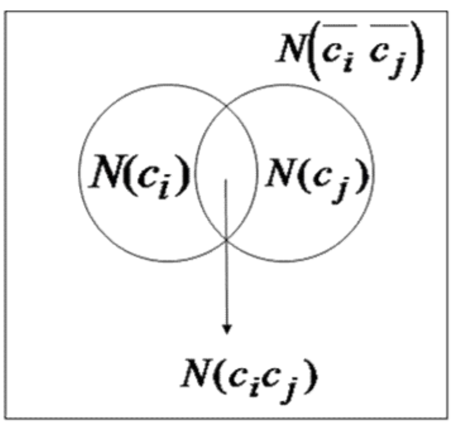
Module 4

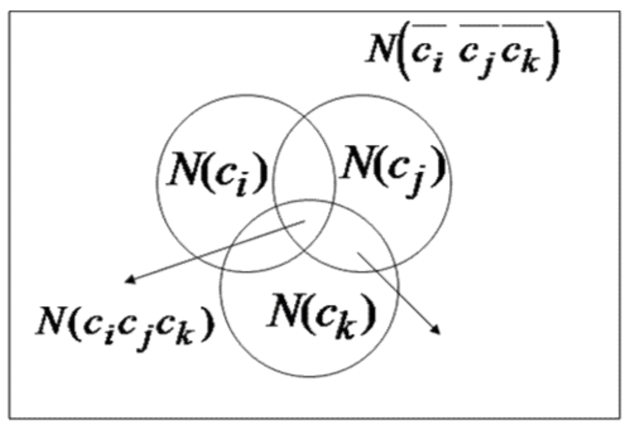
PRINCIPLE OF INCLUSION AND EXCLUSION

The Principle of Inclusion and Exclusion

Let S be a set with $|S|=N$, and let $c_1, c_2, \ldots, c_t$ be a collection of conditions or properties satisfied by some, or all, of the elements of S. Some elements of S may satisfy more than one of the conditions, whereas others may not satisfy any of them.

$N(c_i)$: the number of elements in S that satisfy condition c_i
$N(c_i c_j)$: the number of elements in S that satisfy both of the conditions c_i, c_j, and perhaps some others
$N(\overline{c_i}) = N - N(c_i)$
$N(\overline{c_i}\,\overline{c_j})$: the number of elements in S that do not satisfy either of the conditions c_i *or* c_j $(\neq N(\overline{c_i c_j}))$





$N(\overline{c_i}\ \overline{c_j})$=N-[$N(c_i)$ + $N(c_j)$]

Corollary 8.1 The number of elements in S that satisfy at least one of the conditions is $N - \overline{N}$.

Notations

$$S_0 = N, S_1 = \sum N(c_i), S_2 = \sum N(c_i c_j),$$

$$, S_k = \sum N(c_{i_1} c_{i_2} \quad c_{i_k}), 1 \le k \le t.$$

Ex. 8.1 Determine the number of positive integer n where $1 \le n \le 100$ and n is not divisible by 2,3, or 5.

Here $S = \{1,2, \quad ,100\}$, $N = 100$, c_1 : divisible by 2, c_2 : divisible by 3, c_3 : divisible by 5.

$$\therefore N\left(\overline{c_1 c_2 c_3}\right) = S_0 - S_1 + S_2 - S_3 = 100 - \left(\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor \right) + \left(\left\lfloor \frac{100}{2 \times 3} \right\rfloor + \left\lfloor \frac{100}{2 \times 5} \right\rfloor + \left\lfloor \frac{100}{3 \times 5} \right\rfloor \right) - \left\lfloor \frac{100}{2 \times 3 \times 5} \right\rfloor = 26$$

$$\overline{N} = N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k)$$

$$+ \quad + (-1)^t N(c_1 c_2 \quad c_t)$$

If x satisfies none of the conditions, then x is counted once in $\overline{N}$ and once in N, but not in any of the other terms. Consequently, x contributed a count of 1 to each side. The other possibility is that x satisfies exactly r of the conditions, $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right - hand side, x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \quad + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times.}$$

$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$$

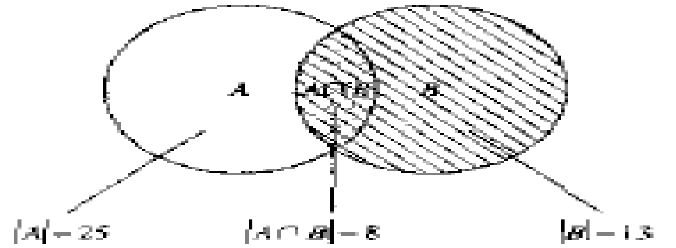


FIGURE 1 The Set of Students in a Discrete Mathematics Class.

How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution: Let A be the set of positive integers not exceeding 1000 that are divisible by 7, and let B be the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of integers not exceeding 1000 that are divisible by either 7 or 11, and $A \cap B$ is the set of integers not exceeding 1000 that are divisible by both 7 and 11. From Example 2 of Section 2.3, we know that among the positive integers not exceeding 1000 there are $\lfloor 1000/7 \rfloor$ integers divisible by 7 and $\lfloor 1000/11 \rfloor$ divisible by 11. Since 7 and 11 are relatively prime, the integers divisible by both 7 and 11 are those divisible by $7 \cdot 11$. Consequently, there are $\lfloor 1000/(11 \cdot 7) \rfloor$ positive integers not exceeding 1000 that are divisible by both 7 and 11. It follows that there are

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor \\ &= 142 + 90 - 12 \\ &= 220 \end{aligned}$$

positive integers not exceeding 1000 that are divisible by either 7 or 11. This computation is illustrated in Figure 2. ■

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$$

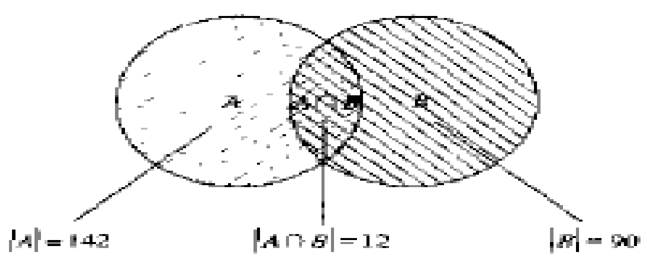


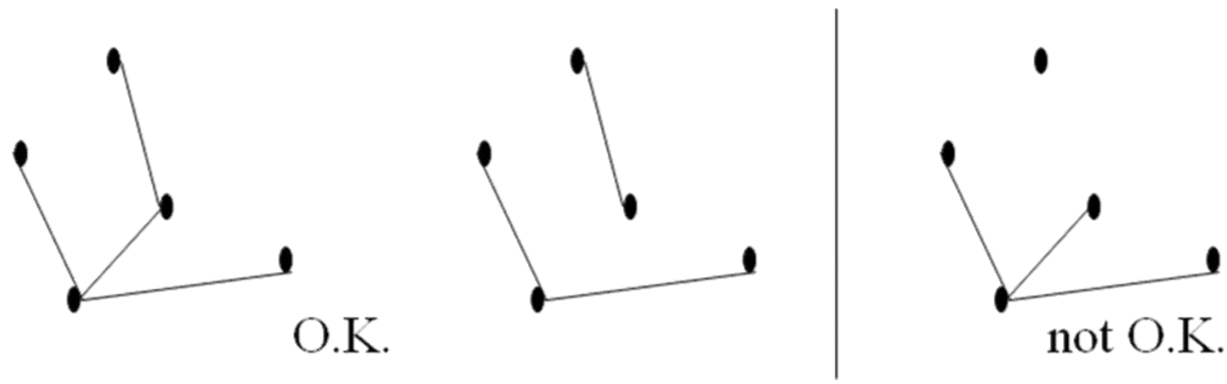
FIGURE 2 The Set of Positive Integers Not Exceeding 1000 Divisible by Either 7 or 11.

S

In general, $\Phi(n) = n\prod_{p|n}\left(1-\dfrac{1}{p}\right)$, where the product is taken over all primes p dividing n. When $n = p$, a prime,

$$\Phi(n) = \Phi(p) = p\left(1-\frac{1}{p}\right) = p-1.$$

Ex. Construct roads for 5 villages such that no village will be isolated. In how many ways can we do this?



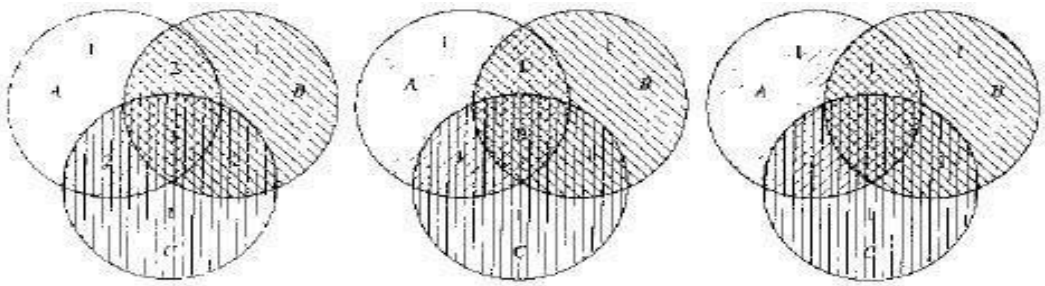
Ex. 8.6 Construct roads for 5 villages such that no village is isolated.

$N = 2^{\binom{5}{2}} = 2^{10}$. c_i : village i is isolated for $1 \le i \le 5$.

$$N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\,\overline{c_5}\right) = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$

$$= 2^{10} - \binom{5}{1}2^{\binom{4}{2}} + \binom{5}{2}2^{\binom{3}{2}} - \binom{5}{3}2^{\binom{2}{2}} + \binom{5}{4}2^{0} - \binom{5}{5}2^{0} = 768.$$

Finding a Formula for the Number of Elements in the Union of Three Sets.



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then

$$|S| = 1232, \qquad |F| = 879, \qquad |R| = 114,$$
$$|S \cap F| = 103, \qquad |S \cap R| = 23, \qquad |F \cap R| = 14,$$

and

$$|S \cup F \cup R| = 2092.$$

Inserting these quantities into the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

gives

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|.$$

Solving for $|S \cap F \cap R|$ shows that $|S \cap F \cap R| = 7$. Therefore, there are seven students who have taken courses in Spanish, French, and Russian. This is illustrated in Figure 5. ■

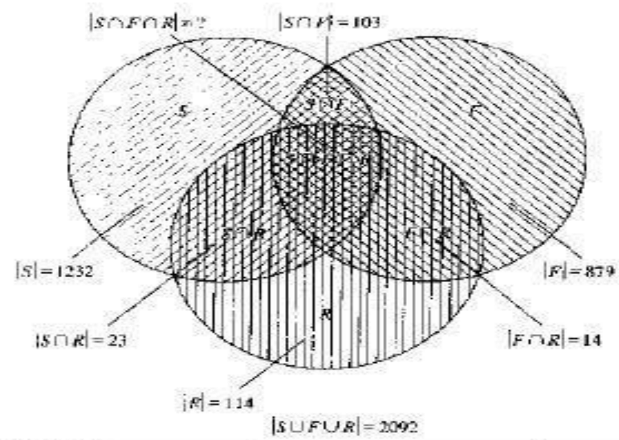


FIGURE 5 The Set of Students Who Have Taken Courses in Spanish, French, and Russian.

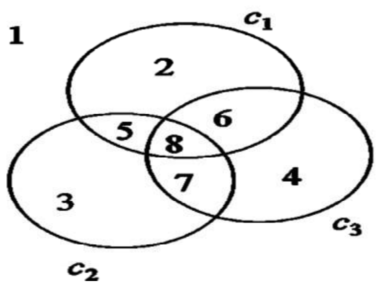
Give a formula for the number of elements in the union of four sets.

Solution: The inclusion–exclusion principle shows that

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ & - |A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

Note that this formula contains 15 different terms, one for each nonempty subset of $\{A_1, A_2, A_3, A_4\}$. ■

If $m \in \mathbb{Z}^+$ and $1 \le m \le t$, we now want to determine E_m , which denotes the number of elements in S that satisfy exactly m of the t conditions. (At present, we can obtain E_0)



E_1: regions 2,3,4
E_2: regions 5,6,7

$$E_1 = S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$$

$$E_2 = S_2 - 3S_3 = S_2 - \binom{3}{1}S_3$$

$$E_3 = S_3$$

Theorem 8.2 For each $1 \le m \le t$, the number of elements in S that satisfy exactly m of the conditions $c_1, c_2, \ldots, c_t$ is given by

$$E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \cdots + (-1)^{t-m}\binom{t}{t-m}S_t$$

Proof : Let $x \in S$, consider the following three cases :

(a) x satisfies fewer than m conditions : it contributes 0 to both side

(b) x satisfies exactly m of the conditions : it contributes 1 to both side (E_m and S_m)

(c) x satisfies r of the conditions, where $m < r \le t$. Then x contributes nothing to E_m. For the right side, x is counted

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$$

times. For $0 \le k \le r - m$,

$$\binom{m+k}{k}\binom{r}{m+k} = \frac{(m+k)!}{k!m!} \cdot \frac{r!}{(m+k)!(r-m-k)!} =$$

$$\frac{r!}{m!}\cdot\frac{1}{k!(r-m-k)!}=\frac{r!}{m!(r-m)!}\cdot\frac{(r-m)!}{k!(r-m-k)!}=\binom{r}{m}\binom{r-m}{k}$$

Consequently, on the right hand side, x is counted

$$\binom{r}{m}\left[\binom{r-m}{0}-\binom{r-m}{1}+\binom{r-m}{2}-\cdots+(-1)^{r-m}\binom{r-m}{r-m}\right]=\binom{r}{m}(1-1)^{r-m}=0 \text{ times.}$$

Let L_m denote the number of elements in S that satisfy at least m of the t conditions. Then we have :

Corollary 8.2 $L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t .$

When $m = 1$, $L_1 = S_1 - \binom{1}{0} S_2 + \binom{2}{0} S_3 - \cdots + (-1)^{t-1} \binom{t-1}{0} S_t$

$= S_1 - S_2 + S_3 - \cdots + (-1)^{t-1} S_t = N - \overline{N}$

How many solutions does

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$?

Solution: To apply the principle of inclusion–exclusion, let a solution have property P_1 if $x_1 > 3$, property P_2 if $x_2 > 4$, and property P_3 if $x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3).$$

Using the same techniques as in Example 6 of Section 4.6, it follows that

- N = total number of solutions $= C(3 + 11 - 1, 11) = 78$,
- $N(P_1)$ = (number of solutions with $x_1 \geq 4$) $= C(3 + 7 - 1, 7) = C(9, 7) = 36$,
- $N(P_2)$ = (number of solutions with $x_2 \geq 5$) $= C(3 + 6 - 1, 6) = C(8, 6) = 28$,
- $N(P_3)$ = (number of solutions with $x_3 \geq 7$) $= C(3 + 4 - 1, 4) = C(6, 4) = 15$,
- $N(P_1P_2)$ = (number of solutions with $x_1 \geq 4$ and $x_2 \geq 5$) $= C(3 + 2 - 1, 2) = C(4, 2) = 6$,
- $N(P_1P_3)$ = (number of solutions with $x_1 \geq 4$ and $x_3 \geq 7$) $= C(3 + 0 - 1, 0) = 1$,
- $N(P_2P_3)$ = (number of solutions with $x_2 \geq 5$ and $x_3 \geq 7$) $= 0$,
- $N(P_1P_2P_3)$ = (number of solutions with $x_1 \geq 4$, $x_2 \geq 5$, and $x_3 \geq 7$) $= 0$.

Inserting these quantities into the formula for $N(P_1'P_2'P_3')$ shows that the number of solutions with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ equals

$$N(P_1'P_2'P_3') = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6. \quad \blacksquare$$

Derangements: Nothing Is in Its Right Place

A **derangement** is a permutation of objects that leaves no object in its original position.

Theorem:

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof: Let a permutation have property P_i if it fixes element i. The number of derangements is the number of permutations having none of the properties P_i for $i = 1, 2, \ldots, n$, or

$$D_n = N(P_1' P_2' \cdots P_n').$$

Using the principle of inclusion–exclusion, it follows that

$$D_n = N - \sum_i N(P_i) + \sum_{i<j} N(P_i P_j) - \sum_{i<j<k} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \cdots P_n),$$

where N is the number of permutations of n elements. This equation states that the number of permutations that fix no elements equals the total number of permutations, less the number that fix at least one element, plus the number that fix at least two elements, less the number that fix at least three elements, and so on. All the quantities that occur on the right-hand side of this equation will now be found.

First, note that $N = n!$, since N is simply the total number of permutations of n elements. Also, $N(P_i) = (n-1)!$. This follows from the product rule, since $N(P_i)$ is the number of permutations that fix element i, so that the ith position of the permutation is determined, but each of the remaining positions can be filled arbitrarily. Similarly,

$$N(P_i P_j) = (n-2)!,$$

since this is the number of permutations that fix elements i and j, but where the other $n - 2$ elements can be arranged arbitrarily. In general, note that

$$N(P_{i_1} P_{i_2} \cdots P_{i_m}) = (n-m)!,$$

because this is the number of permutations that fix elements $i_1, i_2, \ldots, i_m$, but where the other $n - m$ elements can be arranged arbitrarily. Because there are $C(n, m)$ ways to choose m elements from n, it follows that

$$\sum_{1 \le i \le n} N(P_i) = C(n, 1)(n-1)!,$$

$$\sum_{1 \le i < j \le n} N(P_i P_j) = C(n, 2)(n-2)!,$$

and in general,

$$\sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} N(P_{i_1} P_{i_2} \cdots P_{i_m}) = C(n, m)(n-m)!.$$

Consequently, inserting these quantities into our formula for D_n gives

$$D_n = n! - C(n,1)(n-1)! + C(n,2)(n-2)! - \cdots + (-1)^n C(n,n)(n-n)!$$

$$= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!\,0!}0!.$$

Simplifying this expression gives

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right].$$

□

Ex. Find the number of permutations such that 1 is not in the first place, 2 is not in the second place, ..., and 10 is not in the tenth place. (derangements of 1,2,3,...,10)

c_i : i is in the ith place for $1 \le i \le 10$

$$d_{10} = N(\bar{c}_1 \bar{c}_2 \cdots \bar{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \cdots + \binom{10}{10}0!$$

$$= 10!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}\right) \approx 10!\,e^{-1} \text{ since}$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

2! 3!

$$= \sum_{n=0}$$

n

Ex. Assign 7 books to 7 reviewers two times such that everyone gets a different book the second times.
Ans: first time 7!, second time d_7

therefore, 7! d_7

$$n! = \binom{n}{0}d_0 + \binom{n}{1}d_1 + \binom{n}{2}d_2 + \quad + \binom{n}{n}d_n = \sum_{k=0}^{n}\binom{n}{k}d_k$$

d_k = the number of derangements of $1, 2, \quad , k$; $d_0 = 1$

Rook Polynomials

In combinatorial mathematics, a **rook polynomial** is a generating polynomial of the number of ways to place non-attacking rooks on a **board** that looks like a checkerboard; that is, no two rooks may be in the same row or column. The board is any subset of the squares of a rectangular board with m rows and n columns; we think of it as the squares in which one is allowed to put a rook. The board is the ordinary chessboard if all squares are allowed and $m = n = 8$ and a chessboard of any size if all squares are allowed and $m = n$. The coefficient of x^k in the rook polynomial $R_B(x)$ is the number of ways k rooks, none of which attacks another, can be arranged in the squares of B. The rooks are arranged in such a way that there is no pair of rooks in the same row or column. In this sense, an arrangement is the positioning of rooks on a static, immovable board; the arrangement will (usually) be different if the board is rotated or reflected.

The term "rook polynomial" was coined by John Riordan. Despite the name's derivation from chess, the impetus for studying rook polynomials is their connection with counting permutations with restricted position. A board B that is a subset of the $n \times n$ chessboard corresponds to permutations of n objects, which we may take to be the numbers 1, 2, ..., n, such that the number a_j in the j-th position in the permutation must be the column number of an allowed square in row j of B. Famous examples include the number of ways to place n non-attacking rooks on:

- an entire $n \times n$ chessboard, which is an elementary combinatorial problem;
- the same board with its diagonal squares forbidden; this is the derangement or "hat-check" problem;
- the same board without the squares on its diagonal and immediately above its diagonal (and without the bottom left square), which is essential in the solution of the problème des ménages.

Interest in rook placements, i.e., in permutations with restricted position, arises from pure and applied combinatorics, group theory, number theory, and statistical physics. The particular value of rook polynomials comes from the utility of the generating function approach, and also from the fact that the zeroes of the rook polynomial of a board provide valuable information about its coefficients, i.e., the number of non-attacking placements of *k* rooks.

Definition

The **rook polynomial** of a board *B*, $R_B(x)$, is the generating function for the numbers of arrangements of non-attacking rooks:

$$R_B(x) = \sum_{k=0}^{\infty} r_k(B) x^k$$

where r_k is the number of ways to place *k* non-attacking rooks on the board. Despite the notation, this is a finite sum, since the board is finite so there is a maximum number of non-attacking rooks it can hold; indeed, there cannot be more rooks than the smaller of the number of rows and columns in the board.

i. Complete boards

The first few rook polynomials on square $n \times n$ boards are (with $R_n = R_B$):

$$\begin{aligned} R_1(x) &= x+1 \\ R_2(x) &= 2x^2+4x+1 \\ R_3(x) &= 6x^3+18x^2+9x+1 \\ R_4(x) &= 24x^4+96x^3+72x^2+16x+1. \end{aligned}$$

In words, this means that on a 1 × 1 board, 1 rook can be arranged in 1 way, and zero rooks can also be arranged in 1 way (empty board); on a complete 2 × 2 board, 2 rooks can be arranged in 2 ways (on the diagonals), 1 rook can be arranged in 4 ways, and zero rooks can be arranged in 1 way; and so forth for larger boards.

For complete $m \times n$ rectangular boards $B_{m,n}$ we write $R_{m,n} := R_{B_{m,n}}$. The smaller of *m* and *n* can be taken as an upper limit for *k*, since obviously $r_k = 0$ if $k > \min(m,n)$. This is also shown in the formula for $R_{m,n}(x)$.

The rook polynomial of a square chessboard is closely related to the generalized Laguerre polynomial $L_n^{\alpha}(x)$ by the identity:

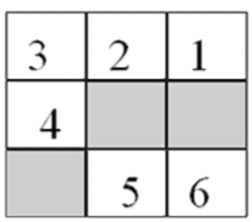
$$R_{m,n}(x) = n! x^n L_n^{(m-n)}(-x^{-1}).$$

ii. Matching polynomials

A rook polynomial is a special case of one kind of matching polynomial, which is the generating function of the number of k-edge matchings in a graph.

The rook polynomial $R_{m,n}(x)$ corresponds to the complete bipartite graph $K_{m,n}$. The rook polynomial of a general board B � $B_{m,n}$ corresponds to the bipartite graph with left vertices v_1, v_2, ..., v_m and right vertices w_1, w_2, ..., w_n and an edge $v_i w_j$ whenever the square (i, j) is allowed, i.e., belongs to B. Thus, the theory of rook polynomials is, in a sense, contained in that of matching polynomials.

We deduce an important fact about the coefficients r_k, which we recall give the number of non-attacking placements of k rooks in B: these numbers are unimodular, i.e., they increase to a maximum and then decrease. This follows (by a standard argument) from the theorem of Heilmann and Lieb about the zeroes of a matching polynomial (a different one from that which corresponds to a rook polynomial, but equivalent to it under a change of variables), which implies that all the zeroes of a rook polynomial are negative real numbers.



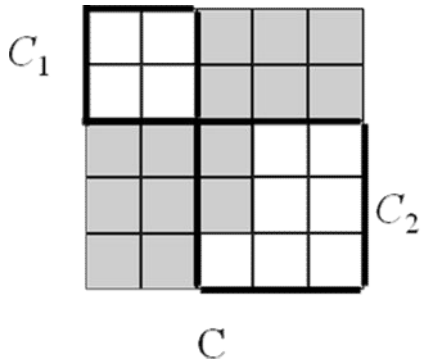
Determine the number of ways in which k rooks can be placed on the chessboard so that no two of them can take each other, i.e., no two of them are in the same row or column of the chessboard. Denote this number by r_k or $r_k(C)$.

$$r_1 = 6, r_2 = 8, r_3 = 2, r_k = 0, \text{ for } k \geq 4$$

With $r_0 = 1$, the rook polynomial

$$r(C,x) = 1 + 6x + 8x^2 + 2x^3$$

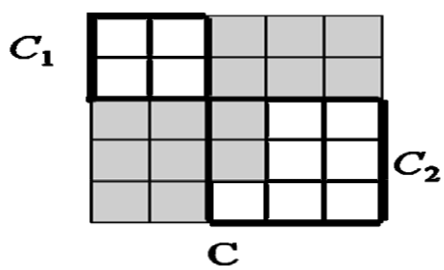
idea: break up a large board into smaller subboards



Did this occur by luck or is something happening here that we should examine more closely?

$r(C_1, x) = 1 + 4x + 2x^2, r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) \cdot r(C_2, x)$

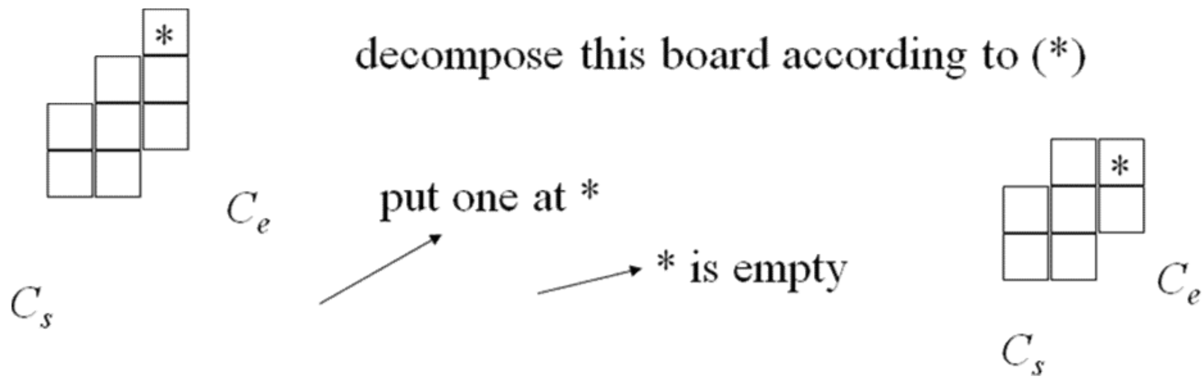


To obtain r_3 for C:
(a) All three rooks are on C_2:(2)(1)=2 ways
(b) Two on C_2 and one on C_1:(10)(4)=40
(c) One on C_2 and two on C_1:(7)(2)=14
total=(2)(1)+(10)(4)+(7)(2)=56

In general, if C is a chessboard made up of pairwise disjoint subboards $C_1, C_2,\quad , C_n$, then $r(C, x) = r(C_1, x) r(C_2, x) \quad r(C_n, x)$.

$r(C_1, x) = 1 + 4x + 2x^2, r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) \cdot r(C_2, x)$



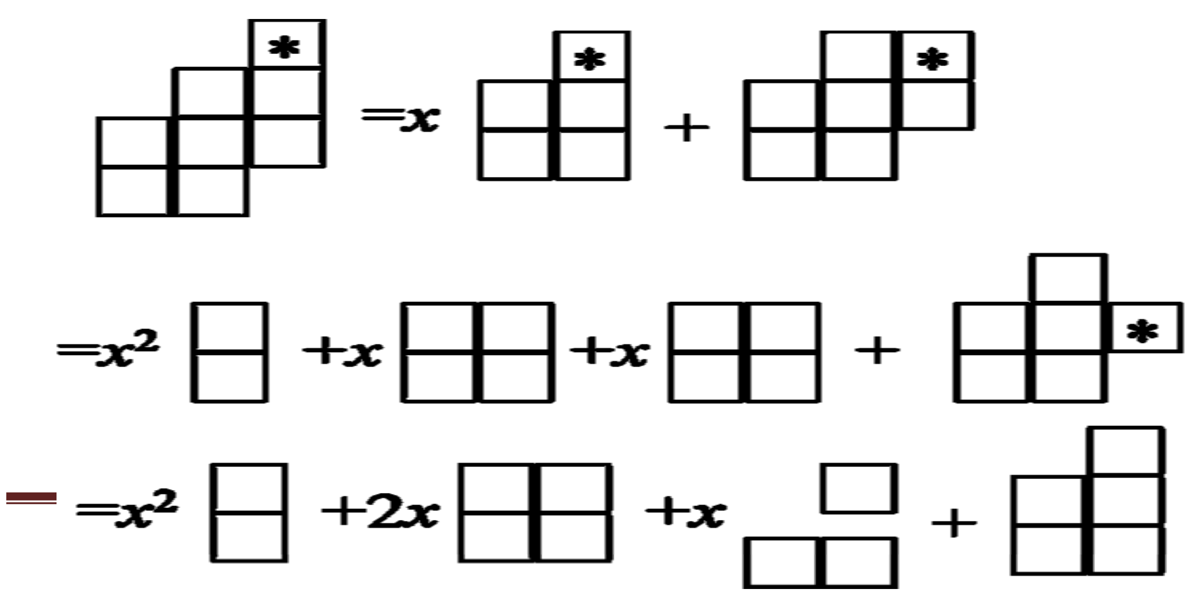
$$r_k(C) = r_{k-1}(C_s) + r_k(C_e)$$

$$r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k$$

$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{n} r_{k-1}(C_s)x^k + \sum_{k=1}^{n} r_k(C_e)x^k$$

$$1+\sum_{k=1}^{n} r_k(C)x^k = x\sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k + 1$$

$$r(C,x) = x \cdot r(C_s,x) + r(C_e,x)$$

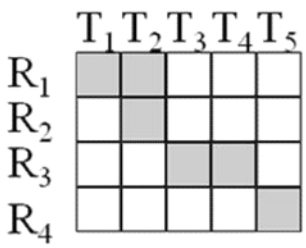


$$= x^2(1+2x) + 2x(1+4x+2x^2) + x(1+3x+x^2) +$$

$$[x(1+2x) + (1+4x+2x^2)] = 1 + 8x + 16x^2 + 7x^3$$

Arrangements with Forbidden Positions

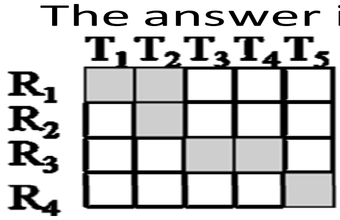
Ex. Arrange 4 persons to sit at five tables such that each one sits at a different table and with the following conditions satisfied:
(a) R_1 will not sit at T_1 or T_2 (b) R_2 will not sit at T_2
(c) R_3 will not sit at T_3 or T_4 (b) R_4 will not sit at T_4 or T_5



condition c_i: R_i is in a forbidden position

It would be easier to work with the shaded area since it is less than the unshaded one.

The answer is $N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\right) = S_0 - S_1 + S_2 - S_3 + S_4.$



condition c_i: R_i is in a forbidden position

condition c_i: R_i is in a forbidden position

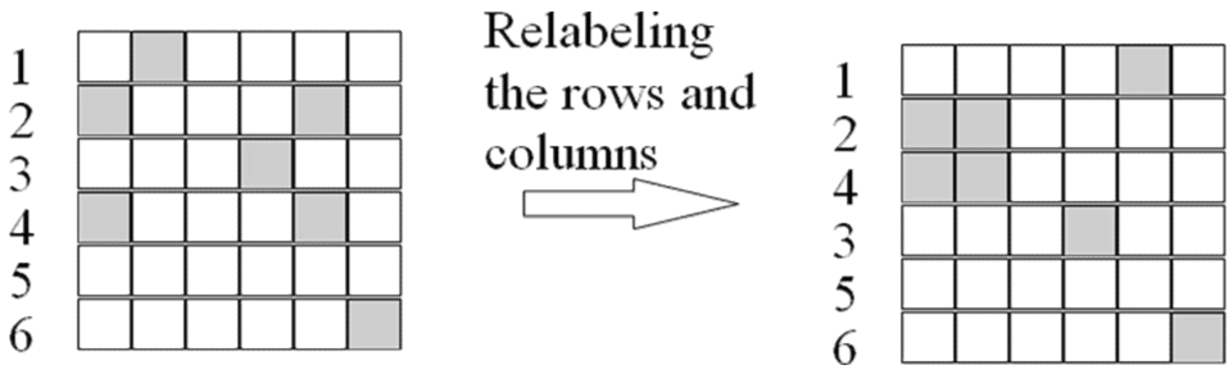
The answer is $N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\right) = S_0 - S_1 + S_2 - S_3 + S_4.$

$S_0 = P(5,4) = 5!$, $S_i = r_i\,(5-i)!$, where r_i is the number of ways in which it is possible to place i nontaking rooks on the shaded chessboard.

$r(C,x) = (1+3x+x^2)(1+4x+3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$

So $N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\right) = \sum_{i=0}^{4} (-1)^i r_i (5-i)! = 25$

Ex. We have a pair of dice; one is red, the other green. We roll these dice six times. What is the *probability* that we obtain all six values on both the red dice and the green die if we know that the ordered pairs (1,2), (2,1),(2,5),(3,4),(4,1),(4,5), and (6,6) did not occur? [(x,y) indicates x on the red die and y on the green.]



For chessboard C of seven shaded squares,

$$r(C,x) = (1+4x+2x^2)(1+x)^3 = 1+7x+17x^2+19x^3+10x^4+2x^5$$

c_i: the condition where, having rolled the dice six times, we find that all six values occur on both the red die and the green die, *but i on the red die is paired with one of the forbidden numbers on the green die*

Then the number of ordered sequences of the six rolls of the dice for the event we are interested in is:

$$6!N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\,\overline{c_5}\,\overline{c_6}) = 6!\sum_{i=0}^{6}(-1)^i S_i =$$

$$6!\sum_{i=0}^{6}(-1)^i r_i (6-i)! = 138{,}240$$

The probability is $138240/(29)^6 \approx 0.00023$

Sequences and Recurrence Relations

Example :

Consider the following two sequences:

$$S_1 : 3, 5, 7, 9, \ldots$$
$$S_2 : 3, 9, 27, 81, \ldots$$

We can find a formula for the nth term of sequences S_1 and S_2 by observing the pattern of the sequences.

$$S_1 : 2 \cdot 1 + 1, 2 \cdot 2 + 1, 2 \cdot 3 + 1, 2 \cdot 4 + 1, \ldots$$
$$S_2 : 3^1, 3^2, 3^3, 3^4, \ldots$$

For S_1, $a_n = 2n + 1$ for $n \geq 1$, and for S_2, $a_n = 3^n$ for $n \geq 1$. This type of formula is called an **explicit formula** for the sequence, because using this formula we can directly find any term of the sequence without using other terms of the sequence. For example, $a_3 = 2 \cdot 3 + 1 = 7$.

Let S denote the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

For this sequence, the explicit formula is not obvious. If we observe closely, however, we find that the pattern of the sequence is such that any term after the second term is the sum of the preceding two terms. Now

$$\begin{aligned}
\text{3rd term} &= 2 = 1 + 1 = \text{1st term} + \text{2nd term} \\
\text{4th term} &= 3 = 1 + 2 = \text{2nd term} + \text{3rd term} \\
\text{5th term} &= 5 = 2 + 3 = \text{3rd term} + \text{4th term} \\
\text{6th term} &= 8 = 3 + 5 = \text{4th term} + \text{5th term} \\
\text{7th term} &= 13 = 5 + 8 = \text{5th term} + \text{6th term}
\end{aligned}$$

Hence, the sequence S can be defined by the equation

$$f_n = f_{n-1} + f_{n-2} \tag{8.1}$$

for all $n \geq 3$ and

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1. \end{aligned} \tag{8.2}$$

Example:

Number of subsets of a finite set. Let s_n denote the number of subsets of a set A with n elements, $n \geq 0$. In Worked-Out Exercise 9 (Chapter 2, page 144), we proved that

$$s_0 = 1,$$
$$s_n = 2s_{n-1}, \quad \text{if } n > 0$$

Hence, a recurrence relation for the sequence $s_0, s_1, s_2, s_3, s_4, \ldots$ is

$$s_n = 2s_{n-1}, \quad n \geq 1$$

and an initial condition is $s_0 = 1$.

and so on. Here $f(n) = nf(n-1)$ for all $n \geq 1$ is the recurrence relation, and $f(0) = 1$ is the initial condition for the function f. Notice that the function f is nothing but the factorial function, i.e., $f(n) = n!$ for all $n \geq 0$.

Sequences and Recurrence Relations

Let us consider the function f as given in (8.3). If we write $a_n = f(n)$, then (8.3) translates into the following equation:

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2.$$

That is, a_n is defined in terms of a_{n-1} and a_{n-2}. As remarked previously, such an equation is called a recurrence relation. Moreover, (8.4) translates into $a_0 = 5$ and $a_1 = 7$. These are called the initial conditions for the recurrence relation.

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ is an equation that relates a_n to some of the terms $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}$ for all integers n with $n \geq k$, where k is a nonnegative integer. The **initial conditions** for the recurrence relation are a set of values that explicitly define some of the members of $a_0, a_1, a_2, \ldots, a_{k-1}$.

The equation

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2,$$

as defined above, relates a_n to a_{n-1} and a_{n-2}. Here $k = 2$. So this is a recurrence relation with initial conditions $a_0 = 5$ and $a_1 = 7$.

Let S be the sequence $\{a_n\}_{n=0}^{\infty}$, where

$$a_n = 7a_{n-1} - 6a_{n-2} \quad \text{for all } n \geq 2. \tag{8.8}$$

Because a_n is defined in terms of the preceding terms a_{n-1} and a_{n-2}, Equation (8.8) is a recurrence relation.

Let us show that $a_n = 5 = 5 + 0 \cdot n$ is a solution of Equation (8.8). Here $a_0 = 5$, $a_1 = 5$, $a_2 = 5, \ldots, a_n = 5$, and so on. Let us evaluate the right side of Equation (8.8), i.e.,

$$7a_{n-1} - 6a_{n-2} = 7 \cdot 5 - 6 \cdot 5 = 35 - 30 = 5 = a_n.$$

Hence, $a_n = 5, n \geq 0$ is a solution of the recurrence relation (8.8).

Now let $a_n = 6^n$. Here $a_0 = 6^0 = 1$, $a_1 = 6^1 = 6$, $a_2 = 6^2 = 36, \ldots, a_{n-2} = 6^{n-2}$, $a_{n-1} = 6^{n-1}$, $a_n = 6^n$, and so on. Let us evaluate the right side of Equation (8.8), using the terms of this sequence. We have

$$\begin{aligned} 7a_{n-1} - 6a_{n-2} &= 7 \cdot 6^{n-1} - 6 \cdot 6^{n-2} \\ &= 7 \cdot 6^{n-1} - 6^{n-1} \\ &= (7-1) \cdot 6^{n-1} \\ &= 6 \cdot 6^{n-1} \\ &= 6^n \\ &= a_n. \end{aligned}$$

Therefore, $a_n = 6^n, n \geq 0$ is also a solution of the recurrence relation (8.8).

Note that the expression $a_n = 2^n$, $n \geq 0$ is not a solution of Equation (8.8).

Linear Homogenous Recurrence Relations

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of numbers. A **linear homogeneous recurrence relation** of order k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \tag{8.31}$$

where $c_k \neq 0$ and $c_1, c_2, c_3, \ldots$, and c_k are constants.

Linear Homogenous Recurrence Relations

Consider the following recurrence relations.

(i) $a_n = 3a_{n-1} + a_{n-2}$
(ii) $a_n = 3a_{n-1} + 5$
(iii) $a_n = 3a_{n-1} + a_{n-2} \cdot a_{n-3}$
(iv) $a_n = 3a_{n-1} + a_{n-2} + \sqrt{2}a_{n-3}$
(v) $a_n = 3a_{n-1} + na_{n-2}$

Recurrence relations (i), (ii), (iii), and (iv) are recurrence relations with constant coefficients. Recurrence relation (v), $a_n = 3a_{n-1} + na_{n-2}$, is not a relation with constant coefficients. Notice that (i) is a linear homogeneous recurrence

Linear Homogenous Recurrence Relations

A sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is said to **satisfy** a linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \quad c_k \neq 0 \qquad (8.32)$$

of order k with constant coefficients if $s_n = c_1 s_{n-1} + c_2 s_{n-2} + c_3 s_{n-3} + \cdots + c_k s_{n-k}$.

If a sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ satisfies a linear homogeneous recurrence relation, then the sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is also called a **solution** of that recurrence relation.

Consider the recurrence relation $a_n = 3a_{n-1}$. This is a linear homogeneous recurrence relation of order 1. Let t be a nonzero number and suppose $a_n = t^n$ for all $n \geq 0$. Then $a_n = 3a_{n-1}$ implies that $t^n = 3t^{n-1}$. Therefore, $t = 3$. Thus, we find that $a_n = 3^n$. Hence, the sequence $1, 3, 3^2, 3^3, \ldots 3^n, \ldots$ is a solution of the recurrence relation $a_n = 3a_{n-1}$.

Example:

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 7a_{n-1} - 10a_{n-2} \tag{8.41}$$

with initial conditions

$$a_0 = 1$$
$$a_1 = 8.$$

The characteristic equation of the given recurrence relation is:

$$t^2 - 7t + 10 = 0.$$

Next, we find the roots of this equation. Now,

$$t^2 - 7t + 10 = (t-5)(t-2)$$

and so

$$(t-5)(t-2) = 0.$$

This implies that the roots of the characteristic equation are $t = 5$, and $t = 2$. The roots are distinct. By Theorem 8.2.10, there exist constants c_1 and c_2, which are to be determined from initial conditions, such that

$$a_n = c_1 5^n + c_2 2^n, \quad n \geq 0.$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

ι with initial conditions

$$a_0 = 4$$
$$a_1 = 12.$$

The characteristic equation of this recurrence relation is the quadratic equation

$$t^2 - 4t + 4 = 0.$$

We find the roots of this equation. Now,

H

$$t^2 - 4t + 4 = (t-2)(t-2)$$

and so

$$(t-2)(t-2) = 0.$$

Linear Nonhomogenous Recurrence Relations

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n.$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = 1$.

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n(n^2 + 6n + 5).$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = n^2 + 6n + 5$.

Linear Nonhomogenous Recurrence Relations

In this example, we use Theorem 8.3.6 to solve the recurrence relation

$$a_n - 4a_{n-1} = 8^n, \quad n \geq 1,$$

with the initial condition

$$a_0 = 1.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n u,$$

Consider the recurrence relation

$$a_n - 3a_{n-1} = 2^n(4n+3), \quad n > 1 \tag{8.94}$$

with initial conditions

$$a_0 = 0,$$
$$a_1 = 14.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n(un + v).$$

Here $d = 3, b = 2, u = 4$, and $v = 3$.

We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0, c_1, and c_2 are constants, which are to be determined from the initial conditions.

Consider the recurrence relation

$$a_n - 3a_{n-1} = 2^n(4n+3), \quad n > 1 \tag{8.94}$$

with initial conditions

$$a_0 = 0,$$
$$a_1 = 14.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n(un + v).$$

Here $d = 3, b = 2, u = 4$, and $v = 3$.

We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0, c_1, and c_2 are constants, which are to be determined from the initial conditions.

Put $n = 2$ in (8.92) to get

$$a_2 - 3a_1 = 2^2(4 \cdot 2 + 3) = 44.$$

Because $a_1 = 14$, we get

$$a_2 = 3 \cdot 14 + 44 = 86.$$

Thus,

$$\begin{aligned} a_0 &= & c_0 + c_1 &= 0 \\ a_1 &= & c_0 \cdot 3 + c_1 \cdot 2 + c_2 \cdot 2 &= 14 \\ a_2 &= & c_0 \cdot 3^2 + c_1 \cdot 2^2 + c_2 \cdot 2 \cdot 2^2 &= 86 \end{aligned}$$

This implies that

$$\begin{aligned} c_0 + c_1 &= 0 \\ 3c_0 + 2c_1 + 2c_2 &= 14 \\ 9c_0 + 4c_1 + 8c_2 &= 86 \end{aligned}$$

We solve these equations for c_0, c_1, and c_2 to obtain $c_0 = 30$, $c_1 = -30$, and $c_2 = -8$. Thus, we find that

$$a_n = 30(3^n) - 30(2^n) - n2^{n+3}, \quad n \geq 0. \tag{8.95}$$