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Numerical Methods: Assignment 6
Fixed Point Iteration

1. a) Let $f(p) = 0$. We get:

$$p^4 + 2p^2 - p - 3 = 0$$
$$p^4 + 2p^2 = p + 3$$
$$p^2(p^2+2) = p+3$$
$$p^2 = \frac{p+3}{p^2+2}$$
$$\pm p = \pm\sqrt{\frac{p+3}{p^2+2}}$$

The positive version of this corresponds to $g(p) = p$. Hence, when $f(p) = 0$, we get that p is a fixed point of $g(x)$.

b) $$P_1 = g(P_0) = \left(\frac{1+3}{3}\right)^{1/2} = \frac{2}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{3}}$$

$$P_2 = \left(\frac{\frac{2\sqrt{3}}{3}+3}{\left(\frac{2\sqrt{3}}{3}\right)^2+2}\right)^{1/2} = \left(\frac{\frac{2\sqrt{3}+9}{3}}{\frac{12}{9}+2}\right)^{1/2} = \left(\frac{\frac{2\sqrt{3}+9}{3}}{\frac{10}{3}}\right)^{1/2}$$

$$= \boxed{\left(\frac{2\sqrt{3}+9}{10}\right)^{1/2}}$$

$$P_3 = \left(\frac{\left(\frac{2\sqrt{3}+9}{10}\right)^{1/2}+3}{\frac{2\sqrt{3}+9}{10}+2}\right)^{1/2} = \left(\frac{\frac{\sqrt{2\sqrt{3}+9}+3\sqrt{10}}{\sqrt{10}}}{\frac{2\sqrt{3}+29}{10}}\right)^{1/2}$$

$$= \left(\frac{\sqrt{10}\cdot\left(\sqrt{2\sqrt{3}+9}+3\sqrt{10}\right)}{2\sqrt{3}+29}\right)^{1/2} = \boxed{\left(\frac{\sqrt{20\sqrt{3}+90}+30}{2\sqrt{3}+29}\right)^{1/2}}$$

$$P_0 = 1,\ P_1 = \frac{2\sqrt{3}}{3},\ P_2 = \left(\frac{2\sqrt{3}+9}{10}\right)^{1/2},\ P_3 = \left(\frac{\sqrt{20\sqrt{3}+90}+30}{2\sqrt{3}+29}\right)^{1/2}$$

2. From calculator, I know that ~~in that~~ when $x \in [0, 2\pi]$, maximum of $g(x)$ is around 3.642 and minimum of $g(x)$ is π. So, we have: that when $x \in [0, 2\pi]$, $g(x) \in [0, 2\pi]$ and $g(x)$ is continous. From these facts, we can conclude that $g(x)$ has at least one fixed point in $[0, 2\pi]$.

$g(x) = \pi + \frac{1}{2}\sin(x/2) \Rightarrow g'(x) = \frac{1}{4}\cos(\frac{x}{2})$.

Maximum of $|g'(x)|$ in $x \in [0, 2\pi]$ is $\frac{1}{4}$.

So, $\exists\, k \in (0,1)$ s.t. $|g'(x)| \leq k$ when x is in the interval. Hence, by the theorem we learned in class, $g(x)$ has a unique fixed point on $[0, 2\pi]$.

3. Normally, if we have to estimate how fast one of these sequences will converge to the fixed point within a given error tolerance, we would use the formula, $\epsilon \leq k^n \frac{|P_1 - P_0|}{1-k}$. Since no error tolerance is provided, I will assume we are calculating for infinite precision. In that case, we just need the value of k, b/c that value has the biggest weight on the expression for

$n \to \infty$. So, the lower the k is, the faster, $k^n \frac{|P_1 - P_0|}{1-k}$, will converge to 0.

a) $g(x) = \dfrac{20x + \frac{20}{x^2}}{21}$

$$|g'(x)| = \frac{20 - \frac{43}{x^3}}{21}$$

We will use the interval $x \in [2,3]$, b/c $2^3 < 21 < 3^3 \Rightarrow 2 < 21^{1/3} < 3$.
Max of $|g'(x)|$ when $x \in [2,3]$ is around 0.878307. k must be larger than that, but since we are just comparing, we can round $\underline{k \approx 0.87307}$.

b) $g(x) = x - \dfrac{x^3 - 21}{3x^2}$

Max of $|g'(x)|$ when $x \in [2,3]$, is around 92. Estimate $k \approx 92$.

c) $g(x) = x - \dfrac{x^4 - 21x}{x^2 - 21}$

Max of $|g'(x)|$ when $x \in [2,3]$, is 9.
$\underline{k \approx 9}$

d) $g(x) = \left(\frac{21}{x}\right)^{1/2}$; Max of $|g'(x)|$ when $x \in [2,3]$ is around 0.81009. $\underline{k \approx 0.81009}$

In order from fastest convergence to slowest (lowest k value to highest k):
(d), (a), (c), (b).

Notes: To find the max of a function in an interval, I used:

In Mathematica

→ FindMaximum [Abs[function'[x]], a ≤ x ≤ b, {x, _}]

where a and b are the edges of the interval and I replace _ with my guess for the x-value of the max of |function'[x]| in $x \in [a,b]$.