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Theoretical Backgrounds of Audio & Graphics

Additional Material: Maths Notes

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Motivation

Practical

- Mathematical models and representations are central ingredient of digital audio & graphics processing — to understand them, know their maths

General

- Learn to use mathematics as a formalistic tool to describe & share ideas
- Identify the underlying concepts in different application scenarios
- Develop an intuition for the language of mathematics

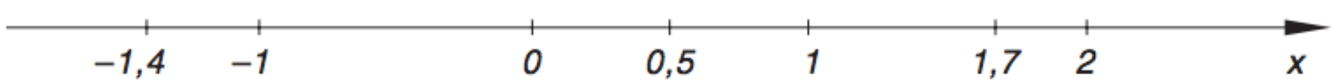
Contents

- The real numbers system
- Sequences & series
- Functions & their properties

The Real Numbers System

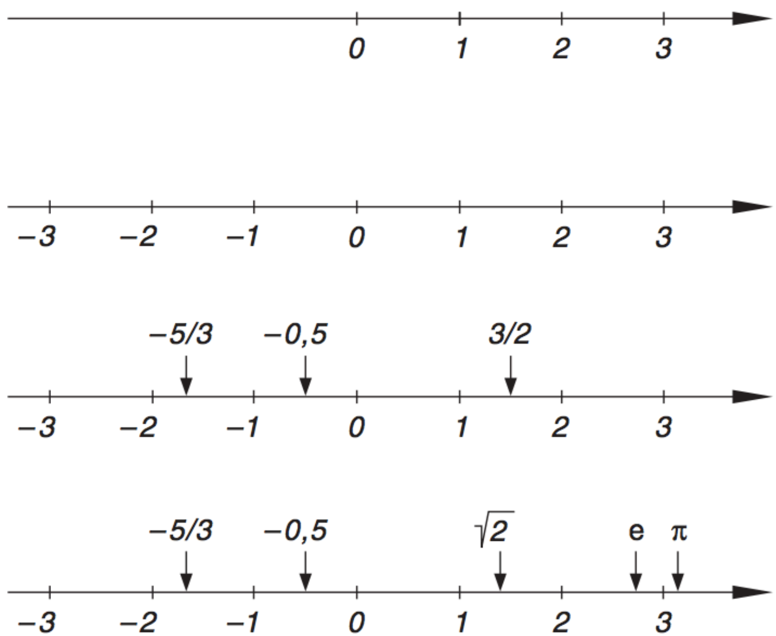
- The real numbers system is defined by a set of numbers ***R*** and a set of operations & properties on ***R***
  - Addition “+”
  - Subtraction “—“
  - Multiplication “ * ”
  - Division “ : ”

- Geometric representation by points on the real axis



Specific Subsets

- Subsets of numbers of the real numbers system



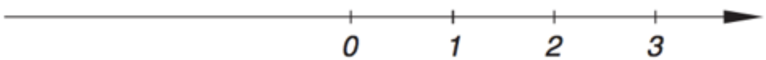
Natural numbers
$\mathbb{N} = \{0, 1, 2, 3, ...\}$ $\quad \mathbb{N}^* = \{1, 2, 3, ...\}$

Integers
$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

Rational numbers
$\mathbb{Q} = \left\{x \,\middle|\, x = \dfrac{a}{b} \text{ with } a \in \mathbb{Z} \;\&\; b \in \mathbb{N}^*\right\}$

Real numbers
$\mathbb{R} = \{x \mid x \text{ is a rational or an irrational number}\}$

Natural Numbers



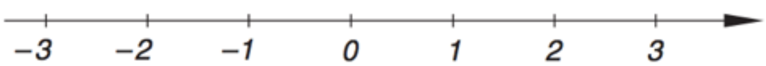
Natural numbers

$\mathbb{N} = \{0, 1, 2, 3, ...\}$ $\quad$ $\mathbb{N}^* = \{1, 2, 3, ...\}$

- Natural numbers are
  - **closed** under addition & multiplication
    - *Adding* natural numbers yields a natural number
    - *Multiplying* natural numbers yields a natural number
  - **not closed** under subtraction & division
    - $\boldsymbol{a} = 2\text{-}5$ — $\boldsymbol{a}$ does not exist in the set of natural numbers
    - $\boldsymbol{b} = 3/6$ — $\boldsymbol{b}$ does not exist in the set of natural numbers

Integers

The Real Numbers System



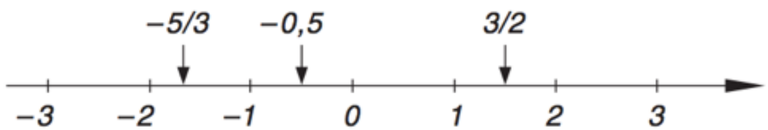
Integers

$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

- Integers are
  - **closed** under addition, subtraction & multiplication
    - *Adding* integers yields an integer
    - *Subtracting* integers yields an integer
    - *Multiplying* natural numbers yields a natural number
  - **not closed** under division
    - $\boldsymbol{b} = 3/6$ — $\boldsymbol{b}$ does not exist in the set of integers

Rational Numbers

The Real Numbers System



Rational numbers

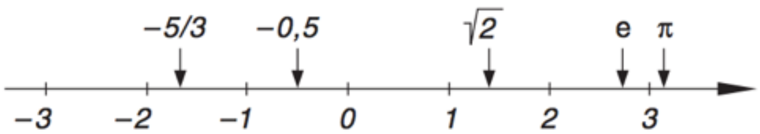
$$\mathbb{Q} = \left\{ x \,\middle|\, x = \frac{a}{b} \text{ with } a \in \mathbb{Z} \;\&\; b \in \mathbb{N}^* \right\}$$

- Rational numbers are
  - **closed** under addition, subtraction, multiplication & division
  - **ratios of integers**, i.e., ***b*** = *3/6* is the ratio of integers *3* & *6*
- Rational numbers can be expressed as **fractions** or **decimals**
  - The decimal expansion either terminates or repeats periodically:

$$\frac{17}{10} = 1{,}7 \qquad \frac{1}{7} = 0{,}142857\ 142857\ \ldots$$

Irrational & Real Numbers

The Real Numbers System



Real numbers
$\mathbb{R} = \{x \mid x \text{ is a rational or an irrational number}\}$

- Irrational numbers are
  - **non-rational** numbers —
    <u>no ratio</u> of any two integers can yield an irrational number
- The decimal expansion of an irrational number **need not** <u>terminate</u> **nor** <u>repeat periodically</u>
- Famous examples of irrational numbers $\sqrt{2} = 1{,}41423...$

$$\pi = 3{,}14159...$$

$$e = 2{,}71828...$$

Irrational Numbers—Example

The Real Numbers System

- The diagonale of a square with side length $\mathbf{1}$ can only be described with the irrational number $\sqrt{2}$
  - Set of Pythagoras: $1^2+1^2 = 2$
  - Length of diagonale: $\sqrt{2}$

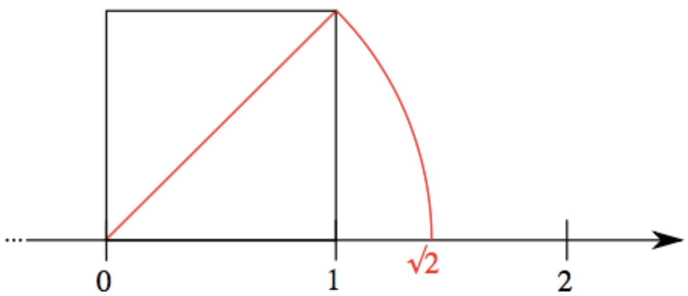


Image credit [1]

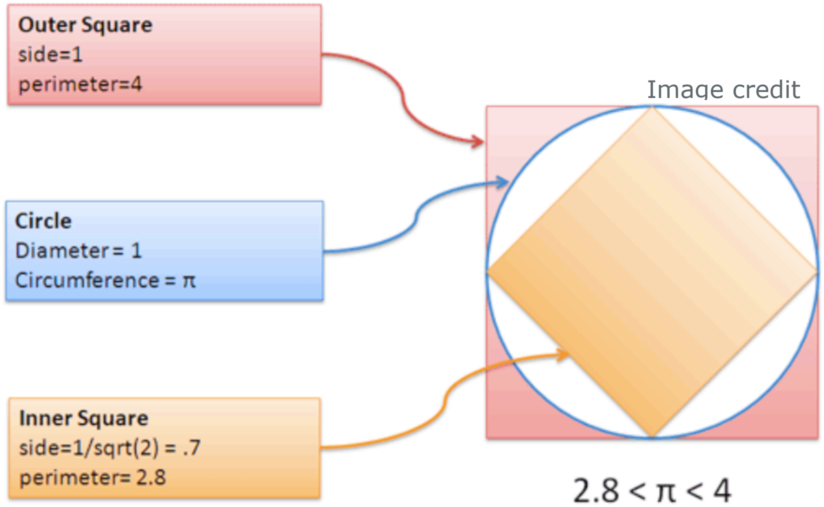
- $\sqrt{2}$ can not be represented by the ratio of two integers — it can only be approached

$$
\begin{array}{lrcccl}
a_n & 1 & < \sqrt{2} < & 2 & b_n \\
a_{n+1} & 1{,}4 & < \sqrt{2} < & 1{,}5 & b_{n+1} \\
a_{n+2} & 1{,}41 & < \sqrt{2} < & 1{,}42 & b_{n+2} \\
a_{n+3} & 1{,}414 & < \sqrt{2} < & 1{,}415 & b_{n+3} \\
 & & \dots & &
\end{array}
$$

Irrational Numbers—Example

The Real Numbers System

- The circumference of a circle can only be described with the irrational number π
- π can be approached, for example, following Archimedes' algorithm:
  - surround the circle with a square
  - place another square inside the circle which has a diagonale equal to the side of the surrounding square
  - calculate the perimeters
  - increase the sides of the inner and outer squares and repeat



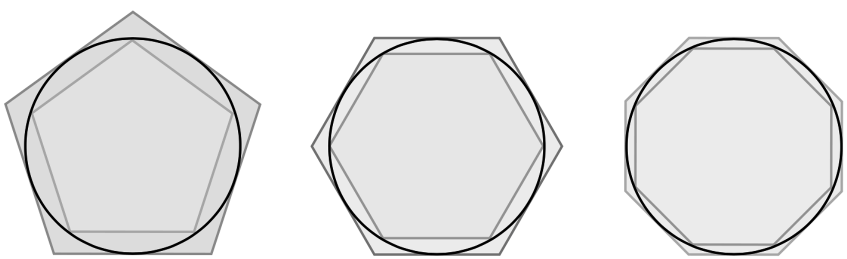


Image credit

Irrational Numbers—Example

- Irrational numbers can be characterized by **nested intervals**
    - Infinite sequences of intervals
    - Each interval is contained in the preceding one
    - The irrational number is approached consecutively

Take Home

- The real numbers system is a formal description to express
  - Existence & non-existence of objects (**natural numbers**)
  - Lack of objects (**integers**)
  - Ratios & fractions of objects (**rational numbers**)
  - Approximations of objects (**real numbers**)

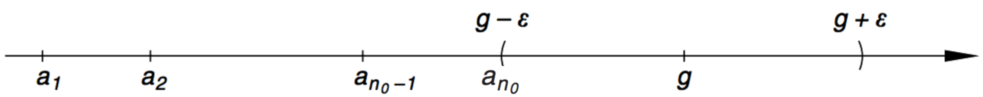
Sequences

Sequences & series

- A **sequence *a*** is a set of numbers, its **terms,** in a definite order of arrangement or **pattern** of the sequence

$$\langle a_n \rangle = a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots \qquad n \in \mathbb{N}^* \quad a_n \in \mathbb{R}$$

- The **limit *g*** of a sequence is a number to which all terms converge



$$|a_n - g| < \varepsilon \text{ for all } n \geq n_0$$

$$\lim_{n\to\infty} a_n = g$$

the pattern of the sequence

$$\langle a_n \rangle = \left\langle 1 - \frac{1}{n} \right\rangle = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \quad \Rightarrow \quad g = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$$

- An (infinite) **series *s*** is a <u>special kind of sequence</u> of the **partial sums** of an (infinite) sequence

$$\langle a_n \rangle = a_1, a_2, a_3, \ldots, a_n, \ldots \qquad n \in \mathbb{N}^* \quad a_n \in \mathbb{R}$$

$$\langle s_n \rangle = s_1, \quad s_2, \quad s_3, \quad s_4, \quad \ldots \qquad n \in \mathbb{N}^* \quad s_n \in \mathbb{R}$$

Partial sums:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\vdots$$

E.g., the nth partial sum is a finite series: $s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k \Rightarrow s_n = \sum_{k=1}^{n} a_k$

$$\vdots$$

(Infinite) series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

Series — Example

Sequences & series

- The **limit** of a series is the value of its sum ***s*** — if one exists

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = s \qquad \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots = s$$

The geometric series $\sum_{k=0}^{\infty} \frac{1}{2^k}$ tends towards ***2***

$$\begin{aligned}
S_1 &= \textstyle\sum_{k=0}^{0} \frac{1}{2^k} &&= 1 &&= 1\\
S_2 &= \textstyle\sum_{k=0}^{1} \frac{1}{2^k} &&= 1 + \tfrac{1}{2} &&= 1{,}5\\
S_3 &= \textstyle\sum_{k=0}^{2} \frac{1}{2^k} &&= 1 + \tfrac{1}{2} + \tfrac{1}{4} &&= 1{,}75\\
S_4 &= \textstyle\sum_{k=0}^{3} \frac{1}{2^k} &&= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= 1{,}875\\
S_5 &= \textstyle\sum_{k=0}^{4} \frac{1}{2^k} &&= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} &&= 1{,}9375\\
&\vdots
\end{aligned}$$

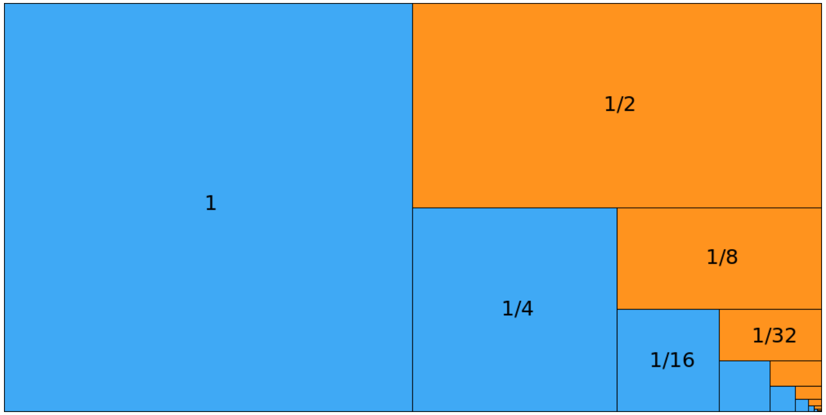


Image credit [4]

Series — Example

- The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

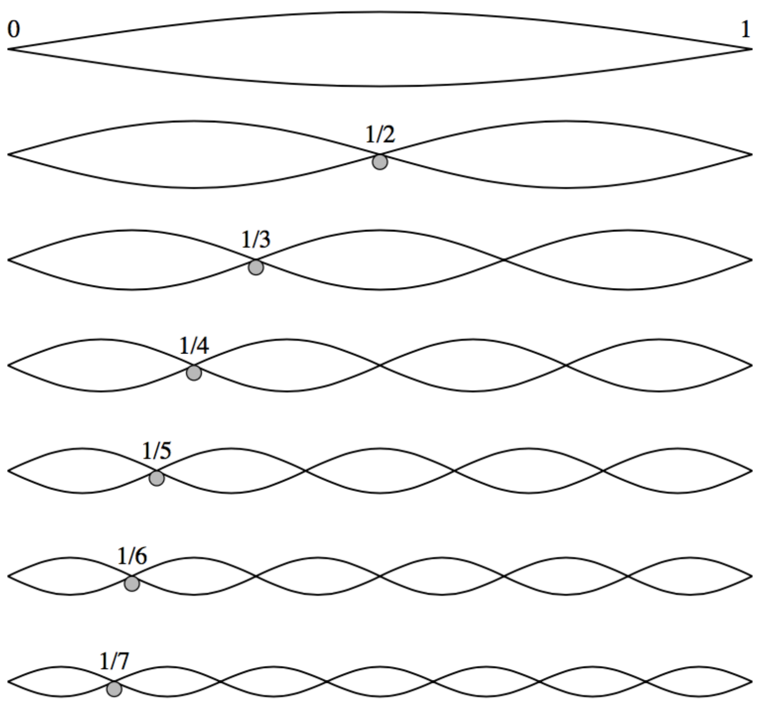


Image credit: https://en.wikipedia.org/wiki/Harmonic_series_(music) (17/10/16)

Take Home

- Sequences allow to
  - **relate numbers** through a **pattern**
  - make assumptions about how the sequence will or will not continue (in the case of a **limit**)
- Series allow to
  - **approximate a value** through a partial sum
  - express how a value is composed of several values

Functions

- Functions are used to clearly describe the functional dependency / relationship between (sets of) numbers

- Sequences & series can also be expressed as **functions**
- Every positive integer $n \in \mathbb{N}^*$ is assigned a real number $a_n \in \mathbb{R}$

$$\langle a_n \rangle = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$$

$$a_n = 1 - \frac{1}{n}$$

$$\Rightarrow \quad a_n = f(n) = \frac{n-1}{n} = 1 - \frac{1}{n} \qquad (n \in \mathbb{N}^*)$$

Functions

- Functions are basically composed of
  - a **domain** & co-domain with elements represented by $\boldsymbol{x}$ & $\boldsymbol{y}$
  - a **range** or image with elements represented by $\boldsymbol{y}$ or $\boldsymbol{f(x)}$
  - a rule of correspondence or **mapping** $f\colon x \mapsto y = f(x)$ or $y = f(x)$

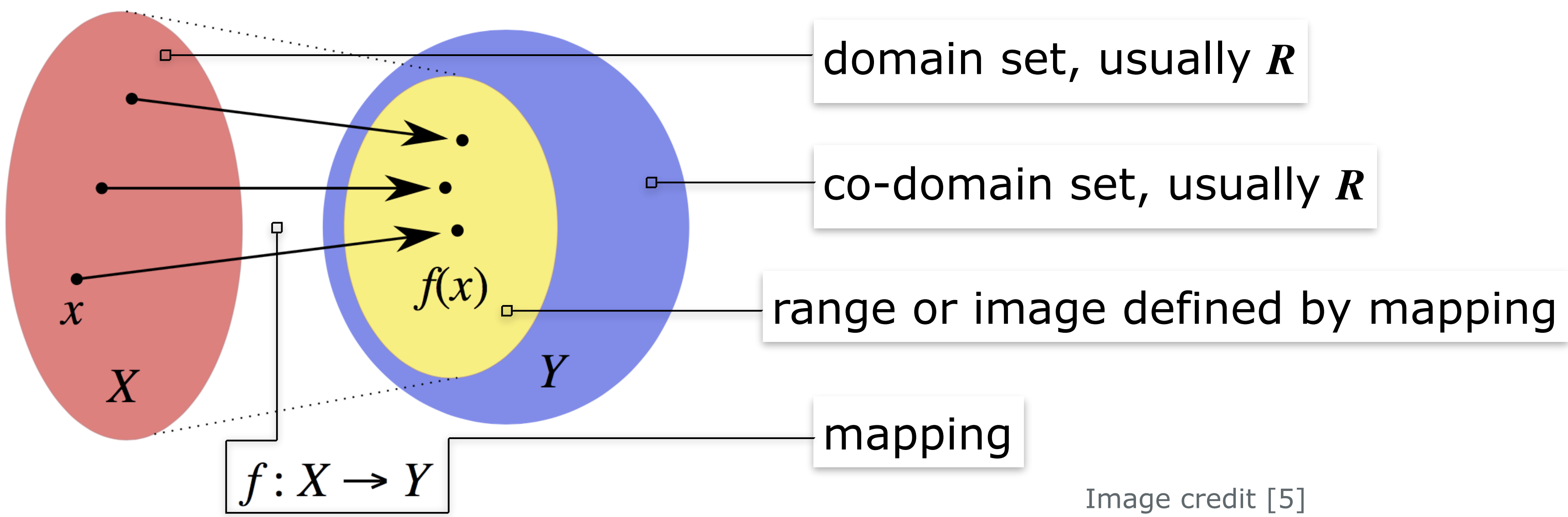


Image credit [5]

Graphical Representations

Functions

- The functional relationship between x & $y = f(x)$ can be visualized graphically with a function graph
- Graphs help us to immediately see specific **curve points** e.g., maxima, minima, zeros, turning points, poles

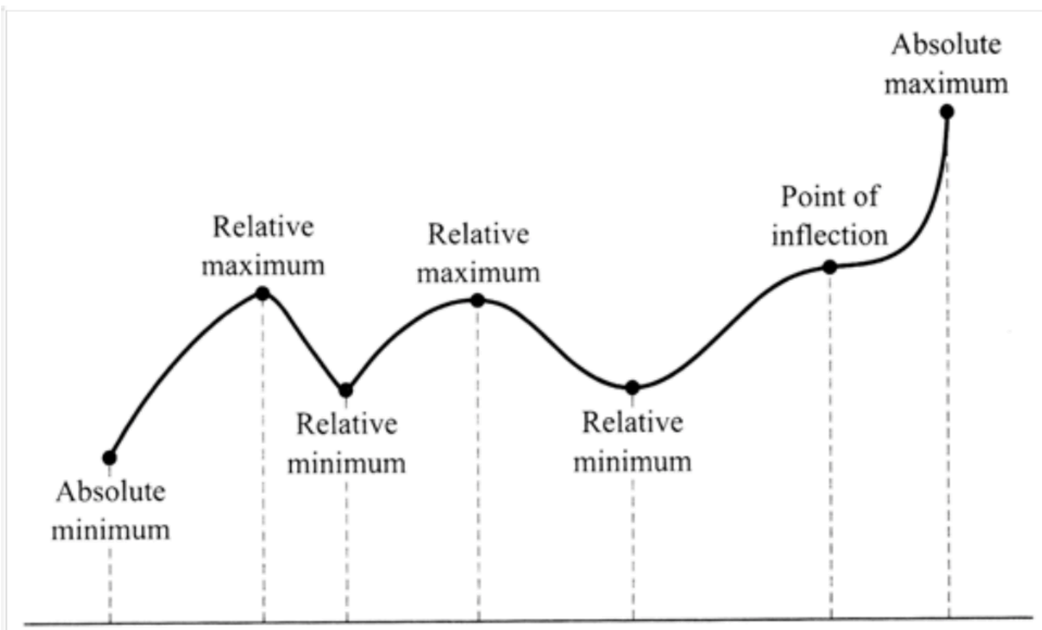
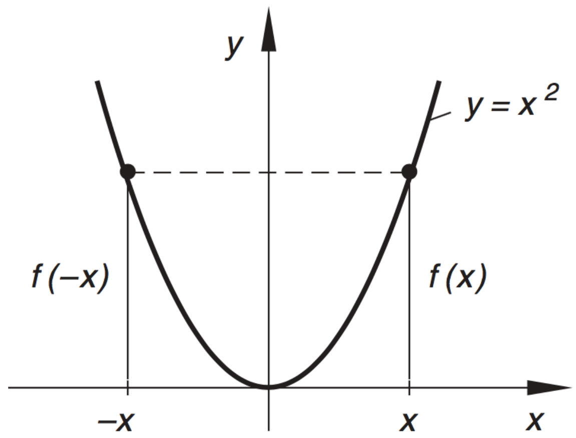


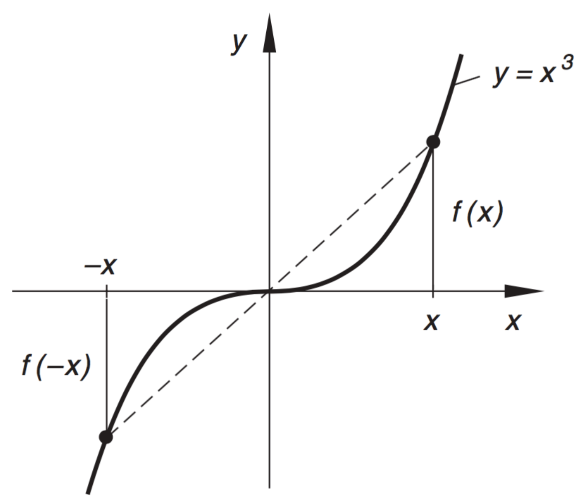
Image source [6]

Properties — Symmetry

- A function is called **even** if $f(-x) = f(x)$
- The function graph is called **mirror-symmetrical**

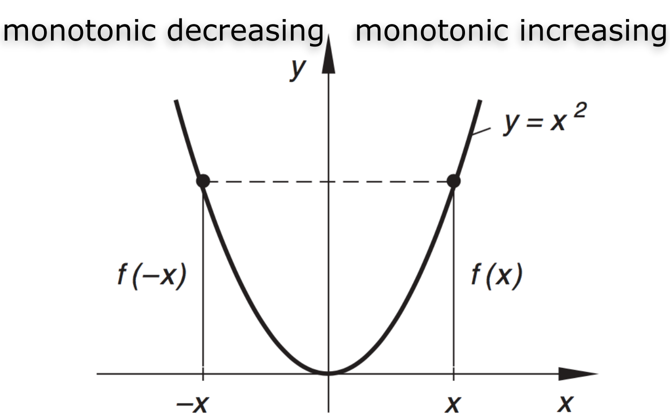


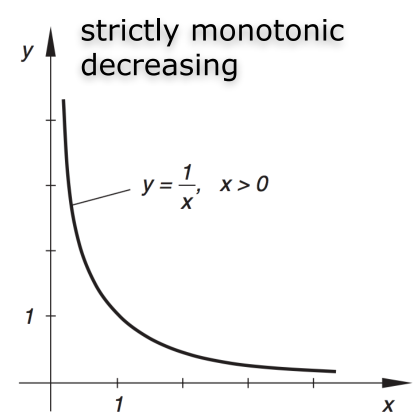
- A function is called **odd** if $f(-x) = -f(x)$
- The function graph is called **point-symmetrical**



- A function is called **monotonic** if it expresses an **ordered set**

monotonically	$f(x_1) \leq f(x_2)$	for all x1, x2 such that	$x_1 \leq x_2$
strictly monotonically	$f(x_1) < f(x_2)$		$x_1 < x_2$
monotonically	$f(x_1) \geq f(x_2)$	for all x1, x2 such that	$x_1 \leq x_2$
strictly monotonically	$f(x_1) > f(x_2)$		$x_1 < x_2$





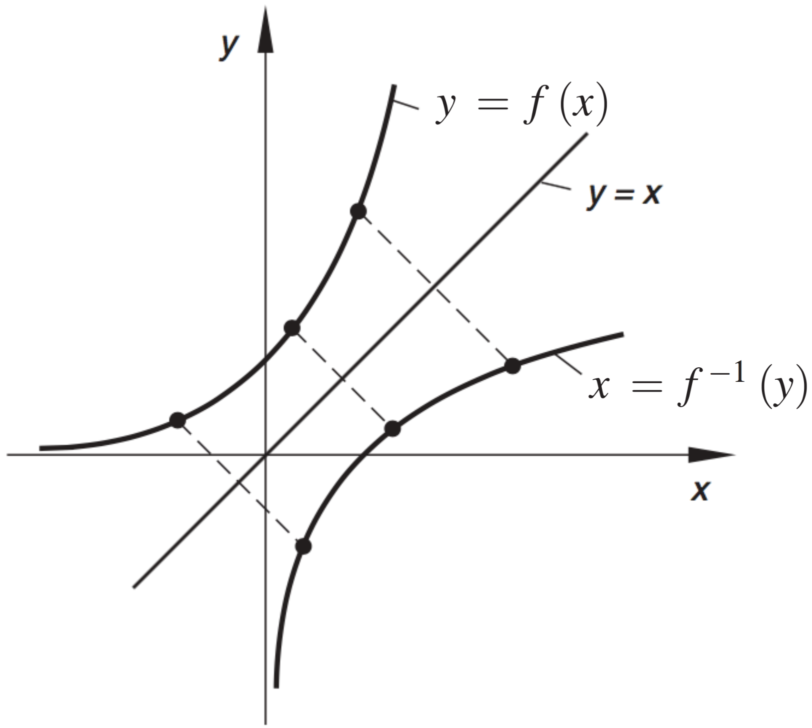
Properties — Inverse

Functions

- Every **strictly monotonic** function has an **inverse**

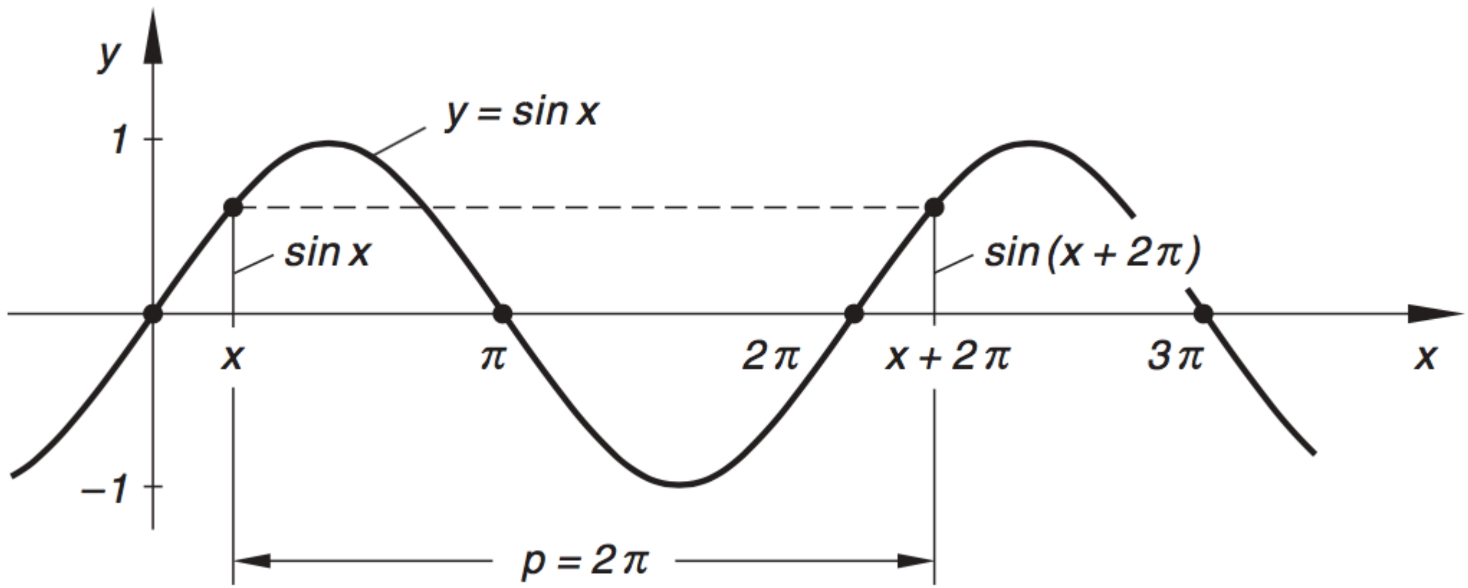
$$y = f(x) \quad \Rightarrow \quad x = f^{-1}(y)$$

- The independent variable **x** is exchanged with the dependent variable **y = f(x)** and the domain and range are changed likewise



- A function is called **periodic** if it expresses repetition

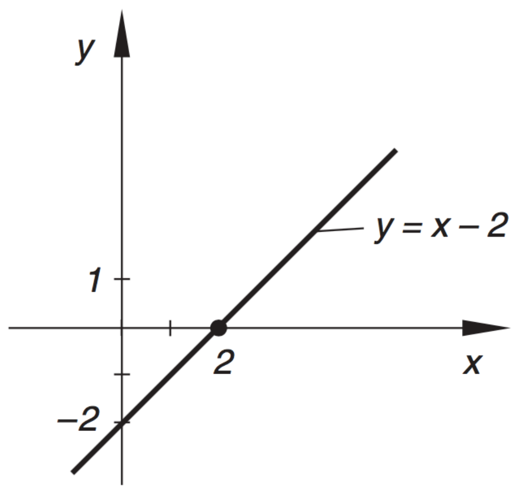
$$f(x \pm p) = f(x)$$



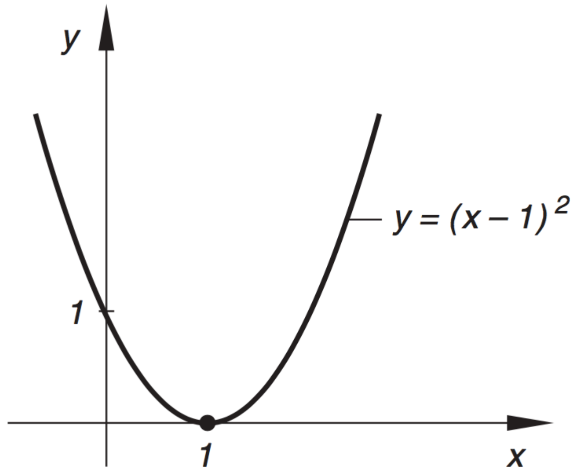
Properties — Zeros

Functions

- The **zero** of a function denotes that at x_0 the value of $f(x_0) = 0$
- The function graph **crosses or touches** the x-axis at x_0
- The zeros of a function can be found by resolving $f(x) = 0$



Zero at $x_1 = 2$



Double zero at $x_1 = 1$

Properties — Limits & Continuity

- The **limit** of a function is used to make an assumption about the function's behavior in a **specific area**
- It denotes that whenever x approaches a number x_0, the value of $f(x)$ approaches the limit g

$$\lim_{n \to \infty} \langle x_n \rangle = x_0 \quad \Rightarrow \quad \lim_{n \to \infty} f(x_n) = g$$

- This is symbolically expressed by writing

$$\lim_{x \to x_0} f(x) = g$$

Properties — Limits & Continuity

- A function is called **continuous** if a limit g exists, the function is defined at x_0 & the value of the function at x_0 equals the limit
- Otherwise, the function is called **discontinuous**

$$\lim_{x \to x_0} f(x) = g \quad \longleftrightarrow$$

The limit g exists for the function when x_0

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \longleftrightarrow$$

The function is defined at x_0 and that value equals the

Properties — Limits & Continuity

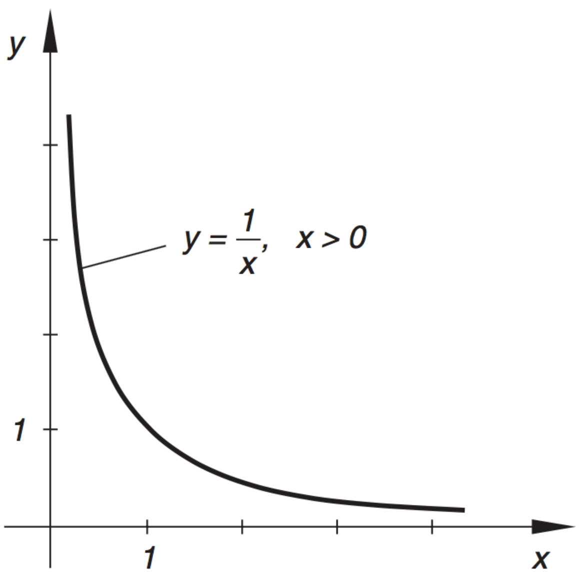
Functions

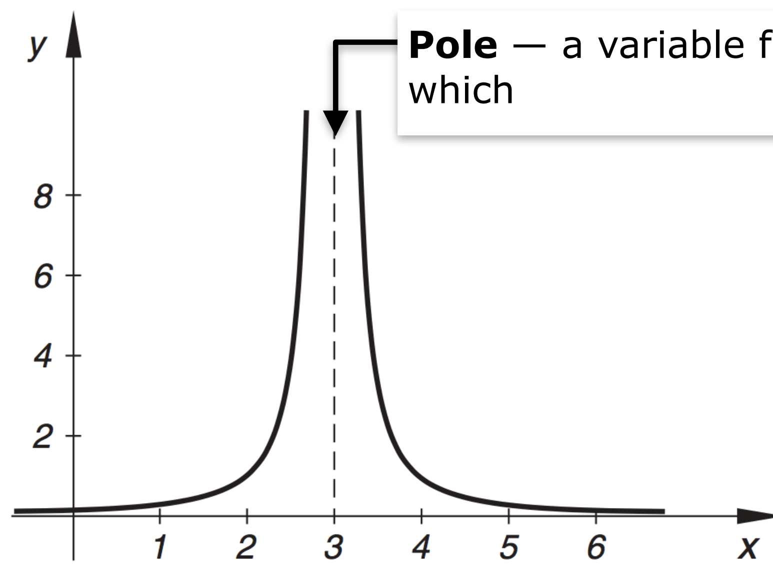
$$f(x) = \frac{1}{x},\ x > 0$$

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$$

$$f(x) = \frac{1}{(x-3)^2}$$

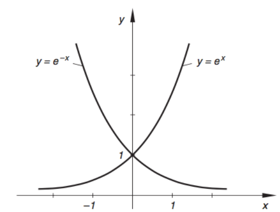
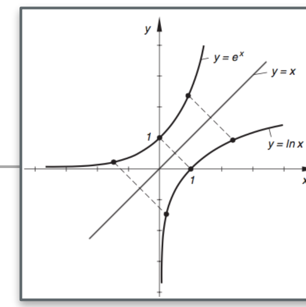
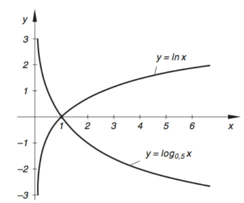
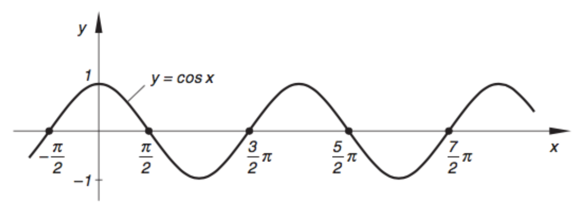
$$\lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty$$





Pole — a variable for which

Types of Functions

Polynomial functions	$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with constan $a_0, \dots, a_n$ and n called the degree of the polynomial
Exponential functions	$f(x) = a^x, a \neq 0$  
Logarithmic functions	$f(x) = \log_a x, a \neq 0$ 
Trigonometric functions	$\sin x, \cos x, \tan x = \dfrac{\sin x}{\cos x}$ 

Take Home

- Functions allow to
  - generalize the concept of a sequence of numbers
  - relate two independent domain sets of numbers usually in ***R*** and create a functional dependency between both of them
  - provide certain properties that further describe the relationship of the two domain sets
- Different types of functions allow to express different types of relationships, i.e., linear, algebraic, geometric relationships, …

Further Reading

Literature used for this lecture:

- Courant, R. & Robbins, H. (1973): **Was ist Mathematik?** 3. Aufl., Berlin u.a.: Springer.

- Papula, Lothar (2014): **Mathematik für Ingenieure & Naturwissenschaftler Band 1.** 14. überarb. Aufl., Wiesbaden: Springer Vieweg.

- Wrede, R. & Spiegel, M. (2010): **Schaum's Outlines — Advanced Calculus**. 3. Ed., New York u.a.: McGrawHill.

- https://betterexplained.com (17/10/22)

Image references

- If not noted otherwise, all image are taken from Papula, Lothar (2014): **Mathematatik für Ingenieure & Naturwissenschaftler Band 1.** 14. überarb. Aufl., Wiesbaden: Springer Vieweg.

- [1] Image credit: https://upload.wikimedia.org/wikipedia/commons/6/69/Construction_of_square_root_of_2_on_the_line_number.svg (17/10/18)
- [2] Image credit: https://betterexplained.com/articles/prehistoric-calculus-discovering-pi/ - 17/10/07
- [3] Image credit: https://en.wikipedia.org/wiki/Pi - 17/10/07
- [4] Image credit: https://en.wikipedia.org/wiki/Geometric_series (17/10/22)
- [5] Image credit: https://en.wikipedia.org/wiki/Codomain (17/10/14)
- [6] Image credit: Wrede, R. & Spiegel, M. (2010): **Schaum's Outlines — Advanced Calculus**. *Page 47*, 3. Ed., New York u.a.: McGrawHill.