CHAPTER

Stability Theory

5

CHAPTER OUTLINES

■ Bounded Input-Bounded Output Stability Continuous Data System; ■ Effect of Location of Poles on Stability; ■ Zero Input and Asymptotic Stability of Continuous Data Systems; ■ Necessary but not Sufficient Conditions for Stability; ■ Methods of Determining Stability; ■ The Routh-Hurwitz Criterion; ■ Application of Routh's Stability Criterion to Control System Analysis; ■ Relative Stability Analysis; ■ Root Locus; ■ Rules for Construction of Root Locii; ■ Determination of K on Root Locii; ■ Effect of Addition of Poles and Zeros on Root Locus; ■ Introduction; ■ Mapping; ■ Mapping of Close Contour and Principle of Argument; ■ Nyquist Path or Nyquist Contour; ■ Nyquist Criterion; ■ General Construction Rules of the Nyquist; ■ Path; ■ Analysis of Stability by Lyapunov's Direct Method; ■ Sylvester's Theorem; ■ How to Obtain Open Loop Transfer Function G(s) H(s) from Characteristic Equation

5.1. BOUNDED INPUT-BOUNDED OUTPUT STABILITY CONTINUOUS DATA SYSTEM

5.1.1. Concept of Stability

The concept of stability is very important to analyse and design the system. A system is said to be stable if its response cannot be made to increase indefinitely by the application of a bounded input excitation. If the output approaches towards infinite value for sufficiently large time, the system is said to be unstable.

A linear time invariant (LTI) system is stable if :

1. The system is excited by a bounded input, the output is bounded (BIBO stability criteria).
2. In the absence of the input, the output tends towards zero (the equilibrium state of the system). This is known as asymptotic stable.

Consider the transfer function

$$\frac{C(s)}{R(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0} \qquad \dots(5.1)$$

The output is given by

$$c(t) = \int_0^\infty g(\tau) r(t-\tau)\, d\tau \qquad \dots(5.2)$$

where $g(\tau) = £^{-1} G(s)$ = impulse response of the system. So, a system is said to be stable if the impulse response approaches zero for sufficiently large time. If the impulse response approaches infinity for sufficiently large time, the system is said to be unstable. If the impulse response approaches a constant value for sufficiently large time, the system is said to be marginally stable.

The absolute value of equation (5.2) given by

$$|c(t)| = \left|\int_0^\infty g(\tau)\, r(t-\tau)\, d\tau\right|$$

As the absolute value of an integral is lesser than the integral of the absolute value of integrand, we have

$$|c(t)| \le \int_0^\infty |g(\tau)|\,|r(t-\tau)|\, d\tau$$

Bounded input means $|r(t)| \le M_1 \le \infty$ and output is bounded implies $|c(t)| \le M_2 \le \infty$

For bounded input, bounded output condition

$$|c(t)| \ge M_1 \int_0^\infty |g(\tau)|\, d\tau \le M_2$$

The impulse response $g(t)$ has a nature dependent on location of the poles of the transfer function $G(s)$ and the roots of the characteristic equation.

Thus for bounded input bounded output stability we can say :

1. If all the roots of characteristic equation have negative real part *i.e.*, they lie in the left half of the s-plane, then the system is bounded input, bounded output stable and the impulse response to be bounded which finally becomes zero *i.e.*,

   $\int_0^\infty |g(\tau)|\, d\tau$ is "finite".

2. If any root of the given characteristic equation has a positive real part *i.e.*, they lies in the right half of s-plane, the impulse response $g(t)$ is bounded and therefore $\int_0^\infty |g(\tau)|\, d\tau$ is "infinite", this makes the system unstable.
3. If there repeated roots on the imaginary axis, the impulse response $g(t)$ is unbounded and $\int_0^\infty |g(\tau)|\, d\tau$ is infinite and the system is unstable.
4. If one or more non-repeated roots of the characteristic equation on the imaginary axis, then the impulse response $g(t)$ is bounded but $\int_0^\infty |g(\tau)|\, d\tau$ is infinite, which makes the system unstable.

Thus, for Bounded input Bounded output stability we can say that a linear time invariant system is stable if :

(*i*) The system is excited by a bounded input, output is also bounded

(*ii*) In the absence of input, output must tends to zero irrespective of the initial conditions.

A linear time invariant system is unstable if :

(*i*) For a bounded input it produces unbounded output.

(*ii*) In absence of the input, output may not return to zero. It shows certain output without input.

5.2. EFFECT OF LOCATION OF POLES ON STABILITY

(a) Poles on Negative Real Axis

Consider a simple pole at $s = -a$ as shown in Fig. 5.1a., the corresponding impulse response is given by

$$g(t) = £^{-1}G(s) = £^{-1}\frac{K}{s+a} = K.e^{-at} \qquad ...(5.3)$$

As the time 't' increases, the response approaches zero and the system is stable. The response is shown in Fig. 5.1(b).

(a) Simple pole on negative real axis

(b) Response

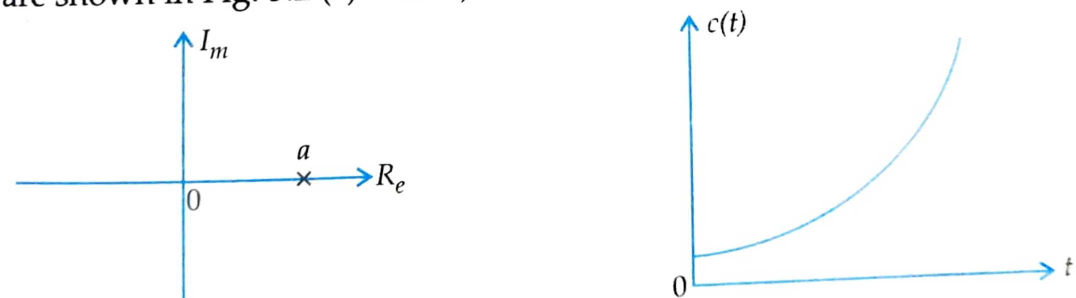
Fig. 5.1.

(b) Pole on Positive Real Axis

Consider a system having simple pole on positive real axis at $s = a$, the corresponding impulse response is given by

$$g(t) = £^{-1}\frac{K}{s-a} = K.e^{at} \qquad ...(5.4)$$

The response increases exponentially with time, hence the system is unstable. The simple pole and response are shown in Fig. 5.2 (a) and (b).



(a) Simple pole on positive real axis

(b) Response

Fig. 5.2.

(c) Pole at the Origin : Consider a Pole at Origin

$$\therefore \quad g(t) = £^{-1}\frac{K}{s} = K \qquad ...(5.5)$$

This is constant value, hence the system is marginally stable. If there are two poles at the origin, the time response would be

$$g(t) = £^{-1}\frac{K}{s^2} = Kt \qquad ...(5.6)$$

(a) Single pole at origin

(b) Response due to single pole

Fig. 5.3.

(a) Double pole at origin

(b) Impulse response

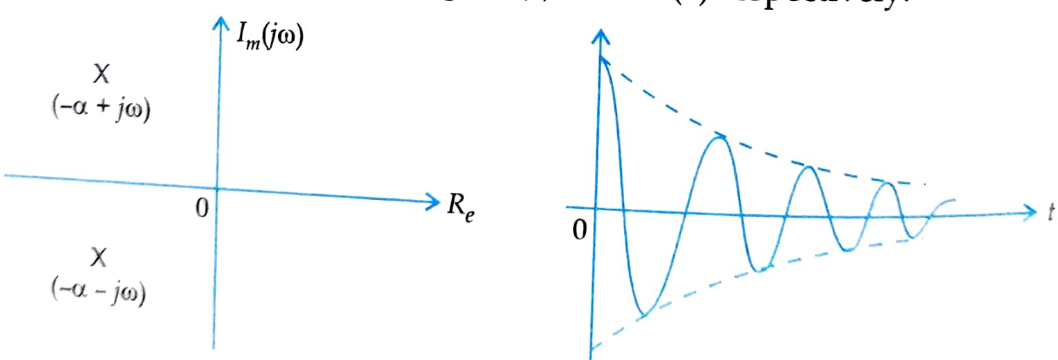
Fig. 5.4.

(d) Complex Pole in the Left Half of s-plane

Let the transfer function has a complex conjugate poles at $s = -\alpha \pm j\omega$. The time response due to the complex conjugate poles is given by

$$g(t) = \pounds^{-1}\left[\frac{K}{s+\alpha-j\omega}+\frac{K}{s+\alpha+j\omega}\right] = \pounds^{-1}\left[\frac{2K\,(s+\alpha)}{(s+\alpha)^2+\omega^2}\right] = 2K\,e^{-\alpha t}\cos\omega t \quad ...(5.7)$$

When t increases $g(t)$ approaches zero and the system is stable. The complex poles and corresponding time response is shown in Fig. 5.5(a) and 5.5(b) respectively.



(a)

(b)

Fig. 5.5.

(e) Complex Poles in the Right Half of s-plane

Suppose the system has complex conjugate poles at $s = \alpha \pm j\omega$. The time response is given by

$$g(t) = \pounds^{-1}\left[\frac{A}{s-\alpha-j\omega}+\frac{A}{s-\alpha+j\omega}\right] = \pounds^{-1}\left[\frac{2A(s-\alpha)}{(s-\alpha)^2+\omega^2}\right] = 2\,A\,e^{\alpha t}\cos\omega t \quad ...(5.8)$$

Hence, the response increases exponentially sinusoid with time and therefore the response is unstable. The poles and time response shown in Fig. 5.6(a) and 5.6(b) respectively.

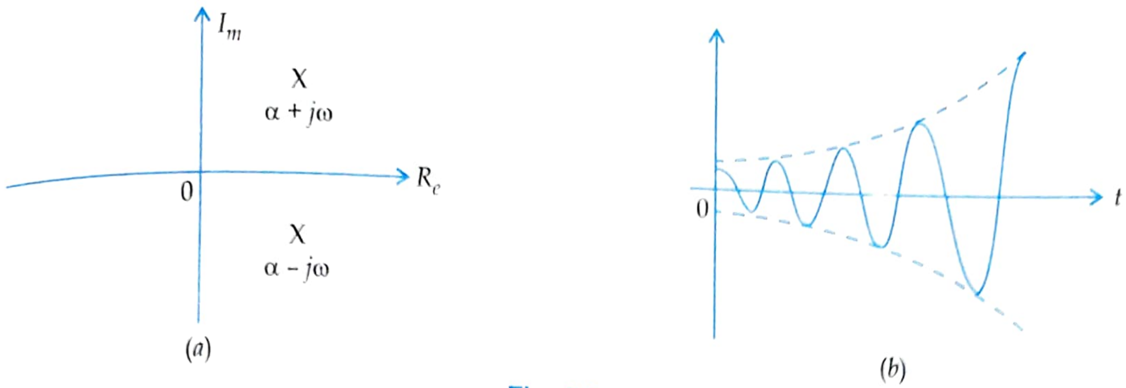


Fig. 5.6.

(f) Poles on $j\omega$-axis

If the system having the complex poles on $j\omega$-axis the corresponding time response would be

$$g(t) = \pounds^{-1}\left[\frac{A}{s+j\omega} + \frac{A}{s-j\omega}\right]$$

$$= \pounds^{-1}\left[\frac{2As}{s^2+\omega^2}\right] = 2A\cos\omega t \quad \text{...(5.9)}$$

The response is marginally stable. The equation (5.9) shows the sustained oscillations of constant amplitude. This situation will also be considered unstable.

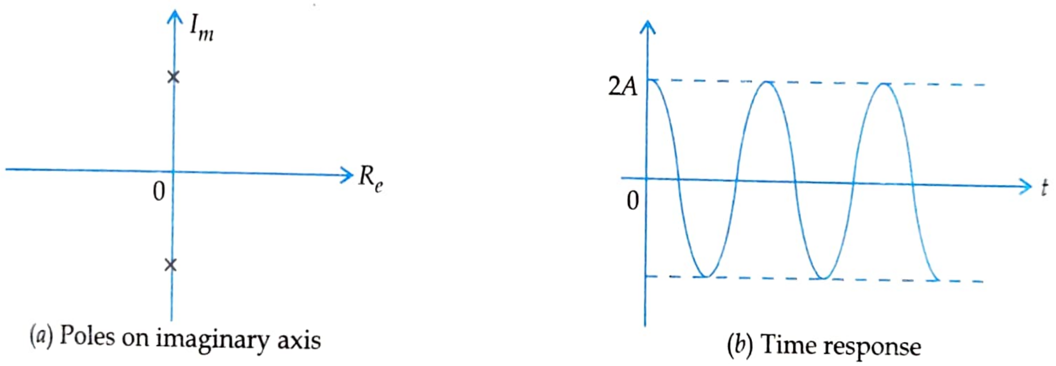


Fig. 5.7.

The overall transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \text{...(5.10)}$$

The characteristic equation is $1 + G(s)H(s) = 0$. ...(5.11)

The necessary and sufficient condition that a feedback system be stable is that all the zeros of the characteristic equation $1 + G(s)H(s) = 0$ have negative real part. Or, in terms of poles we can say that the necessary and sufficient condition that a feedback system be stable is that all the poles of overall transfer function have negative real part.

5.3. ZERO INPUT AND ASYMPTOTIC STABILITY OF CONTINUOUS DATA SYSTEMS

Asymptotic Stability: Consider an RC series circuit with capacitor initially charged to some voltage. This initial voltage is sufficient to operate the system without any external input. This initial voltage drives the current till capacitor gets fully discharge. The stability of such system under zero input but under initial condition is called *asymptotic stability*. The asymtotic stability depends on the closed loop poles of the system, *i.e.*, roots of the characteristic equation.

Asymptotic stability also known as "zero input stability". Consider n^{th} order system with zero input and $c(t)$ is the output due to the initial conditions, mathematically

$$c(t) = \sum_{i=0}^{n-1} g_i(t) C^i(t_0)$$

where,

$$c^i(t_0) = \left.\frac{d^i c(t)}{dt^i}\right|_{t=t_0}$$

where $g_i(t)$ denotes the zero input response due to $C^i(t_0)$ so, the zero input stability can be defined as : If the zero input response $c(t)$ subjected to the finite initial conditions, reaches zero as time t approaches infinity, the system is said to be zero-input stable or asymptotic stable otherwise the system is unstable.

$$\lim_{t\to\infty} |c(t)| = 0$$

The asymptotic stability depends on the closed loop poles of the system, *i.e.*, roots of the characteristic equation. For asymptotic stability, all the roots of the characteristic equation must be located in left half of s-plane.

5.4. NECESSARY BUT NOT SUFFICIENT CONDITIONS FOR STABILITY

Consider a system with characteristic equation

$$a_0 s^m + a_1 s^{m-1} + \ldots.. + a_m = 0 \qquad \ldots(5.12)$$

(*a*) All the coefficients of the equation should have same sign,

(*b*) There should be no missing term.

If above two conditions are not satisfied the system will be unstable. But if all the coefficients have same sign and there is no missing term we have no guarantee that the system will be stable. For stability we use Routh-Hurwitz Criterion.

5.5. METHODS OF DETERMINING STABILITY

Different methods for determining the stability of the system are:

(*i*) Routh-Hurwitz criterion

(*ii*) Bode plot

(*iii*) Nyquist criterion

Routh-Hurwitz criterion is the algebraic method of determining the location of poles of a characteristic equation with respect to the left half and right half of the s-plane without actually solving the equation.

Bode plot is a graphical method for determining the stability in which logarithmic values of magnitude are to be plotted against logarithmic values of frequencies. Bode plot consists of two separate plots. One is a plot of logarithm of magnitude of a sinusoidal transfer function, the other plot is a plot of the phase angle, both plots are plotted against the logarithmic values of frequencies.

Nyquist criterion also graphical method for determining the stability of the system. In this method the encirelements of critical point $(-1 + j_0)$ by Nyquist plot is used to determine the stability of the system.

5.6. THE ROUTH-HURWITZ CRITERION

Consider the following characteristic polynomial

$$a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_n = 0 \qquad \ldots(5.13)$$

where the coefficients $a_0, a_1 \ldots a_n$ are all of the same sign and none is zero.

Step 1 : Arrange all the coefficients of equation (5.13) in two rows

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	....

Step 2 : From these two rows form a third row

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	....
Row 3	b_1	b_3	b_5	....

where,

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}$$

$$b_3 = -\frac{1}{a_1}\begin{matrix} a_0 & a_4 \\ a_1 & a_5 \end{matrix}$$

Step 3 : From second and third row, form a fourth row

Row 1	a_0	a_2	a_4	....
Row 2	a_1	a_3	a_5	.....
Row 3	b_1	b_3	b_5	....
Row 4	c_1	c_3	c_5	....

where,

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$c_3 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_5 \end{vmatrix}$$

$\vdots$

Step 4 : Continue this procedure of forming a new rows.

5.6.1. Statement of Routh-Hurwitz Criterion

Routh-Hurwitz criterion states that the system is stable if and only if all the elements in the first column have the same algebraic sign. If all elements are not of the same sign then the number of sign changes of the elements in first column equals the number of roots of the characteristic equation in the right half of the s-plane (or equals to the number of roots with positive real parts).

EXAMPLE 5.1. Check the stability of the system whose characteristic equation is given by

$$s^4 + 2s^3 + 6s^2 + 4s + 1 = 0$$

Solution : Obtain the array of coefficients as follows

s^4	1	6	1
s^3	2	4	
s^2	4	1	
s^1	3.5		
s^0	1		

$$b_1 = -\frac{1}{2}\begin{matrix} 1 & 6 \\ 2 & 4 \end{matrix} = 4, \qquad c_1 = -\frac{1}{4}\begin{matrix} 2 & 4 \\ 4 & 1 \end{matrix} = 3.5$$

$$b_2 = -\frac{1}{2}\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix} = 1, \qquad d_1 = -\frac{1}{3.5}\begin{vmatrix}4 & 1\\ 3.5 & 0\end{vmatrix} = 1$$

Since, all the coefficients in the first column are of the same sign (positive), the given equation has no roots with positive real parts. Hence, the system is stable.

EXAMPLE 5.2. Determine the stability of the system whose characteristic equation is given by

$$2s^4 + 2s^3 + s^2 + 3s + 2 = 0$$

Solution :

s^4	2	1	2
s^3	2	3	
s^2	–2	2	
s^1	5		
s^0	2		

$$b_1 = -\frac{1}{2}\begin{vmatrix}2 & 1\\ 2 & 3\end{vmatrix} = -2, \qquad c_1 = -\frac{1}{(-2)}\begin{vmatrix}2 & 3\\ -2 & 2\end{vmatrix} = 5$$

$$b_2 = -\frac{1}{2}\begin{vmatrix}2 & 2\\ 2 & 0\end{vmatrix} = 2, \qquad d_1 = -\frac{1}{5}\begin{vmatrix}-2 & 2\\ 5 & 0\end{vmatrix} = 2$$

There are two changes of sign in the first column (from 2 to –2 and from –2 to 5), hence there are two roots in the right half of s-plane. The system is unstable.

EXAMPLE 5.3. Determine the stability of the system having following characteristic equation

$$2s^4 + 5s^3 + 5s^2 + 2s + 1 = 0$$

Solution :

s^4	2	5	1
s^3	5	2	
s^2	4.2	1	
s^1	0.809		
s^0	1		

From the above Routh table :

No. of sign changes in first column = 0

No. of poles on the right hand side of s-plane = 0

Hence, the system is stable.

EXAMPLE 5.4. Check the stability of the system having following characteristic equation.

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Solution :

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1	–6		
s^0	5		

$$b_1 = -\frac{1}{2}\begin{vmatrix}1 & 3\\ 2 & 4\end{vmatrix} = 1, \qquad b_2 = -\frac{1}{2}\begin{vmatrix}1 & 5\\ 2 & 0\end{vmatrix} = 5$$

$$c_1 = -\frac{1}{1}\begin{vmatrix}2 & 4\\ 1 & 5\end{vmatrix} = -6, \qquad d_1 = -\frac{1}{(-6)}\begin{vmatrix}1 & 5\\ -6 & 0\end{vmatrix} = 5$$

From above table :

No. of sign changes in first column = 2

No. of roots in right half of s-plane = 2

Hence, the system is unstable.

EXAMPLE 5.5. A closed loop control system has the characteristic equation given by $s^3 + 4.5s^2 + 3.5s + 1.5 = 0$

Investigate the stability using Routh-Hurwitz criterion. *(R.M.L. University Faizabad, 2001)*

Solution :

s^3	1	3.5
s^2	4.5	1.5
s^1	3.17	
s^0	1.5	

No. of sign changes in first column = 0
No. of roots in right half of s-plane = 0
Hence, system is stable.

EXAMPLE 5.6. Check the stability of the system, having following characteristic equation.

Solution: $s^5 + 6s^4 + 3s^3 + 2s^2 + s + 1 = 0$

s^5	1	3	1
s^4	6	2	1
s^3	2.67	0.83	
s^2	0.135	1	
s^1	–18.95		
s^0	1		

No. of sign change in first column = 2
No. of poles on right half of s-plane = 2
Hence, system is unstable.

SPECIAL CASES

Case 1 : If a first column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then multiply the original equation by a factor $(s + a)$ where 'a' is any positive real number. The simplest value of 'a' is 1 (take $a = 1$). Consider the following example.

EXAMPLE 5.7. Investigate the stability

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

Solution :

s^5	1	2	3
s^4	1	2	5
s^3	0		
s^2			
s^1			
s^0			

Now, multiply the characteristic equation by $(s + 1)$

$(s + 1)(s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5) = 0$

or, $s^6 + 2s^5 + 3s^4 + 4s^3 + 5s^2 + 8s + 5 = 0$

s^6	1	3	5	5
s^5	2	4	8	
s^4	1	1	5	
s^3	2	–2		
s^2	2	5		
s^1	–7			
s^0	1			

From the above table :
No. of sign change in the first column = 2
No. of roots in the right half of s-plane = 2
Hence, system is unstable.

EXAMPLE 5.8. Apply Routh-Hurwitz criterion to the following equation and investigate the stability.

$$s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

Solution :

s^5	1	2	11
s^4	2	4	10
s^3	0		
s^2			
s^1			
s^0			

Since, the third element in first column is zero, so multiply the equation by $(s + 1)$

$$(s + 1)(s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10) = 0$$

or
$$s^6 + 3s^5 + 4s^4 + 6s^3 + 15s^2 + 21s + 10 = 0$$

s^6	1	4	15	10
s^5	3	6	21	
s^4	2	8	10	
s^3	–6	6		
s^2	10	10		
s^1	12			
s^0	1			

From the above table it is clear that there are two changes of sign in first column, therefore there are two roots in the right half of s-plane. The system is unstable.

ALTERNATIVE METHOD

Replace the zero by a small positive quantity ε and continue the procedure. Consider the following example.

EXAMPLE 5.9. Consider the following equation

$$s^3 + s + 2 = 0$$

Solution :

s^3	1	1
s^2	0	2
s^1		
s^0		

Replace 0 by ε and continue the procedure.

s^3	1	1
s^2	ε	2
s^1	$-\frac{1}{\varepsilon}(2-\varepsilon)$	–
s^0	2	

No. of sign changes in first column = 2

No. of roots in right of s-plane = 2

Case 2 : When any one row of Routh table is zero.

When any one row of Routh table is zero, it indicates that the equation has at least one pair of roots which lie radially opposite each other and equidistant from origin. The array can be completed by forming the auxiliary polynomial. The polynomial whose coefficients are the element of the row just above the row of zeros in Routh array is called an auxiliary polynomial. Consider the following example

EXAMPLE 5.10. Consider the equation

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

Solution :

s^5	1	24	– 25
s^4	2	48	– 50
s^3	0	0	
s^2			
s^1			
s^0			

From above table it is clear that the third row is zero. Hence form the auxiliary polynomial. $A(s)$

$$A(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dA(s)}{ds} = 8s^3 + 96s$$

Now the Routh array can be written as

s^5	1	24	– 25
s^4	2	48	–50
s^3	8	96	
s^2	24	– 50	
s^1	112.6		
s^0	– 50		

No. of sign changes in first column = 1

No. of roots in right half of s-plane = 1

The roots of the equation formed by the auxiliary polynomial

$$2s^4 + 48s^2 - 50 = 0$$

are also the roots of the original equation.

The given equation can be written in factored form as

$$(s+1)(s-1)(s+j5)(s-j5)(s+2) = 0$$

It is clear that the original equation has one root with positive real part.

The roots of the auxiliary equation are also the roots of the characteristic equation because auxiliary equation is the part of characteristic equation. Suppose, original characteristic equation is of order six *i.e.*, having six roots and auxiliary equation is of order four *i.e.*, having four roots. These four roots are also the roots of the original characteristic equation and remaining (6 – 4 = 2) two roots will always lie to the left half of s-plane and these two roots do not have significant role for stability because they lie far away from imaginary axis. The roots of the auxiliary equations are dominant roots,* the stability can also be determined from the roots of the auxiliary equation.

Consider the following characteristic equation :

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0
s^2				
s^1				
s^0				

Auxiliary equation $A(s) = 2s^4 + 12s^2 + 16 = 0$, $\frac{dA(s)}{ds} = 8s^3 + 24s$. Since, the order of the characteristic equation is six and order of auxiliary equation is four, the two roots (6 – 4 = 2) lie to the left half of s-plane and four roots of auxiliary equations are dominant roots.

* Dominant roots means the roots are close to the imaginary axis or on the imaginary axis or they are in right half of s-plane.

		8	20	16
s^6	1	12	16	0
s^5	2	12	16	0
s^4	2	24	0	0
s^3	8	16	0	
s^2	6	0		
s^1	2.67			
s^0	16			

No sign change in first column, the system may be stable, since there is a row of zeros it means the roots are lying on imaginary axis. So solve the auxiliary equation $A(s) = 0$.

$$2s^4 + 12s^2 + 16 = 0$$

Put $s^2 = x$

$$2x^2 + 12x + 16 = 0$$

$\therefore$
$$x^2 + 6x + 8 = 0$$

or
$$(x+2)(x+4) = 0$$

$$x + 2 = 0, \qquad x + 4 = 0$$

$$x = -2, \qquad x = -4$$

$$s^2 = -2, \qquad s^2 = -4$$

$$s = \pm j\sqrt{2}, \qquad s = \pm j2$$

Since, the roots are non-repeated on imaginary axis, hence the system is marginally stable.

No. of roots with positive real part = 0

No. of roots with negative real part = 2

No. of roots with zero real part = 4

Important Note : If sign changes in first column, system is unstable. The number of roots with positive real part of characteristics equation will be equal to the number of sign changes i.e., the roots are located in right half of s-plane. But if no sign changes in first column the system may be stable. In such case stability can be determined by solving auxiliary equation for roots and from the location of roots, stability can be determined.

5.7. APPLICATION OF ROUTH'S STABILITY CRITERION TO CONTROL SYSTEM ANALYSIS

Routh stability criterion is also used for the determination of stability of the linear feedback systems. Consider the following example.

EXAMPLE 5.11. The open loop transfer function of unity feedback system is $\dfrac{K}{s(1+0.4s)(1+0.25s)}$.

Find the restriction of K so that the closed loop system is absolutely stable.

(R.M.L. University Faizabad, 2001)

Solution : Given that

$$G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$$

$$H(s) = 1$$

The characteristic equation $1 + G(s)H(s) = 0$

$$1 + \frac{K}{s(1+0.4s)(1+0.25s)} = 0$$

$$s(1 + 0.4s)(1 + 0.25s) + K = 0$$

or $$s^3 + 6.5s^2 + 10s + 10K = 0$$

or

s^3	1	10
s^2	6.5	10K
s^1	$\dfrac{65 - 10K}{6.5}$	
s^0	10K	

For absolute stability, there should be no sign change in the first column *i.e.*, no root of the characteristic equation should lie in right half of *s*-plane. This is possible only when

$K > 0$ and $65 - 10K > 0$

Hence, for closed loop stability

$0 < K < 6.5$

5.8. RELATIVE STABILITY ANALYSIS

Routh stability criterion gives the information about the absolute stability. But we are more interested for the relative stability of the system. Relative stability can be examined by shifting the *s*-plane and then apply Routh stability criterion. The characteristic equation is modified by shifting the origin of *s*-plane to $s_1 = -\sigma$ by the substitution

$$s = z - s_1$$

Now apply the Routh stability criterion, the number of sign change in the first column is equal to the number of roots to the right of the vertical line $s_1 = -\sigma$ or in other words the roots of the original characteristic equation are more negative than $-\sigma_1$.

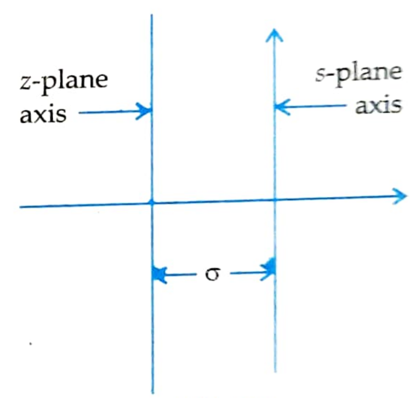


Fig. 5.8.

EXAMPLE 5.12. Check whether all the roots of the equation $s^3 + 7s^2 + 25s + 39 = 0$ have real parts more negative than – 1.

Solution :

s^3	1	25
s^2	7	39
s^1	19.42	
s^0	39	

Since, there is no sign change in the first column, it means all the roots are in the left half of *s*-plane.

Put $s = z - 1$ in the characteristic equation

$$(z-1)^3 + 7(z-1)^2 + 25(z-1) + 39 = 0$$

or, $$z^3 + 4z^2 + 14z + 20 = 0$$

z^3	1	14
z^2	4	20
z^1	9	
z^0	20	

Since, no sign change in the first column, the roots of the characteristic equation lie in the left of *z*-plane, it means all the roots of the original equation in *s*-domain lie to the left of $s = -1$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 5.13. The characteristic equation of a servosystem is given by

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

Determine the conditions which must be satisfied by the coefficient of the characteristic equation for the system to be stable.

Solution :

$$\begin{array}{lll}
s^4 & a_0 & a_2 \quad a_4 \\
s^3 & a_1 & a_3 \\
s^2 & \dfrac{a_1a_2 - a_0a_3}{a_1} & \dfrac{a_4a_1}{a_1} \\
s^1 & \dfrac{a_3\left[\dfrac{a_1a_2 - a_0a_3}{a_1}\right] - \dfrac{a_1^2a_4}{a_1}}{\dfrac{a_1a_2 - a_0a_3}{a_1}} & \\
s^0 & a_4 &
\end{array}$$

For stability there should not be any sign change in first column.

$$\therefore \quad a_1 > 0$$
$$a_1 a_2 - a_0a_3 > 0$$
$$a_1a_2a_3 - a_0 a^2_3 - a_1{}^2a_4 > 0$$
$$a_4 > 0$$

Above are the required conditions.

EXAMPLE 5.14. The characteristic equations for certain feedback control system are given below. In each case, determine the range of values K for the system to be stable.

(a) $s^4 + 20\,Ks^3 + 5s^2 + 10s + 15 = 0$

(b) $s^3 + 2Ks^2 + (K + 2)\,s + 4 = 0$

Solution : (a) Routh array is

$$\begin{array}{llll}
s^4 & 1 & 5 & 15 \\
s^3 & 20K & 10 & \\
s^2 & \dfrac{100K - 10}{20K} & 15 & \\
s^1 & 10 - \dfrac{600K^2}{10K - 1} & & \\
s^0 & 15 & &
\end{array}$$

For stability no root should lie in right half of s-plane.

(a) $20K > 0 \quad$ *i.e.,* $\quad K > 0$

(b) $\dfrac{100K - 10}{20K} > 0$ or $5 - \dfrac{1}{2K} > 0$ *i.e.,* $5 > \dfrac{1}{2K}$ or $K > \dfrac{1}{10}$

(c) $10 - \dfrac{600K^2}{10K - 1} > 0$

i.e., $-600K^2 + 100K - 10 > 0$

This gives the complex roots, hence the system is unstable.

(b) $\quad s^3 + 2Ks^2 + (K + 2)\,s + 4 = 0$

$$\begin{array}{lll}
s^3 & 1 & K + 2 \\
s^2 & 2K & 4 \\
s^1 & \dfrac{K^2 + 2K - 2}{K} & \\
s^0 & 4 &
\end{array}$$

For stability

$2K > 0 \quad$ *i.e.,* $\quad K > 0$

$\dfrac{K^2 + 2K - 2}{K} > 0 \quad$ *i.e.,* $\quad K^2 + 2K - 2 > 0$

$$K > 0.73$$

EXAMPLE 5.21. **The characteristic equation of feedback control system is**

$$s^4 + 20s^3 + 15s^2 + 2s + K = 0$$

(a) **Determine the range of K for the system to be stable.**

(b) **Can the system be marginally stable? If so, find the required value of K and the frequency of sustained oscillation.** (GATE, 1998)

Solution :

$$\begin{array}{llll} s^4 & 1 & 15 & K \\ s^3 & 20 & 2 & \\ s^2 & 14.9 & K & \\ s^1 & \dfrac{29.8 - 20K}{14.9} & & \\ s^0 & K & & \end{array}$$

(a) For stability $K > 0$

$$29.8 - 20K > 0$$

or

$$K < 29.8/20$$

$$K < 1.49$$

Hence, range $0 < K < 1.49$

(b) For marginally stable $K = 1.49$

Auxiliary equation $A(s) = 14.9s^2 + 1.49$

$$14.9s^2 = -1.49$$

$$s^2 = -0.1$$

$\therefore$ $s = \pm j\,0.316$

$$\omega = 0.316 \text{ rad/sec.}$$

$\therefore$ Frequency of sustained oscillation = 0.316 rad/sec.

EXAMPLE 5.22. **The characteristic equation for certain feedback control system is given below. Determine the range of value of K for the system to be stable.**

$$s^4 + 4s^3 + 13s^2 + 36s + K = 0$$

Solution :

$$\begin{array}{llll} s^4 & 1 & 13 & K \\ s^3 & 4 & 36 & \\ s^2 & 4 & K & \\ s^1 & 36-K & & \\ s^0 & K & & \end{array}$$

For stability $K > 0$

$36 - K > 0$ or $K < 36$

Hence, range is $36 > K > 0$ **Ans.**

5.9. ROOT LOCUS

Root locus is a graphical method in which roots of the characteristic equation are plotted in s-plane for the different values of prameter. The locus of the roots of the characteristic equation when gain is varied from zero to infinity is called root locus.

Consider a unity feedback system as shown in Fig. 5.10.

The characteristic equation $1 + G(s)\,H(s) = 0$

$$G(s) = \frac{K}{s\,(s+2)}\ ,\ H(s) = 1$$

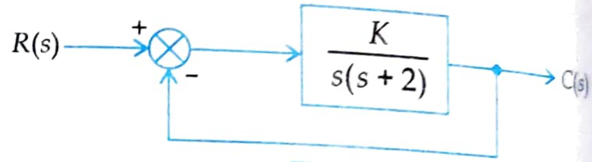


Fig. 5.10.

$$1 + \frac{K}{s\,(s+2)} \cdot 1 = 0$$

or $$s^2 + 2s + K = 0 \qquad ...(5.14)$$

The roots of equation (5.14) are

$$s_1 = -1 + \sqrt{1-K}, \quad s_2 = -1 - \sqrt{1-K}$$

As 'K' is varied, the two roots give the locii in s-plane. For various values of K, the location of the roots are :

1. When $0 < K < 1$, the roots are real and distinct
2. When $K = 0$, the two roots are $s_1 = 0$ and $s_2 = -2$. These are also the open loop poles.
3. When $K = 1$, both roots are real and equal.
4. When $K > 1$, the roots are complex conjugate with real part $= -1$

When 'K' is varying the root locus is shown in the Fig. 5.11.

(*a*) when $K = 0$, two branches of root locus starts from $s = 0$ and $s = -2$.

(*b*) when $K = 1$, both roots meet at $s = -1$

(*c*) when $K > 1$, the roots breakaway from the real axis and become complex conjugate having negative real part equal to -1.

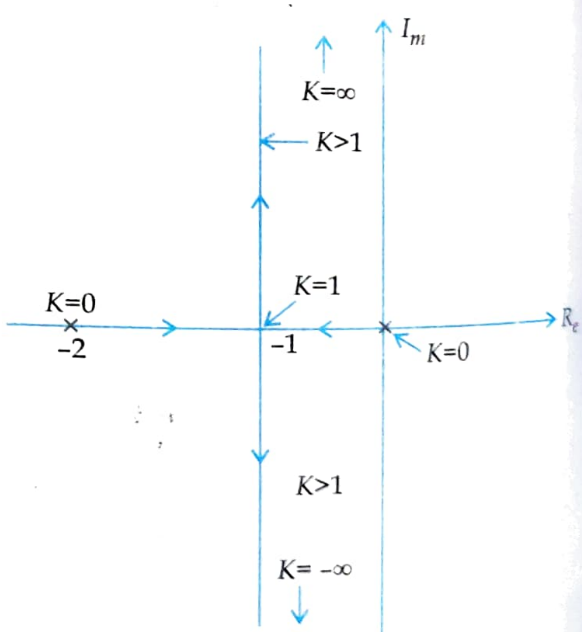


Fig. 5.11.

Consider a closed loop system shown in Fig. 5.12.

The overall transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

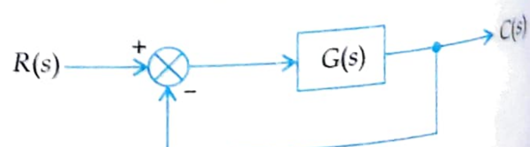


Fig. 5.11 (a)

The characteristic equation is $1 + G(s)\,H(s) = 0$

or, $$G(s)\,H(s) = -1 \qquad ...(5.15)$$

or, $$|G(s)\,H(s)| = -1$$

$$\angle G(s)H(s) = \pm 180(2K + 1) \qquad ...(5.16)$$

$$K = 0, 1, 2, 3, ...$$

Equations (5.15) and (5.16) are the magnitude and angle conditions. The roots of the characteristic equations must satisfied the above two conditions or in other words the values of 's' which satisfies the above said conditions are the roots of the characteristic equation or poles of the closed loop systems. These two conditions are not independent. A plot of the points in 's' plane satisfying the second condition (equation 5.16 known as angle criterion) is the root locus. The value of gain corresponding to a root can be determined from the first condition (equation 5.15 known as magnitude criterion).

Consider a system with $G(s)\,H(s) = \dfrac{K}{s\,(s+4)\,(s+5)}$. Find whether $s = -1$ is on root locus or not using angle condition.

The angle condition is

$$\angle G(s)\,H(s) = \pm\, 180° \,(2\,K + 1).$$

$$\angle G(s)\,H(s)\Big|_{s=-1} = \frac{\angle K + j0}{(-1+j0)\,(3+j0)\,(4+j0)} = \frac{0°}{180°.\,0°.0°} = -\,180°$$

Since $\angle G(s)\,H(s) = -\,180°$ at $s = -\,1$, this satisfies the angle condition, hence the point at $s = -1$ is on the root locus.

Consider above example for magnitude condition. Now we are interested to find the value of K at which $s = -\,1$ is one of the roots of $1 + G(s)\,H(s) = 0$.

$$|\,G(s)\,H(s)\,|_{\,s\,=\,-\,1} = -\,1$$

$$\frac{|K|}{|-1|\,|-1+4|\,|-1+5|} = -\,1$$

$$\therefore \qquad K = 12.$$

Therefore for $K = 12$, one of the three roots is located at $s = -\,1$. Remaining two roots can also be easily calculated.

5.10. RULES FOR CONSTRUCTION OF ROOT LOCII

Following are the rules to sketch the root locus plot.

Rule 1. The root locus is symmetrical about the real axis.

Rule 2. The root locii starts from an open loop pole with $K = 0$ *e.g.*, the system having

$$G(s)H(s) = \frac{K\,(s+3)}{(s+2)} \qquad ...(5.17)$$

Find the starting point of the root locii.

Solution : According to the rule the root locii starts from $s = -\,2$.

Rule 3. The root locii will terminate either on an open loop zeros or on infinity with $K = \infty$ *e.g.*, find the ending point of the root locii given in equation (5.17). According to the rule the root locii will terminate at $s = -\,3$

Rule 4. If

N = No. of separate locii

P = No. of finite poles

Z = No. of finite zeros then

Number of root locii will be equal to the no. of poles if number of poles are more than number of zeros *i.e.*, $P > Z$

$$N = P \;\text{ if }\; P > Z$$

If $Z > P$, then number of root locii will be equal to the number of zeros.

If $P = Z$, then No. of root locii = Poles = Zeros.

For example find the number of separate root locii for the system given by the equation (5.17)

Solution :

$$P = 1$$
$$Z = 1$$
$$\therefore \quad N = 1$$

Rule 5. Root Locii on the Real Axis.

Any point on the real axis is a part of the root locus if and only if the number of poles and zeros to its right is odd.

Rule 6. Asymptotes

The branches of root locus tend to infinity along a set of straight line called asymptotes. These asymptotes making an angle with real axis and is given by

$$\phi = \frac{(2K+1)180°}{P-Z} \qquad \text{where} \quad K = 0, 1, 2, \ldots$$

The total number of asymptotes = $P-Z$

e.g.,

$$\text{If } G(s)H(s) = \frac{K}{s(s^2+6s+10)} \qquad \ldots(5.18)$$

$$P = 3$$
$$z = 0$$

No. of asymptotes = $P - Z = 3 - 0 = 3$

$$K = 0 \qquad \phi_1 = \frac{(2\times0+1)180°}{3} = 60°$$

$$K = 1 \qquad \phi_2 = \frac{(2\times1+1)180°}{3} = 180°$$

$$K = 2 \qquad \phi_2 = \frac{(2\times2+1)180°}{3} = 300°$$

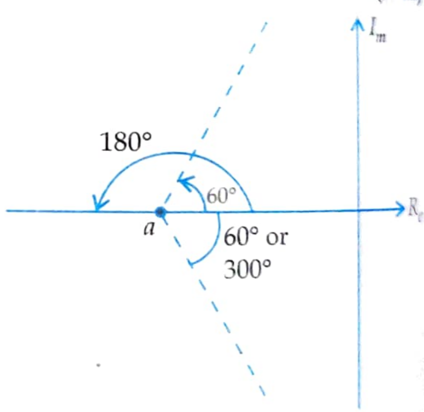


Fig. 5.12.

Rule 7. Centroid of asymptotes

The point of intersection of asymptotes with real axis is called centroid of asymptotes (σ_A) and is given by

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z}$$

For example, find the centroid of asymptotes of the system given by equation (5.18).

Solution : There are three poles at $s_1 = 0$, $s_2 = -3 + j1$, $s_3 = -3 - j1$

No. of zeros = 0

$$\therefore \text{ Centroid} \qquad \sigma_A = \frac{0-3+j1-3-j1-0}{3} = -2$$

The centroid is shown in Fig. 5.12 by the point 'a'.

Rule 8. Angle of departure and angle of arrival of the root locii

The angle of departure of the root locus from a complex pole is given by

$\phi_d = 180°$ – sum of angles of vectors drawn to this pole from other poles + sum of angles of vectors drawn to this pole from the zeros.

The angle of arrival at a complex zero is given by
ϕ_a = 180° – sum of angles of vectors drawn to this zero from other zeros + sum of angles of vectors drawn to this zero from poles.

For example

$$G(s)\,H(s) = \frac{K}{s\,(s+6)\,(s^2+4s+13)}$$

Determine the angle of departure from complex poles.

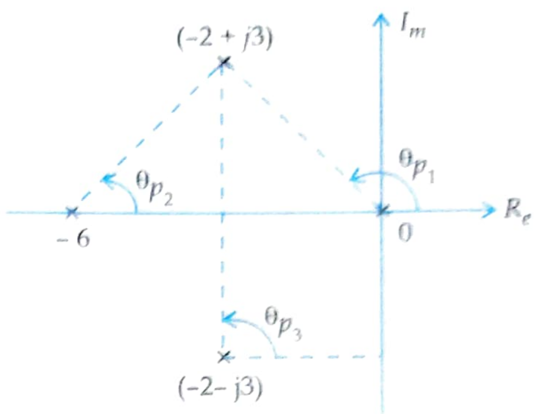


Fig. 5.13.

$$\phi_d = 180° - (\theta_{P_1} + \theta_{P_2} + \theta_{P_3})$$
$$= 180° - (123° + 37° + 90°) = -70°$$

$\therefore$ angle of departure at $(-2 + j3) = -70°$

$\therefore$ angle of departure at $(-2 - j3) = 70°$

Rule 9. Breakaway point on real axis

If the root locus lies between two adjacent open loop poles on the real axis then there will be at least one breakaway point, because the roots move towards each other as K is increased and meet at a point. At this point K is maximum. If we increase the value of K between two poles the root locus breaks in two parts.

Similarly if root locus lies between two adjacent zeros on real axis, then there will be at least one break in point. If the root locus lies between an open loop pole and zero, then there will be no breakaway or breakin point or may be both occur.

The breakaway or break in points can be determined from the roots of

$$\frac{dK}{ds} = 0 \qquad \text{...(5.19)}$$

e.g., if $\quad G(s)H(s) = \dfrac{K}{s\,(s^2+6s+10)}$ determine the breakaway point

$$1 + G(s)\,H(s) = 1 + \frac{K}{s\,(s^2+6s+10)}$$

$$s(s^2 + 6s + 10) + K = 0$$

or
$$K = -s^3 - 6\,s^2 - 10s$$

$$\frac{dK}{ds} = -3s^2 - 12s - 10 = 0$$

or
$$3s^2 + 12s + 10 = 0$$

$s_1 = -1.1835$ and $s_2 = -2.815$ are the breakaway points.

Rule 10. The intersection of root locus branches with $j\omega$-axis can be determined through Routh Hurwitz criterion.

For example, if $G(s)\,H(s) = \dfrac{K}{s\,(s^2+6s+10)}$. Find the intersection of the root locii with the imaginary axis.

Solution : The characteristic equation $s^3 + 6s^2 + 10s + K = 0$

s^3	1	10
s^2	6	K
s^1	$\dfrac{60-K}{6}$	
s^0	K	

Hence, we get a zero row if $K = 60$

The auxiliary equation $A(s) = 6s^2 + K$

$$6s^2 + K = 0$$
$$6s^2 + 60 = 0$$
$$\therefore \quad s = \pm j3.16$$

The root locus branches cross the imaginary axis at $s = \pm j3.16$ for $K = 60$.

5.11. DETERMINATION OF K ON ROOT LOCII

The value of K can be determined by

$$K = \frac{\text{Product of all vector lengths drawn from the poles of } G(s)\,H(s) \text{ to the point}}{\text{Product of all vector lengths drawn from the zeros of } G(s)\,H(s) \text{ to the point}}$$

For example, in Fig 5.14 determine the value of K at the point of intersection of root locus branch with imaginary axis.

$$K = \frac{B.C}{A}$$

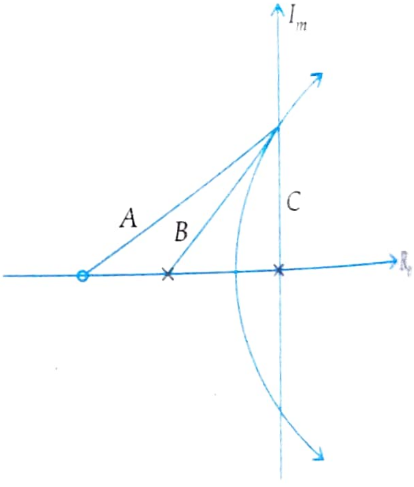


Fig. 5.14.

EXAMPLE 5.23. The forward path transfer function of a unity feedback system is given by $G(s) = \dfrac{K}{s(s+4)(s+5)}$. Sketch the root locus as K varies from zero to infinity.

Solution : Step 1 : Plot the poles and zeros

Poles are at $s = 0, -4$ and -5

No. of poles $P = 3$

No. of zeros $z = 0$

Step 2 : The root locus exists between $s = 0$ and $s = -4$ and to the left of -5. Mark the root locus on real axis.

Step 3 : Number of root locii

$$P = 3$$
$$Z = 0$$

$\therefore$ Number of root locii $(N) = 3$

Step 4 : Centroid of asymptotes

$$\sigma_A = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{(0-4-5)-(0)}{3-0} = -3$$

Step 5 : Angle of asymptotes

$$\phi = \left(\frac{2K+1}{P-Z}\right)180^\circ$$

$$K = 0, \quad \phi_1 = 60^\circ$$
$$K = 1, \quad \phi_2 = 180^\circ$$
$$K = 2, \quad \phi_3 = 300^\circ$$

Step 6 : Caluculation of breakaway point

The characteristic equation $1 + G(s)\,H(s) = 0$

$$1 + \frac{K}{s(s+4)(s+5)} = 0$$

$$s^3 + 9s^2 + 20s + K = 0$$

$$K = -s^3 - 9s^2 - 20s$$

$\therefore$

$$\frac{dK}{ds} = -3s^2 - 18s - 20 = 0$$

$$3s^2 + 18s + 20 = 0$$

$$s_1 = -1.4725,\ s_2 = -4.5275$$

Since, -4 to -5 is not the segment of root locus. Therefore we consider -1.4725 as a breakaway point.

Step 7 : Determination of point of intersection of branches of root locus with imaginary axis by Routh Hurwitz.

The characteristic equation is

$$s^3 + 9s^2 + 20s + K = 0$$

s^3	1	20
s^2	9	K
s^1	$\frac{180-K}{9}$	
s^0	K	

For $K = 180$, the auxiliary equation $A(s) = 9s^2 + K$

$$9s^2 + K = 0$$

$$9s^2 + 180 = 0$$

$$s = \pm j\,4.47$$

The complete root locus is shown in Fig. 5.15.

EXAMPLE 5.24. **For a unity feedback system the open loop transfer function is given by**

$$G(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

(a) Sketch the root locus for $0 \le K \le \infty$.

(b) At what value of 'K' the system becomes unstable.

(c) At this point of instability determine the frequency of oscillation of the system.

Solution : **Step 1** : Plot the poles and zeros

Poles are at $s_1 = 0$, $s_2 = -2$, $s_3 = -3 + j4$, $s_4 = -3 - j4$

$$s^2 + 6s + 25 = 0$$

$$s_3 = -3 + j4$$

$$s_4 = -3 - j4$$

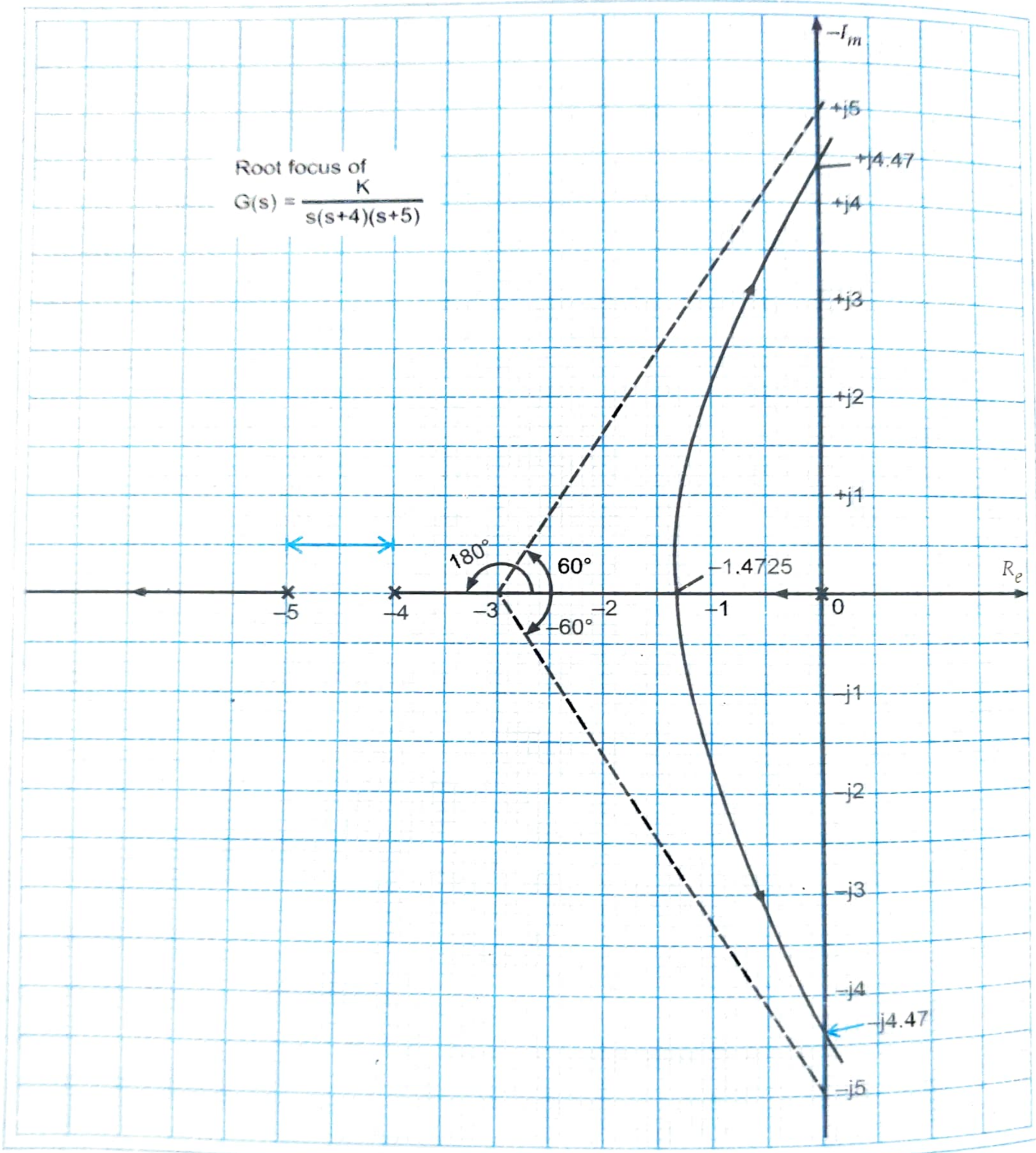
Step 2 : The segment on the real axis between $s = 0$ and $s = -2$ is the part of the root locus.

Step 3 : Number of root locii

Number of Poles $P = 4$

Number of zeros $Z = 0$

No of root locii $N = P = 4$



Root focus of
$G(s) = \dfrac{K}{s(s+4)(s+5)}$
$-I_m$
+j5
+j4.47
+j4
+j3
+j2
+j1
R_e
180°
60°
−1.4725
−5
−4
−3
−2
−1
0
−60°
−j1
−j2
−j3
−j4
−j4.47
−j5

Fig. 5.15.

Step 4 : Centroid of the asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P-Z} = \frac{(0-2-3+j4-3-j4)-(0)}{4-0} = -2$$

Step 5 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}180°$$

$K=0$ $$\phi_1 = \frac{2\times 0+1}{4}180° = 45°$$

$K=1$ $$\phi_2 = \frac{2\times 1+1}{4}180° = 135°$$

$K=2$ $$\phi_3 = \frac{2\times 2+1}{4}180° = 225°$$

$K=3$ $$\phi_4 = \frac{2\times 3+1}{4}180° = 315°$$

Step 6 : Breakaway point

The characteristic equation $1+G(s)H(s)=0$

$$1+\frac{K}{s(s+2)(s^2+6s+25)} = 0$$

or $K + s^4 + 8s^3 + 37s^2 + 50s = 0$

$$K = -(s^4 + 8s^3 + 37s^2 + 50s)$$

$$\therefore \quad \frac{dK}{ds} = -(4s^3 + 24s^2 + 74s + 50) = 0$$

or $4s^3 + 24s^2 + 74s + 50 = 0$

By trial and error method $s = -0.8981$*

Step 7 : Determination of $j\omega$ crossover (by Routh-Hurwitz)

Characteristic equation equation $s^4 + 8s^3 + 37s^2 + 50s + K = 0$

s^4	1	37	K
s^3	8	50	
s^2	30.75	K	
s^1	$\frac{1537.5-8K}{30.75}$		
s^0	K		

$$\frac{1537.5-8K}{30.75} = 0 \qquad \therefore K = 192.18$$

For $K = 192.18$ auxiliary equation

$$30.75\,s^2 + K = 0$$

$$30.75\,s^2 = -192.18$$

$$s = \pm j2.5$$

Step 8 : (a) Angle of departure from upper compex pole

$$\phi_d = 180° - (104° + 90° + 127°)$$

$$\phi_d = -141°$$

* Put the values of 's' between 0 and – 2 because the portion between 0 and – 2 is the part of root locus, if there is any breakaway point then it will lie between these points.

(*b*) The range of values for stability is $0 < K < 192.18$

The closed loop system becomes unstable for $K < 0$ and $K > 192.18$

(*c*) At this point of instability the gain is $K = 192.18$

$$30.75\, s^2 + 192.18 = 0$$

Put $$s = j\omega$$

$$-30.75\, \omega^2 + 192.18 = 0$$

$\therefore$ $$\omega = 2.5 \text{ rad/sec.}$$

$\therefore$ The frequency of oscillation at the point of instability = 2.5 rad/sec.

The root locus plot is shown in Fig. 5.16.

EXAMPLE 5.25. Consider a unity feedback control system with the following feedforward transfer function

$$G(s) = \frac{K}{s(s^2+4s+8)}$$

Plot the root locii for the system.

Solution : Step 1 : Plot the poles and zeros

$$s^2 + 4s + 8 = 0$$

$$s_2, s_3 = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm j2$$

Three poles are at $s_1 = 0$, $s_2 = -2 + j2$ and $s_3 = -2 - j2$

Step 2 : Since there is only one pole at $s = 0$, the entire left half of the real axis is the part of the root locus.

Step 3 : Number of root locii

No. of poles $P = 3$

No. of zeros $Z = 0$

$\therefore$ No. of root locii $N = P = 3$

Step 4 : Centroid of the asymptotes

$$\sigma_A = \frac{\text{sum of poles} - \text{sum of zeros}}{P - Z} = \frac{(0-2+j2-2-j2)-(0)}{3} = -4/3 = -1.33$$

Step 5 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z} 180°$$

$K = 0$ $\quad \phi_1 = 60°$

$K = 1$ $\quad \phi_2 = 180°$

$K = 2$ $\quad \phi_3 = 300°$

Step 6 : Breakaway point

The characteristic equation $1 + G(s)\,H(s) = 0$

$$1 + \frac{K}{s(s^2+4s+8)} = 0$$

$$s^3 + 4s^2 + 8s + K = 0$$

or $$K = -(s^3 + 4s^2 + 8s)$$

$$\frac{dK}{ds} = -(3s^2 + 8s + 8)$$

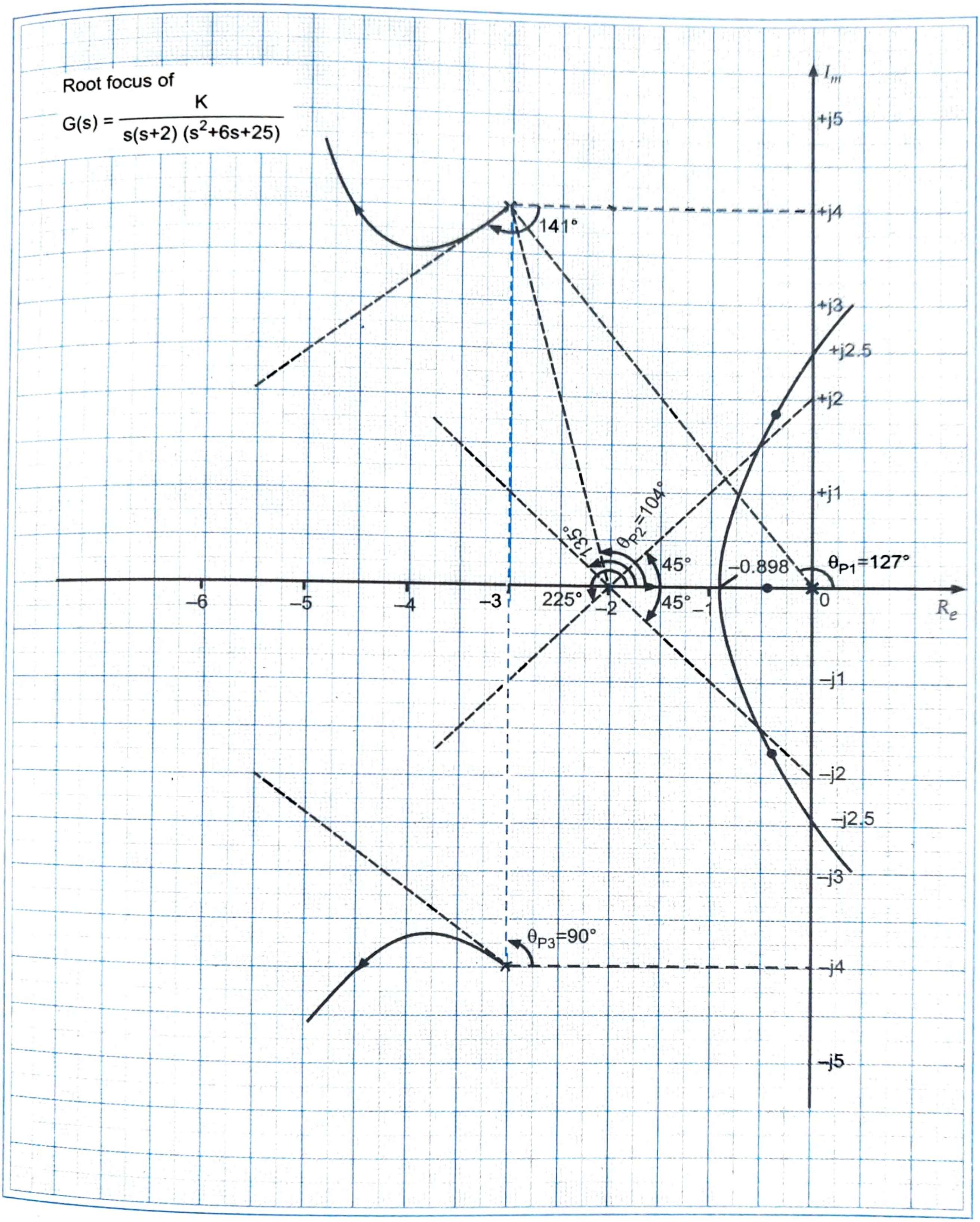

Root focus of
$G(s) = \dfrac{K}{s(s+2)(s^2+6s+25)}$
I_m
+j5
+j4
141°
+j3
+j2.5
+j2
+j1
θ_{P2}=104°
135°
45°
–0.898
θ_{P1}=127°
–6
–5
–4
–3
225°
–2
45°
–1
0
R_e
–j1
–j2
–j2.5
–j3
θ_{P3}=90°
–j4
–j5

Fig. 5.16.

$$3s^2 + 8s + 8 = 0$$

$$\therefore \quad s = \frac{-8 \pm \sqrt{64-96}}{6} = -1.33 \pm j\,0.943$$

Since, at the point $s = -1.33 \pm j\,0.943$, the angle condition is not satisfied. Hence there is no breakaway point on the real axis.

Step 7 : Point of intersection with imaginary axis.

The characteristic equation

$$s^3 + 4s^2 + 8s + K = 0$$

Routh array is

$$\begin{array}{lll} s^3 & 1 & 8 \\ s^2 & 4 & K \\ s^1 & \dfrac{32-K}{4} & \\ s^0 & K & \end{array}$$

For sustained oscillation $32 - K = 0$ or $K = 32$

The auxiliary equation $A(s) = 4s^2 + K$

$$4s^2 + 32 = 0$$

$$\therefore \quad s = \pm j\,2.83$$

Step 8 : The angle of departure from the upper complex pole is

$$\phi_d = 180° - (135° + 90°) = -45°$$

The root locus is shown in Fig. 5.17.

EXAMPLE 5.26. Plot the root locii for the closed loop control system with

$$G(s) = \frac{K}{s\,(s+1)\,(s^2+4s+5)}, \; H(s) = 1.$$

Solution : Step 1 : Plot the poles and zeros $s^2 + 4s + 5$

$$s = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm j1$$

Poles are at $s_1 = 0$, $s_2 = -1$, $s_3 = -2 + j1$ & $s_4 = -2 - j1$

Step 2 : The segment between $s = 0$ and $s = -1$ is the part of the root locus on real axis.

Step 3 : Number of root locii

Number of root locii $N = P = 4$

Step 4 : Centroid of the asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0-1-2+j1-2-j1-0}{4}$$

$$= -5/4 = -1.25$$

Step 5 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}\,180°$$

$K = 0$	$\phi_1 = 45°$
$K = 1$	$\phi_2 = 135°$
$K = 2$	$\phi_3 = 225°$
$K = 3$	$\phi_4 = 315°$

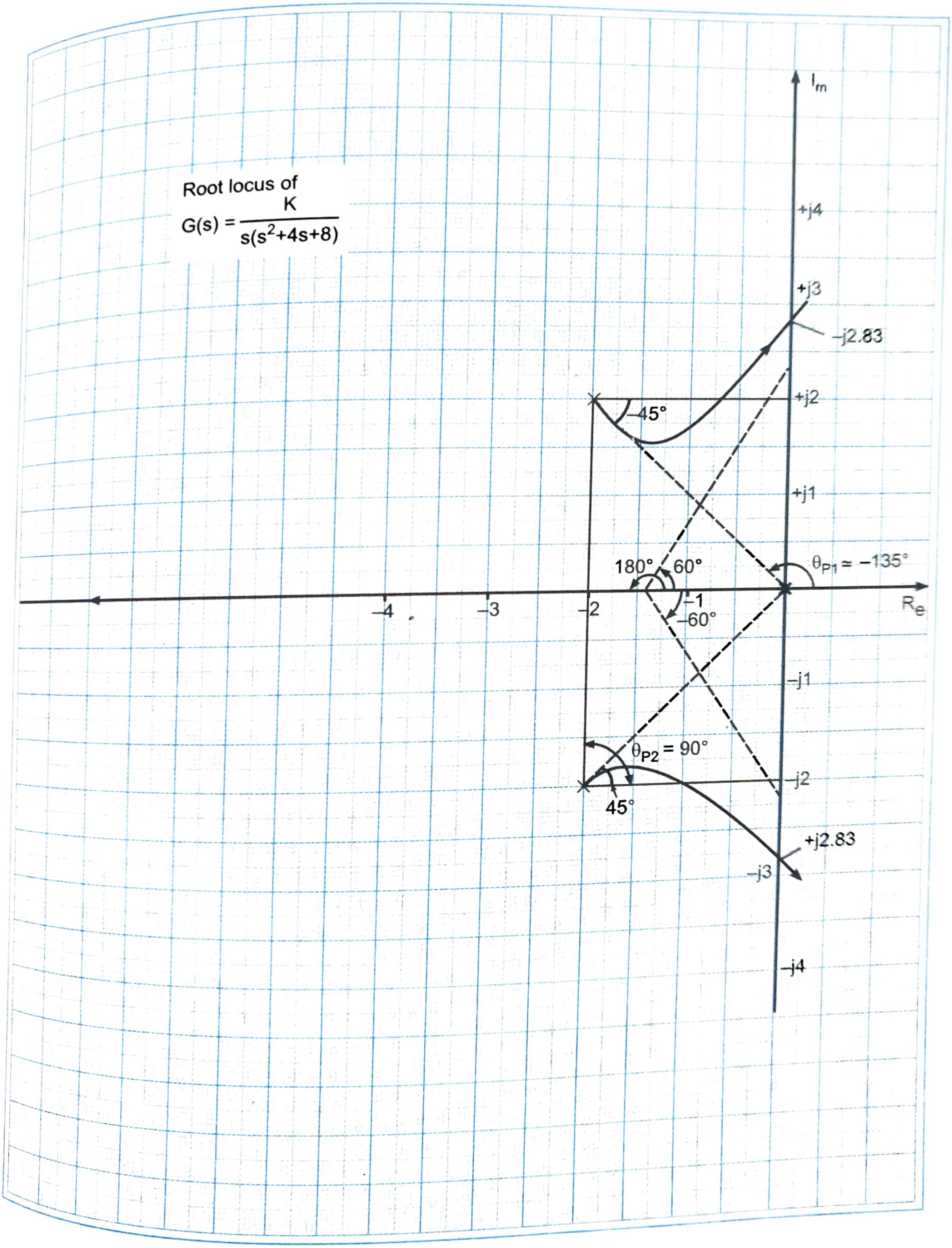

Root locus of
G(s) = K / s(s²+4s+8)
Im
Re
+j4
+j3
–j2.83
+j2
45°
+j1
180°
60°
θP1 ≈ –135°
–4
–3
–2
–1
–60°
–j1
θP2 = 90°
–j2
45°
+j2.83
–j3
–j4

Fig. 5.17.

Step 6 : Breakaway point

The characteristic equation

$$1 + G(s)\,H(s) = 0$$

$$1+\frac{K}{s\,(s+1)\,(s^2+4s+5)} = 0$$

$$s^4 + 5s^3 + 9s^2 + 5s + K = 0$$

$$K = -\,(s^4 + 5s^3 + 9s^2 + 5s)$$

$$\frac{dK}{ds} = -[4s^3 + 15s^2 + 18s + 5]$$

$$4s^3 + 15s^2 + 18s + 5 = 0$$

$\therefore$ Breakaway point is $s = -\,0.4$

Step 7 : Angle of departure at the upper complex pole

$$\phi_d = 180° - (154° + 136° + 90°) = -\,200°$$

Step 8 : Point of intersection on $j\omega$ axis

$$s^4 + 5s^3 + 9s^2 + 5s + K = 0$$

s^4	1	9	K
s^3	5	5	
s^2	8	K	
s^1	$\frac{40-5K}{8}$		
s^0	K		

At $40 - 5\,K = 0$ $\quad K = 8$ (for sustained oscillation)

the auxiliary equation $A(s) = 8s^2 + K$

$$8s^2 + 8 = 0$$

$$s = \pm\, j1$$

The root locus plot is shown in Fig. 5.18.

EXAMPLE 5.27. Sketch the root locii for

$$G(s) = \frac{K\,(s+1)}{s^2\,(s+3.6)}$$

$$H(s) = 1$$

Solution : Step 1 : Plot the poles and zeros

Poles are at $\quad s = 0, 0, -\,3.6$

zeros at $\quad s = -\,1$

Step 2 : The segment between $s = -\,1$ and $-\,3.6$ is the part of the root locus

Step 3 : Centroid of the asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P-Z} = \frac{0+0-3.6+1}{3-1} = -1.3$$

Step 4 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}180°$$

$K = 0 \quad \phi_1 = 90°$

$K = 1 \quad \phi_2 = 270°$

Step 5 : Breakaway point

Characteristic equation $1 + G(s)\,H(s) = 0$

$$1+\frac{K\,(s+1)}{s^2\,(s+3.6)} = 0$$

$$s^2(s + 3.6) + K\,(s + 1) = 0$$

$$K = -\frac{s^3+3.6s^2}{s+1}$$

$$\frac{dK}{ds} = \frac{(s+1)\,(3s^2+7.2s)-(s^3+3.6s^2)}{(s+1)^2} = 0$$

or
$$s^3 + 3.3\,s^2 + 3.6s = 0$$
$$s(s^2 + 3.3s + 3.6) = 0$$

$$s = 0 \text{ and } s = \frac{-3.3\pm\sqrt{3.3^2-4\times1\times3.6}}{2}$$

$$s = 0\,,\; s = -1.65 \pm j\,0.936$$

Point $s = 0$, is the actual breakaway point

$s = -1.65 \pm j\,0.936$ is a complex quantity, this will not be breakaway point.

Step 6 : Point of intersection of root locii on $j\omega$ axis.

The characteristic equation $s^3 + 3.6s^2 + Ks + K = 0$

s^3	1	K
s^2	3.6	K
s^1	$0.72K$	
s^0	K	

For sustained oscillation $K = 0$

$$A(s) = 3.6\,s^2 + K$$
$$3.6\,s^2 + 0 = 0$$
$$s^2 = 0$$

$\therefore$ Root locus branches do not cross the $j\omega$ axis.

The root locus is shown in Fig. 5.19.

EXAMPLE 5.28. A unity feedback control system has an open loop transfer function

$$G(s) = \frac{K}{s\,(s^2+4s+13)}$$

Sketch the root locus plot of the system by determining the following :

(a) Centroid, number and angle of asymptotes.

(b) Angle of departure of root locii from the poles.

(c) Breakaway point if any.

(d) The value of K and the frequency at which the root locii cross $j\omega$ axis.

Solution : Step 1 : Plot the poles and zeros

$$s = 0$$

$$s = \frac{-4\pm\sqrt{16-52}}{2} = -2 \pm j3$$

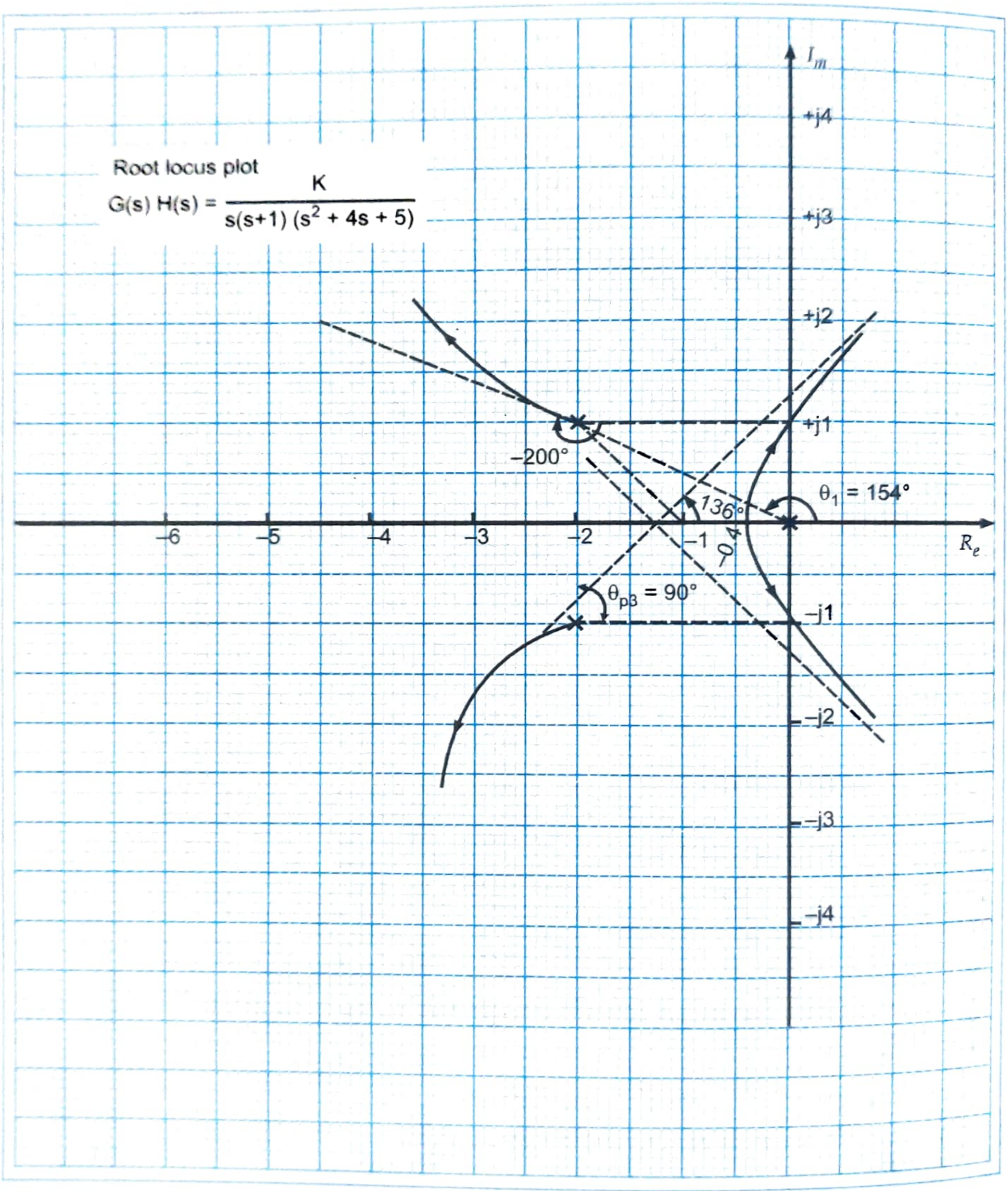

Root locus plot
G(s) H(s) = K / s(s+1) (s² + 4s + 5)
Im
+j4
+j3
+j2
+j1
−j1
−j2
−j3
−j4
Re
−6
−5
−4
−3
−2
−1
−200°
136°
θ1 = 154°
−0.4
θp3 = 90°

Fig. 5.18.

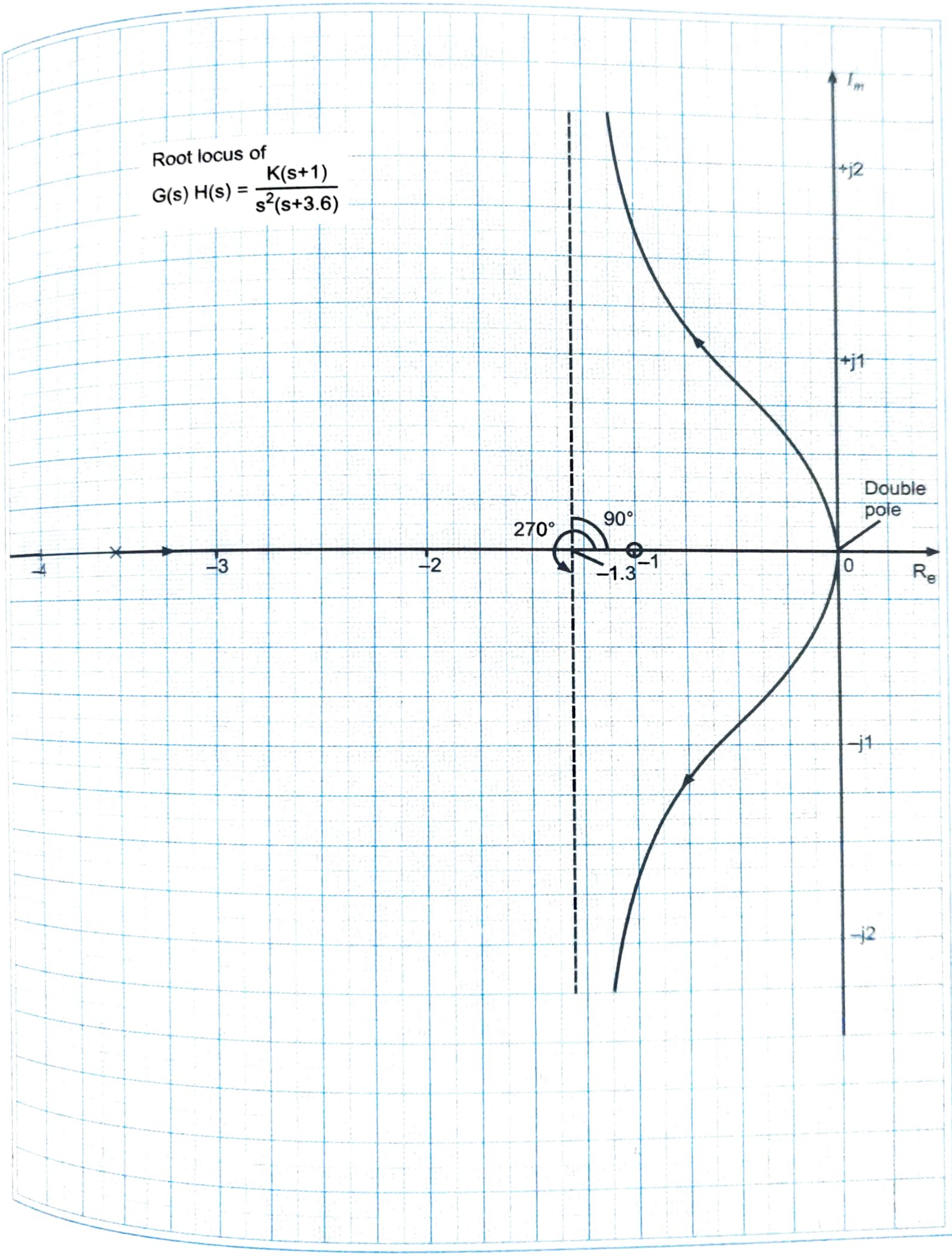

Root locus of
G(s) H(s) = K(s+1) / s²(s+3.6)
Im
+j2
+j1
Double pole
270°
90°
–4
–3
–2
–1.3
–1
0
Re
–j1
–j2

Fig. 5.19.

Step 2 : From $s = 0$ to the left is the part of the root locus

Step 3 : Number of root locii $N = P = 3$

Step 4 : Centroid of the asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P - Z}$$

$$= \frac{0 - 2 + j3 - 2 - j3 - 0}{3 - 0} = -4/3 = -1.33$$

Step 5 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}180°$$

$K = 0$ $\quad \phi_1 = 60°$

$K = 1$ $\quad \phi_2 = 180°$

$K = 2$ $\quad \phi_3 = 300°$

Step 6 : Breakaway point

The characteristic equation $1 + G(s)\,H(s) = 0$

$$1 + \frac{K}{s\,(s^2 + 4s + 13)} = 0$$

$$K = -[s^3 + 4s^2 + 13s]$$

$$\frac{dK}{ds} = -[3s^2 + 8s + 13] = 0$$

$$\therefore \quad s = \frac{-8 \pm \sqrt{64 - 156}}{6} = \frac{-8 \pm j\,9.59}{6} = -1.33 \pm j\,1.59$$

Since, it is a complex root, there will be no breakaway point on real axis.

Step 7 : Point of intersection of root locii on imaginary axis characteristic equation

$$s^3 + 4s^2 + 13s + K = 0$$

s^3	1	13
s^2	4	K
s^1	$\frac{52 - K}{4}$	
s^0	K	

For sustained oscillation $52 - K = 0$ or $K = 52$

Auxiliary equation $A(s) = 4s^2 + K$

$$4s^2 + 52 = 0$$

$$s^2 = -13$$

$$s = \pm j\,3.6$$

$\therefore$ The value of K at the point of intersection on imaginary axis = 52

The frequency at this point = 3.6 rad/sec.

Step 8 : Angle of departure at the upper complex pole

$$\phi_d = 180° - (124° + 90°) = -34°$$

The root locus is shown in Fig. 5.20.

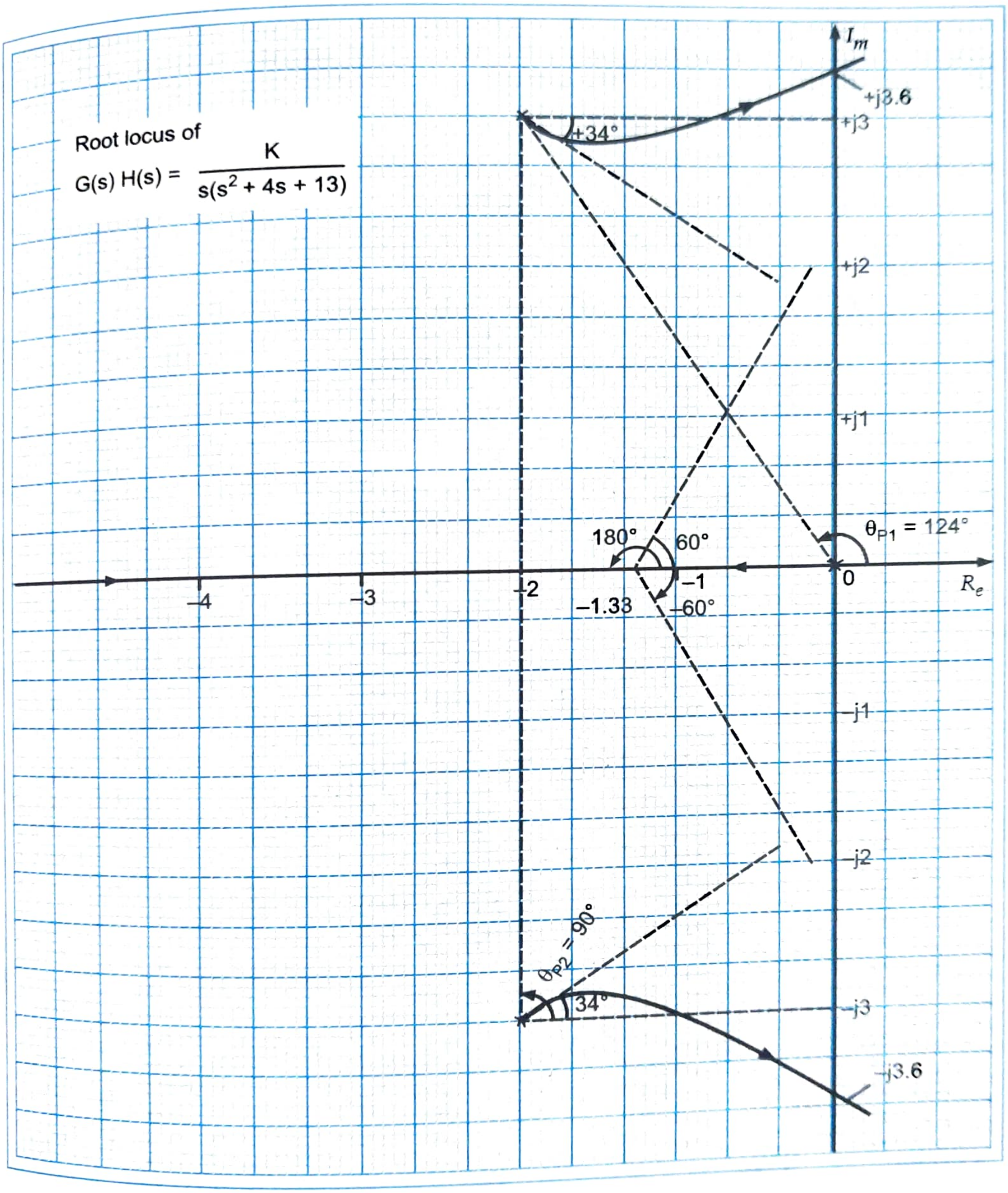

Root locus of
G(s) H(s) = K / s(s² + 4s + 13)
Im
+j3.6
+j3
+j2
+j1
–j1
–j2
–j3
–j3.6
+34°
34°
180°
60°
–60°
θP1 = 124°
θP2 = 90°
–4
–3
–2
–1.33
–1
0
Re

Fig. 5.20.

EXAMPLE 5.29. Sketch the root locus for

$$G(s)\,H(s) = \frac{K}{s\,(s+2)\,(s+4)}$$

and evaluate the value of K at the point where the root locii crosses the imaginary axis. Also determine the frequency. Also, Determine the value of 'K' so that the dominant pair of complex poles of the system has a damping ratio of 0.5.

Solution : Step 1 : Plot the poles and zeros.

Poles are at $s_1 = 0,\ s_2 = -2,\ s_3 = -4$

Step 2 : The segment between $s = 0$ and -2 and from $s = -4$ to the left are the parts of root locus.

Step 3 : Number of root locii $N = P = 3$

Step 4 : Centroid of the asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0-2-4-0}{3-0} = -2$$

Step 5 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}180^\circ$$

$K = 0 \qquad \phi_1 = 60^\circ$

$K = 1 \qquad \phi_2 = 180^\circ$

$K = 2 \qquad \phi_3 = 300^\circ$

Step 6 : Breakaway point

The characteristic equation

$$s(s+2)(s+4) + K = 0$$

$$K = -(s^3 + 6s^2 + 8s)$$

$$\frac{dK}{ds} = -(3s^2 + 12s + 8) = 0$$

$$3s^2 + 12s + 8 = 0$$

$$s = -0.85 \text{ and } -3.15$$

Since, -3.15 is not the part of root locus therefore breakaway point is -0.85.

Step 7 : Point of intersection of root locii on imaginary axis

Characteristic equation

$$s^3 + 6s^2 + 8s + K = 0$$

s^3	1	8
s^2	6	K
s^1	$\frac{48-K}{6}$	
s^0	K	

For sustained oscillation $K = 48$

Auxiliary equation $\quad A(s) = 6s^2 + K$

$$6s^2 + 48 = 0$$

$$s = \pm j\,2.8$$

Put $\quad s = j\omega$

$\therefore \quad \omega = 2.8$

$\therefore$ frequency of oscillation $= 2.8$ rad/sec.

The value of K at the point of intersection of root locii with the imaginary axis = 48

This value of K can also be determined from the graph. Draw the lines from all poles to the point of intersection on imaginary axis (c).

$$K = AC \times BC \times OC$$

From graph

$$AC = 9.8 \text{ cm.} = 4.9$$

$$BC = 6.9 \text{ cm.} = 3.45$$

$$OC = 5.8 \text{ cm.} = 2.9$$

$$\therefore \quad K = 4.9 \times 3.45 \times 2.9 = 49$$

In second quadrant draw a ζ-line at $\theta = \cos^{-1} 0.5 = 60°$ with negative real axis. This line intersect the root locii at point D. The value of 'K' at this point can be obtained as :

Connect all the poles to this point and measure all the distance of all poles to this point

$$OD = 2.6 \text{ cm.} = 1.3$$

$$\therefore \quad BD = 3.6 \text{ cm.} = 1.8$$

$$AD = 7.2 \text{ cm.} = 3.6$$

$$\therefore \quad K = 1.3 \times 1.8 \times 3.6 = 8.424$$

The root locus is shown in the Fig. 5.21.

EXAMPLE 5.30. A unity feedback system has an open loop transfer function

$$G(s) = \frac{K(s+1)}{s(s-1)}$$

Sketch the root locus plot with 'K' as variable parameter and show that the locii of complex roots are part of a circle with (–1, 0) as centre and radius = $\sqrt{2}$.

Solution : Step 1 : Plot the poles and zeros

Poles. $\quad s_1 = 0, s_2 = 1$

zeros. $\quad s_3 = -1$

Step 2 : Centroid of asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0+1-(-1)}{2-1} = 2$$

Step 3 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z} 180°$$

$K = 0$,

$$\phi = \frac{2 \times 0 + 1}{2-1} \times 180° = 180°$$

Step 4 : Breakaway point

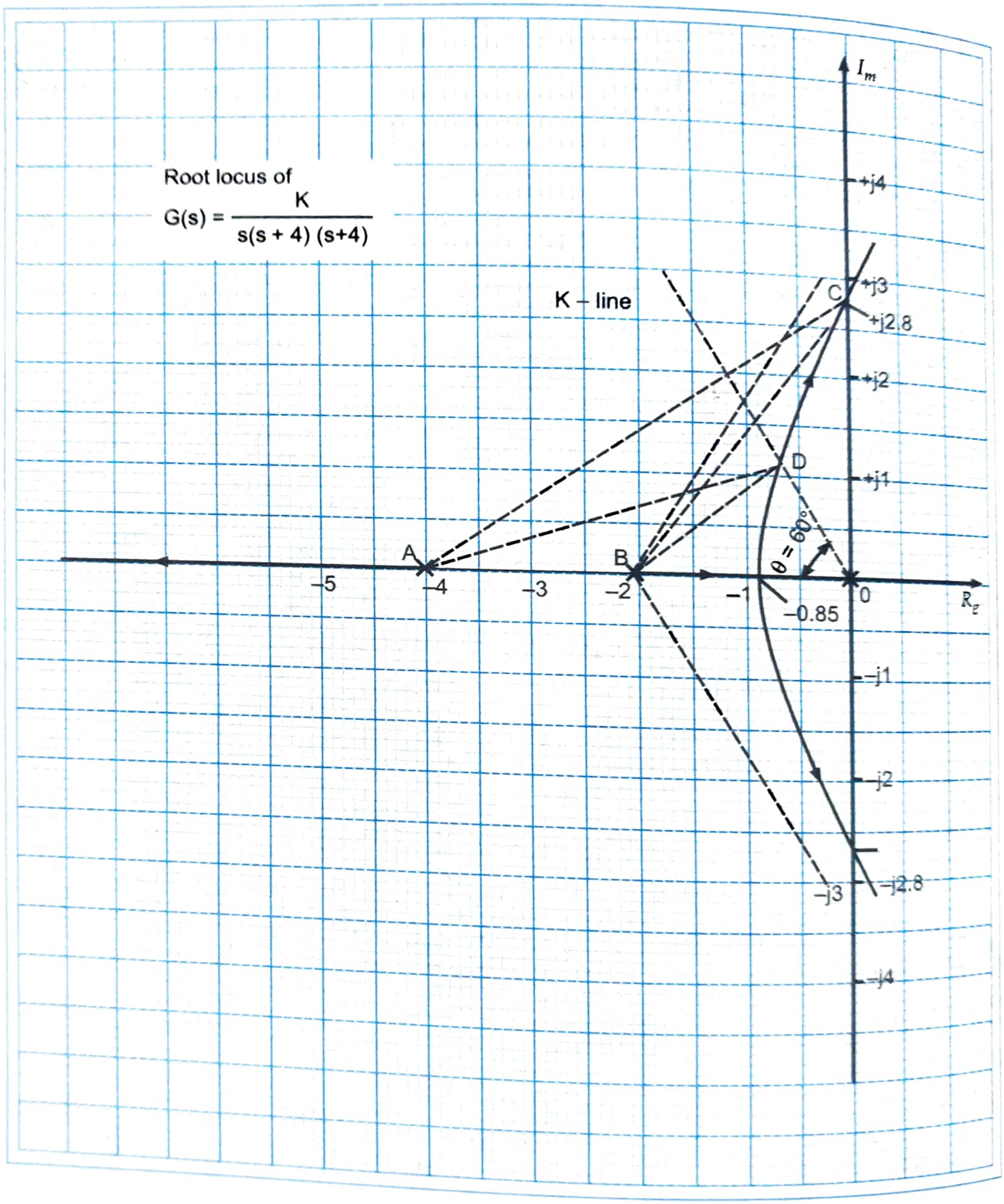
The characteristic equation

$$1 + \frac{K(s+1)}{s(s-1)} = 0$$

$$K = -\frac{s(s-1)}{s+1}$$

$$\frac{dK}{ds} = s^2 + 2s - 1 = 0$$

$$s = -2.414 \text{ and } 0.414$$



Root locus of
G(s) = K / s(s + 4) (s+4)
K – line
I_m
R_e
+j4
+j3
+j2.8
+j2
+j1
–j1
–j2
–j2.8
–j3
–j4
A
B
C
D
–5
–4
–3
–2
–1
–0.85
0
θ = 60°

Fig. 5.21.

This is the equation of the circle with centre at (–1, 0) and radius $\sqrt{2}$ **Proved.**

EXAMPLE 5.36. The open loop transfer function of a system is given by

$$G(s)\,H(s) = \frac{K(s+12)}{s^2\,(s+20)}$$

Sketch the root locus for the system. *(Control system, R.M.L, University Faizabad, 2000)*

Solution : Step 1 : Plot the poles and zero

Poles are at $s_1 = 0,\ s_2 = 0,\ s_3 = -20$

Zero is at $s_4 = -12$

Step 2 : The segment between $s = -20$ and $s = -12$ is the part of the root locus.

Step 3 : Centroid of asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P - Z} = \frac{0+0-20+12}{3-1} = -4$$

Step 4 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}180^\circ$$

$K = 0$ $\qquad \phi_1 = 90^\circ$

$K = 1$ $\qquad \phi_2 = 270^\circ$

Step 5 : Breakaway point. The characteristic equation

$$1+\frac{K(s+12)}{s^2(s+20)} = 0$$

or

$$K = -\frac{(s^3+20s^2)}{s+12}$$

$$\frac{dK}{ds} = -\left[\frac{(s+12)(3s^2+40s)-(s^3+20s^2)}{(s+12)^2}\right] = 0$$

or

$$s^3 + 28s^2 + 240\,s = 0$$

$$s\,(s^2 + 28s + 240) = 0$$

We get $\qquad s = 0, -14 \pm j\,6.63$

Breakaway point $s = 0$, points $-14 \pm j\,6.63$ are neither breakaway point nor breakin point, because the corresponding gain values K becomes complex quantities.

Step 6 : Point of intersection of rod locii with imaginary axis.

The characteristic equation

$$s^3 + 20\,s^2 + Ks + 12\,K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 20\,(j\omega)^2 + K\,(j\omega) + 12\,K = 0$$

$$(12\,K - 20\omega^2) + j\omega\,(K - \omega^2) = 0$$

If $\qquad \omega = 0, \quad K = 0$

Because of double pole at the origin, the root locus is tangent to the imaginary axis at $\omega = 0$.

The root locus is shown in Fig. 5.25.

EXAMPLE 5.37. Sketch the root locus for

$$G(s) = \frac{K}{s\,(s^2+6s+12)}, \; H(s) = 1$$

Solution : Step 1 : Draw the pole zero plot

Poles at $s = 0$

$$s^2 + 6s + 12 = 0$$

$$s = -3 \pm j\,1.73$$

Step 2 : The left portion of the negative real axis from origin is the part of the root locus.

Step 3 : Centroid of asymptotes

$$\sigma_A = \frac{\text{Sum of poles} - \text{sum of zeros}}{P-Z} = \frac{0-3+j1.73-3-j1.73}{3} = -2$$

Step 4 : Angle of asymptotes

$$\phi = \frac{2K+1}{P-Z}180^\circ$$

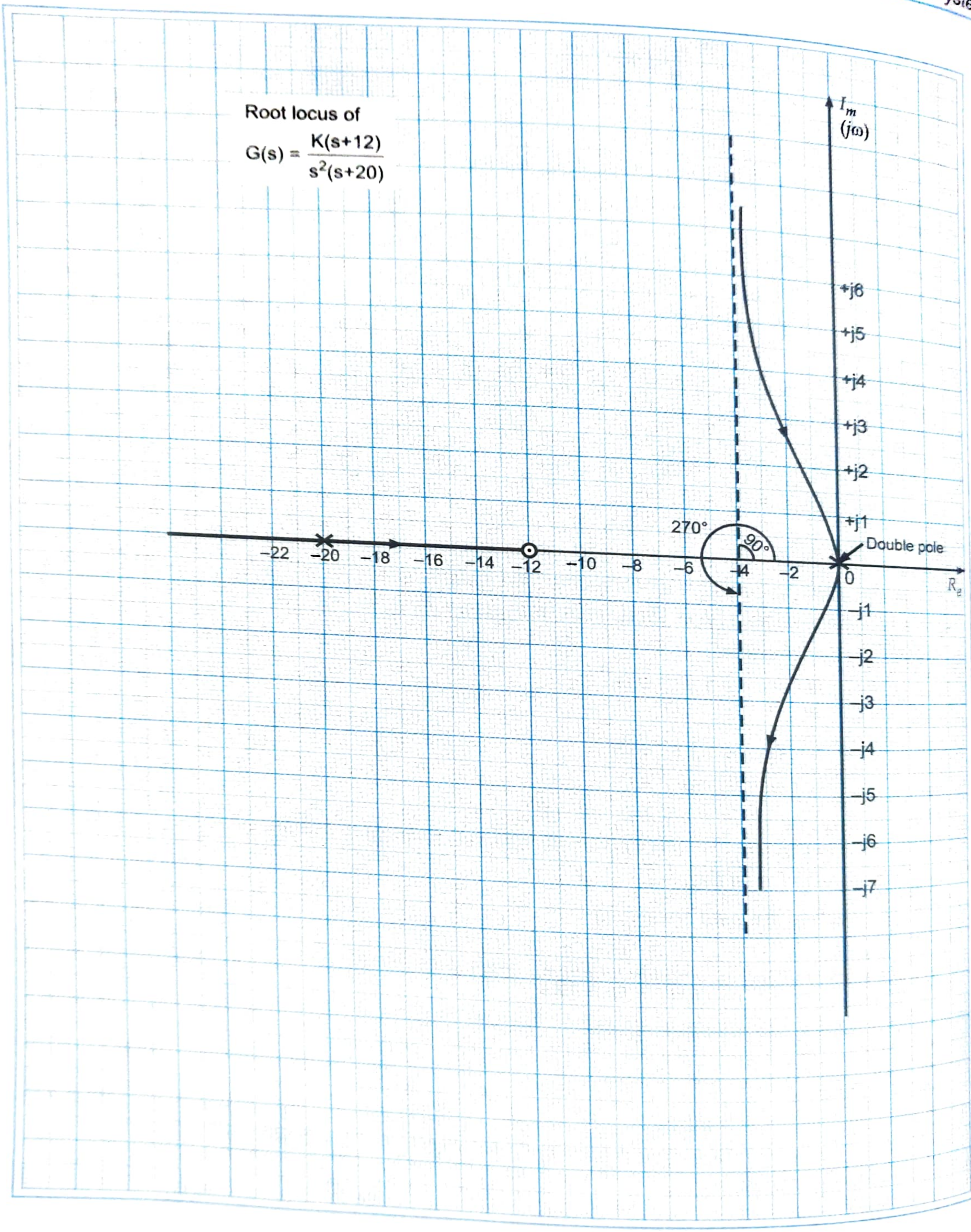

Root locus of
$G(s) = \frac{K(s+12)}{s^2(s+20)}$
I_m $(j\omega)$
+j6
+j5
+j4
+j3
+j2
+j1
270°
90°
Double pole
–22
–20
–18
–16
–14
–12
–10
–8
–6
–4
–2
0
R_e
–j1
–j2
–j3
–j4
–j5
–j6
–j7

Fig. 5.25.

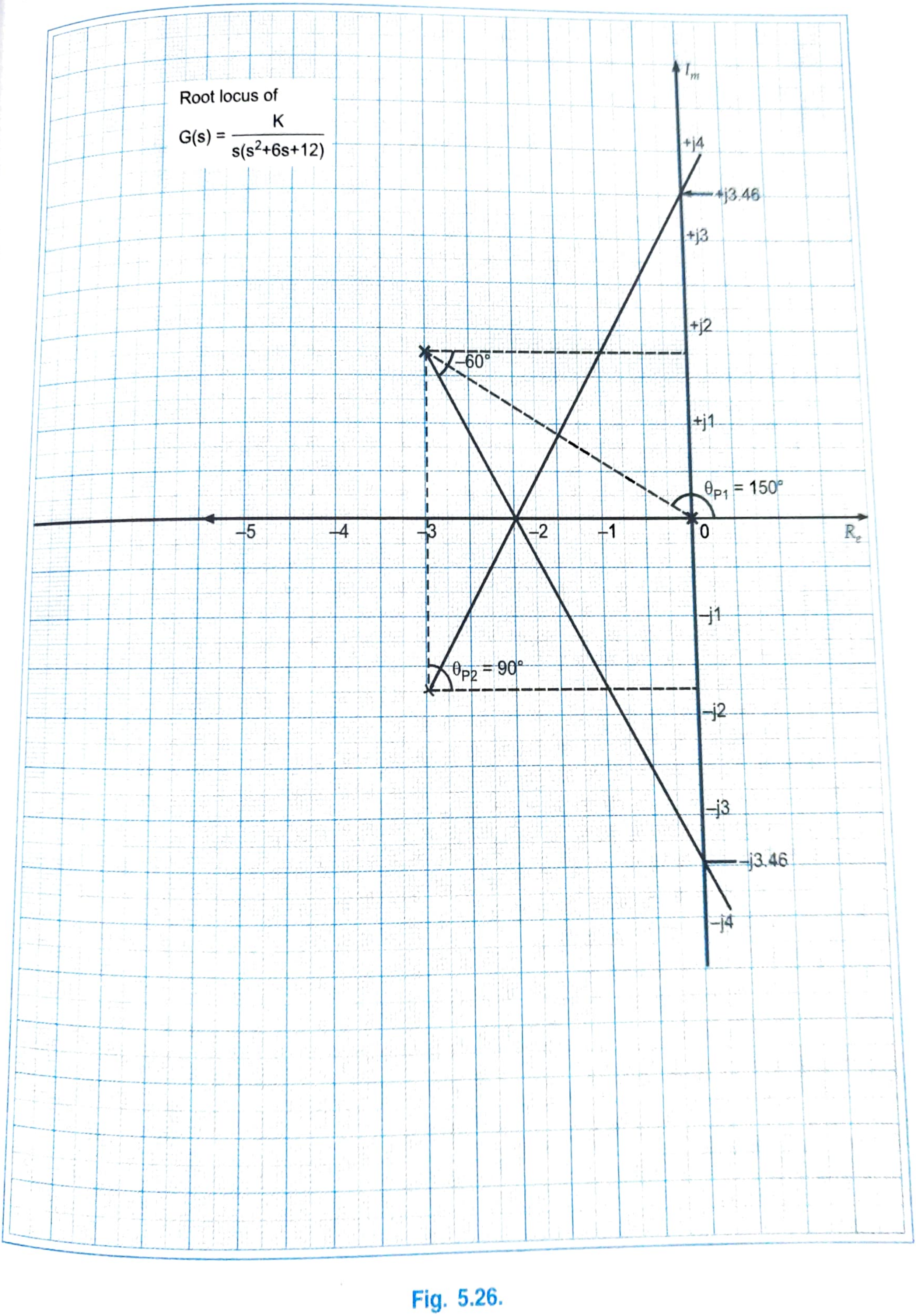

Root locus of
G(s) = K / s(s²+6s+12)
I_m
+j4
+j3.46
+j3
+j2
+j1
−60°
θ_{P1} = 150°
−5
−4
−3
−2
−1
0
R_e
−j1
θ_{P2} = 90°
−j2
−j3
−j3.46
−j4

Fig. 5.26.

$$K = 0 \qquad \phi_1 = 60^\circ$$
$$K = 1 \qquad \phi_2 = 180^\circ$$
$$K = 2 \qquad \phi_3 = 300^\circ$$

Step 5 : Breakaway point

The characteristic equation

$$1 + \frac{K}{s(s^2 + 6s + 12)} = 0$$
$$K = -(s^3 + 6s^2 + 12s)$$
$$\frac{dK}{ds} = -(3s^2 + 12\,s + 12) = 0$$
$$3s^2 + 12s + 12 = 0$$
$$s = -2 \text{ is the breakaway point}$$

Step 6 : Point of intersection of root locii on imaginary axis.

Routh array

$$\begin{array}{lcc} s^3 & 1 & 12 \\ s^2 & 6 & K \\ s^1 & \dfrac{72-K}{6} & \\ s^0 & K & \end{array}$$

For stability $\quad K > 0$

$$\frac{72-K}{6} > 0 \quad \text{or } K < 72$$

For $\quad K = 72$

The auxiliary equation $A(s) = 6s^2 + K$

$$6s^2 + 72 = 0$$
$$s = \pm j\,3.46$$

Step 7 : Angle of departure from upper complex pole

$$\phi_d = 180^\circ - (150^\circ + 90^\circ)$$
$$\phi_d = -60^\circ$$

Similarly from lower complex pole

$$\phi_d = +60^\circ$$

The root locus is shown in Fig. 5.26.

EXAMPLE 5.38. The block diagram of a position control system is shown in Fig. 5.27. Draw the root locus for the system as the