ASSIGNMENT - 4

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download latex-tikz codes from

https://github.com/VikasPatnala/Assignment−4/blob/main/Assignment%20−%204.tex

1 PROBLEM : UGC/MATH 2019, Q.52

Consider the function *f(x)* defined as *f(x)* = ce^{-x^4}, $x \in R$. For what value of c is f a probability density function?

1) $\frac{2}{\Gamma(1/4)}$
2) $\frac{4}{\Gamma(1/4)}$
3) $\frac{3}{\Gamma(1/3)}$
4) $\frac{1}{4\Gamma(4)}$

2 SOLUTION

Consider a continuous random variable X so that the function f can be probability density function if and only if it satisfies the condition

$$\int_{-\infty}^{\infty} f_X(u)du = 1 \tag{2.0.1}$$

Hence by applying the (2.0.1) for the function f we get

$$\int_{-\infty}^{\infty} ce^{-u^4}du = 1 \tag{2.0.2}$$

$$2c\int_{0}^{\infty} e^{-u^4}du = 1 \tag{2.0.3}$$

$$2c\int_{0}^{\infty} e^{-t}\frac{dt}{4t^{\frac{3}{4}}} = 1 \tag{2.0.4}$$

$$\frac{c}{2}\int_{0}^{\infty} e^{-t}t^{-\frac{3}{4}}dt = 1 \tag{2.0.5}$$

We know that gamma function for any real positive α

$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1}e^{-x}dx \tag{2.0.6}$$

Hence by using (2.0.6) in (2.0.5) we get

$$\frac{c}{2}\Gamma(1/4) = 1 \tag{2.0.7}$$

$$c = \frac{2}{\Gamma(1/4)} \tag{2.0.8}$$

Hence $c = \frac{2}{\Gamma(1/4)}$ and option (1) is correct.

The CDF of f by using (2.0.6) when x is positive we get

$$F_X(x) = \int_{-\infty}^{x} f(u)du \tag{2.0.9}$$

$$= \frac{2}{\Gamma(\frac{1}{4})}\int_{-\infty}^{0} e^{-u^4}du + \frac{2}{\Gamma(\frac{1}{4})}\int_{0}^{x} e^{-u^4}du \tag{2.0.10}$$

$$= \frac{2}{4\Gamma(\frac{1}{4})}\int_{0}^{\infty} e^{-t}t^{\frac{-3}{4}}dt + \frac{2}{4\Gamma(\frac{1}{4})}\int_{0}^{x^4} e^{-t}t^{\frac{-3}{4}}dt \tag{2.0.11}$$

$$= \frac{1}{2\Gamma(\frac{1}{4})}\Gamma\left(\frac{1}{4}\right) + \frac{1}{2\Gamma(\frac{1}{4})}\int_{0}^{x^4} e^{-t}t^{\frac{-3}{4}}dt \tag{2.0.12}$$

$$= \frac{1}{2} + \frac{1}{2\Gamma(\frac{1}{4})}\left(\Gamma\left(\frac{1}{4}\right) - \Gamma\left(\frac{1}{4}, x^4\right)\right) \tag{2.0.13}$$

$$= \frac{1}{2} + \frac{1}{2\Gamma(\frac{1}{4})}\gamma\left(\frac{1}{4}, x^4\right) \tag{2.0.14}$$

The CDF of f by using (2.0.6) when x is negative we get

$$F_X(x) = \int_{-\infty}^{x} f(u)du \tag{2.0.15}$$

$$= \int_{-\infty}^{x} \frac{2}{\Gamma(1/4)}e^{-u^4}du \tag{2.0.16}$$

$$= \frac{2}{4\Gamma\left(\frac{1}{4}\right)}\int_{x^4}^{\infty} e^{-t}t^{\frac{-3}{4}}dt \tag{2.0.17}$$

$$= \frac{1}{2\Gamma\left(\frac{1}{4}\right)}\int_{x^4}^{\infty} e^{-t}t^{\frac{-3}{4}}dt \tag{2.0.18}$$

$$= \frac{1}{2\Gamma\left(\frac{1}{4}\right)}\Gamma\left(\frac{1}{4}, x^4\right) \tag{2.0.19}$$

Hence the CDF of f is

$$F_X(x) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2\Gamma(\frac{1}{4})}\gamma\left(\dfrac{1}{4}, x^4\right) & \text{if x>=0} \\ \\ \dfrac{1}{2\Gamma\left(\frac{1}{4}\right)}\Gamma\left(\dfrac{1}{4}, x^4\right) & \text{if x<0} \end{cases} \tag{2.0.20}$$

And the plot of CDF of f is