Theoretical Backgrounds of Audio & Graphics

Affine Transformations

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Recap

- The properties of **linear transformations**
  - Origin remains fixed
  - Lines remain parallel
  - Ratios of distances remain equal
- Matrix multiplication can simplify the overall computational processes substantially
- Matrix multiplication is not commutative!
- **How do we involve translation?**

Translation

To move, or *translate*, an object by shifting all its points the same amount, we need a transform of the form,

$$\begin{aligned} x' &= x + x_t, \\ y' &= y + y_t. \end{aligned}$$

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There is no way to do that by multiplying (x, y) by a 2×2 matrix.

Translation

- How to combine linear transformations (rotation, scaling, shearing, …) with translation (vector addition by a point) into **one matrix**?

Homogeneous Coordinates

Homogeneous Coordinates

The idea is simple:

- Represent the point (x, y) by a 3D vector $[x\ y\ 1]^{\mathrm{T}}$ and use 3 × 3 matrices of the form

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The following are all the same point

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 3x \\ 3y \\ 3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} \frac{x}{2} \\ \frac{y}{2} \\ \frac{1}{2} \end{bmatrix}$$

“Dehomogenisation”

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \end{bmatrix}$$

Homogeneous Coordinates

The extra (third) coordinate will be either 1 or 0 depending on whether we are encoding a position or a direction. Now we can distinguish between locations and other vectors.

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$ is a location and

$$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$ is a direction or displacement

Homogeneous Coordinates

In 3D a fourth coordinate is added accordingly:

$$\begin{bmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + x_t \\ y + y_t \\ z + z_t \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

$$\mathrm{translate}(x, y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Rotation (in 2D) around point $\mathbf{c} = (c_x, c_y)$ with angle $\boldsymbol{\phi}$

Steps: $\mathbf{T}(-\mathbf{c}) \rightarrow \mathbf{R_z}\ \phi \rightarrow \mathbf{T}(\mathbf{c})$

$$\begin{pmatrix} 1 & 0 & c_y \\ 0 & 1 & c_x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -c_y \\ 0 & 1 & -c_x \\ 0 & 0 & 1 \end{pmatrix}$$

Homogeneous Coordinates

Now, a single matrix can implement a linear transformation followed by a translation.

Rotation Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

Linear and Affine Transformations

- Properties of linear transformations
  - Lines remain parallel
  - Origin remains fixed
  - Ratios of distances remain equal
- Properties of **affine transformations**
  - Composition of linear functions adding translation
  - Origin does not remain fixed

References

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