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NOTES

DESIGN AND ANALYSIS OF ALGORITHMS

SUBJECT CODE: 19SCS42

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MODULE 1
INTRODUCTION TO ALGORITHMS

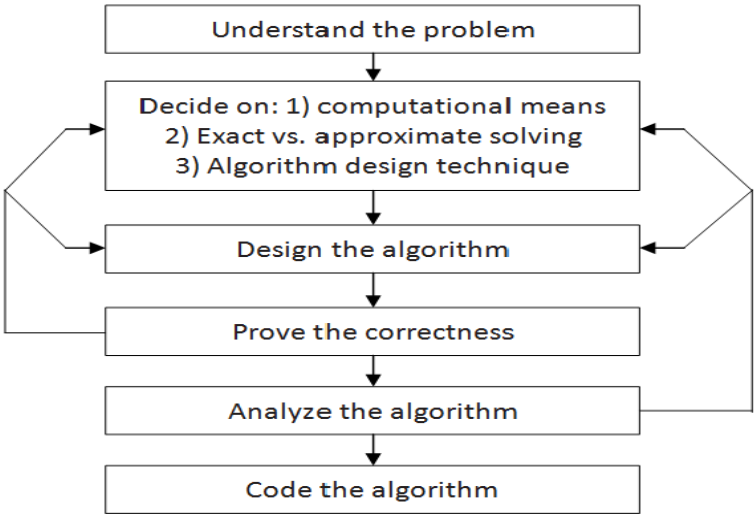
1. Introduction

1.1 What is an Algorithm?

An ***algorithm*** is a finite set of instructions to solve a particular problem. In addition, all algorithms must satisfy the following criteria:

a. ***Input***. Zero or more quantities are externally supplied.
b. ***Output***. At least one quantity is produced.
c. ***Definiteness***. Each instruction is clear and unambiguous. It must be perfectly clear what should be done.
d. ***Finiteness***. If we trace out the instruction of an algorithm, then for all cases, the algorithm terminates after a finite number of steps.
e. ***Effectiveness***. Every instruction must be very basic so that it can be carried out, in principle, by a person using only pencil and paper. It is not enough that each operation be definite as in criterion c; it also must be feasible.

Algorithm design and analysis process - We now briefly discuss a sequence of steps one typically goes through in designing and analyzing an algorithm



(Extra)

- **Understanding the Problem -** From a practical perspective, the first thing you need to do before designing an algorithm is to understand completely the problem given. An input to an algorithm specifies an instance of the problem the algorithm solves. It is very important to specify exactly the set of instances the algorithm needs to handle.
- **Ascertaining the Capabilities of the Computational Device** - Once you completely understand a problem, you need to ascertain the capabilities of the computational device

the algorithm is intended for. Select appropriate model from sequential or parallel programming model.

- **Choosing between Exact and Approximate Problem Solving** - The next principal decision is to choose between solving the problem exactly and solving it approximately. Because, there are important problems that simply cannot be solved exactly for most of their instances and some of the available algorithms for solving a problem exactly can be unacceptably slow because of the problem's intrinsic complexity.
- **Algorithm Design Techniques** - An algorithm design technique (or "strategy" or "paradigm") is a general approach to solving problems algorithmically that is applicable to a variety of problems from different areas of computing. They provide guidance for designing algorithms for new problems, i.e., problems for which there is no known satisfactory algorithm.
- **Designing an Algorithm and Data Structures** - One should pay close attention to choosing data structures appropriate for the operations performed by the algorithm. For example, the sieve of Eratosthenes would run longer if we used a linked list instead of an array in its implementation. ***Algorithms + Data Structures = Programs***
- **Methods of Specifying an Algorithm-** Once you have designed an algorithm; you need to specify it in some fashion. These are the two options that are most widely used nowadays for specifying algorithms. Using a ***natural language*** has an obvious appeal; however, the inherent ambiguity of any natural language makes a concise and clear description of algorithms surprisingly difficult. ***Pseudocode*** is a mixture of a natural language and programming language like constructs. Pseudocode is usually more precise than natural language, and its usage often yields more succinct algorithm descriptions.
- **Proving an Algorithm's Correctness -** Once an algorithm has been specified, you have to prove its correctness. That is, you have to prove that the algorithm yields a required result for every legitimate input in a finite amount of time. For some algorithms, a proof of correctness is quite easy; for others, it can be quite complex. A common technique for proving correctness is to use mathematical induction because an algorithm's iterations provide a natural sequence of steps needed for such proofs.
- **Analyzing an Algorithm -** After correctness, by far the most important is ***efficiency***. In fact, there are two kinds of algorithm efficiency: ***time efficiency***, indicating how fast the algorithm runs, and ***space efficiency***, indicating how much extra memory it uses. Another desirable characteristic of an algorithm is ***simplicity***. Unlike efficiency, which can be precisely defined and investigated with mathematical rigor, simplicity, like beauty, is to a considerable degree in the eye of the beholder.
- **Coding an Algorithm -** Most algorithms are destined to be ultimately implemented as computer programs. Implementing an algorithm correctly is necessary but not sufficient: you would not like to diminish your algorithm's power by an inefficient implementation.

Modern compilers do provide a certain safety net in this regard, especially when they are used in their code optimization mode.

1.1. Algorithm Specification

An algorithm can be specified in

1) Simple English
2) Graphical representation like flow chart
3) Programming language like c++ / java
4) Combination of above methods.

Using the combination of simple English and C++, the algorithm for **selection sort** is specified as follows.

```
for (i=1; i<=n; i++) {
   examine a[i] to a[n] and suppose
   the smallest element is at a[j];
   interchange a[i] and a[j];
}
```

In C++ the same algorithm can be specified as follows

```
void SelectionSort(Type a[], int n)
// Sort the array a[1:n] into nondecreasing order.
{
    for (int i=1; i<=n; i++) {
       int j = i;
       for (int k=i+1; k<=n; k++)
          if (a[k]<a[j]) j=k;
       Type t = a[i]; a[i] = a[j]; a[j] = t;
    }
}
```

Here *Type* is a basic or user defined data type.

Recursive algorithms

An algorithm is said to be **recursive** if the same algorithm is invoked in the body (direct recursive). Algorithm ***A*** is said to be **indirect recursive** if it calls another algorithm which in turn calls A.

Example 1: Factorial computation n! = n * (n-1)!

Example 2: Binomial coefficient computation

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} = \frac{n!}{m!(n-m)!}$$

Example 3: Tower of Hanoi problem

Example 4: Permutation Generator

1.2. Analysis Framework

General framework for analyzing the efficiency of algorithms is discussed here. There are two kinds of efficiency: **time efficiency** and **space efficiency**. Time efficiency indicates how fast an algorithm in question runs; space efficiency deals with the extra space the algorithm requires.

In the early days of electronic computing, both resources **time** and **space** were at a premium. Now the amount of extra space required by an algorithm is typically not of as much concern, In addition, the research experience has shown that for most problems, we can achieve much more spectacular progress in speed than in space. Therefore, following a well-established tradition of algorithm textbooks, we primarily concentrate on time efficiency.

Measuring an Input's Size

It is observed that almost all algorithms **run longer on larger inputs.** For example, it takes longer to sort larger arrays, multiply larger matrices, and so on. Therefore, it is logical to investigate an algorithm's efficiency as a function of some parameter ***n*** indicating the **algorithm's input size**.

There are situations, where the choice of a **parameter indicating an input size does matter**. The choice of an appropriate size metric can be influenced by operations of the algorithm in question. For example, how should we measure an input's size for a spell-checking algorithm? If the algorithm examines individual characters of its input, then we should measure the size by the number of characters; if it works by processing words, we should count their number in the input.

We should make a special note about measuring the size of inputs for algorithms involving **properties of numbers** (e.g., checking whether a given integer n is prime). For such algorithms, computer scientists prefer measuring size by the number ***b*** of bits in the ***n***'s binary representation: $b = \lfloor \log_2 n \rfloor + 1$. This metric usually gives a better idea about the efficiency of algorithms in question.

Units for Measuring Running lime

To measure an algorithm's efficiency, we would like to have a **metric that does not depend on these extraneous factors**. One possible approach is to count the number of times each of the algorithm's operations is executed. This approach is both excessively difficult and, as we shall see, usually unnecessary. The thing to do is to identify the most important operation of the algorithm, called the **basic operation**, the operation contributing the most to the total running time, and compute the number of times the basic operation is executed.

For example, most **sorting** algorithms work by **comparing elements** (keys) of a list being sorted with each other; for such algorithms, the basic operation is a key comparison.

As another example, algorithms for **matrix multiplication** and **polynomial evaluation** require two arithmetic operations: **multiplication and addition**.

Let c_{op} be the execution time of an algorithm's basic operation on a particular computer, and let ***C(n)*** be the number of times this operation needs to be executed for this algorithm. Then we can estimate the running time ***T(n)*** of a program implementing this algorithm on that computer by the formula:

$$T(n) = c_{op}C(n)$$

unless ***n*** is extremely large or very small, the formula can give a reasonable estimate of the algorithm's running time.

It is for these reasons that the efficiency analysis framework ignores multiplicative constants and concentrates on the count's **order of growth** to within a constant multiple for large-size inputs.

Orders of Growth

Why this emphasis on the count's order of growth for large input sizes? Because for large values of n, it is the function's order of growth that counts: just look at table which contains values of a few functions particularly important for analysis of algorithms.

Table: *Values of several functions important for analysis of algorithms*

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

Algorithms that require an exponential number of operations are practical for solving only problems of very small sizes.

Worst-Case, Best-Case, and Average-Case Efficiencies

Definition: The **worst-case efficiency** of an algorithm is its efficiency for the worst-case input of size n, for which the algorithm runs the longest among all possible inputs of that size.

Consider the algorithm for sequential search.

ALGORITHM *SequentialSearch*$(A[0..n-1], K)$

```
//Searches for a given value in a given array by sequential search
//Input: An array A[0..n − 1] and a search key K
//Output: The index of the first element in A that matches K
//        or −1 if there are no matching elements
i ← 0
while i < n and A[i] ≠ K do
    i ← i + 1
if i < n return i
else return −1
```

The running time of above algorithm can be quite different for the same list size n. In the worst case, when there are **no matching elements** or the first **matching element happens to be the last one on the list**, the algorithm makes the largest number of key comparisons among all possible inputs of size n: $C_{worst}(n) = n$.

In general, we analyze the algorithm to see what kind of inputs yield the largest value of the basic operation's count $C(n)$ among all possible inputs of size n and then compute this worst-case value $C_{worst}(n)$. The worst-case analysis provides algorithm's efficiency by bounding its running time from above. Thus it guarantees that for any instance of size n, the running time will not exceed $C_{worst}(n)$, its running time on the worst-case inputs.

Definition: The **best-case efficiency** of an algorithm is its efficiency for the best-case input of size n, for which the algorithm runs the fastest among all possible inputs of that size.

We determine the kind of inputs for which the count $C(n)$ will be the smallest among all possible inputs of size n. For example, for sequential search, best-case inputs are lists of size n with their first elements equal to a search key; $C_{best}(n) = 1$.

The analysis of the best-case efficiency is not nearly as important as that of the worst-case efficiency. Also, neither the worst-case analysis nor its best-case counterpart yields the necessary information about an algorithm's behavior on a "typical" or "random" input. This information is provided by **average-case** efficiency.

Definition: the **average-case complexity** of an algorithm is the amount of time used by the algorithm, averaged over all possible inputs.

Let us consider again sequential search. The standard assumptions are that (a) the probability of a successful search is equal top ($0 \le p \le 1$) and (b) the probability of the first match occurring in the i^{th} position of the list is the same for every i. We can find the average number of key comparisons $C_{avg}(n)$ as follows.

In the case of a successful search, the probability of the first match occurring in the i^{th} position of the list is p/n for every i, and the number of comparisons made by the algorithm in such a situation is obviously i. In the case of an unsuccessful search, the number of comparisons is n with the probability of such a search being $(1-p)$. Therefore,

$$\begin{aligned} C_{avg}(n) &= [1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + \cdots + i \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}] + n \cdot (1 - p) \\ &= \frac{p}{n}[1 + 2 + \cdots + i + \cdots + n] + n(1 - p) \\ &= \frac{p}{n}\frac{n(n+1)}{2} + n(1 - p) = \frac{p(n+1)}{2} + n(1 - p). \end{aligned}$$

Investigation of the average-case efficiency is considerably more difficult than investigation of the worst-case and best-case efficiencies. But there are many important algorithms for which the average case efficiency is much better than the overly pessimistic worst-case efficiency would lead us to believe. Note that average-case efficiency cannot be obtained by taking the average of the worst-case and the best-case efficiencies.

Summary of analysis framework

- Both time and space efficiencies are measured as functions of the algorithm's input size.
- Time efficiency is measured by counting the number of times the algorithm's basic operation is executed. Space efficiency is measured by counting the number of extra memory units consumed by the algorithm.
- The efficiencies of some algorithms may differ significantly for inputs of the same size. For such algorithms, we need to distinguish between the worst-case, average-case, and best-case efficiencies.
- The framework's primary interest lies in the order of growth of the algorithm's running time (or extra memory units consumed) as its input size goes to infinity.

2. Performance Analysis

2.1 Space complexity

Total amount of computer memory required by an algorithm to complete its execution is called as **space complexity** of that algorithm. The Space required by an algorithm is the sum of following components

- A **fixed** part that is independent of the input and output. This includes memory space for codes, variables, constants and so on.
- A **variable** part that depends on the input, output and recursion stack. (We call these parameters as instance characteristics)

Space requirement *S(P)* of an algorithm *P,* $S(P) = c + Sp$ where c is a constant depends on the fixed part, Sp is the instance characteristics

Example-1: Consider following algorithm **abc()**

```
loat a, float b, float c)
(a + b + b*c + (a+b-c)/(a+b) + 4.0);
```

Here fixed component depends on the size of a, b and c. Also instance characteristics *Sp=0*

Example-2: Let us consider the algorithm to find sum of array.

For the algorithm given here the problem instances are characterized by n, the number of elements to be summed. The space needed by $a[\,]$ depends on n. So the space complexity can be written as; $S_{sum}(n) \geq (n+3)$ n for a[ ], One each for n, i and s.

```
float Sum(float a[], int n)
{   float s = 0.0;
    for (int i=1; i<=n; i++)
       s += a[i];
    return s;
}
```

2.2 Time complexity

Usually, the execution time or run-time of the program is refereed as its time complexity denoted by t_p (instance characteristics). This is the sum of the time taken to execute all instructions in the program.

Exact estimation runtime is a complex task, as the number of instruction executed is dependent on the input data. Also different instructions will take different time to execute. So for the estimation of the time complexity **we count only the number of p ogram steps**.

A program step is loosely defined as syntactically or semantically meaning segment of the program that has and execution time that is independent of instance characteristics. For example comment has zero steps; assignment statement has one step and so on.

We can determine the **steps needed by a program** to solve a particular problem instance in two ways.

In the **first method** we introduce a new variable ***count*** to the program which is initialized to zero. We also introduce statements to increment ***count*** by an appropriate amount into the program. So when each time original program executes, the ***count*** also incremented by the step count.

Example-1: Consider the algorithm ***sum( )***. After the introduction of the count the program will be as follows.

```
float Sum(float a[], int n)
{   float s = 0.0;
    count++; // count is global
    for (int i=1; i<=n; i++) {
       count++; // For 'for'
       s += a[i]; count++; // For assignment
    }
    count++; // For last time of 'for'
    count++; // For the return
    return s;
}
```

From the above we can estimate that invocation of ***sum( )*** executes total number of **2n+3** steps.

The **second method** to determine the step count of an algorithm is to build a table in which we list the total number of steps contributed by each statement. An example is shown below.

Statement	s/e	frequency	total steps
`float Sum(float a[], int n)`	0	–	0
`{ float s = 0.0;`	1	1	1
`for (int i=1; i<=n; i++)`	1	$n+1$	$n+1$
`s += a[i];`	1	n	n
`return s;`	1	1	1
`}`	0	–	0
Total			$2n+3$

Example-2: matrix addition

```
void Add(Type a[][SIZE], Type b[][SIZE],
         Type c[][SIZE], int m, int n)
{   for (int i=1; i<=m; i++)
       for (int j=1; j<=n; j++)
          c[i][j] = a[i][j] + b[i][j];
}
```

Statement	s/e	freq	total
void Add(Type a[][SIZE], ...)	0	–	0
{ for (int i=1; i<=m; i++)	1	$m+1$	$m+1$
for (int j=1; j<=n; j++)	1	$m(n+1)$	$mn+m$
c[i][j] = a[i][j]			
+ b[i][j];	1	mn	mn
}	0	–	0
Total			$2mn+2m+1$

The above thod is both excessively difficult and, usually unnecessary. The thing to do is to identify the most important operation of the algorithm, called the **basic operation**, the operation contributing the m st to the total running time, and compute the number of times the basic operation is executed.

Trade-off

There is often a **time-space-tradeoff** involved in a problem, that is, it cannot be solved with few computing time and low memory consumption. One has to make a compromise and to exchange computing time for memory consumption or vice versa, depending on which algorithm one chooses and how one parameterizes it.

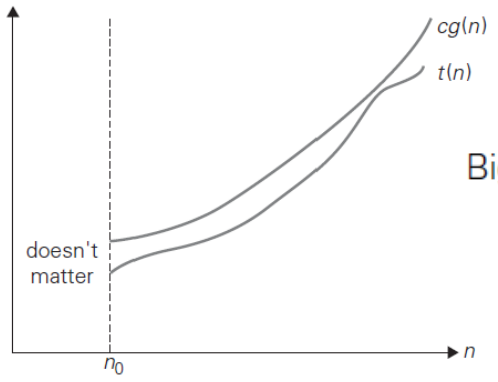
3. Asymptotic Notations

The efficiency analysis framework concentrates on the order of growth of an algorithm's basic operation count as the principal indicator of the algorithm's efficiency. To compare and rank such orders of growth, computer scientists use three notations: O(big oh), Ω(big omega), Θ (big theta) and *o*(little oh)

3.1. Big-Oh notation

Definition: A function *t(n)* is said to be in *O(g(n)),* denoted *t(n)*∈*O(g(n)),* if t (n) is bounded above by some constant multiple of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n_0 such that

$$t(n) \leq c\, g(n) \text{ for all } n \geq n_0.$$



Big-oh notation: $t(n) \in O(g(n))$.

Informally, *O(g(n))* is the set of all functions with a lower or same order of growth as *g(n)*

Examples: $n \in O(n^2)$, $100n + 5 \in O(n^2)$, $\frac{1}{2}n(n-1) \in O(n^2)$.

$n^3 \notin O(n^2)$, $0.00001n^3 \notin O(n^2)$, $n^4 + n + 1 \notin O(n^2)$.

As another example, let us formally prove **100n + 5 ∈ O(n²)**

$100n + 5 \leq 100n + n$ (for all $n \geq 5$) = $101n \leq 101n^2$. *(c=101, n_0=5)*

Note that the definition gives us a lot of freedom in choosing specific values for constants ***c*** and n_0.

Example: To prove $n^2 + n = \underline{O}(n^3)$

Here, we have $f(n) = n^2 + n$, and $g(n) = n^3$

Notice that if $n \geq 1$, $n \leq n^3$ is clear.

Also, notice that if $n \geq 1$, $n^2 \leq n^3$ is clear.

Therefore,

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

We have just shown that

$$n^2 + n \leq 2n^3 \text{ for all } n \geq 1$$

Thus, we have shown that $n^2 + n = O(n^3)$
(by definition of Big-O, with $n_0 = 1$, and $c = 2$.)

Strategies for Big-O Sometimes the easiest way to prove that f(n) = O(g(n)) is to take c to be the sum of the positive coefficients of f(n). We can usually ignore the negative coefficients.

Example: To prove $5n^2 + 3n + 20 = O(n^2)$, we pick $c = 5 + 3 + 20 = 28$. Then if $n \geq n_0 = 1$,

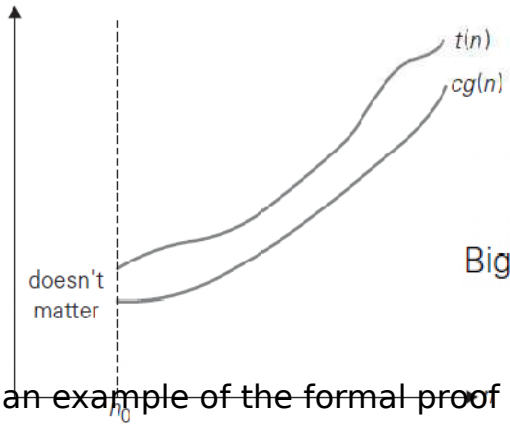
$$5\,n^2 + 3\,n + 20 \leq 5\,n^2 + 3\,n^2 + 20\,n^2 = 28\,n^2,$$

thus $5n^2 + 3n + 20 = O(n^2)$.

3.2. Omega notation

Definition: A function t(n) is said to be in Ω(g(n)), denoted ***t(n)*∈ *Ω(g(n)),*** if t(n) is bounded below by some positive constant multiple of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n_0 such that

$$t(n) \geq c\, g(n) \text{ for all } n \geq n_0.$$



Big-omega notation: $t(n) \in \Omega(g(n))$.

Here is an example of the formal proof that ***n3 ∈Ω(n2): n3 ≥ n2*** for all n ≥ 0,
i.e., we can select c = 1 and n0 = 0.

Example:

Example: To prove n3 + 4n2 = Ω(n2)

$$n^3 \in \Omega(n^2), \qquad \frac{1}{2}n(n-1) \in \Omega(n^2), \qquad \text{but } 100n + 5 \notin \Omega(n^2).$$

Here, we have $f(n) = n^3 + 4n^2$, and $g(n) = n^2$

It is not too hard to see that if $n \geq 0$,

$$n^3 \leq n^3 + 4n^2$$

We have already seen that if $n \geq 1$,

$$n^2 \leq n^3$$

Thus when $n \geq 1$,

$$n^2 \leq n^3 \leq n^3 + 4n^2$$

Therefore,

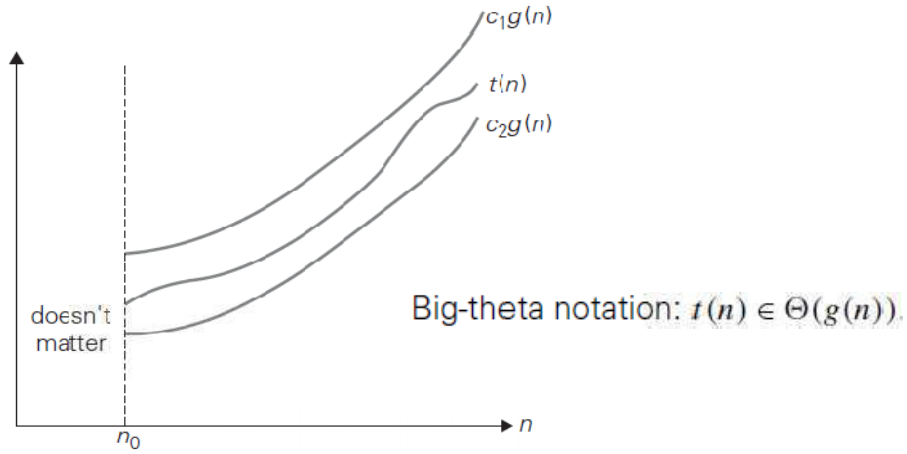
$$1n^2 \leq n^3 + 4n^2 \text{ for all } n \geq 1$$

Thus, we have shown that $n^3 + 4n^2 = \Omega(n^2)$
(by definition of Big-Ω, with $n_0 = 1$, and $c = 1$.)

3.3. Theta notation

A function t(n) is said to be in **Θ(g(n)),** denoted **t(n) ∈ Θ(g(n)),** if t (n) is bounded both above and below by some positive constant multiples of g(n) for all large n, i.e., if there exist some positive constants c_1 and c_2 and some nonnegative integer n_0 such that

c_2 g(n) Š t(n) Š c_1g(n) for all n Š n_0.



For example, let us prove that $\frac{1}{2}n(n-1) \in \Theta(n^2)$. First, we prove the right inequality (the upper bound):

$$\frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \leq \frac{1}{2}n^2 \quad \text{for all } n \geq 0.$$

Second, we prove the left inequality (the lower bound):

$$\frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \geq \frac{1}{2}n^2 - \frac{1}{2}n\frac{1}{2}n \text{ (for all } n \geq 2) = \frac{1}{4}n^2.$$

Hence, we can select $c_2 = \frac{1}{4}$, $c_1 = \frac{1}{2}$, and $n_0 = 2$.

Example: $n^2 + 5n + 7 = \Theta(n^2)$

When $n \geq 1$,

$$n^2 + 5n + 7 \leq n^2 + 5n^2 + 7n^2 \leq 13n^2$$

When $n \geq 0$,

$$n^2 \leq n^2 + 5n + 7$$

Thus, when $n \geq 1$

$$1n^2 \leq n^2 + 5n + 7 \leq 13n^2$$

Thus, we have shown that $n^2 + 5n + 7 = \Theta(n^2)$ (by definition of Big-Θ, with $n_0 = 1$, $c_1 = 1$, and $c_2 = 13$.)

Strategies for Ω and Θ

- Proving that a f(n) = Ω(g(n)) often requires more thought.
  - Quite often, we have to pick c < 1.
  - A good strategy is to pick a value of c which you think will work, and determine which value of n_0 is needed.
  - Being able to do a little algebra helps.
  - We can sometimes simplify by ignoring terms of f(n) with the positive coefficients.
- The following theorem shows us that proving f(n) = Θ(g(n)) is nothing new:

  Theorem: f(n) = Θ(g(n)) if and only if f(n) = O(g(n)) and f(n) = Ω(g(n)).

  Thus, we just apply the previous two strategies.

Show that $\frac{1}{2}n^2 + 3n = \Theta(n^2)$

Notice that if $n \geq 1$,

$$\frac{1}{2}n^2 + 3n \leq \frac{1}{2}n^2 + 3n^2 = \frac{7}{2}n^2$$

Thus,

$$\frac{1}{2}n^2 + 3n = O(n^2)$$

Also, when $n \geq 0$,

$$\frac{1}{2}n^2 \leq \frac{1}{2}n^2 + 3n$$

So

$$\frac{1}{2}n^2 + 3n = \Omega(n^2)$$

Since $\frac{1}{2}n^2 + 3n = O(n^2)$ and $\frac{1}{2}n^2 + 3n = \Omega(n^2)$,

$$\frac{1}{2}n^2 + 3n = \Theta(n^2)$$

Show that $(n \log n - 2n + 13) = \Omega(n \log n)$

Proof: We need to show that there exist positive constants c and n_0 such that

$$0 \leq c\,n \log n \leq n \log n - 2n + 13 \text{ for all } n \geq n_0$$

Since $n \log n - 2n \leq n \log n - 2n + 13$,

we will instead show that

$$c\,n \log n \leq n \log n - 2n,$$

which is equivalent to

$$c \leq 1 - \frac{2}{\log n}, \text{ when } n > 1.$$

If $n \geq 8$, then $2/(\log n) \leq 2/3$, and picking $c = 1/3$ suffices. Thus if $c = 1/3$ and $n_0 = 8$, then for all $n \geq n_0$, we have

$$0 \leq c\,n \log n \leq n \log n - 2n \leq n \log n - 2n + 13.$$

Thus $(n \log n - 2n + 13) = \Omega(n \log n)$.

Show that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

We need to find positive constants c_1, c_2, and n_0 such that

$$0 \leq c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2 \text{ for all } n \geq n_0$$

Dividing by n^2, we get

$$0 \leq c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2$$

3.4. Little Oh The function **f(n) = o(g(n))** [i.e f of n is a little oh of g of n] if and only if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Example: The function $3n + 2 = o(n^2)$ since $\lim_{n\to\infty} \frac{3n-2}{n^2} = 0$. $3n + 2 = o(n \log n)$. $3n + 2 \neq o(n \log \log n)$. $6 * 2^n + n^2 = o(3^n)$. $6 * 2^n + n^2 = o(2^n \log n)$. $3n + 2 \neq o(n)$. $6 * 2^n + n^2 \neq o(2^n)$. □

For comparing the order of growth limit is used

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \begin{cases} 0 & \text{implies that } t(n) \text{ has a smaller order of growth than } g(n) \\ c & \text{implies that } t(n) \text{ has the same order of growth as } g(n), \\ \infty & \text{implies that } t(n) \text{ has a larger order of growth than } g(n). \end{cases}$$

If the case-1 holds good in the above limit, we represent it by little-oh.

EXAMPLE 1 Compare the orders of growth of $\frac{1}{2}n(n-1)$ and n^2. (This is one of the examples we used at the beginning of this section to illustrate the definitions.)

$$\lim_{n\to\infty} \frac{\frac{1}{2}n(n-1)}{n^2} = \frac{1}{2} \lim_{n\to\infty} \frac{n^2 - n}{n^2} = \frac{1}{2} \lim_{n\to\infty} (1 - \frac{1}{n}) = \frac{1}{2}.$$

Since the limit is equal to a positive constant, the functions have the same order of growth or, symbolically, $\frac{1}{2}n(n-1) \in \Theta(n^2)$. ■

EXAMPLE 2 Compare the orders of growth of $\log_2 n$ and $\sqrt{n}$. (Unlike Example 1, the answer here is not immediately obvious.)

$$\lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{(\log_2 n)'}{(\sqrt{n})'} = \lim_{n\to\infty} \frac{(\log_2 e)\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = 2\log_2 e \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Since the limit is equal to zero, $\log_2 n$ has a smaller order of growth than $\sqrt{n}$. (Since $\lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = 0$, we can use the so-called ***little-oh notation***: $\log_2 n \in o(\sqrt{n})$. Unlike the big-Oh, the little-oh notation is rarely used in analysis of algorithms.) ■

3.5. Basic asymptotic Efficiency

Class	*Name*	*Comments*
1	*constant*	Short of best-case efficiencies, very few reasonable examples can be given since an algorithm's running time typically goes to infinity when its input size grows infinitely large.
$\log n$	*logarithmic*	Typically, a result of cutting a problem's size by a constant factor on each iteration of the algorithm (see Section 4.4). Note that a logarithmic algorithm cannot take into account all its input or even a fixed fraction of it: any algorithm that does so will have at least linear running time.
n	*linear*	Algorithms that scan a list of size n (e.g., sequential search) belong to this class.
$n \log n$	*linearithmic*	Many divide-and-conquer algorithms (see Chapter 5), including mergesort and quicksort in the average case, fall into this category.
n^2	*quadratic*	Typically, characterizes efficiency of algorithms with two embedded loops (see the next section). Elementary sorting algorithms and certain operations on $n \times n$ matrices are standard examples.
n^3	*cubic*	Typically, characterizes efficiency of algorithms with three embedded loops (see the next section). Several nontrivial algorithms from linear algebra fall into this class.
2^n	*exponential*	Typical for algorithms that generate all subsets of an n-element set. Often, the term "exponential" is used in a broader sense to include this and larger orders of growth as well.
$n!$	*factorial*	Typical for algorithms that generate all permutations of an n-element set.

3.6. Mathematical Analysis of Non-recursive & Recursive Algorithms

Analysis of Non-recursive Algorithms

General Plan for Analyzing the Time Efficiency of Nonrecursive Algorithms

1. Decide on a parameter (or parameters) indicating an input's size.
2. Identify the algorithm's basic operation. (As a rule, it is located in innermost loop.)
3. Check whether the number of times the basic operation is executed depends only on the size of an input. If it also depends on some additional property, the worst-case, average-case, and, if necessary, best-case efficiencies have to be investigated separately.
4. Set up a sum expressing the number of times the algorithm's basic operation is executed.
5. Using standard formulas and rules of sum manipulation, either find a closedform formula for the count or, at the very least, establish its order of growth.

Example-1: To find maximum element in the given array

Algorithm *MaxElement*$(A[0..n-1])$
//Determines the value of the largest element in a given ai
//Input: An array $A[0..n-1]$ of real numbers
//Output: The value of the largest element in A
$maxval \leftarrow A[0]$
for $i \leftarrow 1$ **to** $n-1$ **do**
    if $A[i] > maxval$
        $maxval \leftarrow A[i]$
return $maxval$

Here comparison is the basic operation.

Note that number of comparisions will be same for all arrays of size n. Therefore, no need to distinguish worst, best and average cases.

Total number of basic operations (comparison) are, $C(n) = \sum_{i=1}^{n-1} 1 = n-1 \in \Theta(n).$

Example-2: To check whether all the elements in the given array are distinct

Algorithm *UniqueElements*$(A[0..n-1])$
//Determines whether all the elements in a given array are distinct
//Input: An array $A[0..n-1]$
//Output: Returns "true" if all the elements in A are distinct
// and "false" otherwise
for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow i+1$ **to** $n-1$ **do**
        if $A[i] = A[j]$ **return false**
return true

Here basic operation is comparison. The maximum no. of comparisons happen in the worst case. (i.e. all the elements in the array are distinct and algorithms return *true*).

Total number of basic operations (comparison) in the worst case are,

$$C_{worst}(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i)$$

$$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \frac{(n-2)(n-1)}{2}$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{(n-1)n}{2} \approx \frac{1}{2} n^2 \in \Theta(n^2).$$

Other than the worst case, the total comparisons are **less than** $\frac{1}{2}n^2$. (For example if the first two elements of the array are equal, only one comparison is computed). So in general **C(n) =O(n^2)**

Example-3: To perform matrix multiplication

Algorithm *MatrixMultiplication*$(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$

```
//Multiplies two square matrices of order n by the definition-based algorithm
//Input: Two n × n matrices A and B
//Output: Matrix C = AB
for i ← 0 to n − 1 do
    for j ← 0 to n − 1 do
        C[i, j] ← 0.0
        for k ← 0 to n − 1 do
            C[i, j] ← C[i, j] + A[i, k] ∗ B[k, j]
return C
```

Number of basic operations (multiplications) is

$$M(n) = \sum_{i-0}^{n-1} \sum_{j-0}^{n-1} \sum_{k-0}^{n-1} 1 = \sum_{i-0}^{n-1} \sum_{j-0}^{n-1} n = \sum_{i-0}^{n-1} n^2 = n^3.$$

Total running time:

$$T(n) \approx c_m M(n) = c_m n^3$$

Suppose if we take into account of addition; Algoritham also have same number of additions $A(n) = n^3$

Total running time: $T(n) \approx c_m M(n) + c_a A(n) = c_m n^3 + c_a n^3 = (c_m + c_a) n^3$

Example-4: To count the bits in the binary representation

Algorithm *Binary*(n)

```
//Input: A positive decimal integer n
//Output: The number of binary digits in n's binary representation
count ← 1
while n > 1 do
    count ← count + 1
    n ← ⌊n/2⌋
return count
```

The basic operation is count=count + 1 repeats $\lfloor \log_2 n \rfloor + 1$ no. of times

Analysis of Recursive Algorithms

General plan for analyzing the time efficiency of recursive algorithms

1. Decide on a parameter (or parameters) indicating an input's size.
2. Identify the algorithm's basic operation.
3. Check whether the number of times the basic operation is executed can vary on different inputs of the same size; if it can, the worst-case, average-case, and best-case efficiencies must be investigated separately. Set up a recurrence relation, with an appropriate initial condition, for the number of times the basic operation is executed.
4. Solve the recurrence or, at least, ascertain the order of growth of its solution.

Example-1 | Compute the factorial function $F(n) = n!$ for an arbitrary nonnegative integer n. Since

$$n! = 1 \cdot \ldots \cdot (n-1) \cdot n = (n-1)! \cdot n \quad \text{for } n \geq 1$$

and $0! = 1$ by definition, we can compute $F(n) = F(n-1) \cdot n$ with the following recursive algorithm.

Algorithm *F*(n)

```
//Computes n! recursively
//Input: A nonnegative integer n
//Output: The value of n!
if n = 0 return 1
else return F(n − 1) ∗ n
```

Since the function *F(n)* is computed according to the formula

$F(n) = F(n-1) \cdot n \quad \text{for } n > 0,$

The number of multiplications *M(n)* needed to compute it must satisfy the equality

$$M(n) = \underbrace{M(n-1)}_{\substack{\text{to compute} \\ F(n-1)}} + \underbrace{1}_{\substack{\text{to multiply} \\ F(n-1) \text{ by } n}} \quad \text{for } n > 0.$$

Such equations are called **recurrence Relations**

Condition that makes the algorithm stop ***if n = 0 return 1.*** Thus recurrence relation and initial condition for the algorithm's number of multiplications M(n) can be stated as

$$M(n) = M(n-1) + 1 \quad \text{for } n > 0,$$
$$M(0) = 0.$$

We can use backward substitutions method to solve this

$$\begin{aligned} M(n) &= M(n-1) + 1 && \text{substitute } M(n-1) = M(n-2) + 1 \\ &= [M(n-2) + 1] + 1 = M(n-2) + 2 && \text{substitute } M(n-2) = M(n-3) + 1 \\ &= [M(n-3) + 1] + 2 = M(n-3) + 3. \\ &\ldots \\ &= M(n-i) + i = \cdots = M(n-n) + n = n. \end{aligned}$$

Example-2: Tower of Hanoi puzzle. In this puzzle, There are ***n*** disks of different sizes that can slide onto any of three pegs. Initially, all the disks are on the first peg in order of size, the largest on the bottom and the smallest on top. The goal is to move all the disks to the third peg, using the second one as an auxiliary, if necessary. We can move only one disk at a time, and it is forbidden to place a larger disk on top of a smaller one.

The problem has an elegant recursive solution, which is illustrated in Figure.

- To move n>1 disks from peg 1 to peg 3 (with peg 2 as auxiliary),
  - we first move recursively n-1 disks from peg 1 to peg 2 (with peg 3 as auxiliary),
  - then move the largest disk directly from peg 1 to peg 3, and,
  - finally, move recursively n-1 disks from peg 2 to peg 3 (using peg 1 as auxiliary).
- If n = 1, we move the single disk directly from the source peg to the destination peg.

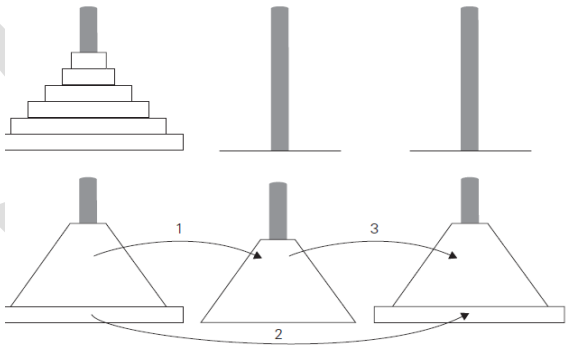


Figure: Recursive solution to the Tower of Hanoi puzzle

The number of moves ***M(n)*** depends only on n. The recurrence equation is

$$M(n) = M(n-1) + 1 + M(n-1) \quad \text{for } n > 1.$$

We have the following recurrence relation for the number of moves M(n):

$$M(n) = 2M(n-1) + 1 \quad \text{for } n > 1$$
$$M(1) = 1.$$

We solve this recurrence by t e same method of backward substitutions:

$$M(n) = 2M(n-1) + 1 \qquad \text{sub. } M(n-1) = 2M(n-2) + 1$$
$$= 2[2M(n-2) + 1] + 1 = 2^2 M(n-2) + 2 + 1 \quad \text{sub. } M(n-2) = 2M(n-3) + 1$$
$$= 2^2[2M(n-3) + 1] + 2 + 1 = 2^3 M(n-3) + 2^2 + 2 + 1.$$

The pattern of the first three sums on the left suggests that the next one will be 2^4 M(n − 4) + 2^3 + 2^2 + 2 + 1, and generally, after *i* substitutions, we get

$$M(n) = 2^i M(n-i) + 2^{i-1} + 2^{i-2} + \cdots + 2 + 1 = 2^i M(n-i) + 2^i - 1$$

Since the initial condition is specified for n = 1, which is achieved for i = n - 1, we get the following formula for the solution to recurrence,

$$M(n) = 2^{n-1} M(n-(n-1)) + 2^{n-1} - 1$$
$$= 2^{n-1} M(1) + 2^{n-1} - 1 = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1$$

Alternatively, by counting the number of nodes in the tree obtained by recursive calls, we can get the total number of calls made by the Tower of Hanoi algorithm:

$$C(n) = \sum_{l=0}^{n-1} 2^l \text{ (where } l \text{ is the level in the tree in Figure)} = 2^n - 1$$

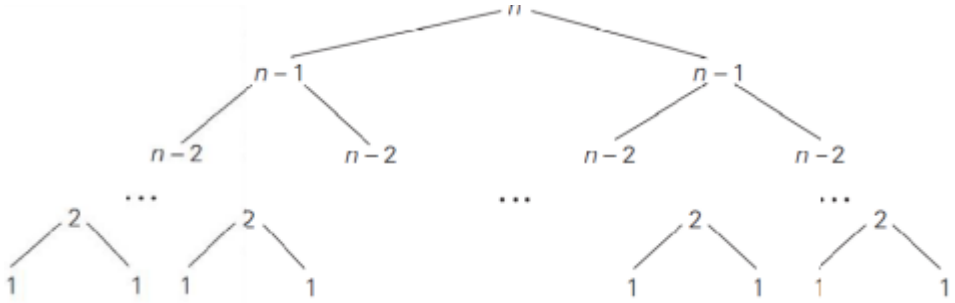


Figure: Tree of recursive calls made by the recursive algorithm for the Tower of Hanoi puzzle.

Example-3

ALGORITHM *BinRec*(*n*)

//Input: A positive decimal integer *n*
//Output: The number of binary digits in *n*'s binary representation
if $n = 1$ **return** 1
else return $BinRec(\lfloor n/2 \rfloor) + 1$

The recurrence relation can be written as

$$A(n) = A(\lfloor n/2 \rfloor) + 1 \quad \text{for } n > 1.$$

Also note that A(1) = 0.

The standard approach to solving such a recurrence is to **solve it only for n = 2^k** and then take advantage of the theorem called the **smoothness rule** which claims that under very broad assumptions the order of growth observed for n = 2^k gives a correct answer about the order of growth for all values of n.

$$A(2^k) = A(2^{k-1}) + 1 \quad \text{for } k > 0,$$
$$A(2^0) = 0.$$

Now backward substitutions encounter no problems:

$$\begin{aligned}
A(2^k) &= A(2^{k-1}) + 1 && \text{substitute } A(2^{k-1}) = A(2^{k-2}) + 1\\
&= [A(2^{k-2}) + 1] + 1 = A(2^{k-2}) + 2 && \text{substitute } A(2^{k-2}) = A(2^{k-3}) + 1\\
&= [A(2^{k-3}) + 1] + 2 = A(2^{k-3}) + 3 && \dots\\
&\dots\\
&= A(2^{k-i}) + i\\
&\dots\\
&= A(2^{k-k}) + k.
\end{aligned}$$

Thus, we end up with

$$A(2^k) = A(1) + k = k,$$

or, after returning to the original variable $n = 2^k$ and hence $k = \log_2 n$,

$$A(n) = \log_2 n \in \Theta(\log n).$$

4. Important Problem Types

In this section, we are going to introduce the most important problem types: Sorting, Searching, String processing, Graph problems, Combinatorial problems.

4.1. Sorting

The sorting problem is to rearrange the items of a given list in **non-decreasing order.** As a practical matter, we usually need to sort lists of numbers, characters from an alphabet or character strings.Although some algorithms are indeed better than others, there is no algorithm that would be the best solution in all situations. Some of the algorithms are simple but relatively slow, while others are faster but more complex; some work better on randomly ordered inputs, while
others do better on almost-s rted lists; some are suitable only for lists residing in the fast memory, while others can be adapted for sorting large files stored on a disk; and so on.

Two properties of sorting algorithms deserve special mention. A sorting algorithm is called ***stable*** if it preserves the relative order of any two equal elements in its input. The second notable feature of a sorting algorithm is the amount of ***extra memory*** the algorithm requires. An algorithm is said to be in-place if it does not require extra memory, except, possibly, for a few memory units.

4.2. Searching

The searching problem deals with finding a given value, called a search key, in a given set. (or a multiset, which permits several elements to have the same value). There are plenty of searching algorithms to choose from. They range from the straightforward **sequential search** to a spectacularly efficient but limited **binary search** and algorithms based on representing the underlying set in a different form more conducive to searching. The latter algorithms are of particular importance for real-world applications because they are indispensable for storing and retrieving information from large databases.

4.3. String Processing

In recent decades, the rapid proliferation of applications dealing with non-numerical data has intensified the interest of researchers and computing practitioners in string-handling algorithms. A string is a sequence of characters from an alphabet. String-processing algorithms have been important for computer science in conjunction with computer languages and compiling issues.

4.4. Graph Problems

One of the oldest and most interesting areas in algorithmics is graph algorithms. Informally, a graph can be thought of as a collection of points called **vertices**, some of which are connected by line segments called **edges**. Graphs can be used for modeling a wide variety of applications, including transportation, communication, social and economic networks, project

scheduling, and games. Studying different technical and social aspects of the Internet in

particular is one of the active areas of current research involving computer scientists, economists, and social scientists.

4.5. Combinatorial Problems

Generally speaking, combinatorial problems are the most difficult problems in computing, from both a theoretical and practical standpoint. Their difficulty stems from the following facts. First, the number of combinatorial objects typically grows extremely fast with a problem's size, reaching unimaginable magnitudes even for moderate-sized instances. Second, there are no known algorithms for solving most such problems exactly in an acceptable amount of time.

Fundamental Data Structures

Since the vast majority of algorithms of interest operate on data, particular ways of organizing data play a critical role in the design and analysis of algorithms. A **data structure** can be defined as a particular scheme of organizing related data items.

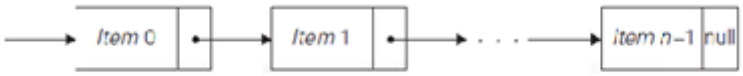
Linear Data Structures

The two most important elementary data structures are the array and the linked list.

A (one-dimensional) **array** is a sequence of n items of the same data type that are stored contiguously in computer memory and made accessible by specifying a value of the array's index.

Array of n elements.

A **linked list** is a sequence of zero or more elements called nodes, each containing two kinds of information: some data and one or more links called pointers to other nodes of the linked list. In a **singly linked list**, each node except the last one contains a single pointer to the next element. Another extension is the structure called the **doubly linked list**, in which every node, except the first and the last, contains pointers to both its successor and its predecessor.



RE 1.4 Singly linked list of n elements.

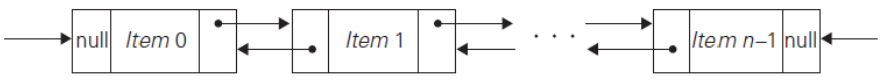


FIGURE 1.5 Doubly linked list of n elements.

A **list** is a finite sequence of data items, i.e., a collection of data items arranged in a certain linear order. The basic operations performed on this data structure are searching for, inserting, and deleting an element. Two special types of lists, stacks and queues, are particularly important.

A **stack** is a list in which insertions and deletions can be done only at the end. This end is called the top because a stack is usually visualized not horizontally but vertically—akin to a stack of plates whose "operations" it mimics very closely.

A **queue**, on the other hand, is a list from which elements are deleted from one end of the structure, called the front (this operation is called dequeue), and new elements are added to the other end, called the rear (this operation is called enqueue). Consequently, a queue operates in a "first-in–first-out" (FIFO) fashion—akin to a queue of customers served by a single teller in a bank. Queues also have many important applications, including several algorithms for graph problems.

Many important applications require selection of an item of the highest priority among a dynamically changing set of candidates. A data structure that seeks to satisfy the needs of such applications is called a **priority queue**. A priority queue is a collection of data items from a totally ordered universe (most often, integer or real numbers). The principal operations on a priority queue are finding its largest element, deleting its largest element, and adding a new element.

Graphs

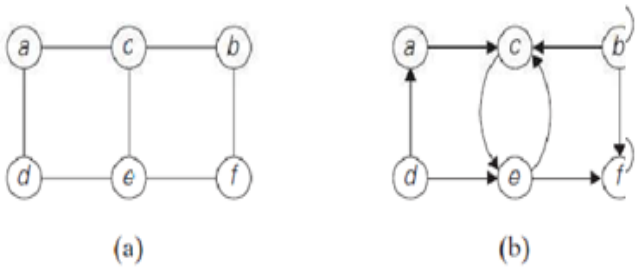
A graph is informally thought of as a collection of points in the plane called "vertices" or nodes," some of them connected by line segments called "edges" or "arcs." A graph G is called **undirected** if every edge in it is undirected. A graph whose every edge is directed is called **directed**. Directed graphs are also called **digraphs**.

The graph depicted in Figure (a) has six vertices and seven undirected edges:

$$V = \{a, b, c, d, e, f\}, E = \{(a, c), (a, d), (b, c), (b, f), (c, e), (d, e), (e, f)\}.$$

The digraph depicted in Figure 1.6b has six vertices and eight directed edges:

$$V = \{a, b, c, d, e, f\}, E = \{(a, c), (b, c), (b, f), (c, e), (d, a), (d, e), (e, c), (e, f)\}.$$



(a) Undirected graph. (b) Digraph.

Graph Representations - Graphs for computer algorithms are usually represented in one of two ways: the ***adjacency matrix*** and ***adjacency lists***.

The **adjacency matrix** of a graph with n vertices is an **n x n** boolean matrix with one row and one column for each of the graph's vertices, in which the element in the i^{th} row and the j^{th} column is equal to 1 if there is an edge from the i^{th} vertex to the j^{th} vertex, and equal to 0 if there is no such edge.

The **adjacency lists** of a graph or a digraph is a collection of linked lists, one for each vertex, that contain all the vertices adjacent to the list's vertex (i.e., all the vertices connected to it by an edge).

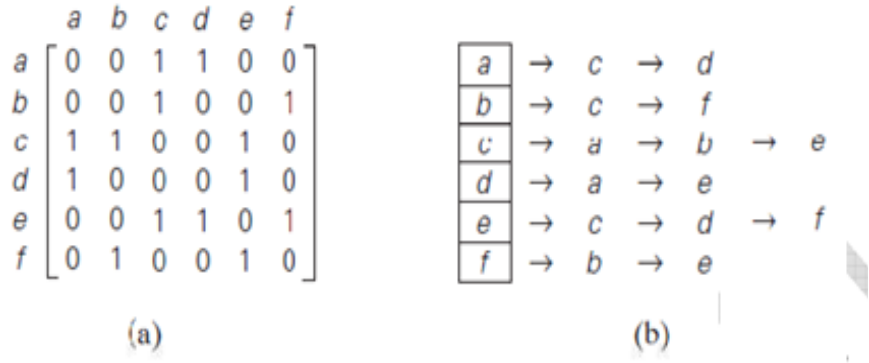
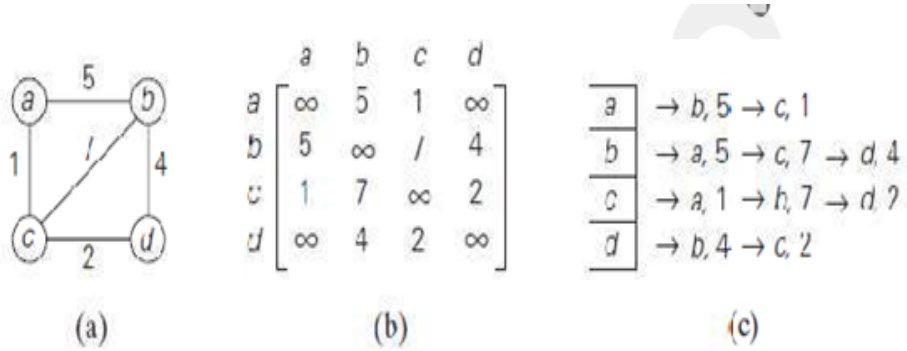


FIGURE 1.7 (a) Adjacency matrix and (b) adjacency lists of the graph in Figure

Weighted Graphs: A weighted graph (or weighted digraph) is a graph (or digraph) with numbers assigned to its edges. These numbers are called weights or costs.

Among the many properties of graphs, two are important for a great number of applications: connectivity and acyclicity. Both are based on the notion of a path. A path from vertex u to vertex v of a graph G can e defined as a sequence of adjacent (connected by an edge) vertices that starts with u and ends with v.

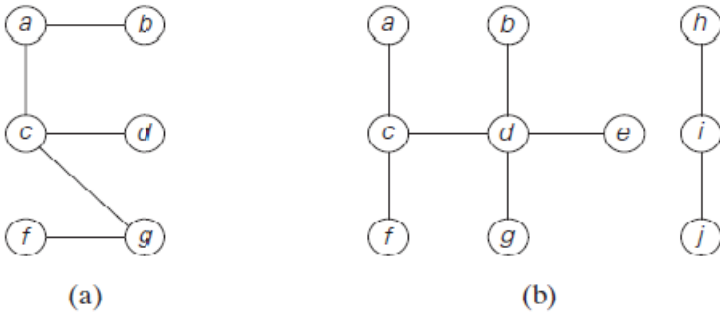


(a) Weighted graph. (b) Its weight matrix. (c) Its adjacency lists.

A graph is said to be **connected** if for every pair of its vertices u and v there is a path from u to v. Graphs with several connected components do happen in real-world applications. It is important to know for many applications whether or not a graph under consideration has cycles. A **cycle** is a path of a positive length that starts and ends at the same vertex and does not traverse the same edge more than once.

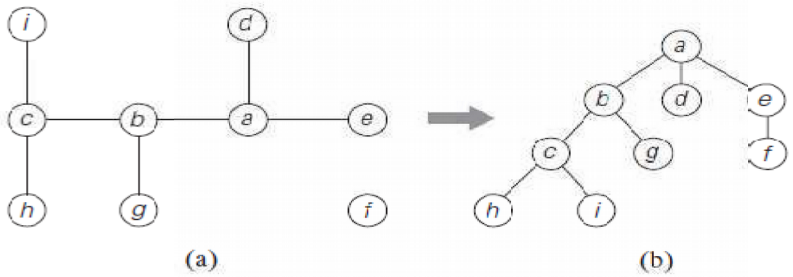
Trees

A **tree** (more accurately, a free tree) is a connected acyclic graph. A graph that has no cycles but is not necessarily connected is called a **forest**: each of its connected components is a tree. Trees have several important properties other graphs do not have. In particular, the number of edges in a tree is always one less than the number of its vertices: |E| = |V| - 1



(a) Tree. (b) Forest

Rooted Trees: Another very important property of trees is the fact that for every two vertices in a tree, there always exists exactly one simple path from one of these vertices to the other. This property makes it possible to select an arbitrary vertex in a free tree and consider it as the root of the so-called rooted tree. A rooted tree is usually depicted by placing its root on the top (level 0 of the tree), the vertices adjacent to the root below it (level 1), the vertices two edges apart from the root still below (level 2), and so on.



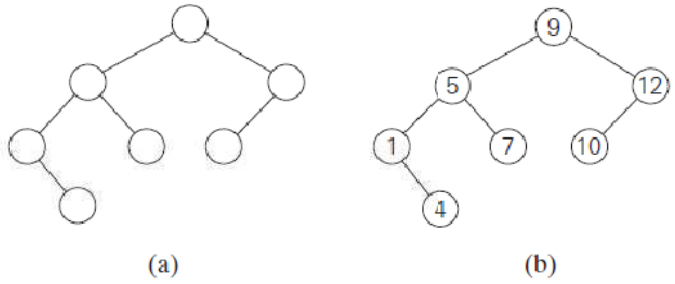
1.11 (a) Free tree. (b) Its transformation into a rooted tree.

The **depth** of a vertex v is the length of the simple path from the root to v. The **height** of a tree is the length of the longest simple path from the root to a leaf.

Ordered Trees- An ordered tree is a rooted tree in which all the children of each vertex are ordered. It is convenient to assume that in a tree's diagram, all the children are ordered left to

right. A **binary tree** can be defined as an ordered tree in which every vertex has no more than two children and each child is designated as either a left child or a right child of its parent; a binary tree may also be empty.

If a number assigned to each parental vertex is larger than all the numbers in its left subtree and smaller than all the numbers in its right subtree. Such trees are called **binary search trees**. Binary trees and binary search trees have a wide variety of applications in computer science.



2 (a) Binary tree. (b) Binary search tree.

Sets and Dictionaries

A **set** can be described as an unordered collection (possibly empty) of distinct items called **elements** of the set. A specific set is defined either by an explicit listing of its elements (e.g., S = {2, 3, 5, 7}) or by specifying a property that all the set's elements and only they must satisfy (e.g., S = {n: n is a prime number smaller than 10}).

The most important **set operations** are: checking membership of a given item in a given set; finding the union of two sets, which comprises all the elements in either or both of them; and finding the intersection of two sets, which comprises all the common elements in the sets.

Sets can be **implemented** in computer applications in two ways. The first considers only sets that are subsets of some large set U, called the universal set. If set U has n elements, then any subset S of U can be represented by a bit string of size n, called a **bit vector**, in which the i^{th} element is 1 if and only if the i^{th} element of U is included in set S.

The second and more common way to represent a set for computing purposes is to use the **list** structure to indicate the set's elements. This is feasible only for finite sets. The requirement for uniqueness is sometimes circumvented by the introduction of a multiset, or bag, an unordered collection of items that are not necessarily distinct. Note that if a set is represented by a list, depending on the application at hand, it might be worth maintaining the list in a sorted order.

Dictionary: In computing, the operations we need to perform for a set or a multiset most often are searching for a given item, adding a new item, and deleting an item from the collection. A data structure that implements these three operations is called the **dictionary**.

An efficient implementation of a dictionary has to strike a compromise between the efficiency of searching and the efficiencies of the other two operations. They range from an unsophisticated use of arrays (sorted or not) to much more sophisticated techniques such as hashing and balanced search trees.

A number of applications in computing require a dynamic partition of some n-element set into a collection of disjoint subsets. After being initialized as a collection of n one-element subsets, the collection is subjected to a sequence of intermixed union and search operations. This problem is called the **set union** problem.