**SRINIVAS UNIVERSITY**

**COLLEGE OF ENGINEERING & TECHNOLOGY**

**MUKKA, MANGALURU**


# DISCRETE MATHEMATICAL STRUCTURES AND GRAPH THEORY

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**Module 1; Fundamentals of Logic: 10 Hours**

- Basic Connectives and Truth Tables,
- Logic Equivalence
- The Laws of Logic,
- Logical Implication
- Rules of Inference
  - The Use of Quantifiers
  - Quantifiers
  - Definitions and Proofs of Theorems

# Module 1 10 Hours

Fundamentals of Logic

SYLLABUS

**Fundamentals of Logic: Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication – Rules of Inference**

**Introduction:**

**Propositions:**

A proposition is a declarative sentence that is either true or false (but not both). For instance, the following are propositions: ―Paris is in France‖ (true), ―London is in Denmark‖ (false), ―2 < 4‖ (true), ―4 = 7 (false)‖. However the following are not propositions: ―what is your name?‖ (this is a question), ―do your homework‖ (this is a command), ―this sentence is false‖ (neither true nor false), ―x is an even number‖ (it depends on what x represents), ―Socrates‖ (it is not even a sentence). The truth or falsehood of a proposition is called its truth value.

## Basic Connectives and Truth Tables:

Connectives are used for making compound propositions. The main ones are the following (p and q represent given propositions):

| Name | Represented | Meaning |
|---|---|---|
| Negation | $\neg p$ | ―not p‖ |
| Conjunction | $p \wedge q$ | ―p and q‖ |
| Disjunction | $p \vee q$ | ―p or q (or both)‖ |
| Exclusive Or | $p \oplus q$ | ―either p or q, but not both‖ |
| Implication | $p \longrightarrow q$ | ―if p then q‖ |
| Biconditional | $p \leftrightarrow q$ | ―p if and only if q‖ |

The truth value of a compound proposition depends only on the value of its components. Writing F for ―false‖ and T for ―true‖, we can summarize the meaning of the connectives in the following way:

| p | q | $\neg p$ | $p \wedge q$ | $p \vee q$ | $p \oplus q$ | $p \longrightarrow$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | T | T | F | T | T |
| T | F | F | F | T | T | F | F |
| F | T | T | F | T | T | T | F |
| F | F | T | F | F | F | T | T |

Note that $\vee$ represents a non-exclusive or, i.e., $p \vee q$ is true when any of p, q is true and also when both are true. On the other hand $\oplus$represents an exclusive or, i.e., $p \oplus q$ is true only when exactly one of p and q is true.

## Tautology, Contradiction, Contingency:

1. A proposition is said to be a tautology if its truth value is T for any assignment of truth values to its components. Example: The proposition $p \vee \neg p$ is a tautology.

2. A proposition is said to be a contradiction if its truth value is F for any assignment of truth values to its components. Example: The proposition $p \wedge \neg p$ is a contradiction.

3. A proposition that is neither a tautology nor a contradiction is called a contingency.

| p | ¬p | p ∨ ¬p | p ∧ ¬p |
|---|---|---|---|
| T | F | T | F |
| T | F | T | F |
| F | T | T | F |
| F | T | T | F |

*

tautology

*

contradiction

**Conditional Propositions**: A proposition of the form ―if p then q‖ or ―p implies q‖, represented ―p $\longrightarrow$ q‖ is called a conditional proposition. For instance: ―if John is from Chicago then John is from Illinois‖. The proposition p is called hypothesis or antecedent, and the proposition q is the conclusion or consequent.

Note that $p \longrightarrow q$ is true always except when p is true and q is false. So, the following sentences are true: ―if 2 < 4 then Paris is in France‖ (true $\longrightarrow$ true), ―if London is in Denmark then 2 < 4‖ (false $\longrightarrow$ true),

―if 4 = 7 then London is in Denmark‖ (false $\longrightarrow$ false). However the following one is false: ―if 2 < 4 then London is in Denmark‖ (true $\longrightarrow$ false).

In might seem strange that ―p $\longrightarrow$ q‖ is considered true when p is false, regardless

of the truth value of q. This will become clearer when we study predicates such as –if x is a multiple of 4 then x is a multiple of 2‖. That implication is obviously true, although for the particular
case x = 3 it becomes –if 3 is a multiple of 4 then 3 is a multiple of 2‖.

The proposition $p \leftrightarrow q$, read ―p if and only if q‖, is called bicon- ditional. It is true precisely when p and q have the same truth value, i.e., they are both true or both false.

**Logical Equivalence:** Note that the compound proposi- tions $p \longrightarrow q$ and $\neg p \vee q$ have the same truth values:

| p | q | ¬p | ¬p ∨ q | p ⟶ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

When two compound propositions have the same truth values no matter what truth value their constituent propositions have, they are called logically equivalent. For instance $p \longrightarrow q$ and $\neg p \vee q$ are logically equivalent, and we write it:

$$p \longrightarrow q \equiv \neg p \vee q$$

Note that that two propositions A and B are logically equivalent precisely when $A \leftrightarrow B$ is a tautology.

Example : De Morgan‘s Laws for Logic. The following propositions are logically equivalent:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

| p | q | ¬p | ¬q | p ∨ q | ¬(p ∨ q) | ¬p ∧ ¬q | p ∧ q | ¬(p ∧ q) | ¬p ∨ ¬q |
|---|---|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F | T | F | F |
| T | F | F | T | T | F | F | F | T | T |
| F | T | T | F | T | F | F | F | T | T |
| F | F | T | T | F | T | T | F | T | T |

Example: The following propositions are logically equivalent:

$p \leftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p)$

Again, this can be checked with the truth tables:

| p | q | $p \longrightarrow$ | $q \longrightarrow$ | $(p \longrightarrow q) \wedge (q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

Exercise: Check the following logical equivalences:

$\neg(p \longrightarrow q) \equiv p \wedge \neg q$
$p \longrightarrow q \equiv \neg q \longrightarrow \neg p$
$\neg(p \leftrightarrow q) \equiv p \oplus q$

**<u>Converse, Contrapositive:</u>** The converse of a conditional proposition $p \longrightarrow q$ is the proposition $q \longrightarrow p$. As we have seen, the bi- conditional proposition is equivalent to the conjunction of a conditional proposition an its converse.

$p \leftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p)$

So, for instance, saying that –John is married if and only if he has a spouse‖ is the same as saying –if John is married then he has a spouse‖ and –if he has a spouse then he is married‖.

Note that the converse is not equivalent to the given conditional proposition, for instance –if John is from Chicago then John is from Illinois‖ is true, but the converse –if John is from Illinois then John is from Chicago‖ may be false.
The contrapositive of a conditional proposition $p \longrightarrow q$ is the propo- sition $\neg q \longrightarrow \neg p$. They are logically equivalent. For instance the con- trapositive of –if John is from Chicago then John is from Illinois‖ is –if John is not from Illinois then John is not from Chicago‖.

**<u>LOGICAL CONNECTIVES:</u>** New propositions are obtained with the aid of word or phrases like –not‖,‖and‖,‖if….then‖,and –if and only if‖. Such words or phrases are called logical connectives. The new propositions obtained by the use of connectives are called compound propositions. The original propositions from which a compound proposition is obtained are called the components or the primitives of the compound proposition. Propositions which do not contain any logical connective are called simple propositions

**NEGATION:** A Proposition obtained by inserting the word –not‖ at an appropriate place in a given proposition is called the negation of the given proposition. The negation of a proposition p is denoted by ~p(read –not p‖)

Ex: p: 3 is a prime number

~p: 3 is not a prime number

Truth Table:

| p | ~p |
|---|---|
| 0 | 1 |
| 1 | 0 |

## CONJUNCTION:

A compound proposition obtained by combining two given propositions by inserting the word –and‖ in between them is called the conjunction of the given proposition.The conjunction of two proposition p and q is denoted by p^q(read –p and q‖).

- The conjunction p^q is true only when p is true and q is true; in all other cases it is false.
- Ex: p: $\sqrt{2}$ is an irational number q: 9 is a prime number

  p^q: $\sqrt{2}$ is an irational number and 9 is a prime number
- Truth table:

| p | q | p^q |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

## DISJUNCTION:

A compound proposition obtained by combining two given propositions by inserting the word –or‖ in between them is called the disjunction of the given proposition.The disjunction of two proposition p and q is denoted by $p \vee q$(read –p or q‖).

- The disjunction $p \vee q$ is false only when p is false and q is false ; in all other cases it is true.

- Ex: p: $\sqrt{2}$ is an irational number q: 9 is a prime number

  $p \vee q$ : $\sqrt{2}$ is an irational number or 9 is a prime number Truth table:

•

| p | q | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

## EXCLUSIVE DISJUNCTION:

• The compound proposition ―p or q‖ to be true only when either p is true or q is true but not both. The exclusive or is denoted by symbol $\underline{v}$.

• Ex: p:√2 is an irrational number q: 2+3=5

$P\underline{v}q$: Either √2 is an irrational number or 2+3=5 but not both.

• Truth Table:

| p | q | $p\underline{v}q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

## CONDITIONAL(or IMPLICATION):

• A compound proposition obtained by combining two given propositions by using the words ―if‖ and ―then‖ at appropriate places is called a conditional or an implication..

• Given two propositions p and q, we can form the conditionals ―if p, then q‖ and ―if q, then p:. The conditional ―if p, then q‖is denoted by $p\rightarrow q$ and the conditional ―if q, then p‖ is denoted by $q \rightarrow p$.

• The conditional $p\rightarrow q$ is false only when p is true and q is false ;in all other cases it is true.

• Ex: p: 2 is a prime number q: 3 is a prime number

$p\rightarrow q$: If 2 is a prime number then 3 is a prime number; it is true

• Truth Table:

| p | q | $p\rightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

## BICONDITIONAL:

- Let p and q be two propositions,then the conjunction of the conditionals $p\rightarrow q$ and $q\rightarrow p$ is called bi-conditional of p and q. It is denoted by $p\leftrightarrow q$.
- $p\leftrightarrow q$ is same as $(p\rightarrow q)\wedge(q\rightarrow p)$. As such $p\leftrightarrow q$ is read as ― if p then q and if q then p‖.
- Ex: p: 2 is a prime number q: 3 is a prime number $p\leftrightarrow q$ are true.

Truth Table:

| p | q | $p\rightarrow q$ | $q\rightarrow p$ | $p\leftrightarrow q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

## COMBINED TRUTH TABLE

| P | q | ~p | p^q | $p\vee q$ | $p\underline{\vee}q$ | $p\rightarrow q$ | $p\leftrightarrow q$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |

## TAUTOLOGIES; CONTRADICTIONS:

A compound proposition which is always true regardless of the truth values of its components is called a tautology.

A compound proposition which is always false regardless of the truth values of its components is called a contradiction or an absurdity.

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency I.e contingency is a compound proposition which is neither a tautology nor a contradiction.

## LOGICAL EQUIVALENCE

- Two propositions _u‘ and _v‘ are said to be logically equivalent whenever u and v have the same truth value, or equivalently .
- Then we write $u\Leftrightarrow v$. Here the symbol $\Leftrightarrow$stands for ―logically equivalent to‖.

- When the propositions u and v are not logically equivalent we write $u \Leftrightarrow v$.

## LAWS OF LOGIC:

To denote a tautology and To denotes a contradiction.

- Law of Double negation: For any proposition p, $(\sim\sim p) \Leftrightarrow p$
- Idempotent laws: For any propositions p, 1) $(p \vee p) \Leftrightarrow p$ 2) $(p \wedge p) \Leftrightarrow p$
- Identity laws: For any proposition p, 1)$(p \vee Fo) \Leftrightarrow p$ 2)$(p \wedge To) \Leftrightarrow p$
- Inverse laws: For any proposition p, 1) $(p \vee \sim p) \Leftrightarrow To$ 2)$(p \wedge \sim p) \Leftrightarrow Fo$
- Commutative Laws: For any proposition p and q, 1)$(p \vee q) \Leftrightarrow (q \vee p)$ 2)$(p \wedge q) \Leftrightarrow (q \wedge p)$
- Domination Laws: For any proposition p, 1) $(p \vee To) \Leftrightarrow To$ 2) $(p \wedge Fo) \Leftrightarrow Fo$
- Absorption Laws: For any proposition p and q,1) $[p \vee (p \wedge q)] \Leftrightarrow p$ 2)$[p \wedge (p \vee q)] \Leftrightarrow p$
- De-Morgan Laws: For any proposition p and q, 1)$\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$ 2) $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$
- Associative Laws : For any proposition p ,q and r, 1) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ 2) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
- Distributive Laws: For any proposition p ,q and r, 1) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ 2) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- Law for the negation of a conditional : Given a conditional $p \rightarrow q$, its negation is obtained by using the following law: $\sim (p \rightarrow q) \Leftrightarrow [p \wedge (\sim q)]$

## NOTE:

- $\sim (p \vee q) \equiv \sim p \wedge \sim q$
- $\sim (p \wedge q) \equiv \sim p \vee \sim q$
- $\sim (p \rightarrow q) \equiv [p \wedge (\sim q)]$
- $(p \rightarrow q) \equiv \sim \sim (p \rightarrow q) \equiv \sim [p \wedge (\sim q)] \equiv \sim p \vee q$

**TRANSITIVE AND SUBSTITUTION RULES** If u,v,w are propositions such that $u \Leftrightarrow v$ and $v \Leftrightarrow w$, then $u \Leftrightarrow w$. (this is transitive rule)

- Suppose that a compound proposition u is a tautology and p is a component of u, we replace each occurrence of p in u by a proposition q, then the resulting compound proposition v is also a tautology(This is called a substitution rule).

- Suppose that u is a compound proposition which contains a proposition p. Let q be a proposition such that q $\Leftrightarrow$p , suppose we replace one or more occurrences of p by q and obtain a compound proposition v. Then u $\Leftrightarrow$v (This is also substitution rule)

## APPLICATION TO SWITCHING NETWORKS

- If a switch p is open, we assign the symbol o to it and if p is closed we assign the symbol 1 to it.

- Ex: current flows from the terminal A to the terminal B if the switch is closed i.e if p is assigned the symbol 1. This network is represented by the symbol p

A P B

- Ex: parallel network consists of 2 switches p and q in which the current flows from the terminal A to the terminal B , if p or q or both are closed i.e if p or q (or both) are assigned the symbol 1. This network is represent by $p \vee q$

Ex: Series network consists of 2 switches p and q in which the current flows from the terminal A to the terminal B if both of p and q are closed; that is if both p and q are assigned the symbol 1. This network is represented by $p \wedge q$

## DUALITY:

Suppose u is a compound proposition that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurrence of $\wedge$ and $\vee$ in u by $\vee$ and $\wedge$ respectively.

Also if u contains To and Fo as components, suppose we replace each occurrence of To and Fo by Fo and To respectively, then the resulting compound proposition is called the dual of u and is denoted by $u^d$.

Ex: u: $p \wedge (q \vee \sim r) \vee (s \wedge T_o)$ $u^d$: $p \vee (q \wedge \sim r) \wedge (s \vee F_o)$

## NOTE:

- $(u^d)^d \Leftrightarrow u$. The dual of the dual of u is logically equivalent to u.

- For any two propositions u and v if u $\Leftrightarrow$v, then $u^d \Leftrightarrow v^d$ . This is known as the principle of duality.

## The connectives NAND and NOR

$(p \uparrow q) = \sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

$(p \downarrow q) = \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

## CONVERSE,INVERSE AND CONTRAPOSITIVE

Consider a conditional $(p \to q)$ , Then :

1) $q \to p$ is called the converse of $p \to q$

2) $\sim p \to \sim q$ is called the inverse of $p \to q$

3) $\sim q \to \sim p$ is called the contrapositive of $p \to q$

# RULES OF INFERENCE:

There exist rules of logic which can be employed for establishing the validity of arguments . These rules are called the Rules of Inference.

1) Rule of conjunctive simplification: This rule states that for any two propositions p and q if $p \wedge q$ is true, then p is true i.e $(p \wedge q) \Rightarrow p$.

2) Rule of Disjunctive amplification: This rule states that for any two proposition p and q if p is true then $p \vee q$ is true i.e $p \Rightarrow (p \vee q)$

3) 3) Rule of Syllogism: This rule states that for any three propositions p,q r if $p \to q$ is true and $q \to r$ is true then $p \to r$ is true. i.e $\{(p \to q) \wedge (q \to)\} \Rightarrow (p \to r)$ In tabular form:

$p \to q$ $q \to r$ $\therefore (p \to r)$

4) 4) Modus pones(Rule of Detachment): This rule states that if p is true and $p \to q$ is true, then q is true, ie $\{p \wedge (p \to q)\} \Rightarrow q$. Tabular form

p $p \to q$ $\therefore q$

5) Modus Tollens: This rule states that if $p \to q$ is true and q is false, then p is false. $\{(p \to q) \wedge \sim q\} \Rightarrow q$ Tabular form: $p \to q$

$\sim q$ $\therefore \sim p$

6) Rule of Disjunctive Syllogism: This rule states that if $p \vee q$ is true and p is false, then q is true i.e. $\{(p \vee q) \wedge \sim p\} \Rightarrow q$ Tabular Form $p \vee q$

$\sim p$ $\therefore q$

## QUANTIFIERS:

The words –ALL‖,‖EVERY‖,‖SOME‖,‖THERE EXISTS‖ are called quantifiers in the proposition

The symbol $\forall$ is used to denote the phrases –FOR ALL‖,‖FOR EVERY‖,‖FOR EACH‖ and –FOR ANY‖.this is called as universal quantifier.

$\exists$ is used to denote the phrases –FOR SOME‖and –THERE EXISTS‖and –FOR ATLEAST ONE‖.this symbol is called existential quantifier.

A proposition involving the universal or the existential quantifier is called a quantified statement

## LOGICAL EQUIVALENCE:

1. $\forall$ x,[p(x)$\wedge$q(x)]$\Leftrightarrow$($\forall$x p(x))$\wedge$($\forall$x,q(x))

2. $\exists$ x, [p(x)$\vee$q(x)]$\Leftrightarrow$($\exists$x p(x)) $\vee$($\exists$x,q(x))

3. $\exists$ x, [p(x)$\rightarrow$q(x)] $\Leftrightarrow$$\exists$x ,[~p(x)$\vee$q(x)]

## RULE FOR NEGATION OF A QUANTIFIED STATEMENT:

~{$\forall$x,p(x)}$\equiv$$\exists$x{~p(x)}        ~{$\exists$x,p(x)}$\equiv$$\forall$x{~p(x)}

## RULES OF INTERFERENCE:

1. RULE OF UNIVERSAL SPECIFICATION
2. RULE OF UNIVERSAL GENERALIZATION

If an open statement p(x) is proved to be true for any (arbitrary)x chosen from a set S, then the quantified statement $\forall$x€S, p(x) is true.

# METHODS OF PROOF AND DISPROOF:

1. DIRECT PROOF:

The direct method of proving a conditional p$\rightarrow$q has the following lines of argument:

a) hypothesis : First assume that p is true

b) Analysis: starting with the hypothesis and employing the rules /laws of logic and other known facts , infer that q is true

c) Conclusion:$p\rightarrow q$ is true.

2. INDIRECT PROOF:

Condition $p\rightarrow q$ and its contrapositive $\sim q\rightarrow\sim p$ are logically equivalent. On basis of this proof, we infer that the conditional $p\rightarrow q$ is true. This method of proving a conditional is called an indirect method of proof.

3.PROOF BY CONTRADICTION

The indirect method of proof is equivalent to what is known as the proof by contradiction. The lines of argument in this method of proof of the statement $p\rightarrow q$ are as follows:

1) Hypothesis: Assume that $p\rightarrow q$ is false i.e assume that p is true and q is false.

2)Analysis: starting with the hypothesis that q is false and employing the rules of logic and other known facts , infer that p is false. This contradicts the assumption that p is true

3)Conculsion: because of the contradiction arrived in the analysis , we infer that $p\rightarrow q$ is true

4.PROOF BY EXHAUSTION:

–∀x *€S*,p(x)‖ is true if p(x)is true for every (each) x in S.If S consists of only a limited number of elements , we can prove that the statement –∀x *€S*,p(x)‖ is true by considering p(a) for each a in S and verifying that p(a) is true .such a method of prove is called method of exhaustion.

5.PROOF OF EXISTENCE:

–∃x *€S*,p(x)‖ is true if any one element a € S such that p(a) is true is exhibited. Hence , the best way of proving a proposition of the form –∃x *€S*,p(x)‖ is to exhibit the existence of one a€S such that p(a) is true. This method of proof is called proof of existence.

6.DISPROOF BY CONTRADICTION :

Suppose we wish to disprove a conditional $p\rightarrow q$. for this propose we start with the hypothesis that p is true and q is true, and end up with a contradiction. In view of the contradiction , we conclude that the conditional $p\rightarrow q$ is false.this method of disproving $p\rightarrow q$ is called DISPROOF BY CONTRADICTION

7.DISPROOF BY COUNTER EXAMPLE:

–$\forall x$ €S,p(x)‖ is false if any one element a€S such that p(a) is false is exhibited hence the best way of disproving a proposition involving the universal quantifiers is to exhibit just one case where the proposition is false. This method of disproof is called DISPROOF BY COUNTER EXAMPLE

# <u>Fundamentals of Logic *contd*.:</u>

## <u>SYLLABUS</u>

**Fundamentals of Logic *contd*.: The Use of Quantifiers, Quantifiers, Definitions and the Proofs of Theorems**

## <u>Predicates, Quantifiers</u>

## <u>Predicates:</u>

A predicate or propositional function is a state- ment containing variables. For instance ―x + 2 = 7‖, ―X is American‖, ―x < y‖, ―p is a prime number‖ are predicates. The truth value of the predicate depends on the value assigned to its variables. For instance if we replace x with 1 in the predicate ―x + 2 = 7‖ we obtain ―1 + 2 = 7‖, which is false, but if we replace it with 5 we get ―5 + 2 = 7‖, which is true. We represent a predicate by a letter followed by the variables enclosed between parenthesis: P (x), Q(x, y), etc. An example for P (x) is a value of x for which P (x) is true. A counterexample is a value of x for which P (x) is false. So, 5 is an example for ―x + 2 = 7‖, while 1 is a counterexample.

Each variable in a predicate is assumed to belong to a universe (or domain) of discourse, for instance in the predicate ―n is an odd integer‖ 'n' represents an integer, so the universe of discourse of n is the set of all integers. In ―X is American‖ we may assume that X is a human being, so in this case the universe of discourse is the set of all human beings.[1]

## <u>Quantifiers</u>:

Given a predicate P (x), the statement ―for some x, P (x)‖ (or ―there is some x such that p(x)‖), represented ―∃ x P (x)‖, has a definite truth value, so it is a proposition in the usual sense. For instance if P (x) is ―x + 2 = 7‖ with the integers as universe of discourse, then ∃ x P (x) is true, since there is indeed an integer, namely 5, such that P (5) is a true statement. However, if

Q(x) is ―2x = 7‖ and the universe of discourse is still the integers, then ∃ x Q(x) is false. On the other hand, ∃ x Q(x) would be true if we extend the universe of discourse to the rational numbers. The symbol

∃ is called the existential quantifier.

Analogously, the sentence –for all x, P (x)‖—also –for any x, P (x)‖, –for every x, P (x)‖, –for each x, P (x)‖—, represented –∀ x P (x)‖, has a definite truth value. For instance, if P (x) is –x + 2 = 7‖ and the universe of discourse is the integers, then ∀ x P (x) is false. However if Q(x) represents –$(x + 1)^2 = x^2 + 2x + 1$‖ then ∀ x Q(x) is true. The symbol ∀ is called the universal quantifier.

In predicates with more than one variable it is possible to use several quantifiers at the same time, for instance ∀ x∀ y∃ z P (x, y, z), meaning –for all x and all y there is some z such that P (x, y, z)‖.

Note that in general the existential and universal quantifiers cannot be swapped, i.e., in general ∀ x∃ y P (x, y) means something different from ∃ y∀ x P (x, y). For instance if x and y represent human beings and P (x, y) represents ―x is a friend of y‖, then ∀ x∃ y P (x, y) means that everybody is a friend of someone, but ∃ y∀ x P (x, y) means that there is someone such that everybody is his or her friend.

A predicate can be partially quantified, e.g. ∀ x∃ y P (x, y, z, t). The variables quantified (x and y in the example) are called bound variables, and the rest (z and t in the example) are called free variables. A partially quantified predicate is still a predicate, but depending on fewer variables.

## Generalized De Morgan Laws for Logic:

If ∃ x P (x) is false then there is no value of x for which P (x) is true, or in other words, P (x) is always false. Hence

¬∃ x P (x) ≡ ∀ x ¬P(x) .

On the other hand, if ∀ x P (x) is false then it is not true that for every x, P (x) holds, hence for some x, P (x) must be false. Thus:

¬∀ x P (x) ≡ ∃ x ¬P(x) .

This two rules can be applied in successive steps to find the negation of a more complex quantified statement, for instance:

$\neg\exists x \forall y\, p(x, y) \equiv \forall x \neg\forall y\, P(x, y) \equiv \forall x \exists y\, \neg P(x, y)$ .

Exercise : Write formally the statement –for every real number there is a greater real number‖. Write the negation of that statement.

Answer : The statement is: $\forall x \exists y (x < y)$ (the universe of discourse is the real numbers). Its negation is: $\exists x \forall y \neg(x < y)$, i.e., $\exists x \forall y (x < y)$. (Note that among real numbers $x < y$ is equivalent to $x \geq y$, but formally they are different predicates.)

# Proofs

## Mathematical Systems, Proofs:

A Mathematical Sys- tem consists of:

1. Axioms: propositions that are assumed true.
2. Definitions : used to create new concepts from old ones.
3. Undefined terms : corresponding to the primitive concepts of the system (for instance in set theory the term –set‖ is undefined).

A theorem is a proposition that can be proved to be true. An argument that establishes the truth of a proposition is called a proof.

Example: Prove that if $x > 2$ and $y > 3$ then $x + y > 5$.

Answer : Assuming $x > 2$ and $y > 3$ and adding the inequalities term by term we get: $x + y > 2 + 3 = 5$.

That is an example of direct proof. In a direct proof we assume the hypothesis together with axioms and other theorems previously proved and we derive the conclusion from them.

An indirect proof or proof by contrapositive consists of proving the contrapositive of the desired implication, i.e., instead of proving $p \longrightarrow q$ we prove $\neg q \longrightarrow \neg p$.

Example: Prove that if $x + y > 5$ then $x > 2$ or $y > 3$.

Answer : We must prove that $x + y > 5 \longrightarrow (x > 2) \vee (y > 3)$. An indirect proof consists of proving $\neg((x > 2) \vee (y > 3)) \longrightarrow \neg(x + y > 5)$. In fact: $\neg((x > 2) \vee (y > 3))$ is the same as $(x \leq 2) \wedge (y \leq 3)$, so adding both inequalities we get $x + y \leq 5$, which is the same as $\neg(x + y > 5)$.

Proof by Contradiction. In a proof by contradiction or (Reductio ad Absurdum ) we assume the hypotheses and the negation of the conclu- sion, and try to derive a contradiction, i.e., a proposition of the form $r \wedge \neg r$.

Example: Prove by contradiction that if $x+ y > 5$ then either $x > 2$ or $y > 3$.

Answer : We assume the hypothesis $x + y > 5$. From here we must conclude that $x > 2$ or $y > 3$. Assume to the contrary that ―$x > 2$ or $y > 3$‖ is false, so $x \leq 2$ and $y \leq 3$. Adding those inequalities we get
$x \leq 2 + 3 = 5$, which contradicts the hypothesis $x + y > 5$. From here we conclude that the assumption ―$x \leq 2$ and $y \leq 3$‖ cannot be right, so ―$x > 2$ or $y > 3$‖ must be true.

Remark : Sometimes it is difficult to distinguish between an indirect proof and a proof by contradiction. In an indirect proof we prove an implication of the form $p \longrightarrow q$ by proving the contrapositive $\neg q \longrightarrow \neg p$. In an proof by contradiction we prove an statement s (which may or may not be an implication) by assuming $\neg s$ and deriving a contradiction. In fact proofs by contradiction are more general than indirect proofs.

Exercise : Prove by contradiction that $\sqrt{2}$ is not a rational number, i.e., there are no integers a, b such that $\sqrt{2} = a/b$.

Answer : Assume that $\sqrt{2}$ is rational, i.e., $\sqrt{2} = a/b$, where a and b are integers and the fraction is written in least terms. Squaring both sides we have $2 = a^2/b^2$, hence $2 b^2 = a^2$. Since the left hand side is even, then $a^2$ is even, but this implies that a itself is even, so $a = 2 a'$. Hence: $2 b^2 = 4 a'^2$, and simplifying: $b^2 = 2 a'^2$. This implies that $b^2$ is even, so b is even: $b = 2b'$. Consequently $a/b = 2a'/2b' = a'/b'$, contradicting the hypothesis that a/b was in least terms.

## Arguments, Rules of Inference:

An argument is a se- quence of propositions $p_1, p_2, \ldots, p_n$ called hypotheses (or premises ) followed by a proposition q called conclusion. An argument is usually written:

$p_1$
$p_2$
.
$p_n$

$\therefore\ q$

or

$p_1, p_2, \ldots, p_n / \therefore\ q$

The argument is called valid if q is true whenever $p_1, p_2, \ldots, p_n$ are true; otherwise it is called invalid.
Rules of inference are certain simple arguments known to be valid and used to make a proof step by step. For instance the following argument is called modus ponens or rule of detachment :

$p \longrightarrow q\ p$

$\therefore\ q$

In order to check whether it is valid we must examine the following truth table:

| p | q | p $\longrightarrow$ | p | q |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | F | F |

If we look now at the rows in which both $p \longrightarrow q$ and p are true (just the first row) we see that also q is true, so the argument is valid.

Other rules of inference are the following:

1. *Modus Ponens* or *Rule of Detachment*:

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

2. *Modus Tollens*:

$$\begin{array}{rl} & p \to q \\ & \neg q \\ \hline \therefore & \neg p \end{array}$$

3. *Addition*:

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$$

4. *Simplification*:

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

5. *Conjunction*:

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

6. *Hypothetical Syllogism*:

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

7. *Disjunctive Syllogism*:

$$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

8. *Resolution*:

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

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Arguments are usually written using three columns. Each row con- tains a label, a statement and the reason that justifies the introduction of that statement in the argument. That justification can be one of the following:

1. The statement is a premise.
2. The statement can be derived from statements occurring earlier in the argument by using a rule of inference.

Example: Consider the following statements: –I take the bus or I walk. If I walk I get tired. I do not get tired. Therefore I take the bus.‖ We can formalize this by calling B = ―I take the bus‖, W = ―I walk‖ and T = ―I get tired‖. The premises are $B \vee W$, $W \longrightarrow T$ and $\neg T$, and the conclusion is B. The argument can be described in the following steps:

| step | statement | reason |
|---|---|---|
| 1) | $W \longrightarrow T$ | Premise |
| 2) | $\neg T$ | Premise |
| 3) | $\neg W$ | 1,2, Modus Tollens |
| 4) | $B \vee W$ | Premise |
| 5) | $\therefore B$ | 4,3, Disjunctive Syllogism |

## Rules of Inference for Quantified Statements:

We state the rules for predicates with one variable, but they can be gener- alized to predicates with two or more variables.

1. Universal Instantiation. If $\forall x\, p(x)$ is true, then p(a) is true for each specific element a in the universe of discourse; i.e.:

$$\begin{array}{l} \forall x\, p(x) \\ \hline \therefore\ p(a) \end{array}$$

For instance, from $\forall x\,(x+1 = 1+x)$ we can derive $7+1 = 1+7$.

2. Existential Instantiation. If $\exists x\, p(x)$ is true, then p(a) is true for some specific element a in the universe of discourse; i.e.:

$$\begin{array}{l} \exists x\, p(x) \\ \hline \therefore\ p(a) \end{array}$$

The difference respect to the previous rule is the restriction in the meaning of a, which now represents some (not any) element of the universe of discourse. So, for instance, from $\exists x\,(x^2 = 2)$ (the universe of discourse is the real numbers) we derive the existence of some element, which we may represent $\pm\sqrt{2}$, such that $(\pm\sqrt{2})^2 = 2$.

3. Universal Generalization. If p(x) is proved to be true for a generic element in the universe of discourse, then $\forall x\, p(x)$ is true; i.e.:

$$\begin{array}{l} p(x) \\ \hline \therefore\ \forall x\, p(x) \end{array}$$

By ―generic‖ we mean an element for which we do not make any assumption other than its belonging to the universe of discourse. So, for instance, we can prove $\forall x\,[(x + 1)^2 = x^2 + 2x + 1]$ (say, for real numbers) by assuming that x is a generic real number and using algebra to prove $(x + 1)^2 = x^2 + 2x + 1$.

4. Existential Generalization. If p(a) is true for some specific ele- ment a in the universe of discourse, then $\exists x\, p(x)$ is true; i.e.:

$$\frac{p(a)}{\therefore\ \exists x\, p(x)}$$

For instance: from $7 + 1 = 8$ we can derive $\exists x\, (x + 1 = 8)$.

Example: Show that a counterexample can be used to disprove a universal statement, i.e., if a is an element in the universe of discourse,

then from $\neg p(a)$ we can derive $\neg\forall x\, p(x)$. Answer: The argument is as follows:

| step statement | reason |
|---|---|
| 1) $\neg p(a)$ | Premise |
| 2) $\exists x\, \neg p(x)$ | Existential Generalization |
| 3) $\neg\forall x\, p(x)$ | Negation of Universal Statement |