Probability

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Contents

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***Abstract*—This book provides solved examples on Probability**

1 June 2019

1.1. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \end{array} \quad (1.1.1)$$

Then which of the following are true?

a) $\lim_{n\to\infty} p_{12}^{(n)} = 0$
b) $\lim_{n\to\infty} p_{12}^{(n)} = \lim_{n\to\infty} p_{21}^{(n)}$
c) $\lim_{n\to\infty} p_{22}^{(n)} = \frac{1}{8}$
d) $\lim_{n\to\infty} p_{21}^{(n)} = \frac{1}{3}$

1.2. A sample of size $n = 2$ is drawn from a population of size $N = 4$ using probability proportional to size without replacement scheme , Where the probabilities proportional to size are The probability of inclusion of unit (1) in

i:	1	2	3	4
p_i	0.4	0.2	0.2	0.2

Table : Probability vs Size

the sample is

a) 0.4 b) 0.6 c) 0.7 d) 0.75

Solution: Let $P_i(j)$ represent the probability for selecting unit (j) as second unit after selecting unit (i)

$$P_i(j) = \frac{p_j}{1 - p_i} \quad (1.2.1)$$

Let $\Pr(i,j)$ be probability of selecting sample {i,j} ,using (**??**) is

$$\Pr(i,j) = P_i(j) + P_j(i) \tag{1.2.2}$$
$$= \left(p_i \times \frac{p_j}{1-p_i}\right) + \left(p_j \times \frac{p_i}{1-p_j}\right) \tag{1.2.3}$$

Total samples(Size $n = 2$)are Let P_i be

Case	1	2	3	4	5	6
Sample(size $n=2$)	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)

TABLE 1.2: list of samples

the probability of inclusion of unit (i) in the sample(size $n = 2$),Now i will calculate P_1 ,Favourable cases for inclusion of unit(1) are case (1,2,3),So

$$P_1 = \Pr(1,2) + \Pr(1,3) + \Pr(1,4) \tag{1.2.4}$$

using (1.2.3) and p_i from question ,

$$P_1 = \frac{7}{30} + \frac{7}{30} + \frac{7}{30} \tag{1.2.5}$$
$$= 0.7 \tag{1.2.6}$$

Therefore Option (3) is correct.

2 December 2018

2.1. Let X and Y be i.i.d random variables uniformly distributed on (0,4).Then $\Pr(X > Y | X < 2Y)$ is

a) 1/3
b) 5/6
c) 1/4
d) 2/3

Solution:
The PDF is given by

$$f_X(x) = f_Y(x) = \begin{cases} \frac{1}{4}, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

The CDF is given by

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$F_X(x) = F_Y(x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{4}, & \text{if } 0 < x < 4 \\ 1, & x \ge 4 \end{cases}$$

Using definition of conditional probability

$$\Pr(X > Y | X < 2Y) = \frac{\Pr(Y < X < 2Y)}{\Pr(X < 2Y)} \tag{2.1.1}$$

Now finding $\Pr(X < 2Y)$

$$\Pr(X < 2y) = F_X(2y) \tag{2.1.2}$$
$$\implies \Pr(X < 2Y) = \int_{-\infty}^{\infty} f_Y(x) \times F_X(2x)dx \tag{2.1.3}$$
$$\implies \Pr(X < 2Y) = \int_0^2 \frac{x}{8}dx + \int_2^4 \frac{1}{4}dx \tag{2.1.4}$$
$$\implies \Pr(X < 2Y) = \frac{3}{4} = 0.75 \tag{2.1.5}$$

Now to find $\Pr(Y < X < 2Y)$

$$\Pr(y < X < 2y) = F_X(2y) - F_X(y) \tag{2.1.6}$$
$$\implies \Pr(Y < X < 2Y) \tag{2.1.7}$$
$$= \int_{-\infty}^{\infty} f_Y(x)(F_X(2x) - F_X(x))dx$$
$$\implies \int_0^2 \frac{1}{4}\left(\frac{x}{2} - \frac{x}{4}\right)dx + \int_2^4 \frac{1}{4}\left(1 - \frac{x}{4}\right)dx \tag{2.1.8}$$
$$\implies \Pr(Y < X < 2Y) = \frac{1}{4} = 0.25 \tag{2.1.9}$$

Now using (2.1.1),(2.1.5) and (2.1.9)

$$\Pr(X > Y | X < 2Y) = \frac{1/4}{3/4} = \frac{1}{3} \tag{2.1.10}$$

Hence final solution is option 1) or 1/3

2.2. Suppose X is a positive random variable with the following probability density function,

$$f(x) = (\alpha x^{\alpha-1} + \beta x^{\beta-1})e^{-x^\alpha - x^\beta}; x > 0$$

for $\alpha > 0, \beta > 0$. Then the hazard function of X for some choices of α and β can be

a) an increasing function.
b) a decreasing function.
c) a constant function.
d) a non monotonic function

Solution:

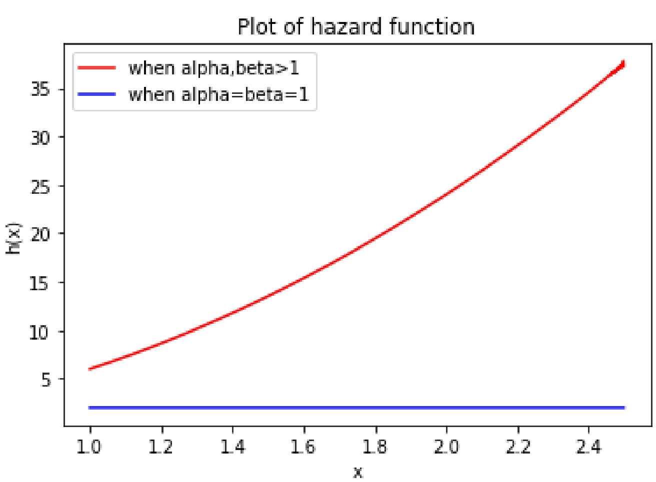
CDF of X,

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{2.2.1}$$
$$= \int_{0}^{x} f(t)dt \qquad \text{as } x > 0 \tag{2.2.2}$$
$$= \int_{-\infty}^{t} \left((\alpha t^{\alpha-1} + \beta t^{\beta-1}) \times e^{-t^{\alpha}-t^{\beta}}\right) dt \tag{2.2.3}$$
$$= -e^{-t^{\alpha}-t^{\beta}}\Big|_{0}^{x} \tag{2.2.4}$$
$$= 1 - e^{-x^{\alpha}-x^{\beta}} \tag{2.2.5}$$

Hazard function,

$$h(x) = \frac{f(x)}{1 - F(x)} \tag{2.2.6}$$
$$= \alpha x^{\alpha-1} + \beta x^{\beta-1} \tag{2.2.7}$$
$$h'(x) = \alpha(\alpha-1)x^{\alpha-2} + \beta(\beta-1)x^{\beta-2} \tag{2.2.8}$$
$$h'(x) = \begin{cases} 0 & \alpha = \beta = 1 \\ > 0 & \text{otherwise} \end{cases} \tag{2.2.9}$$

Thus $h(x)$ can be either constant function or an increasing function.



From the above figure, it is verified that $h(x)$ can be either constant function or an increasing function.

Correct options are 1,3.

2.3. Suppose n units are drawn from a population of N units sequentially as follows. A random sample

$$U_1, U_2, ...U_N \text{ of size N, drawn from } U(0,1) \tag{2.3.1}$$

The k-th population unit is selected if

$$U_k < \frac{n - n_k}{N - k + 1}, k = 1, 2, ..N.\text{where, } n_1 = 0, n_k = \tag{2.3.2}$$

number of units selected out of first k-1 units for each k = 2, 3, ..N. Then,

a) The probability of inclusion of the second unit in the sample

$$\text{is } \frac{n}{N} \tag{2.3.3}$$

b) The probability of inclusion of the first and the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{2.3.4}$$

c) The probability of not including the first and including the second unit in the sample

$$\text{is } \frac{n(N-n)}{N(N-1)} \tag{2.3.5}$$

d) The probability of including the first and not including the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{2.3.6}$$

Solution:

$$\text{Defining random variable } X \in \{0, 1, 2, ..N\} \tag{2.3.7}$$

$$\text{Where, } X = i \text{ when ith unit is included.} \tag{2.3.8}$$

The first unit in the sample is included if

$$U_1 < \frac{n - n_1}{N - 1 + 1} \tag{2.3.9}$$
$$\text{Here, } n_1 = 0 \text{ is given in the qn.} \tag{2.3.10}$$
$$\therefore \Pr(X = 1) = \frac{n}{N} \tag{2.3.11}$$

a) For k=2,

$$n_2 = 1 \text{ when, first unit is included.} \tag{2.3.12}$$

$$U_2 < \frac{n - n_2}{N - 2 + 1}\left(= \frac{n-1}{N-1}\right) \tag{2.3.13}$$

$$\therefore \Pr(X = 2 \mid X = 1) = \frac{n-1}{N-1} \tag{2.3.14}$$

$\Pr(X=1, X=2)$

$$= \Pr(X=2 \mid X=1) \times \Pr(X=1) \quad (2.3.15)$$

$$\therefore \Pr(X=1, X=2) = \frac{n(n-1)}{N(N-1)} \quad (2.3.16)$$

$n_2 = 0$ when, first unit is not included.

(2.3.17)

$$U_2 < \frac{n-n_2}{N-2+1}\left(=\frac{n}{N-1}\right) \quad (2.3.18)$$

$$\therefore \Pr(X=2 \mid X \neq 1) = \frac{n}{N-1} \quad (2.3.19)$$

$\Pr(X \neq 1, X=2)$

$$= \Pr(X=2 \mid X \neq 1) \times \Pr(X \neq 1) \quad (2.3.20)$$

$$\therefore \Pr(X \neq 1, X=2) = \left(1-\frac{n}{N}\right) \times \frac{n}{N-1} \quad (2.3.21)$$

$$\therefore \Pr(X \neq 1, X=2) = \frac{n(N-n)}{N(N-1)} \quad (2.3.22)$$

From (2.3.16) and (2.3.22)

$$\Pr(X=2) = \frac{n(n-1)}{N(N-1)} + \frac{n(N-n)}{N(N-1)} = \frac{n}{N} \quad (2.3.23)$$

Hence, option 1 is correct.

b) From (2.3.16)

$$\Pr(X=1, X=2) = \frac{n(n-1)}{N(N-1)} \quad (2.3.24)$$

Hence, option 2 is correct.

c) From (2.3.22)

$$\Pr(X \neq 1, X=2) = \frac{n(N-n)}{N(N-1)} \quad (2.3.25)$$

Hence, option 3 is correct.

d)

$$\Pr(X=1, X \neq 2) = \frac{n}{N} \times \left(1-\frac{n}{N}\right) = \frac{n(N-n)}{N^2} \quad (2.3.26)$$

Hence, option 4 is incorrect.

Therefore, Options 1, 2, 3 are correct

2.4. Consider a Markov chain with state space 1,2,....,100. Suppose states 2i and 2j communicate with each other and states 2i-1 and 2j-1 communicate with each other for every i,j = 1,2,...,50. Further suppose that $p^{(2)}_{3,3}$ ¿ 0,$p^{(3)}_{4,4}$ ¿ 0 and $p^{(7)}_{2,5}$ ¿ 0. Then

a) The Markov chain is irreducible.
b) The Markov chain is aperiodic.
c) State 8 is recurrent.
d) State 9 is recurrent.

Solution:

2.5. Out of 6 unbiased coins, 5 are tossed independently and they all result in heads. If the 6th coin is now independently tossed, the probability of getting head is:

(a) 1
(b) 0
(c) $\frac{1}{2}$
(d) $\frac{1}{6}$

Solution: Define a random variable $X = \{0, 1\}$ denoting the outcome of the toss of 6th coin with $X = 0$ and $X = 1$ representing tails and head respectively.Therefore,

$$\Pr(X=0) + \Pr(X=1) = 1 \quad (2.5.1)$$

$$\Pr(X=1) = \frac{1}{2} \quad (2.5.2)$$

Hence the correct answer is option (c).

2.6. Let $X_1, X_2, X_3, .., X_n$ be independent random variables follow a common continuous distribution **F**, which is symmetric about 0. For i=1,2,3,..n, define

$$S_i = \begin{cases} 1 & if \;\; X_i > 0 \\ -1 & if \;\; X_i < 0 \;\; and \\ 0 & if \;\; X_i = 0 \end{cases} \quad (1.1)$$

R_i=rank of $|X_i|$ in the set$\{|X_1|, |X_2|, .., |X_n|\}$.Which of the following statements are correct?

a) $S_1, S_2, .., S_n$ are independent and identically distributed.
b) $R_1, R_2, .., R_n$ are independent and identically distributed.
c) $S = (S_1, S_2, .., S_n)$ and $R = (R_1, R_2, .., R_n)$ are independent.

Solution:

A sequence $\{X_i\}$ is an Independent and identical if and only if $F_{X_n}(x) = F_{X_k}(x)$ $\forall$ n,k,x and any subset of terms of the sequence is a set of mutually independent random variables. Where F is the probability density function.

a) As the probability distribution function of $\{X_i\}$ is symmetric about origin we can say that

$$F_{X_i}(-x) = F_{X_i}(x) \forall x \in R \tag{2.1}$$

and the mean of the distribution(μ)

$$\mu = 0 \tag{2.2}$$

The sequence S_i depend on X_i as mention in 1.1, as each S_i depend only on X_i we can say that sequence S_i is independent.

$$\Pr(S_1 = 1, S_2 = 1, ..., S_n = 1) = \Pi_{i=1}^{n} \Pr(S_i = 1) \tag{2.3}$$

Any subset of terms of sequence $\{S_i\}$ is a set of mutually independent random variables and its distribution is identical.

$$F_{S_n}(s) = F_{S_k}(s) \quad \forall s, k, n \tag{2.4}$$

So, the sequence $\{S_i\}$ is independent and identical.

b) **Ranking** refers to the data transformation in which the numerical or ordinary values are replaced by the rank of numerical value when compared to a list of other values.Usually we follow increasing order for ranking.
Ranking of a sequence depend on every elements of the sequence.Let $\{R_i\}$ be the output sequence of the ranking function of $\{|X_i|\}$.

$$R_k = \text{rank of } |X_k| \text{ in the set}\{|X_1|, |X_2|, .., |X_n|\} \tag{2.5}$$

As R_k depend not only on $|X_k|$ but on the rest of the elements of the set$\{|X_1|, |X_2|, .., |X_n|\}$. So the sequence R_i is not independent. Hence R_i is not an independent and identical distribution.

c) As the i^{th} element of sequence R depends only on set$\{|X_1|, |X_2|, .., |X_n|\}$, we can say that sequence S and R are independent.

Answer:A,C

2.7. Let $X_1, X_2, \cdots$ be i.i.d. $N(0, 1)$ random variables.Let $S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1$.Which of the following statements are correct?

a) $\frac{S_n - n}{\sqrt{2}} \sim N(0, 1)$ for all $n \geq 1$

b) For all $\epsilon > 0$,$\Pr\left(\left|\frac{S_n}{n} - 2\right| > \epsilon\right) \to 0$ as $n \to \infty$

c) $\frac{S_n}{n} \to 1$ with probability 1

d) $\Pr(S_n \leq n + \sqrt{n}x) \to \Pr(Y \leq x) \forall x \in R$,where $Y \sim N(0, 2)$

Solution:

Definition 1 (Almost sure convergence)**.** *A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if*

$$\Pr(\omega | X_n(\omega) \to X(\omega)) = 1 \tag{2.7.1}$$

Definition 2 (Convergence in probability)**.** *A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge in probability (denoted by i.p) to X if*

$$\lim_{n \to \infty} \Pr(|X_n - X| > \epsilon) = 0, \forall \epsilon > 0 \tag{2.7.2}$$

Theorem 2.1 (Weak law of large numbers)**.** *Let $X_1, X_2, \cdots$ be i.i.d random variables with same expectation(μ) and finite variance(σ^2).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$*

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \tag{2.7.3}$$

in probability

Theorem 2.2 (Strong law of large numbers)**.** *Let $X_1, X_2, \cdots$ be i.i.d random variables with same expectation(μ) and finite variance(σ^2).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$*

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \tag{2.7.4}$$

almost surely.

Theorem 2.3 (Central limit theorem)**.** *The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger,given that all the samples are equal in size,regardless of the distribution of the individual samples.*

Given $X_1, X_2, \cdots$ follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2}\pi} e^{-\frac{x^2}{2}}, i \in \{1, 2, \cdots\} \tag{2.7.5}$$

As $X_1, X_2, \cdots$ are i.i.d random variables therefore $X_1^2, X_2^2, \cdots$ are also identical and independent. We can write

$$E(X^2) = Var(X) \tag{2.7.6}$$

a)

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = E\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \tag{2.7.7}$$

$$= \frac{\sum_i E(X_i^2 - 1)}{\sqrt{2}} \tag{2.7.8}$$

From (2.7.6) we can write

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = 0 \tag{2.7.9}$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = Var\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \tag{2.7.10}$$

$$= \frac{\sum_i Var(X_i^2 - 1)}{\sqrt{2}} \tag{2.7.11}$$

$$Var(X_i^2 - 1) = \int_{-\infty}^{\infty} (X_i^2 - 1)^2 f_{X_i}(x)dx \tag{2.7.12}$$

$$= \int_{-\infty}^{\infty} (X_i^4 + 1 - 2X_i^2) f_{X_i}(x)dx \tag{2.7.13}$$

$$= 2 \tag{2.7.14}$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = n\sqrt{2} \tag{2.7.15}$$

Hence from theorem 2.2 as $n \to \infty$

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n\sqrt{2}) \tag{2.7.16}$$

Hence **Option A is false.**

b) Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{2.7.17}$$

Hence from theorem 2.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X) \tag{2.7.18}$$

$$\implies \frac{S_n}{n} \xrightarrow{i.p} 1 \tag{2.7.19}$$

in probability.From definition 2 we can write,

$$\implies \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \to 0, \forall \epsilon > 0 \tag{2.7.20}$$

Hence **Option B is false .**

c) Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{2.7.21}$$

Hence from theorem 2.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X) \tag{2.7.22}$$

$$\implies \frac{S_n}{n} \xrightarrow{a.s} 1 \tag{2.7.23}$$

almost surely.From definition 1 we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \tag{2.7.24}$$

with probability 1. Hence **Option C is true**.

d) Consider,

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = 0 \tag{2.7.25}$$

using (2.7.6) and (2.7.8).

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{2n}{\sqrt{n}} \tag{2.7.26}$$

$$= 2\sqrt{n}. \tag{2.7.27}$$

using (2.7.14).From theorem 2.3 we can write,

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \sim N(0, 2\sqrt{n}) \tag{2.7.28}$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) = \Pr\left(S_n \leq n + \sqrt{n}x\right) \tag{2.7.29}$$

Hence using (2.7.28), **Option D is false.**

3 June 2018

3.1. Two students are solving the same problem independently,if the probability of first one solves the problem is $\frac{3}{5}$ and the probability that the second one solves the problem is $\frac{4}{5}$,what is the probability that atleast one of them solves the problem?

a) $\frac{17}{25}$

b) $\frac{19}{25}$

c) $\frac{21}{25}$

d) $\frac{23}{25}$

Solution: Let X,Y be two events representing solving the problem by students A,B respectively.
Given

$$\Pr(X) = \frac{3}{5} \quad (3.1.1)$$

$$\Pr(Y) = \frac{4}{5} \quad (3.1.2)$$

Since students solve the problem independently,So events X and Y are independent, For independent events

$$\Pr(XY) = \Pr(X) \times \Pr(Y) \quad (3.1.3)$$

from (3.1.1) and (3.1.2)

$$\Pr(XY) = \frac{3}{5} \times \frac{4}{5} \quad (3.1.4)$$

$$\Pr(XY) = \frac{12}{25} \quad (3.1.5)$$

Now we have to find probability of solving the problem by atleast one of them i.e $\Pr(X + Y)$. As,

$$\Pr(X + Y) = \Pr(X) + \Pr(Y) - \Pr(XY) \quad (3.1.6)$$

from (3.1.1), (3.1.2), (3.1.5)

$$\Pr(X + Y) = \frac{3}{5} + \frac{4}{5} - \frac{12}{25} \quad (3.1.7)$$

$$\Pr(X + Y) = \frac{23}{25} \quad (3.1.8)$$

Hence the required probability is $\frac{23}{25}$

3.2. A standard fair die is rolled until some face other than 5 or 6 turns up.Let X denote the face value of the last roll.Let A={X is even} and B={X is atmost 2} Then,

a) $\Pr(A \cap B) = 0$

b) $\Pr(A \cap B) = \frac{1}{6}$

c) $\Pr(A \cap B) = \frac{1}{4}$

d) $\Pr(A \cap B) = \frac{1}{3}$

Solution: Let us assume the following table. Let us represent the markov chain diagram in a matrix.Let P_{ij} represent the element of a matrix

Fig. 3.2.1: Markov chain

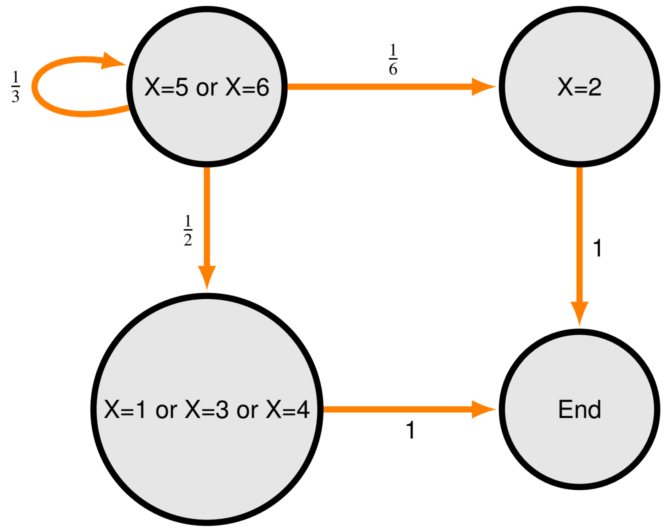


TABLE 3.2.1

state 1	state 2	state 3	state 4
$X = 5$ or $X = 6$	$X = 2$	$X = 1$ or $X = 3$ or $X = 4$	end

which is in i^{th} row and j^{th} column.The value of P_{ij} is equal to probability of transition from state i to state j

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.2.1)$$

We need the probability that $X = 2$.Hence required probability is

$$P_{12} + (P_{12})^2 + \dots + \infty \quad (3.2.2)$$

where P_{12}^n represents the 1st row ,2nd column element in the P^n

$$P^2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.2.3)$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.2.4)$$

$$P^3 = (P^2)(P^1) \tag{3.2.5}$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.6}$$

$$= \begin{bmatrix} \frac{1}{27} & \frac{1}{54} & \frac{1}{18} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.7}$$

From above we can notice that each time P_{12} reduces by $\frac{1}{3}$.Hence from (3.2.2),

$$\sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i \frac{1}{6} \tag{3.2.8}$$

From Geometric progression we can write ,required probability $=\frac{1}{4}$ $\therefore$ **option C is correct**

3.3. Let X and Y be two random variables with joint probability density function

$$f(x.y) = \begin{cases} \frac{1}{\pi} & 0 \le x^2 + y^2 \le 1 \\ 0 & otherwise \end{cases}$$

Which of the following statements are correct?

a) X and Y are independent.
b) $\Pr(X > 0) = \frac{1}{2}$
c) E(Y)=0
d) Cov(X,Y)=0

Solution:

3.4. Let X and Y be two random variables with joint probability density function

$$f(x.y) = \begin{cases} \frac{1}{\pi} & 0 \le x^2 + y^2 \le 1 \\ 0 & otherwise \end{cases}$$

Which of the following statements are correct?

a) X and Y are independent.
b) $\Pr(X > 0) = \frac{1}{2}$
c) E(Y)=0
d) Cov(X,Y)=0

Solution:

a) The marginal PDF of X is given by

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f_{XY}(x,y)dy \tag{3.4.1}$$

$$= \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \frac{1}{\pi}dy \tag{3.4.2}$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{3.4.3}$$

The marginal PDF of Y is given by

$$f_Y(x) = \int_{x=-\infty}^{x=\infty} f_{XY}(x,y)dx \tag{3.4.4}$$

$$= \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{3.4.5}$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{3.4.6}$$

Now,

$$f_X(x) \times f_Y(x) = \frac{2\sqrt{1-x^2}}{\pi} \times \frac{2\sqrt{1-y^2}}{\pi} \tag{3.4.7}$$

$$= \frac{4(1-x^2)(1-y^2)}{\pi^2} \tag{3.4.8}$$

$$\neq \frac{1}{\pi} \tag{3.4.9}$$

$$\neq f_{XY}(x,y) \tag{3.4.10}$$

Therefore, X and Y are not independent.

b) Now,

$$\Pr(X > 0) = \int_{x=0}^{x=\infty} f_X(x)dx \tag{3.4.11}$$

$$= \int_{x=0}^{x=1} \frac{2\sqrt{1-x^2}}{\pi}dx \tag{3.4.12}$$

$$= \left(\frac{\arcsin(x) + x\sqrt{1-x^2}}{\pi}\right)_0^1 \tag{3.4.13}$$

$$= \frac{1}{2} \tag{3.4.14}$$

Therefore, option(2) is correct.

c) Now,

$$E[Y] = \int_{y=-\infty}^{y=\infty} y f_Y(y)dy \tag{3.4.15}$$

$$= \int_{y=-1}^{y=1} \frac{2y\sqrt{1-y^2}}{\pi}dy \tag{3.4.16}$$

$$= \left(\frac{-2(1-y^2)^{\frac{3}{2}}}{3\pi}\right)_{-1}^1 \tag{3.4.17}$$

$$= 0 \tag{3.4.18}$$

Therefore, option(3) is also correct.

d) Now,

$$E[XY] = \int_x \int_y xy f_{XY}(x,y)dydx \tag{3.4.19}$$

$$= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \frac{xy}{\pi}dydx \tag{3.4.20}$$

$$= \frac{x}{\pi}\int_{x=-1}^{x=1} \left(\frac{y^2}{2}\right)_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \tag{3.4.21}$$

$$= 0 \tag{3.4.22}$$

Now,

$$Cov(X,Y) = E[XY] - E[X]E[Y] \tag{3.4.23}$$

$$= 0 - E[X] \times 0 \tag{3.4.24}$$

$$= 0 \tag{3.4.25}$$

Therefore, option(4) is also correct.

3.5. A simple random variable of size n will be drawn from a class of 125 students, and the mean mathematics score of the sample will be computed, If the standard error of the sample mean for "with replacement sampling" is twice as much as the standard error of the sample mean for "without replacement sampling", the value of n is ?

a) 32
b) 63
c) 79
d) 94

Solution: Let N be the population size so, N=120. The given sample size is n. **Notations :** y : student under consideration. y_i : Maths marks of i^{th} student in the sample. Y : student of class. Y_i : Maths marks of i^{th} student in the class. $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$: Average of sample class. $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$: Average of whole class. $S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$: S=Std dev of the class. $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$: Variance of the class. Standard error of sample mean $SE_{mean} = \frac{s}{\sqrt{n}}$.

Where

s = standard deviation of sample mean.

n = sample class size.

Variance of the $\overline{y}$

$$V(\overline{y}) = E(\overline{y} - \overline{Y})^2 \tag{3.5.1}$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{Y})\right]^2 \tag{3.5.2}$$

$$= E\left[\frac{1}{n^2}\sum_{i=1}^{n}(y_i - \overline{Y})^2 + \frac{1}{n^2}\sum\sum_{1\le i\ne j\le n}(y_i - \overline{Y})(y_j - \overline{Y})\right] \tag{3.5.3}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}E(y_i - \overline{Y})^2 + \frac{1}{n^2}\sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y}) \tag{3.5.4}$$

$$\text{Let } K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y}) \tag{3.5.5}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 + \frac{K}{n^2} \tag{3.5.6}$$

$$= \frac{1}{n^2}n\sigma^2 + \frac{K}{n^2} \tag{3.5.7}$$

$$= \frac{N-1}{Nn}S^2 + \frac{K}{n^2} \tag{3.5.8}$$

Finding the value of K in case of Simple random sampling with repetition (SRSWR)and Simple random sampling without repetition(SRSWOR) allows us to calculate the variance of mean. **K value in case of SRSWOR**

$$K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y})$$

Consider

$$E(y_i - \overline{Y})(y_j - \overline{Y}) = \frac{1}{N(N-1)}\sum\sum_{1\le k\ne l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

Since

$$\left[\sum_{k=1}^{N}(y_k - \overline{Y})\right]^2 = \sum_{i=1}^{N}(y_k - \overline{Y})^2 + \sum\sum_{1\le k\ne l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

$$\implies 0 = (N-1)S^2 + \sum\sum_{1\le k\ne l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

$$\implies E(y_i - \overline{Y})(y_j - \overline{Y}) = \frac{1}{N(N-1)}(N-1)(-S^2)$$

$$\implies K = n(n-1)\frac{(-S^2)}{N}$$

Putting this value in (3.5.8) gives us

$$V(\overline{y})_{WOR} = \frac{N-1}{Nn}S^2 + \frac{n-1(-S^2)}{Nn} \tag{3.5.9}$$

$$= \frac{N-n}{Nn}S^2 \tag{3.5.10}$$

K value in case of SRSWR

$$K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y})$$

Since we are selecting the samples with replacements choosing i^{th} and j^{th} sample is independent of each other. So,

$$K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})E(y_j - \overline{Y})$$

$$= 0$$

(Since deviation about mean is 0)

Putting K=0 in (3.5.8) we get

$$V(\overline{y})_{WR} = \frac{N-1}{Nn}S^2 \tag{3.5.11}$$

From equation (3.5.10) standard error of mean of sample class without repetition

$$SE_{WOR} = \frac{s}{\sqrt{n}} \tag{3.5.12}$$

$$= \sqrt{\frac{V(\overline{y})_{WOR}}{n}} \tag{3.5.13}$$

$$= \sqrt{\frac{N-n}{Nn^2}}S \tag{3.5.14}$$

From equation (3.5.11) standard error of mean of sample class with repetition

$$SE_{WR} = \sqrt{\frac{V(\overline{y})_WR}{n}} \tag{3.5.15}$$

$$= \sqrt{\frac{N-1}{Nn^2}}S \tag{3.5.16}$$

Given to find the value of n if $2 \times SE_{WOR} =$

SE_{WR}. From (3.5.14) and (3.5.16) we can write

$$2\sqrt{\frac{N-n}{Nn^2}}S = \sqrt{\frac{N-1}{Nn^2}}S \quad (3.5.17)$$

$$\implies 4(N-n) = N-1 \quad (3.5.18)$$

$$\implies 4N+1-N = 4n \quad (3.5.19)$$

$$\implies 4n = 3(125)+1 \quad (3.5.20)$$

$$\implies n = 94 \quad (3.5.21)$$

Therefore the sample size for the given condition to be met is n=94.(**Option D**)

3.6. Let X and Y be two independent and identically distributed (I.I.D) random variables uniformly distributed in (0,1). Let $Z = max(X,Y)$ and $W = min(X,Y)$, then the probability that $[Z-W > \frac{1}{2}]$ is

(A) $\frac{1}{2}$

(B) $\frac{3}{4}$

(C) $\frac{1}{4}$

(D) $\frac{2}{3}$ **Solution:**
X and Y are two independent random variables. Let

$$f_X(x) = \Pr(X = x) \quad (3.6.1)$$

$$f_Y(y) = \Pr(Y = y) \quad (3.6.2)$$

$$f_V(v) = \Pr(V = v) \quad (3.6.3)$$

be the probability densities of random variables X ,Y and V=X-Y.
The density for X is

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \quad (3.6.4)$$

We have ,

$$V = X - Y \iff v = x - y \iff x = v + y \quad (3.6.5)$$

The density of X can also be represented as,

$$f_X(v+y) = \begin{cases} 1 & 0 \le v+y \le 1 \\ 0 & otherwise \end{cases} \quad (3.6.6)$$

and the density of Y is,

$$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \quad (3.6.7)$$

The density of V i.e. $V = X - Y$ is given by the convolution of $f_X(-v)$ with $f_Y(v)$.

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v+y) f_Y(y)\, dy \quad (3.6.8)$$

From 3.6.6 and 3.6.7 we have,
The integrand is 1 when,

$$0 \le y \le 1 \quad (3.6.9)$$

$$0 \le v+y \le 1 \quad (3.6.10)$$

$$-v \le y \le 1-v \quad (3.6.11)$$

and zero, otherwise.
Now when $-1 \le v \le 0$ we have,

$$f_V(v) = \int_{-v}^{1} dy \quad (3.6.12)$$

$$= (1-(-v)) \quad (3.6.13)$$

$$= 1+v \quad (3.6.14)$$

For $0 \le v \le 1$ we have,

$$f_V(v) = \int_{0}^{1-v} dy \quad (3.6.15)$$

$$= (1-v-(0)) \quad (3.6.16)$$

$$= 1-v \quad (3.6.17)$$

Therefore the density of V is given by

$$f_V(v) = \begin{cases} 1+v & -1 \le v \le 0 \\ 1-v & 0 < v \le 1 \\ 0 & otherwise \end{cases} \quad (3.6.18)$$

The plot for PDF of V can be observed at figure 3.6.1
The CDF of V is defined as,

$$F_V(v) = \Pr(V \le v) \quad (3.6.19)$$

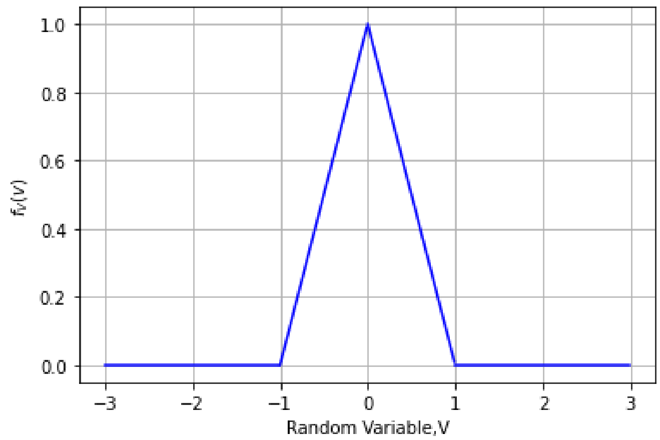


Fig. 3.6.1: The PDF of V

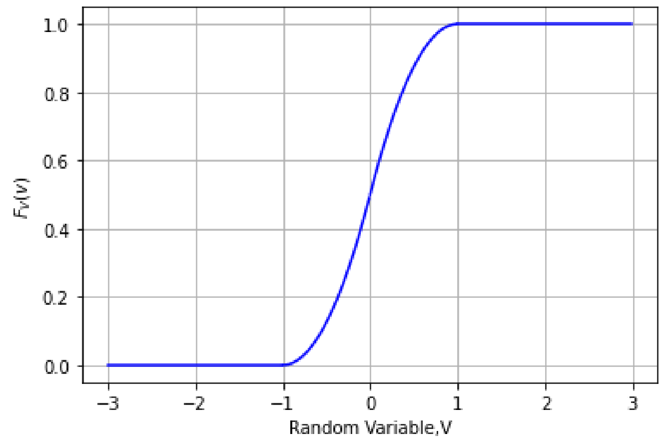


Fig. 3.6.2: The CDF of V

Now for $v \leq 0$,

$$\Pr(V \leq v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{3.6.20}$$
$$= \int_{-1}^{v} (1+v)\,dv \tag{3.6.21}$$
$$= \left(\frac{v^2}{2} + v\right)\Big|_{-1}^{v} \tag{3.6.22}$$
$$= \left(\left(\frac{v^2}{2} + v\right) - \left(\frac{1}{2} - 1\right)\right) \tag{3.6.23}$$
$$= \frac{v^2 + 2v + 1}{2} \tag{3.6.24}$$

Similarly for $v \leq 1$,

$$\Pr(V \leq v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{3.6.25}$$
$$= \frac{1}{2} + \int_{0}^{v} (1-v)\,dz \tag{3.6.26}$$
$$= \frac{-v^2 + 2v + 1}{2} \tag{3.6.27}$$

The CDF is as below:

$$F_V(v) = \begin{cases} 0 & v < -1 \\ \dfrac{v^2+2v+1}{2} & v \leq 0 \\ \dfrac{-v^2+2v+1}{2} & v \leq 1 \\ 1 & v > 1 \end{cases} \tag{3.6.28}$$

The plot for CDF of V can be observed at figure 3.6.2

We need $\Pr\left(Z - W > \frac{1}{2}\right)$ where $Z = max(X,Y)$ and $W = min(X,Y)$. Now,

$$Z - W = \begin{cases} X - Y & \text{for } X \geq Y \\ Y - X & \text{for } X < Y \end{cases} \tag{3.6.29}$$

Therefore,

$$\Pr\left(Z - W > \frac{1}{2}\right) = \Pr\left(X - Y > \frac{1}{2}, X \geq Y\right) + \Pr\left(Y - X > \frac{1}{2}, X < Y\right) \tag{3.6.30}$$
$$= \Pr\left(X - Y > \frac{1}{2}\right) + \Pr\left(Y - X > \frac{1}{2}\right) \tag{3.6.31}$$
$$= \Pr\left(V > \frac{1}{2}\right) + \Pr\left(-V > \frac{1}{2}\right) \tag{3.6.32}$$
$$= 1 - \Pr\left(V \leq \frac{1}{2}\right) + \Pr\left(V < \frac{-1}{2}\right) \tag{3.6.33}$$
$$= 1 - F_V(\frac{1}{2}) + F_V(-\frac{1}{2}) \tag{3.6.34}$$
$$= 1 - \frac{7}{8} + \frac{1}{8} \tag{3.6.35}$$
$$= \frac{1}{4} \tag{3.6.36}$$

Hence the correct answer is option (C).

3.7. Let X_1 and X_2 be i.i.d. with probability mass function $f_\theta(x) = \theta^x (1-\theta)^{1-x}$; $x = 0, 1$ where $\theta \in (0,1)$. Which of the following statements are true?

a) $X_1 + 2X_2$ is a sufficient statistic
b) $X_1 - X_2$ is a sufficient statistic
c) $X_1^2 + X_2^2$ is a sufficient statistic
d) $X_1^2 + X_2$ is a sufficient statistic

Solution: Given that, X_1 and X_2 are i.i.d. with probability mass function

$$f(x) = \begin{cases} (1-\theta) & x = 0 \\ \theta & x = 1 \end{cases} \tag{3.7.1}$$

A statistic $t = T(X)$ is sufficient for a parameter θ if the conditional probability distribution of the data, given the statistic $t = T(X)$ does not depend on the parameter θ. i.e,

$$P_\theta(X_1 = x_1, X_2 = x_2 | T = t) \tag{3.7.2}$$

is independent of θ for all x_1, x_2 and t

a) Let $T = X_1 + 2X_2$
Consider a case where $x_1 = 0, x_2 = 0$ and $t = 0$

$$\Pr(T = 0) = \Pr(X_1 + 2X_2 = 0) \tag{3.7.3}$$
$$= \Pr(X_1 = 0, X_2 = 0) \tag{3.7.4}$$

As X_1 and X_2 are independent

$$\Pr(T = 0) = \Pr(X_1 = 0)\Pr(X_2 = 0)$$
$$= (1-\theta)^2 \tag{3.7.5}$$

The conditional probability,

$$\Pr(X_1 = 0, X_2 = 0 | T = 0)$$
$$= \frac{\Pr((X_1 = 0, X_2 = 0) \cap (T = 0))}{\Pr(T = 0)} \tag{3.7.6}$$

From (3.7.4), $(X_1 = 0, X_2 = 0) \subseteq (T = 0)$

$$= \frac{\Pr(X_1 = 0, X_2 = 0)}{\Pr(T = 0)} = \frac{(1-\theta)^2}{(1-\theta)^2} = 1 \tag{3.7.7}$$

Similarly, conditional probabilities for other values of x_1, x_2 and t are given in table 3.7.1

x_1	x_2	t $t = X_1 + 2X_2$	Conditional probability $P_\theta(X_1 = x_1, X_2 = x_2 \mid T = t)$
0	0	0 otherwise	1 0
1	0	1 otherwise	1 0
0	1	2 otherwise	1 0
1	1	3 otherwise	1 0

TABLE 3.7.1: Conditional Probabilities

From table 3.7.1, all the conditional probabilities are independent of θ
$\therefore X_1 + 2X_2$ is a sufficient statistic.

b) Let $T = X_1 - X_2$
Consider a case where $x_1 = 0, x_2 = 0$ and $t = 0$

$$\Pr(T = 0) = \Pr(X_1 - X_2 = 0)$$
$$= \Pr(X_1 = 0, X_2 = 0) + \Pr(X_1 = 1, X_2 = 1) \tag{3.7.8}$$

As X_1 and X_2 are independent

$$= \Pr(X_1 = 0)\Pr(X_2 = 0)$$
$$+ \Pr(X_1 = 1)\Pr(X_2 = 1) = (1-\theta)^2 + \theta^2 \tag{3.7.9}$$

The conditional probability,

$$\Pr(X_1 = 0, X_2 = 0 | T = 0)$$
$$= \frac{\Pr((X_1 = 0, X_2 = 0) \cap (T = 0))}{\Pr(T = 0)} \tag{3.7.10}$$

From (3.7.8), $(X_1 = 0, X_2 = 0) \subseteq (T = 0)$

$$= \frac{\Pr(X_1 = 0, X_2 = 0)}{\Pr(T = 0)} = \frac{(1-\theta)^2}{(1-\theta)^2 + \theta^2} \tag{3.7.11}$$

depends on θ.
$\therefore X_1 - X_2$ is not a sufficient statistic.

c) Let $T = X_1^2 + X_2^2$
Consider a case where $x_1 = 1, x_2 = 0$ and $t = 1$

$$\Pr(T = 1) = \Pr\left(X_1^2 + X_2^2 = 1\right)$$
$$= \Pr(X_1 = 1, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1)$$
$$= \theta(1-\theta) + (1-\theta)\theta = 2\theta(1-\theta) \tag{3.7.12}$$

The conditional probability,

$$\Pr(X_1 = 1, X_2 = 0 | T = 1) = \frac{\Pr((X_1 = 1, X_2 = 0) \cap (T = 0))}{\Pr(T = 1)} \quad (3.7.13)$$

From (3.7.12), $(X_1 = 1, X_2 = 0) \subseteq (T = 1)$

$$= \frac{\Pr(X_1 = 1, X_2 = 0)}{\Pr(T = 1)} = \frac{\theta(1-\theta)}{2\theta(1-\theta)} = \frac{1}{2} \quad (3.7.14)$$

Similarly, conditional probabilities for other values of x_1, x_2 and t are given in table 3.7.2

x_1	x_2	t $t = X_1^2 + X_2^2$	Conditional probability $P_\theta(X_1 = x_1, X_2 = x_2 \vert T = t)$
0	0	0 otherwise	1 0
1	0	1 otherwise	$\frac{1}{2}$ 0
0	1	1 otherwise	$\frac{1}{2}$ 0
1	1	2 otherwise	1 0

TABLE 3.7.2: Conditional Probabilities

From table 3.7.2, all the conditional probabilities are independent of θ
$\therefore X_1^2 + X_2^2$ is a sufficient statistic.

d) Let $T = X_1^2 + X_2$
Consider a case where $x_1 = 1, x_2 = 0$ and $t = 1$

$$\Pr(T = 1) = \Pr\left(X_1^2 + X_2 = 1\right)$$
$$= \Pr(X_1 = 1, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1)$$
$$= \theta(1-\theta) + (1-\theta)\theta = 2\theta(1-\theta) \quad (3.7.15)$$

The conditional probability,

$$\Pr(X_1 = 1, X_2 = 0 | T = 1) = \frac{\Pr((X_1 = 1, X_2 = 0) \cap (T = 0))}{\Pr(T = 1)} \quad (3.7.16)$$

From (3.7.15), $(X_1 = 1, X_2 = 0) \subseteq (T = 1)$

$$= \frac{\Pr(X_1 = 1, X_2 = 0)}{\Pr(T = 1)} = \frac{\theta(1-\theta)}{2\theta(1-\theta)} = \frac{1}{2} \quad (3.7.17)$$

Similarly, conditional probabilities for other values of x_1, x_2 and t are given in table 3.7.3

x_1	x_2	t $t = X_1^2 + X_2$	Conditional probability $P_\theta(X_1 = x_1, X_2 = x_2 \vert T = t)$
0	0	0 otherwise	1 0
1	0	1 otherwise	$\frac{1}{2}$ 0
0	1	1 otherwise	$\frac{1}{2}$ 0
1	1	2 otherwise	1 0

TABLE 3.7.3: Conditional Probabilities

From table 3.7.3, all the conditional probabilities are independent of θ
$\therefore X_1^2 + X_2$ is a sufficient statistic.

Answer : Options 1,3,4

4 December 2016

4.1. $X_1, X_2, \ldots, X_n$ are independent and identically distributed as $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$. Then

a) $\sum_1^n \frac{(X_i - \bar{X})^2}{n-1}$ is the Minimum Variance Unbiased Estimate of σ^2

b) $\sqrt{\sum_1^n \frac{(X_i - \bar{X})^2}{n-1}}$ is the Minimum Variance Unbiased Estimate of σ

c) $\sum_1^n \frac{(X_i - \bar{X})^2}{n}$ is the Maximum Likelihood Estimate of σ^2

d) $\sqrt{\sum_1^n \frac{(X_i - \bar{X})^2}{n}}$ is the Maximum Likelihood Estimate of σ

Solution: The pdf for each random variable is same as they are all identical and independent Normal Distributions with same μ and σ^2.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(x-\mu)^2}{2\sigma^2} \quad (4.1.1)$$

Let us take our maximum likelihood function for given random variable X_i

$$L(\mu; \sigma | X_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i - \mu)^2}{2\sigma^2} \quad (4.1.2)$$

Since all the random variables are i.i.d

$$L(\mu;\sigma|X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} L(\mu;\sigma|X_i) \quad (4.1.3)$$

Let us denote:

$$L_m : L(\mu;\sigma|X_1, X_2, \dots, X_n) \quad (4.1.4)$$

Substituting (4.1.2) for each Random Variable in (4.1.3)

$$L_m = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i-\mu)^2}{2\sigma^2} \quad (4.1.5)$$

Taking natural log on both sides and simplifying

$$\ln L_m = \frac{-n}{2} \ln 2\pi - n \ln \sigma - \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{2\sigma^2} \quad (4.1.6)$$

In order to find Maximum Likelihood we need to maximise μ and σ w.r.t. all Random variables. Taking partial derivative w.r.t μ and taking σ as constant

$$\frac{\partial \ln L_m}{\partial \mu} = \sum_{i=1}^{n} \frac{(X_i-\mu)}{\sigma^2} \quad (4.1.7)$$

The value for μ at which L_m achieves maximum value is same in $\ln L_m$

$$\because \frac{\partial \ln L_m}{\partial \mu} = 0 \quad (4.1.8)$$

$$\therefore \sum_{i=1}^{n} \frac{(X_i-\mu)}{\sigma^2} = 0 \quad (4.1.9)$$

On simplifying the expression we get:

$$n\mu = \sum_{i=1}^{n} X_i \quad (4.1.10)$$

$$\mu = \frac{1}{n} \sum_{i=1}^{n} X_i \quad (4.1.11)$$

Let us denote the value achieved in (4.1.11) as $\bar{X}$. Taking partial derivative w.r.t σ and taking μ as constant

$$\frac{\partial \ln L_m}{\partial \sigma} = \frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^3} \quad (4.1.12)$$

The value for σ at which L_m achieves maximum value is same in $\ln L_m$

$$\frac{\partial \ln L_m}{\partial \sigma} = 0 \quad (4.1.13)$$

$$\frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^3} = 0 \quad (4.1.14)$$

Upon simplifying the expression

$$\frac{n}{\sigma} = \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^3} \quad (4.1.15)$$

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{n} \quad (4.1.16)$$

Substituting (4.1.11) in (4.1.16)

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n} \quad (4.1.17)$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n}} \quad (4.1.18)$$

Hence **Option 3** and **Option 4** are correct

4.2. There are two boxes. Box-1 contains 2 red balls and 4 green balls. Box-2 contains 4 red balls and 2 green balls. A box is selected at random and a ball is chosen randomly from the selected box. If the ball turns out to be red, what is the probability that Box-1 had been selected?
Solution: Box-1 has 2 red balls and 4 green balls.
Box-2 has 4 red balls and 2 green balls.
Let $B \in \{1, 2\}$ represent a random variable where 1 represents selecting box-1 and 2 represents selecting box-2. From Baye's theorem

$$\Pr(R = 1) = \Pr(R = 1|B = 1) \times \Pr(B = 1) + \Pr(R = 1|B = 2) \times \Pr(B = 2) \quad (4.2.1)$$

Substiting values from table (4.2.1) in (4.2.1)

$$\Pr(R = 1) = \frac{1}{2} \quad (4.2.2)$$

$$\Pr((R = 1)(B = 1)) = \Pr(R = 1|B = 1) \times \Pr(B = 1) \quad (4.2.3)$$

$$= \frac{1}{6} \quad (4.2.4)$$

We need to find $\Pr(B = 1|R = 1)$

Event	definition	value
$\Pr(B=1)$	Probability of selecting Box-1	$\frac{1}{2}$
$\Pr(B=2)$	Probability of selecting Box-2	$\frac{1}{2}$
$\Pr(R=1\|B=1)$	Probability of drawing red ball from Box-1	$\frac{1}{3}$
$\Pr(G=1\|B=1)$	Probability of drawing green ball from Box-1	$\frac{2}{3}$
$\Pr(R=1\|B=2)$	Probability of drawing red ball from Box-2	$\frac{2}{3}$
$\Pr(G=1\|B=2)$	Probability of drawing green ball from Box-2	$\frac{1}{3}$

TABLE 4.2.1: Table 1

$$\Pr(B=1|R=1) = \frac{\Pr((R=1)(B=1))}{\Pr(R=1)} \tag{4.2.5}$$

$$= \frac{1}{3} \tag{4.2.6}$$

$\therefore$ The desired probability that box-1 is selected $= \frac{1}{3}$

4.3. Suppose customers arrive in a shop according to a Poisson process with rate 4 per hour. The shop opens at 10 : 00 am. If it is given that the second customer arrives at 10 : 40 am, what is the probability that no customer arrived before 10 : 30 am?

a) $\frac{1}{4}$
b) e^{-2}
c) $\frac{1}{2}$
d) $e^{\frac{1}{2}}$

Solution: We need to find

Random Variable	Time at which people arrive
X_p	$p = 10:00 - 10:30$
X_q	$q = 10:30 - 10:40$
X_r	$r = 10:00 - 10:40$
Y	$10:40$

TABLE 4.3.1: Random Variables

$$\Pr\left(X_p = 0|Y = 2\right) \tag{4.3.1}$$

In the world where the 2^{nd} person arrives at 10 : 40 am the (4.3.1) becomes:

$$= \frac{\Pr\left(X_p = 0, X_q = 1\right)}{\Pr(X_r = 1)} \tag{4.3.2}$$

$$= \frac{\Pr\left(X_p = 0\right) \times \Pr\left(X_q = 1\right)}{\Pr(X_r = 1)} \tag{4.3.3}$$

The Poisson function distribution for time interval t and rate λ for a random variable X:

$$f_X(x;t) = \frac{(\lambda t)^x \exp(-\lambda t)}{x!}$$

For the time interval p:

$$\lambda = 4, t = 0.5, x = 0 \tag{4.3.4}$$

$$\Pr\left(X_p = 0\right) = f_X\left(0; \frac{1}{2}\right) \tag{4.3.5}$$

$$= e^{-2} \tag{4.3.6}$$

$$\tag{4.3.7}$$

For the time interval q:

$$\lambda = 4, t = \frac{1}{6}, x = 1 \tag{4.3.8}$$

$$\Pr\left(X_q = 1\right) = f_X\left(1; \frac{1}{6}\right) \tag{4.3.9}$$

$$= \frac{2}{3} e^{\frac{-2}{3}} \tag{4.3.10}$$

For the time interval r:

$$\lambda = 4, t = \frac{2}{3}, x = 1 \tag{4.3.11}$$

$$\Pr(X_r = 1) = f_X\left(1; \frac{2}{3}\right) \tag{4.3.12}$$

$$= \frac{8}{3} e^{\frac{-8}{3}} \tag{4.3.13}$$

Substituting (4.3.6) (4.3.10) (4.3.13) in (4.3.3):

$$\Pr\left(X_p = 0|Y = 2\right) = \frac{1}{4} \tag{4.3.14}$$

4.4. A fair die is thrown two times independently. Let X, Y be the outcomes of these two throws and $Z = X+Y$. Let U be the remainder obtained when Z is divided by 6. Then which of the following statement(s) is/are true?

a) X and Z are independent
b) X and U are independent
c) Z and U are independent
d) Y and Z are not independent

Solution: Let $X \in \{1, 2, 3, 4, 5, 6\}$ represent the random variable which represents the outcome of the first throw of a dice. Similarly, $Y \in \{1, 2, 3, 4, 5, 6\}$ represents the random variable which represents the outcome of the second throw of a dice.

$$n(X = i) = 1, \quad i \in \{1, 2, 3, 4, 5, 6\} \tag{4.4.1}$$

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{4.4.2}$$

Similarly,

$$\Pr(Y = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{4.4.3}$$

$$Z = X + Y \tag{4.4.4}$$

$$\text{Let } z \in \{1, 2, \dots, 11, 12\} \tag{4.4.5}$$

$$\Pr(Z = z) = \Pr(X + Y = z) \tag{4.4.6}$$

$$= \sum_{x=0}^{z} \Pr(X = x) \Pr(Y = z - x) \tag{4.4.7}$$

$$= (6 - |z - 7|) \times \frac{1}{6} \times \frac{1}{6} \tag{4.4.8}$$

$$= \frac{6 - |z - 7|}{36} \tag{4.4.9}$$

$$\Pr(Z = z) = \begin{cases} \frac{6-|z-7|}{36} & z \in \{1, 2, \dots, 11, 12\} \\ 0 & \text{otherwise} \end{cases} \tag{4.4.10}$$

U is the remainder obtained when Z is divided by 6.

$$\text{Let } u \in \{0, 1, 2, 3, 4, 5\} \tag{4.4.11}$$

$$\Pr(U = u) = \sum_{k=0}^{2} \Pr(Z = 6k + u) \tag{4.4.12}$$

$$\Pr(U = 0) = \Pr(Z = 0) + \Pr(Z = 6) + \Pr(Z = 12) \tag{4.4.13}$$

$$= 0 + \frac{5}{36} + \frac{1}{36} = \frac{1}{6} \tag{4.4.14}$$

$$\text{for } u \in \{1, 2, 3, 4, 5\} \tag{4.4.15}$$

$$\Pr(U = u) = \Pr(Z = 0 + u) + \Pr(Z = 6 + u) \tag{4.4.16}$$

$$= \frac{6 - |u - 7|}{36} + \frac{6 - |6 + u - 7|}{36} \tag{4.4.17}$$

$$= \frac{6 - (7 - u)}{36} + \frac{6 - (u - 1)}{36} \tag{4.4.18}$$

$$= \frac{u - 1 + 7 - u}{36} = \frac{6}{36} \tag{4.4.19}$$

$$= \frac{1}{6} \tag{4.4.20}$$

$$\Pr(U = u) = \begin{cases} \frac{1}{6} & u \in \{0, 1, 2, 3, 4, 5\} \\ 0 & \text{otherwise} \end{cases} \tag{4.4.21}$$

Now, for checking each option,

a) Checking if X and Z are independent

$$p_1 = \Pr(Z = z, X = x) \tag{4.4.22}$$

$$= \Pr(Y = z - x, X = x) \tag{4.4.23}$$

$$= \Pr(Y = z - x) \times \Pr(X = x) \tag{4.4.24}$$

$$= \begin{cases} \frac{1}{36} & z - x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{4.4.25}$$

$$\Pr(Z = z) \times \Pr(X = x) = \frac{6 - |z - 7|}{36} \times \frac{1}{6} \tag{4.4.26}$$

$$= \frac{6 - |z - 7|}{216} \tag{4.4.27}$$

$$\Pr(Z = z) \Pr(X = x) \neq \Pr(Z = z, X = x) \tag{4.4.28}$$

X and Z are not independent from (4.4.28) and hence option (4.4a) is false.

b) Checking if X and U are independent

$$p_2 = \Pr(U = u, X = x) \tag{4.4.29}$$

$$p_2 = \Pr((Z = u) + (Z = 6 + u) + (Z = 12 + u), X = x) \tag{4.4.30}$$

$$p_2 = \Pr((Y = u - x) + (Y = 6 + u - x) + (Y = 12 + u - x), X = x) \tag{4.4.31}$$

$$p_2 = \frac{1}{6} \times \frac{1}{6} \tag{4.4.32}$$

$$= \frac{1}{36} \tag{4.4.33}$$

$$\Pr(U = u) \times \Pr(X = x) = \frac{1}{6} \times \frac{1}{6} \tag{4.4.34}$$

$$= \frac{1}{36} \tag{4.4.35}$$

$$\Pr(U = u) \Pr(X = x) = \Pr(U = u, X = x) \tag{4.4.36}$$

X and U are independent from (4.4.36) and hence option (4.4b) is true.

c) Checking if Z and U are independent

$$p_3 = \Pr(Z = z | U = u) \tag{4.4.37}$$

$$p_3 = \begin{cases} 1 & u = 1 \text{ and } z = 7 \\ \frac{1}{2} & u = 0 \text{ and } z \in \{6, 12\} \\ \frac{1}{2} & u \in \{2, 3, 4, 5\} \text{ and } \\ & z = u \text{ or } z = 6 + u \\ 0 & \text{otherwise} \end{cases} \tag{4.4.38}$$

$$\Pr(Z = z) = \frac{6 - |z - 7|}{36} \tag{4.4.39}$$

If Z and U are independent, then

$$\Pr(Z = z | U = u) = \frac{\Pr(Z = z, U = u)}{\Pr(U = u)} \tag{4.4.40}$$

$$= \frac{\Pr(Z = z) \Pr(U = u)}{\Pr(U = u)} \tag{4.4.41}$$

$$= \Pr(Z = z) \tag{4.4.42}$$

But,

$$\Pr(Z = z | U = u) \neq \Pr(Z = z) \tag{4.4.43}$$

X and U are not independent from (4.4.43) and hence option (4.4c) is false.

d) Checking if Y and Z are independent

$$p_1 = \Pr(Z = z, Y = y) \tag{4.4.44}$$

$$= \Pr(X = z - y, Y = y) \tag{4.4.45}$$

$$= \Pr(X = z - y) \times \Pr(Y = y) \tag{4.4.46}$$

$$= \begin{cases} \frac{1}{36} & z - y \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{4.4.47}$$

$$\Pr(Z = z) \times \Pr(Y = y) = \frac{6 - |z - 7|}{36} \times \frac{1}{6} \tag{4.4.48}$$

$$= \frac{6 - |z - 7|}{216} \tag{4.4.49}$$

$$\Pr(Z = z) \Pr(Y = y) \neq \Pr(Z = z, Y = y) \tag{4.4.50}$$

X and Z are not independent from (4.4.50) and hence option (4.4d) is true.

Thus, options (4.4b) and (4.4d) are true.

4.5. Let X be a random variable with a certain non-degenerate distribution. Then identify the correct statements

a) If X has an exponential distribution then $median(X) < E(X)$
b) If X has a uniform distribution on an interval $[a, b]$, then $E(X) < median(X)$
c) If X has a Binomial distribution then $V(X) < E(X)$
d) If X has a normal distribution, then $E(X) < V(X)$

Solution: Expected value($E(X)$): It is nothing but weighted average Median($median(X)$): It is the value separating the higher half from the lower half of a data sample Variance($V(X)$): It is the expectation of the squared deviation of a random variable from its mean

a) Let's consider X has an exponential distribution.

$$X \sim Exp(\lambda) \tag{4.5.1}$$

where λ is rate parameter.
Probability function of exponential distribu-

tion,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{4.5.2}$$

The expected value of $X \sim Exp(\lambda)$,

$$E(X) = \frac{1}{\lambda} \tag{4.5.3}$$

The median of $X \sim Exp(\lambda)$,

$$median(X) = \frac{\ln 2}{\lambda} \tag{4.5.4}$$

$$\ln 2 < 1 \tag{4.5.5}$$

$$\frac{\ln 2}{\lambda} < \frac{1}{\lambda} \tag{4.5.6}$$

$$median(X) < E(X) \tag{4.5.7}$$

Hence, option 1 is correct.

b) Let's consider X has a uniform distribution in interval $[a, b]$,

$$X \sim U(a, b) \tag{4.5.8}$$

where, a = lower limit
b = upper limit
Probability function of uniform distribution,

$$f_X(k) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a, x > b \end{cases} \tag{4.5.9}$$

The expected value of $X \sim U(a, b)$,

$$E(X) = \frac{1}{2}(a + b) \tag{4.5.10}$$

The median of $X \sim U(a, b)$,

$$median(X) = \frac{1}{2}(a + b) \tag{4.5.11}$$

$$E(X) = median(X) \tag{4.5.12}$$

Hence, option 2 is incorrect.

c) Let's consider X has a binomial distribution,

$$X \sim B(n, p) \tag{4.5.13}$$

where, n = no. of trails
p = success parameter
Probability function of binomial distribution,

$$f_X(k) = \begin{cases} {}^nC_k p^k (1-p)^{n-k} & 0 \leq k \leq n \\ 0 & otherwise \end{cases} \tag{4.5.14}$$

The expected value of $X \sim B(n, p)$,

$$E(X) = np \tag{4.5.15}$$

The variance of $X \sim B(n, p)$,

$$V(X) = \sigma^2 = np(1-p) \tag{4.5.16}$$

$$1 - p \leq 1 \tag{4.5.17}$$

$$np(1-p) \leq np \tag{4.5.18}$$

$$V(X) \leq E(X) \tag{4.5.19}$$

Hence, option 3 is incorrect.

d) Let's consider X has a normal distribution,

$$X \sim N\left(\mu, \sigma^2\right) \tag{4.5.20}$$

where, μ = mean of distribution
σ^2 = variance
Probability function of normal distribution,

$$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{2\sigma}\right)^2} \tag{4.5.21}$$

The expected value of $X \sim N\left(\mu, \sigma^2\right)$,

$$E(X) = \mu \tag{4.5.22}$$

The variance of $X \sim N\left(\mu, \sigma^2\right)$,

$$V(X) = \sigma^2 \tag{4.5.23}$$

$E(X)$ and $V(X)$ are user defined. So, they can take any value.
Hence, option 4 is incorrect.

4.6. A and B play a game of tossing a fair coin. A starts the game by tossing the coin once and B then tosses the coin twice, followed by A tossing the coin once and B tossing the coin twice and this continues until a head turns up. Whoever gets the first head wins the game. Then,

a) $P(B \text{ Wins}) > P(A \text{ Wins})$
b) $P(B \text{ Wins}) = 2P(A \text{ Wins})$
c) $P(A \text{ Wins}) > P(B \text{ Wins})$
d) $P(A \text{ Wins}) = 1 - P(B \text{ Wins})$

Solution: Given, a fair coin is tossed till heads turns up.

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{104.1}$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4, 5\}$, such that The state transition matrix for the Markov

TABLE 4.6.1: States and their notations

Notation	State
$S = 1$	A's turn
$S = 2$	B's first turn
$S = 3$	B's second turn
$S = 4$	A wins
$S = 4$	B wins

chain is

$$P = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{104.2}$$

Clearly, the states $1, 2, 3$ are transient, while $4, 5$ are absorbing. The standard form of a state transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{104.3}$$

where, Converting (104.2) to standard form, we

TABLE 4.6.2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatices

get

$$P = \begin{array}{c c} & \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix} \end{array} \tag{104.4}$$

From (104.4),

$$R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \tag{104.5}$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \tag{104.6}$$

where,

$$F = (I - Q)^{-1} \tag{104.7}$$

is called the fundamental matrix of P.
On solving, we get

$$\bar{P} = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5714 & 0.4285 & 0 & 0 & 0 \\ 0.1428 & 0.8571 & 0 & 0 & 0 \\ 0.2857 & 0.7142 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{104.8}$$

A element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability in state j, starting from state i. Then,

a) $Pr(A \text{ wins}) = \bar{p}_{14} \approx 0.5714$
b) $Pr(B \text{ wins}) = \bar{p}_{15} \approx 0.4285$

$$\therefore \bar{p}_{14} > \bar{p}_{15} \tag{104.9}$$

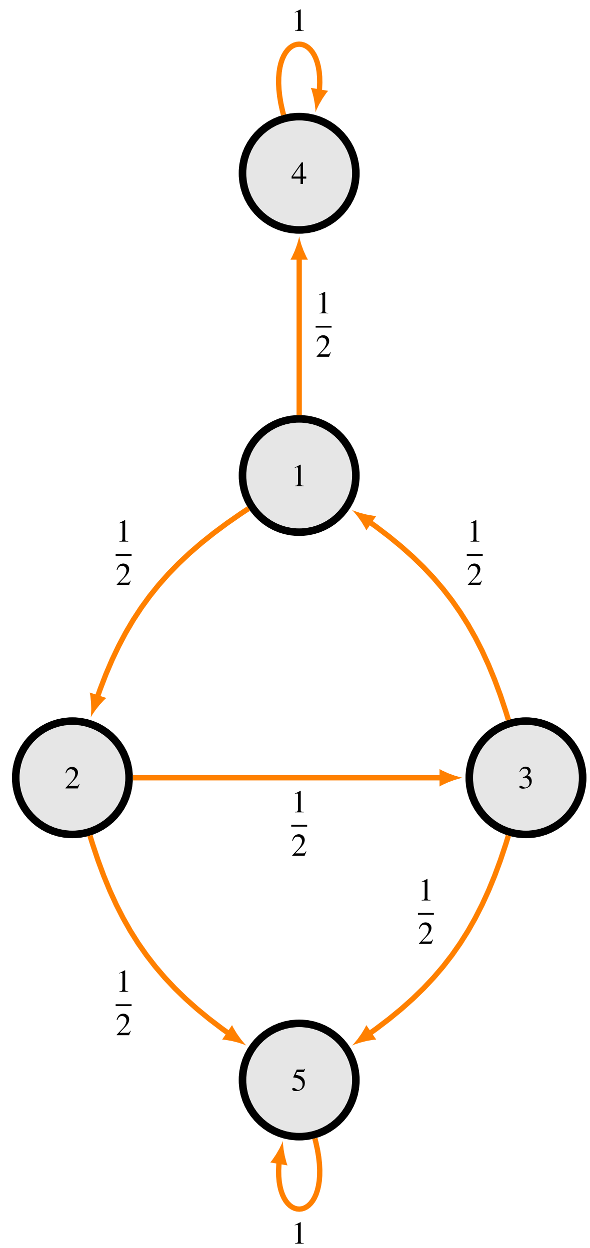
Also, in $\bar{P}$, all the terms in every row should sum to 1.

$$\Rightarrow \bar{p}_{14} + \bar{p}_{15} + 0 + 0 + 0 = 1 \tag{104.10}$$

$$\therefore \bar{p}_{14} = 1 - \bar{p}_{15} \tag{104.11}$$

Therefore, options 3), 4) are correct.

Markov chain diagram



5 June 2016

5.1. The joint probability density function of (X,Y) is

$$f(x, y) = \begin{cases} 6(1 - x) & if \quad 0 < y < x, 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.1.1}$$

Which among the following are correct?

a) X and Y are not independent

b) $f_Y(y) = \begin{cases} 3(y - 1)^2 & if \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

c) X and Y are independent

d) $f_Y(y) = \begin{cases} 3\left(y - \frac{1}{2}y^2\right) & if \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

Solution: Given joint probability density function of X and Y, marginal probability density functions are as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy \tag{5.1.2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx \tag{5.1.3}$$

Calculating $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{5.1.4}$$
$$= \int_0^x 6(1-x)dy \tag{5.1.5}$$

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{5.1.6}$$

Calculating $f_Y(y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{5.1.7}$$
$$= \int_y^1 6(1-x)dx \tag{5.1.8}$$
$$=6x - 3x^2\Big|_y^1 \tag{5.1.9}$$
$$=3 - 6y + 3y^2 \tag{5.1.10}$$
$$=3(y-1)^2 \tag{5.1.11}$$

$$f_Y(y) = \begin{cases} 3(y-1)^2 & 0 < y < 1 \\ 0 & otherwise \end{cases} \tag{5.1.12}$$

To check whether X and Y are independent, we calculate $f_X(x) \times f_Y(y)$. From (5.1.6) and (5.1.12)

$$f_X(x) \times f_Y(y) = \begin{cases} 18x(1-x)(y-1)^2 \\ \qquad 0 < x < 1, 0 < y < 1 \\ 0 \qquad \text{otherwise} \end{cases} \tag{5.1.13}$$
$$\neq f(x,y) \tag{5.1.14}$$

Since f(x,y) and $f_X(x) \times f_Y(y)$ are different, random variables X and Y are not independent.

Options 1 and 2 are correct

5.2. Three types of components are used in electrical circuits 1, 2, 3 as shown below in the figure
Solution: For q_1, the truth table Multiplying

A	B	C	$(AB)+C$
1	1	0	1
1	1	1	1
0	1	1	1
0	0	1	1
1	0	1	1

TABLE 5.2.1: Circuit 1 working

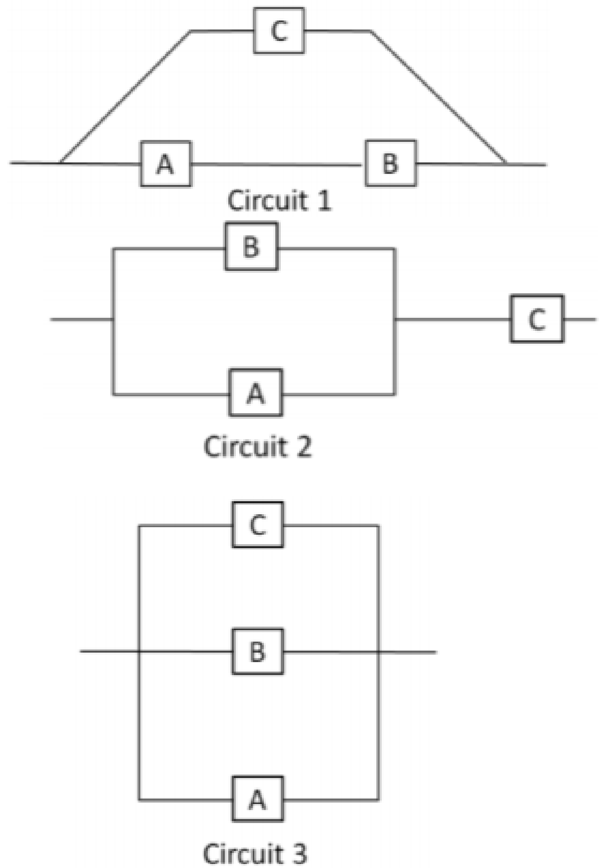


Fig. 5.2.1: Figure

and adding probability for each case of q_1 gives us the value of q_1 as

$$q_1 = p^3 - 2p^2 + 1 \tag{5.2.1}$$

For q_2, the truth table Multiplying and adding

A	B	C	$(A+B)C$
1	1	1	1
1	0	1	1
0	1	1	1

TABLE 5.2.2: Circuit 2 working

probability for each case of q_2 gives us the value of q_2 as

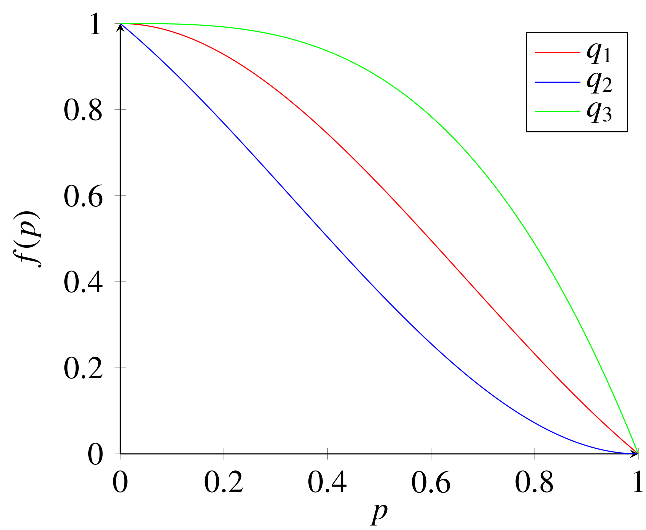
$$q_2 = p^3 - p^2 - p + 1 \tag{5.2.2}$$

For q_3, the truth table Multiplying and adding probability for each case of q_3 gives us the value of q_3 as

$$q_3 = 1 - p^3 \tag{5.2.3}$$

A	B	C	$A+B+C$
1	0	0	1
0	1	0	1
0	0	1	1
1	1	0	1
1	0	1	1
0	1	1	1
1	1	1	1

TABLE 5.2.3: Circuit 3 working



$$\therefore q_3 > q_1 > q_2 \tag{5.2.4}$$

Hence **Option 1** is correct

5.3. Suppose X and Y are independent and identically distributed random variables and let $Z = X + Y$. Then the distribution of Z is in the same family as that of X and Y if X is
Solution:

1) Normal
2) Exponential
3) Uniform
4) Binomial

1) Let X and Y be independent and identically distributed normal random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = e^{j\eta\omega - \sigma^2\omega^2/2} \tag{5.3.1}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{5.3.2}$$
$$= e^{2j\eta\omega - \sigma^2\omega^2} \tag{5.3.3}$$

Thus Z is a normal random variable with parameters 2η and $2\sigma^2$. Thus option (1) is correct.

2) Let X and Y be independent and identically distributed exponential random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = \frac{\lambda}{1 - j\omega} \tag{5.3.4}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{5.3.5}$$
$$= \frac{\lambda^2}{(1 - j\omega)^2} \tag{5.3.6}$$

Thus Z is not an exponential random variable. Therefore option (2) is wrong.

3) Let X and Y be independent and identically distributed uniform random variables such that X, Y $\sim$ U(a,b). Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = \frac{e^{jb\omega} - e^{ja\omega}}{j\omega(b-a)} \tag{5.3.7}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{5.3.8}$$
$$= -\frac{(e^{jb\omega} - e^{ja\omega})^2}{\omega^2(b-a)^2} \tag{5.3.9}$$

Thus Z is not a uniform random variable. Thus option (3) is wrong.

4) Let X and Y be independent and identically distributed binomial random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = (pe^{j\omega} + q)^n \tag{5.3.10}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{5.3.11}$$
$$= (pe^{j\omega} + q)^{2n} \tag{5.3.12}$$

Thus Z is a binomial random variable with parameter 2n. Thus option (4) is correct.

The following figures show the experimental distributions for Z in each case. The simulation length was kept one million.

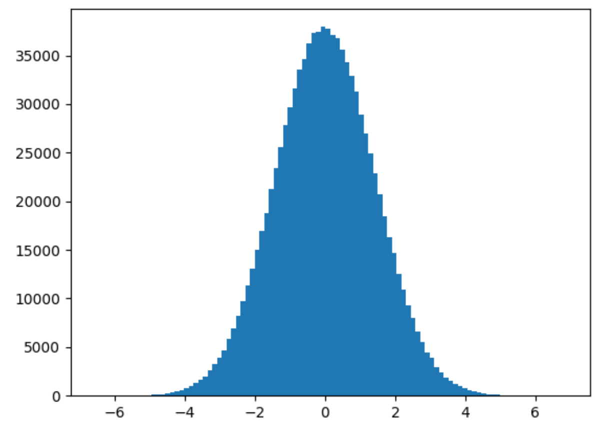


Fig. 5.3.1: Z when X is standard normal

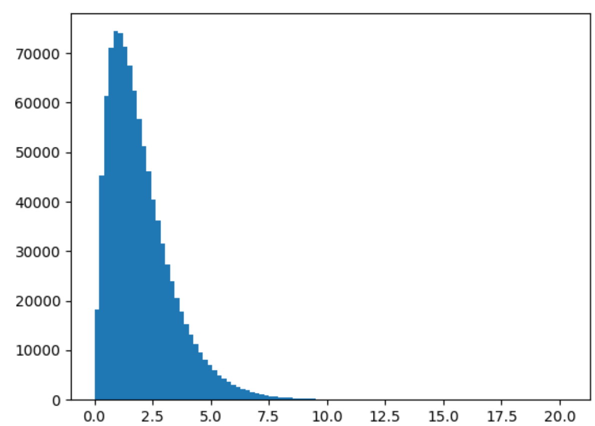


Fig. 5.3.2: Z when X is exponential with $\lambda = 1$

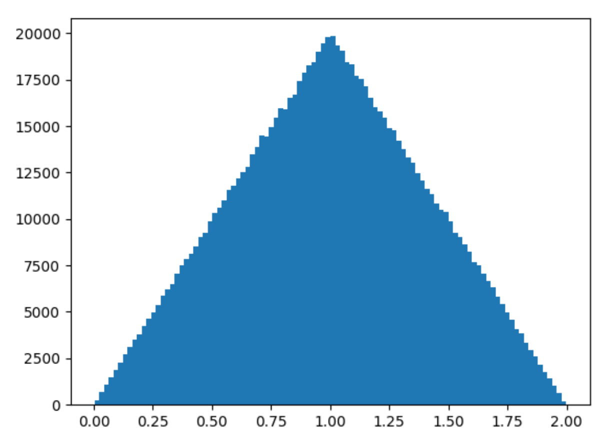


Fig. 5.3.3: Z when X $\sim$ U(0,1)

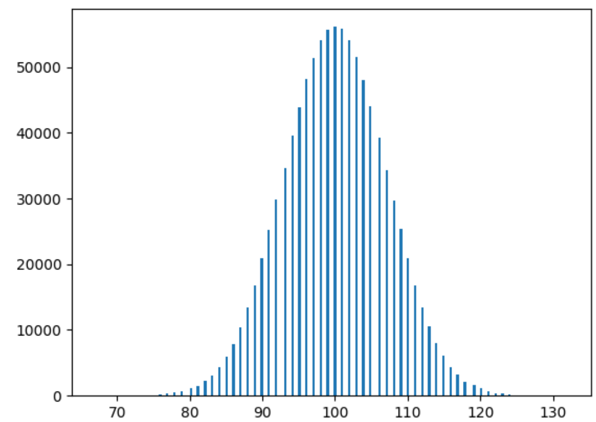


Fig. 5.3.4: Z when X $\sim$ B(100,0.5)

6 December 2015

6.1. The probability that a ticketless traveler is caught during a trip is 0.1. If the traveler makes 4 trips , the probability that he/she will be caught during at least one of the trips is:

a) $1 - (0.9)^4$
b) $(1 - 0.9)^4$
c) $1 - (1 - 0.9)^4$
d) $(0.9)^4$

Solution: Let $X_i \in \{0, 1\}$ represent the ith trip where 1 denotes a ticketless traveller is caught. Given,

$$\Pr(X_i = 1) = p = 0.1 \tag{6.1.1}$$

Let,

$$X = \sum_{i=1}^{n} X_i \tag{6.1.2}$$

where n is the number of trips and X has a binomial distribution.

$$p_X(k) = \begin{cases} {}^nC_k \, p^K (1-p)^{n-k}, & 0 \le k \le n \\ 0, & otherwise \end{cases} \tag{6.1.3}$$

As he/she makes 4 trips in total, Using (6.1.1) and (6.1.3),

$$\Pr(X = 0) = p_X(0) \tag{6.1.4}$$
$$= {}^4C_0 \, p^0 (1-p)^4 \tag{6.1.5}$$
$$\Pr(X = 0) = (0.9)^4 \tag{6.1.6}$$

Then probability of being caught in atleast one trip is,(Using (6.1.6))

$$\Pr(X \geq 1) = 1 - \Pr(X < 1) \tag{6.1.7}$$
$$= 1 - \Pr(X = 0) \tag{6.1.8}$$
$$= 1 - (0.9)^4 \tag{6.1.9}$$

6.2. Suppose that (X, Y) has a joint probability distribution with the marginal distribution of X being N(0,1) and $E(Y|X = x) = x^3$ for all $x \in R$. Then, which of the following statements are true?

a) Corr$(X, Y) = 0$
b) Corr$(X, Y) > 0$
c) Corr$(X, Y) < 0$
d) X and Y are independent

Solution: The following result shall be useful later. For $n \in N$

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \begin{cases} 0 & n \text{ is odd} \\ (n-1) \times ... \times 3 \times 1 & n \text{ is even} \end{cases} \tag{6.2.1}$$

The proof for the above can be found at the end of the solution.

$$Corr(X, Y) = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} \tag{6.2.2}$$

We know $X \sim N(0, 1)$. Thus,

$$f_X(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \tag{6.2.3}$$
$$E(X) = 0 \tag{6.2.4}$$
$$\sigma_X^2 = 1 \tag{6.2.5}$$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 \tag{6.2.6}$$

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx \tag{6.2.7}$$
$$= \int_{-\infty}^{\infty} \frac{x^3 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{6.2.8}$$
$$= 0 \tag{6.2.9}$$

$$E(Y^2) = \int_{-\infty}^{\infty} E(Y^2|X = x) f_X(x) dx \tag{6.2.10}$$
$$= \int_{-\infty}^{\infty} \frac{x^6 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{6.2.11}$$
$$= 15 \tag{6.2.12}$$

Substituting in (6.2.6)

$$\sigma_Y^2 = 15 \tag{6.2.13}$$

$$\sigma_{XY}^2 = E(XY) - E(X)E(Y) \tag{6.2.14}$$

$$E(XY) = \int_{-\infty}^{\infty} E(XY|X = x) f_X(x) dx \tag{6.2.15}$$
$$= \int_{-\infty}^{\infty} E(xY|X = x) f_X(x) dx \tag{6.2.16}$$
$$= \int_{-\infty}^{\infty} x E(Y|X = x) f_X(x) dx \tag{6.2.17}$$
$$= \int_{-\infty}^{\infty} \frac{x^4 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{6.2.18}$$
$$= 3 \tag{6.2.19}$$

Substituting in (6.2.14)

$$\sigma_{XY}^2 = 3 \tag{6.2.20}$$

Substituting in (6.2.2)

$$Corr(X, Y) = \frac{3}{\sqrt{15}} > 0 \tag{6.2.21}$$

Since $Corr(X, Y) \neq 0$, X and Y are dependent. Thus option 2 is the only correct option. **Proof for the integral:** If n is odd, $\frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ is an odd function, thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = 0 \tag{6.2.22}$$

If n is even,

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} (x^{n-1}) \left(\frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \right) dx \tag{6.2.23}$$

Using integration by parts,

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \left(x^{n-1} \int \frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \right) \Big|_{-\infty}^{\infty}$$

$$- (n-1) \int_{-\infty}^{\infty} x^{n-2} \left(\int \frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \right) dx \quad (6.2.24)$$

$$= \left(x^{n-1} \left(- \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\right) \right) \Big|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} \left(- \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\right) dx \quad (6.2.25)$$

$$= (n-1) \int_{-\infty}^{\infty} \frac{x^{n-2} e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \quad (6.2.26)$$

$$= (n-1)(n-3) \int_{-\infty}^{\infty} \frac{x^{n-4} e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \quad (6.2.27)$$

$$= (n-1) \times ... \times 3 \times 1 \int_{-\infty}^{\infty} \frac{x^0 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \quad (6.2.28)$$

$$= (n-1) \times ... \times 3 \times 1 \quad (6.2.29)$$

Alternative proof for the integral:

If n is odd, $\frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ is an odd function, thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = 0 \quad (6.2.30)$$

If n is even, let $n = 2k$. We differentiate the following identity k times w.r.t. α.

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\left(\frac{\pi}{\alpha}\right)} \quad (6.2.31)$$

On differentiating k times, we get

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2} = \frac{1 \times 3 \times ... \times (2k-1)}{2^k} \sqrt{\left(\frac{\pi}{\alpha^{2k+1}}\right)} \quad (6.2.32)$$

On substituting $\alpha = \frac{1}{2}$, we get

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} = 1 \times 3 \times ... \times (n-1) \sqrt{2\pi} \quad (6.2.33)$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = (n-1) \times ... \times 3 \times 1 \quad (6.2.34)$$

6.3. Let $X_1, X_2, ..., X_n$ be independent and identically distributed, each having a uniform distribution on $(0, 1)$. Let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Then, which of the following statements are true?

A) $\frac{S_n}{n \log n} \to 0$ as $n \to \infty$ with probability 1.
B) $\Pr\left(\left(S_n > \frac{2n}{3}\right) \text{occurs for infinitely many n}\right) = 1$
C) $\frac{S_n}{\log n} \to 0$ as $n \to \infty$ with probability 1.
D) $\Pr\left(\left(S_n > \frac{n}{3}\right) \text{occurs for infinitely many n}\right) = 1$

Solution:

Symbol	expression/definition
S_n	$\sum_{i=1}^{n} X_i$
μ_n	$\frac{1}{n} \sum_{i=1}^{n} X_i$
X	Independent continuous random variable identical to $X_1, X_2, ..., X_n$

TABLE 6.3.1: Variables and their definitions

a) Given

$$S_n = \sum_{i=1}^{n} X_i, n \geq 1 \quad (6.3.1)$$

Dividing by n on both sides

$$\frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i = \mu_n \quad (6.3.2)$$

It can be said that $X_1, X_2, ..., X_n$ are the trials of X. By definition

$$E[X] = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = \lim_{n\to\infty} \frac{S_n}{n} \quad (6.3.3)$$

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X] = \frac{1}{2} \quad (6.3.4)$$

$$\therefore \lim_{n\to\infty} \frac{S_n}{n \log n} = 0 \quad (6.3.5)$$

b) Using weak law, (6.3.4), and table (6.3.1)

$$\lim_{n\to\infty} \Pr(|\mu_n - E[X]| > \epsilon) = 0, \forall \epsilon > 0 \quad (6.3.6)$$

$$\lim_{n\to\infty} \Pr\left(S_n = \frac{n}{2}\right) = 1 \quad (6.3.7)$$

It can be easily implied from (6.3.7) that option B is false.

c) It is easy to observe from (6.3.4) that option C is false.

d) Using (6.3.7), we get

$$\Pr\left(\left(S_n > \frac{n}{3}\right)\text{occurs for infinitely many n}\right) = 1 \tag{6.3.8}$$

6.4. A fair coin is tossed repeatedly. Let X be the number of tails before the first heads occurs. Let Y denote the number of tails between the first and second heads. Let $X + Y = N$. Then which of the following are true?

a) X and Y are independent random variables with

$$\Pr(X = k) = \Pr(Y = k) = \begin{cases} 2^{-(k+1)} & k = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \tag{6.4.1}$$

b) N has a probability mass function given by

$$\Pr(N = k) = \begin{cases} (k-1)2^{-k} & k = 2, 3, 4 \dots \\ 0 & otherwise \end{cases} \tag{6.4.2}$$

c) Given $N = n$, the conditional distribution of X and Y are independent

d) Given $N = n$

$$\Pr(X = k) = \begin{cases} \frac{1}{n+1} & n = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \tag{6.4.3}$$

6.5. An urn has 3 red and 6 black balls. Balls are drawn at random one by one without replacement. The probability that second red ball appears on fifth draw is:

a) $\frac{1}{9!}$

b) $\frac{4!}{9!}$

c) $4\left(\frac{6!4!}{9!}\right)$

d) $\frac{6!4!}{9!}$

Solution: To obtain a second red ball at the fifth draw, the first 4 trials should involve drawing only 1 red ball out of the 3 and 3 black balls out of the 6. Probability of this happening:

$$\frac{{}^3C_1 {}^6C_3}{{}^9C_4} \tag{6.5.1}$$

The probability of the fifth ball turning out to be red is:

$$\frac{{}^2C_1}{{}^5C_1} \tag{6.5.2}$$

By Multiplication rule, total probability:

$$\frac{{}^3C_1 {}^6C_3 {}^2C_1}{{}^5C_1 {}^9C_4} = \frac{3! \times 6! \times 2! \times 4! \times 4! \times 5!}{2! \times 3! \times 3! \times 5! \times 9!} \tag{6.5.3}$$

$$= 4\left(\frac{4!6!}{9!}\right) \tag{6.5.4}$$

6.6. Let $X_i's$ be independent random variables such that $X_i's$ are symmetric about 0 and $var(X_i) = 2i - 1$,for $i \geq 1$.then, $\lim_{n\to\infty} \Pr(X_1 + X_2 + \cdots + X_n > n \log n)$

a) does not exist.

b) equals $\frac{1}{2}$.

c) equals 1.

d) equals 0.

Solution: Let $X = X_1 + X_2 + \cdots + X_n$, as $X_i's$ are symmetric about 0. The mean of X is given by,

$$E[X] = 0 \tag{6.6.1}$$

the variance of X is given by,

$$var[X] = \sum_{i=1}^{n}(2i - 1) \tag{6.6.2}$$

$$= \frac{2n(n+1)}{2} - n \tag{6.6.3}$$

$$= n^2 \tag{6.6.4}$$

the standard deviation,

$$\sigma_X = n \tag{6.6.5}$$

Applying Chebyshev's Inequality for the random variable X, for any $k > 0$

$$\Pr(|X - E[X]| > k\sigma_X) \leq \frac{1}{k^2} \tag{6.6.6}$$

let $k = \log n$, using (6.6.1) and (6.6.5) in

(6.6.6),

$$\Pr\left(|X| > n\log n\right) \le \frac{1}{(\log n)^2} \tag{6.6.7}$$

$$\Pr\left(X > n\log n\right) + \Pr\left(X < -n\log n\right) \le \frac{1}{(\log n)^2} \tag{6.6.8}$$

As, X is symmetric about 0,

$$\Pr\left(X > n\log n\right) = \Pr\left(X < -n\log n\right) \tag{6.6.9}$$

using (6.6.9) in (6.6.8),

$$2\Pr\left(X > n\log n\right) \le \frac{1}{(\log n)^2} \tag{6.6.10}$$

$$\Pr\left(X > n\log n\right) \le \frac{1}{2(\log n)^2} \tag{6.6.11}$$

as any probability is greater than 0,

$$0 < \Pr\left(X > n\log n\right) \le \frac{1}{2(\log n)^2} \tag{6.6.12}$$

applying sandwich principle to (6.6.12),

$$\lim_{n\to\infty} 0 < \lim_{n\to\infty} \Pr\left(X > n\log n\right) \le \lim_{n\to\infty} \frac{1}{2(\log n)^2} \tag{6.6.13}$$

$$\lim_{n\to\infty} \Pr\left(X_1 + X_2 + \cdots + X_n > n\log n\right) = 0 \tag{6.6.14}$$

Hence the option.4 is correct.

7 December 2014

7.1. $N, A_1, A_2 \cdots$ are independent real valued random variables such that

$$\Pr\left(N = k\right) = (1-p)p^k, k = 0, 1, 2, 3\cdots \tag{7.1.1}$$

where $0 < p < 1$ and $\{A_i : i = 1, 2, \cdots\}$ is a sequence of independent and identically distributed bounded random variables. Let

$$X(w) = \begin{cases} 0 & \text{if } N(w) = 0 \\ \sum_{j=1}^{k} A_j & \text{if } N(w) = k, k = 1, 2, 3\cdots \end{cases} \tag{7.1.2}$$

Which of the following are necessarily correct?

a) X is a bounded random variable.

b) Moment generating function m_X of X is

$$m_X(t) = \frac{1-p}{1-pm_A(t)}, t \in \mathbb{R}, \tag{7.1.3}$$

where m_A is moment generating function of A_1.

c) Characteristic function φ_X of X is

$$\varphi_X(t) = \frac{1-p}{1-p\varphi_A(t)}, t \in \mathbb{R}, \tag{7.1.4}$$

where φ_A is the characteristic function of A_1.

d) X is symmetric about 0.

7.2. Consider a Markov chain with state space 1,2,....,100. Suppose states 2i and 2j communicate with each other and states 2i-1 and 2j-1 communicate with each other for every i,j = 1,2,...,50. Further suppose that $p^{(2)}_{3,3}$ ¿ 0,$p^{(3)}_{4,4}$ ¿ 0 and $p^{(7)}_{2,5}$ ¿ 0. Then

a) The Markov chain is irreducible.

b) The Markov chain is aperiodic.

c) State 8 is recurrent.

d) State 9 is recurrent.

Solution:

8 June 2013

8.1. Let X be a non-negative integer valued random variable with probability mass function $f(x)$ satisfying $(x+1)f(x+1) = (\alpha + \beta x)f(x)$, $x = 0, 1, 2, ...$; $\beta \neq 1$. You may assume that $E(X)$ and $Var(X)$ exist. Then which of the following statements are true?

a) $E(X) = \dfrac{\alpha}{1-\beta}$

b) $E(X) = \dfrac{\alpha^2}{(1-\beta)(1+\alpha)}$

c) $Var(X) = \dfrac{\alpha^2}{(1-\beta)^2}$

d) $Var(X) = \dfrac{\alpha}{(1-\beta)^2}$

Solution: For a discrete random variable X with P.D.F. $f(x)$ and which can take values from a set $\mathbb{S}$,

$$E(X) = \sum_{x\in\mathbb{S}} xf(x) \tag{8.1.1}$$

And,

$$E(X^2) = \sum_{x\in\mathbb{S}} x^2 f(x) \tag{8.1.2}$$

Also, as $f(x)$ is the P.D.F.,

$$\sum_{x\in\mathbb{S}} f(x) = 1 \tag{8.1.3}$$

Given, for $x \in \mathbb{S} = \{0, 1, 2, ...n\}$,

$$(x+1)f(x+1) = (\alpha + \beta x)f(x) \tag{8.1.4}$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)f(x+1) = \sum_{x=0}^{n}(\alpha + \beta x)f(x) \tag{8.1.5}$$

Replacing $x + 1$ with x in L.H.S. we get,

$$\sum_{x=1}^{n+1} xf(x) = \sum_{x=0}^{n}(\alpha + \beta x)f(x) \tag{8.1.6}$$

Rewriting LHS, we get,

$$\sum_{x=0}^{n} xf(x) + (n+1)f(n+1) = \sum_{x=0}^{n}(\alpha + \beta x)f(x) \tag{8.1.7}$$

But as $x \in \{0, 1, 2...n\}$, $f(n+1) = 0$. So the equation becomes

$$\sum_{x=0}^{n} xf(x) = \alpha\sum_{x=0}^{n} f(x) + \beta\sum_{x=0}^{n} xf(x) \tag{8.1.8}$$

Using (8.1.1) and (8.1.3), we get,

$$E(X) = \alpha(1) + \beta E(X) \tag{8.1.9}$$

So,

$$E(X) = \frac{\alpha}{1-\beta} \tag{8.1.10}$$

Now in (8.1.4), multiplying both sides by $(x + 1)$, we get,

$$(x+1)^2 f(x+1) = (\alpha + \beta x)(x+1)f(x) \tag{8.1.11}$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)^2 f(x+1) = \sum_{x=0}^{n}(\alpha + \beta x)(x+1)f(x) \tag{8.1.12}$$

Replacing $x + 1$ with x in L.H.S. we get,

$$\sum_{x=1}^{n+1} x^2 f(x) = \sum_{x=0}^{n}(\beta x^2 f(x) + (\alpha + \beta)xf(x) + \alpha f(x)) \tag{8.1.13}$$

Rewriting LHS similarly as before, we get,

$$\sum_{x=0}^{n} x^2 f(x) = \beta\sum_{x=0}^{n} x^2 f(x) + (\alpha + \beta)\sum_{x=0}^{n} xf(x) + \alpha\sum_{x=0}^{n} f(x) \tag{8.1.14}$$

Using (8.1.1), (8.1.2) and (8.1.3), we get,

$$E(X^2) = \beta E(X^2) + (\alpha + \beta)E(X) + \alpha(1) \tag{8.1.15}$$

Using (8.1.10)

$$E(X^2)(1-\beta) = \frac{\alpha(\alpha+\beta)}{1-\beta} + \alpha \tag{8.1.16}$$

So,

$$E(X^2) = \frac{\alpha^2 + \alpha}{(1-\beta)^2} \tag{8.1.17}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{8.1.18}$$

Using (8.1.10) and (8.1.17),

$$Var(X) = \frac{\alpha^2 + \alpha}{(1-\beta)^2} - \frac{\alpha^2}{(1-\beta)^2} \tag{8.1.19}$$

So,

$$Var(X) = \frac{\alpha}{(1-\beta)^2} \tag{8.1.20}$$

So, options 1 and 4 are correct.

8.2. Let X be a random variable with probability density function,

$$f(x) = \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^\alpha} \tag{8.2.1}$$

such that $-\infty < \mu < \infty$; $\alpha > 0$; $x > \mu$, The hazard function is:

a) constant for all α
b) an increasing function for some α
c) independent of α
d) independent of μ when $\alpha = 1$

Solution: Given PDF of X,

$$f(x) = \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^\alpha} \tag{8.2.2}$$

Important property(using in (8.2.8) as $x > \mu$): Given $x - y > 0$ and $-\infty < y < \infty$, then

$$\lim_{x\to-\infty} x - y = 0 \tag{8.2.3}$$

CDF of X,

$$F(x) = \int_{-\infty}^{x} f(x)\, dx \quad (8.2.4)$$
$$= \int_{-\infty}^{x} \alpha(x-\mu)^{\alpha-1} e^{-(x-\mu)^{\alpha}} dx \quad (8.2.5)$$
$$= \int_{-\infty}^{x} e^{-(x-\mu)^{\alpha}} d(x-\mu)^{\alpha} \quad (8.2.6)$$
$$= \left[\frac{e^{-(x-\mu)^{\alpha}}}{-1}\right]_{-\infty}^{x} \quad (8.2.7)$$
$$= -e^{-(x-\mu)^{\alpha}} - \lim_{x\to-\infty} \frac{e^{-(x-\mu)^{\alpha}}}{-1} \quad (8.2.8)$$
$$= -e^{-(x-\mu)^{\alpha}} + e^{-(0)^{\alpha}} \quad (8.2.9)$$
$$F(x) = 1 - e^{-(x-\mu)^{\alpha}} \quad (8.2.10)$$

Hazard function $\beta(x)$,(using (8.2.2) and (8.2.10))

$$\beta(x) = \frac{f(x)}{1-F(x)} \quad (8.2.11)$$
$$= \frac{\alpha(x-\mu)^{\alpha-1} e^{-(x-\mu)^{\alpha}}}{1-(1-e^{-(x-\mu)^{\alpha}})} \quad (8.2.12)$$
$$= \frac{\alpha(x-\mu)^{\alpha-1} e^{-(x-\mu)^{\alpha}}}{e^{-(x-\mu)^{\alpha}}} \quad (8.2.13)$$
$$\beta(x) = \alpha(x-\mu)^{\alpha-1} \quad (8.2.14)$$

a) $\beta(x)$ is not constant for all α
b) $\beta(x) = \alpha(x-\mu)^{\alpha-1}$ is an increasing function for $\alpha < 0$ *or* $\alpha > 1$ as given $x-\mu > 0$ for all x.
   Proof: Using first derivative test, A function is increasing iff its first derivative is positive for all x.

$$\frac{d}{dx}\beta(x) = \frac{d}{dx}\alpha(x-\mu)^{\alpha-1} \quad (8.2.15)$$
$$= \alpha(\alpha-1)(x-\mu)^{\alpha-2} \quad (8.2.16)$$

   For (8.2.16) to be positive, (As given $x-\mu > 0$)

$$\alpha(\alpha-1)(x-\mu)^{\alpha-2} > 0 \quad (8.2.17)$$
$$\alpha(\alpha-1) > 0 \quad (8.2.18)$$
$$\implies \alpha \in (-\infty, 0) \cup (1, \infty) \quad (8.2.19)$$

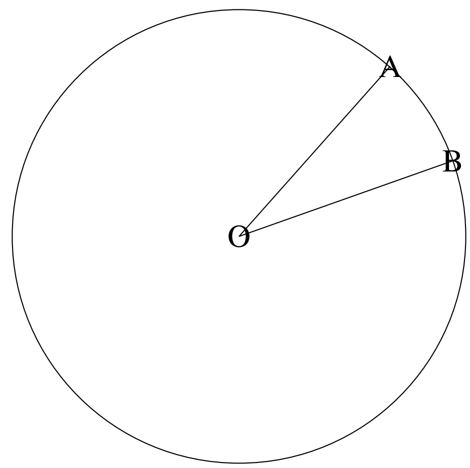
   $\therefore \beta(x)$ an increasing function for some α
c) $\beta(x)$ is dependent of α
d) when $\alpha = 1$,

$$\beta(x) = \alpha(x-\mu)^{0} = \alpha \quad (8.2.20)$$

   Therefore the hazard function is independent of μ when $\alpha = 1$.

ANSWER: (2) and (4)

8.3. A point is chosen at random from a circular disc shown below. What is the probability that the point lies in the sector OAB?



(where ∠AOB = x radians)

a) $\frac{2x}{\pi}$
b) $\frac{x}{\pi}$
c) $\frac{x}{2\pi}$
d) $\frac{x}{4\pi}$

Solution:
Let $X \in \{0, 1\}$ be a random variable such that X=0 means we choose a point lying in sector OAB and X=1 means that we choose a point lying outside sector OAB and inside the circle.
Area of a sector subtending an angle θ at the centre of circle with radius a is given by :

$$A = \frac{1}{2}a^2\theta \quad (8.3.1)$$

where θ is in radians.
Let the radius of circle shown in figure be r. It is given that sector OAB subtends an angle of x radians at the centre of the circle.
Probability that the chosen point lies in sector OAB is:

$$\Pr(X=0) = \frac{\text{Area of sector OAB}}{\text{Area of circle}} \quad (8.3.2)$$
$$= \frac{\frac{1}{2}r^2 x}{\pi r^2} \quad (8.3.3)$$
$$= \frac{x}{2\pi} \quad (8.3.4)$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$.

ALTERNATE SOLUTION

The joint pdf is given by:

$$f_{r\theta}(r,\theta) = \begin{cases} \frac{r}{\pi R^2} & \text{if } 0 < r < R\ ,\ 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{8.3.5}$$

Let A ≡ (R, θ_2) and B ≡ (R, θ_1).
Hence,

$$(\theta_2 - \theta_1) = x \tag{8.3.6}$$

We want $\theta \in (\theta_1\ ,\ \theta_2)$ and r ∈ (0,R) for point to lie in the sector. Let the point to be chosen be (r, θ).
So, Required probability is:

$$\Pr(\theta_1 < \theta < \theta_2, 0 < r < R)$$
$$= \int_{\theta_1}^{\theta_2}\int_0^R \frac{r}{\pi R^2}\, dr\, d\theta \tag{8.3.7}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{\pi R^2}\frac{r^2}{2}\Big|_0^R \tag{8.3.8}$$

$$= \int_{\theta_1}^{\theta_2} \frac{R^2}{2\pi R^2}\, d\theta \tag{8.3.9}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2\pi}\, d\theta \tag{8.3.10}$$

$$= \frac{\theta}{2\pi}\Big|_{\theta_1}^{\theta_2} \tag{8.3.11}$$

$$= \frac{\theta_2 - \theta_1}{2\pi} \tag{8.3.12}$$

$$= \frac{x}{2\pi} \tag{8.3.13}$$

∴The correct answer is **option (3)** $\frac{x}{2\pi}$.

8.4. Let X and Y be independent random variables each following a uniform distribution on $(0,1)$.Let $W = XI_{\{Y \le X^2\}}$,where I_A denotes the indicator function of set A.Then which of the following statements are true?

a) The cumulative distribution function of W is given by

$$F_W(t) = t^2 I_{\{0 \le t \le 1\}} + I_{\{t>1\}} \tag{8.4.1}$$

b) $P[W > 0] = \frac{1}{3}$

c) The cumulative distribution function of W is continuous

d) The cumulative distribution function of W is given by

$$F_W(t) = \left(\frac{2+t^3}{3}\right) I_{\{0 \le t \le 1\}} + I_{\{t>1\}} \tag{8.4.2}$$

Solution:
Given X and Y are two independent random variables.
Given $W = XI_{\{Y \le X^2\}}$
$X \in (0,1)$, $Y \in (0,1)$, $W \in [0,1)$

a) We need to find CDF of W
  i) The PDF for X is
$$p_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{8.4.3}$$
  ii) The CDF for X is
$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & otherwise \end{cases} \tag{8.4.4}$$
  iii) The PDF for Y is
$$p_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & otherwise \end{cases} \tag{8.4.5}$$
  iv) The CDF for Y is
$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ y & 0 < y < 1 \\ 1 & otherwise \end{cases} \tag{8.4.6}$$
  v) $I_{\{Y \le X^2\}}$ is defined as follows
$$I_{\{Y \le X^2\}} = \begin{cases} 1 & y \le x^2 \\ 0 & otherwise \end{cases} \tag{8.4.7}$$
  vi) W is defined as follows
$$W = \begin{cases} x & y \le x^2 \\ 0 & otherwise \end{cases} \tag{8.4.8}$$
From (8.4.8)
$$p_W(W = 0) = \Pr(I_{\{Y \le X^2\}} = 0) \tag{8.4.9}$$
$$= \Pr(x^2 < y) \tag{8.4.10}$$
  vii) Let $Z = X^2 - Y$ be a random variable where $Z \in (-1, 1)$
$$F_{X^2}(u) = \Pr(X^2 \le u) \tag{8.4.11}$$
$$= \Pr(X \le \sqrt{u}) \tag{8.4.12}$$
$$= F_X(\sqrt{u}) \tag{8.4.13}$$
   A) From (8.4.4),The CDF for X^2 is
$$F_{X^2}(u) = \begin{cases} 0 & u \le 0 \\ \sqrt{u} & 0 < u < 1 \\ 1 & otherwise \end{cases} \tag{8.4.14}$$

B) The PDF for X^2 is

$$p_{X^2}(u) = \begin{cases} \frac{1}{2\sqrt{u}} & 0 < u < 1 \\ 0 & otherwise \end{cases} \tag{8.4.15}$$

$$F_{\{-Y\}}(v) = \Pr(-Y \le v) \tag{8.4.16}$$
$$= \Pr(Y \ge -v) \tag{8.4.17}$$
$$= 1 - F_Y(-v) \tag{8.4.18}$$

C) From (8.4.6),The CDF for $(-Y)$ is

$$F_{\{-Y\}}(v) = \begin{cases} 0 & v \le -1 \\ 1 + v & -1 < v < 0 \\ 1 & otherwise \end{cases} \tag{8.4.19}$$

D) The PDF for $(-Y)$ is

$$p_{\{-Y\}}(v) = \begin{cases} 1 & -1 < v < 0 \\ 0 & otherwise \end{cases} \tag{8.4.20}$$

E) $Z = X^2 - Y \implies z = u + v$
Using convolution

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X^2}(z - v) p_{\{-Y\}}(v) \mathrm{dv} \tag{8.4.21}$$

Solving (8.4.21) using (8.4.20),(8.4.15) for $z \in (-1, 1)$, we get PDF of Z as follows

$$p_Z(z) = \begin{cases} \sqrt{z + 1} & -1 < z \le 0 \\ 1 - \sqrt{z} & 0 < z < 1 \\ 0 & otherwise \end{cases} \tag{8.4.22}$$

F) CDF of Z as follows

$$F_Z(z) = \begin{cases} \frac{2}{3}(z + 1)^{\frac{3}{2}} & -1 < z \le 0 \\ z - \frac{2}{3}z^{\frac{3}{2}} & 0 < z < 1 \\ 1 & otherwise \end{cases} \tag{8.4.23}$$

viii) using (8.4.23) to find $p_W(W = 0)$

$$p_W(W = 0) = \Pr(x^2 < y) \tag{8.4.24}$$
$$= F_z(0) \tag{8.4.25}$$
$$= \frac{2}{3} \tag{8.4.26}$$

ix) $W = t \implies X = t$ where $t \in (0, 1)$

$$p_W(t) = \int_{-\infty}^{\infty} p_X(t) I_{\{y \le t^2\}} \mathrm{dy} \tag{8.4.27}$$

$$0 < y < 1 \tag{8.4.28}$$
$$0 < y \le t^2 \tag{8.4.29}$$

For $0 < t < 1$,

$$p_W(t) = \int_0^{t^2} p_X(t) I_{\{y \le t^2\}} \mathrm{dy} \tag{8.4.30}$$

$$= t^2 \tag{8.4.31}$$

x) $\therefore$ PDF of W is as follows

$$p_W(t) = \begin{cases} \frac{2}{3} & t = 0 \\ t^2 & 0 < t < 1 \\ 0 & otherwise \end{cases} \tag{8.4.32}$$

xi) The CDF of W is as follows:

$$F_W(t) = \begin{cases} 0 & t < 0 \\ \frac{2+t^3}{3} & 0 \le t \le 1 \\ 1 & otherwise \end{cases} \tag{8.4.33}$$

b) We need to find $P[W > 0]$

$$\Pr(W > 0) = 1 - F_W(0) \tag{8.4.34}$$

$$= \frac{1}{3} \tag{8.4.35}$$

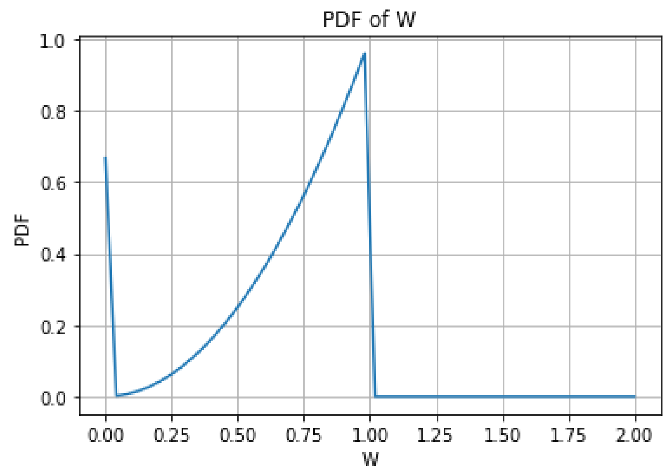
$$\therefore \Pr(W > 0) = \frac{1}{3} \tag{8.4.36}$$

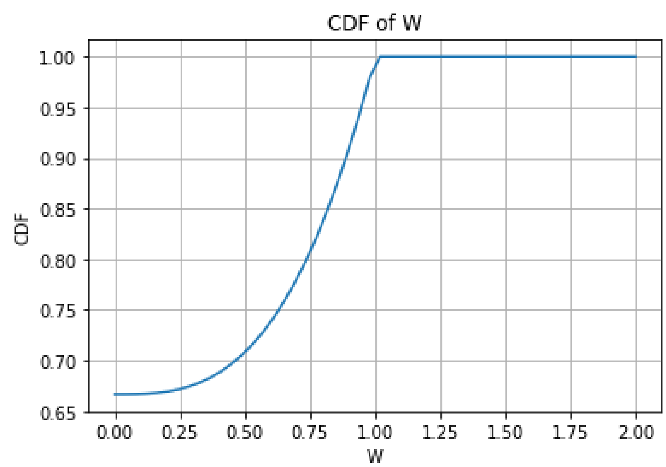
c) CDF of W is discontinuous at $W = 0$.
$\therefore$ option 3 is incorrect.

d) The CDF in (8.4.33) can be written as

$$F_W(t) = \left(\frac{2 + t^3}{3}\right) I_{\{0 \le t \le 1\}} + I_{\{t > 1\}} \tag{8.4.37}$$

$\therefore$ option 2 and 4 are correct.





8.5. Let $U_1, U_2 \ldots, U_n$ be independent and identically distributed random variables each having a uniform distribution on (0,1). Then,$\lim_{n \to +\infty} \Pr\left(U_1 + U_2 \ldots, U_n \le \frac{3}{4}n\right)$

a) does not exist
b) exists and equals 0
c) exists and equals 1

d) exists and equals $\frac{3}{4}$

Solution: We use Weak law for large numbers to solve this problem. Let the collection of identically distributed random variables $U_1, U_2 \ldots, U_n$ have a finite mean μ and finite variance σ^2.

$$\mu = E[U_i] \quad \text{for i} \in (1,2,3\ldots,\text{n}) \tag{8.5.1}$$

Since the distribution is uniform on (0,1), $\mu = 0.5$. Let M_n be the sample mean

$$M_n = \frac{U_1 + U_2 + U_3 \cdots + U_n}{n} \tag{8.5.2}$$

Expected value of M_n (using (8.5.2) and (8.5.1))is

$$E[M_n] = \frac{E[U_1 + U_2 + U_3 + \cdots + U_n]}{E[n]} \tag{8.5.3}$$

$$= \frac{E[U_1] + E[U_2] + \cdots + E[U_n]}{n} \tag{8.5.4}$$

$$= \frac{n \times \mu}{n} \tag{8.5.5}$$

$$= \mu \tag{8.5.6}$$

Variance of M

$$Var(M_n) = \frac{Var(U_1 + U_2 + U_3 \cdots + U_n)}{n^2} \tag{8.5.7}$$

$$= \frac{Var(U_1) + Var(U_2) \cdots + Var}{n^2} \tag{8.5.8}$$

$$= \frac{n \times \sigma^2}{n^2} \tag{8.5.9}$$

$$= \frac{\sigma^2}{n} \tag{8.5.10}$$

From Chebyshev inequality, for any $\epsilon > 0$

$$\Pr(|M_n - \mu| \geq \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} \tag{8.5.11}$$

From (8.5.1) and (8.5.10)

$$\Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n \times \epsilon^2}$$

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right)$$
$$\leq \lim_{n\to\infty} \frac{\sigma^2}{n \times \epsilon^2} \leq 0 \text{ for fixed } \epsilon > 0 \tag{8.5.12}$$

But since Probabilities are always non-negative,

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \to 0 \tag{8.5.13}$$

This is known as the weak law of large numbers

The inverse of (8.5.13) is also true

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \le \epsilon\right) \to 1 \quad (8.5.14)$$

$$\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \le \epsilon \text{ as n} \to \infty \quad (8.5.15)$$

From ϵ, n definition of limits, it is clear that

$$\frac{U_1 + U_2 \cdots + U_n}{n} \to \mu \quad (8.5.16)$$

$$U_1 + U_2 \ldots U_n \to n \times \mu \text{ as n} \to \infty \quad (8.5.17)$$

Since $\mu = \frac{1}{2}$,

$$\lim_{n\to+\infty} U_1 + U_2 \ldots U_n = \frac{1}{2}n < \frac{3}{4}n \quad (8.5.18)$$

So

$$\lim_{n\to+\infty} \Pr\left(U_1 + U_2 \ldots, U_n \le \frac{3}{4}n\right) = 1 \quad (8.5.19)$$

8.6. Consider the quadratic equation $x^2 + 2Ux + V = 0$ where U and V are chosen independently and randomly from $\{1, 2, 3\}$ with equal probabilities. Then probability that the equation has both roots real

a) $\frac{2}{3}$
b) $\frac{1}{2}$
c) $\frac{7}{9}$
d) $\frac{1}{3}$

Solution: Let $U \in \{1.2, 3\}$ and $V \in \{1, 2, 3\}$ For $x^2+2Ux+V = 0$

TABLE 8.6.1: Probability of selecting values for U

k	1	2	3
$\Pr(U = k)$	1/3	1/3	1/3

TABLE 8.6.2: Probability of selecting values for V

k	1	2	3
$\Pr(V = k)$	1/3	1/3	1/3

to have real roots,

$$b^2 - 4ac \ge 0 \quad (8.6.1)$$

$$(2U)^2 - 4(1)(V) \ge 0 \quad (8.6.2)$$

$$U^2 \ge V \quad (8.6.3)$$

$$\Pr\left(U^2 \ge V\right) = 1 - \Pr\left(U^2 < V\right) \quad (8.6.4)$$

The possible pairs (U, V) for $\Pr\left(U^2 < V\right)$,

TABLE 8.6.3: Table for $\Pr\left(U^2 < V\right)$

(U, V) for $U^2 < V$	Probability
$(1, 2)$	$\Pr(U = 1)\Pr(V = 2)$=1/9
$(1, 3)$	$\Pr(U = 1)\Pr(V = 3) = \frac{1}{9}$
Total	$\Pr\left(U^2 < V\right) = \frac{2}{9}$

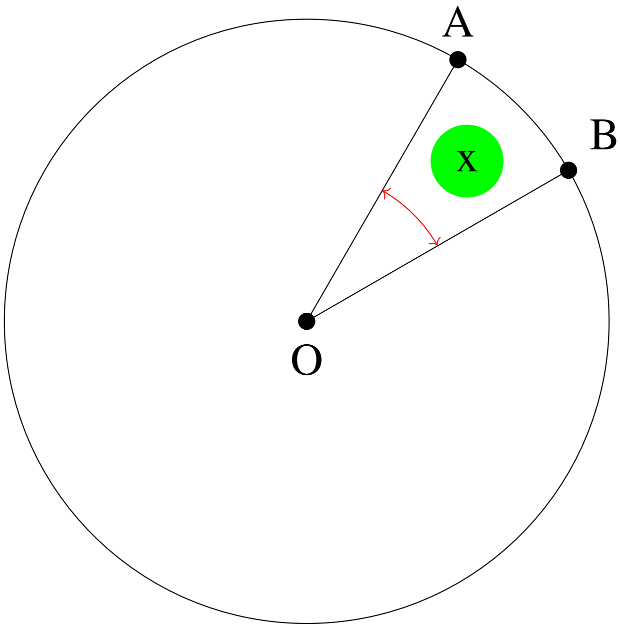
$$\Pr\left(U^2 \ge V\right) = 1 - \frac{2}{9} \quad (8.6.5)$$

$$\Pr\left(U^2 \ge V\right) = \frac{7}{9} \quad (8.6.6)$$

Hence, Option 3 is correct.

8.7. A point is chosen at random from a circular disc shown below.

What is the probability that the point lies in the sector OAB?



(where $\angle$AOB = x radians)

a) $\frac{2x}{\pi}$ c) $\frac{x}{2\pi}$
b) $\frac{x}{\pi}$ d) $\frac{x}{4\pi}$

Solution:
Let $X \in \{0, 1\}$ be a random variable such that X=0 means we choose a point lying in sector OAB and X=1 means that we choose a point lying outside sector OAB and inside the circle.
Area of a sector subtending an angle θ at the centre of circle with radius a is given by :

$$A = \frac{1}{2}a^2\theta \tag{8.7.1}$$

where θ is in radians.
Let the radius of circle shown in figure be r. It is given that sector OAB subtends an angle of x radians at the centre of the circle.
Probability that the chosen point lies in sector OAB is:

$$\Pr(X = 0) = \frac{\text{Area of sector OAB}}{\text{Area of circle}} \tag{8.7.2}$$

$$= \frac{\frac{1}{2}r^2x}{\pi r^2} \tag{8.7.3}$$

$$= \frac{x}{2\pi} \tag{8.7.4}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$. **alternate solution** The joint pdf is given by:

$$f_{r\theta}(r, \theta) = \begin{cases} \frac{r}{\pi R^2} & \text{if } 0 < r < R\ ,\ 0 < \theta < 2 \\ 0 & \text{otherwise} \end{cases} \tag{8.7.5}$$

Let A $\equiv$ (R, θ_2) and B $\equiv$ (R, θ_1).
Hence,

$$(\theta_2 - \theta_1) = x \tag{8.7.6}$$

We want $\theta \in (\theta_1\ , \theta_2)$ and r $\in$ (0,R) for point to lie in the sector. Let the point to be chosen be (r, θ).
So, Required probability is:

$$\Pr(\theta_1 < \theta < \theta_2, 0 < r < R)$$

$$= \int_{\theta_1}^{\theta_2}\int_{0}^{R} \frac{r}{\pi R^2}\, dr\, d \tag{8.7.7}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{\pi R^2} \frac{r^2}{2}\Big|_0^R \tag{8.7.8}$$

$$= \int_{\theta_1}^{\theta_2} \frac{R^2}{2\pi R^2}\, d\theta \tag{8.7.9}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2\pi}\, d\theta \tag{8.7.10}$$

$$= \frac{\theta}{2\pi}\Big|_{\theta_1}^{\theta_2} \tag{8.7.11}$$

$$= \frac{\theta_2 - \theta_1}{2\pi} \tag{8.7.12}$$

$$= \frac{x}{2\pi} \tag{8.7.13}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$.

9 December 2012

9.1. Let X be a binomial random variable with parameters $\left(11, \frac{1}{3}\right)$. At which value(s) of k is $\Pr(X = k)$ maximized?

a) k=2
b) k=3
c) k=4
d) k=5

Solution: X has a binomial distribution :

$$\Pr(X = k) = {}^{n}C_k (q)^{n-k} (p)^k \tag{9.1.1}$$

Where,

- n=11
- $p = \frac{1}{3}$
- $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

$$\Pr(X = k) = {}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k \tag{9.1.2}$$

For Pr(X = k) to be maximized

$$\Pr(X = k) \geq \Pr(X = k+1) \tag{9.1.3}$$

$$\frac{\Pr(X = k)}{\Pr(X = k+1)} = \frac{{}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k}{{}^{11}C_{k+1} \left(\frac{2}{3}\right)^{10-k} \left(\frac{1}{3}\right)^{k+1}} \geq 1 \tag{9.1.4}$$

$$\frac{2(k+1)}{11-k} \geq 1 \tag{9.1.5}$$

$$\implies k \geq 3 \tag{9.1.6}$$

$$\Pr(X = k) \geq \Pr(X = k-1) \tag{9.1.7}$$

$$\frac{\Pr(X = k)}{\Pr(X = k-1)} = \frac{{}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k}{{}^{11}C_{k-1} \left(\frac{2}{3}\right)^{12-k} \left(\frac{1}{3}\right)^{k-1}} \geq 1 \tag{9.1.8}$$

$$\frac{12-k}{2k} \geq 1 \tag{9.1.9}$$

$$\implies k \leq 4 \tag{9.1.10}$$

From (9.1.6) , (9.1.10) and since k is an integer
$\Pr(X = k)$ is maximized for k=3, k=4
Thus options 2) and 3) are correct

9.2. Men arrive in a queue according to a Poisson process with rate λ_1 and women arrive in the same queue according to another Poisson process with rate λ_2. The arrivals of men and women are independent. The probability that the first person to arrive in the queue is a man is:

a) $\frac{\lambda_1}{\lambda_1 + \lambda_2}$

b) $\dfrac{\lambda_2}{\lambda_1 + \lambda_2}$

c) $\dfrac{\lambda_1}{\lambda_2}$

d) $\dfrac{\lambda_2}{\lambda_1}$

Solution: Let X and Y be Poisson random variables, with the values X takes being the number of men joining the queue in an arbitrary time t, and the values Y takes being the number of women joining the queue in an arbitrary time t.

$$Pr(X = i) = \frac{\lambda_1^i \cdot e^{-\lambda_1}}{i!} \tag{9.2.1}$$

$$Pr(Y = i) = \frac{\lambda_2^i \cdot e^{-\lambda_2}}{i!} \tag{9.2.2}$$

For 2 independent Poisson distributions with means λ_1 and λ_2, the simultaneous distribution can be represented by:

$$Pr(X + Y = i) = \frac{(\lambda_1 + \lambda_2)^i \cdot e^{-(\lambda_1+\lambda_2)}}{i!} \tag{9.2.3}$$

Now we take conditional probability that if only one person entered the queue within a certain time t, then the probability of them being a man and not a woman is given by:

$$Pr(X = 1|(X + Y) = 1) = \frac{Pr((X = 1) + (Y = 0))}{Pr(X + Y = 1)} \tag{9.2.4}$$

$$\tag{9.2.5}$$

Since X and Y are independent,

$$Pr(X = 1|(X + Y) = 1) = \frac{Pr(X = 1) \cdot Pr(Y = 0)}{Pr(X + Y = 1)} \tag{9.2.6}$$

$$= \frac{\frac{\lambda_1^1 \cdot e^{-\lambda_1}}{1!} \cdot \frac{\lambda_2^0 \cdot e^{-\lambda_2}}{0!}}{\frac{(\lambda_1+\lambda_2)^1 \cdot e^{-(\lambda_1+\lambda_2)}}{1!}} \tag{9.2.7}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \tag{9.2.8}$$

The probability that the first person to arrive in the queue is a man is option A, i.e $\frac{\lambda_1}{\lambda_1+\lambda_2}$