Linear Programming Algrebra

Abrémod Training

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ABRÉMOD

Overview

- Linear Programming Geometry
- Linear Programming Algebra
- Duality
- Sensitivity Analysis
  - Dual Values
  - Reduced Costs

Linear Programming Geometry

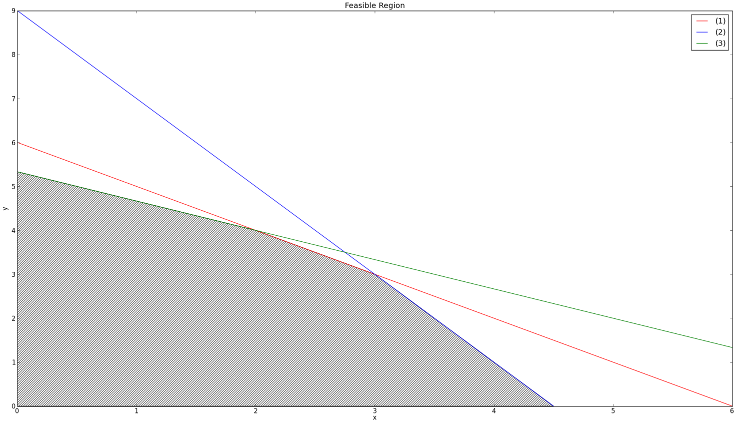
$$\max_{x,y} \quad z = 6x + 4y$$

$$\text{s.t.} \quad x + y \leq 6 \tag{1}$$

$$2x + y \leq 9 \tag{2}$$

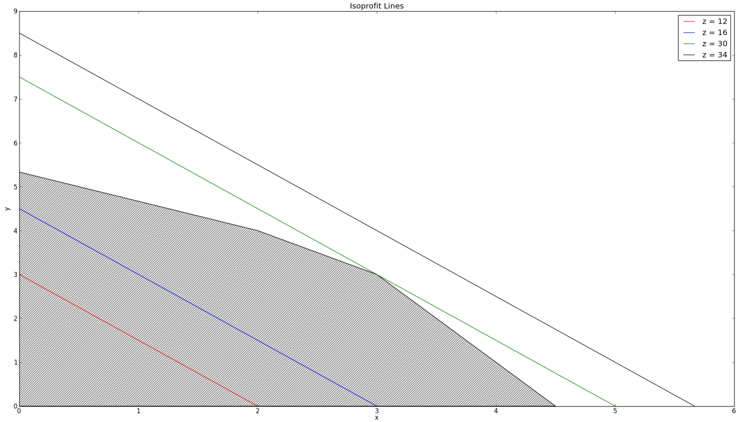
$$2x + 3y \leq 16 \tag{3}$$

$$x, y \geq 0$$



Linear Programming Geometry

- Plot the objective function $z = 6x + 4y$ for some fixed values of z.
- These are the so-called *isoprofit lines* or *objective function contours*.
- Increasing z results in a parallel shift to the right.



Observations

- Feasible region of a linear program is always a convex polyhedron
- At least one optimal solution occurs at a corner point (a.k.a. extreme point or vertex) of this polyhedron
- Infinitely-many points in the feasible region, but only finitely many corner points

Linear Programming Algebra

First, how to solve linear systems of equations?

$$\begin{aligned} 2x_1 + 1x_2 + 1x_3 &= 4 \\ 4x_1 - 6x_2 + 0x_3 &= 2 \\ -2x_1 + 7x_2 + 2x_3 &= 1 \end{aligned}$$

Systematically perform row operations to form equivalent systems

$$\begin{gathered} \text{Row } 1 \leftarrow \tfrac{1}{2} \text{ Row } 1 \\ \text{Row } 2 \leftarrow \text{Row } 2 \text{ - } 4 \text{ Row } 1 \\ \text{Row } 3 \leftarrow \text{Row } 3 + 2 \text{ Row } 1 \\ \vdots \end{gathered}$$

Until we arrive at an equivalent system with an obvious solution

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 2 \\ 0x_1 + 1x_2 + 0x_3 &= 1 \\ 0x_1 + 0x_2 + 1x_3 &= -1 \end{aligned}$$

Linear Programming Algebra

- These row operations correspond to multiplying equations by constants and adding the result to other equations.
- "Systematically" performing row operations means picking an equation and solving for a specific variable, then eliminating that variable in all other equations.
- For a square system that has an equal number of variables and equations, it is relatively easy to decide which equation to solve and which variable to solve for.
- If the system has a unique solution, we can solve for the ith variable in the ith equation, swapping the order of equations as needed.

Linear Programming Algebra

Linear programs typically have:

- More variables than equations.
- More than one feasible solution (almost always).
- More than one optimal solution (more often than you might think).

This being said, we can still solve LPs via systematic row operations, but the variable selection step is a little trickier.

Linear Programming Algebra

$$
\begin{aligned}
\max_{x,y} \quad & 6x + 4y = z \\
\text{s.t.} \quad & x + y \leq 6 \\
& 2x + y \leq 9 \\
& 2x + 3y \leq 16 \\
& x, y \geq 0
\end{aligned}
$$

As a system of linear equations:

$$
\begin{aligned}
\max_{x,y,s} \quad & 6x + 4y + 0s_1 + 0s_2 + 0s_3 = z \\
\text{s.t.} \quad & 1x + 1y + 1s_1 + 0s_2 + 0s_3 = 6 \\
& 2x + 1y + 0s_1 + 1s_2 + 0s_3 = 9 \\
& 2x + 3y + 0s_1 + 0s_2 + 1s_3 = 16 \\
& x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Add the objective as the equation $-z + 6x + 4y = 0$ and write in matrix form:

$$
\left[\begin{array}{rrrrrr|r}
z & x & y & s_1 & s_2 & s_3 & RHS \\
-1 & 6 & 4 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 6 \\
0 & 2 & 1 & 0 & 1 & 0 & 9 \\
0 & 2 & 3 & 0 & 0 & 1 & 16
\end{array}\right]
$$

Linear Programming Algebra

$$\left[\begin{array}{cccccc|c} z & x & y & s_1 & s_2 & s_3 & RHS \\ -1 & 6 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 6 \\ 0 & 2 & 1 & 0 & 1 & 0 & 9 \\ 0 & 2 & 3 & 0 & 0 & 1 & 16 \end{array}\right]$$

Systematically perform row operations

$$\left[\begin{array}{cccccc|c} z & x & y & s_1 & s_2 & s_3 & RHS \\ -1 & 0 & 1 & 0 & -3 & 0 & -27 \\ 0 & 0 & 1/2 & 1 & -1/2 & 0 & 3/2 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 & 9/2 \\ 0 & 0 & 2 & 0 & -1 & 1 & 7 \end{array}\right]$$

Until the solution is obvious

$$\left[\begin{array}{cccccc|c} z & x & y & s_1 & s_2 & s_3 & RHS \\ -1 & 0 & 0 & -2 & -2 & 0 & -30 \\ 0 & 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 & 1 & 0 & 3 \\ 0 & 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

What is the obvious solution? This is the equivalent LP:

$$\begin{aligned} \max_{x,y,s} \quad & 0x + 0y - 2s_1 - 2s_2 + 0s_3 + 30 = z \\ \text{s.t.} \quad & 0x + 1y + 2s_1 - 1s_2 + 0s_3 = 3 \\ & 1x + 0y - 1s_1 + 1s_2 + 0s_3 = 3 \\ & 0x + 0y - 4s_1 + 1s_2 + 1s_3 = 1 \\ & x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

The optimal solution to this transformed LP is $(x, y, s_1, s_2, s_3) = (3, 3, 0, 0, 1)$, $z^* = 30$

Linear Programming Algebra

In more detail

$$\begin{array}{c} \begin{array}{cccccc|c} z & x & y & s_1 & s_2 & s_3 & RHS \end{array} \\ \left[\begin{array}{cccccc|c} -1 & 6 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 6 \\ 0 & 2 & 1 & 0 & 1 & 0 & 9 \\ 0 & 2 & 3 & 0 & 0 & 1 & 16 \end{array}\right] \end{array}$$

Feasible solution $(x, y, s_1, s_2, s_3) = (0, 0, 6, 9, 16)$, $z = 0$. Increase x since it has a positive coefficient.

$$\begin{array}{c} \begin{array}{cccccc|c} z & x & y & s_1 & s_2 & s_3 & RHS \end{array} \\ \left[\begin{array}{cccccc|c} -1 & 0 & 1 & 0 & -3 & 0 & -27 \\ 0 & 0 & 1/2 & 1 & -1/2 & 0 & 3/2 \\ 0 & 1 & 1/2 & 0 & 1/2 & 0 & 9/2 \\ 0 & 0 & 2 & 0 & -1 & 1 & 7 \end{array}\right] \end{array}$$

Feasible solution $(x, y, s_1, s_2, s_3) = (9/2, 0, 3/2, 0, 7)$, $z = 27$. Increase y since it has a positive coefficient.

$$\begin{array}{c} \begin{array}{cccccc|c} z & x & y & s_1 & s_2 & s_3 & RHS \end{array} \\ \left[\begin{array}{cccccc|c} -1 & 0 & 0 & -2 & -2 & 0 & -30 \\ 0 & 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 & 1 & 0 & 3 \\ 0 & 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] \end{array}$$

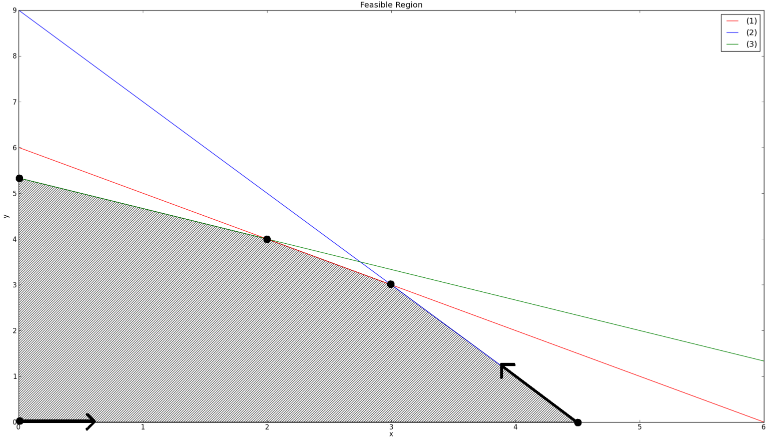
Feasible solution $(x, y, s_1, s_2, s_3) = (3, 3, 0, 0, 1)$, $z = 30$.
Transformed objective has no positive coefficients.
Transformed constraints have an “obvious” solution in which all variables with a negative objective coefficient are zero.
This is a provably optimal solution.

Observations

- Each iteration maintained exactly 3 positive decision variables (one for each of the original structural constraints).
- The set of positive variables is *basis* (and the associated solution is a *basic feasible solution*).
- Each iteration adds a new variable to a basis, and kicks an old variable out.
- The objective improves at each iteration.
- Proof of optimality: transformed objective coefficients are zero for basic variables, non-positive for non-basic variables.
- What would change if we perturbed the original
  - objective function coefficients?
  - right-hand sides?

Connecting the Algebra to the Geometry

We iterated over basic feasible solutions $(0, 0, 6, 9, 16)$, $(9/2, 0, 3/2, 0, 7)$, and $(3, 3, 0, 0, 1)$. Plotting these points in (x, y) space...



Exercises

- Solve the preceding model with Gurobi.
  - Note: Set the model attribute ModelSense to -1 in order to maximize
- Which constraints are tight at the optimal solution?
- Besides the origin, the feasible region has three other extreme points that are suboptimal under the current objective function. How might you change the objective function coefficients so that:
  - the extreme point at $(0, \frac{16}{3})$ is optimal?
  - the extreme point at $(\frac{9}{2}, 0)$ is optimal?
  - all points between $(2, 4)$ and $(3, 3)$ are optimal?

Duality

The diet problem, revisited:

- Let's take the perspective of a supplement vendor, who has pills that contain a single unit of iron or calcium that can be used to replace meals.
- This vendor will attempt sell these pills to a dieter, and must determine the appropriate price to offer.
- We'll assume the dieter knows how to solve the diet problem and will replace her optimal diet with pills but only if her cost does not increase.
- How does the vendor determine pill prices that maximize revenue and are competitive with the food types?

Duality

Food	Iron	Calcium	Cost
1	2	0	20
2	0	1	10
3	3	2	31
4	1	2	11
5	2	1	12

Nutrient requirements:
Iron: 21, Calcium: 12

- Let π_i, π_c be the price to be charged for an iron, calcium pill.
- We wish to maximize total revenue of $v = 21\pi_i + 12\pi_c$.
- We must charge prices that are competitive with the prices of the five food types.
  - $2\pi_i \leq 20$
  - $\pi_c \leq 10$
  - $3\pi_i + 2\pi_c \leq 31$
  - $\pi_i + 2\pi_c \leq 11$
  - $2\pi_i + \pi_c \leq 12$

Duality

The diet problem:

$$
\begin{aligned}
z^* = \min_{x} \quad & 20x_1 + 10x_2 + 31x_3 + 11x_4 + 12x_5 \\
\text{s.t.} \quad & 2x_1 + 0x_2 + 3x_3 + 1x_4 + 2x_5 \geq 21 \\
& 0x_1 + 1x_2 + 2x_3 + 2x_4 + 1x_5 \geq 12 \\
& x_j \geq 0, \;\; j = 1, 2, \ldots, 5
\end{aligned}
$$

The “dual” problem:

$$
\begin{aligned}
v^* = \max_{\pi} \quad & 21\pi_i + 12\pi_c \\
\text{s.t.} \quad & 2\pi_i + 0\pi_c \leq 20 \\
& 0\pi_i + 1\pi_c \leq 10 \\
& 3\pi_i + 2\pi_c \leq 31 \\
& 1\pi_i + 2\pi_c \leq 11 \\
& 2\pi_i + 1\pi_c \leq 12 \\
& \pi_i, \pi_c \geq 0
\end{aligned}
$$

Exercises

- Intuitively, is it possible for $v^* > z^*$?
- Solve the dual of the diet problem with Gurobi. (Maintain a copy of the Model object for the original diet problem for comparison purposes.)
- How are z^* and v^* related?
- How are π_i^* and π_c^* related to solution of the original diet problem?
- Multiply the iron constraint in the original diet problem by π_i^*, the calcium constraint by π_c^*, and add the results.
  - What is the resulting inequality?
  - How can this inequality be used to prove optimality?

Computing Shadow Prices

Gurobi computes π for us even when we solve the primal. Recall that in the optimal solution to the diet problem example, only x_4 and x_5 were non-zero. Letting b_i and b_c be nutrient requirements, we have

$$\begin{aligned} x_4 + 2x_5 &= b_i \\ 2x_4 + x_5 &= b_c \end{aligned}$$

We can solve for x_4 and x_5 as

$$\begin{aligned} x_4 &= -1/3b_i + 2/3b_c \\ x_5 &= 2/3b_i - 1/3b_c \end{aligned}$$

Plugging into the objective, we get

$$\begin{aligned} z &= 11x_4 + 12x_5 \\ &= 11(-1/3b_i + 2/3b_c) + 12(2/3b_i - 1/3b_c) \\ &= 13/3b_i + 10/3b_c \end{aligned}$$

So, $\pi_i = 13/3$ and $\pi_c = 10/3$.

Computing Reduced Costs

- How are the reduced costs related to the shadow prices?
- Consider food 3, which costs 31 per ounce and provides 3 units of iron and 2 units of calcium.
- Iron is priced at $13/3$ per unit, calcium at $10/3$ per unit.
- If we discount the cost of food 3 by the value of the nutrients that it provides, we get $31 - 3 * (13/3) - 2 * (10/3) = 34/3$, which is exactly the reduced cost.
- What is the reduced cost for food 4?
- How cheap would food type 3 need to be in order for it to be in our diet?
- Suppose we introduce a new food that costs 20 per ounce and provides 2 units of iron and 3 units of calcium. Should we include this new food in our diet? Do we need to reoptimize?