CS3383, Winter 2019 Assignment # 2 Sample solutions

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Question 1: See your notes from tutorial (Wednesday, 23/Jan/2019).

Question 2:

a) $T(n) = 9T(n/3) + 3n^2 + 12n - 4$, where $n \geq 1$.

Solution: $3n^2 + 12n - 4 = \Theta(n^2)$ because one can bound this relation as follows:

$$3n^2 \leq 3n^2 + 8 \leq 3n^2 + 12n - 4 \leq 15n^2 - 4 \leq 15n^2; \; \forall n \geq 1.$$

So the given recurrence in (a) can be rewritten as

$$T(n) = 9T(n/3) + \Theta(n^2)$$

By applying Master theorem for $b = 9$, $s = 3$, and $d = 2$. It can be verified that $b = s^d$ which meets the condition of the second case of Master theorem where the total running time is $\Theta(n^2 \log n)$.

b) $T(n) = T(n-1) + n^c$, where $n \geq 1$ and $c > 0$

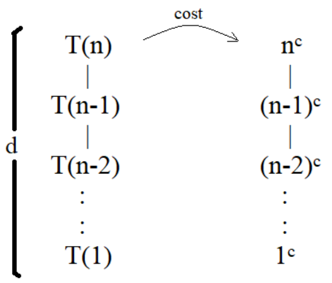

Figure 1: Recursion tree

Solution: The conditions of Master theorem are not met by this recurrence (The size of subproblem $(n-1)$ is not a fraction of n). So, the recursion tree should be employed to solve this recurrence.
By constructing the recursion tree (see Figure 1), the total running time can be computed as follows:

$$T(n) = \sum_{i=0}^{d} c' i^c = \sum_{i=0}^{n} c' i^c = c' \sum_{i=0}^{n} i^c \overset{(*)}{=} c' \frac{1}{c+1} n^{c+1} + \text{some other lower order terms}$$
$$= \Theta(n^{c+1}),$$

where $(*)$ is the sum of powers formula given by:

$$\sum_{k=0}^{n} k^c = \frac{1}{c+1} \sum_{k=0}^{c} \binom{c+1}{k} B_k n^{c+1-k}; \quad \forall n \in \mathbb{N}$$

and B_k is the k^{th} Bernouli number ($B_0 = 1$).

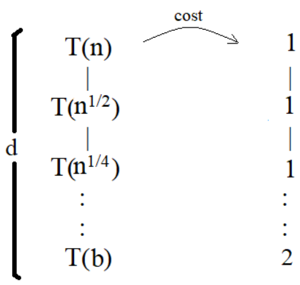

Figure 2: another recursion tree

c) $T(n) = T(\sqrt{n}) + 1$, where $n > b$ and $T(b) = 2$.

Solution: The conditions of Master theorem are not met by this recurrence (The size of sub-problem ($n^{1/2}$) is not a fraction of n). So, we apply the recursion tree method. See the corresponding recursion tree in Figure 2. The depth of tree can be computed as follows:

$$
\begin{aligned}
n^{\frac{1}{2^d}} = b \Rightarrow n = b^{2^d} &\Rightarrow \log n = 2^d \log b \Rightarrow \log(\log n) = \log(2^d \log b) = \log(2^d) + \log(\log b) \\
&\Rightarrow \log(\log n) = d + \log(\log b)d \Rightarrow d = \log(\log n) - \log(\log b) \\
&\Rightarrow d = \Theta(\log(\log n)).
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= c' \sum_{i=0}^{d-1} 1 + T(b) = c' \sum_{i=0}^{\Theta(\log(\log n))-1} 1 + 2 = (\Theta(\log(\log n)) - 1 + 1) + 2 \\
&= \Theta(\log(\log n)) + 2 \\
&= \Theta(\log(\log n)).
\end{aligned}
$$

Question 3: See your notes from tutorial (Wednesday, 23/Jan/2019).