Algorithm Design & Analysis (CS3383)[1]

Unit 1 Cont.: The Master Theorem with applications

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[1]Thanks to Dr. Ptricia Evans and Dr. David Bremner at UNB, Dr. Erik Demaine at MIT for sharing the teaching stuffs

Outline[2]

Divide and Conquer Continued

The Master Theorem

Matrix Multiplication

[2] Reading:

- Main textbook (DPV), Divide and conquer algorithms, Chapter 2 mainly 2.2 and 2.5.
- Algorithms, Cormen, Chapter 4 (4.5 and 4.6)
- Recursive algorithms from Jeff Ericson's Algorithm page `http://jeffe.cs.illinois.edu/teaching/algorithms/notes/99-recurrences.pdf`

Contents

Generic divide and conquer algorithm

```
function SOLVE(P)
    if |P| is small then
        SolveDirectly(P)
    else

P_1 ... P_k = Partition(P)
        for i = 1 ... k do
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- How many times do we recurse?
- what fraction of input in each subproblem?
- How much time to combine results?

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- e.g. one call of $\frac{1}{3}$ and one call of $\frac{2}{3}$,
- partition+combine step $\Theta(n \log n)$. $\longrightarrow \Omega$, O ?

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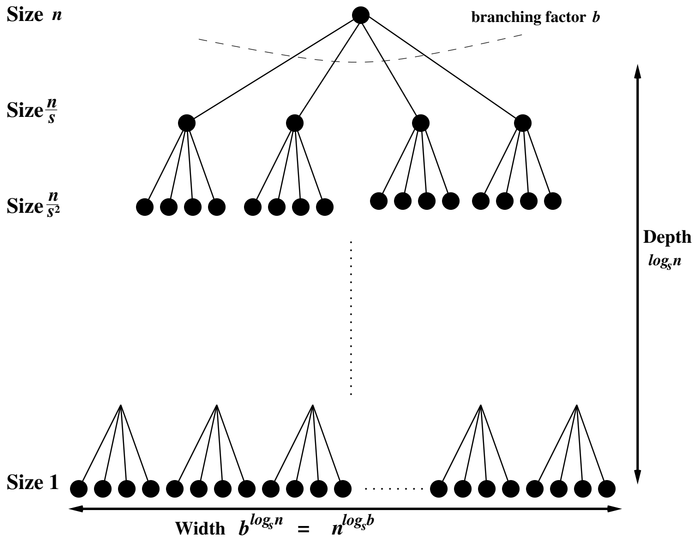
A proof of this follows.

Proof of Master theorem, in pictures



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And so

$$T(n) = \sum_{i=0}^{\log_s n} c \cdot \left(\frac{n}{s^i}\right)^d \cdot b^i$$

Proof of Master theorem, $b = s^d$

$$T(n) = \sum_{i=0}^{\log_s n} c \cdot \left(\frac{n^d}{\left(s^d\right)^i} \right) \cdot b^i \;=\; c \cdot n^d \cdot \left(\sum_{i=0}^{\log_s n} \left(\frac{b}{s^d} \right)^i \right)$$

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If $b = s^d$, then

$$T(n) = c \cdot n^d \cdot \left(\sum_{i=0}^{\log_s n} 1\right) \;=\; c \cdot n^d \log_s n$$

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so $T(n)$ is $\Theta(n^d \log n)$.

Proof of Master Theorem $b \neq s^d$ (1 of 2)

Otherwise ($b \neq s^d$), we have a geometric series,

$$T(n) = c \cdot n^d \cdot \left(\frac{\left(\frac{b}{s^d}\right)^{\log_s n+1} - 1}{\frac{b}{s^d} - 1} \right)$$

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Proof of Master Theorem $b \neq s^d$ (2 of 2)

From rules of powers and logarithms:

$$\left(\frac{b}{s^d}\right)^{\log_s n+1} = \frac{b}{s^d} \cdot \left(\frac{b}{s^d}\right)^{\log_s n} = \frac{b}{s^d} \cdot \frac{b^{\log_s n}}{\left(s^d\right)^{\log_s n}}$$

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Branching versus subproblem size

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Merge Sort

- $T(n) = bT(n/s) + \theta(n^d)$
- b how many recursive calls?
- s what is the the split (denominator of size)
- d degree

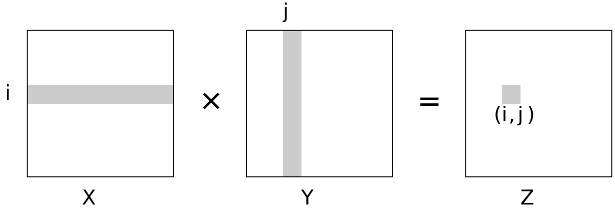
Contents

Matrix Multiplication

The product of two $n \times n$ matrices x and y is a third $n \times n$ matrix $Z = XY$, with

$$Z_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj}$$

where Z_{ij} is the entry in row i and column j of matrix Z.



Calculating Z directly using this formula takes $\Theta(n^3)$ time.

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Eight subinstances $AE, BG, AF, BH, CE, DG, CF, DH$

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(this is not technically "cubic algorithm", input size n^2.)

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where

$$\begin{aligned} P_1 &= A(F - H) \\ P_2 &= (A + B)H \\ P_3 &= (C + D)E \\ P_4 &= D(G - E) \end{aligned} \qquad \begin{aligned} P_5 &= (A + D)(E + H) \\ P_6 &= (B - D)(G + H) \\ P_7 &= (A - C)(E + F) \end{aligned}$$

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Since the input size is $m = n^2$, the algorithm runs in approximately $\Theta(m^{1.404})$ time (versus the $\Theta(m^{1.5})$ of the original).