Homework 3
Problem 2

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$$\tilde{p}(x_n) = \sum_{x_1} \dots \sum_{x_{n-1}} \sum_{x_{n+1}} \dots \sum_{x_N} \psi_{1,2}(x_1,x_2) \dots \psi_{N-1,N}(x_{N-1},x_N) =$$

$$= \sum_{x_1} \dots \sum_{x_{n-1}} \sum_{x_{n+1}} \dots \sum_{x_{N-1}} \psi_{1,2}(x_1,x_2) \times \dots \times \psi_{N-1,N-2}(x_{N-1},x_{N-2}) \underbrace{\sum_{x_N} \psi_{N,N-1}(x_N,x_{N-1})}_{\psi_B(x_{N-1})} =$$

at this point we know that x_n gets forward and backward messages that we shall denote as μ_α and ψ_β respectively. In addition, the chain has a linear (repeating) pattern of connections, thus we will be able to use induction to generalize our conclusions.

$$= \sum_{x_1} \dots \sum_{x_{n-1}} \sum_{x_{n+1}} \dots \sum_{x_{N-2}} \psi_{1,2}(x_1,x_2) \times \dots \underbrace{\sum_{x_{N-1}} \psi_{N-1,N-2}(x_{N-1},x_{N-2}) \cdot \mu_\beta(x_{N-1})}_{\psi_\beta(x_{N-2})}$$

Using induction, we can see that the backward message that will transmit information to x_n will be $\mu_\beta(x_n)$, as the chain has repeating structure.

Now we show the same principle for forward messages:

$$\sum_{x_2} \underbrace{\sum_{x_1} \psi_{1,2}(x_1,x_2)}_{\mu_\alpha(x_2) \leftarrow \text{forward message}} \sum_{x_3} \dots \sum_{x_{N-2}} \times \dots \times \psi_{N-2,N-3}(x_{N-2},x_{N-3}) \cdot \mu_\beta(x_{N-2}) =$$

$$= \underbrace{\sum_{x_2} \psi_{2,3}(x_2,x_3)\, \mu_\alpha(x_2)}_{\mu_\alpha(x_3)} \sum_{x_3} \dots \sum_{x_{n-1}} \sum_{x_{n+1}} \dots \sum_{x_{N-2}} \psi_{N-2,N-3}(x_{N-2},x_{N-3})\, \mu_\beta(x_{N-2})$$

Again, using induction we can see that the last forward message to x_n will be $\mu_\alpha(x_n)$
Now, remembering that potential functions can be unnormalized we normalize the distr to get:

$$p(x_n) = \frac{1}{Z} \mu_\alpha(x_n)\, \psi_\beta(x_n)$$, and by generalizing messages we get:

$$\mu_\alpha(x_k) = \sum_{x_{k-1}} \psi_{k-1,k}(x_{k-1},x_k)\, \mu_\alpha(x_{k-1}) \qquad \psi_\beta(x_k) = \sum_{x_{k+1}} \psi_{k+1,k}(x_{k+1},x_k)\, \psi_\beta(x_{k+1})$$

Problem 3

№1

1) $$\tilde{p}(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4} \tilde{p}(x_1,x_2,x_3,x_4) = \sum_{x_2}\sum_{x_3} f_a(x_1,x_2)\, \underset{(x_2,x_3)}{f_b} \overbrace{\sum_{x_4} f_c(x_2,x_4)}^{\mu_{c\to x_2}(x_2)} =$$

$$= \sum_{x_2} f_a(x_1,x_2) \underbrace{\sum_{x_3} f_b(x_2,x_3)}_{\mu_{b\to x_2}(x_2)} \mu_{c\to x_2}(x_2) = \underbrace{\sum_{x_2} f_a(x_1,x_2) \overbrace{\mu_{b\to x_2}(x_2)\,\mu_{c\to x_2}(x_2)}^{\mu_{x_2\to a}(x_2)}}_{\mu_{a\to x_1}(x_1)} = \mu_{a\to x_1}(x_1)$$

2) One could observe that x_1 and x_3 have symmetric cases, and thus will have sim. results.

$$\tilde{p}(x_3) = \sum_{x_2}\sum_{x_1}\sum_{x_4} \tilde{p}(x_1,x_2,x_3,x_4) = \sum_{x_1}\sum_{x_2} f_a(x_1,x_2)\, f_b(x_2,x_3) \underbrace{\sum_{x_4} f_c(x_2,x_4)}_{\mu_{c\to x_2}(x_2)} =$$

$$= \sum_{x_2} f_b(x_2,x_3) \overbrace{\sum_{x_1} f_a(x_1,x_2)}^{\mu_{a\to x_1}(x_1)} \mu_{c\to x_2}(x_2) = \mu_{b\to x_3}(x_3)$$

№2

$$\tilde{p}(x_1,x_2) = \sum_{x_3}\sum_{x_4} \tilde{p}(x_1,x_2,x_3,x_4) = \sum_{x_3}\sum_{x_4} f_a(x_1,x_2)\, f_b(x_2,x_3)\, f_c(x_2,x_4) =$$

$$= f_a(x_1,x_2) \underbrace{\sum_{x_3} f_b(x_2,x_3)}_{\mu_{b\to x_2}(x_2)} \underbrace{\sum_{x_4} f_c(x_2,x_4)}_{\mu_{c\to x_2}(x_2)} = f_a(x_1,x_2)\, \mu_{b\to x_2}(x_2)\, \mu_{c\to x_2}(x_2) = f_a(x_1,x_2)\,\mu_{x_2\to a}(x_2)$$

This conclusion is supported by 8.72.