Exercise needed for the Analysis of randomized quicksort

January 28, 2019

Exercise) Prove that:

$$\sum_{k \in \{2 \ldots n-1\}} k \lg k \leq (1/2)n^2 \lg n - (1/8)n^2. \tag{1}$$

In the analysis of randomized quicksort, we use the Equation (1) as a fact. Here is the proof which is done by using induction.

For base cases, we can verify for $n = 2$ (the left hand side is 0) and for good measure, for $n = 3$.

We next split off the top term of the sum, and apply (1) inductively to the case $n - 1$.

$$\begin{aligned}(n-1)\lg(n-1) + \sum_{k \in \{2 \ldots n-2\}} k \lg k \leq\ & (n-1)\lg(n-1) \\ & + \frac{1}{2}(n-1)^2 \lg(n-1) - \frac{1}{8}(n-1)^2\end{aligned}$$

Collecting $\lg(n-1)$:

$$= \frac{n^2 - 1}{2} \lg(n-1) - \frac{1}{8}(n-1)^2$$

Expanding the first term:

$$= \frac{n^2}{2} \lg(n-1) - \frac{1}{8}(n-1)^2 - \lg(n-1)$$

Construct a residual:

$$
\begin{aligned}
&= \frac{n^2}{2}\lg n - \frac{1}{8}n^2 \\
&\quad - [\lg n - \lg(n-1)]\frac{1}{2}n^2 \\
&\quad + \frac{1}{8}[n^2 - (n-1)^2] - \lg(n-1)
\end{aligned}
$$

Simplifying, and dropping the trailing $-\lg(n-1)$

$$
\begin{aligned}
&\leq \frac{n^2}{2}\lg n - \frac{1}{8}n^2 \\
&\quad - \left[(\lg(n) - \lg(n-1))\frac{1}{2}n^2 - \frac{n}{4} + \frac{1}{8}\right]
\end{aligned}
$$

Ignoring the $1/8$, in order to show the residual is non-negative it suffices to show.

$$\lg(n) - \lg(n-1) \geq \frac{1}{2n}$$

This is more amenable to induction if we write it as

$$\frac{1}{2n} + \lg(n-1) \leq \lg(n)$$

The base case of $n = 2$ is ok, so we expand repeatedly (telescoping)

$$
\begin{aligned}
\frac{1}{2n} + \lg(n-1) &\leq \frac{1}{2n} + \frac{1}{2(n-1)} + \lg(n-2) \\
&\leq \frac{1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} + \lg(n-3) \\
&\leq \frac{1}{2}H_n
\end{aligned}
$$

Applying the harmonic number integration bound from the last assignment, we get

$$\frac{1}{2n} + \lg(n-1) \leq \frac{\ln n + 1}{2}$$

Using the identity $\ln n = \lg n \ln 2$, And that for $n \geq 2$, $\lg n \geq 1$

$$\leq \frac{\ln 2 \lg n + \lg n}{2}$$

Finally the observation that $\ln 2 < 1$ gives us the bound we wanted.

Note: It is unlikely that I expect you to produce a proof this involved on the exams. However, it is possible for the assignments.