# Probability

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Contents



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***Abstract*—This book provides solved examples on Probability**

## 1 June 2019

1.1. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \end{array} \tag{1.1.1}$$

Then which of the following are true?

a) $\lim_{n\to\infty} p_{12}^{(n)} = 0$
b) $\lim_{n\to\infty} p_{12}^{(n)} = \lim_{n\to\infty} p_{21}^{(n)}$
c) $\lim_{n\to\infty} p_{22}^{(n)} = \frac{1}{8}$
d) $\lim_{n\to\infty} p_{21}^{(n)} = \frac{1}{3}$

1.2. A sample of size $n = 2$ is drawn from a population of size $N = 4$ using probability proportional to size without replacement scheme , Where the probabilities proportional to size are The probability of inclusion of unit (1) in

| i: | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p_i$ | 0.4 | 0.2 | 0.2 | 0.2 |

Table : Probability vs Size

the sample is

a) 0.4 b) 0.6 c) 0.7 d) 0.75

**Solution:** Let $P_i(j)$ represent the probability for selecting unit (j) as second unit after selecting unit (i)

$$P_i(j) = \frac{p_j}{1 - p_i} \tag{1.2.1}$$

Let $\Pr(i,j)$ be probability of selecting sample {i,j} ,using (1.2.1) is

$$\Pr(i,j) = P_i(j) + P_j(i) \tag{1.2.2}$$

$$= \left(p_i \times \frac{p_j}{1-p_i}\right) + \left(p_j \times \frac{p_i}{1-p_j}\right) \tag{1.2.3}$$

Total samples(Size $n = 2$)are  Let $P_i$ be

| Case | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Sample(size $n = 2$) | (1,2) | (1,3) | (1,4) | (2,3) | (2,4) | (3,4) |

TABLE 1.2: list of samples

the probability of inclusion of unit (i) in the sample(size $n = 2$),Now i will calculate $P_1$ ,Favourable cases for inclusion of unit(1) are case (1,2,3),So

$$P_1 = \Pr(1,2) + \Pr(1,3) + \Pr(1,4) \tag{1.2.4}$$

using (1.2.3) and $p_i$ from question ,

$$P_1 = \frac{7}{30} + \frac{7}{30} + \frac{7}{30} \tag{1.2.5}$$

$$= 0.7 \tag{1.2.6}$$

Therefore Option (3) is correct.

1.3. Consider the function *f(x)* defined as $f(x) = ce^{-x^4}$, $x \in R$ . For what value of $c$ is $f$ a probability density function?

a) $\dfrac{2}{\Gamma(1/4)}$
b) $\dfrac{4}{\Gamma(1/4)}$
c) $\dfrac{3}{\Gamma(1/3)}$
d) $\dfrac{1}{4\Gamma(4)}$

**Solution:**

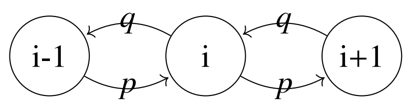
1.4. Consider a simple symmetric random walk on integers ,Where from every state i you to move to i-1 and i+1 with probability half each. Then which of the following are correct?

a) The random walk is aperiodic
b) The random walk is irreducible
c) The random walk is null recurrent
d) The random walk is positive recurrent

**Solution:** This is a Markov Chain ,Where the state space consists of the integers ($i = 0, \pm1, \pm2, \pm3, ...$) and transition probability is given as

$$P_{i,i+1} = p = \frac{1}{2} \tag{1.4.1}$$

$$P_{i,i-1} = q = \frac{1}{2} \tag{1.4.2}$$




Let $P^n_{i,j}$ denotes the probability of being in state j after nth transition starting from state i.

a) We know that for state j in Markov chain to be **aperiodic** ,Then their exist k such that $P^n_{j,j} > 0$ for all $n \geq k$. but for to return to same state j after n transitions ,Number of forward steps should be equal to Backward steps , i.e for odd n in (2m +1)form

$$P^{2m+1}_{j,j} = 0 \tag{1.4.3}$$

when n is even in 2m form

$$P^{2m}_{j,j} = \binom{2m}{m} p^m q^m \tag{1.4.4}$$

$$= \frac{(2m)!}{m!.m!} p^m q^m \tag{1.4.5}$$

,As for odd n $P^n_{j,j} = 0$ ,$P^n_{j,j} > 0$ for all $n \geq k$ is not possible .which implies all states are **Periodic**

Option (1) is **incorrect**.

b) In a Markov Chain for state j to be recurrent then it should satisfy following condition

$$\lim_{t\to\infty} \sum_{n=1}^{t} P^n_{j,j} = \infty \tag{1.4.6}$$

using Stirling approximation in equation (1.4.5)

$$P^{2m}_{j,j} = \frac{((2m)^{2m+\frac{1}{2}}).e^{-2m}.(2\pi)^{\frac{1}{2}}}{m^{m+\frac{1}{2}}.e^{-m}.m^{m+\frac{1}{2}}.e^{-m}.2\pi}.p^m q^m \tag{1.4.7}$$

$$= \frac{(4pq)^{2m}}{(m\pi)^{\frac{1}{2}}} \tag{1.4.8}$$

In this question $p = \frac{1}{2} = q$,then using (1.4.3) and (1.4.8)

$$\lim_{t\to\infty} \sum_{n=1}^{t} P^n_{j,j} = \sum_{n=2k,k=1}^{\infty} \frac{1}{(\frac{n}{2}\pi)^{\frac{1}{2}}} \tag{1.4.9}$$

Since $\frac{1}{n^{\frac{1}{2}}}$ is divergent,

$$\lim_{t\to\infty}\sum_{n=1}^{t} P^n_{j,j} = \infty \tag{1.4.10}$$

Therefore state j is recurrent ,as what we calculated is independent of j ,all states are **recurrent** . The first-passage-time probability, $f_{i,j}(n)$, of a Markov chain is the probability,given as

$$f_{i,j}(n) = \Pr\left(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \dots X_1 \neq j | X_0 = i\right) \tag{1.4.11}$$

The first-passage time $T_{j,j}$from a state j back to itself is of particular importance. It has the PMF $f_{j,j}(n)$ amd Distribution function $F_{j,j}(n)$

$$F_{j,j}(n) = \sum_{k=0}^{n} f_{j,j}(k) \tag{1.4.12}$$

We Know that all states are recurrent .Now i will find whether they are null recurrent or positive recurrent . For positive recurrent

$$\overline{T_{j,j}} < \infty \tag{1.4.13}$$

For null recurrent

$$\overline{T_{j,j}} = \infty \tag{1.4.14}$$

Where $\overline{T_{j,j}}$ is mean time to enter state j starting from j. Now calculating $\overline{T_{j,j}}$ using below formula,

$$\overline{T_{j,j}} = 1 + \sum_{k=0}^{n}(1 - F_{j,j}(k)) \tag{1.4.15}$$

Using (1.4.15) and (1.4.12),We get

$$\overline{T_{j,j}} = \infty \tag{1.4.16}$$

Therefore all states are null recurrent. Option(3) is **correct**

c) Since all states are recurrent,they communicate with each other ,therefore Markov chain is irreducible , option (2) is **correct**
d) As all states are null recurrent , option (4) is **incorrect**

Therefore correct options are **2,3**

## 2 December 2018

2.1. Let X and Y be i.i.d random variables uniformly distributed on (0,4).Then $\Pr(X > Y | X < 2Y)$ is

a) 1/3
b) 5/6
c) 1/4
d) 2/3

**Solution:**

The PDF is given by

$$f_X(x) = f_Y(x) = \begin{cases} \frac{1}{4}, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

The CDF is given by

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$F_X(x) = F_Y(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{4}, & \text{if } 0 < x < 4 \\ 1, & x \geq 4 \end{cases}$$

Using definition of conditional probability

$$\Pr(X > Y | X < 2Y) = \frac{\Pr(Y < X < 2Y)}{\Pr(X < 2Y)} \tag{2.1.1}$$

Now finding $\Pr(X < 2Y)$

$$\Pr(X < 2y) = F_X(2y) \tag{2.1.2}$$

$$\implies \Pr(X < 2Y) = \int_{-\infty}^{\infty} f_Y(x) \times F_X(2x)dx \tag{2.1.3}$$

$$\implies \Pr(X < 2Y) = \int_0^2 \frac{x}{8}dx + \int_2^4 \frac{1}{4}dx \tag{2.1.4}$$

$$\implies \Pr(X < 2Y) = \frac{3}{4} = 0.75 \tag{2.1.5}$$

Now to find $\Pr(Y < X < 2Y)$

$$\Pr(y < X < 2y) = F_X(2y) - F_X(y) \tag{2.1.6}$$

$$\implies \Pr(Y < X < 2Y) \tag{2.1.7}$$

$$= \int_{-\infty}^{\infty} f_Y(x)(F_X(2x) - F_X(x))dx$$

$$\implies \int_0^2 \frac{1}{4}\left(\frac{x}{2} - \frac{x}{4}\right)dx + \int_2^4 \frac{1}{4}\left(1 - \frac{x}{4}\right)dx \tag{2.1.8}$$

$$\implies \Pr(Y < X < 2Y) = \frac{1}{4} = 0.25 \tag{2.1.9}$$

Now using (2.1.1),(2.1.5) and (2.1.9)

$$\Pr(X > Y | X < 2Y) = \frac{1/4}{3/4} = \frac{1}{3} \tag{2.1.10}$$

Hence final solution is option 1) or 1/3

2.2. Suppose $X$ is a positive random variable with the following probability density function,

$$f(x) = (\alpha x^{\alpha-1} + \beta x^{\beta-1})e^{-x^{\alpha}-x^{\beta}}; x > 0$$

for $\alpha > 0, \beta > 0$. Then the hazard function of $X$ for some choices of $\alpha$ and $\beta$ can be

a) an increasing function.
b) a decreasing function.
c) a constant function.
d) a non monotonic function

**Solution:**
CDF of $X$,

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{2.2.1}$$

$$= \int_{0}^{x} f(t)dt \qquad \text{as } x > 0 \tag{2.2.2}$$

$$= \int_{-\infty}^{t} \left((\alpha t^{\alpha-1} + \beta t^{\beta-1}) \times e^{-t^{\alpha}-t^{\beta}}\right) dt \tag{2.2.3}$$

$$= -e^{-t^{\alpha}-t^{\beta}}\Big|_{0}^{x} \tag{2.2.4}$$

$$= 1 - e^{-x^{\alpha}-x^{\beta}} \tag{2.2.5}$$

Hazard function,

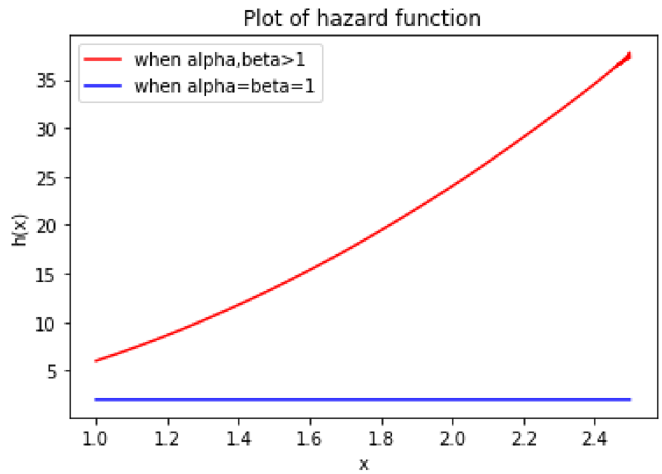
$$h(x) = \frac{f(x)}{1 - F(x)} \tag{2.2.6}$$

$$= \alpha x^{\alpha-1} + \beta x^{\beta-1} \tag{2.2.7}$$

$$h'(x) = \alpha(\alpha-1)x^{\alpha-2} + \beta(\beta-1)x^{\beta-2} \tag{2.2.8}$$

$$h'(x) = \begin{cases} 0 & \alpha = \beta = 1 \\ > 0 & \text{otherwise} \end{cases} \tag{2.2.9}$$

Thus $h(x)$ can be either constant function or an increasing function.




From the above figure, it is verified that $h(x)$ can be either constant function or an increasing function.
Correct options are 1,3.

2.3. Suppose n units are drawn from a population of N units sequentially as follows. A random sample

$$U_1, U_2, ...U_N \text{ of size N, drawn from } U(0,1) \tag{2.3.1}$$

The k-th population unit is selected if

$$U_k < \frac{n - n_k}{N - k + 1}, k = 1, 2, ..N.\text{where, } n_1 = 0, n_k = \tag{2.3.2}$$

number of units selected out of first k-1 units for each k = 2, 3, ..N. Then,

a) The probability of inclusion of the second unit in the sample

$$\text{is } \frac{n}{N} \tag{2.3.3}$$

b) The probability of inclusion of the first and the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{2.3.4}$$

c) The probability of not including the first and including the second unit in the sample

$$\text{is } \frac{n(N-n)}{N(N-1)} \tag{2.3.5}$$

d) The probability of including the first and not including the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{2.3.6}$$

**Solution:**

$$\text{Defining random variable } X \in \{0, 1, 2, ..N\} \tag{2.3.7}$$

$$\text{Where, } X = i \text{ when ith unit is included.} \tag{2.3.8}$$

The first unit in the sample is included if

$$U_1 < \frac{n-n_1}{N-1+1} \quad (2.3.9)$$

Here, $n_1 = 0$ is given in the qn. (2.3.10)

$$\therefore \Pr(X=1) = \frac{n}{N} \quad (2.3.11)$$

a) For k=2,

$n_2 = 1$ when, first unit is included. (2.3.12)

$$U_2 < \frac{n-n_2}{N-2+1}\left(= \frac{n-1}{N-1}\right) \quad (2.3.13)$$

$$\therefore \Pr(X=2 \mid X=1) = \frac{n-1}{N-1} \quad (2.3.14)$$

$\Pr(X=1, X=2)$

$$= \Pr(X=2 \mid X=1) \times \Pr(X=1) \quad (2.3.15)$$

$$\therefore \Pr(X=1, X=2) = \frac{n(n-1)}{N(N-1)} \quad (2.3.16)$$

$n_2 = 0$ when, first unit is not included. (2.3.17)

$$U_2 < \frac{n-n_2}{N-2+1}\left(= \frac{n}{N-1}\right) \quad (2.3.18)$$

$$\therefore \Pr(X=2 \mid X \neq 1) = \frac{n}{N-1} \quad (2.3.19)$$

$\Pr(X \neq 1, X=2)$

$$= \Pr(X=2 \mid X \neq 1) \times \Pr(X \neq 1) \quad (2.3.20)$$

$$\therefore \Pr(X \neq 1, X=2) = \left(1-\frac{n}{N}\right) \times \frac{n}{N-1} \quad (2.3.21)$$

$$\therefore \Pr(X \neq 1, X=2) = \frac{n(N-n)}{N(N-1)} \quad (2.3.22)$$

From (2.3.16) and (2.3.22)

$$\Pr(X=2) = \frac{n(n-1)}{N(N-1)} + \frac{n(N-n)}{N(N-1)} = \frac{n}{N} \quad (2.3.23)$$

Hence, option 1 is correct.

b) From (2.3.16)

$$\Pr(X=1, X=2) = \frac{n(n-1)}{N(N-1)} \quad (2.3.24)$$

Hence, option 2 is correct.

c) From (2.3.22)

$$\Pr(X \neq 1, X=2) = \frac{n(N-n)}{N(N-1)} \quad (2.3.25)$$

Hence, option 3 is correct.

d)

$$\Pr(X=1, X \neq 2) = \frac{n}{N} \times \left(1-\frac{n}{N}\right) = \frac{n(N-n)}{N^2} \quad (2.3.26)$$

Hence, option 4 is incorrect.

Therefore, Options 1, 2, 3 are correct

2.4. Consider a Markov chain with state space 1,2,....,100. Suppose states 2i and 2j communicate with each other and states 2i-1 and 2j-1 communicate with each other for every i,j = 1,2,...,50. Further suppose that $p_{3,3}^{(2)}$ ¿ 0,$p_{4,4}^{(3)}$ ¿ 0 and $p_{2,5}^{(7)}$ ¿ 0. Then

a) The Markov chain is irreducible.
b) The Markov chain is aperiodic.
c) State 8 is recurrent.
d) State 9 is recurrent.

**Solution:**

2.5. Out of 6 unbiased coins, 5 are tossed independently and they all result in heads. If the 6th coin is now independently tossed, the probability of getting head is:

(a) 1
(b) 0
(c) $\frac{1}{2}$
(d) $\frac{1}{6}$

**Solution:** Define a random variable $X = \{0, 1\}$ denoting the outcome of the toss of 6th coin with $X = 0$ and $X = 1$ representing tails and head respectively.Therefore,

$$\Pr(X=0) + \Pr(X=1) = 1 \quad (2.5.1)$$

$$\Pr(X=1) = \frac{1}{2} \quad (2.5.2)$$

Hence the correct answer is option (c).

2.6. Let $X_1, X_2, X_3, .., X_n$ be independent random variables follow a common continuous distribution **F**, which is symmetric about 0. For i=1,2,3,..n, define

$$S_i = \begin{cases} 1 & if \ X_i > 0 \\ -1 & if \ X_i < 0 \ and \\ 0 & if \ X_i = 0 \end{cases} \quad (1.1)$$

$R_i$=rank of $|X_i|$ in the set$\{|X_1|, |X_2|, .., |X_n|\}$.Which of the following statements are correct?

a) $S_1, S_2, .., S_n$ are independent and identically distributed.
b) $R_1, R_2, .., R_n$ are independent and identically distributed.
c) $S = (S_1, S_2, .., S_n)$ and $R = (R_1, R_2, .., R_n)$ are independent.

**Solution:**

A sequence $\{X_i\}$ is an Independent and identical if and only if $F_{X_n}(x) = F_{X_k}(x)$ $\forall$ n,k,x and any subset of terms of the sequence is a set of mutually independent random variables. Where F is the probability density function.

a) As the probability distribution function of $\{X_i\}$ is symmetric about origin we can say that

$$F_{X_i}(-x) = F_{X_i}(x) \forall x \in R \quad (2.1)$$

and the mean of the distribution($\mu$)

$$\mu = 0 \quad (2.2)$$

The sequence $S_i$ depend on $X_i$ as mention in 1.1, as each $S_i$ depend only on $X_i$ we can say that sequence $S_i$ is independent.

$$\Pr(S_1 = 1, S_2 = 1, ..., S_n = 1) = \Pi_{i=1}^{n} \Pr(S_i = 1) \quad (2.3)$$

Any subset of terms of sequence $\{S_i\}$ is a set of mutually independent random variables and its distribution is identical.

$$F_{S_n}(s) = F_{S_k}(s) \quad \forall s, k, n \quad (2.4)$$

So, the sequence $\{S_i\}$ is independent and identical.

b) **Ranking** refers to the data transformation in which the numerical or ordinary values are replaced by the rank of numerical value when compared to a list of other values.Usually we follow increasing order for ranking.

Ranking of a sequence depend on every elements of the sequence.Let $\{R_i\}$ be the output sequence of the ranking function of $\{|X_i|\}$.

$$R_k = \text{rank of } |X_k| \text{ in the set}\{|X_1|, |X_2|, .., |X_n|\} \quad (2.5)$$

As $R_k$ depend not only on $|X_k|$ but on the rest of the elements of the set$\{|X_1|, |X_2|, .., |X_n|\}$. So the sequence $R_i$ is not independent. Hence $R_i$ is not an independent and identical distribution.

c) As the $i^{th}$ element of sequence R depends only on set$\{|X_1|, |X_2|, .., |X_n|\}$, we can say that sequence S and R are independent.

Answer:A,C

2.7. Let $X_1, X_2, \cdots$ be i.i.d. $N(0, 1)$ random variables.Let $S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1$.Which of the following statements are correct?

a) $\frac{S_n - n}{\sqrt{2}} \sim N(0, 1)$ for all $n \geq 1$

b) For all $\epsilon > 0$,$\Pr\left(\left|\frac{S_n}{n} - 2\right| > \epsilon\right) \to 0$ as $n \to \infty$

c) $\frac{S_n}{n} \to 1$ with probability 1

d) $\Pr(S_n \leq n + \sqrt{n}x) \to \Pr(Y \leq x) \forall x \in R$ ,where $Y \sim N(0, 2)$

**Solution:**

**Definition 1** (Almost sure convergence)**.** *A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if*

$$\Pr(\omega | X_n(\omega) \to X(\omega)) = 1 \quad (2.7.1)$$

**Definition 2** (Convergence in probability)**.** *A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge in probability (denoted by i.p) to X if*

$$\lim_{n \to \infty} \Pr(|X_n - X| > \epsilon) = 0, \forall \epsilon > 0 \quad (2.7.2)$$

**Theorem 2.1** (Weak law of large numbers)**.** *Let $X_1, X_2, \cdots$ be i.i.d random variables with same expectation($\mu$) and finite variance($\sigma^2$).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$*

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \quad (2.7.3)$$

*in probability*

**Theorem 2.2** (Strong law of large numbers)**.** *Let $X_1, X_2, \cdots$ be i.i.d random vari-*

*ables with same expectation($\mu$) and finite variance($\sigma^2$).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$*

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \tag{2.7.4}$$

*almost surely.*

**Theorem 2.3** (Central limit theorem)**.** *The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger,given that all the samples are equal in size,regardless of the distribution of the individual samples.*

Given $X_1, X_2, \cdots$ follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in \{1, 2, \cdots\} \tag{2.7.5}$$

As $X_1, X_2, \cdots$ are i.i.d random variables therefore $X_1^2, X_2^2, \cdots$ are also identical and independent. We can write

$$E(X^2) = Var(X) \tag{2.7.6}$$

a)

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = E\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \tag{2.7.7}$$

$$= \frac{\sum_i E(X_i^2 - 1)}{\sqrt{2}} \tag{2.7.8}$$

From (2.7.6) we can write

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = 0 \tag{2.7.9}$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = Var\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \tag{2.7.10}$$

$$= \frac{\sum_i Var(X_i^2 - 1)}{\sqrt{2}} \tag{2.7.11}$$

$$Var(X_i^2 - 1) = \int_{-\infty}^{\infty} (X_i^2 - 1)^2 f_{X_i}(x) dx \tag{2.7.12}$$

$$= \int_{-\infty}^{\infty} (X_i^4 + 1 - 2X_i^2) f_{X_i}(x) dx \tag{2.7.13}$$

$$= 2 \tag{2.7.14}$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = n\sqrt{2} \tag{2.7.15}$$

Hence from theorem 2.2 as $n \to \infty$

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n\sqrt{2}) \tag{2.7.16}$$

Hence **Option A is false.**

b) Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{2.7.17}$$

Hence from theorem 2.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X) \tag{2.7.18}$$

$$\implies \frac{S_n}{n} \xrightarrow{i.p} 1 \tag{2.7.19}$$

in probability.From definition 2 we can write,

$$\implies \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \to 0, \forall \epsilon > 0 \tag{2.7.20}$$

Hence **Option B is false .**

c) Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{2.7.21}$$

Hence from theorem 2.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X) \tag{2.7.22}$$

$$\implies \frac{S_n}{n} \xrightarrow{a.s} 1 \tag{2.7.23}$$

almost surely.From definition 1 we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \tag{2.7.24}$$

with probability 1. Hence **Option C is true**.

d) Consider,

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = 0 \tag{2.7.25}$$

using (2.7.6) and (2.7.8).

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{2n}{\sqrt{n}} \tag{2.7.26}$$

$$= 2\sqrt{n}. \tag{2.7.27}$$

using (2.7.14).From theorem 2.3 we can

write,

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \sim N(0, 2\sqrt{n}) \quad (2.7.28)$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \le x\right) = \Pr\left(S_n \le n + \sqrt{n}x\right) \quad (2.7.29)$$

Hence using (2.7.28), **Option D is false.**

## 3 June 2018

3.1. Two students are solving the same problem independently,if the probability of first one solves the problem is $\frac{3}{5}$ and the probability that the second one solves the problem is $\frac{4}{5}$ ,what is the probability that atleast one of them solves the problem?

a) $\frac{17}{25}$

b) $\frac{19}{25}$

c) $\frac{21}{25}$

d) $\frac{23}{25}$

**Solution:** Let X,Y be two events representing solving the problem by students A,B respectively.
Given

$$\Pr(X) = \frac{3}{5} \quad (3.1.1)$$

$$\Pr(Y) = \frac{4}{5} \quad (3.1.2)$$

Since students solve the problem independently,So events X and Y are independent, For independent events

$$\Pr(XY) = \Pr(X) \times \Pr(Y) \quad (3.1.3)$$

from (3.1.1) and (3.1.2)

$$\Pr(XY) = \frac{3}{5} \times \frac{4}{5} \quad (3.1.4)$$

$$\Pr(XY) = \frac{12}{25} \quad (3.1.5)$$

Now we have to find probability of solving the problem by atleast one of them i.e $\Pr(X + Y)$. As,

$$\Pr(X + Y) = \Pr(X) + \Pr(Y) - \Pr(XY) \quad (3.1.6)$$

from (3.1.1), (3.1.2), (3.1.5)

$$\Pr(X + Y) = \frac{3}{5} + \frac{4}{5} - \frac{12}{25} \quad (3.1.7)$$

$$\Pr(X + Y) = \frac{23}{25} \quad (3.1.8)$$

Hence the required probability is $\frac{23}{25}$

3.2. A standard fair die is rolled until some face other than 5 or 6 turns up.Let X denote the face value of the last roll.Let A={X is even} and B={X is atmost 2} Then,

a) $\Pr(A \cap B) = 0$

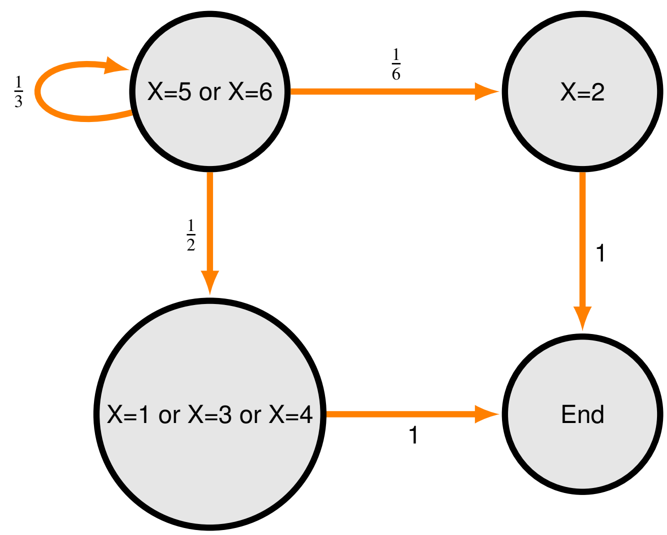
b) $\Pr(A \cap B) = \frac{1}{6}$

c) $\Pr(A \cap B) = \frac{1}{4}$

d) $\Pr(A \cap B) = \frac{1}{3}$

**Solution:** Let us assume the following table.

Fig. 3.2.1: Markov chain




Let us represent the markov chain diagram in a

TABLE 3.2.1

| state 1 | state 2 | state 3 | state 4 |
|---|---|---|---|
| $X = 5$ or $X = 6$ | $X = 2$ | $X = 1$ or $X = 3$ or $X = 4$ | end |

matrix.Let $P_{ij}$ represent the element of a matrix which is in $i^{th}$ row and $j^{th}$ column.The value of $P_{ij}$ is equal to probability of transition from state $i$ to state $j$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.2.1)$$

We need the probability that $X = 2$.Hence required probability is

$$P_{12} + (P_{12})^2 + \cdots + \infty \tag{3.2.2}$$

where $P_{12}^n$ represents the 1st row ,2nd column element in the $P^n$

$$P^2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.3}$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.4}$$

$$P^3 = (P^2)(P^1) \tag{3.2.5}$$

$$= \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.6}$$

$$= \begin{bmatrix} \frac{1}{27} & \frac{1}{54} & \frac{1}{18} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2.7}$$

From above we can notice that each time $P_{12}$ reduces by $\frac{1}{3}$.Hence from (3.2.2),

$$\sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i \frac{1}{6} \tag{3.2.8}$$

From Geometric progression we can write ,required probability $=\frac{1}{4}$ $\therefore$ **option C is correct**

3.3. Let X and Y be two random variables with joint probability density function

$$f(x.y) = \begin{cases} \frac{1}{\pi} & 0 \le x^2 + y^2 \le 1 \\ 0 & otherwise \end{cases}$$

Which of the following statements are correct?

a) X and Y are independent.

b) $\Pr(X > 0) = \frac{1}{2}$

c) E(Y)=0

d) Cov(X,Y)=0

**Solution:**

3.4. Let X and Y be two random variables with joint probability density function

$$f(x.y) = \begin{cases} \frac{1}{\pi} & 0 \le x^2 + y^2 \le 1 \\ 0 & otherwise \end{cases}$$

Which of the following statements are correct?

a) X and Y are independent.

b) $\Pr(X > 0) = \frac{1}{2}$

c) E(Y)=0

d) Cov(X,Y)=0

**Solution:**

a) The marginal PDF of X is given by

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f_{XY}(x, y)dy \tag{3.4.1}$$

$$= \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \frac{1}{\pi}dy \tag{3.4.2}$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{3.4.3}$$

The marginal PDF of Y is given by

$$f_Y(x) = \int_{x=-\infty}^{x=\infty} f_{XY}(x, y)dx \tag{3.4.4}$$

$$= \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{3.4.5}$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{3.4.6}$$

Now,

$$f_X(x) \times f_Y(x) = \frac{2\sqrt{1-x^2}}{\pi} \times \frac{2\sqrt{1-y^2}}{\pi} \tag{3.4.7}$$

$$= \frac{4(1-x^2)(1-y^2)}{\pi^2} \tag{3.4.8}$$

$$\neq \frac{1}{\pi} \tag{3.4.9}$$

$$\neq f_{XY}(x, y) \tag{3.4.10}$$

Therefore, X and Y are not independent.

b) Now,

$$\Pr(X > 0) = \int_{x=0}^{x=\infty} f_X(x)dx \tag{3.4.11}$$
$$= \int_{x=0}^{x=1} \frac{2\sqrt{1-x^2}}{\pi}dx \tag{3.4.12}$$
$$= \left(\frac{\arcsin(x) + x\sqrt{1-x^2}}{\pi}\right)_0^1 \tag{3.4.13}$$
$$= \frac{1}{2} \tag{3.4.14}$$

Therefore, option(2) is correct.

c) Now,

$$E[Y] = \int_{y=-\infty}^{y=\infty} y f_Y(y)dy \tag{3.4.15}$$
$$= \int_{y=-1}^{y=1} \frac{2y\sqrt{1-y^2}}{\pi}dy \tag{3.4.16}$$
$$= \left(\frac{-2(1-y^2)^{\frac{3}{2}}}{3\pi}\right)_{-1}^{1} \tag{3.4.17}$$
$$= 0 \tag{3.4.18}$$

Therefore, option(3) is also correct.

d) Now,

$$E[XY] = \int_x \int_y xy f_{XY}(x,y)dydx \tag{3.4.19}$$
$$= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \frac{xy}{\pi}dydx \tag{3.4.20}$$
$$= \frac{x}{\pi}\int_{x=-1}^{x=1} \left(\frac{y^2}{2}\right)_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \tag{3.4.21}$$
$$= 0 \tag{3.4.22}$$

Now,

$$Cov(X,Y) = E[XY] - E[X]E[Y] \tag{3.4.23}$$
$$= 0 - E[X] \times 0 \tag{3.4.24}$$
$$= 0 \tag{3.4.25}$$

Therefore, option(4) is also correct.

3.5. A simple random variable of size n will be drawn from a class of 125 students, and the mean mathematics score of the sample will be computed, If the standard error of the sample mean for ”with replacement sampling” is twice as much as the standard error of the sample mean for ”without replacement sampling”, the value of n is ?

a) 32
b) 63
c) 79
d) 94

**Solution:** Let N be the population size so, N=120. The given sample size is n. **Notations :** y : student under consideration. $y_i$ : Maths marks of $i^{th}$ student in the sample. Y : student of class. $Y_i$ : Maths marks of $i^{th}$ student in the class. $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ : Average of sample class. $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ : Average of whole class. $S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$ : S=Std dev of the class. $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$ : Variance of the class. Standard error of sample mean $SE_{mean} = \frac{s}{\sqrt{n}}$.
Where

$s$ = standard deviation of sample mean.
$n$ = sample class size.

**Variance of the $\overline{y}$**

$$V(\overline{y}) = E(\overline{y} - \overline{Y})^2 \tag{3.5.1}$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{Y})\right]^2 \tag{3.5.2}$$

$$= E\left[\frac{1}{n^2}\sum_{i=1}^{n}(y_i - \overline{Y})^2 + \frac{1}{n^2}\sum\sum_{1\le i\neq j\le n}(y_i - \overline{Y})(y_j - \overline{Y})\right] \tag{3.5.3}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}E(y_i - \overline{Y})^2 + \frac{1}{n^2}\sum\sum_{1\le i\neq j\le n}E(y_i - \overline{Y})(y_j - \overline{Y}) \tag{3.5.4}$$

Let $K = \sum\sum_{1\le i\neq j\le n}E(y_i - \overline{Y})(y_j - \overline{Y})$ (3.5.5)

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 + \frac{K}{n^2} \tag{3.5.6}$$

$$= \frac{1}{n^2}n\sigma^2 + \frac{K}{n^2} \tag{3.5.7}$$

$$= \frac{N-1}{Nn}S^2 + \frac{K}{n^2} \tag{3.5.8}$$

Finding the value of K in case of Simple random sampling with repetition (SRSWR)and Simple random sampling without repetition(SRSWOR) allows us to calculate the variance of mean. **K value in case of SRSWOR**

$$K = \sum\sum_{1\le i\neq j\le n}E(y_i - \overline{Y})(y_j - \overline{Y})$$

Consider

$$E(y_i - \overline{Y})(y_j - \overline{Y}) = \frac{1}{N(N-1)}\sum\sum_{1\le k\neq l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

Since

$$\left[\sum_{k=1}^{N}(y_k - \overline{Y})\right]^2 = \sum_{i=1}^{N}(y_k - \overline{Y})^2 + \sum\sum_{1\le k\neq l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

$$\implies 0 = (N-1)S^2 + \sum\sum_{1\le k\neq l\le n}E(y_k - \overline{Y})(y_l - \overline{Y})$$

$$\implies E(y_i - \overline{Y})(y_j - \overline{Y}) = \frac{1}{N(N-1)}(N-1)(-S^2)$$

$$\implies K = n(n-1)\frac{(-S^2)}{N}$$

Putting this value in (3.5.8) gives us

$$V(\overline{y})_{WOR} = \frac{N-1}{Nn}S^2 + \frac{n-1(-S^2)}{Nn} \tag{3.5.9}$$

$$= \frac{N-n}{Nn}S^2 \tag{3.5.10}$$

**K value in case of SRSWR**

$$K = \sum\sum_{1\le i\neq j\le n}E(y_i - \overline{Y})(y_j - \overline{Y})$$

Since we are selecting the samples with replacements choosing $i^{th}$ and $j^{th}$ sample is independent of each other. So,

$$K = \sum\sum_{1\le i\neq j\le n}E(y_i - \overline{Y})E(y_j - \overline{Y})$$

$$= 0$$

(Since deviation about mean is 0)

Putting K=0 in (3.5.8) we get

$$V(\overline{y})_{WR} = \frac{N-1}{Nn}S^2 \tag{3.5.11}$$

From equation (3.5.10) standard error of mean of sample class without repetition

$$SE_{WOR} = \frac{s}{\sqrt{n}} \tag{3.5.12}$$

$$= \sqrt{\frac{V(\overline{y})_{WOR}}{n}} \tag{3.5.13}$$

$$= \sqrt{\frac{N-n}{Nn^2}}S \tag{3.5.14}$$

From equation (3.5.11) standard error of mean of sample class with repetition

$$SE_{WR} = \sqrt{\frac{V(\overline{y})_WR}{n}} \tag{3.5.15}$$

$$= \sqrt{\frac{N-1}{Nn^2}}S \tag{3.5.16}$$

Given to find the value of n if $2 \times SE_{WOR} =$

$SE_{WR}$. From (3.5.14) and (3.5.16) we can write

$$2\sqrt{\frac{N-n}{Nn^2}}S = \sqrt{\frac{N-1}{Nn^2}}S \tag{3.5.17}$$

$$\implies 4(N-n) = N-1 \tag{3.5.18}$$

$$\implies 4N+1-N = 4n \tag{3.5.19}$$

$$\implies 4n = 3(125)+1 \tag{3.5.20}$$

$$\implies n = 94 \tag{3.5.21}$$

Therefore the sample size for the given condition to be met is n=94.(**Option D**)

3.6. Let X and Y be two independent and identically distributed (I.I.D) random variables uniformly distributed in (0,1). Let $Z = max(X,Y)$ and $W = min(X,Y)$ , then the probability that $[Z - W > \frac{1}{2}]$ is

(A) $\frac{1}{2}$

(B) $\frac{3}{4}$

(C) $\frac{1}{4}$

(D) $\frac{2}{3}$ **Solution:**
X and Y are two independent random variables. Let

$$f_X(x) = \Pr(X = x) \tag{3.6.1}$$

$$f_Y(y) = \Pr(Y = y) \tag{3.6.2}$$

$$f_V(v) = \Pr(V = v) \tag{3.6.3}$$

be the probability densities of random variables X ,Y and V=X-Y.
The density for X is

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{3.6.4}$$

We have ,

$$V = X - Y \iff v = x - y \iff x = v + y \tag{3.6.5}$$

The density of X can also be represented as,

$$f_X(v+y) = \begin{cases} 1 & 0 \le v+y \le 1 \\ 0 & otherwise \end{cases} \tag{3.6.6}$$

and the density of Y is,

$$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{3.6.7}$$

The density of V i.e. $V = X - Y$ is given by the convolution of $f_X(-v)$ with $f_Y(v)$.

$$f_V(v) = \int_{-\infty}^{\infty} f_X(v+y) f_Y(y)\, dy \tag{3.6.8}$$

From 3.6.6 and 3.6.7 we have,
The integrand is 1 when,

$$0 \le y \le 1 \tag{3.6.9}$$

$$0 \le v + y \le 1 \tag{3.6.10}$$

$$-v \le y \le 1 - v \tag{3.6.11}$$

and zero, otherwise.
Now when $-1 \le v \le 0$ we have,

$$f_V(v) = \int_{-v}^{1} dy \tag{3.6.12}$$

$$= (1 - (-v)) \tag{3.6.13}$$

$$= 1 + v \tag{3.6.14}$$

For $0 \le v \le 1$ we have,

$$f_V(v) = \int_{0}^{1-v} dy \tag{3.6.15}$$

$$= (1 - v - (0)) \tag{3.6.16}$$

$$= 1 - v \tag{3.6.17}$$

Therefore the density of V is given by

$$f_V(v) = \begin{cases} 1+v & -1 \le v \le 0 \\ 1-v & 0 < v \le 1 \\ 0 & otherwise \end{cases} \tag{3.6.18}$$

The plot for PDF of $V$ can be observed at figure 3.6.1
The CDF of V is defined as,

$$F_V(v) = \Pr(V \le v) \tag{3.6.19}$$

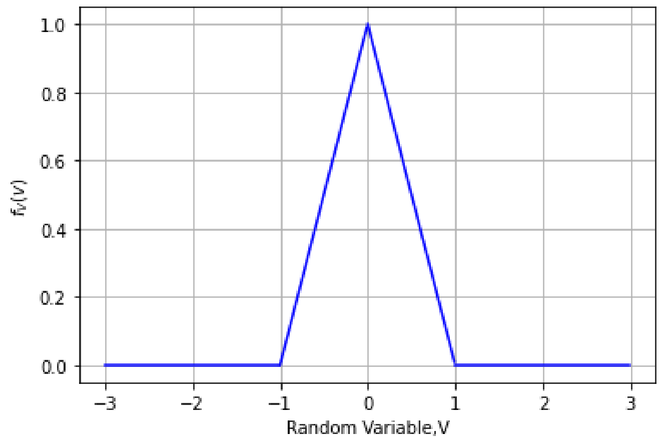



Fig. 3.6.1: The PDF of V

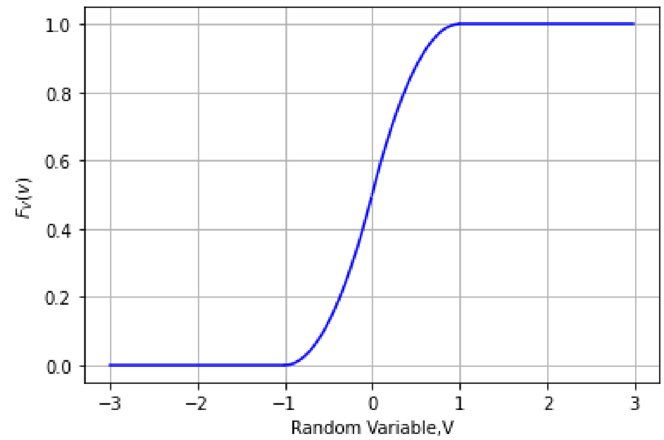



Fig. 3.6.2: The CDF of V

Now for $v \leq 0$,

$$\Pr(V \leq v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{3.6.20}$$
$$= \int_{-1}^{v} (1+v)\,dv \tag{3.6.21}$$
$$= \left(\frac{v^2}{2} + v\right)\Bigg|_{-1}^{v} \tag{3.6.22}$$
$$= \left(\left(\frac{v^2}{2} + v\right) - \left(\frac{1}{2} - 1\right)\right) \tag{3.6.23}$$
$$= \frac{v^2 + 2v + 1}{2} \tag{3.6.24}$$

Similarly for $v \leq 1$,

$$\Pr(V \leq v) = \int_{-\infty}^{v} f_V(v)\,dv \tag{3.6.25}$$
$$= \frac{1}{2} + \int_{0}^{v} (1-v)\,dz \tag{3.6.26}$$
$$= \frac{-v^2 + 2v + 1}{2} \tag{3.6.27}$$

The CDF is as below:

$$F_V(v) = \begin{cases} 0 & v < -1 \\ \dfrac{v^2 + 2v + 1}{2} & v \leq 0 \\ \dfrac{-v^2 + 2v + 1}{2} & v \leq 1 \\ 1 & v > 1 \end{cases} \tag{3.6.28}$$

The plot for CDF of $V$ can be observed at figure 3.6.2

We need $\Pr\left(Z - W > \frac{1}{2}\right)$ where $Z = max(X, Y)$ and $W = min(X, Y)$. Now,

$$Z - W = \begin{cases} X - Y & \text{for } X \geq Y \\ Y - X & \text{for } X < Y \end{cases} \tag{3.6.29}$$

Therefore,

$$\Pr\left(Z - W > \frac{1}{2}\right) = \Pr\left(X - Y > \frac{1}{2}, X \geq Y\right) + \Pr\left(Y - X > \frac{1}{2}, X < Y\right) \tag{3.6.30}$$
$$= \Pr\left(X - Y > \frac{1}{2}\right) + \Pr\left(Y - X > \frac{1}{2}\right) \tag{3.6.31}$$
$$= \Pr\left(V > \frac{1}{2}\right) + \Pr\left(-V > \frac{1}{2}\right) \tag{3.6.32}$$
$$= 1 - \Pr\left(V \leq \frac{1}{2}\right) + \Pr\left(V < \frac{-1}{2}\right) \tag{3.6.33}$$
$$= 1 - F_V(\frac{1}{2}) + F_V(-\frac{1}{2}) \tag{3.6.34}$$
$$= 1 - \frac{7}{8} + \frac{1}{8} \tag{3.6.35}$$
$$= \frac{1}{4} \tag{3.6.36}$$

Hence the correct answer is option (C).

3.7. Let $X_1$ and $X_2$ be i.i.d. with probability mass function $f_\theta(x) = \theta^x (1-\theta)^{1-x}$; $x = 0, 1$ where $\theta \in (0, 1)$. Which of the following statements are true?

a) $X_1 + 2X_2$ is a sufficient statistic
b) $X_1 - X_2$ is a sufficient statistic
c) $X_1^2 + X_2^2$ is a sufficient statistic
d) $X_1^2 + X_2$ is a sufficient statistic

**Solution:** Given that, $X_1$ and $X_2$ are i.i.d. with probability mass function

$$f(x) = \begin{cases} (1-\theta) & x = 0 \\ \theta & x = 1 \end{cases} \tag{3.7.1}$$

A statistic $t = T(X)$ is sufficient for a parameter $\theta$ if the conditional probability distribution of the data, given the statistic $t = T(X)$ does not depend on the parameter $\theta$. i.e,

$$P_\theta(X_1 = x_1, X_2 = x_2 | T = t) \tag{3.7.2}$$

is independent of $\theta$ for all $x_1, x_2$ and $t$

a) Let $T = X_1 + 2X_2$
Consider a case where $x_1 = 0, x_2 = 0$ and $t = 0$

$$\Pr(T = 0) = \Pr(X_1 + 2X_2 = 0) \tag{3.7.3}$$
$$= \Pr(X_1 = 0, X_2 = 0) \tag{3.7.4}$$

As $X_1$ and $X_2$ are independent

$$\Pr(T = 0) = \Pr(X_1 = 0)\Pr(X_2 = 0) = (1-\theta)^2 \tag{3.7.5}$$

The conditional probability,

$$\Pr(X_1 = 0, X_2 = 0 | T = 0) = \frac{\Pr((X_1 = 0, X_2 = 0) \cap (T = 0))}{\Pr(T = 0)} \tag{3.7.6}$$

From (3.7.4), $(X_1 = 0, X_2 = 0) \subseteq (T = 0)$

$$= \frac{\Pr(X_1 = 0, X_2 = 0)}{\Pr(T = 0)} = \frac{(1-\theta)^2}{(1-\theta)^2} = 1 \tag{3.7.7}$$

Similarly, conditional probabilities for other values of $x_1, x_2$ and $t$ are given in table 3.7.1

| $x_1$ | $x_2$ | t <br> $t = X_1 + 2X_2$ | Conditional probability <br> $P_\theta(X_1 = x_1, X_2 = x_2 \| T = t)$ |
|---|---|---|---|
| 0 | 0 | 0 <br> otherwise | 1 <br> 0 |
| 1 | 0 | 1 <br> otherwise | 1 <br> 0 |
| 0 | 1 | 2 <br> otherwise | 1 <br> 0 |
| 1 | 1 | 3 <br> otherwise | 1 <br> 0 |

TABLE 3.7.1: Conditional Probabilities

From table 3.7.1, all the conditional probabilities are independent of $\theta$
$\therefore X_1 + 2X_2$ is a sufficient statistic.

b) Let $T = X_1 - X_2$
Consider a case where $x_1 = 0, x_2 = 0$ and $t = 0$

$$\Pr(T = 0) = \Pr(X_1 - X_2 = 0)$$
$$= \Pr(X_1 = 0, X_2 = 0) + \Pr(X_1 = 1, X_2 = 1) \tag{3.7.8}$$

As $X_1$ and $X_2$ are independent

$$= \Pr(X_1 = 0)\Pr(X_2 = 0) + \Pr(X_1 = 1)\Pr(X_2 = 1) = (1-\theta)^2 + \theta^2 \tag{3.7.9}$$

The conditional probability,

$$\Pr(X_1 = 0, X_2 = 0 | T = 0) = \frac{\Pr((X_1 = 0, X_2 = 0) \cap (T = 0))}{\Pr(T = 0)} \tag{3.7.10}$$

From (3.7.8), $(X_1 = 0, X_2 = 0) \subseteq (T = 0)$

$$= \frac{\Pr(X_1 = 0, X_2 = 0)}{\Pr(T = 0)} = \frac{(1-\theta)^2}{(1-\theta)^2 + \theta^2} \tag{3.7.11}$$

depends on $\theta$.
$\therefore X_1 - X_2$ is not a sufficient statistic.

c) Let $T = X_1^2 + X_2^2$
Consider a case where $x_1 = 1, x_2 = 0$ and $t = 1$

$$\Pr(T = 1) = \Pr\left(X_1^2 + X_2^2 = 1\right)$$
$$= \Pr(X_1 = 1, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1)$$
$$= \theta(1-\theta) + (1-\theta)\theta = 2\theta(1-\theta) \tag{3.7.12}$$

The conditional probability,

$$\Pr(X_1 = 1, X_2 = 0 | T = 1) = \frac{\Pr((X_1 = 1, X_2 = 0) \cap (T = 0))}{\Pr(T = 1)} \quad (3.7.13)$$

From (3.7.12), $(X_1 = 1, X_2 = 0) \subseteq (T = 1)$

$$= \frac{\Pr(X_1 = 1, X_2 = 0)}{\Pr(T = 1)} = \frac{\theta(1-\theta)}{2\theta(1-\theta)} = \frac{1}{2} \quad (3.7.14)$$

Similarly, conditional probabilities for other values of $x_1, x_2$ and $t$ are given in table 3.7.2

| $x_1$ | $x_2$ | t<br>$t = X_1^2 + X_2^2$ | Conditional probability<br>$P_\theta(X_1 = x_1, X_2 = x_2 \mid T = t)$ |
|---|---|---|---|
| 0 | 0 | 0<br>otherwise | 1<br>0 |
| 1 | 0 | 1<br>otherwise | $\frac{1}{2}$<br>0 |
| 0 | 1 | 1<br>otherwise | $\frac{1}{2}$<br>0 |
| 1 | 1 | 2<br>otherwise | 1<br>0 |

TABLE 3.7.2: Conditional Probabilities

From table 3.7.2, all the conditional probabilities are independent of $\theta$
$\therefore X_1^2 + X_2^2$ is a sufficient statistic.

d) Let $T = X_1^2 + X_2$
Consider a case where $x_1 = 1, x_2 = 0$ and $t = 1$

$$\Pr(T = 1) = \Pr\left(X_1^2 + X_2 = 1\right)$$
$$= \Pr(X_1 = 1, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1)$$
$$= \theta(1-\theta) + (1-\theta)\theta = 2\theta(1-\theta) \quad (3.7.15)$$

The conditional probability,

$$\Pr(X_1 = 1, X_2 = 0 | T = 1) = \frac{\Pr((X_1 = 1, X_2 = 0) \cap (T = 0))}{\Pr(T = 1)} \quad (3.7.16)$$

From (3.7.15), $(X_1 = 1, X_2 = 0) \subseteq (T = 1)$

$$= \frac{\Pr(X_1 = 1, X_2 = 0)}{\Pr(T = 1)} = \frac{\theta(1-\theta)}{2\theta(1-\theta)} = \frac{1}{2} \quad (3.7.17)$$

Similarly, conditional probabilities for other values of $x_1, x_2$ and $t$ are given in table 3.7.3

| $x_1$ | $x_2$ | t<br>$t = X_1^2 + X_2$ | Conditional probability<br>$P_\theta(X_1 = x_1, X_2 = x_2 \mid T = t)$ |
|---|---|---|---|
| 0 | 0 | 0<br>otherwise | 1<br>0 |
| 1 | 0 | 1<br>otherwise | $\frac{1}{2}$<br>0 |
| 0 | 1 | 1<br>otherwise | $\frac{1}{2}$<br>0 |
| 1 | 1 | 2<br>otherwise | 1<br>0 |

TABLE 3.7.3: Conditional Probabilities

From table 3.7.3, all the conditional probabilities are independent of $\theta$
$\therefore X_1^2 + X_2$ is a sufficient statistic.

Answer : Options 1,3,4

3.8. Let X and Y be two random variables satisfying $X \geq 0, Y \geq 0, E(X) = 3, Var(X) = 9, E(Y) = 2$ and $Var(Y) = 4$. Which of the following statements are correct?

A) $0 \leq Cov(X, Y) \leq 4$
B) $E(XY) \leq 3$
C) $Var(X + Y) \leq 25$
D) $E(X + Y)^2 \geq 25$

**Solution:**

$$E(X^2) = Var(X) + (E(X))^2 = 18 \quad (3.8.1)$$

Similarly,

$$E(Y^2) = Var(Y) + (E(Y))^2 = 8 \quad (3.8.2)$$

We can use the Covariance inequality for this question,

$$(Cov(X, Y))^2 \leq Var(X)Var(Y) \quad (3.8.3)$$

The proof of this inequality is as shown,

$$Var\left(\frac{X}{\sigma_X} \pm \frac{Y}{\sigma_Y}\right) = Var\left(\frac{X}{\sigma_X}\right) + Var\left(\frac{\pm Y}{\sigma_Y}\right) + 2Cov\left(\frac{X}{\sigma_X}, \frac{\pm Y}{\sigma_Y}\right) \quad (3.8.4)$$

$$= \frac{1}{{\sigma_X}^2} Var(X) + \frac{1}{{\sigma_Y}^2} Var(Y) + 2Cov\left(\frac{X}{\sigma_X}, \frac{\pm Y}{\sigma_Y}\right) \quad (3.8.5)$$

$$= 2 \pm 2\frac{Cov(X,Y)}{\sigma_X \sigma_Y} \quad (3.8.6)$$

Since Variance is always positive,

$$Var\left(\frac{X}{\sigma_X} \pm \frac{Y}{\sigma_Y}\right) \geq 0 \quad (3.8.7)$$

$$2 \pm 2\frac{Cov(X,Y)}{\sigma_X \sigma_Y} \geq 0 \quad (3.8.8)$$

$$1 \pm 1\frac{Cov(X,Y)}{\sigma_X \sigma_Y} \geq 0 \quad (3.8.9)$$

$$|(Cov(X,Y))| \leq (\sigma_X)(\sigma_Y) \quad (3.8.10)$$

$$(Cov(X,Y))^2 \leq Var(X)Var(Y) \quad (3.8.11)$$

a) Substituting values of variance we get,

$$-6 \leq Cov(X,Y) \leq 6 \quad (3.8.12)$$

**Therefore, option A is incorrect**.

b) From equation (3.8.12),

$$Cov(X,Y) = E(XY) - E(X)E(Y) \quad (3.8.13)$$

$$-6 \leq E(XY) - E(X)E(Y) \leq 6 \quad (3.8.14)$$

$$0 \leq E(XY) \leq 12 \quad (3.8.15)$$

Also, if X and Y are independent,

$$E(XY) = E(X)E(Y) = 6 \quad (3.8.16)$$

**Therefore, Option B is incorrect.**

c) Now,

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) \quad (3.8.17)$$

$$= 13 + 2Cov(X,Y) \quad (3.8.18)$$

From equation (3.8.12),

$$1 \leq Var(X+Y) \leq 25 \quad (3.8.19)$$

**Therefore, Option C is correct.**

d) Now,

$$E(X+Y)^2 = E(X^2) + E(Y^2) + 2E(XY) \quad (3.8.20)$$

$$E(X+Y)^2 = 26 + 2E(XY) \quad (3.8.21)$$

From equation (3.8.15),

$$26 \leq E(X+Y)^2 \leq 50 \quad (3.8.22)$$

**Therefore, Option D is correct.**

## 4 December 2017

4.1. Consider a Markov chain with five states $\{1,2,3,4,5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{5}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix} \quad (4.1.1)$$

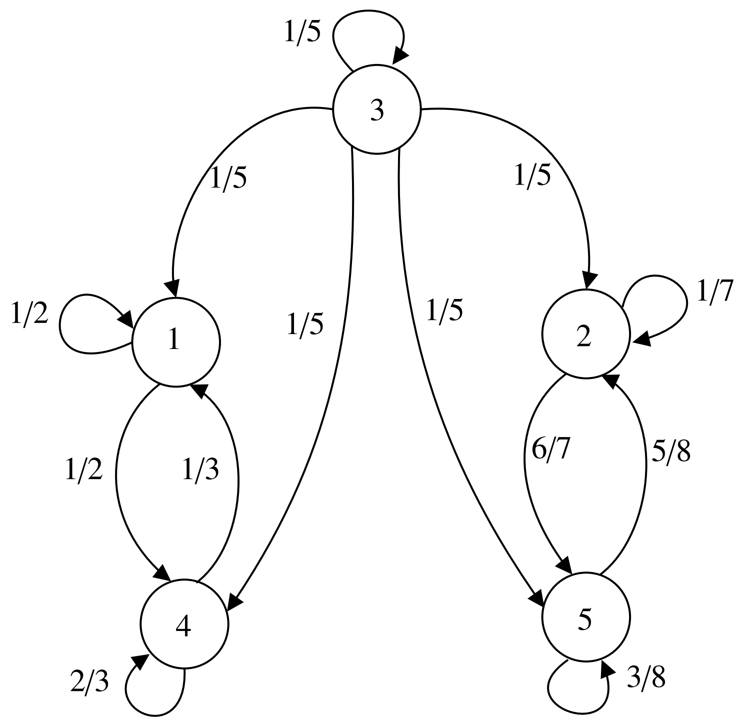
Which of the following are true?

a) 3 and 1 are in the same communicating class
b) 1 and 4 are in the same communicating class
c) 4 and 2 are in the same communicating class
d) 2 and 5 are in the same communicating class

**Solution:** See Tables 4.1.1 and 4.1.2

| | |
|---|---|
| Accessibility of states in Markov's chain | We say that state $j$ is accessible from state $i$, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. Every state is accessible from itself since $p_{ii}^{(0)} = 1$ |
| Communication between states | Two states $i$ and $j$ are said to communicate, written as $i \leftrightarrow j$, if they are accessible from each other. In other words, $i \leftrightarrow j$ means $i \to j$ and $j \to i$. |
| Communicating class | For each Markov chain, there exists a unique decomposition of the state space $S$ into a sequence of disjoint subsets $C_1, C_2, ...,$ $S = \bigcup_{i=1}^{\infty} C_i$ in which each subset has the property that all states within it communicate. Each such subset is called a communication class of the Markov chain. |

TABLE 4.1.1: Definition and Result used

| | |
|---|---|
| Drawing Transition diagram |   |
| Checking whether the states 3 and 1 are in the | Here,<br>State 1 is accessible from the state 3. |

| | |
|---|---|
| same communicating class | But, State 3 is not accessible from the state 1<br>$$\text{i.e. } 3 \to 1,\ 1 \nrightarrow 3$$<br>$$\implies \boxed{3 \nleftrightarrow 1}$$<br>Therefore, 3 and 1 are not in the same communicating class. |
| Checking whether the states 1 and 4 are in the same communicating class | Here,<br>State 1 is accessible from the state 4.<br>Also, State 4 is accessible from the state 1<br>$$\text{i.e. } 3 \to 1,\ 1 \to 3$$<br>$$\implies \boxed{3 \leftrightarrow 1}$$<br>Therefore, 1 and 4 are in the same communicating class. |
| Checking whether the states 4 and 2 are in the same communicating class | Here,<br>State 2 is not accessible from the state 4.<br>Also, State 4 is not accessible from the state 2<br>$$\text{i.e. } 4 \nrightarrow 2,\ 2 \nrightarrow 4$$ |
| | $$\implies \boxed{4 \nleftrightarrow 2}$$<br>Therefore, 4 and 2 are not in the same communicating class. |
| Checking whether the states 2 and 5 are in the same communicating class | Here,<br>State 2 is accessible from the state 5.<br>Also, State 5 is accessible from the state 2<br>$$\text{i.e. } 5 \to 2,\ 2 \to 5$$<br>$$\implies \boxed{2 \leftrightarrow 5}$$<br>Therefore, 2 and 5 are in the same communicating class. |
| Conclusion | Communication classes are:<br>$$\boxed{S = \{1,4\} \cup \{3\} \cup \{2,5\}}$$<br>Option 2) and 4) are true. |

TABLE 4.1.2: Solution

4.2. Let $X$ and $Y$ be independent exponential random variables. If $E[X] = 1$ and $E[Y] = \frac{1}{2}$ then $\Pr(X > 2Y|X > Y)$ is

1. $\frac{1}{2}$
2. $\frac{1}{3}$
3. $\frac{2}{3}$
4. $\frac{3}{4}$

**Solution:** Since $X$ and $Y$ are exponential random variables with means'

$$E[X] = 1 \text{ and } E[Y] = \frac{1}{2} \tag{4.2.1}$$

Marginal PDFs of $X$ and $Y$ are given by

$$f_X(x) = e^{-x}, x > 0 \tag{4.2.2}$$
$$f_Y(y) = 2e^{-2y}, y > 0 \tag{4.2.3}$$

CDFs for $X$ and $Y$ are

$$F_X(b) = \int_0^b f_X(x)\,d_x \tag{4.2.4}$$
$$= \int_0^b e^{-x}\,d_x \tag{4.2.5}$$
$$= 1 - e^{-b} \tag{4.2.6}$$

$$F_Y(b) = \int_0^b f_Y(y)\,d_y \tag{4.2.7}$$
$$= \int_0^b 2e^{-2y}\,d_y \tag{4.2.8}$$
$$= \left[-e^{-2y}\right]_0^b \tag{4.2.9}$$
$$= 1 - e^{-2b} \tag{4.2.10}$$

Now,

$$\Pr(X > 2Y | X > Y) = \frac{\Pr(X > 2Y\,,\,X > Y)}{\Pr(X > Y)} \tag{4.2.11}$$
$$= \frac{\Pr(X > 2Y)}{\Pr(X > Y)} \tag{4.2.12}$$

$$\Pr(X > Y) = \Pr(Y < X) \tag{4.2.13}$$
$$= E[F_Y(X)] \tag{4.2.14}$$
$$= \int_0^\infty F_Y(X)\, f_X(x)\, d_x \tag{4.2.15}$$
$$= \int_0^\infty (1 - e^{-2x})\, e^{-x} d_x \tag{4.2.16}$$
$$= \left[\frac{e^{-x}}{-1} - \frac{e^{-3x}}{-3}\right]_0^\infty \tag{4.2.17}$$
$$= (0 + 1) + \frac{1}{3}(0 - 1) \tag{4.2.18}$$
$$= \frac{2}{3} \tag{4.2.19}$$

$$\Pr(X > 2Y) = \Pr\left(Y < \frac{X}{2}\right) \tag{4.2.20}$$
$$= E[F_Y(X/2)] \tag{4.2.21}$$
$$= \int_0^\infty F_Y(X/2)\, f_X(x)\, d_x \tag{4.2.22}$$
$$= \int_0^\infty (1 - e^{-x})\, e^{-x} d_x \tag{4.2.23}$$
$$= \left[\frac{e^{-x}}{-1} - \frac{e^{-2x}}{-2}\right]_0^\infty \tag{4.2.24}$$
$$= (0 + 1) + \frac{1}{2}(0 - 1) \tag{4.2.25}$$
$$= \frac{1}{2} \tag{4.2.26}$$

Putting (4.2.19) and (4.2.26) in (4.2.12)

$$\Pr(X > 2Y \mid X > Y) = \frac{1/2}{2/3} \tag{4.2.27}$$
$$= \frac{3}{4} \tag{4.2.28}$$

$\therefore$ Option 4 is the correct answer.

4.3. Let $X_1$ and $X_2$ be a random sample of size two from a distribution with probability density function

$$f_\theta(x) = \theta\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}x^2} + (1-\theta)\left(\frac{1}{2}\right)e^{-|x|},$$

$-\infty < x < \infty$,
where $\theta \in \left\{0, \frac{1}{2}, 1\right\}$. If the observed values of $X_1$ and $X_2$ are 0 and 2, respectively, then the maximum likelihood estimate of $\theta$ is

a) 0
b) $\frac{1}{2}$
c) 1

d) not unique

**Solution:** Given $X_1 = 0$, $X_2 = 2$, n=2 and

$$f_\theta(x) = \theta\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}x^2} + (1-\theta)\left(\frac{1}{2}\right)e^{-|x|} \tag{4.3.1}$$

Then log of likelihood function is given by

$$l(\theta) = \sum_{i=1}^{i=n} \log f_\theta(x_i) \tag{4.3.2}$$

$$= \log f_\theta(x_1) + \log f_\theta(x_2) \tag{4.3.3}$$

$$= \log\left(\theta\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}0^2} + (1-\theta)\left(\frac{1}{2}\right)e^{-|0|}\right) + \log\left(\theta\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}2^2} + (1-\theta)\left(\frac{1}{2}\right)e^{-|2|}\right) \tag{4.3.4}$$

$$= \log\left(\theta\left(\frac{1}{\sqrt{2\pi}}\right) + (1-\theta)\left(\frac{1}{2}\right)\right) + \log\left(\theta\left(\frac{1}{\sqrt{2\pi}}\right)e^{-2} + (1-\theta)\left(\frac{1}{2}\right)e^{-2}\right) \tag{4.3.5}$$

$$= 2\log\left(\theta\left(\frac{1}{\sqrt{2\pi}}\right) + (1-\theta)\left(\frac{1}{2}\right)\right) - 2 \tag{4.3.6}$$

Since likelihood $L(\theta) = e^{l(\theta)}$.

Likelihood function $L(\theta)$ at $\theta = 0, \frac{1}{2}, 1$ is given by

a) At $\theta = 0$ $\quad L(\theta = 0) = \frac{1}{4}e^{-2} = 0.0338$

b) At $\theta = 1$ $\quad L(\theta = 1) = \frac{1}{2\pi}e^{-2} = 0.0215$

c) At $\theta = \frac{1}{2}$ $\quad L(\theta = \frac{1}{2}) = \left(\frac{1}{2\sqrt{2\pi}} + \frac{1}{4}\right)^2 e^{-2} = 0.0273$

Hence the maximum likelihood estimate of $\theta$ is at $\theta = 0$

## 5 June 2017

5.1. X and Y are independent random variables each having the density

$$f(t) = \frac{1}{\pi}\frac{1}{1+t^2} - \infty < t < +\infty \tag{5.1.1}$$

Then the density function of $\dfrac{X+Y}{3}$ for $-\infty < t < +\infty$ is

a) $\frac{6}{\pi}\frac{1}{4+9t^2}$

b) $\frac{6}{\pi}\frac{1}{9+4t^2}$

c) $\frac{3}{\pi}\frac{1}{1+9t^2}$

d) $\frac{3}{\pi}\frac{1}{9+t^2}$

**Solution:** Let us consider the random variables X and Y. The Characteristic function of the probability density $f(t)$ is

$$g(w) = \int_{-\infty}^{\infty} f(t)e^{iwt}dt \tag{5.1.2}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+t^2}e^{iwt}dt \tag{5.1.3}$$

$$= e^{-|w|} \quad ,-\infty < w < \infty \tag{5.1.4}$$

The product of the Characteristic function of probability density of X and Y is

$$h(w) = g_1(w) \times g_2(w) = e^{-2|w|} \tag{5.1.5}$$

To get the probability density of X+Y, we find the inverse characteristic function of h(w). But since there is a one to one correspondence between a function and its fourier transform and $h(w) = g(2w)$

$$F_{X+Y}(t) = \frac{1}{2}f\left(\frac{t}{2}\right) \tag{5.1.6}$$

$$= \frac{1}{2\pi}\frac{4}{4+t^2} \quad ,-\infty < t < \infty \tag{5.1.7}$$

We know that if a random variable M has a probability density $f_M(x)$, then the probability density of random variable kM is

$$f_{kM}\,(x) = \frac{1}{|k|}f_M\left(\frac{x}{|k|}\right) \tag{5.1.8}$$

Probability density of $Z = \dfrac{X+Y}{3}$ given $F_{X+Y}(t)$ is

$$F_Z(t) = 3 \times f_{X+Y}(3t) \tag{5.1.9}$$

$$= \frac{6}{\pi}\frac{1}{4+9t^2} \tag{5.1.10}$$

5.2. Let $\{X_n, n \geq 1\}$ be i.i.d. uniform (-1,2) random variables. Which of the following statements are true?

a) $\frac{1}{n}\sum_{i=1}^{n} X_i \to 0$ almost surely

b) $\left\{\frac{1}{2n}\sum_{i=1}^{n} X_{2i} - \frac{1}{2n}\sum_{i=1}^{n} X_{2i-1}\right\} \to 0$ almost surely

c) $\sup\{X_1, X_2, \ldots\} = 2$ almost surely

d) $\inf\{X_1, X_2, \ldots\} = -1$ almost surely

**Solution:** We using convergence in almost surely and Strong law of large number (SLLN)

a) *Almost sure convergence :* Let $X_1, X_2, \ldots$ be an infinite sequence of random variables. We shall say that the sequence $\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable $Y$, if

$$\Pr\left(\lim_{n\to\infty} X_n = Y\right) = 1 \tag{5.2.1}$$

$$\text{and we write}, X_n \xrightarrow{a.s.} Y \tag{5.2.2}$$

b) *SLLN :* Let $X_n$ be i.i.d with $\mathbf{E}[|X_1|] < \infty$. Then, as $n \to \infty$ , we have

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{\text{P}} \mathbf{E}[X_1] \tag{5.2.3}$$

$$, \text{where}\, S_n = X_1 + \cdots + X_n \tag{5.2.4}$$

also,

$$X_i \xrightarrow{a.s.} X \implies g(X_i) \xrightarrow{a.s.} g(X) \tag{5.2.5}$$

a)

$$\frac{1}{n}(X_1 + \cdots + X_n) \to E(X) \in (-1, 2) \tag{5.2.6}$$

$$\text{as}\, n \to \infty, \tag{5.2.7}$$

according to strong law of large numbers (SLLN).
So, option (A) is incorrect.

b) using this 5.2.5 , we solve as

$$\left\{\frac{1}{2n}\sum_{i=1}^{n} X_{2i} - \frac{1}{2n}\sum_{i=1}^{n} X_{2i-1}\right\} \xrightarrow{a.s.} \left\{\frac{nX}{2n} - \frac{nX}{2n}\right\} \tag{5.2.8}$$

$$= 0 \tag{5.2.9}$$

option (B) is correct.

c) Similarly, Let $M = \sup(S)$. Then,

$$x \le M, \quad \forall x \in S \tag{5.2.10}$$

$$\forall \epsilon > 0, \quad (M - \epsilon, M] \cap S \neq \emptyset \tag{5.2.11}$$

where, $S$ be a nonempty subset of $\mathbb{R}$ with an upper bound. Using $X_i \xrightarrow{a.s.} X$ this , we conclude that

$$\sup\{X_1, X_2, \ldots\} = 2 \text{ almost surely} \tag{5.2.12}$$

d) Let $m = \inf(S)$. Then

$$x \ge m, \quad \forall x \in S \tag{5.2.13}$$

$$\forall \epsilon > 0, \quad [m, m + \epsilon] \cap S \neq \emptyset \tag{5.2.14}$$

where, $S$ be a nonempty subset of $\mathbb{R}$ with an lower bound. Again using $X_i \xrightarrow{a.s.} X$ this , we conclude that

$$\inf\{X_1, X_2, \ldots\} = -1 \text{ almost surely} \tag{5.2.15}$$

Hence (B), (C) and (D) are correct options.

5.3. $X_1, X_2, \cdots$ are independent identically distributed random variables having common density $f$.

Assume $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Which of the following statements is correct?

a) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ in probability

b) $\frac{1}{n}(X_1 + \cdots + X_n) \to 0$ almost surely

c) $\Pr\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) \to \frac{1}{2}$

d) $\sum_{i=1}^{n} X_i$ has the same distribution as $\sum_{i=1}^{n}(-1)^i X_i$

**Solution:** We using

a) (1) *Convergence in probability :* Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, and let $Y$ be another random variable. Then the sequence $\{X_n\}$ converges in probability to $Y$, if

$$\forall \epsilon > 0, \lim_{n\to\infty} \Pr(|X_n - Y| \geq \epsilon) = 0, \tag{5.3.1}$$

and we write

$$X_n \xrightarrow{P} Y. \tag{5.3.2}$$

(2) *Convergence in almost surely :* Let $X_1, X_2, \ldots$ be an infinite sequence of random variables. We shall say that the sequence $\{X_i\}$ converges with probability 1 (or converges almost surely (a.s.)) to a random variable $Y$, if

$$\Pr\left(\lim_{n\to\infty} X_n = Y\right) = 1 \tag{5.3.3}$$

and we write

$$X_n \xrightarrow{a.s.} Y \tag{5.3.4}$$

(3) *Strong law of large number(SLLN) :* Let $X_1, X_2, \ldots$ be an infinite sequence of random variables, If $\mathbf{E}[|X_1|] < \infty$. Then, as $n \to \infty$ , we have

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbf{E}[X_1] \implies \frac{S_n}{n} \xrightarrow{P} \mathbf{E}[X_1], \tag{5.3.5}$$

$$\text{where, } S_n = X_1 + \cdots + X_n \tag{5.3.6}$$

using SLLN, (B) are incorrect option.

b) *Relation between in probability and almost surely :* Let $Z, Z_1, Z_2, \ldots$ be random variables. Suppose $Z_n \to Z$ with probability 1. Then ,we say

$$Z_n \xrightarrow{a.s.} Z \implies Z_n \xrightarrow{P} Z. \tag{5.3.7}$$

(5.3.5), also in probability also hold this equation. Hence (A) is incorrect option.

c) *Central Limit Theorem :* Let $X_1, X_2, \ldots$ be i.i.d. with finite mean $\mu$ and finite variance $\sigma^2$. Let $Z \sim N(0, 1)$. Set $S_n = X_1 + \cdots + X_n$, and

$$Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \tag{5.3.8}$$

Then as $n \to \infty$, the sequence $\{Z_n\}$ converges in distribution to the $Z$, i.e., $Z_n \xrightarrow{D} Z$.
Consider,

$$Y = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \tag{5.3.9}$$

So,

$$E(Y) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = 0 \tag{5.3.10}$$

$$V(Y) = V\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\right) = \frac{1}{n}2n = 2 \tag{5.3.11}$$

$$Y \sim N[0, 2] \tag{5.3.12}$$

we know that,

$$f(x) = f(-x) \implies \text{Symmetry about Zero,} \tag{5.3.13}$$

So,

$$\Pr(Y < 0) = \frac{1}{2} \tag{5.3.14}$$

$$\Pr\left(\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) < 0\right) = \frac{1}{2} \tag{5.3.15}$$

Hence, (C) is incorrect option.

d) *Characteristic function :* For a scalar random variable $X$ the characteristic function is defined as the expected value of $e^{it\times}$, where $i$ is the imaginary unit, and $t \in \mathbf{R}$ is the argument of the characteristic function:

$$\begin{cases} \varphi_X : \mathbb{R} \to \mathbb{C} \\ \varphi_X(t) = \mathrm{E}\left[e^{itX}\right] = \int_{\mathbb{R}} e^{itx} dF_X(x) \\ = \int_{\mathbb{R}} e^{itx} f_X(x)dx = \int_0^1 e^{itQ_X(p)}dp \end{cases} \tag{5.3.16}$$

Here $F_X$ is the cumulative distribution function of $X$,Consider, $\phi_x(t)$ is characteristic function of $X_i, i = 1, \ldots, n$.

$$f(x) = f(-x) \implies \phi_x(t) = \phi_{-x}(t) \tag{5.3.17}$$

Therefore,

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{X_1+\ldots+X_n}(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) \tag{5.3.18}$$

$$= [\phi_x(t)]^n \tag{5.3.19}$$

similarly,

$$\phi_{\sum_{i=1}^n (-1)^i X_i}(t) = \phi_{-X_1} + \phi_{X_2} + \ldots + \phi_{(-1)^n X_n}(t) \tag{5.3.20}$$

$$= \phi_{-X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{(-1)^n X_n}(t) \tag{5.3.21}$$

$$= [\phi_x(t)]^n \tag{5.3.22}$$

$$\phi_{\sum_{i=1}^n X_i}(t) = \phi_{\sum_{i=1}^n (-1)^i X_i}(t) \tag{5.3.23}$$

$\therefore \sum_{i=1}^n X_i$ has same distribution as $\sum_{i=1}^n (-1)^i X_i$.

Hence, only (D) is correct option.

5.4. Suppose the random variable X has the following probability density funtion

$$f(x) = \begin{cases} \alpha\,(x-\mu)^{\alpha-1}\, e^{-(x-\mu)^\alpha}; & x > \mu \\ 0 & x \le \mu \end{cases}$$

where $\alpha > 0, -\infty < \mu < \infty$. Which of the following are correct? The hazard function of $X$ is

a) an increasing function for all $\alpha > 0$
b) a decreasing function for all $\alpha > 0$
c) an increasing function for some $\alpha > 0$
d) a decreasing function for some $\alpha > 0$

**Solution:**
For the random variable $X$, the CDF is

$$F(x) = solutions/2017/june/118/figures/\int_0^x f(y)dy \tag{5.4.1}$$

$$= solutions/2017/june/118/figures/\int_0^\mu 0dy + solutions/2017/june/118/figures/\int_\mu^x \alpha\,(y-\mu)^{\alpha-1}\, e^{-(y-\mu)^\alpha} \tag{5.4.2}$$

$$= 0 - e^{-(y-\mu)^\alpha}\Big|_\mu^x \tag{5.4.3}$$

$$= 1 - e^{-(x-\mu)^\alpha} \tag{5.4.4}$$

For X, the hazard function $H(y)$ is defined as

$$H(y) = \frac{f(y)}{1-F(y)}$$

$$\implies H(y) = \begin{cases} \frac{\alpha(y-\mu)^{\alpha-1}e^{-(y-\mu)^\alpha}}{1-\left(1-e^{-(y-\mu)^\alpha}\right)}; & y > \mu \\ 0 & y \le \mu \end{cases}$$

$$= \begin{cases} \alpha\,(y-\mu)^{\alpha-1}; & y > \mu \\ 0 & y \le \mu \end{cases}$$

Differentiating $H(y)$ w.r.t. $y$

$$H'(y) = \begin{cases} \alpha\,(\alpha-1)\,(y-\mu)^{\alpha-2}; & y > \mu \\ 0 & y \le \mu \end{cases}$$

When $y \le \mu$ then $H'(y)$ is 0. When $y > \mu$ then $(y-\mu)^{\alpha-2}$ is positive. This implies that the sign for $H'(y)$ for $y > \mu$ is decided by the sign of $\alpha\,(\alpha-1)$.

$$\alpha\,(1-\alpha) < 0 \implies 0 < \alpha < 1$$

$$\alpha\,(1-\alpha) > 0 \implies \alpha > 1 \qquad \text{(ignoring } \alpha < 0\text{)}$$

$\therefore$ The Hazard function of $X$ is decreasing when $\alpha \in (0,1)$ and increasing when $\alpha \in (1,\infty)$

**Solution:** Options 3, 4

## 6 December 2016

6.1. $X_1, X_2, \ldots, X_n$ are independent and identically distributed as $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$. Then

a) $\sum_1^n \frac{(X_i-\bar{X})^2}{n-1}$ is the Minimum Variance Unbiased Estimate of $\sigma^2$

b) $\sqrt{\sum_1^n \frac{(X_i-\bar{X})^2}{n-1}}$ is the Minimum Variance Unbiased Estimate of $\sigma$

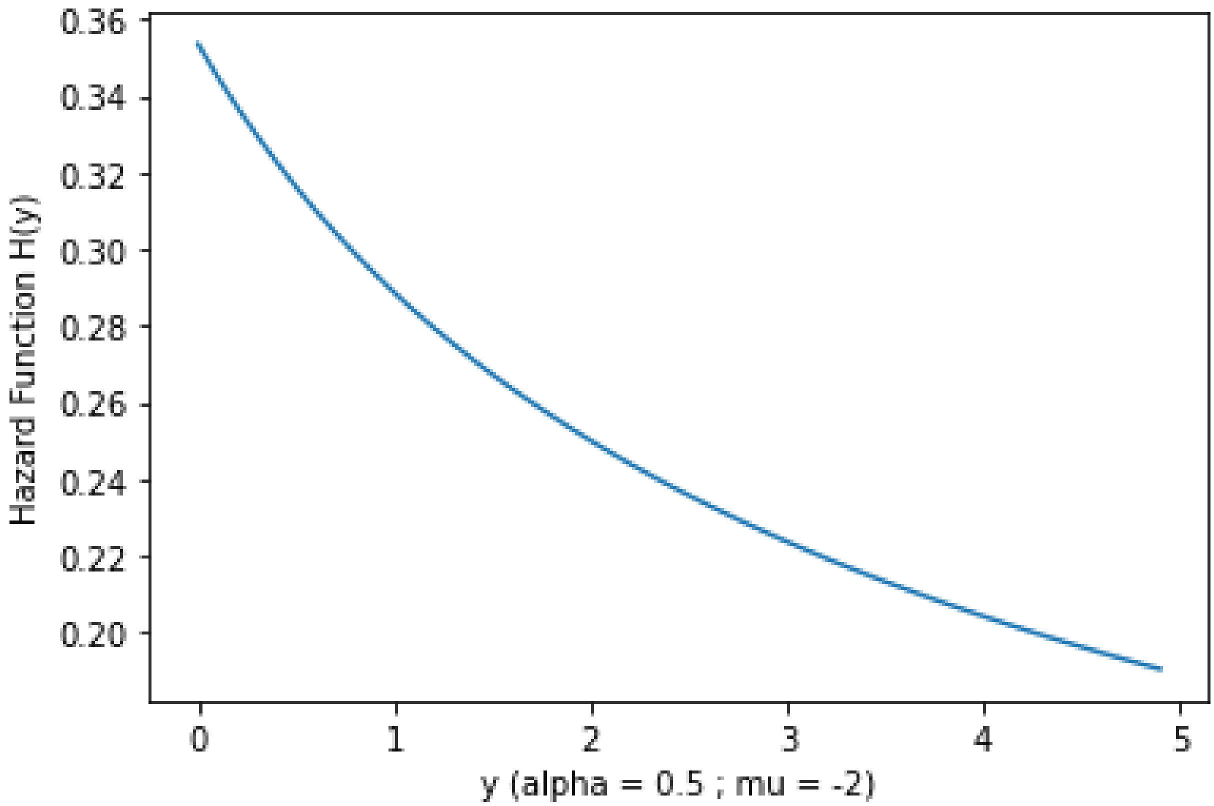



Fig. 5.4.1: Decreasing Hazard Function

c) $\sum_1^n \frac{(X_i-\bar{X})^2}{n}$ is the Maximum Likelihood Estimate of $\sigma^2$

d) $\sqrt{\sum_1^n \frac{(X_i-\bar{X})^2}{n}}$ is the Maximum Likelihood Estimate of $\sigma$

**Solution:** The pdf for each random variable is same as they are all identical and independent Normal Distributions with same $\mu$ and $\sigma^2$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(x-\mu)^2}{2\sigma^2} \tag{6.1.1}$$

Let us take our maximum likelihood function for given random variable $X_i$

$$L(\mu; \sigma|X_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i-\mu)^2}{2\sigma^2} \tag{6.1.2}$$

Since all the random variables are i.i.d

$$L(\mu; \sigma|X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} L(\mu; \sigma|X_i) \tag{6.1.3}$$

Let us denote:

$$L_m : L(\mu; \sigma|X_1, X_2, \ldots, X_n) \tag{6.1.4}$$

Substituting (6.1.2) for each Random Variable in (6.1.3)

$$L_m = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{(X_i-\mu)^2}{2\sigma^2} \tag{6.1.5}$$

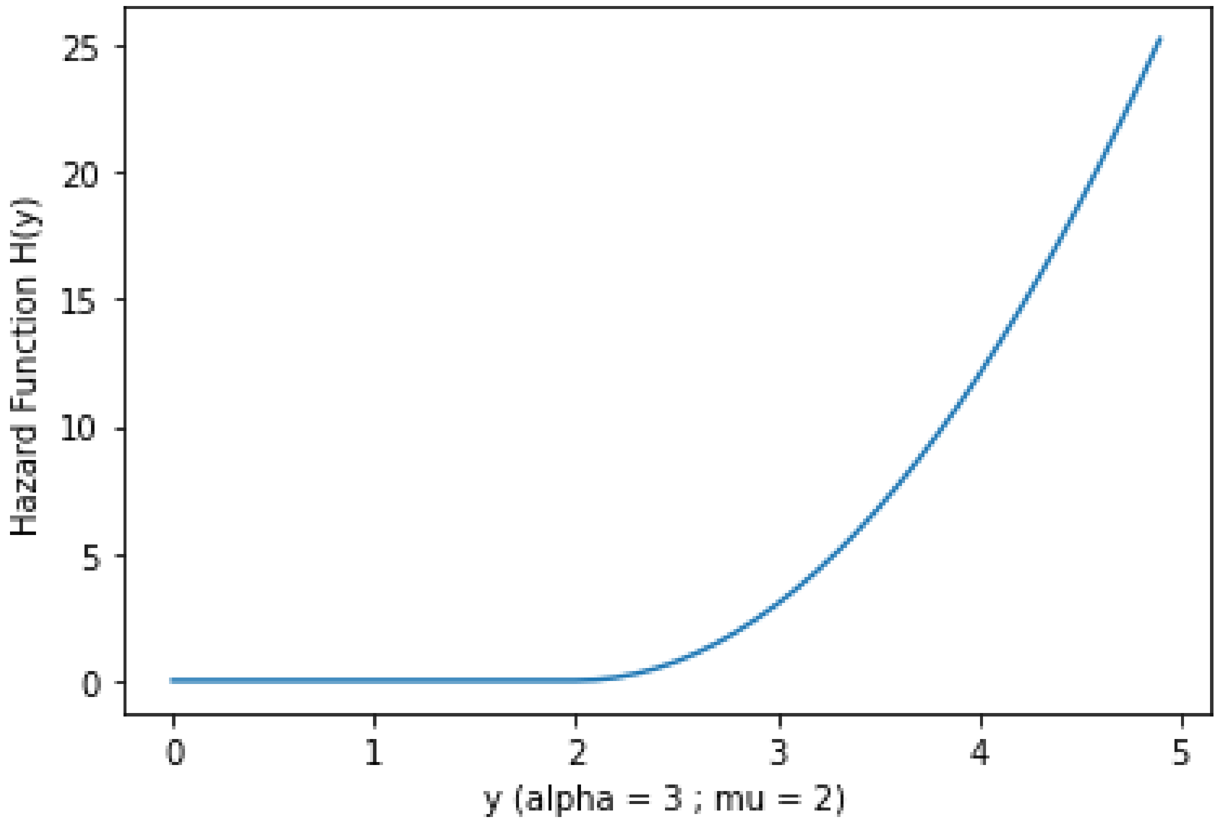



Fig. 5.4.2: Increasing Hazard Function

Taking natural log on both sides and simplifying

$$\ln L_m = \frac{-n}{2}\ln 2\pi - n\ln\sigma - \sum_{i=1}^{n}\frac{(X_i-\mu)^2}{2\sigma^2} \tag{6.1.6}$$

In order to find Maximum Likelihood we need to maximise $\mu$ and $\sigma$ w.r.t. all Random variables. Taking partial derivative w.r.t $\mu$ and taking $\sigma$ as constant

$$\frac{\partial \ln L_m}{\partial\mu} = \sum_{i=1}^{n}\frac{(X_i-\mu)}{\sigma^2} \tag{6.1.7}$$

The value for $\mu$ at which $L_m$ achieves maximum value is same in $\ln L_m$

$$\because \frac{\partial \ln L_m}{\partial\mu} = 0 \tag{6.1.8}$$

$$\therefore \sum_{i=1}^{n}\frac{(X_i-\mu)}{\sigma^2} = 0 \tag{6.1.9}$$

On simplifying the expression we get:

$$n\mu = \sum_{i=1}^{n} X_i \tag{6.1.10}$$

$$\mu = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{6.1.11}$$

Let us denote the value achieved in (6.1.11) as $\bar{X}$. Taking partial derivative w.r.t $\sigma$ and taking $\mu$ as constant

$$\frac{\partial \ln L_m}{\partial \sigma} = \frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^3} \tag{6.1.12}$$

The value for $\sigma$ at which $L_m$ achieves maximum value is same in $\ln L_m$

$$\frac{\partial \ln L_m}{\partial \sigma} = 0 \tag{6.1.13}$$

$$\frac{-n}{\sigma} + \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^3} = 0 \tag{6.1.14}$$

Upon simplifying the expression

$$\frac{n}{\sigma} = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^3} \tag{6.1.15}$$

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{n} \tag{6.1.16}$$

Substituting (6.1.11) in (6.1.16)

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n} \tag{6.1.17}$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n}} \tag{6.1.18}$$

Hence **Option 3** and **Option 4** are correct

6.2. There are two boxes. Box-1 contains 2 red balls and 4 green balls. Box-2 contains 4 red balls and 2 green balls. A box is selected at random and a ball is chosen randomly from the selected box. If the ball turns out to be red, what is the probability that Box-1 had been selected? **Solution:** Box-1 has 2 red balls and 4 green balls.
Box-2 has 4 red balls and 2 green balls.
Let $B \in \{1, 2\}$ represent a random variable where 1 represents selecting box-1 and 2 represents selecting box-2. From Baye's theorem

$$\begin{aligned}\Pr(R = 1) = \Pr(R = 1 | B = 1) \times \Pr(B = 1) \\ + \Pr(R = 1 | B = 2) \times \Pr(B = 2)\end{aligned} \tag{6.2.1}$$

Substiting values from table (6.2.1) in (6.2.1)

$$\Pr(R = 1) = \frac{1}{2} \tag{6.2.2}$$

$$\begin{aligned}\Pr((R = 1)(B = 1)) = \Pr(R = 1 | B = 1) \\ \times \Pr(B = 1)\end{aligned} \tag{6.2.3}$$

$$= \frac{1}{6} \tag{6.2.4}$$

We need to find $\Pr(B = 1 | R = 1)$

| Event | definition | value |
|---|---|---|
| $\Pr(B = 1)$ | Probability of selecting Box-1 | $\frac{1}{2}$ |
| $\Pr(B = 2)$ | Probability of selecting Box-2 | $\frac{1}{2}$ |
| $\Pr(R = 1\|B = 1)$ | Probability of drawing red ball from Box-1 | $\frac{1}{3}$ |
| $\Pr(G = 1\|B = 1)$ | Probability of drawing green ball from Box-1 | $\frac{2}{3}$ |
| $\Pr(R = 1\|B = 2)$ | Probability of drawing red ball from Box-2 | $\frac{2}{3}$ |
| $\Pr(G = 1\|B = 2)$ | Probability of drawing green ball from Box-2 | $\frac{1}{3}$ |

TABLE 6.2.1: Table 1

$$\Pr(B = 1|R = 1) = \frac{\Pr((R = 1)(B = 1))}{\Pr(R = 1)} \tag{6.2.5}$$

$$= \frac{1}{3} \tag{6.2.6}$$

$\therefore$ The desired probability that box-1 is selected $= \frac{1}{3}$

6.3. Suppose customers arrive in a shop according to a Poisson process with rate 4 per hour. The shop opens at 10 : 00 am. If it is given that the second customer arrives at 10 : 40 am, what is the probability that no customer arrived before 10 : 30 am?

a) $\frac{1}{4}$
b) $e^{-2}$
c) $\frac{1}{2}$
d) $e^{\frac{1}{2}}$

**Solution:** We need to find

$$\Pr\left(X_p = 0|Y = 2\right) \tag{6.3.1}$$

| Random Variable | Time at which people arrive |
| --- | --- |
| $X_p$ | $p = 10:00 - 10:30$ |
| $X_q$ | $q = 10:30 - 10:40$ |
| $X_r$ | $r = 10:00 - 10:40$ |
| $Y$ | $10:40$ |

TABLE 6.3.1: Random Variables

In the world where the $2^{nd}$ person arrives at $10:40$ am the (6.3.1) becomes:

$$= \frac{\Pr\left(X_p = 0, X_q = 1\right)}{\Pr\left(X_r = 1\right)} \tag{6.3.2}$$

$$= \frac{\Pr\left(X_p = 0\right) \times \Pr\left(X_q = 1\right)}{\Pr\left(X_r = 1\right)} \tag{6.3.3}$$

The Poisson function distribution for time interval $t$ and rate $\lambda$ for a random variable $X$:

$$f_X(x;t) = \frac{(\lambda t)^x \exp\left(-\lambda t\right)}{x!}$$

For the time interval $p$:

$$\lambda = 4, t = 0.5, x = 0 \tag{6.3.4}$$

$$\Pr\left(X_p = 0\right) = f_X\left(0; \frac{1}{2}\right) \tag{6.3.5}$$

$$= e^{-2} \tag{6.3.6}$$

$$\tag{6.3.7}$$

For the time interval $q$:

$$\lambda = 4, t = \frac{1}{6}, x = 1 \tag{6.3.8}$$

$$\Pr\left(X_q = 1\right) = f_X\left(1; \frac{1}{6}\right) \tag{6.3.9}$$

$$= \frac{2}{3} e^{\frac{-2}{3}} \tag{6.3.10}$$

For the time interval $r$:

$$\lambda = 4, t = \frac{2}{3}, x = 1 \tag{6.3.11}$$

$$\Pr\left(X_r = 1\right) = f_X\left(1; \frac{2}{3}\right) \tag{6.3.12}$$

$$= \frac{8}{3} e^{\frac{-8}{3}} \tag{6.3.13}$$

Substituting (6.3.6) (6.3.10) (6.3.13) in (6.3.3):

$$\Pr\left(X_p = 0|Y = 2\right) = \frac{1}{4} \tag{6.3.14}$$

6.4. A fair die is thrown two times independently. Let $X, Y$ be the outcomes of these two throws and $Z = X + Y$. Let $U$ be the remainder obtained when $Z$ is divided by 6. Then which of the following statement(s) is/are true?

a) $X$ and $Z$ are independent
b) $X$ and $U$ are independent
c) $Z$ and $U$ are independent
d) $Y$ and $Z$ are not independent

**Solution:** Let $X \in \{1, 2, 3, 4, 5, 6\}$ represent the random variable which represents the outcome of the first throw of a dice. Similarly, $Y \in \{1, 2, 3, 4, 5, 6\}$ represents the random variable which represents the outcome of the second throw of a dice.

$$n(X = i) = 1, \quad i \in \{1, 2, 3, 4, 5, 6\} \tag{6.4.1}$$

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.4.2}$$

Similarly,

$$\Pr(Y = i) = \begin{cases} \frac{1}{6} & i \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.4.3}$$

$$Z = X + Y \tag{6.4.4}$$

$$\text{Let } z \in \{1, 2, \dots, 11, 12\} \tag{6.4.5}$$

$$\Pr(Z = z) = \Pr(X + Y = z) \tag{6.4.6}$$

$$= \sum_{x=0}^{z} \Pr(X = x) \Pr(Y = z - x) \tag{6.4.7}$$

$$= (6 - |z - 7|) \times \frac{1}{6} \times \frac{1}{6} \tag{6.4.8}$$

$$= \frac{6 - |z - 7|}{36} \tag{6.4.9}$$

$$\Pr(Z = z) = \begin{cases} \frac{6-|z-7|}{36} & z \in \{1, 2, \dots, 11, 12\} \\ 0 & \text{otherwise} \end{cases} \tag{6.4.10}$$

$U$ is the remainder obtained when $Z$ is divided by 6.

$$\text{Let } u \in \{0,1,2,3,4,5\} \tag{6.4.11}$$

$$\Pr(U = u) = \sum_{k=0}^{2} \Pr(Z = 6k + u) \tag{6.4.12}$$

$$\Pr(U = 0) = \Pr(Z = 0) + \Pr(Z = 6) + \Pr(Z = 12) \tag{6.4.13}$$

$$= 0 + \frac{5}{36} + \frac{1}{36} = \frac{1}{6} \tag{6.4.14}$$

$$\text{for } u \in \{1,2,3,4,5\} \tag{6.4.15}$$

$$\Pr(U = u) = \Pr(Z = 0 + u) + \Pr(Z = 6 + u) \tag{6.4.16}$$

$$= \frac{6 - |u - 7|}{36} + \frac{6 - |6 + u - 7|}{36} \tag{6.4.17}$$

$$= \frac{6 - (7 - u)}{36} + \frac{6 - (u - 1)}{36} \tag{6.4.18}$$

$$= \frac{u - 1 + 7 - u}{36} = \frac{6}{36} \tag{6.4.19}$$

$$= \frac{1}{6} \tag{6.4.20}$$

$$\Pr(U = u) = \begin{cases} \frac{1}{6} & u \in \{0,1,2,3,4,5\} \\ 0 & \text{otherwise} \end{cases} \tag{6.4.21}$$

Now, for checking each option,

a) Checking if $X$ and $Z$ are independent

$$p_1 = \Pr(Z = z, X = x) \tag{6.4.22}$$

$$= \Pr(Y = z - x, X = x) \tag{6.4.23}$$

$$= \Pr(Y = z - x) \times \Pr(X = x) \tag{6.4.24}$$

$$= \begin{cases} \frac{1}{36} & z - x \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.4.25}$$

$$\Pr(Z = z) \times \Pr(X = x) = \frac{6 - |z - 7|}{36} \times \frac{1}{6} \tag{6.4.26}$$

$$= \frac{6 - |z - 7|}{216} \tag{6.4.27}$$

$$\Pr(Z = z)\Pr(X = x) \neq \Pr(Z = z, X = x) \tag{6.4.28}$$

$X$ and $Z$ are not independent from (6.4.28) and hence option (6.4a) is false.

b) Checking if $X$ and $U$ are independent

$$p_2 = \Pr(U = u, X = x) \tag{6.4.29}$$

$$p_2 = \Pr((Z = u) + (Z = 6 + u) + (Z = 12 + u), X = x) \tag{6.4.30}$$

$$p_2 = \Pr((Y = u - x) + (Y = 6 + u - x) + (Y = 12 + u - x), X = x) \tag{6.4.31}$$

$$p_2 = \frac{1}{6} \times \frac{1}{6} \tag{6.4.32}$$
$$= \frac{1}{36} \tag{6.4.33}$$

$$\Pr(U = u) \times \Pr(X = x) = \frac{1}{6} \times \frac{1}{6} \tag{6.4.34}$$
$$= \frac{1}{36} \tag{6.4.35}$$
$$\Pr(U = u) \Pr(X = x) = \Pr(U = u, X = x) \tag{6.4.36}$$

$X$ and $U$ are independent from (6.4.36) and hence option (6.4b) is true.

c) Checking if $Z$ and $U$ are independent

$$p_3 = \Pr(Z = z|U = u) \tag{6.4.37}$$
$$p_3 = \begin{cases} 1 & u = 1 \text{ and } z = 7 \\ \frac{1}{2} & u = 0 \text{ and } z \in \{6, 12\} \\ \frac{1}{2} & u \in \{2, 3, 4, 5\} \text{ and} \\ & z = u \text{ or } z = 6 + u \\ 0 & \text{otherwise} \end{cases} \tag{6.4.38}$$
$$\Pr(Z = z) = \frac{6 - |z - 7|}{36} \tag{6.4.39}$$

If $Z$ and $U$ are independent, then

$$\Pr(Z = z|U = u) = \frac{\Pr(Z = z, U = u)}{\Pr(U = u)} \tag{6.4.40}$$
$$= \frac{\Pr(Z = z)\Pr(U = u)}{\Pr(U = u)} \tag{6.4.41}$$
$$= \Pr(Z = z) \tag{6.4.42}$$

But,

$$\Pr(Z = z|U = u) \neq \Pr(Z = z) \tag{6.4.43}$$

$X$ and $U$ are not independent from (6.4.43) and hence option (6.4c) is false.

d) Checking if $Y$ and $Z$ are independent

$$p_1 = \Pr(Z = z, Y = y) \tag{6.4.44}$$
$$= \Pr(X = z - y, Y = y) \tag{6.4.45}$$
$$= \Pr(X = z - y) \times \Pr(Y = y) \tag{6.4.46}$$
$$= \begin{cases} \frac{1}{36} & z - y \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.4.47}$$

$$\Pr(Z = z) \times \Pr(Y = y) = \frac{6 - |z - 7|}{36} \times \frac{1}{6} \tag{6.4.48}$$
$$= \frac{6 - |z - 7|}{216} \tag{6.4.49}$$
$$\Pr(Z = z)\Pr(Y = y) \neq \Pr(Z = z, Y = y) \tag{6.4.50}$$

$X$ and $Z$ are not independent from (6.4.50) and hence option (6.4d) is true.

Thus, options (6.4b) and (6.4d) are true.

6.5. Let $X$ be a random variable with a certain non-degenerate distribution. Then identify the correct statements

a) If $X$ has an exponential distribution then $median(X) < E(X)$
b) If $X$ has a uniform distribution on an interval $[a, b]$, then $E(X) < median(X)$
c) If $X$ has a Binomial distribution then $V(X) < E(X)$
d) If $X$ has a normal distribution, then $E(X) < V(X)$

**Solution:** Expected value($E(X)$): It is nothing but weighted average Median($median(X)$): It is the value separating the higher half from the lower half of a data sample Variance($V(X)$): It is the expectation of the squared deviation of a random variable from its mean

a) Let's consider $X$ has an exponential distribution.

$$X \sim Exp(\lambda) \tag{6.5.1}$$

where $\lambda$ is rate parameter.
Probability function of exponential distribution,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.5.2}$$

The expected value of $X \sim Exp(\lambda)$,

$$E(X) = \frac{1}{\lambda} \tag{6.5.3}$$

The median of $X \sim Exp(\lambda)$,

$$median(X) = \frac{\ln 2}{\lambda} \tag{6.5.4}$$

$$\ln 2 < 1 \tag{6.5.5}$$
$$\frac{\ln 2}{\lambda} < \frac{1}{\lambda} \tag{6.5.6}$$
$$median(X) < E(X) \tag{6.5.7}$$

Hence, option 1 is correct.

b) Let's consider $X$ has a uniform distribution in interval $[a, b]$,

$$X \sim U(a, b) \tag{6.5.8}$$

where, $a$ = lower limit
$b$ = upper limit
Probability function of uniform distribution,

$$f_X(k) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a, x > b \end{cases} \tag{6.5.9}$$

The expected value of $X \sim U(a, b)$,

$$E(X) = \frac{1}{2}(a + b) \tag{6.5.10}$$

The median of $X \sim U(a, b)$,

$$median(X) = \frac{1}{2}(a + b) \tag{6.5.11}$$

$$E(X) = median(X) \tag{6.5.12}$$

Hence, option 2 is incorrect.

c) Let's consider $X$ has a binomial distribution,

$$X \sim B(n, p) \tag{6.5.13}$$

where, $n$ = no. of trails
$p$ = success parameter
Probability function of binomial distribution,

$$f_X(k) = \begin{cases} {}^nC_k p^k (1-p)^{n-k} & 0 \le k \le n \\ 0 & otherwise \end{cases} \tag{6.5.14}$$

The expected value of $X \sim B(n, p)$,

$$E(X) = np \tag{6.5.15}$$

The variance of $X \sim B(n, p)$,

$$V(X) = \sigma^2 = np(1-p) \tag{6.5.16}$$

$$1 - p \le 1 \tag{6.5.17}$$

$$np(1-p) \le np \tag{6.5.18}$$

$$V(X) \le E(X) \tag{6.5.19}$$

Hence, option 3 is incorrect.

d) Let's consider $X$ has a normal distribution,

$$X \sim N\left(\mu, \sigma^2\right) \tag{6.5.20}$$

where, $\mu$ = mean of distribution
$\sigma^2$ = variance
Probability function of normal distribution,

$$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{2\sigma}\right)^2} \tag{6.5.21}$$

The expected value of $X \sim N\left(\mu, \sigma^2\right)$,

$$E(X) = \mu \tag{6.5.22}$$

The variance of $X \sim N\left(\mu, \sigma^2\right)$,

$$V(X) = \sigma^2 \tag{6.5.23}$$

$E(X)$ and $V(X)$ are user defined. So, they can take any value.
Hence, option 4 is incorrect.

6.6. $A$ and $B$ play a game of tossing a fair coin. $A$ starts the game by tossing the coin once and $B$ then tosses the coin twice, followed by $A$ tossing the coin once and $B$ tossing the coin twice and this continues until a head turns up. Whoever gets the first head wins the game. Then,

a) $P(B \text{ Wins}) > P(A \text{ Wins})$
b) $P(B \text{ Wins}) = 2P(A \text{ Wins})$
c) $P(A \text{ Wins}) > P(B \text{ Wins})$
d) $P(A \text{ Wins}) = 1 - P(B \text{ Wins})$

**Solution:** Given, a fair coin is tossed till heads turns up.

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{104.1}$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4, 5\}$, such that The state

TABLE 6.6.1: States and their notations

| Notation | State |
|---|---|
| $S = 1$ | $A$'s turn |
| $S = 2$ | $B$'s first turn |
| $S = 3$ | $B$'s second turn |
| $S = 4$ | $A$ wins |
| $S = 4$ | $B$ wins |

transition matrix for the Markov chain is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{104.2}$$

Clearly, the states $1, 2, 3$ are transient, while $4, 5$ are absorbing. The standard form of a state transition matrix is

$$P = \begin{array}{c} \\ A \\ N \end{array} \begin{array}{c} \begin{array}{cc} A & N \end{array} \\ \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{104.3}$$

where, Converting (104.2) to standard form, we get

TABLE 6.6.2: Notations and their meanings

| Notation | Meaning |
|---|---|
| $A$ | All absorbing states |
| $N$ | All non-absorbing states |
| $I$ | Identity matrix |
| $O$ | Zero matrix |
| $R, Q$ | Other submatices |

$$P = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix} \end{array} \tag{104.4}$$

From (104.4),

$$R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix} \tag{104.5}$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \tag{104.6}$$

where,

$$F = (I - Q)^{-1} \tag{104.7}$$

is called the fundamental matrix of $P$.
On solving, we get

$$\bar{P} = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5714 & 0.4285 & 0 & 0 & 0 \\ 0.1428 & 0.8571 & 0 & 0 & 0 \\ 0.2857 & 0.7142 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{104.8}$$

A element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability in state $j$, starting from state $i$. Then,

a) $Pr(A \text{ wins}) = \bar{p}_{14} \approx 0.5714$
b) $Pr(B \text{ wins}) = \bar{p}_{15} \approx 0.4285$

$$\therefore \bar{p}_{14} > \bar{p}_{15} \tag{104.9}$$

Also, in $\bar{P}$, all the terms in every row should sum to 1.

$$\Rightarrow \bar{p}_{14} + \bar{p}_{15} + 0 + 0 + 0 = 1 \tag{104.10}$$

$$\therefore \bar{p}_{14} = 1 - \bar{p}_{15} \tag{104.11}$$

Therefore, options 3), 4) are correct.

## 7 June 2016

7.1. The joint probability density function of (X,Y) is

$$f(x,y) = \begin{cases} 6(1-x) & if \quad 0 < y < x, 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.1.1}$$

Which among the following are correct?

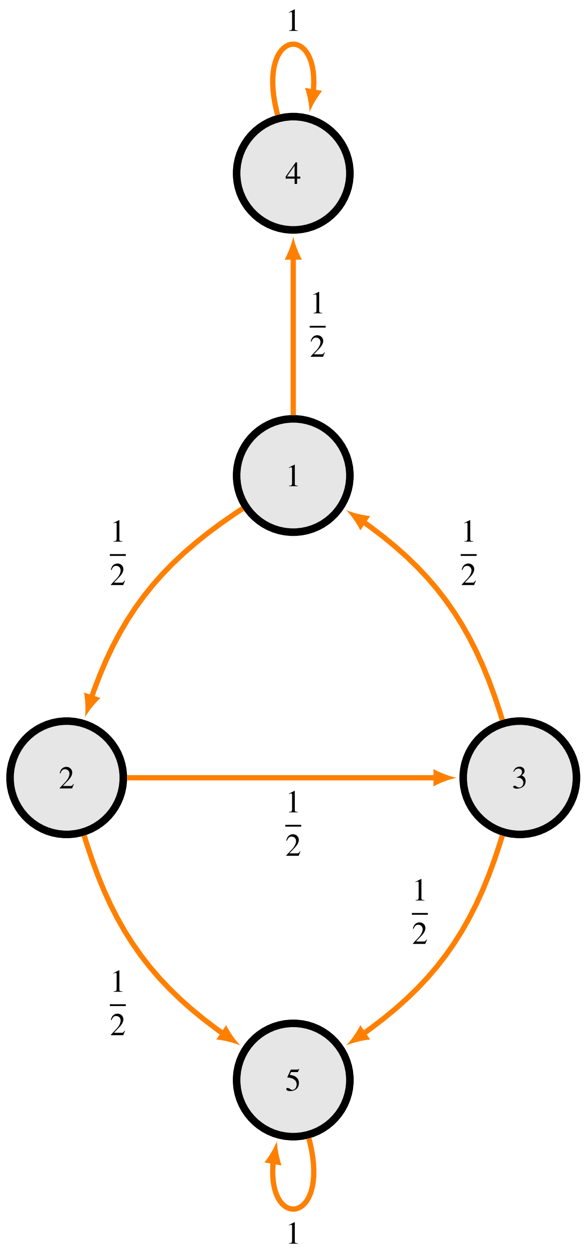
a) X and Y are not independent

b) $f_Y(y) = \begin{cases} 3\,(y-1)^2 & if \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

c) X and Y are independent

d) $f_Y(y) = \begin{cases} 3\left(y - \dfrac{1}{2}y^2\right) & if \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

**Markov chain diagram**




**Solution:** Given joint probability density function of X and Y, marginal probability density functions are as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy \tag{7.1.2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx \tag{7.1.3}$$

Calculating $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{7.1.4}$$
$$= \int_{0}^{x} 6(1-x)dy \tag{7.1.5}$$

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{7.1.6}$$

Calculating $f_Y(y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{7.1.7}$$
$$= \int_{y}^{1} 6(1-x)dx \tag{7.1.8}$$
$$=6x - 3x^2\Big|_y^1 \tag{7.1.9}$$
$$=3 - 6y + 3y^2 \tag{7.1.10}$$
$$=3(y-1)^2 \tag{7.1.11}$$

$$f_Y(y) = \begin{cases} 3(y-1)^2 & 0 < y < 1 \\ 0 & otherwise \end{cases} \tag{7.1.12}$$

To check whether X and Y are independent, we calculate $f_X(x) \times f_Y(y)$. From (7.1.6) and (7.1.12)

$$f_X(x) \times f_Y(y) = \begin{cases} 18x(1-x)(y-1)^2 & \\ & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.1.13}$$
$$\neq f(x,y) \tag{7.1.14}$$

Since f(x,y) and $f_X(x) \times f_Y(y)$ are different, random variables X and Y are not independent.

Options 1 and 2 are correct

7.2. Three types of components are used in electrical circuits 1, 2, 3 as shown below in the figure **Solution:** For $q_1$, the truth table Multiplying and adding probability for each case of $q_1$ gives us the value of

| $A$ | $B$ | $C$ | $(AB)+C$ |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

TABLE 7.2.1: Circuit 1 working

$q_1$ as

$$q_1 = p^3 - 2p^2 + 1 \tag{7.2.1}$$

For $q_2$, the truth table Multiplying and adding probability for each case of $q_2$ gives us the value of

| $A$ | $B$ | $C$ | $(A+B)C$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |

TABLE 7.2.2: Circuit 2 working

$q_2$ as

$$q_2 = p^3 - p^2 - p + 1 \tag{7.2.2}$$

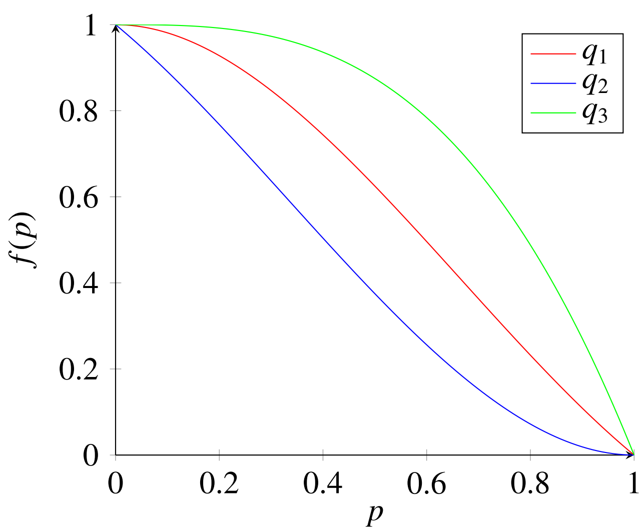
For $q_3$, the truth table Multiplying and adding probability for each case of $q_3$ gives us the value of

| $A$ | $B$ | $C$ | $A+B+C$ |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |

TABLE 7.2.3: Circuit 3 working

$q_3$ as

$$q_3 = 1 - p^3 \tag{7.2.3}$$




$$\therefore q_3 > q_1 > q_2 \tag{7.2.4}$$

Hence **Option 1** is correct

7.3. Suppose X and Y are independent and identically distributed random variables and let $Z = X + Y$. Then the distribution of $Z$ is in the same family as that of $X$ and $Y$ if X is **Solution:**

1) Normal 2) Exponential
3) Uniform 4) Binomial

1) Let X and Y be independent and identically distributed normal random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = e^{j\eta\omega-\sigma^2\omega^2/2} \tag{7.3.1}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{7.3.2}$$
$$= e^{2j\eta\omega-\sigma^2\omega^2} \tag{7.3.3}$$

Thus Z is a normal random variable with parameters $2\eta$ and $2\sigma^2$. Thus option (1) is correct.

2) Let X and Y be independent and identically distributed exponential random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = \frac{\lambda}{1 - j\omega} \tag{7.3.4}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{7.3.5}$$
$$= \frac{\lambda^2}{(1 - j\omega)^2} \tag{7.3.6}$$

Thus Z is not an exponential random variable. Therefore option (2) is wrong.

3) Let X and Y be independent and identically distributed uniform random variables such that X, Y ~ U(a,b). Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = \frac{e^{jb\omega} - e^{ja\omega}}{j\omega(b - a)} \tag{7.3.7}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{7.3.8}$$
$$= -\frac{(e^{jb\omega} - e^{ja\omega})^2}{\omega^2(b - a)^2} \tag{7.3.9}$$

Thus Z is not a uniform random variable. Thus option (3) is wrong.

4) Let X and Y be independent and identically distributed binomial random variables. Then the characteristic function of X and Y is given by

$$\Phi_X(\omega) = (pe^{j\omega} + q)^n \tag{7.3.10}$$

The characteristic function of Z is given by

$$\Phi_Z(\omega) = \Phi_X^2(\omega) \tag{7.3.11}$$
$$= (pe^{j\omega} + q)^{2n} \tag{7.3.12}$$

Thus Z is a binomial random variable with parameter 2n. Thus option (4) is correct.

The following figures show the experimental distributions for Z in each case. The simulation length was kept one million.

## 8 December 2015

8.1. The probability that a ticketless traveler is caught during a trip is 0.1. If the traveler makes 4 trips , the probability that he/she will be caught during at least one of the trips is:

a) $1-(0.9)^4$
b) $(1-0.9)^4$
c) $1-(1-0.9)^4$
d) $(0.9)^4$

**Solution:** Let $X_i \in \{0,1\}$ represent the ith trip where 1 denotes a ticketless traveller is caught. Given,

$$\Pr(X_i = 1) = p = 0.1 \tag{8.1.1}$$

Let,

$$X = \sum_{i=1}^{n} X_i \tag{8.1.2}$$

where n is the number of trips and X has a binomial distribution.

$$p_X(k) = \begin{cases} {}^nC_k\, p^K (1-p)^{n-k}, & 0 \le k \le n \\ 0, & otherwise \end{cases} \tag{8.1.3}$$

As he/she makes 4 trips in total, Using (8.1.1) and (8.1.3),

$$\Pr(X = 0) = p_X(0) \tag{8.1.4}$$
$$= {}^4C_0\, p^0 (1-p)^4 \tag{8.1.5}$$
$$\Pr(X = 0) = (0.9)^4 \tag{8.1.6}$$

Then probability of being caught in atleast one trip is,(Using (8.1.6))

$$\Pr(X \ge 1) = 1 - \Pr(X < 1) \tag{8.1.7}$$
$$= 1 - \Pr(X = 0) \tag{8.1.8}$$
$$= 1 - (0.9)^4 \tag{8.1.9}$$

8.2. Suppose that $(X, Y)$ has a joint probability distribution with the marginal distribution of $X$ being N(0,1) and $E(Y|X = x) = x^3$ for all $x \in R$. Then, which of the following statements are true?

a) $\text{Corr}(X, Y) = 0$
b) $\text{Corr}(X, Y) > 0$
c) $\text{Corr}(X, Y) < 0$
d) X and Y are independent

**Solution:** The following result shall be useful later. For $n \in N$

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \begin{cases} 0 & n \text{ is odd} \\ (n-1) \times \ldots \times 3 \times 1 & n \text{ is even} \end{cases} \tag{8.2.1}$$

The proof for the above can be found at the end of the solution.

$$Corr(X, Y) = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y} \tag{8.2.2}$$

We know $X \sim N(0, 1)$. Thus,

$$f_X(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \tag{8.2.3}$$
$$E(X) = 0 \tag{8.2.4}$$
$$\sigma_X^2 = 1 \tag{8.2.5}$$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 \tag{8.2.6}$$

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx && (8.2.7)\\ &= \int_{-\infty}^{\infty} \frac{x^3 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx && (8.2.8)\\ &= 0 && (8.2.9) \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{\infty} E(Y^2|X = x) f_X(x) dx && (8.2.10)\\ &= \int_{-\infty}^{\infty} \frac{x^6 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx && (8.2.11)\\ &= 15 && (8.2.12) \end{aligned}$$

Substituting in (8.2.6)

$$\sigma_Y^2 = 15 \tag{8.2.13}$$

$$\sigma_{XY}^2 = E(XY) - E(X)E(Y) \tag{8.2.14}$$

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} E(XY|X = x) f_X(x) dx && (8.2.15)\\ &= \int_{-\infty}^{\infty} E(xY|X = x) f_X(x) dx && (8.2.16)\\ &= \int_{-\infty}^{\infty} xE(Y|X = x) f_X(x) dx && (8.2.17)\\ &= \int_{-\infty}^{\infty} \frac{x^4 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx && (8.2.18)\\ &= 3 && (8.2.19) \end{aligned}$$

Substituting in (8.2.14)

$$\sigma_{XY}^2 = 3 \tag{8.2.20}$$

Substituting in (8.2.2)

$$Corr(X, Y) = \frac{3}{\sqrt{15}} > 0 \tag{8.2.21}$$

Since $Corr(X, Y) \neq 0$, X and Y are dependent. Thus option 2 is the only correct option. **Proof for the integral:** If n is odd, $\frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ is an odd function, thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = 0 \tag{8.2.22}$$

If n is even,

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} (x^{n-1})(\frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}) dx \tag{8.2.23}$$

Using integration by parts,

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = \left( x^{n-1} \int \frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \right) \Bigg|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} \left( \int \frac{x e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \right) dx \tag{8.2.24}$$

$$= \left( x^{n-1} (-\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}) \right) \Bigg|_{-\infty}^{\infty} - (n-1) \int_{-\infty}^{\infty} x^{n-2} (-\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}) dx \tag{8.2.25}$$

$$= (n-1) \int_{-\infty}^{\infty} \frac{x^{n-2} e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{8.2.26}$$

$$= (n-1)(n-3) \int_{-\infty}^{\infty} \frac{x^{n-4} e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{8.2.27}$$

$$= (n-1) \times ... \times 3 \times 1 \int_{-\infty}^{\infty} \frac{x^0 e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx \tag{8.2.28}$$

$$= (n-1) \times ... \times 3 \times 1 \tag{8.2.29}$$

Alternative proof for the integral:

If n is odd, $\frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ is an odd function, thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = 0 \tag{8.2.30}$$

If n is even, let $n = 2k$. We differentiate the following identity k times w.r.t. $\alpha$.

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\left(\frac{\pi}{\alpha}\right)} \tag{8.2.31}$$

On differentiating k times, we get

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2} = \frac{1 \times 3 \times ... \times (2k-1)}{2^k} \sqrt{\left(\frac{\pi}{\alpha^{2k+1}}\right)} \tag{8.2.32}$$

On substituting $\alpha = \frac{1}{2}$, we get

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} = 1 \times 3 \times ... \times (n-1) \sqrt{2\pi} \tag{8.2.33}$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^n e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx = (n-1) \times ... \times 3 \times 1 \tag{8.2.34}$$

8.3. Let $X_1, X_2, ..., X_n$ be independent and identically distributed, each having a uniform distribution on $(0, 1)$. Let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Then, which of the following statements are true?

A) $\frac{S_n}{n \log n} \to 0$ as $n \to \infty$ with probability 1.

B) $\Pr\left(\left(S_n > \frac{2n}{3}\right) \text{occurs for infinitely many n}\right) = 1$

C) $\frac{S_n}{\log n} \to 0$ as $n \to \infty$ with probability 1.

D) $\Pr\left(\left(S_n > \frac{n}{3}\right) \text{occurs for infinitely many n}\right) = 1$

**Solution:**

| Symbol | expression/definition |
|---|---|
| $S_n$ | $\sum_{i=1}^{n} X_i$ |
| $\mu_n$ | $\frac{1}{n}\sum_{i=1}^{n} X_i$ |
| $X$ | Independent continuous random variable identical to $X_1, X_2, \ldots, X_n$ |

TABLE 8.3.1: Variables and their definitions

a) Given

$$S_n = \sum_{i=1}^{n} X_i, n \geq 1 \tag{8.3.1}$$

Dividing by $n$ on both sides

$$\frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i = \mu_n \tag{8.3.2}$$

It can be said that $X_1, X_2, ..., X_n$ are the trials of $X$. By definition

$$E[X] = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = \lim_{n\to\infty} \frac{S_n}{n} \tag{8.3.3}$$

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X] = \frac{1}{2} \tag{8.3.4}$$

$$\therefore \lim_{n\to\infty} \frac{S_n}{n \log n} = 0 \tag{8.3.5}$$

b) Using weak law, (8.3.4), and table (8.3.1)

$$\lim_{n\to\infty} \Pr\left(|\mu_n - E[X]| > \epsilon\right) = 0, \forall \epsilon > 0 \tag{8.3.6}$$

$$\lim_{n\to\infty} \Pr\left(S_n = \frac{n}{2}\right) = 1 \tag{8.3.7}$$

It can be easily implied from (8.3.7) that option B is false.

c) It is easy to observe from (8.3.4) that option C is false.

d) Using (8.3.7), we get

$$\Pr\left(\left(S_n > \frac{n}{3}\right) \text{occurs for infinitely many n}\right) = 1 \tag{8.3.8}$$

8.4. A fair coin is tossed repeatedly. Let $X$ be the number of tails before the first heads occurs. Let $Y$ denote the number of tails between the first and second heads. Let $X + Y = N$. Then which of the following are true?

a) X and Y are independent random variables with

$$\Pr(X = k) = \Pr(Y = k) = \begin{cases} 2^{-(k+1)} & k = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \tag{8.4.1}$$

b) $N$ has a probability mass function given by

$$\Pr(N = k) = \begin{cases} (k-1)2^{-k} & k = 2, 3, 4 \dots \\ 0 & otherwise \end{cases} \tag{8.4.2}$$

c) Given $N = n$, the conditional distribution of X and Y are independent
d) Given $N = n$

$$\Pr(X = k) = \begin{cases} \frac{1}{n+1} & n = 0, 1, 2 \dots \\ 0 & otherwise \end{cases} \tag{8.4.3}$$

8.5. An urn has 3 red and 6 black balls. Balls are drawn at random one by one without replacement. The probability that second red ball appears on fifth draw is:

a) $\frac{1}{9!}$

b) $\frac{4!}{9!}$

c) $4\left(\frac{6!4!}{9!}\right)$

d) $\frac{6!4!}{9!}$

**Solution:** To obtain a second red ball at the fifth draw, the first 4 trials should involve drawing only 1 red ball out of the 3 and 3 black balls out of the 6. Probability of this happening:

$$\frac{{}^3C_1{}^6C_3}{{}^9C_4} \tag{8.5.1}$$

The probability of the fifth ball turning out to be red is:

$$\frac{{}^2C_1}{{}^5C_1} \tag{8.5.2}$$

By Multiplication rule, total probability:

$$\frac{{}^3C_1{}^6C_3{}^2C_1}{{}^5C_1{}^9C_4} = \frac{3! \times 6! \times 2! \times 4! \times 4! \times 5!}{2! \times 3! \times 3! \times 5! \times 9!} \tag{8.5.3}$$

$$= 4\left(\frac{4!6!}{9!}\right) \tag{8.5.4}$$

8.6. Let $X_i's$ be independent random variables such that $X_i's$ are symmetric about 0 and $var(X_i) = 2i-1$,for $i \geq 1$.then, $\lim_{n\to\infty} \Pr(X_1 + X_2 + \cdots + X_n > n \log n)$

a) does not exist.

b) equals $\frac{1}{2}$.

c) equals 1.

d) equals 0.

**Solution:** Let $X = X_1 + X_2 + \cdots + X_n$, as $X_i's$ are symmetric about 0. The mean of X is given by,

$$E[X] = 0 \tag{8.6.1}$$

the variance of X is given by,

$$var[X] = \sum_{i=1}^{n}(2i-1) \tag{8.6.2}$$
$$= \frac{2n(n+1)}{2} - n \tag{8.6.3}$$
$$= n^2 \tag{8.6.4}$$

the standard deviation,

$$\sigma_X = n \tag{8.6.5}$$

Applying Chebyshev's Inequality for the random variable X, for any $k > 0$

$$\Pr(|X - E[X]| > k\sigma_X) \leq \frac{1}{k^2} \tag{8.6.6}$$

let $k = \log n$ , using (8.6.1) and (8.6.5) in (8.6.6),

$$\Pr(|X| > n\log n) \leq \frac{1}{(\log n)^2} \tag{8.6.7}$$
$$\Pr(X > n\log n) + \Pr(X < -n\log n) \leq \frac{1}{(\log n)^2} \tag{8.6.8}$$

As, X is symmetric about 0,

$$\Pr(X > n\log n) = \Pr(X < -n\log n) \tag{8.6.9}$$

using (8.6.9) in (8.6.8),

$$2\Pr(X > n\log n) \leq \frac{1}{(\log n)^2} \tag{8.6.10}$$
$$\Pr(X > n\log n) \leq \frac{1}{2(\log n)^2} \tag{8.6.11}$$

as any probability is greater than 0,

$$0 < \Pr(X > n\log n) \leq \frac{1}{2(\log n)^2} \tag{8.6.12}$$

applying sandwich principle to (8.6.12),

$$\lim_{n\to\infty} 0 < \lim_{n\to\infty} \Pr(X > n\log n) \leq \lim_{n\to\infty} \frac{1}{2(\log n)^2} \tag{8.6.13}$$
$$\lim_{n\to\infty} \Pr(X_1 + X_2 + \cdots + X_n > n\log n) = 0 \tag{8.6.14}$$

Hence the option.4 is correct.

8.7. Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a random vector such that the marginal distribution of $X$ and the marginal distribution of $Y$ are the same and each is normally distributed with mean 0 and variance 1. Then, which of the following conditions imply independence of $X$ and $Y$?

a) $\text{Cov}(X, Y) = 0$
b) $aX + bY$ is normally distributed with mean 0 and variance $a^2 + b^2$ for all real $a$ and $b$
c) $\Pr(X \leq 0, Y \leq 0) = \frac{1}{4}$
d) $E\left[e^{itX+isY}\right] = E\left[e^{itX}\right]E\left[e^{isY}\right]$ for all real $s$ and $t$

**Solution:**

An important property of dirac delta function that will be used at multiple ocassions in this solution is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{8.7.1}$$

Given $X \sim N(0,1)$, $Y \sim N(0,1)$

a)

$$Cov(X,Y) = 0 \tag{8.7.2}$$

$$E[XY] - E[X]E[Y] = 0 \tag{8.7.3}$$

$$E[XY] = 0 \tag{8.7.4}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy = 0 \tag{8.7.5}$$

This doesn't imply independence. Counter example given below

Lets consider a case where $X$ and $Y$ are dependent based on the following relation, $Y$ being independent of $K$

$$X = KY \tag{8.7.6}$$

PMF for $K$ is given as

$$p_K(k) = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = -1 \\ 0 & \text{otherwise} \end{cases} \tag{8.7.7}$$

A simulation is given below, Y is gaussian, then X also follows gaussian Theoretically it can be proved in the following manner, Since $K$ and $Y$ are independent

$$f_X(x) = \Pr(K=1) f_Y(x) + \Pr(K=-1) f_Y(-x) \tag{8.7.8}$$

$$= \frac{1}{2}\left(f_Y(x) + f_Y(-x)\right) \tag{8.7.9}$$

$$= f_Y(x) \tag{8.7.10}$$

Therefore, $X$ follows identical but not independent distribution as $Y$, An alternative proof is given below as a proof for marginal probability

Now consider that $X$ is normally distributed, we will establish $Y$ is also normally distributed. The joint probability distribution is therefore

$$f_{XY}(x,y) = f_{X|Y}(x|y) f_X(x)$$

$$= f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y)) \tag{8.7.11}$$

The marginal probability distribution function for $X$ is given as

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y) + \delta(x-y))dy \tag{8.7.12}$$

Using (8.7.1), we get

$$\int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y)+\delta(x-y))dy = f_X(x) \tag{8.7.13}$$

We know that $X \sim N(0,1)$, $f_X(x)$ represents gaussian probability distribution function.
Futher, using symmetry of (8.7.6), we can establish that marginal distribution of $Y$ is gaussian. Here is a proof anyways

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\frac{1}{2}(\delta(x+y)+\delta(x-y))dx \tag{8.7.14}$$

Using (8.7.1), we get

$$f_Y(y) = \frac{1}{2}\left(f_X(y)+f_X(-y)\right) = f_X(y) \tag{8.7.15}$$

Since $Y$ has identical probability distribution function, $Y \sim N(0,1)$
The covariance is given as

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[XY] \tag{8.7.16}$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dydx \tag{8.7.17}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)\frac{1}{2}(\delta(x+y)+\delta(x-y))dydx \tag{8.7.18}$$

$$= \int_{-\infty}^{\infty} xf_X(x)\int_{-\infty}^{\infty} y\frac{1}{2}(\delta(x+y)+\delta(x-y))dydx \tag{8.7.19}$$

Using (8.7.1)

$$E[XY] = \int_{-\infty}^{\infty} xf_X(x)\frac{1}{2}(x-x)dx = 0 \tag{8.7.20}$$

b) Defining the following matrices/vectors Given

| **vector/matrix** | **expression** |
|---|---|
| $\mathbf{Z}$ | $\begin{pmatrix} X & Y \end{pmatrix}^\top$ |
| $\mathbf{C}$ | $\begin{pmatrix} a & b \end{pmatrix}^\top$ |
| $\boldsymbol{\mu}$ | $\begin{pmatrix} 0 & 0 \end{pmatrix}^\top$ |
| $\boldsymbol{\Sigma}$ | $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ |

TABLE 8.7.1: vectors/matrices and their expressions

$$\mathbf{C}^\top\mathbf{Z} \sim N\left(0, a^2+b^2\right) \tag{8.7.21}$$

Since this is true for all $a$ and $b$, it is equivalent to $X$ and $Y$ being jointly gaussian

$$\mathbf{Z} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{8.7.22}$$

For correlated random variables $X$ and $Y$ in bivariate normal distribution, we have

$$\sigma_Z^2 = \sum_{i,j} \Sigma_{ij} \tag{8.7.23}$$

$$a^2 + b^2 = a^2 + b^2 + 2\rho ab \tag{8.7.24}$$

$$\therefore \rho = 0 \tag{8.7.25}$$

The joint distribution is given as

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}\left(z - \boldsymbol{\mu}\right)^\top \boldsymbol{\Sigma}^{-1}\left(z - \boldsymbol{\mu}\right)\right)}{\sqrt{(2\pi)^2 \left|\boldsymbol{\Sigma}\right|}} \tag{8.7.26}$$

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}\begin{pmatrix} x & y \end{pmatrix} I_2 \begin{pmatrix} x & y \end{pmatrix}^\top\right)}{\sqrt{(2\pi)^2}} \tag{8.7.27}$$

Where $I_2$ is the identity matrix of order 2

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} x & y \end{pmatrix}^\top\right)}{\sqrt{(2\pi)^2}} \tag{8.7.28}$$

$$f_{\mathbf{Z}}(x, y) = \frac{\exp\left(-\frac{1}{2}\left(x^2 + y^2\right)\right)}{\sqrt{(2\pi)^2}} = f_X(x) f_Y(y) \tag{8.7.29}$$

$\therefore$ **Option(2) is correct**, A simulation for bivariate gaussian is given below

c)

$$\Pr\left(X \le 0, Y \le 0\right) = \frac{1}{4} \tag{8.7.30}$$

This doesn't imply independence, it can be true even for dependent $X$ and $Y$, the counter example is (8.7.11), the joint probability function is symmetric across all 4 quadrants

$$\therefore \Pr\left(X \le 0, Y \le 0\right) = \frac{1}{4} \tag{8.7.31}$$

Alternatively, here is proof

$$\Pr\left(X \le 0\right) = F_X(0) = \frac{1}{2} \tag{8.7.32}$$

Using (8.7.6)

$$\Pr\left(Y \le 0 | X \le 0\right) = \frac{1}{2} \tag{8.7.33}$$

Using (8.7.32) and (8.7.33)

$$\Pr\left(X \le 0, Y \le 0\right) = \frac{1}{4} \tag{8.7.34}$$

d)

$$E\left[e^{itX+isY}\right] = E\left[e^{itX}\right]E\left[e^{isY}\right] \tag{8.7.35}$$

$$E\left[e^{itX+isY}\right] = \varphi_X(t)\varphi_Y(s) \tag{8.7.36}$$

The inverse is given as

$$f_{XY}(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-itX-isY}E\left[e^{itX+isY}\right]dsdt \tag{8.7.37}$$

Using (8.7.36)

$$f_{XY}(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-itX-isY}\varphi_X(t)\varphi_Y(s)dsdt \tag{8.7.38}$$

$$f_{XY}(x,y) = f_X(x)f_Y(y) \tag{8.7.39}$$

$\therefore$ **Option(4) is correct**

## 9 June 2015

9.1. Assume that $X \sim Binomial(n,p)$ for some $n \geq 1$ and $0 < p < 1$ and $Y \sim poisson(\lambda)$for some $\lambda > 0$.Suppose E[X]=E[Y].Then

a) $var(X) = Var(Y)$

b) $var(X) < Var(Y)$

c) $var(Y) < Var(X)$

d) $Var(X)$ may be larger or smaller than $Var(Y)$ depending on the values of n,p and $\lambda$

**Solution:** For the random variable

$$X \sim Binomial(n,p)$$

As we know,

$$E[X] = np \tag{9.1.1}$$

$$Var(X) = np(1-p) \tag{9.1.2}$$

for the random variable

$$Y \sim poisson(\lambda)$$

As we know,

$$E[Y] = \lambda \tag{9.1.3}$$

$$Var(Y) = \lambda \tag{9.1.4}$$

given that,

$$E[X] = E[Y] \tag{9.1.5}$$

$$np = \lambda \tag{9.1.6}$$

from (9.1.2),

$$Var(X) = np(1-p) \tag{9.1.7}$$

using (9.1.6),

$$Var(X) = \lambda(1-p) \tag{9.1.8}$$

using (9.1.4),

TABLE 9.1.1: Mean and Variance for random variables X and Y

| | $X$ | $Y$ |
|---|---|---|
| E | $\lambda$ | $\lambda$ |
| $var$ | $\lambda(1-p)$ | $\lambda$ |

$$Var(X) = Var(Y)(1-p) \tag{9.1.9}$$

$$\frac{Var(X)}{Var(Y)} = 1-p \tag{9.1.10}$$

as,

$$1-p < 1 \tag{9.1.11}$$

$$\frac{Var(X)}{Var(Y)} < 1 \tag{9.1.12}$$

$$Var(X) < Var(Y) \tag{9.1.13}$$

$\therefore$ $Var(Y) > Var(X)$, independent of n,p and $\lambda$.

a) $var(X) = Var(Y)$
using TABLE 9.1.1,

$$\lambda(1-p) = \lambda \tag{9.1.14}$$

$$1-p = 1 \tag{9.1.15}$$

$$p = 0 \tag{9.1.16}$$

which is wrong as per the question($0 < p < 1$). hence the option is incorrect.

b) $var(X) < Var(Y)$
using TABLE 9.1.1,

$$\lambda(1-p) < \lambda \tag{9.1.17}$$

$$1-p < 1 \tag{9.1.18}$$

$$p > 0 \tag{9.1.19}$$

which is true as per the question($0 < p < 1$). hence the option is correct.

c) $var(Y) < Var(X)$
using TABLE 9.1.1,

$$\lambda(1-p) > \lambda \tag{9.1.20}$$

$$1-p > 1 \tag{9.1.21}$$

$$p < 0 \tag{9.1.22}$$

which is wrong as per the question($0 < p < 1$). hence the option is incorrect.

d) $Var(X)$ may be larger or smaller than $Var(Y)$ depending on the values of n,p and $\lambda$.
Wrong, since we have shown that irrespective of the values of lambda, n, and p, $var(y) > var(x)$

# 10 December 2014

10.1. $N, A_1, A_2 \cdots$ are independent real valued random variables such that

$$\Pr(N = k) = (1-p)p^k, k = 0, 1, 2, 3 \cdots \tag{10.1.1}$$

where $0 < p < 1$ and $\{A_i : i = 1, 2, \cdots\}$ is a sequence of independent and identically distributed bounded random variables. Let

$$X(w) = \begin{cases} 0 & \text{if } N(w) = 0 \\ \sum_{j=1}^{k} A_j & \text{if } N(w) = k, k = 1, 2, 3 \cdots \end{cases} \tag{10.1.2}$$

Which of the following are necessarily correct?

a) $X$ is a bounded random variable.
b) Moment generating function $m_X$ of $X$ is

$$m_X(t) = \frac{1-p}{1 - p m_A(t)}, t \in \mathbb{R}, \tag{10.1.3}$$

where $m_A$ is moment generating function of $A_1$.
c) Characteristic function $\varphi_X$ of $X$ is

$$\varphi_X(t) = \frac{1-p}{1 - p\varphi_A(t)}, t \in \mathbb{R}, \tag{10.1.4}$$

where $\varphi_A$ is the characteristic function of $A_1$.
d) $X$ is symmetric about 0.

10.2. Consider a Markov chain with state space 1,2,....,100. Suppose states 2i and 2j communicate with each other and states 2i-1 and 2j-1 communicate with each other for every i,j = 1,2,...,50. Further suppose that $p_{3,3}^{(2)}$ ¿ 0,$p_{4,4}^{(3)}$ ¿ 0 and $p_{2,5}^{(7)}$ ¿ 0. Then
a) The Markov chain is irreducible.
b) The Markov chain is aperiodic.
c) State 8 is recurrent.
d) State 9 is recurrent.

**Solution:**

10.3. Suppose $X_1$,$X_2$,$X_3$ and $X_4$ are independent and identically distributed random variables, having density function f. Then,
a) $\Pr(X_4 > Max(X_1, X_2) > X_3) = \frac{1}{6}$
b) $\Pr(X_4 > Max(X_1, X_2) > X_3) = \frac{1}{8}$
c) $\Pr(X_4 > X_3 > Max(X_1, X_2)) = \frac{1}{12}$
d) $\Pr(X_4 > X_3 > Max(X_1, X_2)) = \frac{1}{6}$

**Solution:** The probability density function (pdf) f(x) of a random variable X is defined as the derivative of the cdf F(x):

$$f(x) = \frac{d}{dx}F(x).$$

It is sometimes useful to consider the cdf F(x) in terms of the pdf f(x):

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

The PDF of X is,

$$F_X(x) = \int_{-\infty}^{\infty} f(x)dx \tag{10.3.1}$$

a) $\Pr(X_2 > X_1)$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{x} f_X(t)\ dtdx \tag{10.3.2}$$

$$= \int_{-\infty}^{\infty} f_X(x) F_X(x)dx \tag{10.3.3}$$

$$= \left.\frac{F_X^2(x)}{2}\right|_{-\infty}^{\infty} \tag{10.3.4}$$

$$= \frac{1}{2}. \tag{10.3.5}$$

b) $\Pr(X_4 > Max(X_1, X_2) > X_3)$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{x} f_X(t).^2C_1.$$

$$\left[\int_{-\infty}^{t} f_X(w)dw\right] \int_{-\infty}^{t} f_X(z)dzdtdx \tag{10.3.6}$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{x} 2f_X(t)F_X^2(t)dtdx \tag{10.3.7}$$

$$= \int_{-\infty}^{\infty} f_X(x).\frac{2}{3}F_X^3(x)dx \tag{10.3.8}$$

$$= \frac{2}{3}\left.\frac{F_X^4(x)}{4}\right|_{-\infty}^{\infty} \tag{10.3.9}$$

$$= \frac{1}{6}. \tag{10.3.10}$$

c) $\Pr\left(X_4 > X_3 > Max(X_1, X_2)\right)$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{x} f_X(t) \int_{-\infty}^{t} f_X(z).{}^2C_1.$$

$$\left[\int_{-\infty}^{t} f_X(w)dw\right] dzdtdx \tag{10.3.11}$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{x} f_X(t) \int_{-\infty}^{t} 2f_X(z)F_X(t)\ dzdtdx \tag{10.3.12}$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{x} f_X(t)F_X^2(t)dtdx \tag{10.3.13}$$

$$= \int_{-\infty}^{\infty} f_X(x).\frac{1}{3}F_X^3(x)dx \tag{10.3.14}$$

$$= \frac{1}{3}\frac{F_X^4(x)}{4}\bigg|_{-\infty}^{\infty} \tag{10.3.15}$$

$$= \frac{1}{12}. \tag{10.3.16}$$

$\therefore$ $\boxed{\textbf{Option 1,3 are correct answers.}}$

## 11 June 2013

11.1. Let $X$ be a non-negative integer valued random variable with probability mass function $f(x)$ satisfying $(x+1)f(x+1) = (\alpha + \beta x)f(x),\ x = 0, 1, 2, ...;\ \beta \neq 1$. You may assume that $E(X)$ and $Var(X)$ exist. Then which of the following statements are true?

a) $E(X) = \dfrac{\alpha}{1-\beta}$

b) $E(X) = \dfrac{\alpha^2}{(1-\beta)(1+\alpha)}$

c) $Var(X) = \dfrac{\alpha^2}{(1-\beta)^2}$

d) $Var(X) = \dfrac{\alpha}{(1-\beta)^2}$

**Solution:** For a discrete random variable $X$ with P.D.F. $f(x)$ and which can take values from a set $\mathbb{S}$,

$$E(X) = \sum_{x\in\mathbb{S}} xf(x) \tag{11.1.1}$$

And,

$$E(X^2) = \sum_{x\in\mathbb{S}} x^2 f(x) \tag{11.1.2}$$

Also, as $f(x)$ is the P.D.F.,

$$\sum_{x\in\mathbb{S}} f(x) = 1 \tag{11.1.3}$$

Given, for $x \in \mathbb{S} = \{0, 1, 2, ...n\}$,

$$(x+1)f(x+1) = (\alpha + \beta x)f(x) \tag{11.1.4}$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)f(x+1) = \sum_{x=0}^{n}(\alpha + \beta x)f(x) \tag{11.1.5}$$

Replacing $x+1$ with $x$ in L.H.S. we get,

$$\sum_{x=1}^{n+1} xf(x) = \sum_{x=0}^{n}(\alpha + \beta x)f(x) \tag{11.1.6}$$

Rewriting LHS, we get,

$$\sum_{x=0}^{n} xf(x) + (n+1)f(n+1) = \sum_{x=0}^{n}(\alpha + \beta x)f(x) \tag{11.1.7}$$

But as $x \in \{0, 1, 2...n\}$, $f(n+1) = 0$. So the equation becomes

$$\sum_{x=0}^{n} xf(x) = \alpha \sum_{x=0}^{n} f(x) + \beta \sum_{x=0}^{n} xf(x) \tag{11.1.8}$$

Using (11.1.1) and (11.1.3), we get,

$$E(X) = \alpha(1) + \beta E(X) \tag{11.1.9}$$

So,

$$E(X) = \frac{\alpha}{1-\beta} \tag{11.1.10}$$

Now in (11.1.4), multiplying both sides by $(x+1)$, we get,

$$(x+1)^2 f(x+1) = (\alpha + \beta x)(x+1)f(x) \tag{11.1.11}$$

Summing both sides for $x \in \mathbb{S}$ we get,

$$\sum_{x=0}^{n}(x+1)^2 f(x+1) = \sum_{x=0}^{n}(\alpha + \beta x)(x+1)f(x) \tag{11.1.12}$$

Replacing $x+1$ with $x$ in L.H.S. we get,

$$\sum_{x=1}^{n+1} x^2 f(x) = \sum_{x=0}^{n}(\beta x^2 f(x) + (\alpha + \beta)xf(x) + \alpha f(x)) \tag{11.1.13}$$

Rewriting LHS similarly as before, we get,

$$\begin{aligned}\sum_{x=0}^{n} x^2 f(x) = \beta \sum_{x=0}^{n} x^2 f(x) + \\ (\alpha + \beta)\sum_{x=0}^{n} xf(x) + \alpha \sum_{x=0}^{n} f(x)\end{aligned} \tag{11.1.14}$$

Using (11.1.1), (11.1.2) and (11.1.3), we get,

$$E(X^2) = \beta E(X^2) + (\alpha + \beta)E(X) + \alpha(1) \tag{11.1.15}$$

Using (11.1.10)

$$E(X^2)(1-\beta) = \frac{\alpha(\alpha+\beta)}{1-\beta} + \alpha \tag{11.1.16}$$

So,

$$E(X^2) = \frac{\alpha^2+\alpha}{(1-\beta)^2} \tag{11.1.17}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{11.1.18}$$

Using (11.1.10) and (11.1.17),

$$Var(X) = \frac{\alpha^2+\alpha}{(1-\beta)^2} - \frac{\alpha^2}{(1-\beta)^2} \tag{11.1.19}$$

So,

$$Var(X) = \frac{\alpha}{(1-\beta)^2} \tag{11.1.20}$$

So, options 1 and 4 are correct.

11.2. Let X be a random variable with probability density function,

$$f(x) = \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^\alpha} \tag{11.2.1}$$

such that $-\infty < \mu < \infty$ ; $\alpha > 0$ ; $x > \mu$, The hazard function is:

a) constant for all $\alpha$
b) an increasing function for some $\alpha$
c) independent of $\alpha$
d) independent of $\mu$ when $\alpha = 1$

**Solution:** Given PDF of X,

$$f(x) = \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^\alpha} \tag{11.2.2}$$

**Important property**(using in (11.2.8) as $x > \mu$): Given $x - y > 0$ and $-\infty < y < \infty$, then

$$\lim_{x\to-\infty} x - y = 0 \tag{11.2.3}$$

CDF of X,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx \tag{11.2.4}$$

$$= \int_{-\infty}^{x} \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^\alpha}dx \tag{11.2.5}$$

$$= \int_{-\infty}^{x} e^{-(x-\mu)^\alpha}d(x-\mu)^\alpha \tag{11.2.6}$$

$$= \left[\frac{e^{-(x-\mu)^\alpha}}{-1}\right]_{-\infty}^{x} \tag{11.2.7}$$

$$= -e^{-(x-\mu)^\alpha} - \lim_{x\to-\infty}\frac{e^{-(x-\mu)^\alpha}}{-1} \tag{11.2.8}$$

$$= -e^{-(x-\mu)^\alpha} + e^{-(0)^\alpha} \tag{11.2.9}$$

$$F(x) = 1 - e^{-(x-\mu)^\alpha} \tag{11.2.10}$$

Hazard function $\beta(x)$,(using (11.2.2) and (11.2.10))

$$\beta(x) = \frac{f(x)}{1 - F(x)} \tag{11.2.11}$$

$$= \frac{\alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^{\alpha}}}{1-(1-e^{-(x-\mu)^{\alpha}})} \tag{11.2.12}$$

$$= \frac{\alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^{\alpha}}}{e^{-(x-\mu)^{\alpha}}} \tag{11.2.13}$$

$$\beta(x) = \alpha(x-\mu)^{\alpha-1} \tag{11.2.14}$$

a) $\beta(x)$ is not constant for all $\alpha$
b) $\beta(x) = \alpha(x-\mu)^{\alpha-1}$ is an increasing function for $\alpha < 0$ *or* $\alpha > 1$ as given $x - \mu > 0$ for all x.
**Proof:** Using first derivative test, A function is increasing iff its first derivative is positive for all x.

$$\frac{d}{dx}\beta(x) = \frac{d}{dx}\alpha(x-\mu)^{\alpha-1} \tag{11.2.15}$$

$$= \alpha(\alpha-1)(x-\mu)^{\alpha-2} \tag{11.2.16}$$

For (11.2.16) to be positive, (As given $x - \mu > 0$)

$$\alpha(\alpha-1)(x-\mu)^{\alpha-2} > 0 \tag{11.2.17}$$

$$\alpha(\alpha-1) > 0 \tag{11.2.18}$$

$$\implies \alpha \in (-\infty, 0) \cup (1, \infty) \tag{11.2.19}$$

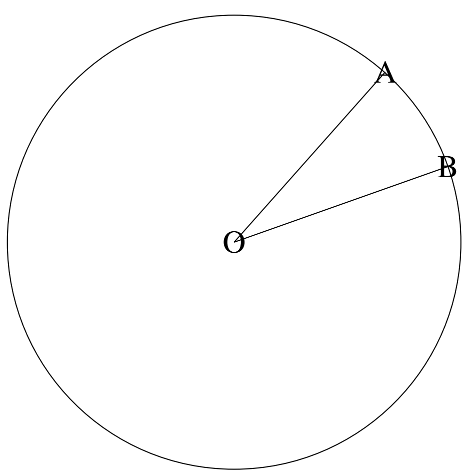
$\therefore \beta(x)$ an increasing function for some $\alpha$
c) $\beta(x)$ is dependent of $\alpha$
d) when $\alpha = 1$,

$$\beta(x) = \alpha(x-\mu)^{0} = \alpha \tag{11.2.20}$$

Therefore the hazard function is independent of $\mu$ when $\alpha = 1$.

**ANSWER: (2) and (4)**

11.3. A point is chosen at random from a circular disc shown below. What is the probability that the point lies in the sector OAB?




( where $\angle$AOB = x radians )

a) $\frac{2x}{\pi}$

b) $\frac{x}{\pi}$
c) $\frac{x}{2\pi}$
d) $\frac{x}{4\pi}$

**Solution:**

Let $X \in \{0, 1\}$ be a random variable such that X=0 means we choose a point lying in sector OAB and X=1 means that we choose a point lying outside sector OAB and inside the circle.

Area of a sector subtending an angle $\theta$ at the centre of circle with radius a is given by :

$$A = \frac{1}{2}a^2\theta \tag{11.3.1}$$

where $\theta$ is in radians.

Let the radius of circle shown in figure be r. It is given that sector OAB subtends an angle of x radians at the centre of the circle.

Probability that the chosen point lies in sector OAB is:

$$\Pr(X = 0) = \frac{\text{Area of sector OAB}}{\text{Area of circle}} \tag{11.3.2}$$

$$= \frac{\frac{1}{2}r^2x}{\pi r^2} \tag{11.3.3}$$

$$= \frac{x}{2\pi} \tag{11.3.4}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$.

**ALTERNATE SOLUTION**

The joint pdf is given by:

$$f_{r\theta}(r, \theta) = \begin{cases} \frac{r}{\pi R^2} & \text{if } 0 < \text{r} < \text{R} \text{ , } 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{11.3.5}$$

Let A $\equiv$ (R, $\theta_2$) and B $\equiv$ (R, $\theta_1$).

Hence,

$$(\theta_2 - \theta_1) = x \tag{11.3.6}$$

We want $\theta \in (\theta_1 , \theta_2)$ and r $\in$ (0,R) for point to lie in the sector. Let the point to be chosen be (r, $\theta$). So, Required probability is:

$$\Pr(\theta_1 < \theta < \theta_2, 0 < r < R)$$

$$= \int_{\theta_1}^{\theta_2} \int_{0}^{R} \frac{r}{\pi R^2}\, dr\, d\theta \tag{11.3.7}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{\pi R^2} \frac{r^2}{2} \bigg|_0^R \tag{11.3.8}$$

$$= \int_{\theta_1}^{\theta_2} \frac{R^2}{2\pi R^2} \, d\theta \tag{11.3.9}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2\pi} \, d\theta \tag{11.3.10}$$

$$= \frac{\theta}{2\pi} \bigg|_{\theta_1}^{\theta_2} \tag{11.3.11}$$

$$= \frac{\theta_2 - \theta_1}{2\pi} \tag{11.3.12}$$

$$= \frac{x}{2\pi} \tag{11.3.13}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$.

11.4. Let $X$ and $Y$ be independent random variables each following a uniform distribution on $(0, 1)$.Let $W = XI_{\{Y \le X^2\}}$,where $I_A$ denotes the indicator function of set $A$.Then which of the following statements are true?

a) The cumulative distribution function of $W$ is given by

$$F_W(t) = t^2 I_{\{0 \le t \le 1\}} + I_{\{t > 1\}} \tag{11.4.1}$$

b) $P[W > 0] = \frac{1}{3}$

c) The cumulative distribution function of $W$ is continuous

d) The cumulative distribution function of $W$ is given by

$$F_W(t) = \left(\frac{2 + t^3}{3}\right) I_{\{0 \le t \le 1\}} + I_{\{t > 1\}} \tag{11.4.2}$$

**Solution:**

Given $X$ and $Y$ are two independent random variables.

Given $W = XI_{\{Y \le X^2\}}$

$X \in (0, 1)$ , $Y \in (0, 1)$ , $W \in [0, 1)$

a) We need to find CDF of $W$

i) The PDF for $X$ is

$$p_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{11.4.3}$$

ii) The CDF for $X$ is

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & otherwise \end{cases} \tag{11.4.4}$$

iii) The PDF for $Y$ is

$$p_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & otherwise \end{cases} \tag{11.4.5}$$

iv) The CDF for $Y$ is

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & otherwise \end{cases} \tag{11.4.6}$$

v) $I_{\{Y \leq X^2\}}$ is defined as follows

$$I_{\{Y \leq X^2\}} = \begin{cases} 1 & y \leq x^2 \\ 0 & otherwise \end{cases} \tag{11.4.7}$$

vi) $W$ is defined as follows

$$W = \begin{cases} x & y \leq x^2 \\ 0 & otherwise \end{cases} \tag{11.4.8}$$

From (11.4.8)

$$p_W(W = 0) = \Pr(I_{\{Y \leq X^2\}} = 0) \tag{11.4.9}$$

$$= \Pr(x^2 < y) \tag{11.4.10}$$

vii) Let $Z = X^2 - Y$ be a random variable where $Z \in (-1, 1)$

$$F_{X^2}(u) = \Pr(X^2 \le u) \tag{11.4.11}$$
$$= \Pr(X \le \sqrt{u}) \tag{11.4.12}$$
$$= F_X(\sqrt{u}) \tag{11.4.13}$$

A) From (11.4.4),The CDF for $X^2$ is

$$F_{X^2}(u) = \begin{cases} 0 & u \le 0 \\ \sqrt{u} & 0 < u < 1 \\ 1 & otherwise \end{cases} \tag{11.4.14}$$

B) The PDF for $X^2$ is

$$p_{X^2}(u) = \begin{cases} \frac{1}{2\sqrt{u}} & 0 < u < 1 \\ 0 & otherwise \end{cases} \tag{11.4.15}$$

$$F_{\{-Y\}}(v) = \Pr(-Y \le v) \tag{11.4.16}$$
$$= \Pr(Y \ge -v) \tag{11.4.17}$$
$$= 1 - F_Y(-v) \tag{11.4.18}$$

C) From (11.4.6),The CDF for $(-Y)$ is

$$F_{\{-Y\}}(v) = \begin{cases} 0 & v \le -1 \\ 1 + v & -1 < v < 0 \\ 1 & otherwise \end{cases} \tag{11.4.19}$$

D) The PDF for $(-Y)$ is

$$p_{\{-Y\}}(v) = \begin{cases} 1 & -1 < v < 0 \\ 0 & otherwise \end{cases} \tag{11.4.20}$$

E) $Z = X^2 - Y \implies z = u + v$

Using convolution

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X^2}(z - v) p_{\{-Y\}}(v) \mathrm{dv} \tag{11.4.21}$$

Solving (11.4.21) using (11.4.20),(11.4.15) for $z \in (-1, 1)$, we get

PDF of $Z$ as follows

$$p_Z(z) = \begin{cases} \sqrt{z+1} & -1 < z \le 0 \\ 1 - \sqrt{z} & 0 < z < 1 \\ 0 & otherwise \end{cases} \quad (11.4.22)$$

F) CDF of $Z$ as follows

$$F_Z(z) = \begin{cases} \frac{2}{3}(z+1)^{\frac{3}{2}} & -1 < z \le 0 \\ z - \frac{2}{3}z^{\frac{3}{2}} & 0 < z < 1 \\ 1 & otherwise \end{cases} \quad (11.4.23)$$

viii) using (11.4.23) to find $p_W(W = 0)$

$$p_W(W = 0) = \Pr(x^2 < y) \quad (11.4.24)$$

$$= F_z(0) \quad (11.4.25)$$

$$= \frac{2}{3} \quad (11.4.26)$$

ix) $W = t \implies X = t$ where $t \in (0, 1)$

$$p_W(t) = \int_{-\infty}^{\infty} p_X(t) I_{\{y \le t^2\}} \text{dy} \quad (11.4.27)$$

$$0 < y < 1 \quad (11.4.28)$$

$$0 < y \le t^2 \quad (11.4.29)$$

For $0 < t < 1$,

$$p_W(t) = \int_0^{t^2} p_X(t) I_{\{y \le t^2\}} \text{dy} \quad (11.4.30)$$

$$= t^2 \quad (11.4.31)$$

x) $\therefore$ PDF of $W$ is as follows

$$p_W(t) = \begin{cases} \frac{2}{3} & t = 0 \\ t^2 & 0 < t < 1 \\ 0 & otherwise \end{cases} \quad (11.4.32)$$

xi) The CDF of $W$ is as follows:

$$F_W(t) = \begin{cases} 0 & t < 0 \\ \frac{2+t^3}{3} & 0 \le t \le 1 \\ 1 & otherwise \end{cases} \tag{11.4.33}$$

b) We need to find $P[W > 0]$

$$\Pr(W > 0) = 1 - F_W(0) \tag{11.4.34}$$

$$= \frac{1}{3} \tag{11.4.35}$$

$$\therefore \Pr(W > 0) = \frac{1}{3} \tag{11.4.36}$$

c) CDF of $W$ is discontinuous at $W = 0$.
$\therefore$ option 3 is incorrect.

d) The CDF in (11.4.33) can be written as

$$F_W(t) = \left(\frac{2+t^3}{3}\right) I_{\{0 \le t \le 1\}} + I_{\{t>1\}} \tag{11.4.37}$$

$\therefore$ option 2 and 4 are correct.

11.5. Let $U_1, U_2 \ldots, U_n$ be independent and identically distributed random variables each having a uniform distribution on (0,1). Then,$\lim_{n\to+\infty} \Pr\left(U_1 + U_2 \ldots, U_n \le \frac{3}{4}n\right)$

a) does not exist
b) exists and equals 0
c) exists and equals 1
d) exists and equals $\frac{3}{4}$

**Solution:** We use Weak law for large numbers to solve this problem. Let the collection of identically distributed random variables $U_1, U_2 \ldots, U_n$ have a finite mean $\mu$ and finite variance $\sigma^2$.

$$\mu = E[U_i] \quad \text{for i} \in (1,2,3\ldots,\text{n}) \tag{11.5.1}$$

Since the distribution is uniform on (0,1), $\mu = 0.5$. Let $M_n$ be the sample mean

$$M_n = \frac{U_1 + U_2 + U_3 \cdots + U_n}{n} \tag{11.5.2}$$

Expected value of $M_n$ (using (11.5.2) and (11.5.1))is

$$E[M_n] = \frac{E[U_1 + U_2 + U_3 + \cdots + U_n]}{E[n]} \tag{11.5.3}$$

$$= \frac{E[U_1] + E[U_2] + \cdots + E[U_n]}{n} \tag{11.5.4}$$

$$= \frac{n \times \mu}{n} \tag{11.5.5}$$

$$= \mu \tag{11.5.6}$$

Variance of M

$$Var(M_n) = \frac{Var(U_1 + U_2 + U_3 \cdots + U_n)}{n^2} \tag{11.5.7}$$

$$= \frac{Var(U_1) + Var(U_2) \cdots + Var(U_n)}{n^2} \tag{11.5.8}$$

$$= \frac{n \times \sigma^2}{n^2} \tag{11.5.9}$$

$$= \frac{\sigma^2}{n} \tag{11.5.10}$$

From Chebyshev inequality, for any $\epsilon > 0$

$$\Pr(|M_n - \mu| \geq \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} \tag{11.5.11}$$

From (11.5.1) and (11.5.10)

$$\Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n \times \epsilon^2}$$

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n \times \epsilon^2} \leq 0 \text{ for fixed } \epsilon > 0 \tag{11.5.12}$$

But since Probabilities are always non-negative,

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \geq \epsilon\right) \to 0 \tag{11.5.13}$$

This is known as the weak law of large numbers
The inverse of (11.5.13) is also true

$$\lim_{n\to\infty} \Pr\left(\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \leq \epsilon\right) \to 1 \tag{11.5.14}$$

$$\left|\frac{U_1 + U_2 \cdots + U_n}{n} - \mu\right| \leq \epsilon \text{ as n} \to \infty \tag{11.5.15}$$

From $\epsilon$, n definition of limits, it is clear that

$$\frac{U_1 + U_2 \cdots + U_n}{n} \to \mu \tag{11.5.16}$$

$$U_1 + U_2 \ldots U_n \to n \times \mu \text{ as n} \to \infty \tag{11.5.17}$$

Since $\mu = \frac{1}{2}$,

$$\lim_{n\to+\infty} U_1 + U_2 \ldots U_n = \frac{1}{2}n < \frac{3}{4}n \tag{11.5.18}$$

So

$$\lim_{n\to+\infty} \Pr\left(U_1 + U_2 \ldots, U_n \leq \frac{3}{4}n\right) = 1 \tag{11.5.19}$$

11.6. Consider the quadratic equation $x^2 + 2Ux + V = 0$ where $U$ and $V$ are chosen independently and randomly from $\{1, 2, 3\}$ with equal probabilities. Then probability that the equation has both roots real

a) $\frac{2}{3}$
b) $\frac{1}{2}$
c) $\frac{7}{9}$
d) $\frac{1}{3}$

**Solution:** Let $U \in \{1.2, 3\}$ and $V \in \{1, 2, 3\}$ For $x^2 + 2Ux + V = 0$ to

TABLE 11.6.1: Probability of selecting values for $U$

| $k$ | 1 | 2 | 3 |
|---|---|---|---|
| $\Pr(U = k)$ | $1/3$ | $1/3$ | $1/3$ |

TABLE 11.6.2: Probability of selecting values for $V$

| $k$ | 1 | 2 | 3 |
|---|---|---|---|
| $\Pr(V = k)$ | $1/3$ | $1/3$ | $1/3$ |

have real roots,

$$b^2 - 4ac \geq 0 \tag{11.6.1}$$

$$(2U)^2 - 4(1)(V) \geq 0 \tag{11.6.2}$$

$$U^2 \geq V \tag{11.6.3}$$

$$\Pr\left(U^2 \geq V\right) = 1 - \Pr\left(U^2 < V\right) \tag{11.6.4}$$

The possible pairs $(U, V)$ for $\Pr\left(U^2 < V\right)$,

TABLE 11.6.3: Table for $\Pr\left(U^2 < V\right)$

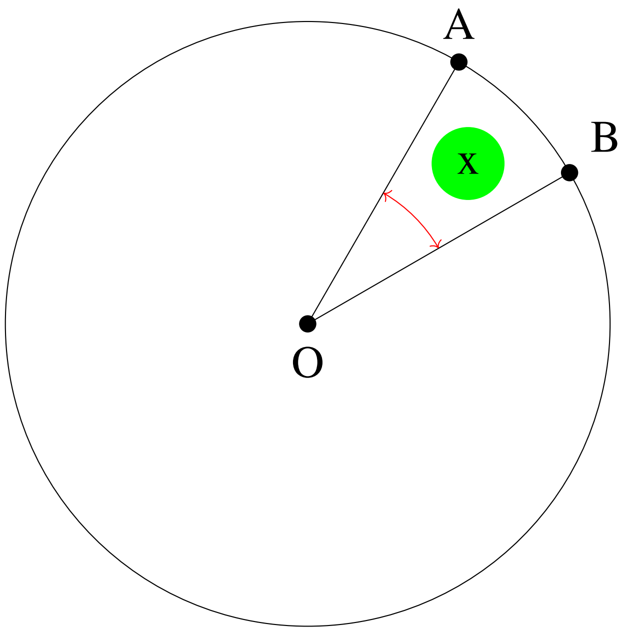
| $(U, V)$ for $U^2 < V$ | Probability |
|---|---|
| $(1, 2)$ | $\Pr(U = 1)\Pr(V = 2)=1/9$ |
| $(1, 3)$ | $\Pr(U = 1)\Pr(V = 3) = \frac{1}{9}$ |
| Total | $\Pr\left(U^2 < V\right) = \frac{2}{9}$ |

$$\Pr\left(U^2 \geq V\right) = 1 - \frac{2}{9} \tag{11.6.5}$$

$$\Pr\left(U^2 \geq V\right) = \frac{7}{9} \tag{11.6.6}$$

Hence, Option 3 is correct.

11.7. A point is chosen at random from a circular disc shown below. What is the probability that the point lies in the sector OAB?




( where ∠AOB = x radians )

a) $\frac{2x}{\pi}$

b) $\frac{x}{\pi}$

c) $\frac{x}{2\pi}$

d) $\frac{x}{4\pi}$

**Solution:**

Let $X \in \{0, 1\}$ be a random variable such that X=0 means we choose a point lying in sector OAB and X=1 means that we choose a point lying outside sector OAB and inside the circle.

Area of a sector subtending an angle $\theta$ at the centre of circle with radius a is given by :

$$A = \frac{1}{2}a^2\theta \tag{11.7.1}$$

where $\theta$ is in radians.

Let the radius of circle shown in figure be r. It is given that sector OAB subtends an angle of x radians at the centre of the circle.

Probability that the chosen point lies in sector OAB is:

$$\Pr(X = 0) = \frac{\text{Area of sector OAB}}{\text{Area of circle}} \tag{11.7.2}$$

$$= \frac{\frac{1}{2}r^2 x}{\pi r^2} \tag{11.7.3}$$

$$= \frac{x}{2\pi} \tag{11.7.4}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$. **alternate solution** The joint pdf is given by:

$$f_{\mathrm{r}\theta}(r, \theta) = \begin{cases} \frac{r}{\pi R^2} & \text{if } 0 < \mathrm{r} < \mathrm{R} \text{ , } 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{11.7.5}$$

Let A $\equiv$ (R, $\theta_2$) and B $\equiv$ (R, $\theta_1$).
Hence,

$$(\theta_2 - \theta_1) = x \tag{11.7.6}$$

We want $\theta \in (\theta_1 \text{ , } \theta_2)$ and r $\in$ (0,R) for point to lie in the sector. Let the point to be chosen be (r, $\theta$).
So, Required probability is:

$$\Pr(\theta_1 < \theta < \theta_2, 0 < r < R)$$

$$= \int_{\theta_1}^{\theta_2} \int_0^R \frac{r}{\pi R^2} \, dr \, d\theta \tag{11.7.7}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{\pi R^2} \frac{r^2}{2}\bigg|_0^R \tag{11.7.8}$$

$$= \int_{\theta_1}^{\theta_2} \frac{R^2}{2\pi R^2}\, d\theta \tag{11.7.9}$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2\pi}\, d\theta \tag{11.7.10}$$

$$= \frac{\theta}{2\pi}\bigg|_{\theta_1}^{\theta_2} \tag{11.7.11}$$

$$= \frac{\theta_2 - \theta_1}{2\pi} \tag{11.7.12}$$

$$= \frac{x}{2\pi} \tag{11.7.13}$$

$\therefore$The correct answer is **option (3)** $\frac{x}{2\pi}$.

11.8. Consider a parallel system with two components. The lifetimes of the two components are independent and identically distributed random variables each following an exponential distribution with mean 1. The expected lifetime of the system is:

A) 1

B) $\frac{1}{2}$

C) $\frac{3}{2}$

D) 2

**Solution:**

Consider two random variables X and Y which represent the lifetime of the two components namely A and B.

$$X \sim Exp(\lambda_X) \tag{11.8.1}$$

$$Y \sim Exp(\lambda_Y) \tag{11.8.2}$$

Let $f_X(x)$ denote the probability distribution function for random variable X.

$$f_X(x) = \begin{cases} \lambda_X e^{-\lambda_X x} & x \geq 0 \\ 0 & otherwise \end{cases} \tag{11.8.3}$$

Let $f_Y(y)$ denote the probability distribution function for random variable Y.

$$f_Y(y) = \begin{cases} \lambda_Y e^{-\lambda_Y y} & y \geq 0 \\ 0 & otherwise \end{cases} \tag{11.8.4}$$

Let $F_X(x)$ denote the cumulative distribution function for random variable X.

$$F_X(x) = \begin{cases} 1 - e^{-\lambda_X x} & x \geq 0 \\ 0 & otherwise \end{cases} \tag{11.8.5}$$

Let $F_Y(y)$ denote the cumulative distribution function for random variable Y.

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda_Y y} & y \geq 0 \\ 0 & otherwise \end{cases} \tag{11.8.6}$$

$$E(X) = \frac{1}{\lambda_X} \tag{11.8.7}$$

$$E(Y) = \frac{1}{\lambda_Y} \tag{11.8.8}$$

From 11.8.7 and 11.8.8,

$$\lambda_X = \lambda_Y = 1 \tag{11.8.9}$$

Let Z be a random variable such that $Z = max(X, Y)$

$$\begin{aligned} P(Z \leq z) &= P(max(X, Y) \leq z) && (11.8.10) \\ &= P(X \leq z, Y \leq z) && (11.8.11) \\ &= P(X \leq z)P(Y \leq z) && (11.8.12) \\ &= (F_X(z) - F_X(0))(F_Y(z) - F_Y(0)) && (11.8.13) \\ &= 1 - e^{-(\lambda_X)z} - e^{-(\lambda_Y)z} + e^{-(\lambda_X+\lambda_Y)z} && (11.8.14) \end{aligned}$$

$P(Z \leq z)$ denotes the probability that the system dies in the first $z$ seconds.

$$Expectation = \int_0^\infty z\, d(P(Z \le z)) \tag{11.8.15}$$

$$= \int_0^\infty z(\lambda_X e^{-(\lambda_X)z} + \lambda_Y e^{-(\lambda_Y)z}$$

$$- (\lambda_X + \lambda_Y)e^{-(\lambda_X+\lambda_Y)z})\, dz \tag{11.8.16}$$

$$= \frac{1}{\lambda_X} + \frac{1}{\lambda_Y} - \frac{1}{\lambda_X + \lambda_Y} \tag{11.8.17}$$

From 11.8.9,

$$Expectation = \frac{3}{2} \tag{11.8.18}$$

Therefore, option C correct.

11.9. Let $X_1$,$X_2$,... be independent random variables each following exponential distribution with mean 1. Then which of the following statements are correct?

a) P($X_n > \log n$ for infinitely many $n \ge 1$) = 1
b) P($X_n > 2$ for infinitely many $n \ge 1$) = 1
c) P($X_n > \frac{1}{2}$ for infinitely many $n \ge 1$) = 0
d) P($X_n > \log n, X_{n+1} > \log(n+1)$ for infinitely many $n \ge 1$) = 0

**Solution:** PDF of $X_i$ is

$$f_{X_i}(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Mean of $X_i$ is expressed as

$$E[X_i] = \int_{-\infty}^{\infty} x f_{X_i}(x) dx$$

$$= \int_{-\infty}^{0} 0 dx + \int_{0}^{\infty} x\lambda_i e^{-\lambda_i x}$$

$$= \frac{1}{\lambda_i} \tag{11.9.1}$$

From (11.9.1)and $E[X_i] = 1$, we have $\lambda_i = 1 \forall i \ge 1$ Now, for some constant $c \geq 0$

$$\begin{aligned}
\Pr(X_n > c) &= \int_c^{\infty} f_{X_n}(x)dx \\
&= \int_c^{\infty} e^{-x}dx \\
&= e^{-c}
\end{aligned} \tag{11.9.2}$$

**Borel-Cantelli Lemma**:

Let $E_1$,$E_2$,... be a sequence of events in some probability space. The Borel–Cantelli lemma states that, if the sum of the probabilities of the events $E_n$ is finite

$$\sum_{n=1}^{\infty} \Pr(E_n) < \infty \tag{11.9.3}$$

then the probability that infinitely many of them occur is 0

$$\Pr\left(\lim_{n\to\infty} \sup E_n\right) = 0 \tag{11.9.4}$$

**Second Borel-Cantelli Lemma**:

If the events $E_n$ are independent and the sum of the probabilities of the $E_n$ diverges to infinity, then the probability that infinitely many of them occur is 1. If for independent events $E_1, E_2, ...$

$$\sum_{n=1}^{\infty} \Pr(E_n) = \infty \tag{11.9.5}$$

Then

$$\Pr\left(\lim_{n\to\infty} \sup E_n\right) = 1 \tag{11.9.6}$$

a) **Option 1:** We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.

From (11.9.2)

$$\begin{aligned}\sum_{n=1}^{\infty} \Pr(X_n > \log n) &= \sum_{n=1}^{\infty} e^{-\log n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty \text{ (Cauchy's Criterion)}\end{aligned}$$

Now, from second Borel-Cantelli lemma

$$\begin{aligned}&\Pr(X_n > \log n \text{ for infinitely many } n \geq 1) \\ &= \Pr\left(\lim_{n\to\infty} \sup X_n > \log n\right) \\ &= 1\end{aligned}$$

$\therefore$ Option 1 is correct.

b) **Option 2:** We can say the events $X_n > 2$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.
From (11.9.2)

$$\begin{aligned}\sum_{n=1}^{\infty} \Pr(X_n > 2) &= \sum_{n=1}^{\infty} e^{-2} \\ &= \infty\end{aligned}$$

Now, from second Borel-Cantelli lemma

$$\begin{aligned}&\Pr(X_n > 2 \text{ for infinitely many } n \geq 1) \\ &= \Pr\left(\lim_{n\to\infty} \sup X_n > 2\right) \\ &= 1\end{aligned}$$

$\therefore$ Option 2 is correct.

c) **Option 3:** We can say the events $X_n > \frac{1}{2}$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.

From (11.9.2)

$$\sum_{n=1}^{\infty} \Pr\left(X_n > \frac{1}{2}\right) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}}$$
$$= \infty$$

Now, from second Borel-Cantelli lemma

$$\Pr\left(X_n > \frac{1}{2} \text{ for infinitely many } n \geq 1\right)$$
$$= \Pr\left(\lim_{n\to\infty} \sup X_n > \frac{1}{2}\right)$$
$$= 1$$

$\therefore$ Option 3 is incorrect.

d) **Option 4:** We can say the events $X_n > \log n$ are independent $\forall n \geq 1$ as $X_n$ are independent random variable.
Let the event $X_n > \log n, X_{n+1} > \log(n+1)$ be represented by $E_n$'
From (11.9.2)

$$\sum_{n=1}^{\infty} \Pr(E_n)$$
$$= \sum_{n=1}^{\infty} \Pr(X_n > \log n) \Pr(X_{n+1} > \log(n+1))$$
$$= \sum_{n=1}^{\infty} e^{-\log n} e^{-\log(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$
$$= 1 \tag{11.9.7}$$

Now, from Borel-Cantelli lemma

$$\begin{aligned}
&\Pr\left(E_n \text{ for infinitely many } n \geq 1\right) \\
&= \Pr\left(\lim_{n\to\infty} \sup(X_n > \log n, X_{n+1} > \log(n+1))\right) \\
&= 0
\end{aligned}$$

$\therefore$ Option 4 is correct.

$$\boxed{\textbf{Solution: } \text{Options 1, 2, 4}}$$

11.10. Suppose $X_1$ and $X_2$ are independent and identically distributed random variables each following an exponential distribution with mean $\theta$, i.e., the common pdf is given by $f_\theta(x) = \frac{1}{\theta} e^{\frac{-x}{\theta}}, 0 < x < \infty, 0 < \theta < \infty$. Then which of the following is true? Conditional distribution of $X_2$ given $X_1 + X_2 = t$ is

a) exponential with mean $\frac{t}{2}$ and hence $X_1 + X_2$ is sufficient for $\theta$
b) exponential with mean $\frac{t\theta}{2}$ and hence $X_1 + X_2$ is not sufficient for $\theta$
c) uniform$(0, t)$ and hence $X_1 + X_2$ is sufficient for $\theta$
d) uniform$(0, t\theta)$ and hence $X_1 + X_2$ is not sufficient for $\theta$

**Solution:** Let $f_{X_1,X_2}(x_1, x_2)$ denote the joint probability distribution of random variables $X_1$ and $X_2$. Let $Z$ be another random variable such that $Z = X_1 + X_2$. Let $\Phi_{X_1}(\omega)$ and $\Phi_Z(\omega)$ be the characteristic functions of the probability density functions $f_{X_1}(x)$ and $f_Z(x)$ respectively. The conditional probability density function of $X_2$ can be defined by:

$$f_{X_2|(X_1+X_2=t)}(x_2) = \begin{cases} \frac{f_{X_1,X_2}(x_1,x_2)}{f_{(X_1+X_2)}(t)} & \text{if } x_1 + x_2 = t \\ 0 & \text{otherwise} \end{cases} \tag{11.10.1}$$

$$x_1 + x_2 = t \tag{11.10.2}$$

$$0 < x_1, x_2 < \infty \tag{11.10.3}$$

$$x_1 = t - x_2 \tag{11.10.4}$$

$$t - x_2 > 0 \tag{11.10.5}$$

$$x_2 < t \tag{11.10.6}$$

From equations (11.10.3) and (11.10.6), we can conclude that $x_2 \in (0, t)$

if $x_1 + x_2 = t$. Also, given in the question,

$$0 < \theta < \infty \tag{11.10.7}$$

$$f_{X_1}(x_1) = \frac{1}{\theta} e^{\frac{-x_1}{\theta}}, 0 < x_1 < \infty \tag{11.10.8}$$

$$f_{X_2}(x_2) = \frac{1}{\theta} e^{\frac{-x_2}{\theta}}, 0 < x_2 < \infty \tag{11.10.9}$$

Since $X_1$ and $X_2$ are independent,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) \times f_{X_2}(x_2) \tag{11.10.10}$$

$$= \frac{1}{\theta} e^{\frac{-x_1}{\theta}} \times \frac{1}{\theta} e^{\frac{-x_2}{\theta}} \tag{11.10.11}$$

$$= \frac{1}{\theta^2} e^{\frac{-(x_1+x_2)}{\theta}} \tag{11.10.12}$$

$$\Phi_{X_1}(\omega) = \frac{1}{\theta} \int_0^\infty e^{i\omega x} e^{\frac{-x}{\theta}}\, dx \tag{11.10.13}$$

$$= \frac{1}{\theta} \times \frac{1}{i\omega - \frac{1}{\theta}} \left( e^{x(i\omega - \frac{1}{\theta})} \right) \Big|_0^\infty \tag{11.10.14}$$

$$= \frac{1}{1 - i\omega\theta} - \frac{\lim_{x\to\infty} \left( e^{x(i\omega - \frac{1}{\theta})} \right)}{1 - i\omega\theta} \tag{11.10.15}$$

$$= \frac{1}{1 - i\omega\theta} - 0 = \frac{1}{1 - i\omega\theta} \tag{11.10.16}$$

$$\Phi_Z(\omega) = \left( \frac{1}{1 - i\omega\theta} \right)^2 \tag{11.10.17}$$

$$f_Z(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{\left( \frac{1}{1 - i\omega\theta} \right)^2}\, d\omega \tag{11.10.18}$$

The equation (11.10.18) is the characteristic function expression of a gamma random variable with k=2. Thus,

$$f_Z(x) = \frac{x^{k-1}e^{\frac{-x}{\theta}}}{\Gamma(k)\theta^k} \tag{11.10.19}$$

$$= \frac{x^{2-1}e^{\frac{-x}{\theta}}}{\Gamma(2)\theta^2} \tag{11.10.20}$$

$$= \frac{xe^{\frac{-x}{\theta}}}{\theta^2} \tag{11.10.21}$$

$$f_{X_2|(X_1+X_2=t)}(x_2) = \begin{cases} \frac{f_{X_1,X_2}(x_1,x_2)}{f_Z(t)} & x_2 \in (0,t) \\ 0 & \text{otherwise} \end{cases} \tag{11.10.22}$$

Let $x_2 \in (0,t)$.

$$f_{X_2|(X_1+X_2=t)}(x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_Z(t)} \tag{11.10.23}$$

$$= \frac{\frac{1}{\theta^2}e^{\frac{-(x_1+x_2)}{\theta}}}{\frac{1}{\theta^2}e^{\frac{-t}{\theta}}t} \tag{11.10.24}$$

$$= \frac{e^{\frac{-(t)}{\theta}}}{e^{\frac{-t}{\theta}}t} \tag{11.10.25}$$

$$= \frac{1}{t} \quad \forall x_2 \in (0,t) \tag{11.10.26}$$

The obtained pdf is uniform$(0,t)$. Any distribution is sufficient for underlying parameter $\theta$ if the conditional probability distribution of the data does not depend on the parameter $\theta$. And since the conditional distribution of $X_2$ does not depend on $\theta$ for any value of $t$, $X_1 + X_2$ is sufficient for $\theta$. Verifying the pdf,

$$\text{total probability} = \int_0^t f_{X_2|(X_1+X_2=t)}(x_2)\,dx_2 \tag{11.10.27}$$

$$= \int_0^t \frac{1}{t}\,dx_2 \tag{11.10.28}$$

$$= 1 \tag{11.10.29}$$

Hence, the correct answer is option (11.10c)

11.11. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables each following a uniform distribution on (0,1). Denote $T_n = max\{X_1, X_2, \ldots, X_n\}$. Then, which of the following statements are true?

a) $T_n$ converges to 1 in probability.
b) $n(1 - T_n)$ converges in distribution.
c) $n^2(1 - T_n)$ converges in distribution.
d) $\sqrt{n}(1 - T_n)$ converges to 0 in probability.

**Solution:** The PDF, CDF of each $X_1, X_2, X_3, \ldots$ is

$$f_{X_i}(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases} \tag{72.1}$$

$$F_{X_i}(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \tag{72.2}$$

$\forall i \in \mathbb{N}$. Then, as $T_n = max\{X_1, X_2, \ldots, X_n\}$,

$$f_{T_n}(x) = \begin{cases} nx^{n-1}, & 0 < x < 1 \\ 0, & otherwise \end{cases} \tag{72.3}$$

$$F_{T_n}(x) = \begin{cases} x^n, & 0 < x < 1 \\ 1, & x \geq 1 \\ 0, & otherwise \end{cases} \tag{72.4}$$

NOTE : If $Y = aX + b$ and $a < 0$, then

$$F_Y(y) = 1 - F_X\left(\frac{y - b}{a}\right) \tag{72.5}$$

a) OPTION-1:
Convergence in Probability :
A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in probability

to a random variable $X$, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{72.6}$$

To evaluate : $\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right), \forall \epsilon > 0$

$$\lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = \lim_{n\to\infty} \Pr\left(1 - T_n \geq \epsilon\right) \tag{72.7}$$

$$= \lim_{n\to\infty} \Pr\left(T_n \leq 1 - \epsilon\right) = \lim_{n\to\infty} F_{T_n}(1-\epsilon) \tag{72.8}$$

$$F_{T_n}(1-\epsilon) = \begin{cases} (1-\epsilon)^n, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \tag{72.9}$$

$$\because \lim_{n\to\infty} (1-\epsilon)^n = 0 \text{ for } 0 < \epsilon < 1 \tag{72.10}$$

$$\therefore \lim_{n\to\infty} \Pr\left(|T_n - 1| \geq \epsilon\right) = 0, \forall \epsilon > 0 \tag{72.11}$$

$\therefore T_n$ converges to 1 in probability.

b) OPTION-2:

Convergence in Distribution :

A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable $X$, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{72.12}$$

for all $x$ at which $F_X(x)$ is continuous.

To evaluate : $\lim_{n\to\infty} F_{n(1-T_n)}(x)$

Substituting $a = -n, b = n$ in (72.5),

$$F_{n(1-T_n)}(x) = 1 - F_{T_n}\left(1 - \frac{x}{n}\right) \tag{72.13}$$

$$F_{T_n}\left(1 - \frac{x}{n}\right) = \begin{cases} \left(1 - \frac{x}{n}\right)^n, & 0 < x < n \\ 1, & x \leq 0 \\ 0, & x \geq n \end{cases} \tag{72.14}$$

$$\because \lim_{n\to\infty}\left(1-\frac{y}{n}\right)^n = e^{-y} \tag{72.15}$$

$$\therefore \lim_{n\to\infty} F_{T_n}\left(1-\frac{x}{n}\right) = \begin{cases} e^{-x}, & 0 < x < n \\ 1, & x \le 0 \\ 0, & x \ge n \end{cases} \tag{72.16}$$

$$\therefore F_{n(1-T_n)}(x) = \begin{cases} 1-e^{-x}, & 0 < x < n \\ 0, & x \le 0 \\ 1, & x \ge n \end{cases} \tag{72.17}$$

$\therefore n(1-T_n)$ converges in distribution to a random variable with CDF in (72.17).

c) OPTION-3:

Convergence in Distribution :

A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in distribution to a random variable $X$, shown by $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{72.18}$$

for all $x$ at which $F_X(x)$ is continuous.

To evaluate : $\lim\limits_{n\to\infty} F_{n^2(1-T_n)}(x)$

Substituting $a = -n^2, b = n^2$ in (72.5),

$$F_{n^2(1-T_n)}(x) = 1 - F_{T_n}\left(1-\frac{x}{n^2}\right) \tag{72.19}$$

$$F_{T_n}\left(1-\frac{x}{n^2}\right) = \begin{cases} \left(1-\frac{x}{n^2}\right)^n, & 0 < x < n^2 \\ 1, & x \le 0 \\ 0, & x \ge n^2 \end{cases} \tag{72.20}$$

$$\because \lim_{n\to\infty}\left(1-\frac{y}{n^2}\right)^n \text{ is not defined} \tag{72.21}$$

$\therefore n^2(1 - T_n)$ does not converge in distribution.

d) OPTION-4:

Convergence in Probability :

A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in probability to a random variable $X$, shown by $X_n \xrightarrow{p} X$, if

$$\lim_{n\to\infty} \Pr\left(|X_n - X| \geq \epsilon\right) = 0, \forall\epsilon > 0 \tag{72.22}$$

To evaluate :

$\lim_{n\to\infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon\right), \forall\epsilon > 0$

$$= \lim_{n\to\infty} \Pr\left(1 - T_n \geq \frac{\epsilon}{\sqrt{n}}\right) \tag{72.23}$$

$$= \lim_{n\to\infty} \Pr\left(T_n \leq 1 - \frac{\epsilon}{\sqrt{n}}\right) \tag{72.24}$$

$$= \lim_{n\to\infty} F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) \tag{72.25}$$

$$F_{T_n}\left(1 - \frac{\epsilon}{\sqrt{n}}\right) = \begin{cases} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n, & 0 < \epsilon < \sqrt{n} \\ 0, & \epsilon \geq \sqrt{n} \end{cases} \tag{72.26}$$

$$\because \lim_{n\to\infty} \left(1 - \frac{\epsilon}{\sqrt{n}}\right)^n = 0 \text{ for } 0 < \epsilon < \sqrt{n} \tag{72.27}$$

$$\therefore \lim_{n\to\infty} \Pr\left(|\sqrt{n}(1 - T_n) - 0| \geq \epsilon\right) = 0, \forall\epsilon > 0 \tag{72.28}$$

$\therefore \sqrt{n}(1 - T_n)$ converges to 0 in probability.

$\boxed{\text{Hence, options 1), 2), 4) are correct.}}$

## 12 December 2012

12.1. Let X be a binomial random variable with parameters $\left(11, \frac{1}{3}\right)$. At which value(s) of k is $\Pr(X = k)$ maximized?

a) k=2
b) k=3
c) k=4

d) k=5

**Solution:** X has a binomial distribution :

$$\Pr(X = k) = {}^{n}C_k (q)^{n-k} (p)^k \tag{12.1.1}$$

Where,

- n=11
- $p = \frac{1}{3}$
- $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

$$\Pr(X = k) = {}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k \tag{12.1.2}$$

For Pr(X = k) to be maximized

$$\Pr(X = k) \ \geq \ \Pr(X = k + 1) \tag{12.1.3}$$

$$\frac{\Pr(X = k)}{\Pr(X = k + 1)} = \frac{{}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k}{{}^{11}C_{k+1} \left(\frac{2}{3}\right)^{10-k} \left(\frac{1}{3}\right)^{k+1}} \geq 1 \tag{12.1.4}$$

$$\frac{2(k + 1)}{11 - k} \geq 1 \tag{12.1.5}$$

$$\implies k \geq 3 \tag{12.1.6}$$

$$\Pr(X = k) \ \geq \ \Pr(X = k - 1) \tag{12.1.7}$$

$$\frac{\Pr(X = k)}{\Pr(X = k - 1)} = \frac{{}^{11}C_k \left(\frac{2}{3}\right)^{11-k} \left(\frac{1}{3}\right)^k}{{}^{11}C_{k-1} \left(\frac{2}{3}\right)^{12-k} \left(\frac{1}{3}\right)^{k-1}} \geq 1 \tag{12.1.8}$$

$$\frac{12 - k}{2k} \geq 1 \tag{12.1.9}$$

$$\implies k \leq 4 \tag{12.1.10}$$

From (12.1.6) , (12.1.10) and since k is an integer
$\Pr(X = k)$ is maximized for k=3, k=4
Thus options 2) and 3) are correct

12.2. Men arrive in a queue according to a Poisson process with rate $\lambda_1$ and women arrive in the same queue according to another Poisson process with rate $\lambda_2$. The arrivals of men and women are independent. The probability that the first person to arrive in the queue is a man is:

a) $\frac{\lambda_1}{\lambda_1 + \lambda_2}$

b) $\frac{\lambda_2}{\lambda_1 + \lambda_2}$

c) $\frac{\lambda_1}{\lambda_2}$

d) $\frac{\lambda_2}{\lambda_1}$

**Solution:** Let $X$ and $Y$ be Poisson random variables, with the values X takes being the number of men joining the queue in an arbitrary time $t$, and the values Y takes being the number of women joining the queue in an arbitrary time $t$.

$$Pr(X = i) = \frac{\lambda_1^i \cdot e^{-\lambda_1}}{i!} \tag{12.2.1}$$

$$Pr(Y = i) = \frac{\lambda_2^i \cdot e^{-\lambda_2}}{i!} \tag{12.2.2}$$

For 2 independent Poisson distributions with means $\lambda_1$ and $\lambda_2$, the simultaneous distribution can be represented by:

$$Pr(X + Y = i) = \frac{(\lambda_1 + \lambda_2)^i \cdot e^{-(\lambda_1+\lambda_2)}}{i!} \tag{12.2.3}$$

Now we take conditional probability that if only one person entered the queue within a certain time t, then the probability of them being a man and not a woman is given by:

$$Pr(X = 1|(X + Y) = 1) = \frac{Pr((X = 1) + (Y = 0))}{Pr(X + Y = 1)} \tag{12.2.4}$$

$$\tag{12.2.5}$$

Since X and Y are independent,

$$Pr(X = 1|(X + Y) = 1) = \frac{Pr(X = 1) \cdot Pr(Y = 0)}{Pr(X + Y = 1)} \tag{12.2.6}$$

$$= \frac{\frac{\lambda_1^1 \cdot e^{-\lambda_1}}{1!} \cdot \frac{\lambda_2^0 \cdot e^{-\lambda_2}}{0!}}{\frac{(\lambda_1+\lambda_2)^1 \cdot e^{-(\lambda_1+\lambda_2)}}{1!}} \tag{12.2.7}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \tag{12.2.8}$$

The probability that the first person to arrive in the queue is a man is option A, i.e $\frac{\lambda_1}{\lambda_1+\lambda_2}$

## 13 June 2012

13.1. Let $X_1, X_2, X_3, ...., X_n$ be i.i.d observations from a distribution with continuous probability density function f which is symmetric around $\theta$ i.e

$$f(x - \theta) = f(\theta - x) \tag{13.1.1}$$

for all real x.Consider the test $H_0 : \theta = 0$ vs $H_1 : \theta > 0$ and the sign test statistic

$$S_n = \sum_{i=1}^{n} sign(X_i) \tag{13.1.2}$$

where

$$sign(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{13.1.3}$$

Let $z_\alpha$ be the upper $100(1 - \alpha)^{th}$ percentile of the standard normal distribution where $0 < \alpha < 1$. Which of the following is/are correct?

a) If $\theta = 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n}z_\alpha\right\} = 1$

b) If $\theta = 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n}z_\alpha\right\} = \alpha$

c) If $\theta > 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n} z_\alpha\right\} = 1$

d) If $\theta > 0$ then $\lim_{n\to\infty} P\left\{S_n > \sqrt{n} z_\alpha\right\} = \alpha$

**Solution:** $H_0 : \theta = 0$ Assume hypothesis $H_0 : \theta = 0$ is true.

a) Given X is symmetric around zero.

$$f_X(x) = f_X(-x) \tag{13.1.4}$$

$$\int_0^\infty f_X(x)dx = \int_0^\infty f_X(-x)dx \tag{13.1.5}$$

i) Solving LHS of (13.1.5)

$$\int_0^\infty f_X(x)dx = \Pr(X \geq 0) \tag{13.1.6}$$

ii) Solving RHS of (13.1.5)

$$\int_0^\infty f_X(-x)dx \tag{13.1.7}$$

Changing $-x \to x$ we get

$$\int_0^\infty f_X(-x)dx = \int_{-\infty}^0 f_X(x)dx \tag{13.1.8}$$

$$= \Pr(X \leq 0) \tag{13.1.9}$$

but

$$\int_{-\infty}^0 f_X(x)dx + \int_0^\infty f_X(x)dx = 1 \tag{13.1.10}$$

from (13.1.5) , (13.1.8) and (13.1.10)

$$\int_{-\infty}^0 f_X(x)dx = \int_0^\infty f_X(x)dx = \frac{1}{2} \tag{13.1.11}$$

$$\implies \Pr(X \leq 0) = \Pr(X \geq 0) = \frac{1}{2} \tag{13.1.12}$$

b) Let Y be a random variable such that

$$Y = sign(X) \tag{13.1.13}$$

$$Y = \begin{cases} 1 & X > 0 \\ -1 & X < 0 \end{cases} \tag{13.1.14}$$

From (13.1.12) and (13.1.14) we have

$$\Pr(Y = -1) = \Pr(Y = 1) = \frac{1}{2} \tag{13.1.15}$$

So $Y = sign(X)$ is also symmetric around zero.

$$\implies \mu_y = 0 \tag{13.1.16}$$

and variance is

$$\sigma_y^2 = (-1)^2\left(\frac{1}{2}\right) + (1)^2\left(\frac{1}{2}\right) \tag{13.1.17}$$

$$= 1 \tag{13.1.18}$$

c) Given

$$S_n = \sum_{i=1}^{n} sign(X_i) \tag{13.1.19}$$

$$S_n(\theta = 0) = \sum_{i=1}^{n} Y_i \tag{13.1.20}$$

From central limit theorem

$$Z = \lim_{n\to\infty} \sqrt{n}\left(\frac{\frac{S_n}{n} - \mu_y}{\sigma_y}\right) \tag{13.1.21}$$

$$= \lim_{n\to\infty} \sqrt{n}\left(\frac{S_n}{n}\right) \tag{13.1.22}$$

$$= \lim_{n\to\infty}\left(\frac{S_n}{\sqrt{n}}\right) \tag{13.1.23}$$

where Z is a standard normal variable N(0,1).

i) Given

$$\alpha = P\{Z > z_\alpha\} \tag{13.1.24}$$

So from (13.1.23) and (13.1.24)

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} > z_\alpha\right\} = \alpha \tag{13.1.25}$$

$$\implies \lim_{n\to\infty} P\left\{S_n > \sqrt{n}z_\alpha\right\} = \alpha \tag{13.1.26}$$

$H_1 : \theta > 0$ is true

a) Given X is symmetric around $\theta > 0$.Let us assume $\theta = \theta_0 > 0$.

$$f_X(\theta_0 - x) = f_X(\theta_0 + x) \tag{13.1.27}$$

$$\int_{\theta_0}^{\infty} f_X(\theta_0 - x)dx = \int_{\theta_0}^{\infty} f_X(\theta_0 + x)dx \tag{13.1.28}$$

i) Solving LHS of (13.1.28).Changinng $(\theta_0 - x) \to t$

$$\int_{\theta_0}^{\infty} f_X(\theta_0 - x)dx = \int_{-\infty}^{0} f_X(t)dt \tag{13.1.29}$$

$$= \Pr(X \leq 0) \tag{13.1.30}$$

ii) Solving RHS of (13.1.28).Changing $(\theta_0 + x) \to t$

$$\int_{\theta_0}^{\infty} f_X(\theta_0 + x)dx = \int_{2\theta_0}^{\infty} f_X(t)dt \tag{13.1.31}$$

$$= \int_{0}^{\infty} f_X(t)dt - \int_{0}^{2\theta_0} f_X(t)dt \tag{13.1.32}$$

$$= \Pr(X \geq 0) - k \tag{13.1.33}$$

where

$$k = \int_{0}^{2\theta_0} f_X(t)dt > 0 \tag{13.1.34}$$

From (13.1.28),(13.1.14) and (13.1.33)

$$\Pr(X \geq 0) > \Pr(X \leq 0) \tag{13.1.35}$$

b) So

$$\Pr(Y = 1) > \Pr(Y = -1) \tag{13.1.36}$$

Therefore,if we perform the experiment and find the value of $\left(\frac{S_n}{\sqrt{n}}\right)$,it is most likely to occur on the right side of the distribution of $\left(\frac{S_n}{\sqrt{n}}\right)$.In (13.1.23) it is shown that distrubution of the random variable $\left(\frac{S_n}{\sqrt{n}}\right)$ is $N(0,1)$ when n is very large.So

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} > Z_\alpha\right\} = 1 \tag{13.1.37}$$

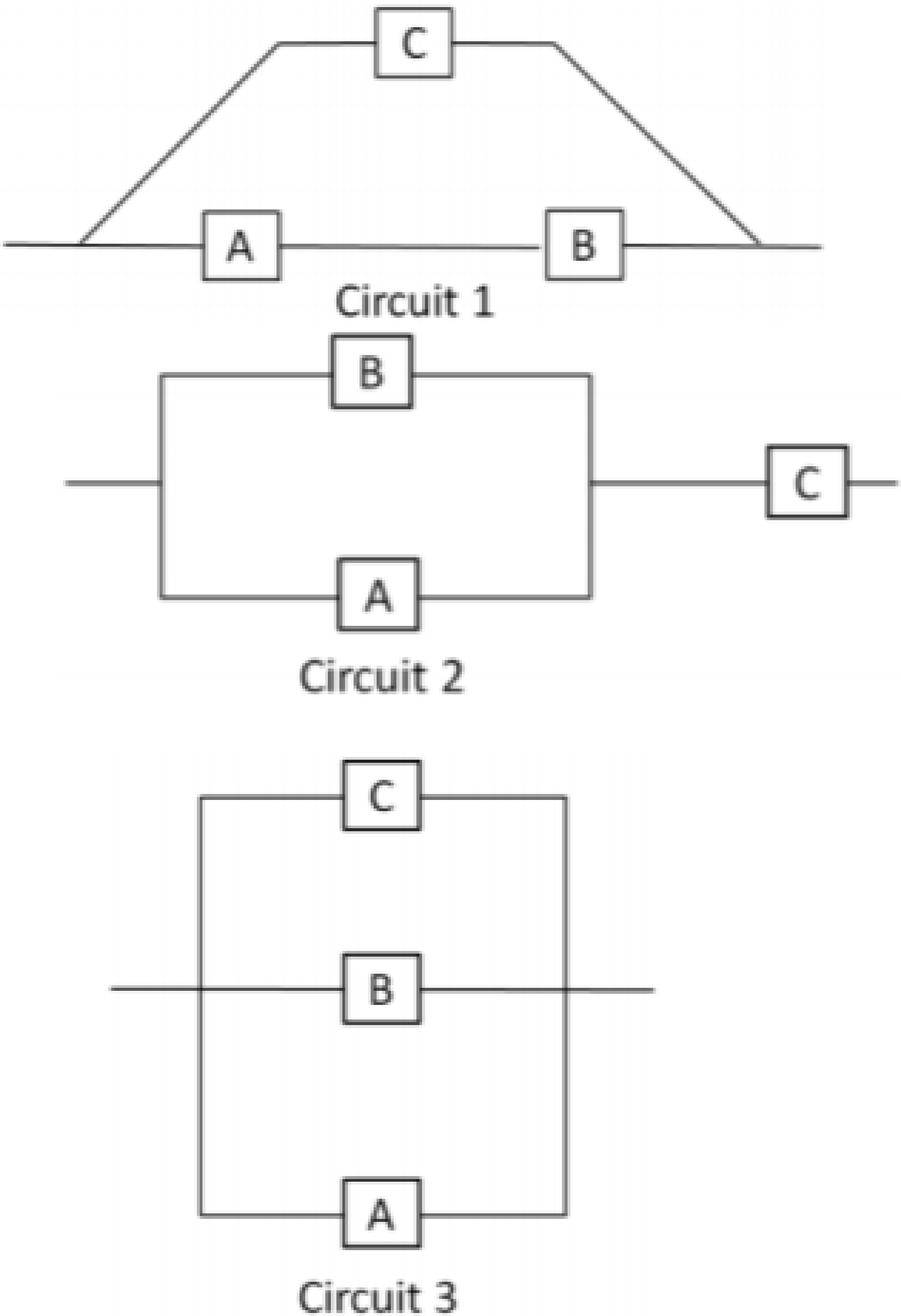



Fig. 7.2.1: Figure

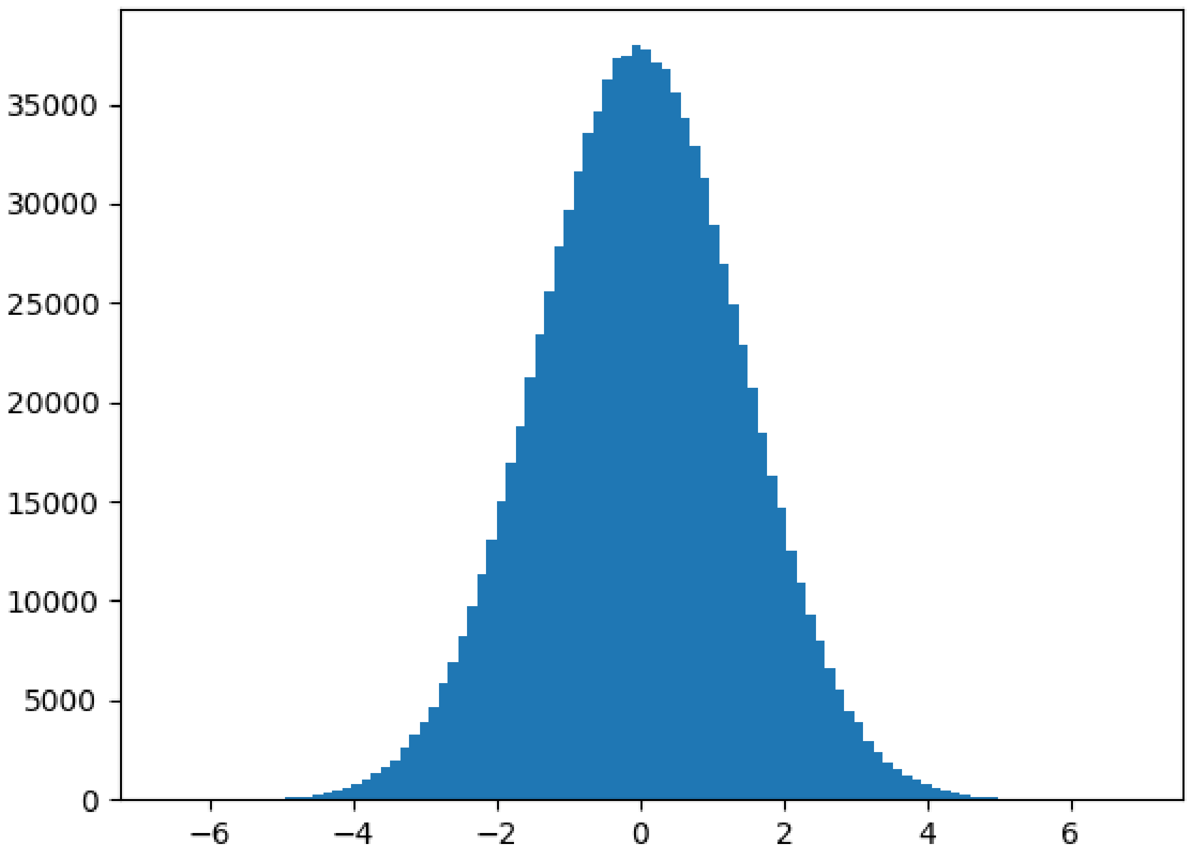



Fig. 7.3.1: Z when X is standard normal

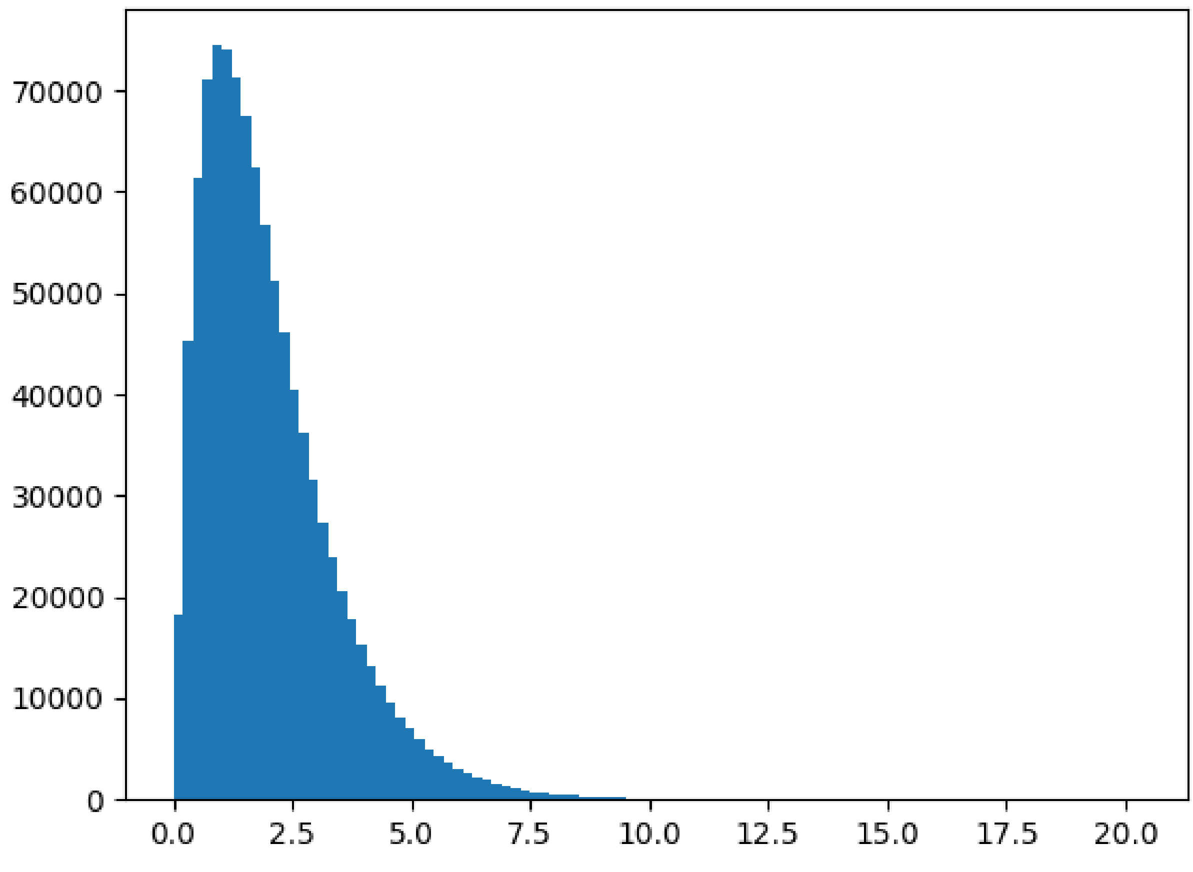



Fig. 7.3.2: Z when X is exponential with $\lambda = 1$

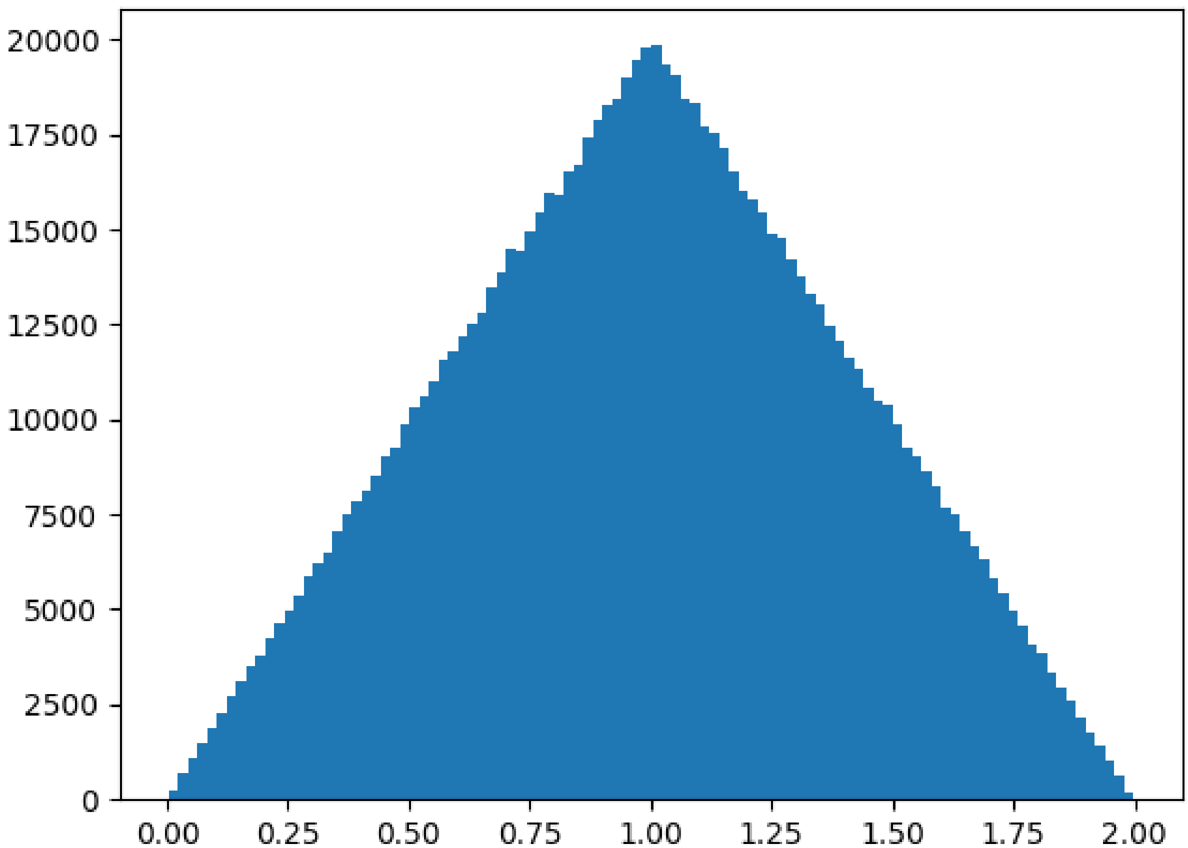



Fig. 7.3.3: Z when X $\sim$ U(0,1)

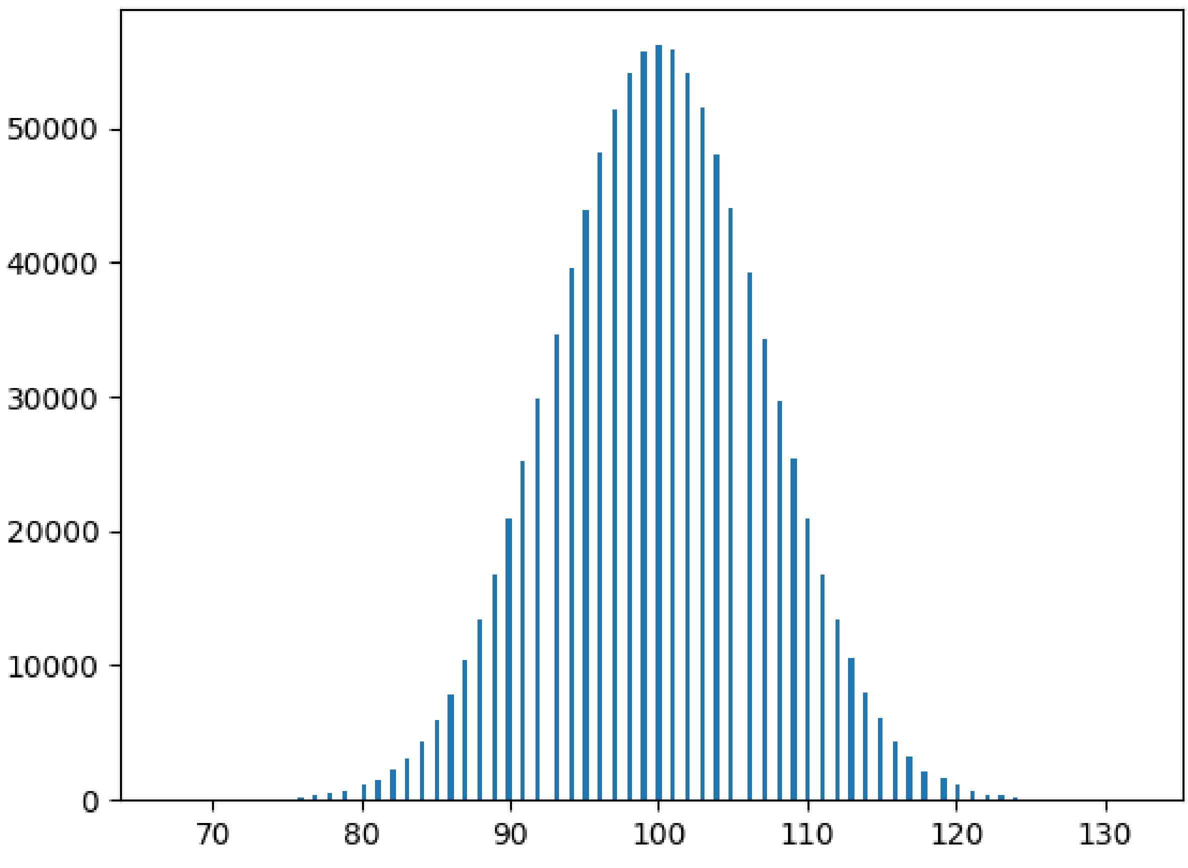



Fig. 7.3.4: Z when X ~ B(100,0.5)

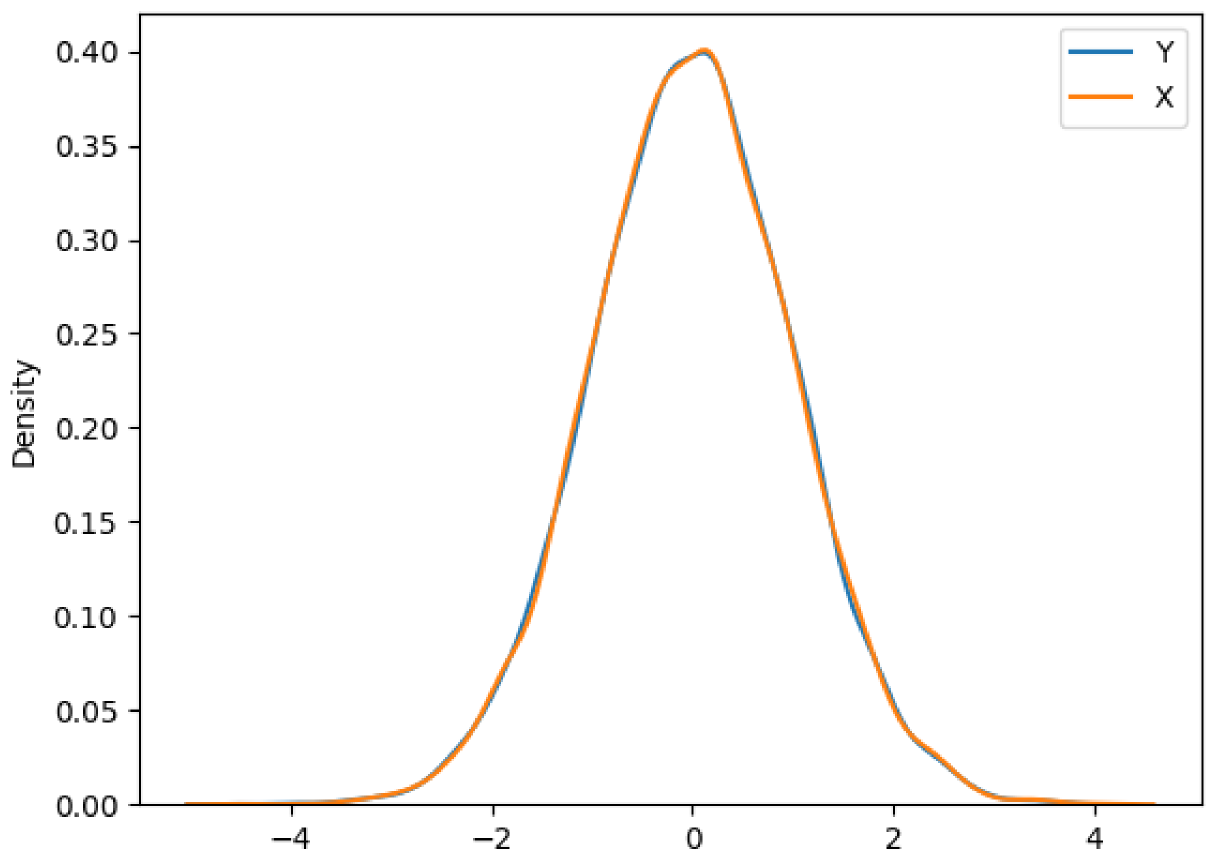



Fig. 8.7.1: X and Y, if Y is normal

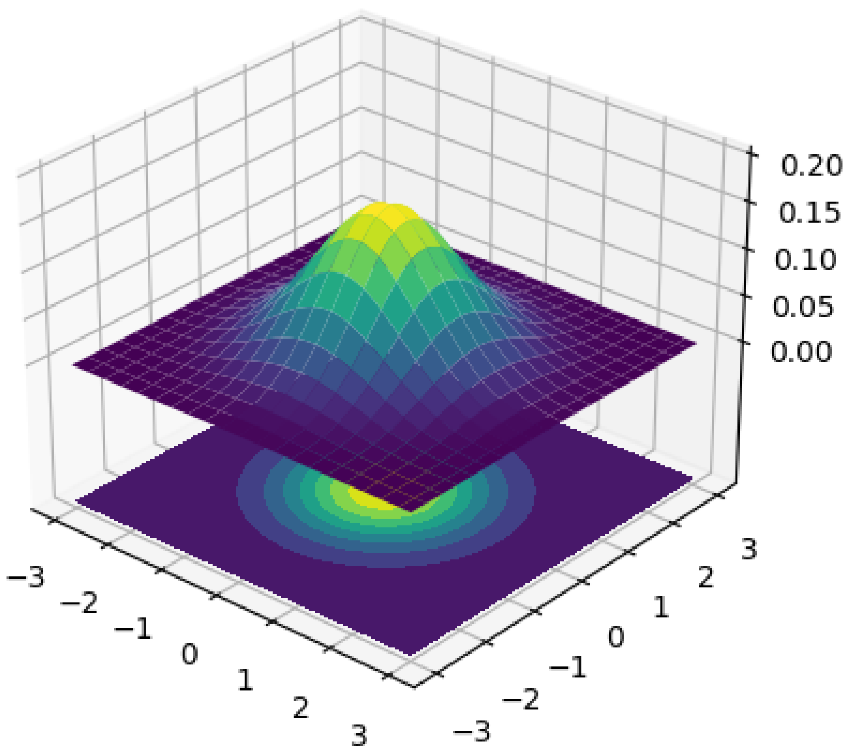
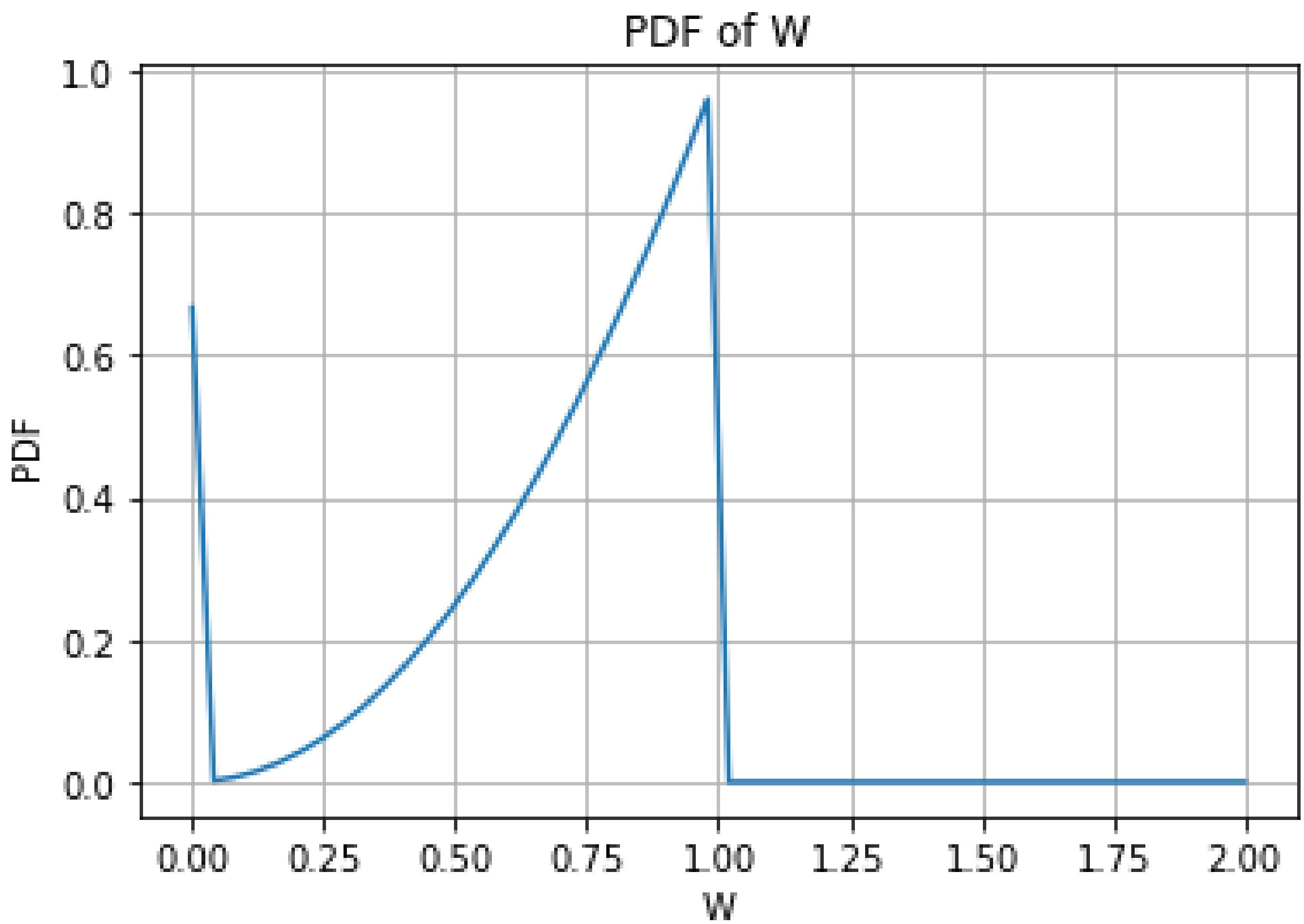
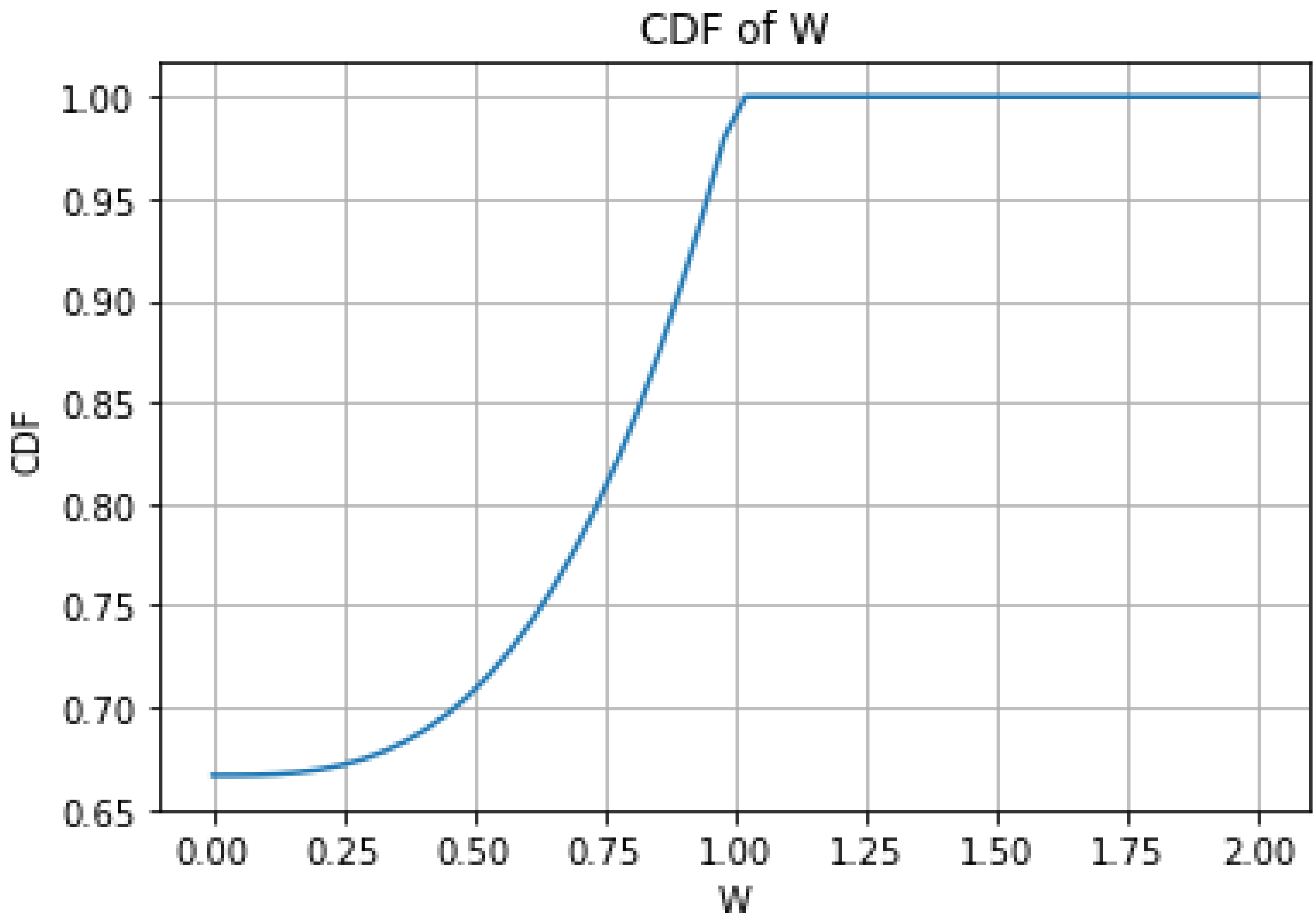



Fig. 8.7.2: bivariate gaussian while 0 mean vector and identity covariance matrix


PDF of W
PDF
W
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00


CDF of W
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
CDF
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
W

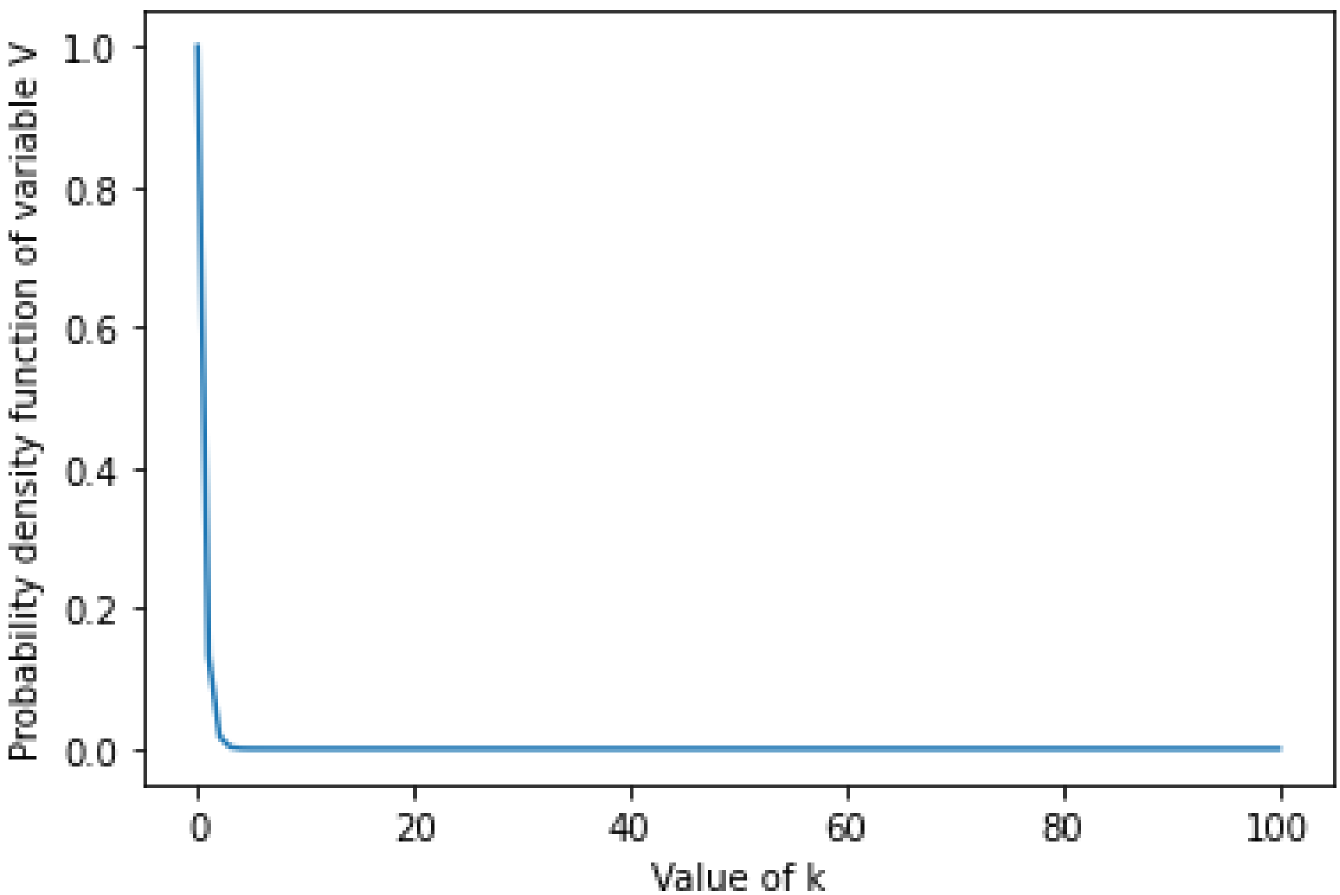



Fig. 11.8.1: Parallel system

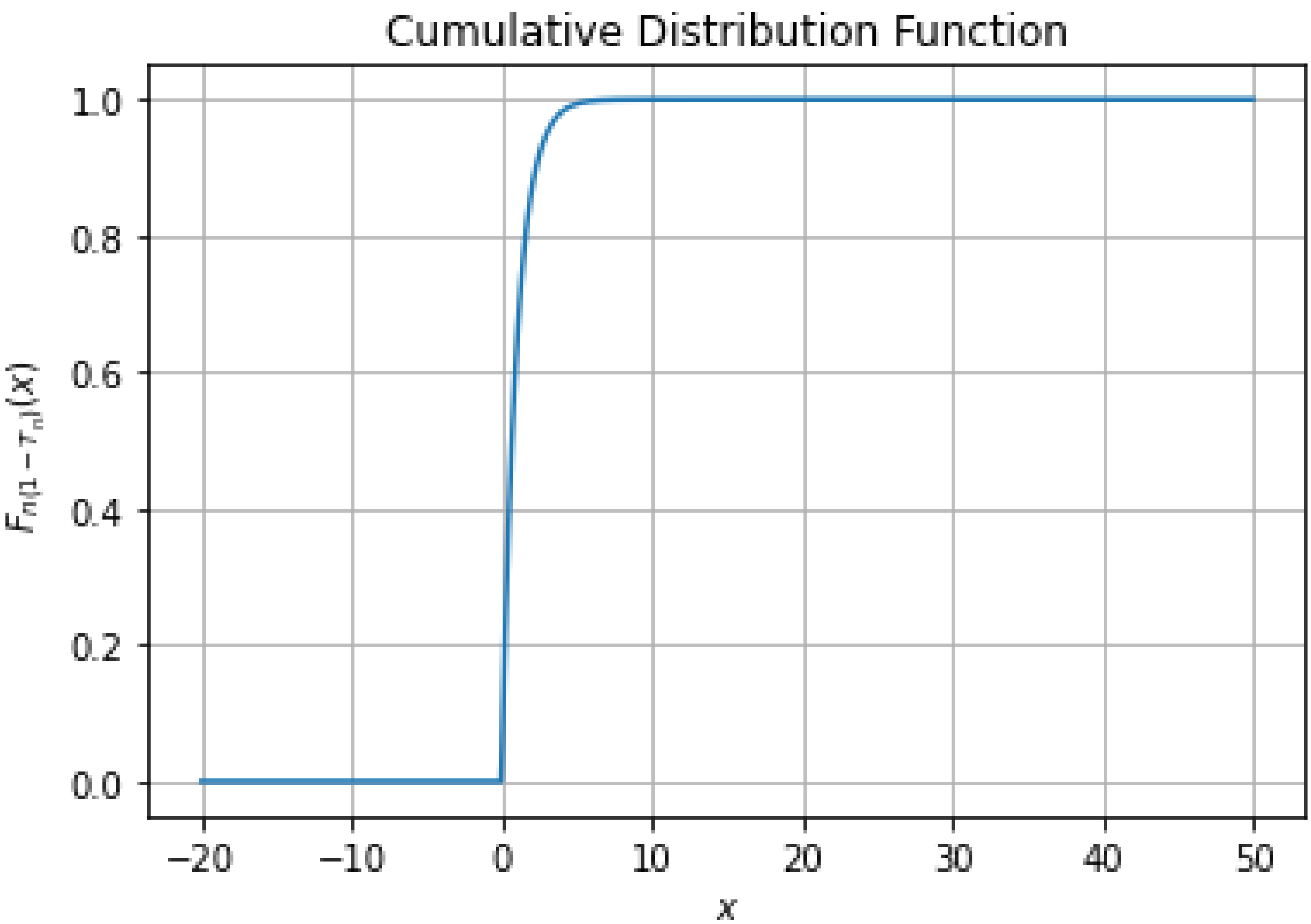



Fig. 11.11.1: CDF