MODULE-2 Properties of the Integers: 10 Hours

- Mathematical Induction
- The Well Ordering Principle
- Mathematical Induction
- Recursive Definitions

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Properties of the Integers

SYLLABUS

Properties of the Integers: Mathematical Induction, The Well Ordering Principle – Mathematical Induction, Recursive Definitions

MATHEMATICAL INDUCTION:

The method of mathematical induction is based on a principle called the induction principle .

INDUCTION PRINCIPLE:

The induction principle states as follows : let S(n) denote an open statement that involves a positive integer n .suppose that the following conditions hold ;

1. S(1) is true
2. If whenever S(k) is true for some particular , but arbitrarily chosen k $€Z^+$, then S(k+1) is true. Then S(n) is true for all n € Z+ . Z+ denotes the set of all positive integers .

METHOD OFMATHEMATICAL INDUCTION

Suppose we wish to prove that a certain statement S(n) is true for all integers n ≥1 , the method of proving such a statement on the basis of the induction principle is calledd the method of mathematical induction. This method consist of the following two steps, respectively called the basis step and the induction step

1) Basis step: verify that the statement S(1) is true ; i.e. verify that S(n) is true for n=1.
2) Induction step: assuming that S(k) is true , where k is an integer≥1, show that S(k+1) is true.

Many properties of positive integers can be proved by mathematical induction.

Principle of Mathematical Induction:

Let P be a prop- erty of positive integers such that:

1. Basis Step: P (1) is true, and

2. Inductive Step: if P (n) is true, then P (n + 1) is true.

Then P (n) is true for all positive integers.

Remark : The premise P (n) in the inductive step is called Induction Hypothesis.

The validity of the Principle of Mathematical Induction is obvious. The basis step states that P (1) is true. Then the inductive step implies that P (2) is also true. By the inductive step again we see that P (3) is true, and so on. Consequently the property is true for all positive integers.

Remark : In the basis step we may replace 1 with some other integer m. Then the conclusion is that the property is true for every integer n greater than or equal to m.

Example: Prove that the sum of the n first odd positive integers is n^2, i.e., $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Answer: Let $S(n) = 1 + 3 + 5 + \cdots + (2n - 1)$. We want to prove by induction that for every positive integer n, $S(n) = n^2$.

1. Basis Step: If $n = 1$ we have $S(1) = 1 = 1^2$, so the property is true for 1.
2. Inductive Step: Assume (Induction Hypothesis) that the prop- erty is true for some positive integer n, i.e.: $S(n) = n^2$. We must prove that it is also true for n + 1, i.e., $S(n + 1) = (n + 1)^2$. In fact:

   $S(n + 1) = 1 + 3 + 5 + \cdots + (2n + 1) = S(n) + 2n + 1$.

   But by induction hypothesis, $S(n) = n^2$, hence:

   $S(n + 1) = n^2 + 2n + 1 = (n + 1)^2$.

This completes the induction, and shows that the property is true for all positive integers.

Example: Prove that $2n + 1 \leq 2^m$ for $n \geq 3$.

Answer : This is an example in which the property is not true for all positive integers but only for integers greater than or equal to 3.

1. Basis Step: If $n = 3$ we have $2n + 1 = 2 \cdot 3 + 1 = 7$ and $2^m = 2^3 = 8$, so the property is true in this case.

2. Inductive Step: Assume (Induction Hypothesis) that the prop- erty is true for some positive integer n, i.e.: $2n + 1 \le 2^m$. We must prove that it is also true for n + 1, i.e., $2(n + 1) + 1 \le 2^{m+1}$. By the induction hypothesis we know that $2n \le 2^m$, and we also have that $3 \le 2^m$ if $n \ge 3$, hence

$$2(n + 1) + 1 = 2n + 3 \le 2^m + 2^m = 2^{m+1} .$$

This completes the induction, and shows that the property is true for all $n \ge 3$.

Exercise: Prove the following identities by induction:

- $1 + 2 + 3 + \cdots + n = \dfrac{n (n + 1)}{2}$.
- $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n (n + 1) (2n + 1)}{6}$.
- $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

Strong Form of Mathematical Induction:

Let P be a property of positive integers such that:

1. Basis Step: P (1) is true, and

2. Inductive Step: if P (k) is true for all $1 \le k \le n$ then P (n + 1) is true.

Then P (n) is true for all positive integers.

Example: Prove that every integer $n \ge 2$ is prime or a product of primes. Answer :

1. Basis Step: 2 is a prime number, so the property holds for

$n = 2$.

2. Inductive Step: Assume that if $2 \le k \le n$, then k is a prime number or a product of primes. Now, either $n + 1$ is a prime number or it is not. If it is a prime number then it verifies the property. If it is not a prime number, then it can be written as the product of two positive integers, $n + 1 = k_1 k_2$, such that $1 < k_1, k_2 < n + 1$. By induction hypothesis each of k_1 and k_2 must be a prime or a product of primes, hence $n + 1$ is a product of primes.

This completes the proof.

The Well-Ordering Principle

Every nonempty set of positive integers has a smallest element.

Example : Prove that $\sqrt{2}$ is irrational (i.e., $\sqrt{2}$ cannot be written as a quotient of two positive integers) using the well-ordering principle.

Answer : Assume that $\sqrt{2}$ is rational, i.e., $\sqrt{2} = a/b$, where a and b are integers. Note that since $\sqrt{2} > 1$ then $a > b$. Now we have $2 = a^2/b^2$, hence $2b^2 = a^2$. Since the left hand side is even, then a^2 is even, but this implies that a itself is even, so $a = 2a'$. Hence: $2b^2 = 4a'^2$, and simplifying: $b^2 = 2a'^2$. From here we see that $\sqrt{2} = b/a'$.

Hence starting with a fractional representation of $\sqrt{2} = a/b$ we end up with another fractional representation $\sqrt{2} = b/a'$ with a smaller numerator $b < a$. Repeating the same argument with the fraction b/a' we get another fraction with an even smaller numerator, and so on. So the set of possible numerators of a fraction representing $\sqrt{2}$ cannot have a smallest element, contradicting the well-ordering principle. Consequently, our assumption that $\sqrt{2}$ is rational has to be false.

Reccurence relations

Here we look at recursive definitions under a different point of view. Rather than definitions they will be considered as equations that we must solve. The point is that a recursive definition is actually a def- inition when there is one and only one object satisfying it, i.e., when the equations involved in that definition have a unique solution. Also, the solution to those equations may provide a closed-form (explicit) formula for the object defined.

The recursive step in a recursive definition is also called a recurrence relation. We will focus on kth-order linear recurrence relations, which are of the form

$$C_0 x_m + C_1 x_{m-1} + C_2 x_{m-2} + \cdots + C_k x_{m-k} = b_m ,$$

where $C_0 = 0$. If $b_m = 0$ the recurrence relation is called homogeneous. Otherwise it is called non-homogeneous.

The basis of the recursive definition is also called initial conditions of the recurrence. So, for instance, in the recursive definition of the Fibonacci sequence, the recurrence is

$$F_m = F_{m-1} + F_{m-2}$$

or

$$F_m - F_{m-1} - F_{m-2} = 0 ,$$

and the initial conditions are

$$F_0 = 0, F_1 = 1 .$$

One way to solve some recurrence relations is by iteration, i.e., by using the recurrence repeatedly until obtaining a explicit close-form formula. For instance consider the following recurrence relation:

$$x_m = r\, x_{m-1} \qquad (n > 0) ; \quad x_0 = A .$$

By using the recurrence repeatedly we get:

$$x_m = r\, x_{m-1} = r^2 x_{m-2} = r^3 x_{m-3} = \cdots = r^m x_0 = A\, r^m ,$$

hence the solution is $x_m = A\, r^m$.

In the following we assume that the coefficients $C_0, C_1, \ldots, C_k$ are constant.

First Order Recurrence Relations. The homogeneous case can be written in the following way:

$$x_n = r\, x_{n-1} \qquad (n > 0) ; \quad x_0 = A.$$

Its general solution is

$x_n = A\, r^n$,

which is a geometric sequence with ratio r.

The non-homogeneous case can be written in the following way:

$x_n = r\, x_{n-1} + c_n \qquad (n > O)\ ; \qquad x_0 = A$.

Using the summation notation, its solution can be expressed like this:

$$x_n = A\, r^n + \sum_{k=1}^{n} c_k\, r^{n-k} .$$

We examine two particular cases. The first one is

$x_n = r\, x_{n-1} + c \qquad (n > O); \qquad x_0 = A.$

where c is a constant. The solution is

$$x_n = A\, r^n + c \sum_{k=1}^{n} r^{n-k} = A\, r^n + c\, \frac{r^n - 1}{r - 1} \qquad \text{if } r = 1 ,$$

and

$$x_n = A + c\, n \qquad \text{if } r = 1 .$$

Example : Assume that a country with currently 1OO million people has a population growth rate (birth rate minus death rate) of 1% per year, and it also receives 1OO thousand immigrants per year (which are quickly assimilated and reproduce at the same rate as the native population). Find its population in 1O years from now. (Assume that all the immigrants arrive in a single batch at the end of the year.)

Answer: If we call x_n = population in year n from now, we have:

$x_n = 1.O1\, x_{n-1} + 1OO,OOO \qquad (n > O); \qquad x_0 = 1OO,OOO,OOO$.

This is the equation above with $r = 1.O1$, $c = 1OO,OOO$ and $A = 1OO,OOO,OO$, hence:

$$x_n = 1OO,OOO,OOO \cdot 1.O1^n + 1OO,OOO\, \frac{1.O1^n - 1}{1.O1 - 1}$$

$$= 1OO,OOO,OOO \cdot 1.O1^n + 1OOO\, (1.O1^n - 1) .$$

So:

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The second particular case is for $r = 1$ and $c_m = c + d\,n$, where c and d are constant (so c_m is an arithmetic sequence):

$$x_m = x_{m-1} + c + d\,n \qquad (n > O)\,; \qquad x_0 = A.$$

The solution is now

$$x_m = A + \sum_{k=1}^{m} (c + d\,k) = A + c\,n + \frac{d\,n\,(n+1)}{2}.$$

Second Order Recurrence Relations. Now we look at the recurrence relation

$$C_0\,x_m + C_1\,x_{m-1} + C_2\,x_{m-2} = O\,.$$

First we will look for solutions of the form $x_m = c\,r^m$. By plugging in the equation we get:

$$C_0\,c\,r^m + C_1\,c\,r^{m-1} + C_2\,c\,r^{m-2} = O\,,$$

hence r must be a solution of the following equation, called the characteristic equation of the recurrence:

$$C_0\,r^2 + C_1\,r + C_2 = O\,.$$

Let r_1 , r_2 be the two (in general complex) roots of the above equation. They are called characteristic roots. We distinguish three cases:

1. Distinct Real Roots. In this case the general solution of the recurrence relation is

   $$x_m = c_1\,r_1^m + c_2\,r_2^m,$$

   where c_1, c_2 are arbitrary constants.

2. Double Real Root. If $r_1 = r_2 = r$, the general solution of the recurrence relation is

$$x_m = c_1 r^m + c_2 n r^m ,$$

where c_1, c_2 are arbitrary constants.

3. Complex Roots. In this case the solution could be expressed in the same way as in the case of distinct real roots, but in order to avoid the use of complex numbers we write $r_i — r e^{\alpha i}$,

$r_2 — r e^{—\alpha i}$, $k_i — c_i + c_2$, $k_2 — (c_i — c_2)$ i, which yields:[i]

$$x_m — k_i r^m \cos n\alpha + k_2 r^m \sin n\alpha .$$

Example: Find a closed-form formula for the Fibonacci sequence defined by:

$$F_{m+i} — F_m + F_{m—i} \quad (n > 0) ; \qquad F_0 — 0, F_i — 1 .$$

Answer: The recurrence relation can be written

$$F_m — F_{m—i} — F_{m—2} — 0 .$$

The characteristic equation is

$$r^2 — r — 1 — 0 .$$

Its roots are:[2]

$$r_i — \varphi — \frac{1 + \sqrt{5}}{2} ; \qquad r_2 —— \varphi^{—i} — \frac{1 — \sqrt{5}}{2} .$$

They are distinct real roots, so the general solution for the recurrence is:

$$F_m — c_i \varphi^m + c_2 (—\varphi^{—i})^m .$$

Using the initial conditions we get the value of the constants:

$$(n — 0) \quad c_i + c_2 \quad — \; 0 \qquad c_i \; — \; 1/\sqrt{5}$$

RECURSIVE DEFINITIONS:

RECURRENCE RELATIONS:- The important methods to express the recurrance formula in explict form are

1) BACKTRACKING METHOD
2) CHARACTERISTIC EQUATION METHOD

BACKTRACKING METHOD:

This is suitable method for linear non-homogenous recurrence relation of the type

$$x_n = r\, x_{n-1} + s$$

The general method to find explicit formula

$xn = r^{n-1} x_1 + s(r^{n-1}-1)/(r-1)$ where $r \neq 1$ is the general explicit

CHARACTERISTIC EQUATION METHOD:

This is suitable method to find an explicit formula for a linear homogenous recurrance relation

LINEAR HOMOGENOUS RELATION :

A recurrence relation of the type $a_n = r_1 a_{n-1} + r_2 a_{n-2} + \ldots\ldots + r_k a_{n-k}$ where r_i _s' are constants is a linear homogenous recurrence relation (LHRR) of degree k

1) A relation $c_n = -2\, c_{n-1}$ is a LHRR of degree 1 .
2) A relation $x_n = 4\, x_{n-1} + 5$ is a linear non HRR because 2nd term in RHS is a constant . It doesn't contain x_{n-2} factor .
3) A relation $x_n = x_{n-1} + 2x_{n-2}$ is a LHRR of degree 2
4) A relation $x_n = x^2_{n-1} + x_{n-2}$ is a non linear , non HRR because the 1st term in RHS is a second degree term.

CHARACTERISTIC EQUATION:

$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \ldots + r_k a_{n-k}$..(1) is a LHRR of degree K .

$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \ldots + r_k$ is called characteristic equation.

- Let $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ be LHRR of degree 2. its characteristic equation is $x^2 = r_1 x + r_2$ or $x^2 - r_1 x - r_2 = 0$. if the characteristic equation has 2 distinct roots e_1 , e_2 then the explicit formula of the recurrence relation in $a_n = u\, e^n{}_1 + v\, e^n{}_2$ where u and v depends on the initial values.
- Let $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ be a LHRR of degree 2 . Its characteristic equation is $x^2 - r_1 x - r_2 = 0$ if the characteristic equation has repeated roots e, then the explicit formula is $a_n = u\, e^n + v\, n\, e^n$ where u and v depends on the initial values.