CHAPTER

2

Time-Domain Analysis

CHAPTER OUTLINES

■ Introduction; ■ Test Input Signals for Transient Analysis; ■ Time Response of a First Order System; ■ Time Response of Second Order System; ■ Transient Response Specifications of Second Order System; ■ Time Response of Second Order System with Unit Impulse Input; ■ Time Response of Second Order System with Unit Ramp Input; ■ Time Response of Higher Order Control System; ■ Effect of Feedback on Time Constant of a Control System; ■ Effect of Adding a Zero to Second Order System

2.1. INTRODUCTION

Any system containing energy storing element like inductor, capacitor, mass and inertia etc., possess certain energy. These energy storing elements are the part of the control system and cannot be avoided. If the energy state of the system is disturbed then it takes a certain time to change from one state to another state. This disturbance sometimes occurs at input, sometimes occurs at output and sometimes at both ends. The time required to change from one state to another state is known as transient time and the values of current and voltages during this period is called transient response. These transient may have oscillations which may be either sustained or decaying in nature. This will depend upon the parameters of the system. For any system we obtain a linear differential equation. The solution of linear differential equation gives the response of the system.

Thus, the time response of a control system is divided into two parts : (*a*) transient response, (*b*) steady state response.

From the Fig. 2.1 it is clear that the transient response is the part of the response which goes to zero as time increases and steady state response is the part of the total response after transient has died. If the steady state response of the output does not match with the input then the system has steady state error.

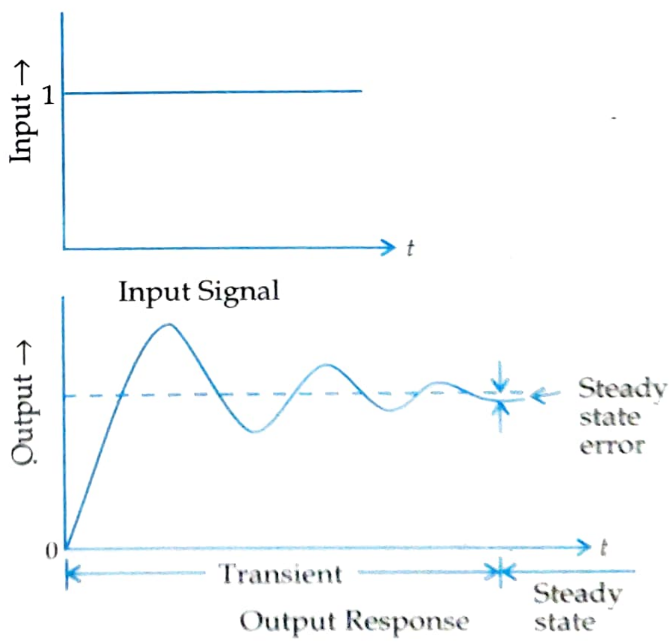

Fig. 2.1.

2.2. TEST INPUT SIGNALS FOR TRANSIENT ANALYSIS

For the analysis of time response of a control system, the following input signals are used.

1. Step Function

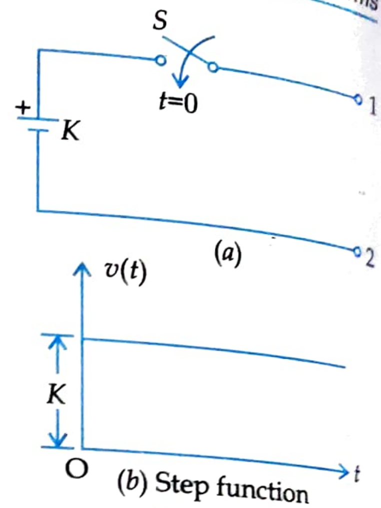

Fig. 2.2.

Consider an independent voltage source is in series with a switch 's'. When switch was open the voltage at terminals 1 – 2 is zero. Mathematically

$$V(t) = 0 \qquad -\infty < t < 0$$

When switch is closed at $t = 0$

$$V(t) = K \qquad 0 < t < \infty$$

Combining above two equations

$$\begin{aligned} V(t) &= 0; \qquad -\infty < t < 0 \\ &= K; \qquad 0 < t < \infty \end{aligned}$$

A unit step function is denoted by $u(t)$ and is defined as

$$\begin{aligned} u(t) &= 0; \qquad t \le 0 \\ &= 1; \qquad 0 \le t \end{aligned}$$

Laplace Transform : let $f(t)$ be defined in the interval $0 \le t \le \infty$

Laplace transform is obtained by multiplying $f(t)$ by e^{-st} and integrate between the limit 0 to ∞.

$$£ f(t) = \int_0^\infty f(t)e^{-st} = \int_0^\infty u(t)e^{-st}\, dt = \int_0^\infty 1.\, e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

Step function is also called displacement function. Step function can be described as sudden application of input signal to a system.

If input is $R(s)$, then

$$R(s) = \frac{1}{s}$$

2. Ramp Function

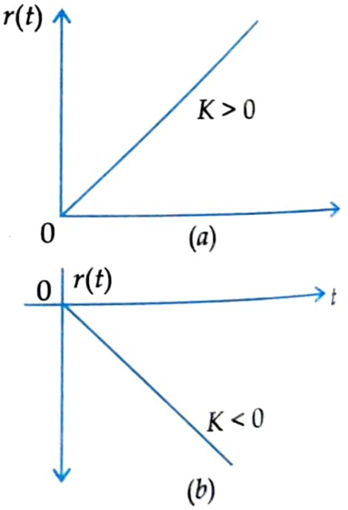

Fig. 2.3. Ramp function

Ramp function starts from origin and increases or decreases linearly with time, as shown in Fig. 2.3, let $r(t)$ be the ramp function then

$$\begin{aligned} r(t) &= 0 \quad ; \quad t < 0 \\ &= Kt \quad ; \quad t > 0 \end{aligned}$$

where 'K' is the slope of the line. For positive value of 'K' the slope is upward and the slope is downwards for negative slope.

Laplace Transform :

$$£\, r(t) = \int_0^\infty r(t)\, e^{-st}\, dt = \int_0^\infty Kt\, e^{-st}\, dt = \frac{K}{s^2}$$

$$R(s) = \frac{K}{s^2}$$

For, unit ramp $K = 1$

3. Parabolic Function

The value of $r(t)$ is zero for $t < 0$ and is quadratic function of time for $t > 0$. The parabolic function is defined as

$$\begin{aligned} r(t) &= 0 \quad ; \quad t < 0 \\ &= \frac{Kt^2}{2} \quad ; \quad t > 0 \end{aligned}$$

where 'K' is the constant. For unit parabolic function $K = 1$. The unit parabolic function is defined as

$$\begin{aligned} r(t) &= 0 \quad ; \quad t < 0 \\ &= \frac{t^2}{2} \quad ; \quad t > 0 \end{aligned}$$

Lalpace Transform :

$$\pounds\, r(t) = \int_0^\infty r(t)\, e^{-st}\, dt = \int_0^\infty \frac{Kt^2}{2} e^{-st}\, dt = \frac{K}{s^3}$$

$$R(S) = \frac{K}{s^3}$$

The parabolic function is shown in Fig. 2.4

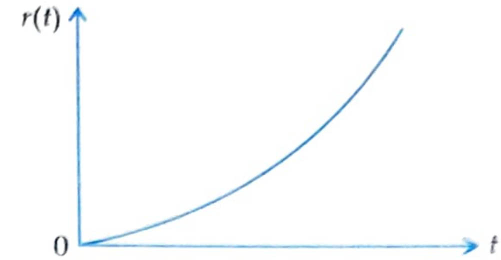

Fig. 2.4. Parabolic function

4. Impulse Function

Consider the Fig. 2.5, the first pulse has a width T and height $\frac{1}{T}$ such that area of the pulse is $T \times \frac{1}{T} = 1$. If we halve the duration and double the amplitude we get second pulse. The area under the second pulse is also unity. We can say the duration of the pulse approaches zero, the amplitude approaches infinity but the area of the pulse is unity. The pulse for which the duration tends to zero and amplitude tends to infinity is called the impulse function. Impulse function is also known as delta function,

A unit impulse function is defined as

$$\delta(t) = \begin{cases} 0 & ; \quad t \neq 0 \\ \infty & ; \quad t = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Thus, we can say that the impulse function has zero value everywhere except at $t = 0$, where the amplitude is infinite. Thus, at $t = 0$, unit impulse has infinite amplitude and area equal to unity.

Mathematically, an impulse function is the derivative of a step function *i.e.,*

$$\delta(t) = \dot{u}(t)$$

$$\pounds\, \delta(t) = \pounds \frac{d}{dt}[u(t)] = s\, \pounds\,(\text{unit step function}) = s \cdot \frac{1}{s} = 1$$

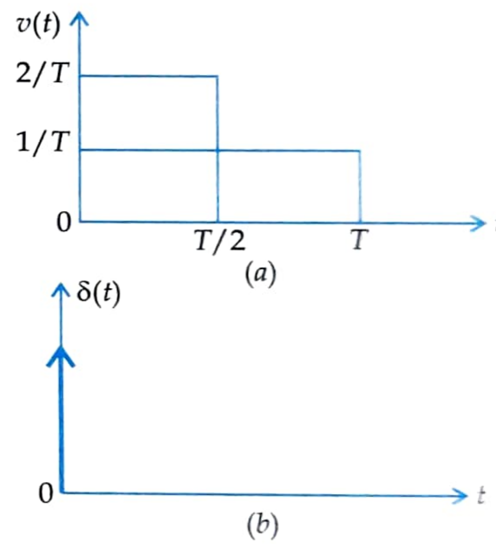

Fig. 2.5.

2.3. TIME RESPONSE OF A FIRST ORDER SYSTEM

Consider a first order system with unity feedback as shown in Fig. 2.6.

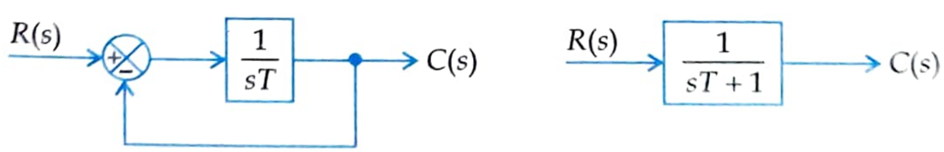

Fig. 2.6. Block diagram of first order system

$$G(s) = \frac{1}{sT}$$

$$H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{1/sT}{1 + \frac{1}{sT} \cdot 1} = \frac{1/sT}{(sT+1)/sT} = \frac{1}{sT+1}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{1}{sT+1} \qquad \text{...(2.1)}$$

2.3.1. Response of the First Order System with Unit Step Input

$$\frac{C(s)}{R(s)} = \frac{1}{sT+1}$$

$$\therefore \quad C(s) = \frac{1}{sT+1} R(s) \qquad ...(2.2)$$

Since, input is the unit step function

$$\therefore \quad R(s) = \frac{1}{s}$$

Put the value of $R(S)$ in equation (2.2)

$$C(s) = \frac{1}{s(sT+1)} \qquad ...(2.3)$$

After partial fraction equation (2.3) can be written as

$$C(s) = \frac{1}{s} - \frac{T}{1+sT} \qquad ...(2.4)$$

$$\text{or,} \quad C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}} \qquad ...(2.5)$$

Inverse Laplace of equation (2.5)

$$£^{-1} C(s) = £^{-}\frac{1}{s} - £^{-1}\frac{1}{s+1/T}$$

$$c(t) = 1-e^{-t/T} \qquad ...(2.6)$$

When $t = T$ in equation (2.6)

$$c(t) = 1-e^{-T/T} = 1-e^{-1} = 0.632 \text{ or } 63.2\%$$

where, 'T' is known as time constant and it is defined as the time required for the signal to attain 63.2% of final or steady state value. Time constant indicates how fast the system reaches the final value. Smaller the time constant, faster is the system response. A large time constant corresponds to a sluggish system (slow moving).

Since, the output increases exponentially from zero to final value the slope of the curve at $t=0$ is

$$\left.\frac{dc}{dt}\right|_{t=0} = \frac{1}{T}e^{-t/T}$$

$$\text{at } t = 0 \quad \frac{dc}{dt} = \frac{1}{T}e^{o} = \frac{1}{T}$$

From the exponential curve it is clear that, the magnitude of the output response equals to 63.2% of final value in one time constant (T). In two time constant the magnitude of output reaches 86.4% of final value and approximately 98% in four time constant ($4T$). When the actual output reaches within 2% of the desired output it is said that steady state has reached. The time $4T$ is known as setling time (t_s).

The error is given by $\quad e(t) = r(t) - c(t) = 1-(1-e^{-t/T}) = e^{-t/T}$

$$\text{Steady state error} = \lim_{t\to\infty} e^{-t/T} = 0$$

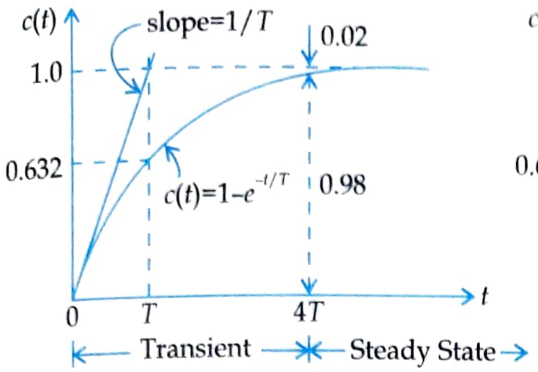

Fig. 2.7. Exponential curve

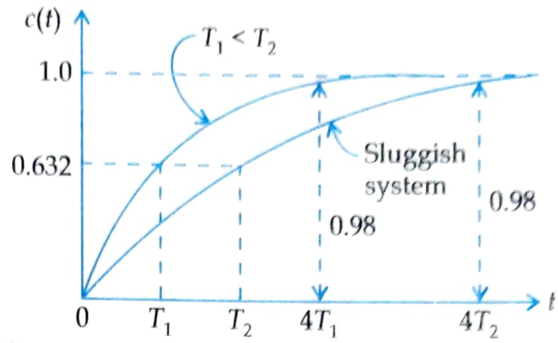

Fig. 2.8. Exponential curves for two systems having diffferent time constants

2.3.2. Response of the First Order System with Unit Ramp Function

Since, input is the unit ramp $R(s) = \frac{1}{s^2}$

Put the value of $R(s)$ in equation (2.2)

$$C(s) = \frac{1}{sT+1} \cdot \frac{1}{s^2}$$

$$C(s) = \frac{1}{s^2(1+sT)} \qquad \text{...(2.7)}$$

After partial fraction equation (2.7) can be written as

$$C(s) = \frac{1-sT}{s^2} + \frac{T^2}{1+sT}$$

or,

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + T \cdot \frac{1}{s+\frac{1}{T}} \qquad \text{...(2.8)}$$

Inverse Laplace of equation (2.8)

$$£^{-1} C(s) = £^{-1}\frac{1}{s^2} - £^{-1}\frac{T}{s} + £^{-1} T \cdot \frac{1}{s+1/T}$$

$$C(t) = t - T + T.\, e^{-t/T} \qquad \text{For} \quad t \geq 0 \qquad \text{...(2.9)}$$

The error signal $e(t)$ will be

$$e(t) = r(t) - c(t) = t - [t - T + Te^{-t/T}]$$

$$e(t) = T(1 - e^{-t/T}) \qquad \text{...(2.10)}$$

$$\text{Steady state error} = \lim_{t \to \infty} (T - T\, e^{-t/T}) = T \qquad \text{...(2.11)}$$

The steady state error is equal to 'T', where 'T' is the time constant of the system. For smaller time constant steady state error will be small and the speed of the response will increase.

2.3.3. Response of the First Order System with Unit Impulse Function

Input is unit impulse function *i.e.*, $R(s) = 1$

Put the value of $R(s)$ in equation (2.2)

$$C(s) = \frac{1}{sT+1}\cdot R(s)$$

$$C(s) = \frac{1}{sT+1}\cdot 1$$

or $$C(s) = \frac{1}{T}\cdot\frac{1}{s+1/T} \qquad \text{...(2.12)}$$

Inverse Laplace of equation (2.12)

$$£^{-1} C(s) = £^{-1}\left[\frac{1}{T}\cdot\frac{1}{s+1/T}\right]$$

$$c(t) = \frac{1}{T}e^{-t/T} \qquad \text{...(2.13)}$$

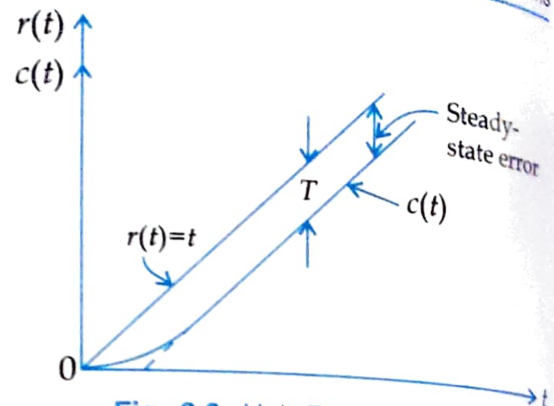

Fig. 2.9. Unit Ramp response of the system

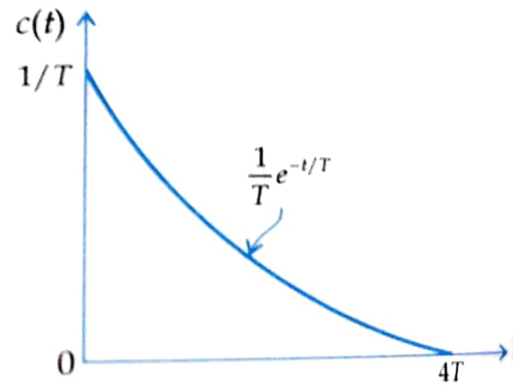

Fig. 2.10. Unit impulse response of the system

The curve for equation (2.13) is shown in Fig. (2.10)
Compare the equations(2.6), (2.9) and (2.13), that is

For unit ramp input $R(s) = \frac{1}{s^2}$ $\quad C(t) = t - T + Te^{-t/T}$

For unit step input $R(s) = \frac{1}{s}$ $\quad C(t) = 1 - e^{-t/T}$

For unit impulse $R(s) = 1$ $\quad C(t) = \frac{1}{T}e^{-t/T}$

From above three equations, it is clear that the unit step input is the derivative of unit ramp input and unit impulse input is the derivative of unit step input. Thus, the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. This is the property of the linear time-invariant systems.

2.4. TIME RESPONSE OF SECOND ORDER SYSTEM

The block diagram of second order control system is shown in Fig. 2.11.

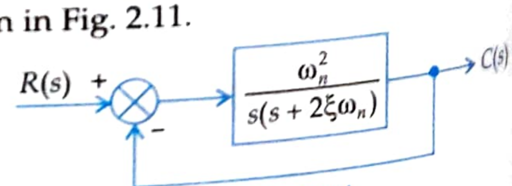

Fig. 2.11.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}, \qquad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\omega_n^2/s(s+2\xi\omega_n)}{1+\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \qquad \text{...(2.14)}$$

2.4.1. Time Response of Second Order System with Unit Step Input

From equation (2.14)

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} . R(s)$$

For unit step input $\quad R(s) = \frac{1}{s}$

$$\therefore \quad C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad ...(2.15)$$

Replace $s^2 + 2\xi\omega_n s + \omega^2_n$ by $(s + \xi\omega_n)^2 + \omega^2_n (1 - \xi^2)$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)} \qquad ...(2.16)$$

Break the equation (2.16) by partial fraction, put $\omega^2_d = \omega^2_n (1 - \xi^2)$

$$\frac{\omega_n^2}{s[(s+\xi\omega_n)^2 + \omega_d^2]} = \frac{A}{s} + \frac{B}{(s+\xi\omega_n)^2 + \omega_d^2} \qquad ...(2.17)$$

Multiply equation (2.17) by s and put $s = 0$

$$\frac{\omega_n^2}{\xi^2\omega_n^2 + \omega_d^2} = A$$

But $\quad \omega^2_d = \omega^2_n(1 - \xi^2) = \omega^2_n - \omega^2_n \xi^2$

$$\therefore \quad A = \frac{\omega_n^2}{\xi^2\omega_n^2 + \omega_n^2 - \xi^2\omega_n^2} = 1 \qquad \therefore \quad A = 1$$

Multiply equation (2.17) both side by $[(s + \xi\omega_n)^2 + \omega^2_d]$ and put

$$s = -\xi\omega_n - j\omega_d$$

$$\frac{\omega_n^2}{s} = B$$

$$B = \frac{\omega_n^2}{-\xi\omega_n - j\omega_d} = \frac{-\omega_n^2}{\xi\omega_n + j\omega_d}$$

$$= \frac{-\omega_n^2(\xi\omega_n - j\omega_d)}{(\xi\omega_n + j\omega_d)(\xi\omega_n - j\omega_d)} = -(\xi\omega_n - j\omega_d)$$

Put $\quad -j\omega_d = s + \xi\omega_n$

$$B = -(\xi\omega_n + s + \xi\omega_n) = -(s + 2\xi\omega_n)$$

Equation 2.16 can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s+\xi\omega_n)^2 + w_d^2} = \frac{1}{s} - \frac{s+\xi\omega_n + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}$$

or

$$C(s) = \frac{1}{s} - \left[\frac{s+\xi\omega_n}{(s+\xi w_n)^2 + \omega_d^2} + \frac{\xi\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2}\right] \qquad ...(2.18)$$

Laplace inverse of equation (2.18)

$$c(t) = 1 - \left[e^{-\xi\omega_n t}.\cos\omega_d t + \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}.\sin\omega_d t \right] \quad ...(2.19)$$

Put $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \cdot \sin\omega_d t \right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}.\cos\omega_d t + \xi\sin\omega_d t \right]$$

Put $\sqrt{1-\xi^2} = \sin\phi$

$\therefore \quad \cos\phi = \xi, \tan\phi = \dfrac{\sqrt{1-\xi^2}}{\xi}$

$$\therefore \quad c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}[\sin\phi \cdot \cos\omega_d t + \cos\phi\sin\omega_d t] = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)$$

Put the values of ω_d and ϕ *i.e.,*

$$\omega_d = \omega_n\sqrt{1-\xi^2}, \phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \right] \quad ...(2.20)$$

The error signal for the system

$$e(t) = r(t) - c(t)$$

$$\therefore \quad e(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \right] \quad ...(2.21)$$

The steady state value of $c(t)$

$$e_{ss} = \lim_{t\to\infty} c(t) = 1$$

Therefore at steady-state there is no error between input and output.

ω_n = natural frequency of oscillation or undamped natural frequency

ω_d = damped frequency of oscillation

$\xi\omega_n$ = damping factor or actual damping or damping coefficient.

(a) Underdamped case ($0 < \xi < 1$)

From the expression (2.20), it is clear that the time constant is $1/\xi\omega_n$ and the response having damped oscillation with overshoot and undershoot. Such response is known as underdamped response. The response is shown in Fig. 2.12.

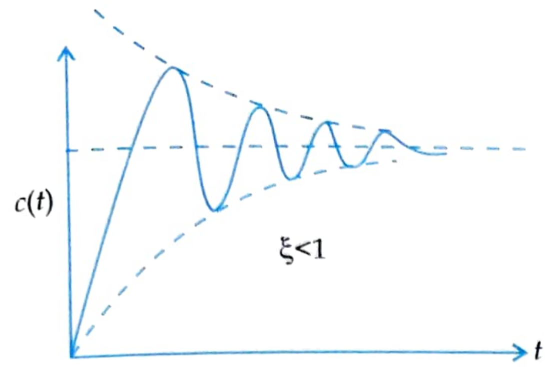

Fig. 2.12. Under damped oscillation

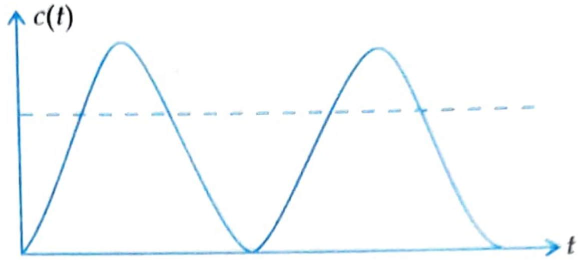

Fig. 2.13. Sustained oscillation

(b) When $\xi = 0$, undamped case

The equation (2.20) will be

$$C(t) = 1 - \sin(\omega_n t + {}^{\pi}/_{2})$$

$$C(t) = 1 - \cos \omega_n t \qquad ...(2.22)$$

Thus at ω_n the system will oscillate (with $\xi = 0$). The damped frequency always less than the undamped frequency (ω_n) because of factor ξ. If the system having certain value of ξ then it is not possible to measure undamped natural frequency experimentally. The observed frequency is the damped frequency (ω_d) which is equal to $\omega_n\sqrt{1-\xi^2}$. The response is shown in Fig. 2.13.

(c) $\xi = 1$ critically damped case :

Put $\xi = 1$ in equation (2.15)

$$C(s) = \frac{1}{s}\cdot\frac{\omega^2{}_n}{s^2+2\omega_n s+\omega^2{}_n}$$

or

$$C(s) = \frac{1}{s}\cdot\frac{\omega_n{}^2}{(s+\omega_n)^2} \qquad ...(2.23)$$

Break the equation (2.23) by partial fraction

$$\frac{\omega_n{}^2}{s(s+\omega_n)^2} = \frac{A}{s}+\frac{B}{s+\omega_n}+\frac{C}{(s+\omega_n)^2} \qquad ...(2.24)$$

Multiply both the sides by $(s + \omega_n)^2$ and put $s = -\omega_n$

$$\frac{\omega^2{}_n}{s} = \frac{A(s+\omega_n)^2}{s} + B(s+\omega_n)+C \qquad ...(2.25)$$

$$\boxed{C = -\omega_n}$$

Multiply both the sides by 's' and put $s = 0$

$$\frac{\omega^2{}_n}{(s+\omega_n)^2} = A+\frac{Bs}{s+\omega_n}+\frac{Cs}{(s+\omega_n)^2}$$

$$\boxed{A = 1}$$

Differentiate equation (2.25) and put $s = -\omega_n$

$$\frac{-\omega_n{}^2}{s^2} = A\,\frac{s+\omega_n}{s} + B$$

$$\boxed{B = -1}$$

$$\therefore \quad \frac{w^2{}_n}{s(s+\omega_n)^2} = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2} \quad \text{...(2.25 a)}$$

Inverse Laplace of equation (2.25)

$$c\,(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2} \quad \text{...(2.26)}$$

$$\pounds^{-1} c\,(s) = \pounds^{-1}\left(\tfrac{1}{s}\right) - \pounds^{-1}\left(\frac{1}{s+\omega_n}\right) - \pounds^{-1}\left(\frac{\omega_n}{(s+\omega_n)^2}\right) \quad \text{...(2.27)}$$

$$\pounds^{-1}\left(\tfrac{1}{s}\right) = 1$$

$$\pounds^{-1}\left(\tfrac{1}{s+\omega_n}\right) = e^{-\omega_n t}$$

Inverse Laplace of $\frac{\omega_n}{(s+\omega_n)^2}$ can be obtained by the method of residues.

$$\pounds^{-1} \frac{\omega_n}{(s+\omega_n)^2} = \frac{1}{\underline{|2-1}} \frac{d}{ds}\left[(s+\omega_n)\, e^{st} \frac{\omega_n}{(s+\omega_n)^2}\right]_{s=-\omega_n}$$

$$= \omega_n\,[e^{st}.t]_{s=-\omega_n}$$

$$= t\omega_n\, e^{-\omega_n t}$$

Put all values in equation (2.27)

$$\therefore \quad \pounds^{-1} C(s) = c\,(t) = 1 - e^{-\omega_n t} - t\,\omega_n e^{-\omega_n t}$$

$$= 1 - e^{-\omega_n t}\,(1 + \omega_n t) \quad \text{...(2.28)}$$

From the equation (2.20), it is clear that $\xi\omega_n$ is the actual damping for $\xi = 1$, the actual damping $= \omega_n$. This actual damping when $\xi = 1$ is known as CRITICAL DAMPING. At the value of critical damping the oscillations just disappeared. The ratio of actual damping to the critical damping is known as damping ratio that is

$$\frac{\text{Actual damping}}{\text{Critical damping}} = \frac{\xi\omega_n}{\omega_n} = \xi$$

From equation (2.28), the time constant of the system $= 1/\xi\omega_n$

The response of the equation (2.28) is shown in Fig. 2.14.

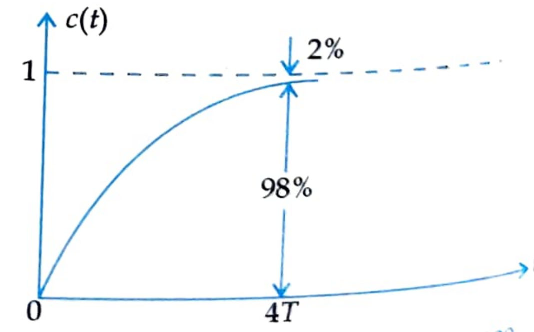

Fig. 2.14. Critically damped response

(d) $\xi > 1$ **overdamped case**

For $\xi > 1$, equation (2.16) can be written as

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n{}^2}{(s+\xi\omega_n)^2 - \omega^2{}_n(\xi^2 - 1)} \quad \text{...(2.29)}$$

put

$$\omega^2{}_d = \omega_n{}^2\,(\xi^2 - 1)$$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n{}^2}{(s+\xi\omega_n)^2 - \omega^2{}_d} \quad \text{...(2.30)}$$

Break the equation (2.30) by partial fraction

Equation (2.30) can be written as

$$C(s) = \frac{\omega^2_n}{s(s+\xi\omega_n+\omega_d)(s+\xi\omega_n-\omega_d)}$$

$$\frac{\omega^2_n}{s(s+\xi\omega_n+\omega_d)(s+\xi\omega_n-\omega_d)} = \frac{A}{s}+\frac{B}{s+\xi\omega_n+\omega_d}+\frac{C}{s+\xi\omega_n-\omega_d} \quad ...(2.31)$$

Multiply both the sides by 's' and put $s = 0$

$$\boxed{A = 1}$$

Multiply both the sides of equation (2.31) by $s + \xi\omega_n + \omega_d$ and put $S = -\xi\omega_n - \omega_d$

$$\frac{\omega_n^2}{s(s+\xi\omega_n-\omega_d)} = B$$

$$\therefore \quad B = \frac{\omega_n^2}{(-\xi\omega_n-\omega_d)(-\xi\omega_n-\omega_d+\xi\omega_n-\omega_d)} = \frac{\omega_n^2}{(\xi\omega_n+\omega_d)(2\omega_d)}$$

Put $\omega^2_d = \omega^2_n(\xi^2-1)$ and simplify for B

$$B = \frac{1}{2\xi\sqrt{\xi^2-1}+2(\xi^2-1)} = \frac{1}{2\xi\sqrt{\xi^2-1}+2\sqrt{\xi^2-1}\sqrt{\xi^2-1}}$$

$$= \frac{1}{2\sqrt{\xi^2-1}\left[\xi+\sqrt{\xi^2-1}\right]}$$

Similarly, $$C = \frac{-1}{2\sqrt{\xi^2-1}\left[\xi-\sqrt{\xi^2-1}\right]}$$

Put the values of A, B and C in equation (2.31)

$$C(s) = \frac{1}{s}+\frac{1}{2\sqrt{\xi^2-1}\left[\xi+\sqrt{\xi^2-1}\right][s+\xi\omega_n+\omega_d]}-\frac{1}{2\sqrt{\xi^2-1}\left[\xi-\sqrt{\xi^2-1}\right][s+\xi\omega_n-\omega_d]}$$

Put the value of ω_d.

$$C(s) = \frac{1}{s}+\frac{1}{2\sqrt{\xi^2-1}\left[\xi+\sqrt{\xi^2-1}\right]\left[s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right]} - \frac{1}{2\sqrt{\xi^2-1}\left[\xi-\sqrt{\xi^2-1}\right]\left[s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{1}{s}+\frac{1}{2\sqrt{\xi^2-1}\left[\xi+\sqrt{\xi^2-1}\right]\left[s+\omega_n\left(\xi+\sqrt{\xi^2-1}\right)\right]} - \frac{1}{2\sqrt{\xi^2-1}\left[\xi-\sqrt{\xi^2-1}\right]\left[s+\omega_n\left(\xi-\sqrt{\xi^2-1}\right)\right]} \quad ...(2.32)$$

Inverse Laplace of equation (2.32)

$$C(t) = 1+\frac{e^{-\left(\xi+\sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)} - \frac{e^{-\left(\xi-\sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)} \quad ...(2.33)$$

From equation (2.33) we get two time constant

$$T_1 = \frac{1}{\left(\xi + \sqrt{\xi^2 - 1}\right)\omega_n}, \quad T_2 = \frac{1}{\left(\xi - \sqrt{\xi^2 - 1}\right)\omega_n}$$

From equation (2.33) we observe that when ξ is greater than one there are two exponential term, the first term has a time constant T_1 which is smaller than the time constant of other exponential term (having time constant T_2), in other words we can say that the first exponential term decaying much faster than other exponential term. So, for time response we can neglect it, then

$$C(t) = 1 - \frac{e^{-\left(\xi - \sqrt{\xi^2-1}\right)\omega_n t}}{2\sqrt{\xi^2 - 1}\left(\xi - \sqrt{\xi^2 - 1}\right)} \quad \text{...(2.34)}$$

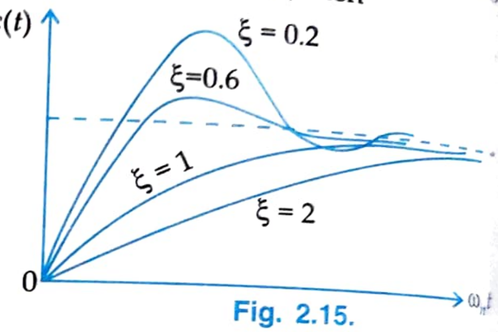

Fig. 2.15.

and time constant

$$= \frac{1}{(\xi - \sqrt{\xi^2 - 1})\omega_n} \quad \text{...(2.35)}$$

For different values of ξ the curves of $C(t)$ is shown in Fig. 2.15.

Form the curve it is clear that the overdamped systems are sluggish.

2.4.2. Location of Roots of Characteristic Equation and Time Response

The characteristic equation of

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

is $\quad s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ ...(2.36)

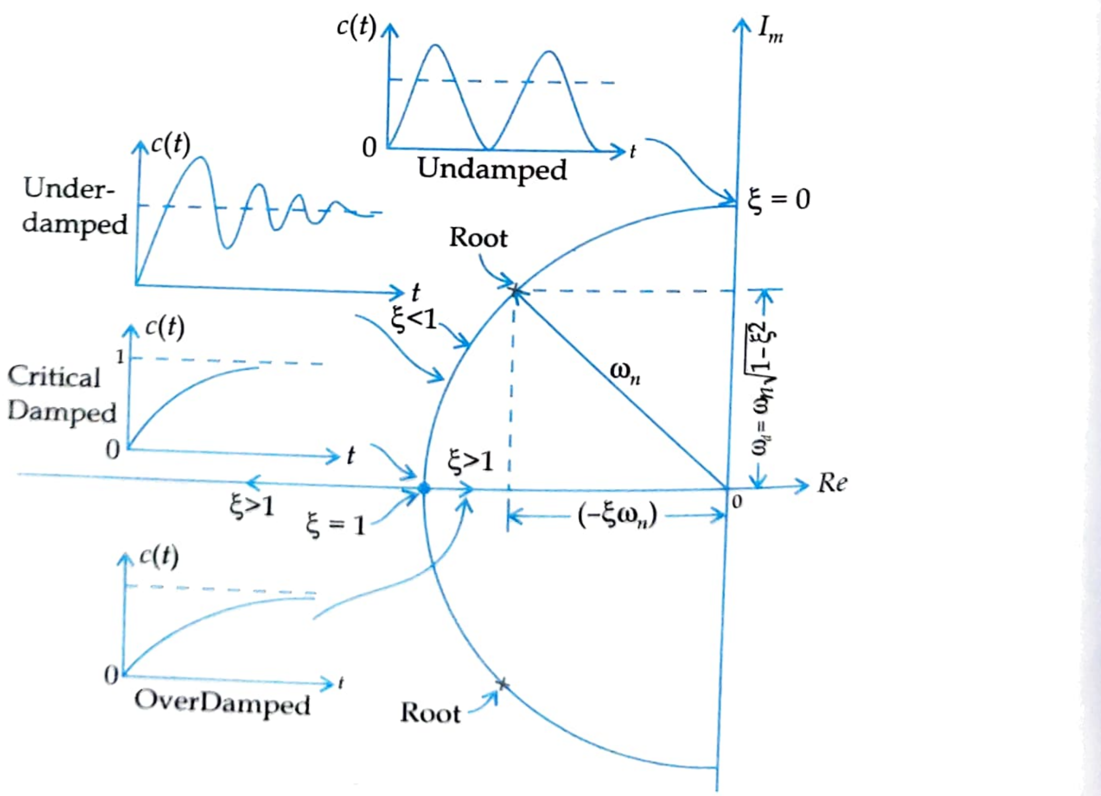

Fig. 2.16.

The roots of equation (2.36)

$$S_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$S_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$$

The real part of the roots ($-\xi\omega_n$) represents the damping and imaginary part $\left(\omega_n\sqrt{1-\xi^2}\right)$ represents the damped frequency.

From the Fig. 2.16

1. ω_n is the distance of root from origin.
2. $\cos\theta = \dfrac{\xi\omega_n}{\omega_n} = \xi$ *i.e.*, $\theta = \cos^{-1}\xi$ (when the roots are in left-half of *s*-plane).
3. ω_d is the imaginary part of the root and is known as damped frequency or conditional frequency.

2.5. TRANSIENT RESPONSE SPECIFICATIONS OF SECOND ORDER SYSTEM

The performance of a control system are express in terms of the transient response to a unit step input because it is easy to generate. The transient response of a control system to a unit step input depends upon the initial conditions. Consider a second order system with unit step input and the system initially at rest *i.e.*, all initial conditions are zero. The following are the common transient response characteristics.

1. Delay time (t_d)
2. Rise time (t_r)
3. Peak time (t_p)
4. Maximum overshoot (M_p)
5. Settling time (t_s)
6. Steady-state error (e_{ss})

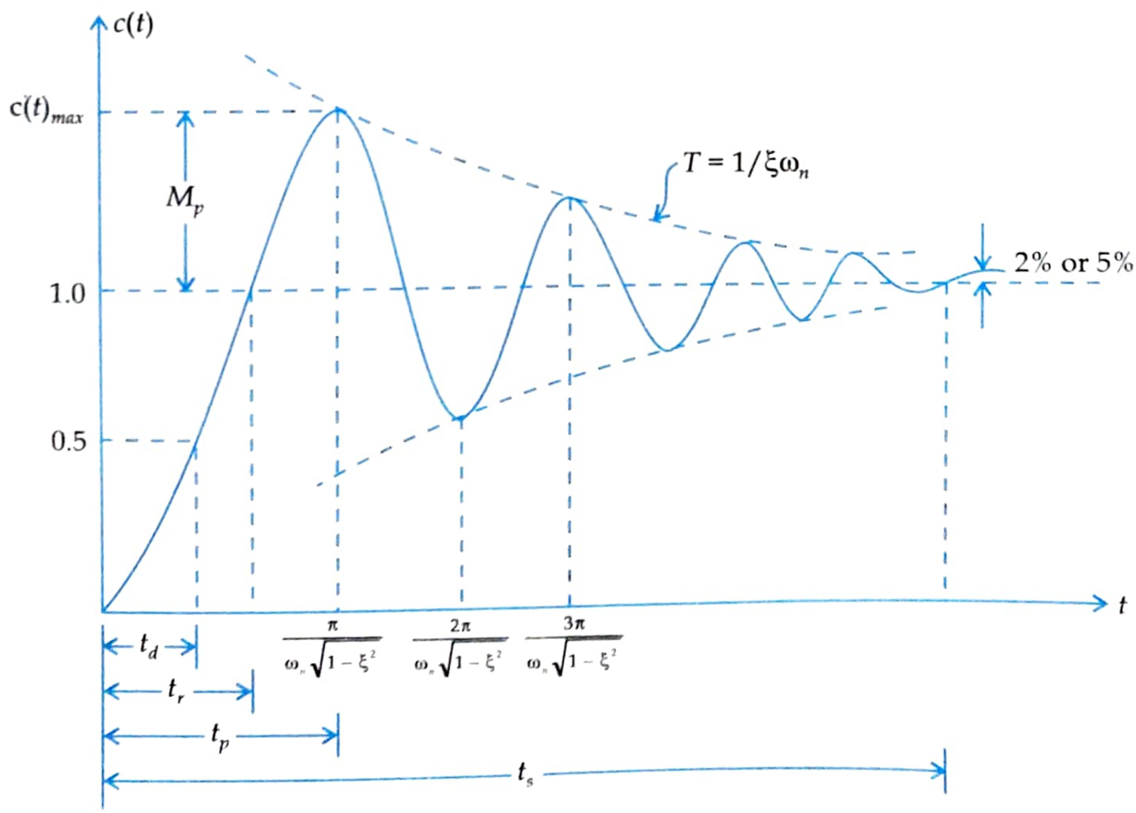

Fig. 2.17.

1. **Delay Time (t_d) :** The delay time is the time required for the response to reach 50% of the final value in first time.
2. **Rise Time (t_r) :** It is the time required for the response to rise from 10% to 90% of its final value for overdamped systems and 0 to 100% for underdamped systems.
3. **Peak Time (t_p) :** The peak time is the time required for the response to reach the first peak of the time response or first peak overshoot.
4. **Maximum Overshoot (M_p) :** It is the normalized difference between the peak of the time response and steady output. The maximum percent overshoot is defined by

$$\text{Maximum percent overshoot} = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100$$

5. **Settling Time (t_s) :** The settling time is the time required for the response to reach and stay within the specified range (2% to 5%) of its final value.
6. **Steady State Error (e_{ss}) :** It is the difference between actual output and desired output as time 't' tends to infinity.

$$e_{ss} = \lim_{t \to \infty}[r(t) - C(t)]$$

Expression for Rise Time (t_r) :

From the equation (2.20)

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]$$

where
$$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

Let response reaches 100% of desired value. Put $C(t) = 1$

$$1 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]$$

or
$$\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] = 0$$

Since, $e^{-\xi\omega nt} \neq 0$

$$\therefore \quad \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] = 0, \text{ or } \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] = \sin n\pi$$

Put $n = 1$

$$\therefore \quad \left(\omega_n\sqrt{1-\xi^2}\right)t_r + \phi = \pi$$

or
$$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\xi^2}}$$

or
$$\boxed{t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}} \quad ...(2.37)$$

Expression for Peak Time t_p :

Since, $$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]$$

For maximum $\frac{dC(t)}{dt} = 0$

$$\frac{dC(t)}{dt} = \frac{-e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\omega_n\sqrt{1-\xi^2} + \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\frac{\xi\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t} \quad ...(2.38)$$

Since $e^{-\xi\omega_n t} \neq 0$

Equation (2.38) can be written as

$$\cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\sqrt{1-\xi^2} = \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\xi \quad ...(2.39)$$

Put $\sqrt{1-\xi^2} = \sin\phi$ and $\xi = \cos\phi$

Equation (2.39) becomes

$$\cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\sin\phi = \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\cos\phi$$

or, $$\frac{\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]}{\cos\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]\sin\phi} = \frac{\sin\phi}{\cos\phi}$$

or, $$\tan\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right] = \tan\phi$$

the time to various peaks

$$\left(\omega_n\sqrt{1-\xi^2}\right)t_p = n\pi$$

where $n = 0, 1, 2, 3, ...$

Maximum overshoot identified by putting $n = 1$, therefore the peak time to the first overshoot

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \qquad \text{or,} \quad \omega_n\sqrt{1-\xi^2}\,t + \phi = n\pi + \phi$$

$$\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}} \quad ...(2.40)$$

The first minimum (undershoot) occurs at $n = 2$

or, $$\omega_n\sqrt{1-\xi^2}\,t = n\pi$$

$$t_p = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$t_{min} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} \quad \text{...(2.41)}$$

Expression for Maximum Overshoot M_p :

$$c(t) = 1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t+\phi\right] \quad \text{...(2.42)}$$

Maximum overshoot occurs at peak time *i.e.*, $t = t_p$

Put $\quad t = t_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$ in equation (2.42)

$$c(t) = 1-\frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}\sin\left[\omega_n\sqrt{1-\xi^2}\cdot\frac{\pi}{\omega_n\sqrt{1-\xi^2}}+\phi\right]$$

$$= 1-\frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}\sin(\pi+\phi) \quad \text{...(2.43)}$$

Since $\quad \phi = \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$ then $\quad \sin\phi = \sqrt{1-\xi^2}$ and $\sin(\pi+\phi) = -\sin\phi$

$$\therefore \quad c(t) = 1+\frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}}(\sin\phi) \qquad \text{put } \sin\theta = \sqrt{1-\xi^2}$$

$$c(t)_{max} = 1+e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$\therefore \quad M_P = 1(t)_{max} - 1$$

$$M_P = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\% M_P = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100 \quad \text{....(2.44)}$$

Settling Time t_s :

As shown in the Fig. 2.17, the curves for $1 \pm \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$ are the envelope curves of the transient response for unit step input. The time constant of these envelope curves is $\dfrac{1}{\xi\omega_n}$. The speed of the decay depends upon the time constant. The settling time for a second order system is approximately four times the time constant $(1/\xi\omega_n)$

$$\therefore \quad t_s = \frac{4}{\xi\omega_n} \quad \text{...(2.45)}$$