CHAPTER 8

State Space Analysis of Control Systems

CHAPTER OUTLINES

■ Analysis of Systems; ■ Advantages of State Space Techniques; ■ Some Important Definitions; ■ State Space Representation; ■ Solution of the Time-invariant State Equation; ■ Transfer Matrix; ■ Computation of State Transition Matrix : e^{At}; ■ Block Diagram of a Linear System in State Variable Form; ■ Controllability and Observability; ■ Time Varying System; ■ Controllability and Observability (Time Variant Systems); ■ Similarity Transformation; ■ State Space Representation of Transfer Function Systems; ■ Decomposition Transfer Function; ■ Effect of Pole-zero Cancellation in Transfer Function

8.1. ANALYSIS OF SYSTEMS

The procedure for determining the state of a system is called state variable analysis. The state of a dynamic system is the smallest set of variables such that the knowledge of these variables at $t = t_0$ with the knowledge of the input for $t \geq t_0$ completely determines the behaviour of the system for any time $t \geq t_0$. This set of variables is called state variables.

In earlier chapters we studied the linear system by transfer function, block diagram etc. The transfer function has some drawbacks *e.g.*, transfer function is only defined under zero initial conditions and also it is applicable to linear time invariant systems.* Therefore due to these limitation state variable approach is developed. This technique can be used for analysis and design of linear and non-linear, time invariant or time variant and multi-input multi-output systems.** The state space analysis involves the description of the system in terms of *I* order differential equations by selecting suitable state variables, the first order derivatives are arranged on left hand side and on right hand side the terms are free from derivatives. The state space techniques have many advantages (Given in next article *i.e.*, 8.2).

8.2. ADVANTAGES OF STATE SPACE TECHNIQUES

This technique has the following advantages :

1. This approach can be applied to linear or non-linear, time variant or time invariant systems.
2. It is easier to apply where the Laplace transform cannot be applied.
3. n^{th} order differential equations can be expressed as 'n' equation of first order whose solutions are easier.
4. It is a time domain approach.

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* If the characteristic of a system does not change with time, then the system is said to be time invariant.

** A system is said to be a single variable system if and only if it has only one input terminal and only one output terminal. A system is said to be multivariable system if and only if it has more than one input terminal or more than one output terminal.

5. This method is suitable for digital computer computation because this is a time domain approach.
6. The system can be designed for optimal conditions with respect to given performance indices.

8.3. SOME IMPORTANT DEFINITIONS

State: The state of a system at any time 't_0' is the minimum set of numbers $x_1, x_2, \ldots, x_n$ which along with the input to the system for time $t \geq t_0$ is sufficient to determine the behaviour of the system for all $t \geq t_0$. In other words, the state of a system represents the minimum amount of information that we need to know about a system at "t_0" such that its future behaviour can be determined without reference to the input before 't_0'. The state can also be defined as the state of a system at time t_0 is the amount of information at t_0, that, together with input $u(t_0, \infty)$ determines the unique behaviour of the system for all $t \geq t_0$. By the behaviour of the system, we mean all responses, including the state of the system. If the system is a network we mean the voltage and current of every branch of the network.

Consider the network shown in Fig. 8.1 if the initial current through the inductor and initial voltage across the capacitor are known, then for any driving voltage the behaviour of the network can be determined. Hence, the inductor current and capacitor voltage can be considered as the state of the network.

Fig. 8.1.

State Variables : The definition is given in Article 8.1.

State Vector : If we need n variables to completely describe the behaviour of a given system, then these n state variables may be considered as n component of a vector x. Such a vector is called state vector, A state vector is thus a vector which determines uniquely the system state $x(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

State space : The n-dimensional space whose coordinate axes consists of the x_1 axis, x_2 axis ......, x_n axis is called state space. Any state can be represented by a point in the state space.

8.3.1. State Space Equations

Consider the following R-C Networks.

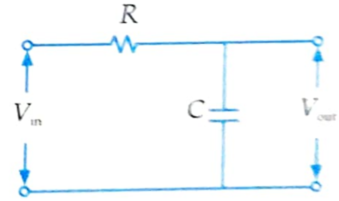

Fig. 8.2.

For analysis of network shown in Fig. 8.2., initial conditions must be known. If we are interested to find the output voltage, the initial capacitor voltage must be known, only input voltage is not sufficient. The systems in which output depends upon input as well as on initial conditions are called *Dynamic system* or *Systems with memory*. So, the dynamic system must have elements that memorize the values of input for $t \geq 0$.

The systems in which output depends upon only applied input are called *Static systems* or *zero memory systems*.

For state space analysis three variables are involved : input variable, output variables and state variables. The input variables are represented by $u_1(t)$, $u_2(t)$, $u_3(t)$ .... $u_m(t)$, output variables are represented by $y_1(t)$, $y_2(t)$, $y_3(t)$ .... $y_p(t)$ and state variables are represented by $x_1(t)$, $x_2(t)$, $x_3(t)$ .... $x_n(t)$.

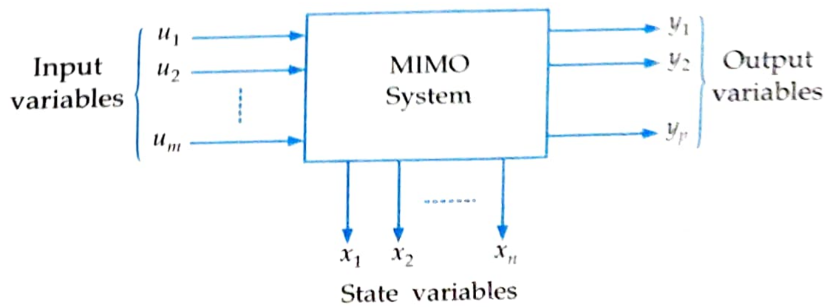

Fig. 8.3.

Consider multiple input multiple output, n^{th} order system shown in Fig. 8.3.

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

Input vector $u(t)$, output vector $y(t)$ and state vector $x(t)$ having orders $(m \times 1)$, $(p \times 1)$ and $(n \times 1)$ respectively.

The state variable represetation can be arranged in the form of n-first order differential equations.

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n;\ u_1, u_2, \ldots u_m) \\ \frac{dx_2}{dt} &= \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n;\ u_1, u_2, \ldots u_m) \\ &\vdots \\ \frac{dx_n}{dt} &= \dot{x}_n = f_n(x_1, x_2, \ldots, x_n;\ u_1, u_2, \ldots u_m) \end{aligned}\right\} \quad \ldots(8.1)$$

where, 'f' is the functional operator.

Integrating above equation

$$x_i(t) = x_i(0) + \int_{t_0}^{t} f_i(x_1, x_2, \ldots x_n;\ u_1, u_2, \ldots u_m)\, dt$$

$$i = 1, 2, 3, \ldots, n$$

Thus, if each state variable is known at $t = t_0$ and all control inputs are known, then n state variables, hence state vector at any time 't' can be determined uniquely.

The 'n' differential equations for time-invariant systems may be written as,

$$\dot{x}(t) = f(x(t), u(t))$$

For time varying systems, the function 'f' is dependent on time, the equation may be written as

$$\dot{x}(t) = f(x(t), u(t), t)$$

For linear time-invariant systems, the state equations are a set of first-order differential equations and the functional equations can be expressed in terms of linear combination of system state and inputs *i.e.*,

$$\left.\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots\ldots + b_{1m}u_m \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots\ldots + b_{2m}u_m \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots\ldots + b_{nm}u_m \end{aligned}\right\} \quad \ldots(8.2)$$

For linear time-invariant systems the coefficients a_{ij} and b_{ij} are constant. Thus all equations can be written as,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where, 'A' is system matrix or evolution matrix of order $n \times n$ and B is input matrix or control matrix of order $n \times m$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

Similarly, the output variables at time t are the linear combination of input and state variables at time t. Then

$$\left.\begin{aligned} y_1(t) &= c_{11}x_1(t) + c_{12}x_2(t) + \ldots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + \ldots + d_{1m}u_m(t) \\ &\vdots \\ y_p(t) &= c_{p1}x_1(t) + c_{p2}x_2(t) + \ldots + c_{pn}x_n(t) + d_{p1}u(t) + d_{p2}u_2(t) + \ldots + d_{pm}u_m(t) \end{aligned}\right\} \quad \ldots(8.3)$$

where, the coefficient c_{ij} and d_{ij} are constant then,

$$y(t) = C\,x(t) + D\,u(t)$$

$$C = \text{Output or observation matrix of order } p \times n = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}$$

$$D = \text{Transmission matrix of order} = p \times m = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$$

The two vector equations together is called state model of linear time-invariant systems.

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{state equation}$$
$$y(t) = Cx(t) + Du(t) \qquad \text{output equation}$$

For linear time-variant systems

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

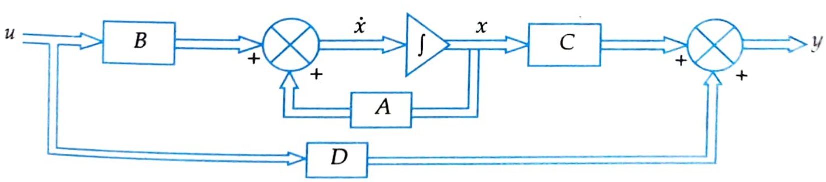

Fig. 8.4. Block diagram of state model of MIMO system

8.3.2. State Model for Single Input Single Output System

In MIMO system, if $m = 1$ and $n = 1$, we get state model for single input single output linear system.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + du(t)$$

where,

A = $n \times n$ matrix B = $n \times 1$ matrix C = $1 \times n$ matrix

d = constant u = scalar control variable

8.4. STATE SPACE REPRESENTATION

8.4.1. State Space Representation for Electrical Network (Physical Variable Form)

Consider an *RLC* network shown in Fig. 8.5. Let, the current at time $t = 0$ be $i_L(0)$ and capacitor voltage at time $t = 0$ be $V_c(0)$. Thus, the state of the network at time $t = 0$ is specified by the inductor current and capacitor voltage. Hence, the pair $i_L(0)$, $V_c(0)$ is called the initial state of the network. Similarly at time 't', the pair $i_L(t)$, $V_c(t)$ is called the state of the network at 't'. The variable i_L and V_c are called state variables of the network.

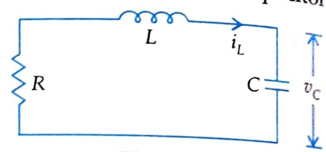

Fig. 8.5.

Apply KVL

$$Ri_L + L\frac{di_L}{dt} + V_c = 0 \qquad \text{...(8.4)}$$

Also,
$$i_c = i_L = C\frac{dV_c}{dt} \qquad \text{...(8.5)}$$

From equation (8.4)
$$\frac{di_L}{dt} = -\frac{R}{L}i_L - \frac{1}{L}v_C \qquad \text{...(8.6)}$$

$$\frac{dv_C}{dt} = \frac{1}{C}i_L \qquad \text{...(8.7)}$$

Equations of this form are called state equations. In such equations all the variables present are state variables.

Equations (8.6) and (8.7) can be written in matrix form as

$$\begin{bmatrix} \dfrac{di_L}{dt} \\ \dfrac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} \qquad \text{...(8.8)}$$

This equation gives the rate of change of state variables. These equations are called state equations.

$$y(t) = v_c \qquad \text{...(8.9)}$$

In matrix form,
$$y(t) = [1 \quad 0]\begin{bmatrix} v_c \\ 0 \end{bmatrix} \qquad \text{...(8.10)}$$

This equation is known as output equation.

The state equation and output equation together are called dynamic equations of the system. They are known as state model of the system.

From equation (8.8)

$$A = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix}, \quad x(t) = \begin{bmatrix} i_L \\ v_C \end{bmatrix}, \quad \dot{x}(t) = \begin{bmatrix} \dfrac{di_L}{dt} \\ \dfrac{dv_C}{dt} \end{bmatrix}$$

Consider another network

Here, v_C and i_L are state variables (in Fig. 8.6).

Apply KVL in mesh (1)

$$-i_2 R_1 + v_C = 0$$

$$v_C = i_2 R_1 \quad \ldots(8.11)$$

Apply KCL in mesh (2)

$$-L\frac{di_L}{dt} - R_2 i_L - v_C = 0$$

$$\frac{di_L}{dt} = -\frac{R_2}{L} i_L - \frac{1}{L} v_C \quad \ldots(8.12)$$

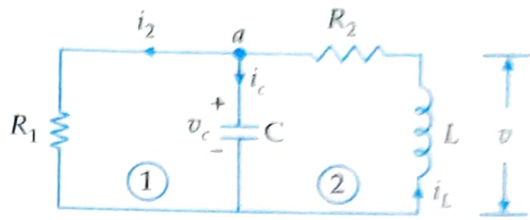

Fig. 8.6.

or,

Apply KCL at node 'a'

$$i_2 + i_c = i_L$$

$$i_2 = i_L - i_c = i_L - C\frac{dv_C}{dt} \quad \ldots(8.13)$$

∴

Put the value of i_2 in equation (8.11) we get

$$v_c = i_L R_1 - R_1 C\frac{dv_C}{dt}$$

∴

$$\frac{dv_C}{dt} = \frac{1}{C} i_L - \frac{1}{R_1 C} v_C \quad \ldots(8.14)$$

From equation (8.12) and (8.14)

$$\begin{bmatrix} \dfrac{di_L}{dt} \\ \dfrac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_2}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{R_1 C} \end{bmatrix} = \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

This is the required state equation.

Now select current i_o through inductor L and voltage v_o across inductor L as the output variables, then

$$i_o = i_L$$

$$v_o = v$$

Then,

$$\begin{bmatrix} i_o \\ v_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ v \end{bmatrix}.$$

If the state variables are selected as i_L is x_1, v_c is x_2, i_o is y_1 and v_o is y_2 then the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R_2}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{R_1 C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

8.4.2. State Space Representation of *n*th Order Differential Equations

Consider the following examples :

(*a*) For *n*th Order Differential Equation

EXAMPLE 8.1. A system is described by the differential equation

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 10y = 8\,u(t)$$

where y is the output and u is the input to the system. Obtain state space representation of the system.

(R.M.L., University, Faizabad, 2003)

Solution : Select the state variables as

$$x_1 = y,\ x_2 = \dot{y} \text{ and } x_3 = \ddot{y} \text{ then}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 8\,u(t) - 10\,x_1 - 11x_2 - 6x_3$$

The last equation is obtained from the given equation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u(t) \qquad \text{...(8.15)}$$

Compare equation (8.15) with equation (8.9) we get

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix},\ x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(*b*) State Space Representation of n^{th} Order Linear System with *r* Forcing Function

Consider the following example :

EXAMPLE 8.2. A system is described by the following differential equation. Represent the system in state space.

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = u_1(t) + 3u_2(t) + 4\,u_3(t)$$

and outputs are :

$$y_1 = 4\frac{dx}{dt} + 3u_1$$

$$y_2 = \frac{d^2x}{dt^2} + 4u_2 + u_3$$

Solution : Select the state variables as

$$x_1 = x$$

$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u_1(t) + 3u_2(t) + 4u_3(t) - 3x_3 - 4x_2 - 4x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Outputs :

$$y_1 = 4x_2 + 3u_1$$
$$y_2 = x_3 + 4u_2 + u_3$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

8.4.3. State Space Representation Using Phase Variables

In this method of state space represention phase variables are state variables. Phase variables are one of the system variables and its derivatives. Generally, system output is the system variable and rest of the variables are then derivatives of the output. A link can be established through phase variables between transfer function design and time domain design.

Consider general form of n^{th} order linear continuous-time system

$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots + a_n y = b_0 u^n + b_1 u^{n-1} + \dots + b_n u$.

The transfer function can be represented as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

They are two type of cases :

Case 1 : Transfer Function has no Zeros

Consider the transfer function

$$G(s) = \frac{k}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

Let the first state variable x_1 equal to output variable y. Second state variable x_2 equal to the first derivative of output variable and so on.

$$x_1 = y$$
$$x_2 = \dot{x}_1 = \dot{y}$$
$$x_3 = \dot{x}_2 = \ddot{y}$$
$$\vdots$$
$$x_n = y^{(n-1)}$$

This can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \dots - a_1 x_n + bu$$

In vector form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ b \end{bmatrix} u$$

$$\dot{x} = Ax + Bu$$

From the matrix A we can see that the last row is negative coefficients of differential equation and remaining elements are zero. Such matrix A is known as bush form or companion form.

Since $y = x_1$, the output equation can be written as

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$y = Cx(t)$$

Here, $C = [1 \;\; 0 \; .... \; 0]$

Case 2 : Transfer Function having Both Poles and Zeros

Let, $$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + ... + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n}$$

Break the above equation in two parts

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Y(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)}$$

where, $$\frac{X_1(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n}$$

$$\frac{Y(s)}{X_1(s)} = b_0 s^n + b_1 s^{n-1} + ... + b_{n-1} s + b_n$$

$\frac{X_1(s)}{U(s)}$ is without zeros, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

For $\frac{Y(s)}{X_1(s)}$

$$\begin{aligned} y &= b_0 x^n + b_1 x_1^{n-1} + ... + b_n x_1 \\ &= b_0 (-a_n x_1 - a_{n-1} x_2 .... - a_1 x_n + u) + b_1 x_n + b_2 x_{n-1} + ... + b_n x_1 \\ &= (b_n - a_n b_0)\, x_1 + (b_{n-1} - a_{n-1} b_0) x_2 .... + b_0 u \end{aligned}$$

Output equation can be written as

$$y = [(b_n - a_n b_0)\ (b_{n-1} - a_{n-1}\, b_0)\ \ldots\ ]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Consider the following example.

EXAMPLE 8.3. For the given transfer function, obtain the state model.

$$G(s) = \frac{y(s)}{u(s)} = \frac{K}{s^3 + a_3 s^2 + a_2 s + a_1}$$

Solution : This transfer function has no zeros.

$$(s^3 + a_3 s^2 + a_2 s + a_1)\, y(s) = Ku(s)$$

or

$$s^3 y(s) + a_3 s^2 y(s) + a_2 s\, y(s) + a_1 y(s) = Ku(s)$$

Taking inverse Laplace

$$\dddot{y}(t) + a_3\, \ddot{y}(t) + a_2\, \dot{y}(t) + a_1\, y(t) = K\, u(t)$$

or

$$\dddot{y}(t) = Ku(t) - a_3\, \ddot{y}(t) - a_2\, \dot{y}(t) - a_1 y(t)$$

Select the state variables as, first state variable as output

$$y(t) = x_1$$
$$\dot{y}(t) = \dot{x}_1 = x_2$$
$$\ddot{y}(t) = \dot{x}_2 = x_3$$
$$\dddot{y}(t) = \dot{x}_3$$

$\therefore$

$$\dddot{y}(t) = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku(t)$$

Rewriting the equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u(t)$$

$$y(t) = [1\ 0\ 0]\ x_1(t)$$

BLOCK DIAGRAM :

The block diagram of the given transfer function is shown in Fig. 8.7.

Now consider another case when the transfer function has zeros.

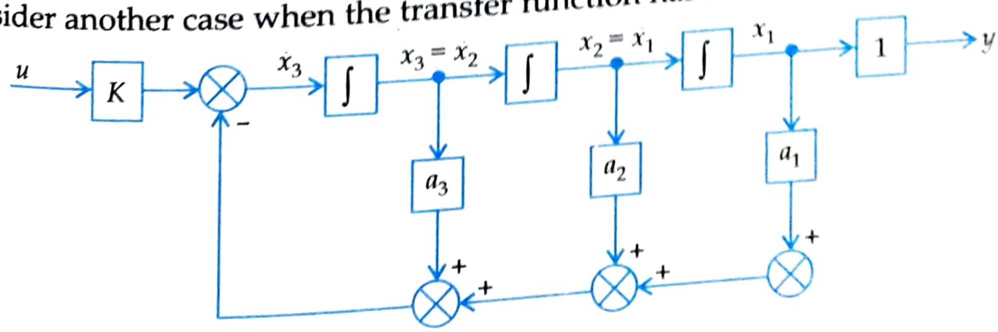

Fig. 8.7.