

Column Generation and Quadratic Expressions

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Column Generation

- Coefficient matrix is often sparse (Low proportion of variables have nonzero values)
- Give Gurobi a small portion of the variables
 - ▶ Variables left out of model are effectively fixed to 0
- Iteratively add additional columns
 - ▶ Solve LP with a subset of all possible variables
 - ▶ Identify and add variables that would improve the solution
- More columns than rows
 - ▶ Number of potential variables may even be exponential
 - ▶ Ratio of rows to columns should be close to 0

Column Generation Theory

$$\begin{aligned} z^* = \min_x \quad & \sum_{j \in J} c_j x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j = b_i : \pi_i, \quad i \in I \\ & x_j \geq 0, \quad j \in J. \end{aligned}$$

- The reduced cost of variable j can be computed as: $c_j - \sum_i a_{ij} \pi_i$
- If $c_j - \sum_i a_{ij} \pi_i \geq 0$ for all $j \in J$, we have a proof of optimality.
- Can compute the reduced cost even for variables not yet added to the model.

What you need to know

- Models can be solved iteratively
 - ▶ Model remains in memory after a call to `Model.optimize()`.
 - ▶ Can then call `addVar` to add new decision variables, then re-optimize.
- Variables are added after constraints are added.
 - ▶ Pass a `Column` object in to `Model.addVar` to back-populate constraints.
- Compute reduced costs from dual prices.
 - ▶ Choose variable(s) with negative reduced costs to add to the model.
 - ▶ If all reduced costs are non-negative, then the existing variables produced an optimal solution.

Columnwise modeling in Gurobi

- Rowwise Pattern

- ▶ Create variables (columns)
- ▶ Create LinExpr list of (coef, Var) pairs
- ▶ Create constraints (rows) in terms of LinExpr

- Columnwise Pattern

- ▶ Create constraints (rows)
- ▶ Create Column list of (coef, Constr) pairs
- ▶ Create variables in terms of Column

Gurobi API for column generation

- `Model.addVar(lb, ub, obj, vtype, name, column)`
- `Column(coeffs, constra)`
 - ▶ `col = Column(3, c1)`
 - ▶ `col = Column([1, 2], [c1, c2])`
- Dual values come from `Pi` attribute of `Constr` objects.

Transportation Problem

$$\min_x \quad \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \quad (\text{minimize shipping costs})$$

$$\text{s.t.} \quad \sum_{i \in I} x_{ij} = d_j, \quad j \in J \quad (\text{satisfy demand})$$

$$\sum_{j \in J} x_{ij} \leq u_i, \quad i \in I \quad (\text{don't exceed capacity})$$

$$x_{ij} \geq 0, \quad i \in I, j \in J \quad (\text{ship nonnegative quantities})$$

Consider x_{ij} for a particular warehouse i and customer j .

```
col = Column([1, 1], [capacity_constrs[i], demand_constrs[j]])  
var = model.addVar(obj=ship_costs[i][j], column=col)
```

Exercise

- Write addColumn method in FacilityLocationColumn.
- Problem should solve exactly like the row-oriented model

Column Generation Loop

- Can add any columns with negative reduced costs
- Need to add at least one column per iteration
 - ▶ Add columns with negative reduced costs
 - ▶ Can add multiple columns per iteration to reduce number of iterations

Exercise

- Write `getRC(warehouse, customer)` method
- Write `addColumnns()` method
- Experiment with different strategies
 - ▶ Add first variable with negative reduced cost
 - ▶ Add all variables with negative reduced costs
 - ▶ Add variable with most negative reduced cost

What is a Convex Program?

Definition

$$\begin{array}{ll}\text{minimize:} & f(x_1, x_2, \dots, x_n) \\ \text{subject to:} & g_i(x_1, x_2, \dots, x_n) \leq b_i, \quad i = 1, \dots, m \\ & x_j \geq 0, \quad j = 1, \dots, n\end{array}$$

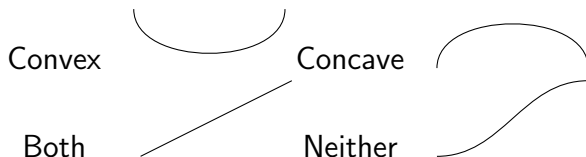
f, g_i are all convex.

Convexity

- Averages are better than extremes.
- If x and y are feasible, then $\frac{x+y}{2}$ is feasible.
- The objective function evaluated at $\frac{x+y}{2}$ must be better than the average of x and y .
- All local optimum are globally optimal

What does convex look like?

- Slopes are non-decreasing as we move from left to right.
- Not as obvious in multiple dimensions.



Convex Quadratic Programming

$$\begin{aligned} &\text{minimize: } \sum_i c_i x_i + \sum_{i,j} q_{ij} x_i x_j \\ &\text{subject to: } \sum_i a_{ik} x_i + \sum_{i,j} d_{ijk} x_i x_j \leq b_k \end{aligned}$$

- All functions must be convex (for minimization, \leq constraints).
- Unlike in the linear case, minimize/maximize; \leq , \geq are not interchangeable.

Feasible Region

Linear



Quadratic



What you can't do with Quadratic Expressions

- nonlinear equality constraints
- $x_i(1 - x_i) \leq 0$ to simulate binary variables.
- “Bilinear” programming ($\sum_i x_i \cdot y_i$).
 - ▶ pricing and allocations (price and sales variables)
 - ▶ blending problems (concentrations and quantities)

What is allowed

- $x^t Q x$ Hessian Matrix Q must be positive definite.
 - ▶ All eigenvalues are positive.
- $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$
- Maximizing a concave function
- lower bound on concave function ($f(x) \geq b$ for concave f).

Convexity

- $f(x)$ convex $\equiv -f(x)$ concave
- if $f(x)$ and $-f(x)$ are both convex then $f(x)$ is linear
 - ▶ $f(x) = b \equiv f(x) \leq b$ **and** $f(x) \geq b$
 - ▶ only linear equality constraints are allowed
- Hard to verify in general
 - ▶ Harder than actually optimizing
 - ▶ Your responsibility to give Gurobi convex problems
 - ▶ Gurobi ErrorCodes if it discovers non-convexity
 - ★ QCP_EQUALITY_CONSTRAINT
 - ★ Q_NOT_PSD

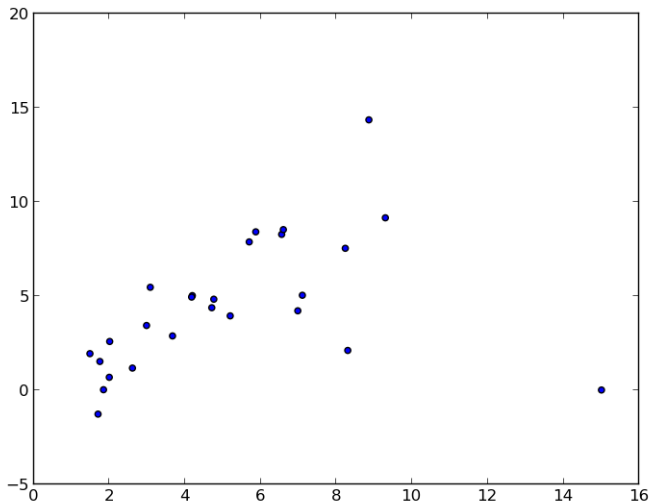
Building a Quadratic Model with Gurobi

- GRBQuadExpr
 - ▶ GRBQuadExpr.addTerm(coef, var1, var2)
 - ▶ GRBQuadExpr.addTerm(coef, var)

Example: Linear Regression

- Input: (x, y) pairs
- Regression Model: $y_i = \text{slope} \cdot x_i + \text{intercept} + \text{residual}_i$
- Minimize: $\sum_i \text{residual}_i^2$

Example: Sample Data



Exercise

- Solve regression as a QP
- Compare results with Minimize: $|\sum_i \text{residual}_i|$