# Theoretical Backgrounds of Audio & Graphics

## Linear Transformations

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# Linear Transformations

A *linear transformation* uses a 2 × 2 matrix to change, or transform, a 2D vector:

$$\underbrace{\boxed{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}}_{\text{Transform}} \underbrace{\boxed{\begin{bmatrix} x \\ y \end{bmatrix}}}_{\text{Vector}} = \underbrace{\boxed{\begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}}}_{\text{Transformed Vector}}$$

This simple formula achieves a variety of useful transformations, depending on the entries of the matrix.

# Linear Transformations

- A linear transformation can also denote a 3 x 3 matrix which is used to transform a vector in 3d

# Linear Transformations

- Linear transformations can be used to transform geometry
- Typical linear **geometric transformations** are:

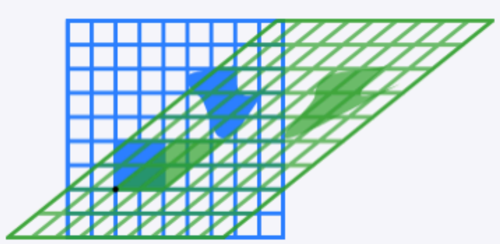
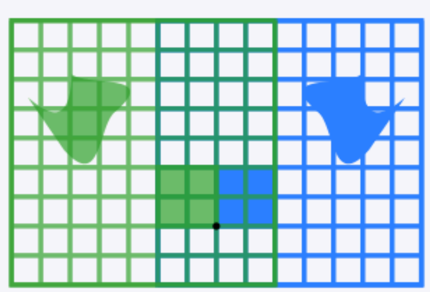
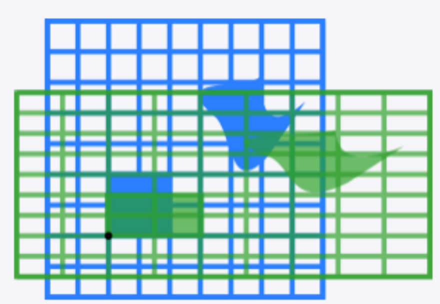
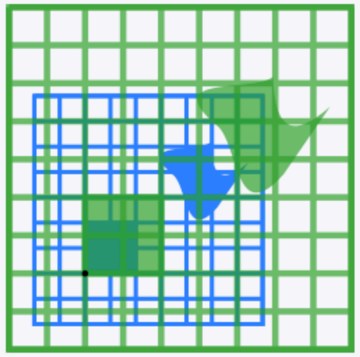
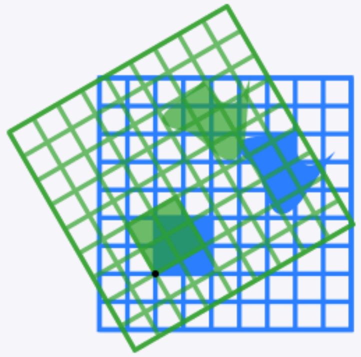
| Horizontal shear with m = 1.25. | Reflection through the vertical axis | Squeeze mapping with r = 3/2 | Scaling by a factor of 3/2 | Rotation by π/6 = 30° |
|---|---|---|---|---|
| $\begin{bmatrix} 1 & 1.25 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$ | $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ | $\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix}$ |
|  |  |  |  |  |

Image source: https://en.wikipedia.org/wiki/Matrix_(mathematics)

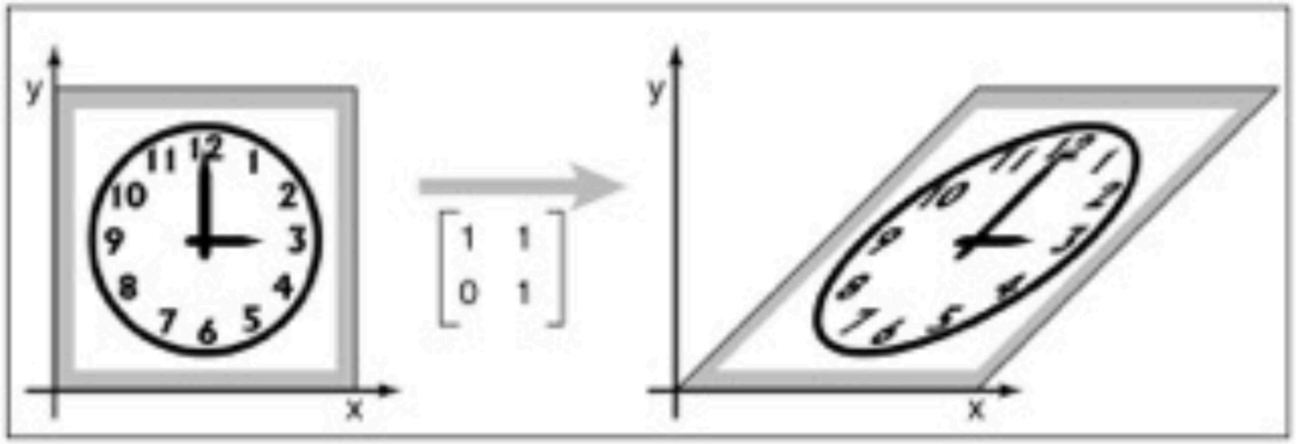
# Shearing

The horizontal and vertical shear matrices are:

$$\text{shear-x}(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \quad \text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

- An x-shear matrix moves points to the right in proportion to their y-coordinate:

$$\text{shear-x}(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



# Scaling

The most basic transform is a scale along the coordinate axes.

- This transform can change length and possibly direction:

$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

# Shearing and Scaling in 3D

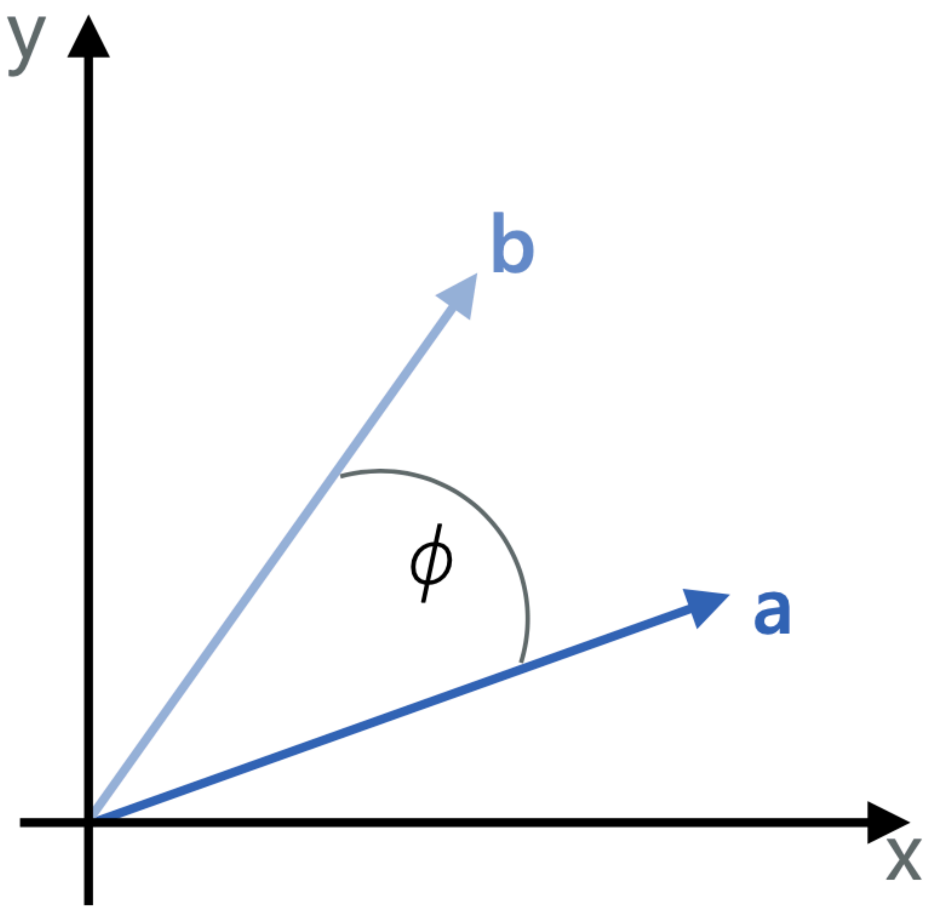
Transformations so far work similarly in 3D

$$\text{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

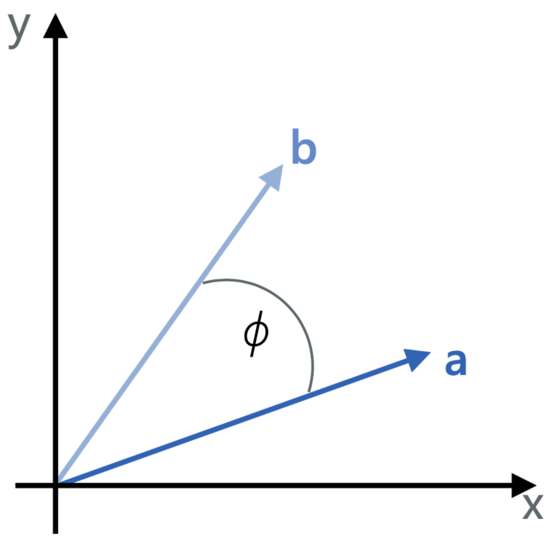
$$\text{shear-x}(d_y, d_z) = \begin{bmatrix} 1 & d_y & d_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Rotation

Suppose we want to rotate a vector **a** by an angle $\phi$ counterclockwise to get vector **b**.



# Rotation



# Rotation in 3D

- Rotation is considerably more complicated in 3D than in 2D, because there are more possible axes of rotation.
- For now we simply want to rotate about one specific axis
  - This will only change the other two coordinates and we can use the 2D rotation matrix with no operation on the rotation axis:

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \quad \text{rotate-y}(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \quad \text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Now What?

So far all transforms have the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{21} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{aligned} x' &= m_{11}x + m_{12}y, \\ y' &= m_{21}x + m_{22}y. \end{aligned}$$

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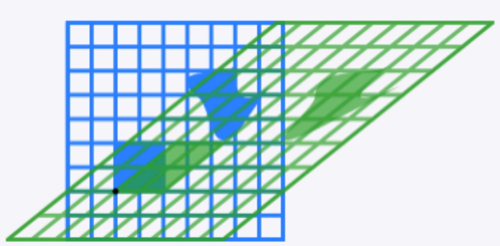
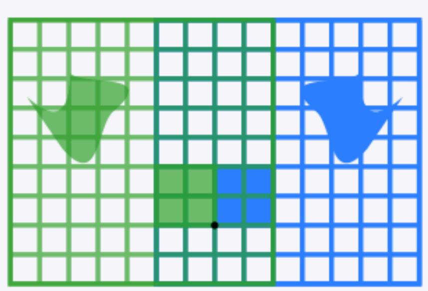
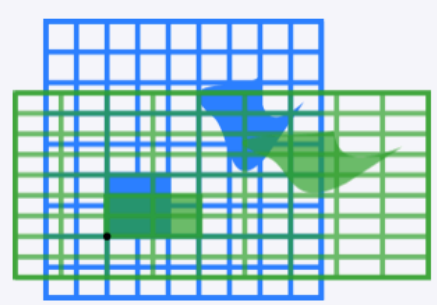
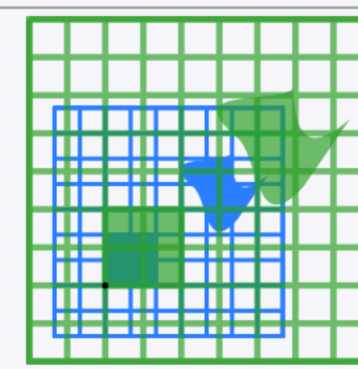
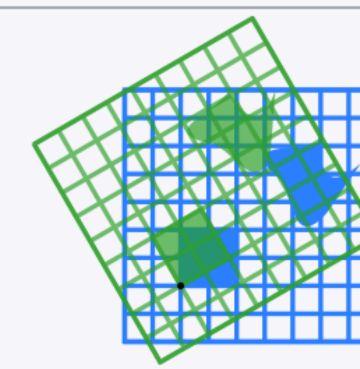
|  |  |  |  |  |

Image source: https://en.wikipedia.org/wiki/Matrix_(mathematics)

# Now What?

- The properties of these kinds of transformations — **linear transformations** — are
  - Origin remains fixed
  - Lines remain parallel
  - Ratios of distances remain equal

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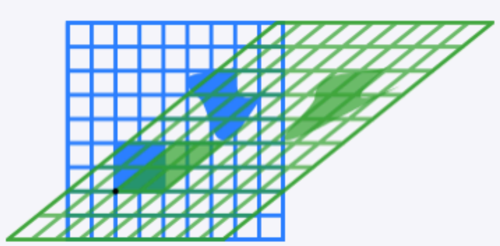
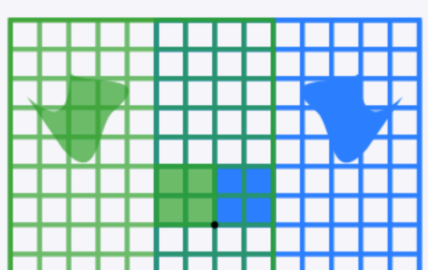
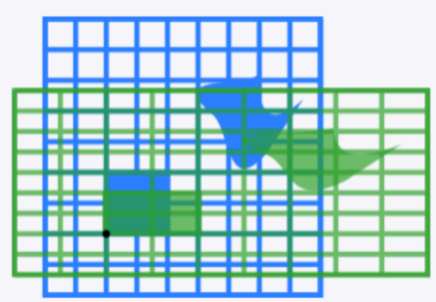
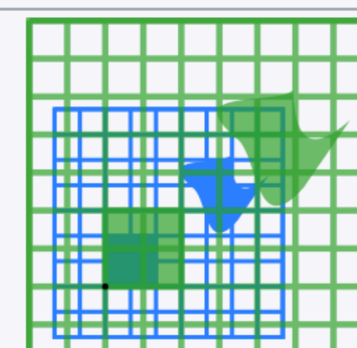
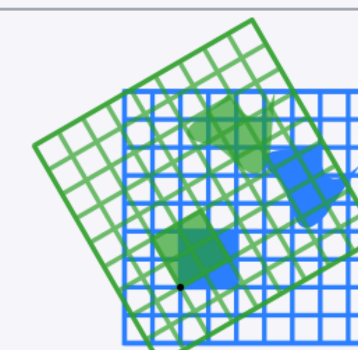
|  |  |  |  |  |

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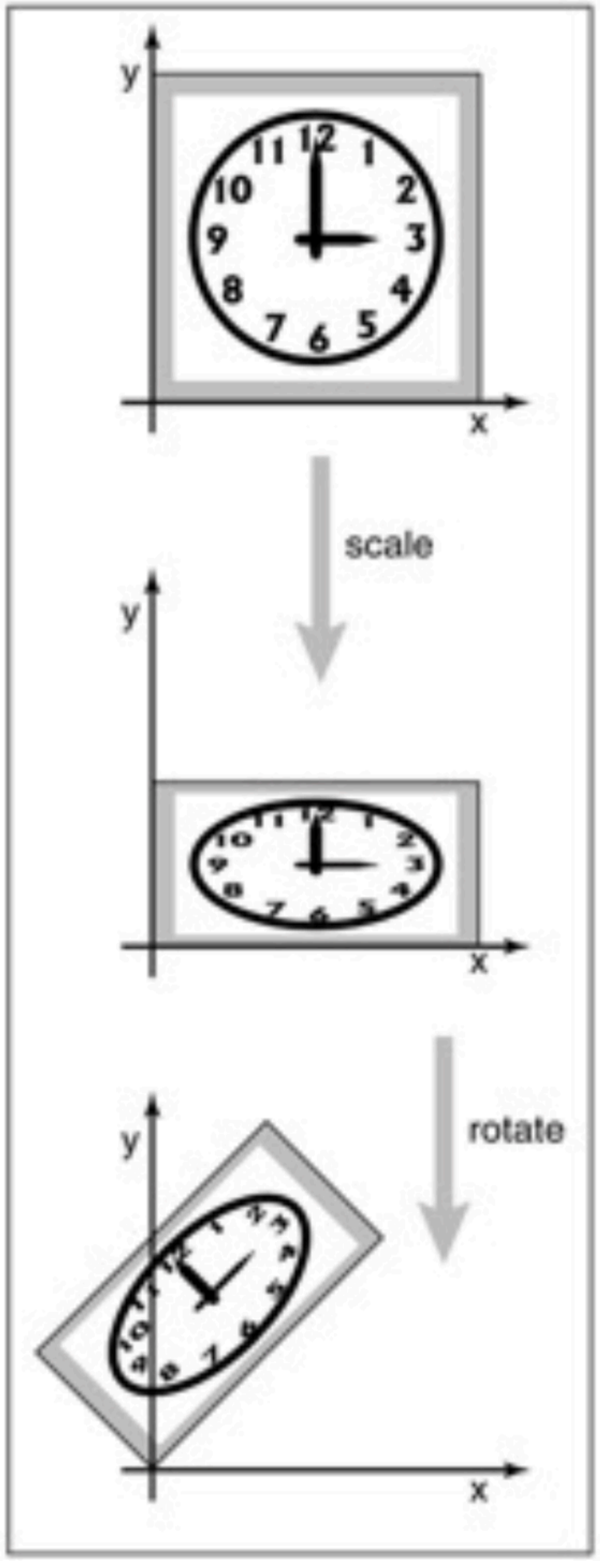


# Matrix Composition

The effects of transforming a vector by two matrices in sequence (e.g. scale **S**, rotation **R**) can be done multiplying the two transformation matrices to a single matrix of the same size:

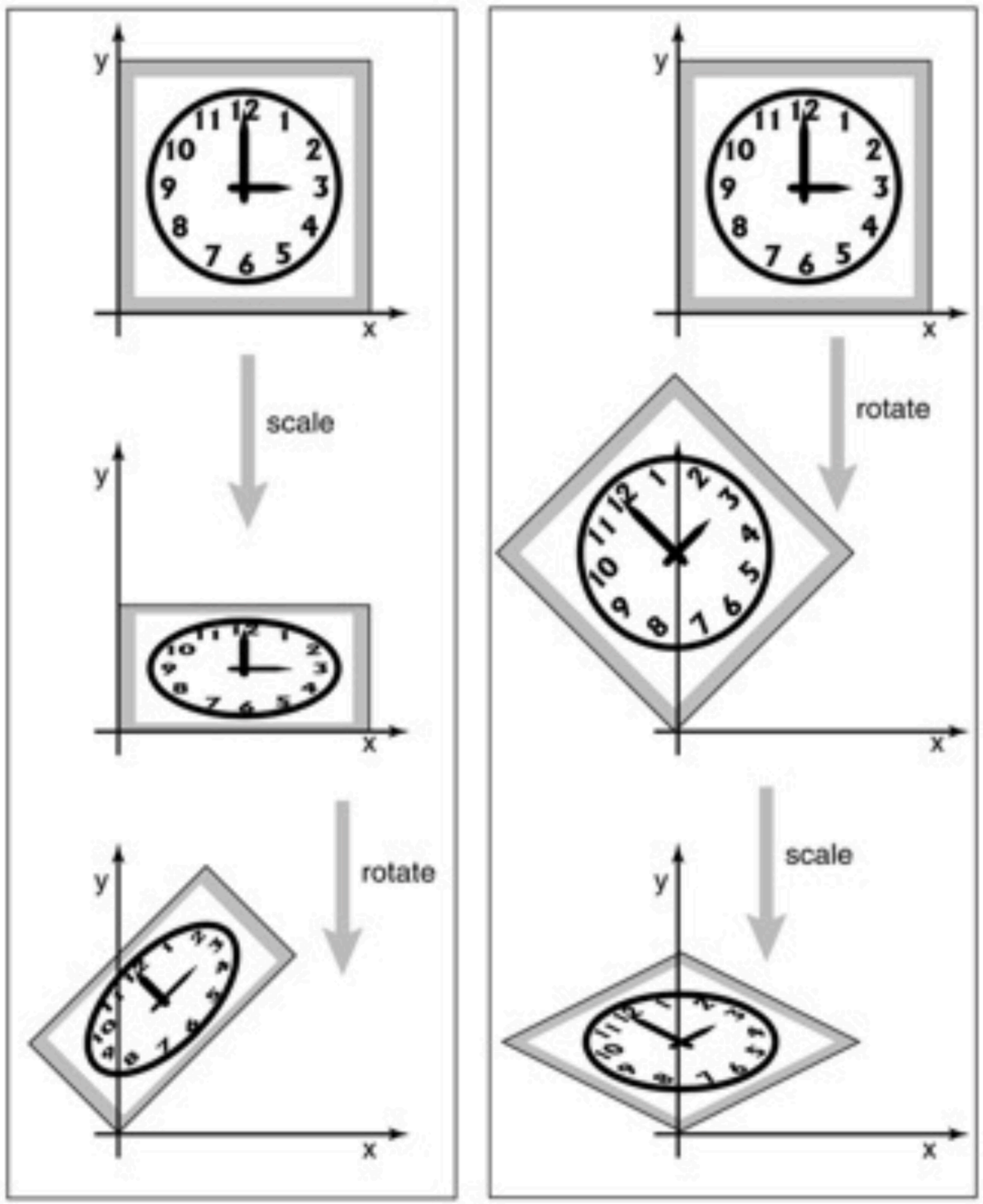
$$\mathbf{M} = \mathbf{R}\mathbf{S}$$

It is very important to remember that **these transforms are applied from the right side first**.

- The matrix **M** = **RS** first applies **S** and then **R**

Matrix multiplication is not commutative. So the order of transforms does matter!

# Matrix Composition



# Recap

- The properties of **linear transformations**
  - Origin remains fixed
  - Lines remain parallel
  - Ratios of distances remain equal

- Matrix multiplication can simplify the overall computational processes substantially
- Matrix multiplication is not commutative!

# Recap

- The properties of **linear transformations**
  - Origin remains fixed
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- Matrix multiplication can simplify the overall computational processes substantially
- Matrix multiplication is not commutative!

- **How do we involve translation?**

# Translation

To move, or *translate*, an object by shifting all its points the same amount, we need a transform of the form,

$$\begin{aligned} x' &= x + x_t, \\ y' &= y + y_t. \end{aligned}$$

# References


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