Proof

Let G be a valid flow network with capacity constraints and let f be a flow in G that satisfies the conservation constraints. Let A be a subset of vertices in G.

We want to show that the total flow out of A is equal to the total flow into A.

Summing over all edges e with both ends in A, we have:

$$\sum_e f(e) = \sum_e f(e)$$

Re-arranging, we get:

$$\sum_e f(e) = \sum_e f(e)$$

Now, consider edges e with one end in A:

- Case 1: Edge e is entirely in A, so both ends are in A.
- Case 2: Edge e is leaving A, so it starts at a vertex in A and ends at a vertex in $V \setminus A$.
- Case 3: Edge e is entering A, so it starts at a vertex in $V \setminus A$ and ends at a vertex in A.
- Case 4: Edge e is entirely in $V \setminus A$, so it does not appear in the sum.

If edge $e = (a, b)$ falls under Case 1, then $f(e)$ appears on the LHS when b is the vertex and $f(e)$ also appears on the RHS when a is the vertex.

Hence:

$$\sum_{e \text{ into } A} f(e) = \sum_{e \text{ within } A} f(e)$$

Therefore,

$$\sum_{e \text{ into } A} f(e) = 2 \sum_e f(e) + \sum_e f(e)$$

This shows that the total flow into A is equal to the total flow out of A.