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Nagubandi Krishna Sai
MS20BTECH11014

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Question

Consider a Markov chain with state space $\{1,2,....,100\}$. Suppose states 2i and 2j communicate with each other and states 2i-1 and 2j-1 communicate with each other for every i,j = 1,2,...,50. Further suppose that $p_{3,3}^{(2)} > 0, p_{4,4}^{(3)} > 0$ and $p_{2,5}^{(7)} > 0$. Then

1. The Markov chain is irreducible.
2. The Markov chain is aperiodic.
3. State 8 is recurrent.
4. State 9 is recurrent.

Prerequisites

According to the question, all even and odd positioned states communicate with each other.

Definition

$$p_{i,j}^{(n)} > 0; n \geq 0$$
$$p_{i,j}^{n} = Pr[X_t = j | X_{t-1} = i]; \forall n \geq 0.$$

Where X is the collection of random variables and **index** t represents time. X(t) represents the **state** of the process at time t. This is the probability that the chain moves from state i to state j in exactly m steps.

Note

If $p_{i,j}^{(n)} > 0$, for some n, then we say that the state j is accessible from state i.

Definition 1

We say that **Markov chain** is **irreducible** if and only if all states belong to one communication class and all states communicate with each other.

Definition 2

In an **irreducible chain** all states belong to a single communicating class. This means that, if one of the states in an irreducible Markov chain is **aperiodic**. Then, all the remaining states are also aperiodic.

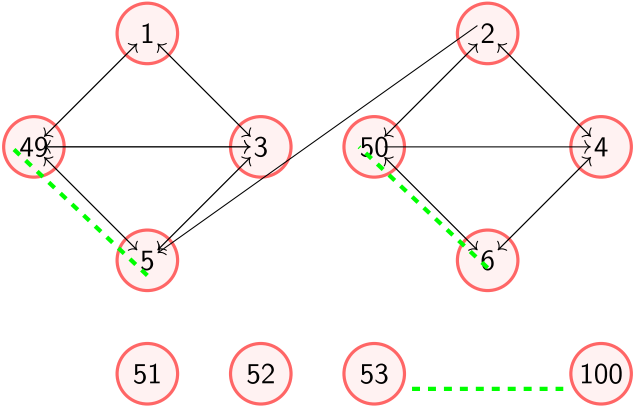
Condition

$$\mathrm{d(K)} = \mathrm{gcd}\{\mathrm{m} \geq 1 : Pm_{k,k} > 0\}$$

If $\mathrm{d(K)} = 1$, then the state K is aperiodic.
If $\mathrm{d(K)} = 0$, then the state K is periodic.

Solution

Consider, S $= \{1,2,3,....,100\}$



Solution

Classification of states

Communication class	set of elements
$C_1(1)$	$\{1,3,5,7,....,49\}$
$C_1(2)$	$\{2,4,6,8,....,50\}$
$C_1(51)$	$\{51\}$
$C_1(52)$	$\{52\}$
$\vdots$	$\vdots$
$C_1(100)$	$\{100\}$

Table: Communication class

Solution

$\therefore$ As there are 52 communication classes, the given Markov chain is reducible.

Periodicity of states

Periodicity of states	set of states
d(1)	{1,3,5,7,.....,49}
d(2)	{2,4,6,8,.....,50}

Table: Periodicity of some of elements of set S

Solution

Periodicity

$$d(1) = d(2) = \gcd\{2,3,....,50\} = 1.$$

$\therefore$ The states 1 to 50 are **Aperiodic**.

Periodicity

$$d(51) = d(52) = ... = d(100) = 0.$$

$\therefore$ The states 51 to 100 are **Periodic**.

Solution

Explanation

Hence, by the above figure, states from 1 to 50 are recurrent. Because the states 5 is recurrent, as it is given in question $p_{2,5}^{(7)} > 0$, this means state 5 is accessible from state 2. This means state 2 after communicating with even states, returns to state 5, and with state 5 all the odd states return to their states again.

Explanation

From above, we can concluded that $\{1,3,5,7,...,49,51,52,53,....,100\}$ are recurrent states. And $\{2,4,6,8,....,50\}$ are transient states.

Conclusion

Option 1

As there are more than two communicating classes, The given **Markov chain** is **reducible**.

$$\therefore \boxed{\textbf{Option 1} \text{ is a } \textbf{incorrect} \text{ answer}}$$

Option 2

As there are communicating classes of aperiodic and periodic, The given **Markov chain** is not a **aperiodic** chain.

$$\therefore \boxed{\textbf{Option 2} \text{ is a } \textbf{incorrect} \text{ answer}}$$

Conclusion

Option 3

After accessing all states, from state 2, the process continues with state 5, while not returning to state 2 again. This means, state 8 is a transient state.

$$\therefore \boxed{\textbf{Option 3} \text{ is a } \textbf{incorrect} \text{ answer}}$$

Option 4

As said above, state 5 re-occurs continuously. So, state 9 is a recurrent state.

$$\therefore \boxed{\textbf{Option 4} \text{ is a } \textbf{correct} \text{ answer}}$$