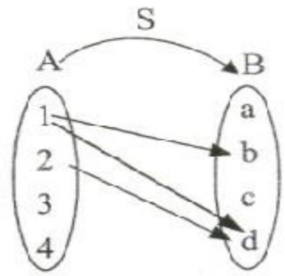
Value of f: If x is an element of A and y is an element of B assigned to x, written y = f(x) and call function value of f at x. The element y is called the image of x under f.

Example: A = {1, 2, 3, 4} and B= {a, b, c, d}

R = {(1, a), (2, b), (3, c), {4, d)}

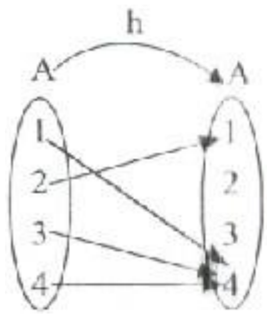
$S = \{(I, b), (I, d), (2, d)\}$





Therefore, R is a function and S is not a function. Since the element 1has two images band d, S is not a function.

Example: Let A = {1, 2, 3, 4} determine whether or not the following relations on A are



functions.

1. f= {(2, 3), (1, 4), (2, 1), (312), (4, 4)}

(Since element 2 has 2 images 3 and 1, f is not a function.)

2. g={(3,1),(4,2),(1,1)}

g is a function

3. h={(2,1),(3,4),(1,4),(2,1),(4,4)}

h is a function

4. Let A= {0, ±1, ±2, 3}. Consider the function F: A→ R, where R is the set of all real numbers, defined by f(x) $=x^3 -2x^2 +3x+1$ for $x\in A$. Find the range of f.

f (0) =1

f (1) =1-2+3+1=3

f (-1) =-1-2-3+1=-5

f (2) =8-8-6+1=7

f (-2) =-8-8-6+1= -21

f (3) =27-18+9+1=19

$\therefore$ Range = {1, 3,-5, 7,-21, 19}

5. If A= {0, ±1, ±2} and f: A$\rightarrow$ R is defined by f(x) =x^2-x+1, x$\in$A find the range.

f (0) =1

f (1) =1-1+1=1

f (-1) =1+1+1=3

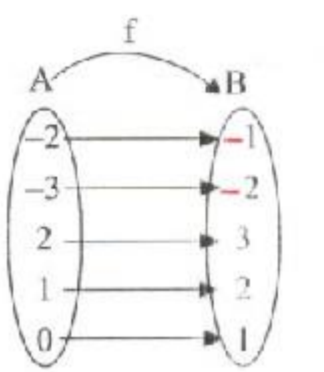
f (2) =4-2+1=3

f (-2) =4+2+1=7

$\therefore$ Range = {1, 3, 7}

Types of functions:

1. ***Everywhere defined* -2**
   A function f: A ~ B is everywhere defined if domain of f equal to A (dom f = A)



Example: Y =f(x) =x+1

Here, dom f=A

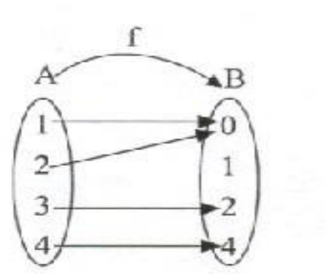
2. ***Onto or surjection function***

A function f: A →B is onto or surjection if Range of f = B. In other words, a function f is surjection or onto if for any value *l in* B, there is at least one element x in A for which f(x) = y.

3. Many to one function

A function F is said to be a many-to-one function if a *:f*= b, f(a) = f(b), where (a, b) E A.

Example:



Here, *1: f*=2 but f (1) = f (2), where 1,2 E A

4. One-to-one function or injection

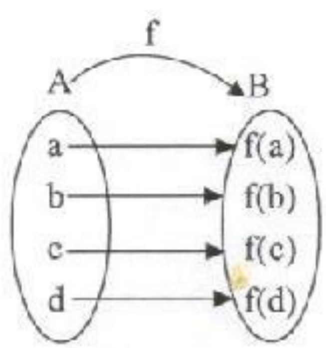
A function f: A →B is one-to-one or injection if (a) =f (b) then a = b, where a, b E A.

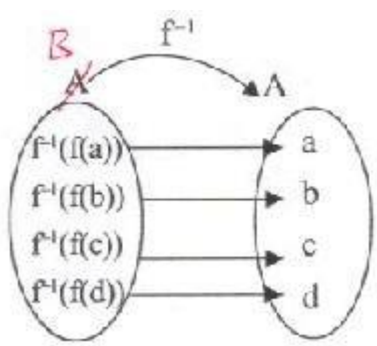
In other words if a*: f*=b then f (a): f= f (b).

5. Bijection function

A function f: A →B is Bijection if it is both onto and one-to-one.

6. Invertible function

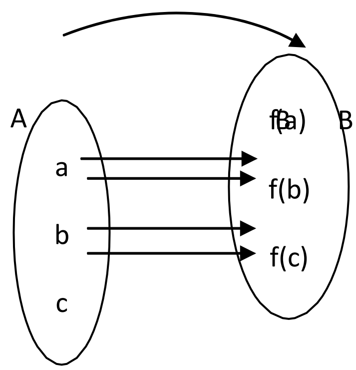


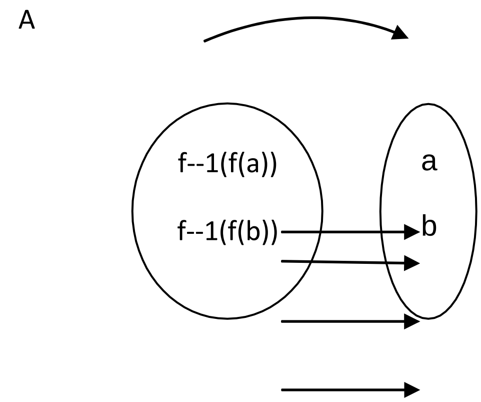


A function f: A ---+ B is said to be an invertible function if its inverse relation, f-I is a function from B → A.

If f: A →B is Bijection, then [-I: B ---+A exists, f is said to be invertible.

Example: f f^{-1}





Here f: A → B f^{-1} B → A

A = {a_1, a_2, a_3} B = {b_1, b_2, b_3} C = { c_1, c_2} D = {d_1, d_2, d_3, d_4}

Let f_1: A → B, f_2: A → D, f_3: B → C, f4: D → B be functions defined as follows,

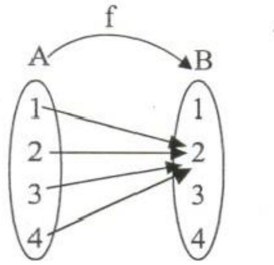
1. f1 = {(a1, b2) (a2, b3) (a3 ,b1)}
2. f_2 = {(a_1, d_2) (a_2, d_1) (a_3 , d_4)}
3. f_3 = {(b_1, c_2)(b_2, c_2) (b_3, c_1)}
4. f_4 = { (d_1, b_1) (d_2, b_2) (d_3,b_1)}

Identity function

A function f: A~ A such that f (a) = a, 'if a Є A is called the identity function or identity mapping on A. Dom (t) = Ran (t) = A

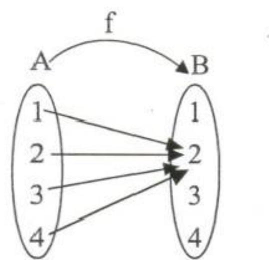
Constant function

A function **f**: A → B such that f (a) =c, ∀a∈ dom (f) where c is a fixed element of B, is called a constant function.



Into function

A function f: A $\rightarrow$ B is said to be an into function if there exist some b in B which is not the image of any a in A under f.



3 is not the image of any element.

One-to-one correspondence

If f: A $\rightarrow$ B is everywhere defined and is Bijective, then corresponding to every $a \in A$ there is an unique $b \in B$ such that b=f(a) and corresponding to every $b \in B$ there is an unique $a \in A$ such that f(a)=b. for this reason a everywhere defined bijection function from A $\rightarrow$ B is called as one-one correspondence from A $\rightarrow$ B

Composition of function

Let f: A (B and g: B (C is any 2 functions, and then the composition of f and g is a function g o f: A (C defined as, g of (a) =g [f (a)] (C, (a (dom f).

Inverse function

Consider a function f: A (B. Then f is a relation from A to B with Dom (f) (A and Ran (f) (B. Its inverse, f -1, is a relation from B to A which is such that if whenever (a, b) (f then (b, a) (f -1)

Also, Dom (f -1) = Ran (f)

Ran (f -1) =Dom (f) and

$(f^{-1})^{-1} = f$

Definition

A function f: A (B is invertible if it is inverse relation f^{-1} is a function from B to A. Then, f^{-1} is called the inverse function of f.

Ex: let A = {a, b, c, d} and B = {e, f, g, h} and f: A (B be a function defined by

f (a) = =e, f (b) = e, f (c) = h, f (d) = g

Then, as a relation from A to B, f reads

f = {(a, e), (b, e), (c, h), (d, g)}

And f^{-1} is a relation from B to A, given by

f^{-1} = {(e, a), (e, b), (h, c), (g, d)}

Now, Dom (f^{-1}) = [e, h, g} = Ran(f) and

Ran (f^{-1}) = {a, b, c, d} = A = Dom (f)

Also, $(f^{-1})^{-1} = f$

Although f^{-1} is a relation from B to A, it is not function from B to A, because e is related to two elements _a‘ and _b‘ under f^{-1}.

Let A = {1,2,3,4} and B = {5,6,7,8} and the function f: A (B defined by

f (1) = 6, f(2) = =8, f(3) = 5, f(4) = 7

Then, f = {(1, 6), (2, 8), (3, 5), (4, 7)}

∴ f^{-1} = {(6 , 1), (8 , 2), (3 , 5), (7 , 4)}

In this case, f^{-1} is not only a relation from B to A but a function as well.

Characteristic function

Introduction

Characteristic function is a special type of function. It is very useful in the field of computer science. Through this function one can tell whether an element present in the set or not. If the function has the value 1 then the particular element belongs to the set and if it has value 0 then the element is not present in the set.

Definition

Associated with the subset a of $\cup$ we can define a characteristic function of A over $\cup$ as $f: \cup \to \{0, 1\}$ where

$f_A(x) =$ 1 if $x \in A$

0 if $x \notin A$

Properties of the characteristics function

1. $f_{\mathbf{A\,n\,B}}(x) = f_{\mathbf{A}}(x).\ f_{\mathbf{B}}(x)$

Proof:

i. if $x \in AnB$ then $x \in A$ and $x \in B$

$\Rightarrow f_A(x) = 1$ and $f_B(x) = 1$

$\therefore$ $f_{AnB}(x) = 1 = f_{\mathbf{A}}(x).\ f_{\mathbf{B}}(x)$

ii. if $x \notin AnB$ then $f_{AnB}(x) = 0$. but if $x \notin AnB$ then $x \notin A$ and $x \notin B$

$\Rightarrow f_A(x) = 0$ and $f_B(x) = 0$

$\therefore$ $f_{AnB}(x) = 0 = f_{\mathbf{A}}(x).\ f_{\mathbf{B}}(x)$

$\therefore$ From case 1 and 2

$f_{AnB}(x) = f_{\mathbf{A}}(x).\ f_{\mathbf{B}}(x)$

2. $f_{\mathbf{AUB}}(x) = f_{\mathbf{A}}(x) + f_{\mathbf{B}}(x) - f_{\mathbf{A}}(x).\ f_{\mathbf{B}}(x)$

Proof:

i. Let $x \in AUB$ then $f_{AUB}(x) = 1$. But if $x \in AUB$ then there are three cases

case1: let $x \in A$ but not in B then $f_A(x) = 1$ and $f_B(x) = 0$ $\Rightarrow f_{AUB}(x) = 1 = f_A(x) + f_A(x) - f_A(x) . f_B(x)$

[Because 1+0+0]

case2: let $x \in B$ but not in A

Then $f_B(x) = 1$ and $f_A(x) = 0$

$\Rightarrow$ $f_{AUB}(x) = 1 = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
[Because 0+1-0]

case3: let $x \in A$ and $x \in B$

Then $f_A(x) = 1$ and $f_B(x) = 1$

$\Rightarrow$ $f_{AUB}(x) = 1 = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
[Because 1+1-1]

$\therefore$ $f_{AUB}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$

ii. Let $x \notin A \cup B$ then $f_{AUB}(x) = 0$

If $x \notin A \cup B$ then $x \notin A$ and $x \notin B$ then

$\therefore$ $f_A(x) = 0$ and $f_B(x) = 0$

$\Rightarrow$ $f_{AUB}(x) = 0 = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
[because 0+0-1]

$\therefore$ From case i and ii.
$\therefore$ $f_{AUB}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$

A symmetric difference is associative on sets

To prove $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ we have to prove $f_{(A \oplus B) \oplus C}(x) = f_{A \oplus (B \oplus C)}(x) \quad \forall x$

LHS = $f_{(A \oplus B) \oplus C}$

= $f_{(D \oplus C)}$ where $D = A \oplus B$

$= f_D + f_c - 2 f_D f_c$

$= f_c + f_D (1\text{-}2 f_c)$

$= f_D + f_{A \oplus B} (1\text{-}2 f_c)$

$= f_c + (f_A + f_B - 2 f_A f_B) (1 - 2 f_c)$

$= f_c + f_A + f_B - 2 f_A f_B - 2 f_A f_C - 2 f_B f_C + 4 f_A f_B f_C$

$= f_A + (f_B + f_c - 2 f_B f_C) - 2 f_A (f_B + f_C - 2 f_B f_C)$

$= f_A + f_B + f_C - 2 f_B f_C (1\text{-} 2 f_A)$

$= f_A + f_{B \oplus C} (1 - 2 f_A)$

$= f_A + f_{B \oplus C} - 2 f_A f_{B \oplus C}$

$= f_{A \oplus (B \oplus C)}$

= RHS

$(A \oplus B) \oplus C = A \oplus (B \oplus C)$