Probability

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Contents

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***Abstract*—This book provides solved examples on Probability**

1 December 2018

1.1. Let X and Y be i.i.d random variables uniformly distributed on (0,4).Then $\Pr(X > Y|X < 2Y)$ is

a) 1/3
b) 5/6
c) 1/4
d) 2/3

Solution:
The PDF is given by

$$f_X(x) = f_Y(x) = \begin{cases} \frac{1}{4}, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

The CDF is given by

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$F_X(x) = F_Y(x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{4}, & \text{if } 0 < x < 4 \\ 1, & x \ge 4 \end{cases}$$

Using definition of conditional probability

$$\Pr(X > Y|X < 2Y) = \frac{\Pr(Y < X < 2Y)}{\Pr(X < 2Y)} \tag{1.1.1}$$

Now finding $\Pr(X < 2Y)$

$$\Pr(X < 2y) = F_X(2y) \tag{1.1.2}$$

$$\implies \Pr(X < 2Y) = \int_{-\infty}^{\infty} f_Y(x) \times F_X(2x)dx \tag{1.1.3}$$

$$\implies \Pr(X < 2Y) = \int_{0}^{2} \frac{x}{8}dx + \int_{2}^{4} \frac{1}{4}dx \tag{1.1.4}$$

$$\implies \Pr(X < 2Y) = \frac{3}{4} = 0.75 \tag{1.1.5}$$

Now to find $\Pr(Y < X < 2Y)$

$$\Pr(y < X < 2y) = F_X(2y) - F_X(y) \tag{1.1.6}$$

$$\implies \Pr(Y < X < 2Y) \tag{1.1.7}$$

$$= \int_{-\infty}^{\infty} f_Y(x)(F_X(2x) - F_X(x))dx$$

$$\implies \int_0^2 \frac{1}{4}\left(\frac{x}{2} - \frac{x}{4}\right)dx + \int_2^4 \frac{1}{4}\left(1 - \frac{x}{4}\right)dx \tag{1.1.8}$$

$$\implies \Pr(Y < X < 2Y) = \frac{1}{4} = 0.25 \tag{1.1.9}$$

Now using (1.1.1),(1.1.5) and (1.1.9)

$$\Pr(X > Y|X < 2Y) = \frac{1/4}{3/4} = \frac{1}{3} \tag{1.1.10}$$

Hence final solution is option 1) or 1/3

1.2. Suppose X is a positive random variable with the following probability density function,

$$f(x) = (\alpha x^{\alpha-1} + \beta x^{\beta-1})e^{-x^{\alpha}-x^{\beta}}; x > 0$$

for $\alpha > 0, \beta > 0$. Then the hazard function of X for some choices of α and β can be

a) an increasing function.
b) a decreasing function.
c) a constant function.
d) a non monotonic function

Solution:
CDF of X,

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{1.2.1}$$

$$= \int_0^x f(t)dt \qquad \text{as } x > 0 \tag{1.2.2}$$

$$= \int_{-\infty}^{t} \left((\alpha t^{\alpha-1} + \beta t^{\beta-1}) \times e^{-t^{\alpha}-t^{\beta}}\right)dt \tag{1.2.3}$$

$$= -e^{-t^{\alpha}-t^{\beta}}\Big|_0^x \tag{1.2.4}$$

$$= 1 - e^{-x^{\alpha}-x^{\beta}} \tag{1.2.5}$$

Hazard function,

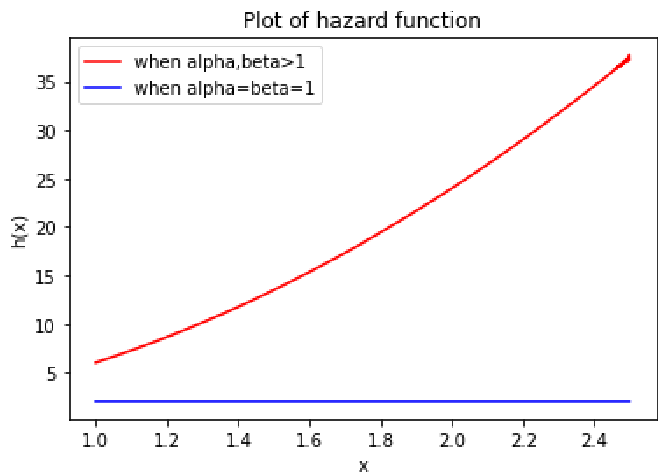
$$h(x) = \frac{f(x)}{1 - F(x)} \tag{1.2.6}$$

$$= \alpha x^{\alpha-1} + \beta x^{\beta-1} \tag{1.2.7}$$

$$h'(x) = \alpha(\alpha-1)x^{\alpha-2} + \beta(\beta-1)x^{\beta-2} \tag{1.2.8}$$

$$h'(x) = \begin{cases} 0 & \alpha = \beta = 1 \\ > 0 & \text{otherwise} \end{cases} \tag{1.2.9}$$

Thus $h(x)$ can be either constant function or an increasing function.



From the above figure, it is verified that $h(x)$ can be either constant function or an increasing function.
Correct options are 1,3.

1.3. Suppose n units are drawn from a population of N units sequentially as follows. A random sample

$$U_1, U_2, ...U_N \text{ of size N, drawn from } U(0,1) \tag{1.3.1}$$

The k-th population unit is selected if

$$U_k < \frac{n - n_k}{N - k + 1}, k = 1, 2, ..N.\text{where, } n_1 = 0, n_k = \tag{1.3.2}$$

number of units selected out of first k-1 units for each k = 2, 3, ..N. Then,

a) The probability of inclusion of the second unit in the sample

$$\text{is } \frac{n}{N} \tag{1.3.3}$$

b) The probability of inclusion of the first and the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{1.3.4}$$

c) The probability of not including the first and including the second unit in the sample

$$\text{is } \frac{n(N-n)}{N(N-1)} \tag{1.3.5}$$

d) The probability of including the first and not including the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{1.3.6}$$

Solution:

Defining random variable $X \in \{0, 1, 2, ..N\}$ (1.3.7)

Where, $X = i$ when ith unit is included. (1.3.8)

The first unit in the sample is included if

$$U_1 < \frac{n - n_1}{N - 1 + 1} \tag{1.3.9}$$

Here, $n_1 = 0$ is given in the qn. (1.3.10)

$$\therefore \Pr(X = 1) = \frac{n}{N} \tag{1.3.11}$$

a) For k=2,

$n_2 = 1$ when, first unit is included. (1.3.12)

$$U_2 < \frac{n - n_2}{N - 2 + 1}\left(= \frac{n-1}{N-1}\right) \tag{1.3.13}$$

$$\therefore \Pr(X = 2 \mid X = 1) = \frac{n-1}{N-1} \tag{1.3.14}$$

$\Pr(X = 1, X = 2)$

$$= \Pr(X = 2 \mid X = 1) \times \Pr(X = 1) \tag{1.3.15}$$

$$\therefore \Pr(X = 1, X = 2) = \frac{n(n-1)}{N(N-1)} \tag{1.3.16}$$

$n_2 = 0$ when, first unit is not included. (1.3.17)

$$U_2 < \frac{n - n_2}{N - 2 + 1}\left(= \frac{n}{N-1}\right) \tag{1.3.18}$$

$$\therefore \Pr(X = 2 \mid X \neq 1) = \frac{n}{N-1} \tag{1.3.19}$$

$\Pr(X \neq 1, X = 2)$

$$= \Pr(X = 2 \mid X \neq 1) \times \Pr(X \neq 1) \tag{1.3.20}$$

$$\therefore \Pr(X \neq 1, X = 2) = \left(1 - \frac{n}{N}\right) \times \frac{n}{N-1} \tag{1.3.21}$$

$$\therefore \Pr(X \neq 1, X = 2) = \frac{n(N-n)}{N(N-1)} \tag{1.3.22}$$

From (1.3.16) and (1.3.22)

$$\Pr(X = 2) = \frac{n(n-1)}{N(N-1)} + \frac{n(N-n)}{N(N-1)} = \frac{n}{N} \tag{1.3.23}$$

Hence, option 1 is correct.

b) From (1.3.16)

$$\Pr(X = 1, X = 2) = \frac{n(n-1)}{N(N-1)} \tag{1.3.24}$$

Hence, option 2 is correct.

c) From (1.3.22)

$$\Pr(X \neq 1, X = 2) = \frac{n(N-n)}{N(N-1)} \tag{1.3.25}$$

Hence, option 3 is correct.

d)

$$\Pr(X = 1, X \neq 2) = \frac{n}{N} \times \left(1 - \frac{n}{N}\right) = \frac{n(N-n)}{N^2} \tag{1.3.26}$$

Hence, option 4 is incorrect.

Therefore, Options 1, 2, 3 are correct

1.4. Consider a Markov chain with state space 1,2,....,100. Suppose states 2i and 2j communicate with each other and states 2i-1 and 2j-1 communicate with each other for every i,j = 1,2,...,50. Further suppose that $p_{3,3}^{(2)}$ ¿ 0,$p_{4,4}^{(3)}$ ¿ 0 and $p_{2,5}^{(7)}$ ¿ 0. Then

a) The Markov chain is irreducible.
b) The Markov chain is aperiodic.

c) State 8 is recurrent.
d) State 9 is recurrent.

Solution:

2 June 2018

2.1. Two students are solving the same problem independently,if the probability of first one solves the problem is $\frac{3}{5}$ and the probability that the second one solves the problem is $\frac{4}{5}$,what is the probability that atleast one of them solves the problem?

a) $\frac{17}{25}$

b) $\frac{19}{25}$

c) $\frac{21}{25}$

d) $\frac{23}{25}$

Solution: Let X,Y be two events representing solving the problem by students A,B respectively.
Given

$$\Pr(X) = \frac{3}{5} \tag{2.1.1}$$

$$\Pr(Y) = \frac{4}{5} \tag{2.1.2}$$

Since students solve the problem independently,So events X and Y are independent, For independent events

$$\Pr(XY) = \Pr(X) \times \Pr(Y) \tag{2.1.3}$$

from (2.1.1) and (2.1.2)

$$\Pr(XY) = \frac{3}{5} \times \frac{4}{5} \tag{2.1.4}$$

$$\Pr(XY) = \frac{12}{25} \tag{2.1.5}$$

Now we have to find probability of solving the problem by atleast one of them i.e $\Pr(X + Y)$. As,

$$\Pr(X + Y) = \Pr(X) + \Pr(Y) - \Pr(XY) \tag{2.1.6}$$

from (2.1.1), (2.1.2), (2.1.5)

$$\Pr(X + Y) = \frac{3}{5} + \frac{4}{5} - \frac{12}{25} \tag{2.1.7}$$

$$\Pr(X + Y) = \frac{23}{25} \tag{2.1.8}$$

Hence the required probability is $\frac{23}{25}$

2.2. A standard fair die is rolled until some face other than 5 or 6 turns up.Let X denote the face value of the last roll.Let A={X is even} and B={X is atmost 2} Then,

a) $\Pr(A \cap B) = 0$

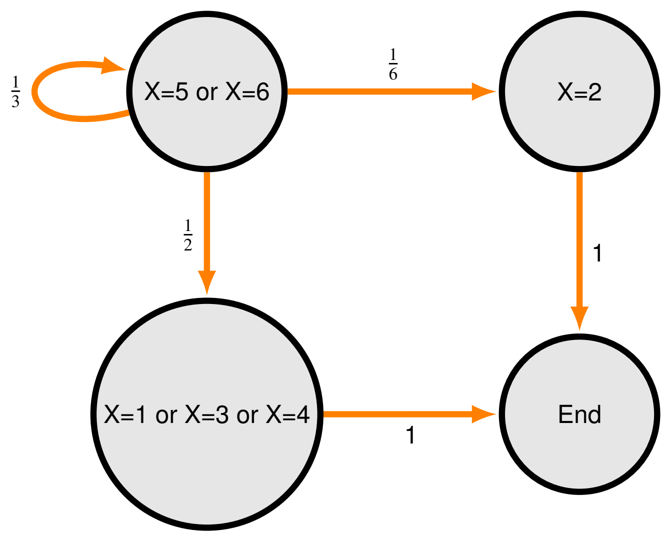
b) $\Pr(A \cap B) = \frac{1}{6}$

c) $\Pr(A \cap B) = \frac{1}{4}$

d) $\Pr(A \cap B) = \frac{1}{3}$

Solution: Let us assume the following table.

Fig. 2.2.1: Markov chain



Let us represent the markov chain diagram in a

TABLE 2.2.1

state 1	state 2	state 3	state 4
$X = 5$ or $X = 6$	$X = 2$	$X = 1$ or $X = 3$ or $X = 4$	end

matrix.Let P_{ij} represent the element of a matrix which is in i^{th} row and j^{th} column.The value of P_{ij} is equal to probability of transition from state i to state j

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.2.1}$$

We need the probability that $X = 2$.Hence required probability is

$$P_{12} + (P_{12})^2 + \cdots + \infty \tag{2.2.2}$$

where P_{12}^n represents the 1st row ,2nd column element in the P^n

$$P^2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.2.3}$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.2.4}$$

$$P^3 = (P^2)(P^1) \tag{2.2.5}$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.2.6}$$

$$= \begin{bmatrix} \frac{1}{27} & \frac{1}{54} & \frac{1}{18} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.2.7}$$

From above we can notice that each time P_{12} reduces by $\frac{1}{3}$.Hence from (2.2.2),

$$\sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i \frac{1}{6} \tag{2.2.8}$$

From Geometric progression we can write ,required probability $=\frac{1}{4}$ $\therefore$ **option C is correct**

2.3. Let X and Y be two random variables with joint probability density function

$$f(x.y) = \begin{cases} \frac{1}{\pi} & 0 \le x^2 + y^2 \le 1 \\ 0 & otherwise \end{cases}$$

Which of the following statements are correct?

a) X and Y are independent.

b) $\Pr(X > 0) = \dfrac{1}{2}$

c) E(Y)=0

d) Cov(X,Y)=0

Solution:

2.4. Let X and Y be two random variables with joint probability density function

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Which of the following statements are correct?

a) X and Y are independent.

b) $\Pr(X > 0) = \dfrac{1}{2}$

c) E(Y)=0

d) Cov(X,Y)=0

Solution:

a) The marginal PDF of X is given by

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f_{XY}(x,y)dy \tag{2.4.1}$$

$$= \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \frac{1}{\pi}dy \tag{2.4.2}$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{2.4.3}$$

The marginal PDF of Y is given by

$$f_Y(x) = \int_{x=-\infty}^{x=\infty} f_{XY}(x,y)dx \tag{2.4.4}$$

$$= \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{2.4.5}$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{2.4.6}$$

Now,

$$f_X(x) \times f_Y(x) = \frac{2\sqrt{1-x^2}}{\pi} \times \frac{2\sqrt{1-y^2}}{\pi} \tag{2.4.7}$$

$$= \frac{4(1-x^2)(1-y^2)}{\pi^2} \tag{2.4.8}$$

$$\neq \frac{1}{\pi} \tag{2.4.9}$$

$$\neq f_{XY}(x,y) \tag{2.4.10}$$

Therefore, X and Y are not independent.

b) Now,

$$\Pr(X > 0) = \int_{x=0}^{x=\infty} f_X(x)dx \tag{2.4.11}$$
$$= \int_{x=0}^{x=1} \frac{2\sqrt{1-x^2}}{\pi}dx \tag{2.4.12}$$
$$= \left(\frac{\arcsin(x) + x\sqrt{1-x^2}}{\pi}\right)_0^1 \tag{2.4.13}$$
$$= \frac{1}{2} \tag{2.4.14}$$

Therefore, option(2) is correct.

c) Now,

$$E[Y] = \int_{y=-\infty}^{y=\infty} y f_Y(y)dy \tag{2.4.15}$$
$$= \int_{y=-1}^{y=1} \frac{2y\sqrt{1-y^2}}{\pi}dy \tag{2.4.16}$$
$$= \left(\frac{-2(1-y^2)^{\frac{3}{2}}}{3\pi}\right)_{-1}^1 \tag{2.4.17}$$
$$= 0 \tag{2.4.18}$$

Therefore, option(3) is also correct.

d) Now,

$$E[XY] = \int_x \int_y xy f_{XY}(x,y)dydx \tag{2.4.19}$$
$$= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \frac{xy}{\pi}dydx \tag{2.4.20}$$
$$= \frac{x}{\pi}\int_{x=-1}^{x=1} \left(\frac{y^2}{2}\right)_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \tag{2.4.21}$$
$$= 0 \tag{2.4.22}$$

Now,

$$Cov(X,Y) = E[XY] - E[X]E[Y] \tag{2.4.23}$$
$$= 0 - E[X] \times 0 \tag{2.4.24}$$
$$= 0 \tag{2.4.25}$$

Therefore, option(4) is also correct.

2.5. A simple random variable of size n will be drawn from a class of 125 students, and the mean mathematics score of the sample will be computed, If the standard error of the sample mean for "with replacement sampling" is twice as much as the standard error of the sample mean for "without replacement sampling", the value of n is ?

a) 32
b) 63
c) 79
d) 94

Solution: Let N be the population size so, N=120. The given sample size is n. **Notations :** y : student under consideration. y_i : Maths marks of i^{th} student in the sample. Y : student of class. Y_i : Maths marks of i^{th} student in the class. $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$: Average of sample class. $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$: Average of whole class. $S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$: S=Std dev of the class. $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$: Variance of the class. Standard error of sample mean $SE_{mean} = \frac{s}{\sqrt{n}}$.
Where

s = standard deviation of sample mean.

n = sample class size.

Variance of the $\overline{y}$

$$V(\overline{y}) = E(\overline{y} - \overline{Y})^2 \tag{2.5.1}$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{Y})\right]^2 \tag{2.5.2}$$

$$= E\left[\frac{1}{n^2}\sum_{i=1}^{n}(y_i - \overline{Y})^2 + \frac{1}{n^2}\sum\sum_{1\le i\ne j\le n}(y_i - \overline{Y})(y_j - \overline{Y})\right] \tag{2.5.3}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}E(y_i - \overline{Y})^2 + \frac{1}{n^2}\sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y}) \tag{2.5.4}$$

$$\text{Let } K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y}) \tag{2.5.5}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 + \frac{K}{n^2} \tag{2.5.6}$$

$$= \frac{1}{n^2}n\sigma^2 + \frac{K}{n^2} \tag{2.5.7}$$

$$= \frac{N-1}{Nn}S^2 + \frac{K}{n^2} \tag{2.5.8}$$

Finding the value of K in case of Simple random sampling with repetition (SRSWR)and Simple random sampling without repetition(SRSWOR) allows us to calculate the variance of mean. **K value in case of SRSWOR**

$$K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y})$$

Consider

$$E(y_i - \overline{Y})(y_j - \overline{Y}) = \frac{1}{N(N-1)}\sum\sum_{1\le k\ne l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

Since

$$\left[\sum_{k=1}^{N}(y_k - \overline{Y})\right]^2 = \sum_{i=1}^{N}(y_k - \overline{Y})^2 + \sum\sum_{1\le k\ne l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

$$\implies 0 = (N-1)S^2 + \sum\sum_{1\le k\ne l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

$$\implies E(y_i - \overline{Y})(y_j - \overline{Y}) = \frac{1}{N(N-1)}(N-1)(-S^2)$$

$$\implies K = n(n-1)\frac{(-S^2)}{N}$$

Putting this value in (2.5.8) gives us

$$V(\overline{y})_{WOR} = \frac{N-1}{Nn}S^2 + \frac{n-1(-S^2)}{Nn} \tag{2.5.9}$$

$$= \frac{N-n}{Nn}S^2 \tag{2.5.10}$$

K value in case of SRSWR

$$K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})(y_j - \overline{Y})$$

Since we are selecting the samples with replacements choosing i^{th} and j^{th} sample is independent of each other. So,

$$K = \sum\sum_{1\le i\ne j\le n}E(y_i - \overline{Y})E(y_j - \overline{Y})$$

$$= 0$$

(Since deviation about mean is 0)

Putting K=0 in (2.5.8) we get

$$V(\overline{y})_{WR} = \frac{N-1}{Nn}S^2 \tag{2.5.11}$$

From equation (2.5.10) standard error of mean of sample class without repetition

$$SE_{WOR} = \frac{s}{\sqrt{n}} \tag{2.5.12}$$

$$= \sqrt{\frac{V(\overline{y})_{WOR}}{n}} \tag{2.5.13}$$

$$= \sqrt{\frac{N-n}{Nn^2}}S \tag{2.5.14}$$

From equation (2.5.11) standard error of mean of sample class with repetition

$$SE_{WR} = \sqrt{\frac{V(\overline{y})_W R}{n}} \tag{2.5.15}$$

$$= \sqrt{\frac{N-1}{Nn^2}}S \tag{2.5.16}$$

Given to find the value of n if $2 \times SE_{WOR} =$

SE_{WR}. From (2.5.14) and (2.5.16) we can write

$$2\sqrt{\frac{N-n}{Nn^2}}S = \sqrt{\frac{N-1}{Nn^2}}S \tag{2.5.17}$$
$$\implies 4(N-n) = N-1 \tag{2.5.18}$$
$$\implies 4N+1-N = 4n \tag{2.5.19}$$
$$\implies 4n = 3(125)+1 \tag{2.5.20}$$
$$\implies n = 94 \tag{2.5.21}$$

Therefore the sample size for the given condition to be met is n=94.(**Option D**)

2.6. Let X and Y be two independent and identically distributed (I.I.D) random variables uniformly distributed in (0,1). Let $Z = max(X,Y)$ and $W = min(X,Y)$, then the probability that $[Z - W > \frac{1}{2}]$ is

(A) $\frac{1}{2}$

(B) $\frac{3}{4}$

(C) $\frac{1}{4}$

(D) $\frac{2}{3}$ **Solution:**

X and Y are two independent random variables. Let

$$f_X(x) = \Pr(X = x) \tag{2.6.1}$$
$$f_Y(y) = \Pr(Y = y) \tag{2.6.2}$$
$$f_V(v) = \Pr(V = v) \tag{2.6.3}$$

be the probability densities of random variables X ,Y and V=X-Y.
The density for X is

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{2.6.4}$$

We have ,

$$V = X - Y \iff v = x - y \iff x = v + y \tag{2.6.5}$$

The density of X can also be represented as,

$$f_X(v+y) = \begin{cases} 1 & 0 \le v+y \le 1 \\ 0 & otherwise \end{cases} \tag{2.6.6}$$

and the density of Y is,

$$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{2.6.7}$$

The density of V i.e. $V = X - Y$ is given by the convolution of $f_X(-v)$ with $f_Y(v)$.

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v+y) f_Y(y)\, dy \tag{2.6.8}$$

From 2.6.6 and 2.6.7 we have,
The integrand is 1 when,

$$0 \le y \le 1 \tag{2.6.9}$$
$$0 \le v+y \le 1 \tag{2.6.10}$$
$$-v \le y \le 1-v \tag{2.6.11}$$

and zero, otherwise.
Now when $-1 \le v \le 0$ we have,

$$f_V(v) = \int_{-v}^{1} dy \tag{2.6.12}$$
$$= (1-(-v)) \tag{2.6.13}$$
$$= 1+v \tag{2.6.14}$$

For $0 \le v \le 1$ we have,

$$f_V(v) = \int_{0}^{1-v} dy \tag{2.6.15}$$
$$= (1-v-(0)) \tag{2.6.16}$$
$$= 1-v \tag{2.6.17}$$

Therefore the density of V is given by

$$f_V(v) = \begin{cases} 1+v & -1 \le v \le 0 \\ 1-v & 0 < v \le 1 \\ 0 & otherwise \end{cases} \tag{2.6.18}$$

The plot for PDF of V can be observed at figure 2.6.1
The CDF of V is defined as,

$$F_V(v) = \Pr(V \le v) \tag{2.6.19}$$

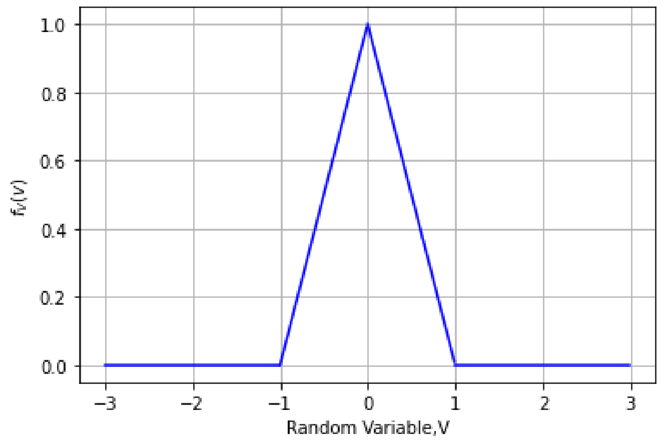


Fig. 2.6.1: The PDF of V

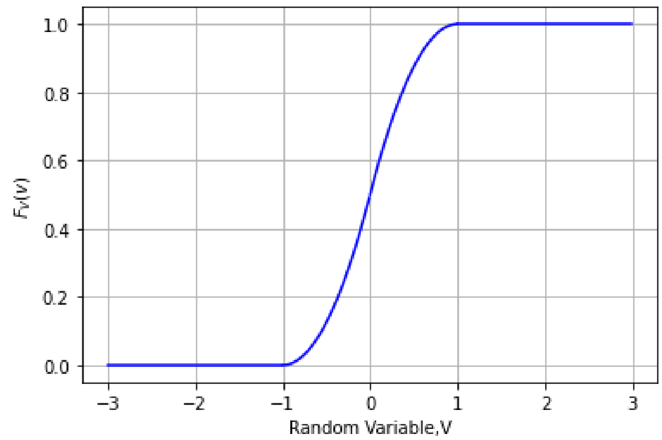


Fig. 2.6.2: The CDF of V

Now for $v \leq 0$,

$$\Pr(V \leq v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{2.6.20}$$

$$= \int_{-1}^{v} (1+v)\,dv \tag{2.6.21}$$

$$= \left(\frac{v^2}{2} + v\right)\Big|_{-1}^{v} \tag{2.6.22}$$

$$= \left(\left(\frac{v^2}{2} + v\right) - \left(\frac{1}{2} - 1\right)\right) \tag{2.6.23}$$

$$= \frac{v^2 + 2v + 1}{2} \tag{2.6.24}$$

Similarly for $v \leq 1$,

$$\Pr(V \leq v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{2.6.25}$$

$$= \frac{1}{2} + \int_{0}^{v} (1-v)\,dz \tag{2.6.26}$$

$$= \frac{-v^2 + 2v + 1}{2} \tag{2.6.27}$$

The CDF is as below:

$$F_V(v) = \begin{cases} 0 & v < -1 \\ \dfrac{v^2+2v+1}{2} & v \leq 0 \\ \dfrac{-v^2+2v+1}{2} & v \leq 1 \\ 1 & v > 1 \end{cases} \tag{2.6.28}$$

The plot for CDF of V can be observed at figure 2.6.2

We need $\Pr\left(Z - W > \frac{1}{2}\right)$ where $Z = max(X, Y)$ and $W = min(X, Y)$. Now,

$$Z - W = \begin{cases} X - Y & \text{for } X \geq Y \\ Y - X & \text{for } X < Y \end{cases} \tag{2.6.29}$$

Therefore,

$$\Pr\left(Z - W > \frac{1}{2}\right) = \Pr\left(X - Y > \frac{1}{2}, X \geq Y\right) + \Pr\left(Y - X > \frac{1}{2}, X < Y\right) \tag{2.6.30}$$

$$= \Pr\left(X - Y > \frac{1}{2}\right) + \Pr\left(Y - X > \frac{1}{2}\right) \tag{2.6.31}$$

$$= \Pr\left(V > \frac{1}{2}\right) + \Pr\left(-V > \frac{1}{2}\right) \tag{2.6.32}$$

$$= 1 - \Pr\left(V \leq \frac{1}{2}\right) + \Pr\left(V < \frac{-1}{2}\right) \tag{2.6.33}$$

$$= 1 - F_V(\frac{1}{2}) + F_V(-\frac{1}{2}) \tag{2.6.34}$$

$$= 1 - \frac{7}{8} + \frac{1}{8} \tag{2.6.35}$$

$$= \frac{1}{4} \tag{2.6.36}$$

Hence the correct answer is option (C).

3 December 2016

3.1. $X_1, X_2, \ldots, X_n$ are independent and identically distributed as $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$. Then

a) $\sum_1^n \frac{(X_i-\bar{X})^2}{n-1}$ is the Minimum Variance Unbiased Estimate of σ^2

b) $\sqrt{\sum_1^n \frac{(X_i-\bar{X})^2}{n-1}}$ is the Minimum Variance Unbiased Estimate of σ

c) $\sum_1^n \frac{(X_i-\bar{X})^2}{n}$ is the Maximum Likelihood Estimate of σ^2

d) $\sqrt{\sum_1^n \frac{(X_i-\bar{X})^2}{n}}$ is the Maximum Likelihood Estimate of σ

Solution: The pdf for each random variable is same as they are all identical and independent Normal Distributions with same μ and σ^2.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(x-\mu)^2}{2\sigma^2} \tag{3.1.1}$$

Let us take our maximum likelihood function for given random variable X_i

$$L(\mu;\sigma|X_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i-\mu)^2}{2\sigma^2} \tag{3.1.2}$$

Since all the random variables are i.i.d

$$L(\mu;\sigma|X_1,X_2,\dots,X_n) = \prod_{i=1}^{n} L(\mu;\sigma|X_i) \tag{3.1.3}$$

Let us denote:

$$L_m : L(\mu;\sigma|X_1,X_2,\dots,X_n) \tag{3.1.4}$$

Substituting (3.1.2) for each Random Variable in (3.1.3)

$$L_m = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i-\mu)^2}{2\sigma^2} \tag{3.1.5}$$

Taking natural log on both sides and simplifying

$$\ln L_m = \frac{-n}{2} \ln 2\pi - n\ln\sigma - \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{2\sigma^2} \tag{3.1.6}$$

In order to find Maximum Likelihood we need to maximise μ and σ w.r.t. all Random variables. Taking partial derivative w.r.t μ and taking σ as constant

$$\frac{\partial \ln L_m}{\partial\mu} = \sum_{i=1}^{n} \frac{(X_i-\mu)}{\sigma^2} \tag{3.1.7}$$

The value for μ at which L_m achieves maximum value is same in $\ln L_m$

$$\because \frac{\partial \ln L_m}{\partial\mu} = 0 \tag{3.1.8}$$

$$\therefore \sum_{i=1}^{n} \frac{(X_i-\mu)}{\sigma^2} = 0 \tag{3.1.9}$$

On simplifying the expression we get:

$$n\mu = \sum_{i=1}^{n} X_i \tag{3.1.10}$$

$$\mu = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{3.1.11}$$

Let us denote the value achieved in (3.1.11) as $\bar{X}$. Taking partial derivative w.r.t σ and taking μ as constant

$$\frac{\partial \ln L_m}{\partial\sigma} = \frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^3} \tag{3.1.12}$$

The value for σ at which L_m achieves maximum value is same in $\ln L_m$

$$\frac{\partial \ln L_m}{\partial\sigma} = 0 \tag{3.1.13}$$

$$\frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^3} = 0 \tag{3.1.14}$$

Upon simplifying the expression

$$\frac{n}{\sigma} = \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^3} \tag{3.1.15}$$

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i-\mu)^2}{n} \tag{3.1.16}$$

Substituting (3.1.11) in (3.1.16)

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n} \tag{3.1.17}$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n}} \tag{3.1.18}$$

Hence **Option 3** and **Option 4** are correct

3.2. There are two boxes. Box-1 contains 2 red balls and 4 green balls. Box-2 contains 4 red balls and 2 green balls. A box is selected at random and a ball is chosen randomly from the selected box. If the ball turns out to be red, what is the probability that Box-1 had been selected?

Solution: Box-1 has 2 red balls and 4 green balls.
Box-2 has 4 red balls and 2 green balls.
Let $B \in \{1, 2\}$ represent a random variable where 1 represents selecting box-1 and 2 represents selecting box-2. From Baye's theorem

Event	definition	value
$\Pr(B = 1)$	Probability of selecting Box-1	$\frac{1}{2}$
$\Pr(B = 2)$	Probability of selecting Box-2	$\frac{1}{2}$
$\Pr(R = 1\|B = 1)$	Probability of drawing red ball from Box-1	$\frac{1}{3}$
$\Pr(G = 1\|B = 1)$	Probability of drawing green ball from Box-1	$\frac{2}{3}$
$\Pr(R = 1\|B = 2)$	Probability of drawing red ball from Box-2	$\frac{2}{3}$
$\Pr(G = 1\|B = 2)$	Probability of drawing green ball from Box-2	$\frac{1}{3}$

TABLE 3.2.1: Table 1

$$\Pr(R = 1) = \Pr(R = 1|B = 1) \times \Pr(B = 1) + \Pr(R = 1|B = 2) \times \Pr(B = 2) \tag{3.2.1}$$

Substiting values from table (3.2.1) in (3.2.1)

$$\Pr(R = 1) = \frac{1}{2} \tag{3.2.2}$$

$$\Pr((R = 1)(B = 1)) = \Pr(R = 1|B = 1) \times \Pr(B = 1) \tag{3.2.3}$$

$$= \frac{1}{6} \tag{3.2.4}$$

We need to find $\Pr(B = 1|R = 1)$

$$\Pr(B = 1|R = 1) = \frac{\Pr((R = 1)(B = 1))}{\Pr(R = 1)} \tag{3.2.5}$$

$$= \frac{1}{3} \tag{3.2.6}$$

$\therefore$ The desired probability that box-1 is selected $= \frac{1}{3}$

3.3. Suppose customers arrive in a shop according to a Poisson process with rate 4 per hour. The shop opens at 10 : 00 am. If it is given that the second customer arrives at 10 : 40 am, what is the probability that no customer arrived before 10 : 30 am?
a) $\frac{1}{4}$
b) e^{-2}
c) $\frac{1}{2}$
d) $e^{\frac{1}{2}}$

Solution: We need to find

Random Variable	Time at which people arrive
X_p	$p = 10 : 00 - 10 : 30$
X_q	$q = 10 : 30 - 10 : 40$
X_r	$r = 10 : 00 - 10 : 40$
Y	$10 : 40$

TABLE 3.3.1: Random Variables

$$\Pr\left(X_p = 0|Y = 2\right) \tag{3.3.1}$$

In the world where the 2^{nd} person arrives at 10 : 40 am the (3.3.1) becomes:

$$= \frac{\Pr\left(X_p = 0, X_q = 1\right)}{\Pr(X_r = 1)} \tag{3.3.2}$$

$$= \frac{\Pr\left(X_p = 0\right) \times \Pr\left(X_q = 1\right)}{\Pr(X_r = 1)} \tag{3.3.3}$$

The Poisson function distribution for time interval t and rate λ for a random variable X:

$$f_X(x; t) = \frac{(\lambda t)^x \exp(-\lambda t)}{x!}$$

For the time interval p:

$$\lambda = 4, t = 0.5, x = 0 \tag{3.3.4}$$

$$\Pr\left(X_p = 0\right) = f_X\left(0; \frac{1}{2}\right) \tag{3.3.5}$$

$$= e^{-2} \tag{3.3.6}$$

$$\tag{3.3.7}$$

For the time interval q:

$$\lambda = 4, t = \frac{1}{6}, x = 1 \tag{3.3.8}$$

$$\Pr\left(X_q = 1\right) = f_X\left(1; \frac{1}{6}\right) \tag{3.3.9}$$

$$= \frac{2}{3} e^{\frac{-2}{3}} \tag{3.3.10}$$

For the time interval r:

$$\lambda = 4, t = \frac{2}{3}, x = 1 \tag{3.3.11}$$

$$\Pr(X_r = 1) = f_X\left(1; \frac{2}{3}\right) \tag{3.3.12}$$

$$= \frac{8}{3}e^{\frac{-8}{3}} \tag{3.3.13}$$

Substituting (3.3.6) (3.3.10) (3.3.13) in (3.3.3):

$$\Pr\left(X_p = 0|Y = 2\right) = \frac{1}{4} \tag{3.3.14}$$

3.4. A fair die is thrown two times independently. Let X, Y be the outcomes of these two throws and $Z = X+Y$. Let U be the remainder obtained when Z is divided by 6. Then which of the following statement(s) is/are true?

a) X and Z are independent
b) X and U are independent
c) Z and U are independent
d) Y and Z are not independent

Solution: Let $X \in \{1, 2, 3, 4, 5, 6\}$ represent the random variable which represents the outcome of the first throw of a dice. Similarly, $Y \in \{1, 2, 3, 4, 5, 6\}$ represents the random variable which represents the outcome of the second throw of a dice.

$$n(X = i) = 1, \quad i \in \{1, 2, 3, 4, 5, 6\} \tag{3.4.1}$$

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{3.4.2}$$

Similarly,

$$\Pr(Y = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{3.4.3}$$

$$Z = X + Y \tag{3.4.4}$$

$$\text{Let } z \in \{1, 2, \ldots, 11, 12\} \tag{3.4.5}$$

$$\Pr(Z = z) = \Pr(X + Y = z) \tag{3.4.6}$$

$$= \sum_{x=0}^{z} \Pr(X = x)\Pr(Y = z - x) \tag{3.4.7}$$

$$= (6 - |z - 7|) \times \frac{1}{6} \times \frac{1}{6} \tag{3.4.8}$$

$$= \frac{6 - |z - 7|}{36} \tag{3.4.9}$$

$$\Pr(Z = z) = \begin{cases} \frac{6-|z-7|}{36} & z \in \{1, 2, \ldots, 11, 12\} \\ 0 & \text{otherwise} \end{cases} \tag{3.4.10}$$

U is the remainder obtained when Z is divided by 6.

$$\text{Let } u \in \{0, 1, 2, 3, 4, 5\} \tag{3.4.11}$$

$$\Pr(U = u) = \sum_{k=0}^{2} \Pr(Z = 6k + u) \tag{3.4.12}$$

$$\Pr(U = 0) = \Pr(Z = 0) + \Pr(Z = 6) + \Pr(Z = 12) \tag{3.4.13}$$

$$= 0 + \frac{5}{36} + \frac{1}{36} = \frac{1}{6} \tag{3.4.14}$$

$$\text{for } u \in \{1, 2, 3, 4, 5\} \tag{3.4.15}$$

$$\Pr(U = u) = \Pr(Z = 0 + u) + \Pr(Z = 6 + u) \tag{3.4.16}$$

$$= \frac{6 - |u - 7|}{36} + \frac{6 - |6 + u - 7|}{36} \tag{3.4.17}$$

$$= \frac{6 - (7 - u)}{36} + \frac{6 - (u - 1)}{36} \tag{3.4.18}$$

$$= \frac{u - 1 + 7 - u}{36} = \frac{6}{36} \tag{3.4.19}$$

$$= \frac{1}{6} \tag{3.4.20}$$

$$\Pr(U = u) = \begin{cases} \frac{1}{6} & u \in \{0, 1, 2, 3, 4, 5\} \\ 0 & \text{otherwise} \end{cases} \tag{3.4.21}$$

Now, for checking each option,

a) Checking if X and Z are independent

$$p_1 = \Pr(Z = z, X = x) \tag{3.4.22}$$
$$= \Pr(Y = z - x, X = x) \tag{3.4.23}$$
$$= \Pr(Y = z - x) \times \Pr(X = x) \tag{3.4.24}$$
$$= \begin{cases} \frac{1}{36} & z - x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{3.4.25}$$

$$\Pr(Z = z) \times \Pr(X = x) = \frac{6 - |z-7|}{36} \times \frac{1}{6} \tag{3.4.26}$$
$$= \frac{6 - |z-7|}{216} \tag{3.4.27}$$
$$\Pr(Z = z)\Pr(X = x) \neq \Pr(Z = z, X = x) \tag{3.4.28}$$

X and Z are not independent from (3.4.28) and hence option (3.4a) is false.

b) Checking if X and U are independent

$$p_2 = \Pr(U = u, X = x) \tag{3.4.29}$$

$$p_2 = \Pr((Z = u) + (Z = 6 + u) + (Z = 12 + u), X = x) \tag{3.4.30}$$

$$p_2 = \Pr((Y = u - x) + (Y = 6 + u - x) + (Y = 12 + u - x), X = x) \tag{3.4.31}$$

$$p_2 = \frac{1}{6} \times \frac{1}{6} \tag{3.4.32}$$
$$= \frac{1}{36} \tag{3.4.33}$$

$$\Pr(U = u) \times \Pr(X = x) = \frac{1}{6} \times \frac{1}{6} \tag{3.4.34}$$
$$= \frac{1}{36} \tag{3.4.35}$$
$$\Pr(U = u)\Pr(X = x) = \Pr(U = u, X = x) \tag{3.4.36}$$

X and U are independent from (3.4.36) and hence option (3.4b) is true.

c) Checking if Z and U are independent

$$p_3 = \Pr(Z = z | U = u) \tag{3.4.37}$$
$$p_3 = \begin{cases} 1 & u = 1 \text{ and } z = 7 \\ \frac{1}{2} & u = 0 \text{ and } z \in \{6, 12\} \\ \frac{1}{2} & u \in \{2,3,4,5\} \text{ and} \\ & z = u \text{ or } z = 6 + u \\ 0 & \text{otherwise} \end{cases} \tag{3.4.38}$$
$$\Pr(Z = z) = \frac{6 - |z-7|}{36} \tag{3.4.39}$$

If Z and U are independent, then

$$\Pr(Z = z | U = u) = \frac{\Pr(Z = z, U = u)}{\Pr(U = u)} \tag{3.4.40}$$
$$= \frac{\Pr(Z = z)\Pr(U = u)}{\Pr(U = u)} \tag{3.4.41}$$
$$= \Pr(Z = z) \tag{3.4.42}$$

But,

$$\Pr(Z = z | U = u) \neq \Pr(Z = z) \tag{3.4.43}$$

X and U are not independent from (3.4.43) and hence option (3.4c) is false.

d) Checking if Y and Z are independent

$$p_1 = \Pr(Z = z, Y = y) \tag{3.4.44}$$
$$= \Pr(X = z - y, Y = y) \tag{3.4.45}$$
$$= \Pr(X = z - y) \times \Pr(Y = y) \tag{3.4.46}$$
$$= \begin{cases} \frac{1}{36} & z - y \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{3.4.47}$$

$$\Pr(Z = z) \times \Pr(Y = y) = \frac{6 - |z-7|}{36} \times \frac{1}{6} \tag{3.4.48}$$
$$= \frac{6 - |z-7|}{216} \tag{3.4.49}$$
$$\Pr(Z = z)\Pr(Y = y) \neq \Pr(Z = z, Y = y) \tag{3.4.50}$$

X and Z are not independent from (3.4.50) and hence option (3.4d) is true.

Thus, options (3.4b) and (3.4d) are true.

3.5. Let X be a random variable with a certain non-degenerate distribution. Then identify the correct statements

a) If X has an exponential distribution then $median(X) < E(X)$
b) If X has a uniform distribution on an interval $[a,b]$, then $E(X) < median(X)$
c) If X has a Binomial distribution then $V(X) < E(X)$
d) If X has a normal distribution, then $E(X) < V(X)$

Solution: Expected value($E(X)$): It is nothing but weighted average Median($median(X)$): It is the value separating the higher half from the lower half of a data sample Variance($V(X)$): It is the expectation of the squared deviation of a random variable from its mean

a) Let's consider X has an exponential distribution.

$$X \sim Exp(\lambda) \tag{3.5.1}$$

where λ is rate parameter.
Probability function of exponential distribution,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{3.5.2}$$

The expected value of $X \sim Exp(\lambda)$,

$$E(X) = \frac{1}{\lambda} \tag{3.5.3}$$

The median of $X \sim Exp(\lambda)$,

$$median(X) = \frac{\ln 2}{\lambda} \tag{3.5.4}$$

$$\ln 2 < 1 \tag{3.5.5}$$

$$\frac{\ln 2}{\lambda} < \frac{1}{\lambda} \tag{3.5.6}$$

$$median(X) < E(X) \tag{3.5.7}$$

Hence, option 1 is correct.

b) Let's consider X has a uniform distribution in interval $[a,b]$,

$$X \sim U(a,b) \tag{3.5.8}$$

where, a = lower limit
b = upper limit
Probability function of uniform distribution,

$$f_X(k) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a, x > b \end{cases} \tag{3.5.9}$$

The expected value of $X \sim U(a,b)$,

$$E(X) = \frac{1}{2}(a+b) \tag{3.5.10}$$

The median of $X \sim U(a,b)$,

$$median(X) = \frac{1}{2}(a+b) \tag{3.5.11}$$

$$E(X) = median(X) \tag{3.5.12}$$

Hence, option 2 is incorrect.

c) Let's consider X has a binomial distribution,

$$X \sim B(n,p) \tag{3.5.13}$$

where, n = no. of trails
p = success parameter
Probability function of binomial distribution,

$$f_X(k) = \begin{cases} {}^nC_k p^k (1-p)^{n-k} & 0 \leq k \leq n \\ 0 & otherwise \end{cases} \tag{3.5.14}$$

The expected value of $X \sim B(n,p)$,

$$E(X) = np \tag{3.5.15}$$

The variance of $X \sim B(n,p)$,

$$V(X) = \sigma^2 = np(1-p) \tag{3.5.16}$$

$$1 - p \leq 1 \tag{3.5.17}$$

$$np(1-p) \leq np \tag{3.5.18}$$

$$V(X) \leq E(X) \tag{3.5.19}$$

Hence, option 3 is incorrect.

d) Let's consider X has a normal distribution,

$$X \sim N\left(\mu, \sigma^2\right) \tag{3.5.20}$$

where, μ = mean of distribution
σ^2 = variance
Probability function of normal distribution,

$$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{2\sigma}\right)^2} \tag{3.5.21}$$

The expected value of $X \sim N\left(\mu, \sigma^2\right)$,

$$E(X) = \mu \tag{3.5.22}$$

The variance of $X \sim N\left(\mu, \sigma^2\right)$,

$$V(X) = \sigma^2 \tag{3.5.23}$$

$E(X)$ and $V(X)$ are user defined. So, they can take any value.
Hence, option 4 is incorrect.

3.6. A and B play a game of tossing a fair coin. A starts the game by tossing the coin once and B then tosses the coin twice, followed by A tossing the coin once and B tossing the coin twice and this continues until a head turns up. Whoever gets the first head wins the game. Then,

a) $P(B \text{ Wins}) > P(A \text{ Wins})$
b) $P(B \text{ Wins}) = 2P(A \text{ Wins})$
c) $P(A \text{ Wins}) > P(B \text{ Wins})$
d) $P(A \text{ Wins}) = 1 - P(B \text{ Wins})$

Solution: Given, a fair coin is tossed till heads turns up.

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{104.1}$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4, 5\}$, such that The state transition matrix for the Markov chain is

TABLE 3.6.1: States and their notations

Notation	State
$S = 1$	A's turn
$S = 2$	B's first turn
$S = 3$	B's second turn
$S = 4$	A wins
$S = 4$	B wins

$$P = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{104.2}$$

Clearly, the states $1, 2, 3$ are transient, while $4, 5$ are absorbing. The standard form of a state transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{104.3}$$

where, Converting (104.2) to standard form, we get

TABLE 3.6.2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatices

$$P = \begin{array}{c c} & \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix} \end{array} \tag{104.4}$$

From (104.4),

$$R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \tag{104.5}$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \tag{104.6}$$

where,

$$F = (I - Q)^{-1} \tag{104.7}$$

is called the fundamental matrix of P.
On solving, we get

$$\bar{P} = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5714 & 0.4285 & 0 & 0 & 0 \\ 0.1428 & 0.8571 & 0 & 0 & 0 \\ 0.2857 & 0.7142 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{104.8}$$

A element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability in state j, starting from state i. Then,

a) $Pr(A \text{ wins}) = \bar{p}_{14} \approx 0.5714$
b) $Pr(B \text{ wins}) = \bar{p}_{15} \approx 0.4285$

$$\therefore \bar{p}_{14} > \bar{p}_{15} \tag{104.9}$$

Also, in $\bar{P}$, all the terms in every row should sum to 1.

$$\Rightarrow \bar{p}_{14} + \bar{p}_{15} + 0 + 0 + 0 = 1 \tag{104.10}$$
$$\therefore \bar{p}_{14} = 1 - \bar{p}_{15} \tag{104.11}$$

Therefore, options 3), 4) are correct.

4 December 2015

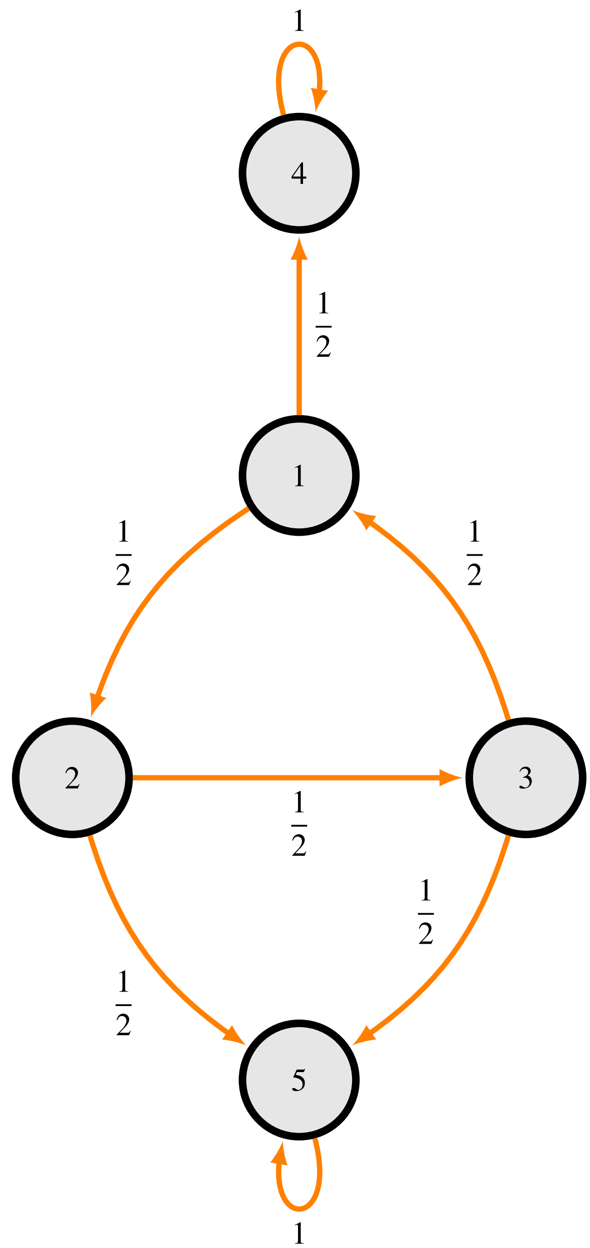
4.1. The probability that a ticketless traveler is caught during a trip is 0.1. If the traveler makes 4 trips , the probability that he/she will be caught during at least one of the trips is:

a) $1 - (0.9)^4$
b) $(1 - 0.9)^4$
c) $1 - (1 - 0.9)^4$
d) $(0.9)^4$

Solution: Let $X_i \in \{0, 1\}$ represent the ith trip where 1 denotes a ticketless traveller is caught. Given,

$$\Pr(X_i = 1) = p = 0.1 \tag{4.1.1}$$

Markov chain diagram



Let,

$$X = \sum_{i=1}^{n} X_i \tag{4.1.2}$$

where n is the number of trips and X has a binomial distribution.

$$p_X(k) = \begin{cases} {}^nC_k \, p^K (1-p)^{n-k}, & 0 \le k \le n \\ 0, & otherwise \end{cases} \tag{4.1.3}$$

As he/she makes 4 trips in total, Using (4.1.1) and (4.1.3),

$$\Pr(X=0) = p_X(0) \tag{4.1.4}$$
$$= {}^4C_0\, p^0(1-p)^4 \tag{4.1.5}$$
$$\Pr(X=0) = (0.9)^4 \tag{4.1.6}$$

Then probability of being caught in atleast one trip is,(Using (4.1.6))

$$\Pr(X \geq 1) = 1 - \Pr(X < 1) \tag{4.1.7}$$
$$= 1 - \Pr(X = 0) \tag{4.1.8}$$
$$= 1 - (0.9)^4 \tag{4.1.9}$$

4.2. Suppose that (X, Y) has a joint probability distribution with the marginal distribution of X being N(0,1) and $E(Y|X = x) = x^3$ for all $x \in R$. Then, which of the following statements are true?

a) $\text{Corr}(X, Y) = 0$
b) $\text{Corr}(X, Y) > 0$
c) $\text{Corr}(X, Y) < 0$
d) X and Y are independent

Solution: The following result shall be useful later. For $n \in N$

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \begin{cases} 0 & n \text{ is odd} \\ (n-1) \times ... \times 3 \times 1 & n \text{ is even} \end{cases} \tag{4.2.1}$$

The proof for the above can be found at the end of the solution.

$$Corr(X, Y) = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} \tag{4.2.2}$$

We know $X \sim N(0, 1)$. Thus,

$$f_X(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \tag{4.2.3}$$
$$E(X) = 0 \tag{4.2.4}$$
$$\sigma_X^2 = 1 \tag{4.2.5}$$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 \tag{4.2.6}$$

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx \tag{4.2.7}$$
$$= \int_{-\infty}^{\infty} \frac{x^3 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{4.2.8}$$
$$= 0 \tag{4.2.9}$$

$$E(Y^2) = \int_{-\infty}^{\infty} E(Y^2|X = x) f_X(x) dx \tag{4.2.10}$$
$$= \int_{-\infty}^{\infty} \frac{x^6 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{4.2.11}$$
$$= 15 \tag{4.2.12}$$

Substituting in (4.2.6)

$$\sigma_Y^2 = 15 \tag{4.2.13}$$

$$\sigma_{XY}^2 = E(XY) - E(X)E(Y) \tag{4.2.14}$$

$$E(XY) = \int_{-\infty}^{\infty} E(XY|X = x) f_X(x) dx \tag{4.2.15}$$
$$= \int_{-\infty}^{\infty} E(xY|X = x) f_X(x) dx \tag{4.2.16}$$
$$= \int_{-\infty}^{\infty} x E(Y|X = x) f_X(x) dx \tag{4.2.17}$$
$$= \int_{-\infty}^{\infty} \frac{x^4 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{4.2.18}$$
$$= 3 \tag{4.2.19}$$

Substituting in (4.2.14)

$$\sigma_{XY}^2 = 3 \tag{4.2.20}$$

Substituting in (4.2.2)

$$Corr(X, Y) = \frac{3}{\sqrt{15}} > 0 \tag{4.2.21}$$

Since $Corr(X, Y) \neq 0$, X and Y are dependent. Thus option 2 is the only correct option. **Proof for the integral:** If n is odd, $\frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ is an odd function, thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = 0 \tag{4.2.22}$$

If n is even,

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} (x^{n-1})\left(\frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\right) dx \tag{4.2.23}$$

Using integration by parts,

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \left(x^{n-1} \int \frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \right) \Bigg|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} \left(\int \frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \right) dx \quad (4.2.24)$$

$$= \left(x^{n-1} \left(-\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\right) \right) \Bigg|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} \left(-\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\right) dx \quad (4.2.25)$$

$$= (n-1) \int_{-\infty}^{\infty} \frac{x^{n-2} e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \quad (4.2.26)$$

$$= (n-1)(n-3) \int_{-\infty}^{\infty} \frac{x^{n-4} e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \quad (4.2.27)$$

$$= (n-1) \times ... \times 3 \times 1 \int_{-\infty}^{\infty} \frac{x^{0} e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \quad (4.2.28)$$

$$= (n-1) \times ... \times 3 \times 1 \quad (4.2.29)$$

Alternative proof for the integral:

If n is odd, $\frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ is an odd function, thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = 0 \quad (4.2.30)$$

If n is even, let $n = 2k$. We differentiate the following identity k times w.r.t. α.

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\left(\frac{\pi}{\alpha}\right)} \quad (4.2.31)$$

On differentiating k times, we get

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2} = \frac{1 \times 3 \times ... \times (2k-1)}{2^k} \sqrt{\left(\frac{\pi}{\alpha^{2k+1}}\right)} \quad (4.2.32)$$

On substituting $\alpha = \frac{1}{2}$, we get

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} = 1 \times 3 \times ... \times (n-1) \sqrt{2\pi} \quad (4.2.33)$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = (n-1) \times ... \times 3 \times 1 \quad (4.2.34)$$

4.3. Let $X_1, X_2, ..., X_n$ be independent and identically distributed, each having a uniform distribution on $(0, 1)$. Let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Then, which of the following statements are true?

A) $\frac{S_n}{n \log n} \to 0$ as $n \to \infty$ with probability 1.

B) $\Pr\left(\left(S_n > \frac{2n}{3}\right) \text{occurs for infinitely many n}\right) = 1$

C) $\frac{S_n}{\log n} \to 0$ as $n \to \infty$ with probability 1.

D) $\Pr\left(\left(S_n > \frac{n}{3}\right) \text{occurs for infinitely many n}\right) = 1$

Solution:

Symbol	**expression/definition**
S_n	$\sum_{i=1}^{n} X_i$
μ_n	$\frac{1}{n} \sum_{i=1}^{n} X_i$
X	Independent continuous random variable identical to $X_1, X_2, ..., X_n$

TABLE 4.3.1: Variables and their definitions

a) Given

$$S_n = \sum_{i=1}^{n} X_i, n \geq 1 \quad (4.3.1)$$

Dividing by n on both sides

$$\frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i = \mu_n \quad (4.3.2)$$

It can be said that $X_1, X_2, ..., X_n$ are the trials of X. By definition

$$E[X] = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = \lim_{n\to\infty} \frac{S_n}{n} \quad (4.3.3)$$

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X] = \frac{1}{2} \quad (4.3.4)$$

$$\therefore \lim_{n\to\infty} \frac{S_n}{n \log n} = 0 \quad (4.3.5)$$

b) Using weak law, (4.3.4), and table (4.3.1)

$$\lim_{n\to\infty} \Pr\left(|\mu_n - E[X]| > \epsilon\right) = 0, \forall \epsilon > 0 \quad (4.3.6)$$

$$\lim_{n\to\infty} \Pr\left(S_n = \frac{n}{2}\right) = 1 \quad (4.3.7)$$

It can be easily implied from (4.3.7) that option B is false.

c) It is easy to observe from (4.3.4) that option C is false.

d) Using (4.3.7), we get

$$\Pr\left(\left(S_n > \frac{n}{3}\right)\text{occurs for infinitely many n}\right) = 1 \quad (4.3.8)$$

4.4. A fair coin is tossed repeatedly. Let X be the number of tails before the first heads occurs. Let Y denote the number of tails between the first and second heads. Let $X + Y = N$. Then which of the following are true?

a) X and Y are independent random variables with

$$\Pr(X = k) = \Pr(Y = k) = \begin{cases} 2^{-(k+1)} & k = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \quad (4.4.1)$$

b) N has a probability mass function given by

$$\Pr(N = k) = \begin{cases} (k-1)2^{-k} & k = 2, 3, 4 \dots \\ 0 & otherwise \end{cases} \quad (4.4.2)$$

c) Given $N = n$, the conditional distribution of X and Y are independent

d) Given $N = n$

$$\Pr(X = k) = \begin{cases} \frac{1}{n+1} & n = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \quad (4.4.3)$$

4.5. An urn has 3 red and 6 black balls. Balls are drawn at random one by one without replacement. The probability that second red ball appears on fifth draw is:

a) $\frac{1}{9!}$

b) $\frac{4!}{9!}$

c) $4\left(\frac{6!4!}{9!}\right)$

d) $\frac{6!4!}{9!}$

Solution: To obtain a second red ball at the fifth draw, the first 4 trials should involve drawing only 1 red ball out of the 3 and 3 black balls out of the 6. Probability of this happening:

$$\frac{{}^3C_1{}^6C_3}{{}^9C_4} \quad (4.5.1)$$

The probability of the fifth ball turning out to be red is:

$$\frac{{}^2C_1}{{}^5C_1} \quad (4.5.2)$$

By Multiplication rule, total probability:

$$\frac{{}^3C_1{}^6C_3{}^2C_1}{{}^5C_1{}^9C_4} = \frac{3! \times 6! \times 2! \times 4! \times 4! \times 5!}{2! \times 3! \times 3! \times 5! \times 9!} \quad (4.5.3)$$

$$= 4\left(\frac{4!6!}{9!}\right) \quad (4.5.4)$$

5 June 2013

5.1. Let X be a non-negative integer valued random variable with probability mass function $f(x)$ satisfying $(x+1)f(x+1) = (\alpha + \beta x)f(x)$, $x = 0, 1, 2, \dots;\ \beta \neq 1$. You may assume that $E(X)$ and $Var(X)$ exist. Then which of the following statements are true?

a) $E(X) = \dfrac{\alpha}{1-\beta}$

b) $E(X) = \dfrac{\alpha^2}{(1-\beta)(1+\alpha)}$

c) $Var(X) = \dfrac{\alpha^2}{(1-\beta)^2}$

d) $Var(X) = \dfrac{\alpha}{(1-\beta)^2}$

Solution: For a discrete random variable X with P.D.F. $f(x)$ and which can take values from a set $\mathbb{S}$,

$$E(X) = \sum_{x\in\mathbb{S}} x f(x) \quad (5.1.1)$$

And,

$$E(X^2) = \sum_{x\in\mathbb{S}} x^2 f(x) \quad (5.1.2)$$

Also, as $f(x)$ is the P.D.F.,

$$\sum_{x\in\mathbb{S}} f(x) = 1 \quad (5.1.3)$$

Given, for $x \in \mathbb{S} = \{0, 1, 2, \dots n\}$,

$$(x+1)f(x+1) = (\alpha + \beta x)f(x) \quad (5.1.4)$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)f(x+1) = \sum_{x=0}^{n}(\alpha + \beta x)f(x) \quad (5.1.5)$$

Replacing $x+1$ with x in L.H.S. we get,

$$\sum_{x=1}^{n+1} xf(x) = \sum_{x=0}^{n}(\alpha+\beta x)f(x) \quad (5.1.6)$$

Rewriting LHS, we get,

$$\sum_{x=0}^{n} xf(x) + (n+1)f(n+1) = \sum_{x=0}^{n}(\alpha+\beta x)f(x) \quad (5.1.7)$$

But as $x \in \{0, 1, 2...n\}$, $f(n+1) = 0$. So the equation becomes

$$\sum_{x=0}^{n} xf(x) = \alpha\sum_{x=0}^{n} f(x) + \beta\sum_{x=0}^{n} xf(x) \quad (5.1.8)$$

Using (5.1.1) and (5.1.3), we get,

$$E(X) = \alpha(1) + \beta E(X) \quad (5.1.9)$$

So,

$$E(X) = \frac{\alpha}{1-\beta} \quad (5.1.10)$$

Now in (5.1.4), multiplying both sides by $(x+1)$, we get,

$$(x+1)^2 f(x+1) = (\alpha+\beta x)(x+1)f(x) \quad (5.1.11)$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)^2 f(x+1) = \sum_{x=0}^{n}(\alpha+\beta x)(x+1)f(x) \quad (5.1.12)$$

Replacing $x+1$ with x in L.H.S. we get,

$$\sum_{x=1}^{n+1} x^2 f(x) = \sum_{x=0}^{n}(\beta x^2 f(x) + (\alpha+\beta)xf(x) + \alpha f(x)) \quad (5.1.13)$$

Rewriting LHS similarly as before, we get,

$$\sum_{x=0}^{n} x^2 f(x) = \beta\sum_{x=0}^{n} x^2 f(x) + (\alpha+\beta)\sum_{x=0}^{n} xf(x) + \alpha\sum_{x=0}^{n} f(x) \quad (5.1.14)$$

Using (5.1.1), (5.1.2) and (5.1.3), we get,

$$E(X^2) = \beta E(X^2) + (\alpha+\beta)E(X) + \alpha(1) \quad (5.1.15)$$

Using (5.1.10)

$$E(X^2)(1-\beta) = \frac{\alpha(\alpha+\beta)}{1-\beta} + \alpha \quad (5.1.16)$$

So,

$$E(X^2) = \frac{\alpha^2+\alpha}{(1-\beta)^2} \quad (5.1.17)$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \quad (5.1.18)$$

Using (5.1.10) and (5.1.17),

$$Var(X) = \frac{\alpha^2+\alpha}{(1-\beta)^2} - \frac{\alpha^2}{(1-\beta)^2} \quad (5.1.19)$$

So,

$$Var(X) = \frac{\alpha}{(1-\beta)^2} \quad (5.1.20)$$

So, options 1 and 4 are correct.

5.2. Let X be a random variable with probability density function,

$$f(x) = \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^\alpha} \quad (5.2.1)$$

such that $-\infty < \mu < \infty$; $\alpha > 0$; $x > \mu$, The hazard function is:

a) constant for all α
b) an increasing function for some α
c) independent of α
d) independent of μ when $\alpha = 1$

Solution: Given PDF of X,

$$f(x) = \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^\alpha} \quad (5.2.2)$$

Important property(using in (5.2.8) as $x > \mu$): Given $x - y > 0$ and $-\infty < y < \infty$, then

$$\lim_{x\to-\infty} x - y = 0 \quad (5.2.3)$$

CDF of X,

$$F(x) = \int_{-\infty}^{x} f(x)\, dx \tag{5.2.4}$$
$$= \int_{-\infty}^{x} \alpha(x-\mu)^{\alpha-1} e^{-(x-\mu)^{\alpha}} dx \tag{5.2.5}$$
$$= \int_{-\infty}^{x} e^{-(x-\mu)^{\alpha}} d(x-\mu)^{\alpha} \tag{5.2.6}$$
$$= \left[\frac{e^{-(x-\mu)^{\alpha}}}{-1}\right]_{-\infty}^{x} \tag{5.2.7}$$
$$= -e^{-(x-\mu)^{\alpha}} - \lim_{x\to-\infty} \frac{e^{-(x-\mu)^{\alpha}}}{-1} \tag{5.2.8}$$
$$= -e^{-(x-\mu)^{\alpha}} + e^{-(0)^{\alpha}} \tag{5.2.9}$$
$$F(x) = 1 - e^{-(x-\mu)^{\alpha}} \tag{5.2.10}$$

Hazard function $\beta(x)$,(using (5.2.2) and (5.2.10))

$$\beta(x) = \frac{f(x)}{1-F(x)} \tag{5.2.11}$$
$$= \frac{\alpha(x-\mu)^{\alpha-1} e^{-(x-\mu)^{\alpha}}}{1-(1-e^{-(x-\mu)^{\alpha}})} \tag{5.2.12}$$
$$= \frac{\alpha(x-\mu)^{\alpha-1} e^{-(x-\mu)^{\alpha}}}{e^{-(x-\mu)^{\alpha}}} \tag{5.2.13}$$
$$\beta(x) = \alpha(x-\mu)^{\alpha-1} \tag{5.2.14}$$

a) $\beta(x)$ is not constant for all α
b) $\beta(x) = \alpha(x-\mu)^{\alpha-1}$ is an increasing function for $\alpha < 0$ *or* $\alpha > 1$ as given $x-\mu > 0$ for all x.

Proof: Using first derivative test, A function is increasing iff its first derivative is positive for all x.

$$\frac{d}{dx}\beta(x) = \frac{d}{dx}\alpha(x-\mu)^{\alpha-1} \tag{5.2.15}$$
$$= \alpha(\alpha-1)(x-\mu)^{\alpha-2} \tag{5.2.16}$$

For (5.2.16) to be positive, (As given $x-\mu > 0$)

$$\alpha(\alpha-1)(x-\mu)^{\alpha-2} > 0 \tag{5.2.17}$$
$$\alpha(\alpha-1) > 0 \tag{5.2.18}$$
$$\implies \alpha \in (-\infty, 0) \cup (1, \infty) \tag{5.2.19}$$

$\therefore \beta(x)$ an increasing function for some α

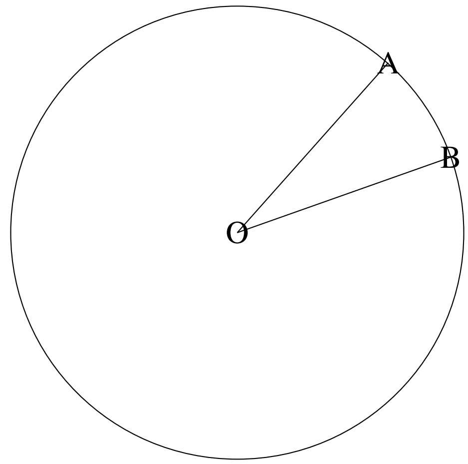
c) $\beta(x)$ is dependent of α
d) when $\alpha = 1$,

$$\beta(x) = \alpha(x-\mu)^{0} = \alpha \tag{5.2.20}$$

Therefore the hazard function is independent of μ when $\alpha = 1$.

ANSWER: (2) and (4)

5.3. A point is chosen at random from a circular disc shown below. What is the probability that the point lies in the sector OAB?



(where ∠AOB = x radians)

a) $\frac{2x}{\pi}$
b) $\frac{x}{\pi}$
c) $\frac{x}{2\pi}$
d) $\frac{x}{4\pi}$

Solution:

SOLUTION

Let $X \in \{0, 1\}$ be a random variable such that X=0 means we choose a point lying in sector OAB and X=1 means that we choose a point lying outside sector OAB and inside the circle.

Area of a sector subtending an angle θ at the centre of circle with radius a is given by :

$$A = \frac{1}{2}a^2\theta \tag{5.3.1}$$

where θ is in radians.

Let the radius of circle shown in figure be r. It is given that sector OAB subtends an angle of x radians at the centre of the circle.

Probability that the chosen point lies in sector

OAB is:

$$\Pr(X=0) = \frac{\text{Area of sector OAB}}{\text{Area of circle}} \tag{5.3.2}$$

$$= \frac{\frac{1}{2}r^2 x}{\pi r^2} \tag{5.3.3}$$

$$= \frac{x}{2\pi} \tag{5.3.4}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$.

ALTERNATE SOLUTION

The joint pdf is given by:

$$f_{r\theta}(r,\theta) = \begin{cases} \frac{r}{\pi R^2} & \text{if } 0 < \text{r} < \text{R} \text{ , } 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{5.3.5}$$

Let A $\equiv$ (R, θ_2) and B $\equiv$ (R, θ_1).
Hence,

$$(\theta_2 - \theta_1) = x \tag{5.3.6}$$

We want $\theta \in (\theta_1 \text{ , } \theta_2)$ and r $\in$ (0,R) for point to lie in the sector. Let the point to be chosen be (r, θ).

So, Required probability is:

$$\Pr(\theta_1 < \theta < \theta_2, 0 < r < R)$$

$$= \int_{\theta_1}^{\theta_2} \int_0^R \frac{r}{\pi R^2} \, dr \, d\theta \tag{5.3.7}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{\pi R^2} \frac{r^2}{2}\Big|_0^R \tag{5.3.8}$$

$$= \int_{\theta_1}^{\theta_2} \frac{R^2}{2\pi R^2} \, d\theta \tag{5.3.9}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2\pi} \, d\theta \tag{5.3.10}$$

$$= \frac{\theta}{2\pi}\Big|_{\theta_1}^{\theta_2} \tag{5.3.11}$$

$$= \frac{\theta_2 - \theta_1}{2\pi} \tag{5.3.12}$$

$$= \frac{x}{2\pi} \tag{5.3.13}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$**.**

6 December 2012

6.1. Let X be a binomial random variable with parameters $\left(11, \frac{1}{3}\right)$. At which value(s) of k is $\Pr(X = k)$ maximized?

a) k=2
b) k=3
c) k=4
d) k=5

Solution: X has a binomial distribution :

$$\Pr(X = k) = {}^nC_k (q)^{n-k} (p)^k \tag{6.1.1}$$

Where,

- n=11
- $p = \frac{1}{3}$
- $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

$$\Pr(X = k) = {}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k \tag{6.1.2}$$

For Pr(X = k) to be maximized

$$\Pr(X = k) \ \geq \ \Pr(X = k+1) \tag{6.1.3}$$

$$\frac{\Pr(X = k)}{\Pr(X = k+1)} = \frac{{}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k}{{}^{11}C_{k+1} \left(\frac{2}{3}\right)^{10-k} \left(\frac{1}{3}\right)^{k+1}} \geq 1 \tag{6.1.4}$$

$$\frac{2(k+1)}{11-k} \geq 1 \tag{6.1.5}$$

$$\implies k \geq 3 \tag{6.1.6}$$

$$\Pr(X = k) \geq \Pr(X = k-1) \tag{6.1.7}$$

$$\frac{\Pr(X=k)}{\Pr(X=k-1)} = \frac{{}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k}{{}^{11}C_{k-1} \left(\frac{2}{3}\right)^{12-k} \left(\frac{1}{3}\right)^{k-1}} \geq 1 \tag{6.1.8}$$

$$\frac{12-k}{2k} \geq 1 \tag{6.1.9}$$

$$\implies k \leq 4 \tag{6.1.10}$$

From (6.1.6) , (6.1.10) and since k is an integer
$\Pr(X = k)$ is maximized for k=3, k=4
Thus options 2) and 3) are correct

6.2. Men arrive in a queue according to a Poisson process with rate λ_1 and women arrive in the same queue according to another Poisson process with rate λ_2. The arrivals of men and women are independent. The probability that the first person to arrive in the queue is a man is:

a) $\dfrac{\lambda_1}{\lambda_1+\lambda_2}$
b) $\dfrac{\lambda_2}{\lambda_1+\lambda_2}$
c) $\dfrac{\lambda_1}{\lambda_2}$
d) $\dfrac{\lambda_2}{\lambda_1}$

Solution: Let X and Y be Poisson random variables, with the values X takes being the number of men joining the queue in an arbitrary time t, and the values Y takes being the number of women joining the queue in an arbitrary time t.

$$Pr(X = i) = \frac{\lambda_1^i \cdot e^{-\lambda_1}}{i!} \tag{6.2.1}$$

$$Pr(Y = i) = \frac{\lambda_2^i \cdot e^{-\lambda_2}}{i!} \tag{6.2.2}$$

For 2 independent Poisson distributions with means λ_1 and λ_2, the simultaneous distribution can be represented by:

$$Pr(X+Y = i) = \frac{(\lambda_1+\lambda_2)^i \cdot e^{-(\lambda_1+\lambda_2)}}{i!} \tag{6.2.3}$$

Now we take conditional probability that if only one person entered the queue within a certain time t, then the probability of them being a man and not a woman is given by:

$$Pr(X = 1|(X+Y) = 1) = \frac{Pr((X = 1) + (Y = 0))}{Pr(X+Y = 1)} \tag{6.2.4}$$

$$\tag{6.2.5}$$

Since X and Y are independent,

$$Pr(X = 1|(X+Y) = 1) = \frac{Pr(X = 1) \cdot Pr(Y = 0)}{Pr(X+Y = 1)} \tag{6.2.6}$$

$$= \frac{\frac{\lambda_1^1 \cdot e^{-\lambda_1}}{1!} \cdot \frac{\lambda_2^0 \cdot e^{-\lambda_2}}{0!}}{\frac{(\lambda_1+\lambda_2)^1 \cdot e^{-(\lambda_1+\lambda_2)}}{1!}} \tag{6.2.7}$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2} \tag{6.2.8}$$

The probability that the first person to arrive in the queue is a man is option A, i.e $\frac{\lambda_1}{\lambda_1+\lambda_2}$