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# Interactions of waves in the speed-gradient traffic flow model

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**Abstract**

In this paper, we use the speed-gradient (SG) model proposed by Jiang et al. [Transportation Research Part B, 36 (2002) 405–419] to study the initial value problem of traffic flow. The evolution of multi-traffic waves is examined through setting piecewise continuous initial density. Numerical results show that distinct shocks and smooth transition layers appear instantaneously for some initial densities, and the shock front is smoothed by the smooth transition layer ahead of or behind it if, in the initial density, the length of shock section or the length of lower-density section before the shock is very short. We thus verify that the SG model can satisfyingly reproduce the evolution of multi-traffic waves with various initial densities.


*Keywords:* Traffic flow; Partial differential equation; Initial value problem; Shock wave; Smooth transition layer

## 1. Introduction

An increasing number of traffic models have been developed to study various complex traffic phenomena, such as local clusters, stop-and-go, ghost jams, wave properties (shocks, rarefaction waves, soliton waves and kink waves), phase transitions, lane changing and overtaking [1–10]. The macro ones of these traffic flow models can be reduced as a partial differential equation (PDE) system with some initial value conditions. It is well known that the initial value problem is very important for successively applying the PDE system. Most researchers, however, have only concerned the following initial value problem in which the initial density has only one discontinuous point [11,13,14]

$$\rho(x,0) = \begin{cases} \rho_{\mathrm{u}}, & x > x_0, \\ \rho_{\mathrm{d}}, & \text{otherwise}, \end{cases} \tag{1}$$

where $\rho$ is the traffic density, $\rho_{\mathrm{u}}(\rho_{\mathrm{d}})$ is the upstream (downstream) density and $x$ represents the spatial coordinate. In reality, the initial density may have several discontinuous points. For simplicity, we assume in

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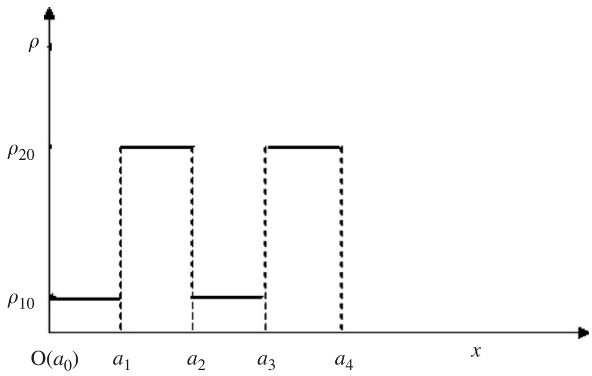


Fig. 1. The initial density scheme.

this study that the initial density is piecewise continuous as follows:

$$\rho(x,0)=\rho_i(x,0), \quad \forall a_i \leqslant x < a_{i+1},\ i=0,1,2,\ldots, \tag{2}$$

where $\rho_i(x,0)$ is the initial density for spatial section $[a_i, a_{i+1})$. Eq. (2) means that multi-shock waves exist in the system initially. It is expected that the use of Eq. (2) can reveal some new phenomena, which has never been exploited when adopting the initial condition with single shock only.

In this paper, we use the speed-gradient (SG) model [11,13] to study the initial value problem of traffic flow. The initial density has the following form:

$$\rho(x,0)=\begin{cases}\rho_{10}, & a_i \leqslant x < a_{i+1},\ i=0,2,4,\ldots,\\ \rho_{20}, & a_i \leqslant x < a_{i+1},\ i=1,3,5,\ldots,\end{cases} \tag{3}$$

where $\rho_{10}$ and $\rho_{20}$ are two positive constants and $|\rho_{10}-\rho_{20}|$ is relatively large for representing waves distinctly, as shown in Fig. 1. Our purpose is to demonstrate the evolution of Eq. (3) in the SG traffic flow model and investigate the impacts by different spatial lengths $(a_{i+1}-a_i)$ on the evolution.

## 2. The model

A typical PDE system used to formulate the traffic flow on a highway is the LWR model [15,16], which can be expressed by a conservation equation

$$\rho_t + (\rho v_e(\rho))_x = 0, \tag{4}$$

where $\rho$ is the traffic density and $v_e(\rho)$ is the speed function in equilibrium. It has been known that the LWR model can reproduce the formation of shock [17], but does not faithfully describe the non-equilibrium flow because the speed is assumed to be always determined by the function $v_e(\rho)$. A lot of researchers have improved the LWR model through adding a momentum equation into the PDE system. These improved models can be grouped into two classes, namely the density-gradient (DG) models and the SG models. The representative of the DG models [18–22] has the following form:

$$\begin{cases}\rho_t + (\rho v)_x = 0,\\ v_t + v v_x = \dfrac{v_e - v}{\tau} - \dfrac{c^2(\rho)}{\rho}\rho_x + \mu(\rho)\rho_{xx},\end{cases} \tag{5}$$

where $\tau$ is the relaxation time, $c(\rho)>0$ is the sonic speed and $\mu(\rho)$ is the viscous coefficient. The SG model [11,12,14,23,24] can be written as

$$\begin{cases}\rho_t + (\rho v)_x = 0,\\ v_t + v v_x = \dfrac{v_e - v}{\tau} + c(\rho) v_x,\end{cases} \tag{6}$$

where $c(\rho)>0$ is the propagation speed of a small perturbation. We can easily show that the single-shock wave (i.e., rarefaction wave) can be reproduced when using the DG model or the SG model to study the evolution of Eq. (1). In addition, the DG and SG models can reveal such non-equilibrium properties as local clusters, stop-and-go, ghost jams, wave properties, phase transitions and others.

Daganzo [25] used the DG model to examine the evolution of traffic density under the following initial and boundary conditions:

$$\begin{cases} v=0,\ \rho=\rho_j H(x),\ x\leqslant A,\ t=0, \\ v=0,\ x=A,\ t>0, \end{cases} \tag{7}$$

where $H(x)$ is a Heaviside function with $H(x)=\{0, x<0,\ 1, x\geqslant 0\}$, and $\rho_j$ is the jam density. He found that the vehicles' backward motion appears under the above condition. Through analyzing the characteristic speeds of the DG and SG models, Jiang et al. [13] found that the DG model generates a characteristic speed larger than the speed $v$. This leads to backward motion under some condition. In their SG model, the characteristic speeds

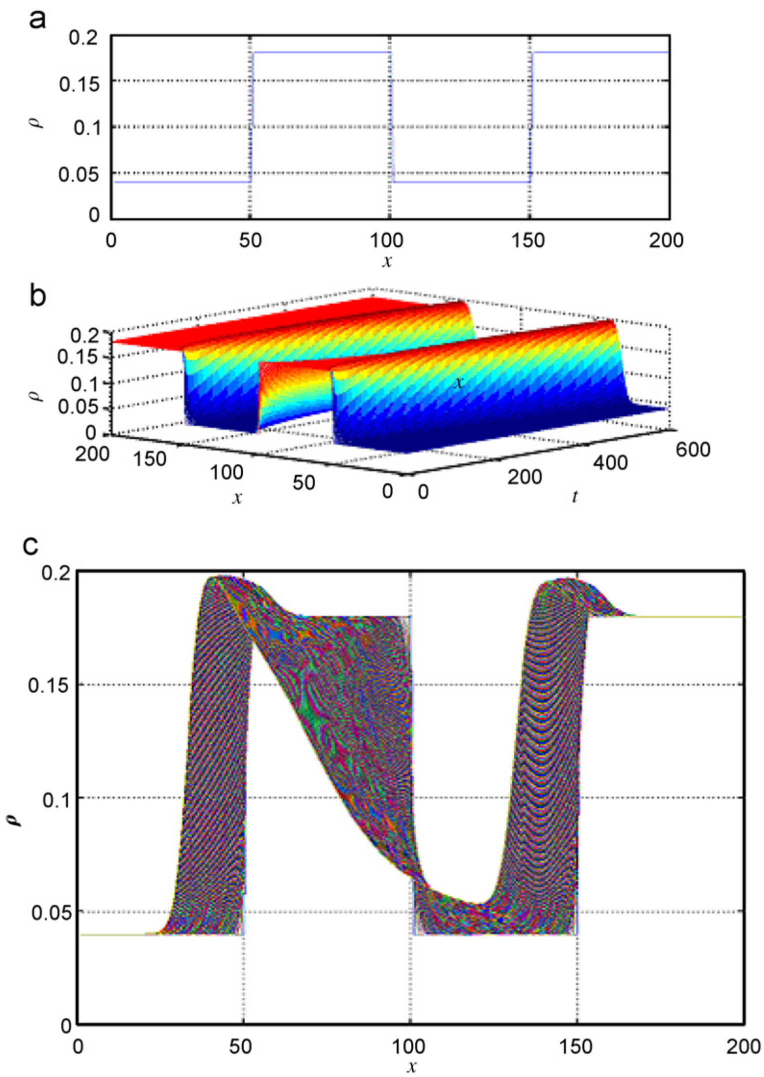


Fig. 2. The time–space evolution and the profile of two shocks ($x=50$ and 150) and one smooth transition layer around $x=100$: (a) initial density, (b) density evolution and (c) density profile.

are always less than $v$, so the backward motion never occurs. However, the SG model [11,13] cannot describe the inner relationship between the propagation speed of small perturbation and the density because the propagation speed is set to be constant. The SG model can easily be rewritten in a conservative form. We thus use this model to study the evolution of Eq. (3). In fact, similar results can also be obtained if other higher-order models are used. The SG model adopted in this study take the following form:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + v v_x = \dfrac{v_e - v}{\tau} + c_0 v_x, \end{cases} \tag{8}$$

where $c_0$ is a constant independent on the density $\rho$.

## 3. Simulation

To simulate the evolution of Eq. (3), we here use the conservative difference scheme to approximate Eq. (8). This says, Eq. (8) is first converted into the following conservative form [26]:

$$u_t + f(u)_x = s(u), \tag{9}$$

where $u = (\rho, v)^{\mathrm{T}}$, $f(u) = (\rho v, 0.5v^2 - c_0 v)^{\mathrm{T}}$, $s(u) = \left(0, (v_e(\rho) - v)/\tau\right)^{\mathrm{T}}$. Then, Eq. (9) is approximated by Ref. [27]

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n\right) + s(u_i^n)\Delta t, \tag{10}$$

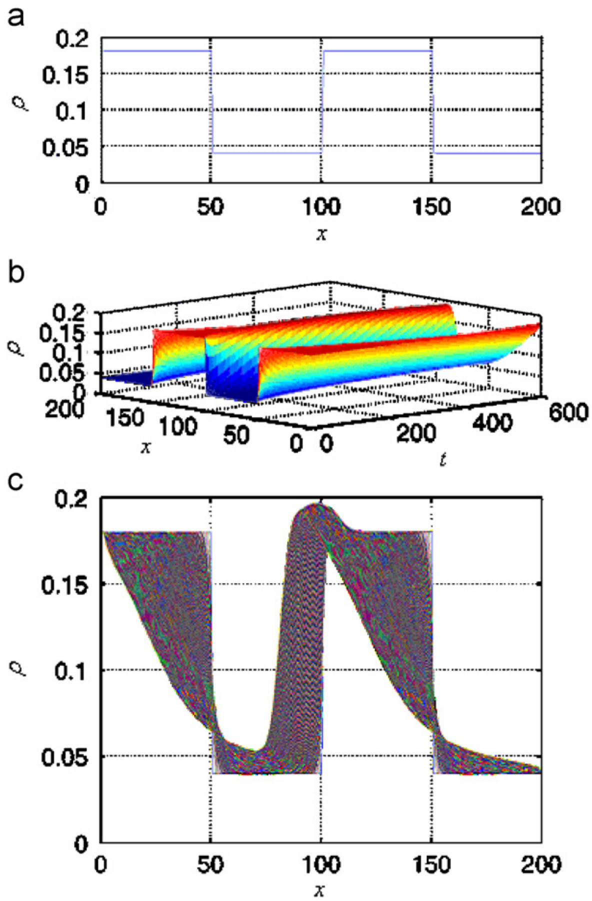


Fig. 3. The time–space evolution and the profile of one shock at $x = 100$ and two smooth transition layers (around $x = 50$ and 150): (a) initial density, (b) density evolution and (c) density profile.

where index $i$ denotes the road section, index $n$ the time interval, $\Delta x$ the space step and $\Delta t$ the time step. In Eq. (10), $\hat{f}^n_{i+1/2} = \hat{f}(u^n_i, u^n_{i+1})$, which can be defined as the local Lax–Friedrichs numerical flux, i.e.,

$$\hat{f}\left(u^n_i, u^n_{i+1}\right) = 0.5\left(f(u^n_i) + f(u^n_{i+1}) - \alpha(u^n_{i+1} - u^n_i)\right), \tag{11}$$

where $\alpha = \Delta x/\Delta t^{(n)}$, here $\Delta t^{(n)}$ is the time step at the $n$th step. For simplicity, we set $\alpha = 3v_{\mathrm{f}}$, herein $v_{\mathrm{f}}$ is the free-flow speed.

For better reproducing the multi-initial shock waves given by Eq. (3), a stability condition for traffic flow is employed in the simulation, i.e.,

$$v - c_0 \leqslant c = \frac{\mathrm{d}(\rho v_e(\rho))}{\mathrm{d}\rho} \leqslant v, \tag{12}$$

where $v-c_0$ and $v$ are the propagating speeds of the second-order waves, $c$ is the propagating speed of the first-order wave. Stop-and-go traffic waves will appear if Eq. (12) is not satisfied. We intend to study the propagation speeds of all traveling waves produced by Eq. (3). Based on the analysis by Jiang et al. [13], we can easily obtain the propagation speeds $U_i$ of traveling waves for the two solution states $\rho_i$ and $\rho_{i+1}$, i.e.,

$$U_i = \frac{q_{\mathrm{e},i+1} - q_{\mathrm{e},i}}{\rho_{i+1} - \rho_i} = \frac{\rho_{i+1} v_{\mathrm{e}}(\rho_{i+1}) - \rho_i v_{\mathrm{e}}(\rho_i)}{\rho_{i+1} - \rho_i}, \quad i = 0, 1, 2, \ldots . \tag{13}$$

As Eq. (8) consists of two types of partial differential equations, it should have two types of traveling waves, whereas Eq. (13) only gives the propagation speed of one type of traveling wave. We assume that $\tau > \tau_0$ for a

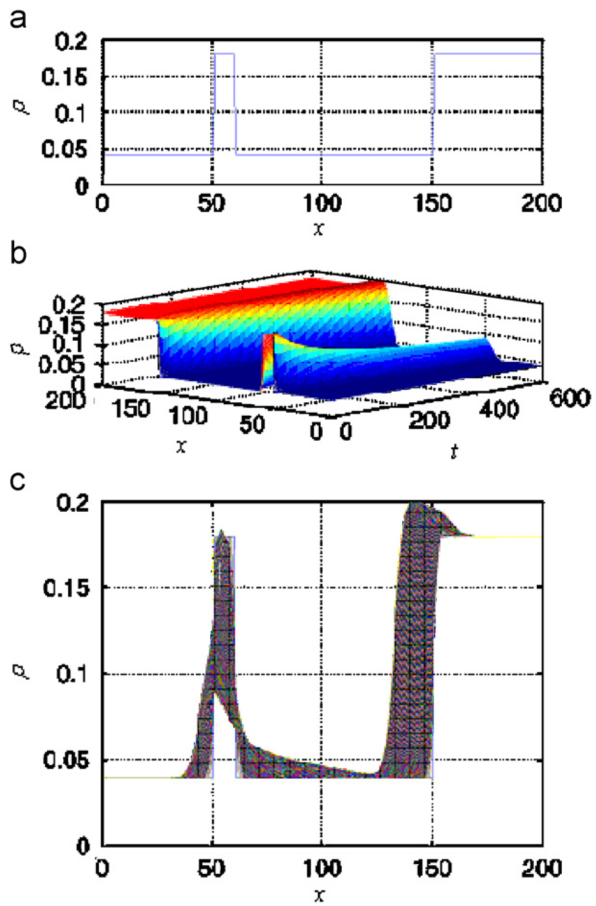


Fig. 4. The time–space evolution and the profile of the traffic density when the first shock section $[a_1, a_2) = [50,60)$: (a) initial density, (b) density evolution and (c) density profile.

certain $\tau_0 > 0$, so $s(u)$ in Eq. (9) is smooth and bounded. With this condition, we can show that the Rankine–Hugoniot conditions (R–H conditions) of the PDE system are applicable to Eq. (8) and the same with that for $s(u) = 0$. On the road section $[a_i, a_{i+1}]$, we assume that the two solution states, $u_i$ and $u_{i+1}$, are separated by a discontinuity with wave speed $\tilde{U}_i = [U_{i,1}, U_{i,2}]^{\mathrm{T}}$. Then, the R–H conditions can be written as

$$f(u_i) - f(u_{i+1}) = \tilde{U}_i(u_i - u_{i+1}). \tag{14}$$

Note that the shock structure described by Eq. (14) constitutes a traveling wave solution [26,28]. Substituting $u = (\rho, v)^{\mathrm{T}}$ and $f(u) = (\rho v, 0.5v^2 - c_0 v)^{\mathrm{T}}$ into Eq. (13), we can immediately obtain two wave speeds $U_{i,1}$ and $U_{i,2}$, among which one coincides with Eq. (13) and another is

$$U_{i,2} = 0.5\big(v_{\mathrm{e}}(\rho_i) + v_{\mathrm{e}}(\rho_{i+1})\big) - c_0. \tag{15}$$

Now, we set parameters for simulating the evolution of Eq. (3). The highway is 40 km long and is divided into four parts, i.e., $[a_i, a_{i+1})$, $i = 0, 1, 2, 3, 4$, where $a_0 = 0$ and the last section is $[a_3, a_4]$ ($a_4 = 40$ km). The following equilibrium speed–density function developed by Castillo and Benitez [29] is employed

$$v_{\mathrm{e}} = v_{\mathrm{f}}\left(1 - \exp\left(1 - \exp\left(\frac{c_{\mathrm{m}}}{v_{\mathrm{f}}}\left(\frac{\rho_j}{\rho} - 1\right)\right)\right)\right), \tag{16}$$

where $c_{\mathrm{m}}$ is the kinematic wave speed at jam density. In addition, the free-boundary conditions are adopted in this simulation, i.e., $\partial\rho/\partial x$ and $\partial v/\partial x$ are zero at the boundaries. Note that our attention is given to the interaction among multi-traffic waves, so the equilibrium speed–density function developed by Kerner and

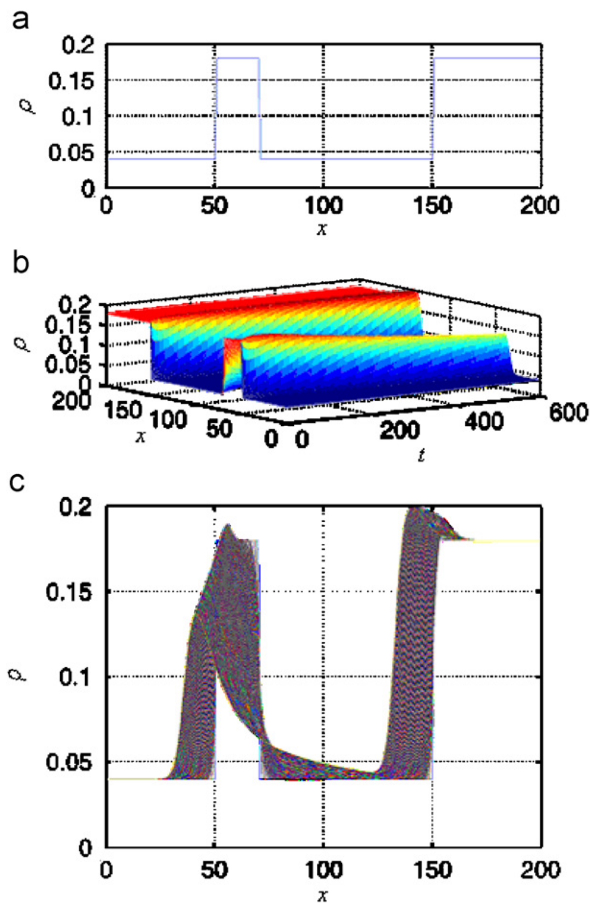


Fig. 5. The time–space evolution and the profile of the density when the first shock section $[a_1, a_2) = [50,70)$: (a) initial density, (b) density evolution and (c) density profile.

Konhäuser [30,31] is not used in our simulation. Other imputer parameters are: $\Delta x = 200\,\mathrm{m}$, $\Delta t = 1\,\mathrm{s}$, $v_f = 30\,\mathrm{m/s}$, $\rho_j = 0.2\,\mathrm{vel/m}$, $\tau = 10\,\mathrm{s}$ and $c_0 = c_m = 11\,\mathrm{m/s}$. Substituting Eq. (16) and all parameter values into Eqs. (12), (13) and (15), we can obtain the propagation speeds of the two traveling waves, i.e., $U_{i,1} = -6.6951\,\mathrm{m/s}$ (the wave will propagate forward) and $U_{i,2} = 4.0766\,\mathrm{m/s}$ (the wave will spreads backward). It is easy to verify that Eq. (12) is satisfied, this says, Eq. (3) cannot produce stop-and-go waves.

All simulation results are shown in Figs. 2–7, which correspond to different initial value problems. From these figures, we have the following findings:

(1) The interaction between shock and smooth transition layer is very small (this says, distinct shocks and smooth transition layers appear instantaneously) if the length of section $[a_i, a_{i+1})$ in the initial density is constant and long enough. Hence, the SG model [11,13] is able to perfectly reproduce the evolution of multi-traffic waves for these initial densities (see Figs. 2 and 3).
(2) The shock front is smoothed by the smooth transition layer ahead of it if the length of shock section, i.e., the value of $(a_2 - a_1)$ in Fig. 4(a), is short enough (see Fig. 4). This shock front becomes distinct gradually when the value of $(a_2 - a_1)$ increases (from Figs. 4 to 5). This is because both the shock front and the smooth transition layer propagate backward, but the latter moves faster than the former. In this sense, the two propagating speeds of the second-order wave associated with the smooth transition layer are larger than that of the second-order wave associated with the shock. So, it is possible for the smooth transition layer to catch up with and smooth the shock.

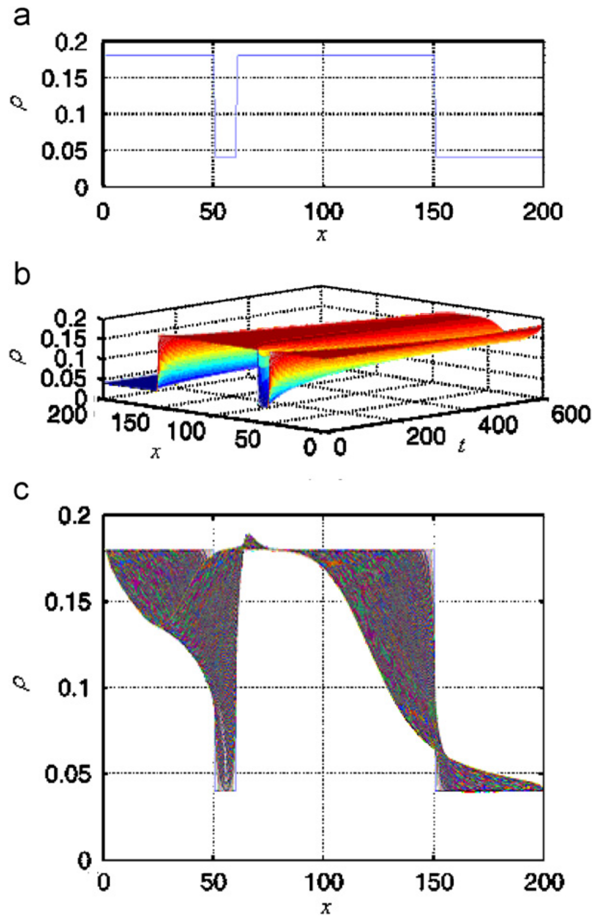


Fig. 6. The time–space evolution and the profile of the density when the length of section $[a_1, a_2) = [50,60)$: (a) initial density, (b) density evolution and (c) density profile.

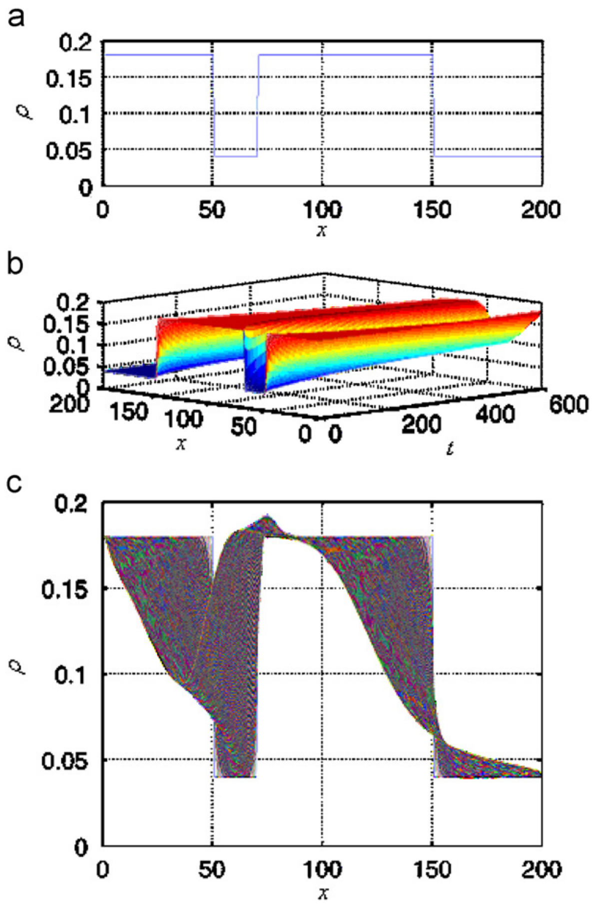


Fig. 7. The time–space evolution and the profile of the density when the length of section $[a_1, a_2) = [50,70)$: (a) initial density, (b) density evolution and (c) density profile.

(3) The shock front is smoothed by the smooth transition layer behind it if the length of lower-density section before the shock, i.e., the value of $(a_2 - a_1)$ in Fig .6(a), is short enough. This shock front becomes distinct gradually when the value of $(a_2 - a_1)$ increases (from Figs. 6 to 7). For these initial densities, the shock propagates backward at a speed $U_{i,2} = 4.0766$ m/s while the smooth transition layer propagates forward at a speed $U_{i,1} = -6.6951$ m/s, so the smooth transition layer can easily overtake the shock.

Therefore, the SG model [11,13] can reproduce such process that a shock front is smoothed by the smooth transition layer ahead of or behind it when $(a_{i+1} - a_i)$ is rather short. With the increase of $(a_{i+1} - a_i)$, the boundary between shock and smooth transition layer becomes more distinct.

## 4. Conclusions

Most researchers have only studied the evolution of Eq. (1) using the PDE system, seldom analyzed the initial value problem with several discontinuous points. In this paper, we use the SG model to investigate the initial value problem of traffic flow. We have mainly studied the evolution of Eq. (3). By simulation, we found that distinct shocks and smooth transition layers appear instantaneously for some initial densities, and the shock front is smoothed by the smooth transition layer ahead of or behind it if, in the initial density, the length of shock section or the length of lower-density section before the shock is very short. We thus verify that the SG model can satisfyingly reproduce the evolution of multi-traffic waves with various initial densities.

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