**Algorithm Design and Analysis (CS3383)**
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**Recall 1:**

- $(U^n(x))' = nU^{n-1}(x)U'(x)$
- $\left(\frac{U(x)}{V(x)}\right)' = \left(\frac{U'(x)V(x) - V'(x)U(x)}{V^2(x)}\right)$
- $(\log_a U(x))' = \left(\frac{U'(x)}{U(x)Ln(a)}\right)$

**Example from Page 13-Lecture 1)** Prove that $\sqrt{n} = \omega(\log_2 n)$, for $(n_0 = 1 + \frac{1}{c})$.

*Proof*: Refering to the definition of $\omega$, we should prove that

$$\forall c > 0, \exists n_0 > 0 \;\; S.t. \;\; c \log_2 n < \sqrt{n}, \; \forall n \geq n_0, \tag{1}$$

where $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$.
Since the $n_0$ is given, we only need to show that the given $n_0$ satisfies (1). Equivalently, it is enough to show that (2) holds.

$$\forall c > 0, \;\; c \log_2 \left(1 + \frac{1}{c}\right) < \sqrt{\left(1 + \frac{1}{c}\right)} \tag{2}$$

**Recall 2**. One way to show that $f(x) < g(x)$ for all $x$ in some range $\mathcal{R}$ is to prove that $\frac{f(x)}{g(x)}$ is an increasing function for all $x$ in $\mathcal{R}$.

**Recall 3** One way to show that a function $h(x)$ is always increasing for all $x$ in a range $\mathcal{R}$ is enough to prove that $h'(x) > 0$ for all $x$ in $\mathcal{R}$.

To show that (2) holds, from **Recall 2** and **Recall 3**, it is enough to show that

$$\left(\frac{\sqrt{(1 + \frac{1}{c})}}{c \log_2 (1 + \frac{1}{c})}\right)' > o \, , \forall c > 0. \tag{3}$$

Using the formulas in **Recall 1**, we compute the derivative in the Equation (3).

$$\left(\frac{\sqrt{(1 + \frac{1}{c})}}{c \log_2 (1 + \frac{1}{c})}\right)' = \left(\frac{(1 + \frac{1}{c})^{\frac{1}{2}}}{\log_2 (1 + \frac{1}{c})^c}\right)' = \frac{\frac{1}{2}\left((1 + \frac{1}{c})^{-\frac{1}{2}}\right)\left(\frac{-1}{c^2}\right)}{\frac{c\left((1+\frac{1}{c})^{c-1}\right)\left(\frac{-1}{c^2}\right)}{(1+\frac{1}{c})^c Ln2}} = \frac{\frac{1}{2}\left((1 + \frac{1}{c})^{-\frac{1}{2}}\right)}{\frac{c\left((1+\frac{1}{c})^{c-1}\right)}{(1+\frac{1}{c})^c Ln2}}$$

$$= \frac{Ln2}{2c}\left(1 + \frac{1}{c}\right)^{c - \frac{1}{2} - c + 1} = \frac{Ln2}{2c}\left(1 + \frac{1}{c}\right)^{\frac{1}{2}}$$

Since $c > 0$ and $Ln2 \approx 0.693$, it is obvious that

$$\frac{Ln2}{2c}\left(1 + \frac{1}{c}\right)^{\frac{1}{2}} > 0. \tag{4}$$

Equation (4) is enough to terminate the proof.