Theoretical Backgrounds of Audio & Graphics

Affine Transformations

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Recap

- The properties of linear transformations
 - Origin remains fixed
 - Lines remain parallel
 - Ratios of distances remain equal
- Matrix multiplication can simplify the overall computational processes substantially
- Matrix multiplication is not commutative!
- How do we involve translation?

Translation

To move, or *translate*, an object by shifting all its points the same amount, we need a transform of the form,

$$\begin{array}{rcl}
x' & = & x & + & x_t, \\
y' & = & y & + & y_t.
\end{array}$$

Translation

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\end{array}$$

There is no way to do that by multiplying (x, y) by a 2 × 2 matrix.

Translation

 How to combine linear transformations (rotation, scaling, shearing, ...) with translation (vector addition by a point) into one matrix?

The idea is simple:

• Represent the point (x, y) by a 3D vector $[x \ y \ 1]^T$ and use 3 × 3 matrices of the form

$$egin{bmatrix} m_{11} & m_{12} & x_t \ m_{21} & m_{22} & y_t \ 0 & 0 & 1 \end{bmatrix} egin{bmatrix} x \ y \ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

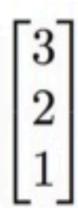
The following are all the same point

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 3x \\ 3y \\ 3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} \frac{x}{2} \\ \frac{y}{2} \\ \frac{1}{2} \end{bmatrix}$$

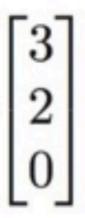
"Dehomogenisation"

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$
 \longrightarrow $\begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \end{bmatrix}$

The extra (third) coordinate will be either 1 or 0 depending on whether we are encoding a position or a direction. Now we can distinguish between locations and other vectors.



is a location and



is a direction or displacement

In 3D a fourth coordinate is added accordingly:

$$\begin{bmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + x_t \\ y + y_t \\ z + z_t \\ 1 \end{bmatrix}$$

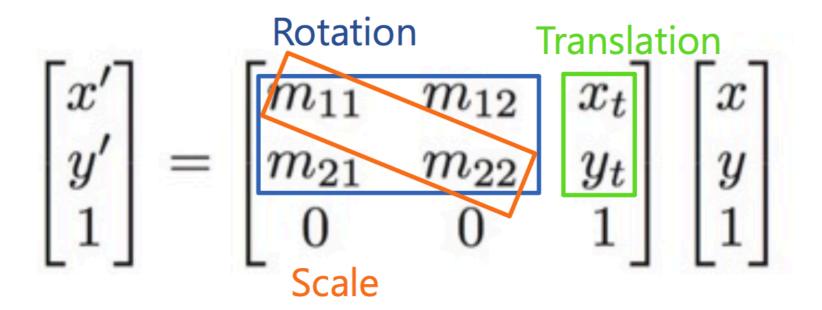
translate
$$(x, y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Rotation (in 2D) around point $\mathbf{c} = (c_x, c_y)$ with angle $\boldsymbol{\phi}$

Steps: $\mathbf{T}(-\mathbf{c}) \rightarrow \mathbf{R_z} \phi \rightarrow \mathbf{T}(\mathbf{c})$

$$\begin{pmatrix} 1 & 0 & c_y \\ 0 & 1 & c_x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -c_y \\ 0 & 1 & -c_x \\ 0 & 0 & 1 \end{pmatrix}$$

Now, a single matrix can implement a linear transformation followed by a translation.



Linear and Affine Transformations

- Properties of linear transformations
 - Lines remain parallel
 - Origin remains fixed
 - Ratios of distances remain equal
- Properties of affine transformations
 - Composition of linear functions adding translation
 - Origin does not remain fixed

References

- Buss, S. (2003): 3D Computer Graphics—A Mathematical introduction with OpenGL.
 Cambridge University Press, New York, NY, USA.
- Angel, E. and Shreiner, D. (2012): Interactive Computer Graphics—A Top-down Approach with Shader-based OpenGL. Addison-Wesley, USA.
- Shiffman, D. (2012): Nature of Code. https://natureofcode.com/book/
- Lecture slides
 - Gieseke, L. Mathematics for Audio & Graphics, 2017/18, Film University
 - **Note:** If not indicated differently, all images, equations, and calculations have been taken from Lena Gieseke's slides.
 - Jarosz, W., Computer Graphics, Fall 2016, Dartmouth College
 - Marschner S., Computer Graphics, Spring 2018, Cornell University
 - Schlechtweg, S. (2010/11), "Computergraphik Grundlagen", Hochschule Anhalt (FH)
 Köthen.