

Quiz

Question #1: What is the value of the following expression?

{+ 1 2}

Quiz

Question #1: What is the value of the following expression?

{+ 1 2}

Wrong answer: 0

Quiz

Question #1: What is the value of the following expression?

$\{+ \ 1 \ 2\}$

Wrong answer: 0

Wrong answer: 42

Quiz

Question #1: What is the value of the following expression?

$\{+ \ 1 \ 2\}$

Wrong answer: 0

Wrong answer: 42

Answer: 3

Quiz

Question #2: What is the value of the following expression?

`{+ fun 17 8}`

Quiz

Question #2: What is the value of the following expression?

`{+ fun 17 8}`

Wrong answer: **error**

Quiz

Question #2: What is the value of the following expression?

`{+ fun 17 8}`

Wrong answer: **error**

Answer: Trick question! `{+ fun 17 8}` is not an expression

Language Grammar for Quiz

```
<MFAE> ::= <num>
        | true
        | false
        | {+ <MFAE> <MFAE>}
        | {- <MFAE> <MFAE>}
        | {= <MFAE> <MFAE>}
        | <id>
        | {fun {<id>*} <MFAE>}
        | {<MFAE> <MFAE>*}
        | {if <MFAE> <MFAE> <MFAE>}
```


Quiz

Question #3: Is the following an expression?

`{{fun {} 1} 7}`

Quiz

Question #3: Is the following an expression?

`{{fun {} 1} 7}`

Wrong answer: **No**

Quiz

Question #3: Is the following an expression?

`{{fun {} 1} 7}`

Wrong answer: **No**

Answer: **Yes** (according to our grammar)

Quiz

Question #4: What is the value of the following expression?

```
{{fun {} 1} 7}
```

Quiz

Question #4: What is the value of the following expression?

```
{{fun {} 1} 7}
```

Answer: **1** (according to some interpreters)

Quiz

Question #4: What is the value of the following expression?

`{{fun {} 1} 7}`

Answer: **1** (according to some interpreters)

But no *real* language would accept it

Quiz

Question #4: What is the value of the following expression?

`{{fun {} 1} 7}`

Answer: **1** (according to some interpreters)

But no *real* language would accept it

Let's agree to call `{{fun {} 1} 7}` an ***ill-formed expression*** because `{fun {} 1}` should be used with only zero arguments

Let's agree to never evaluate ill-formed expressions

Quiz

Question #5: What is the value of the following expression?

```
{{fun {} 1} 7}
```


Quiz

Question #5: What is the value of the following expression?

`{{fun {} 1} 7}`

Answer: **None** — the expression is ill-formed

Quiz

Question #6: Is the following a well-formed expression?

{+ {fun {} 1} 8}

Quiz

Question #6: Is the following a well-formed expression?

{+ {fun {} 1} 8}

Answer: **Yes (at least grammatically)**

Quiz

Question #7: What is the value of the following expression?

`{+ {fun {} 1} 8}`

Quiz

Question #7: What is the value of the following expression?

`{+ {fun {} 1} 8}`

Answer: **None** — it produces an error:

+: expects a numV, given a closureV

Quiz

Question #7: What is the value of the following expression?

`{+ {fun {} 1} 8}`

Answer: **None** — it produces an error:

+: expects a numV, given a closureV

Let's agree that a **fun** expression cannot be inside a **+** form

Quiz

Question #8: Is the following a well-formed expression?

{+ {fun {} 1} 8}

Quiz

Question #8: Is the following a well-formed expression?

{+ {fun {} 1} 8}

Answer: **No**

Quiz

Question #9: Is the following a well-formed expression?

{+ {{fun {x} x} 7} 5}

Quiz

Question #9: Is the following a well-formed expression?

`{+ {{fun {x} x} 7} 5}`

Answer: Depends on what we meant by *inside* in our most recent agreement

- *Anywhere inside* — **No**
- *Immediately inside* — **Yes**

Quiz

Question #9: Is the following a well-formed expression?

`{+ {{fun {x} x} 7} 5}`

Answer: Depends on what we meant by *inside* in our most recent agreement

- *Anywhere inside* — **No**
- *Immediately inside* — **Yes**

Since substitution produces **12**, and since that result makes sense, let's agree on *immediately inside*

Quiz

Question #10: Is the following a well-formed expression?

`{+ {{fun {x} x} {fun {y} y}}} 5}`

Quiz

Question #10: Is the following a well-formed expression?

`{+ {{fun {x} x} {fun {y} y}}} 5}`

Answer: **Yes**, but we don't want it to be!

Quiz

Question #11: Is it possible to define ***well-formed*** (as a decidable property) so that we reject all expressions that produce errors?

Quiz

Question #11: Is it possible to define ***well-formed*** (as a decidable property) so that we reject all expressions that produce errors?

Answer: **Yes:** reject *all* expressions!

Quiz

Question #12: Is it possible to define ***well-formed*** (as a decidable property) so that we reject *only* expressions that produce errors?

Quiz

Question #12: Is it possible to define ***well-formed*** (as a decidable property) so that we reject *only* expressions that produce errors?

Answer: **No**

Quiz

Question #12: Is it possible to define *well-formed* (as a decidable property) so that we reject *only* expressions that produce errors?

Answer: No

```
{+ 1 {if ... 1 {fun {x} x}}}
```

If we always knew whether ... produces true or false, we could solve the halting problem

Types

Solution to our dilemma:

In the process of rejecting expressions that are certainly bad, also reject some good expressions

```
{+ 1 {if {prime? 131101}  
1  
{fun {x} x}}}}
```

Types

Solution to our dilemma:

In the process of rejecting expressions that are certainly bad, also reject some good expressions

```
{+ 1 {if {prime? 131101}
          1
          {fun {x} x}}}}
```

Overall strategy:

- Assign a ***type*** to each expression
without evaluating
- Compute the type of a complex expression based on the types of its subexpressions

Types

1 : *num*

true : *bool*

Types

1 : *num*

true : *bool*

{+ 1 2}

Types

1 : *num*

true : *bool*

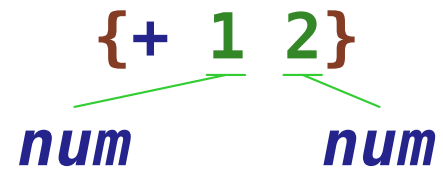
{+ 1 2}
num

Types

1 : *num*

true : *bool*

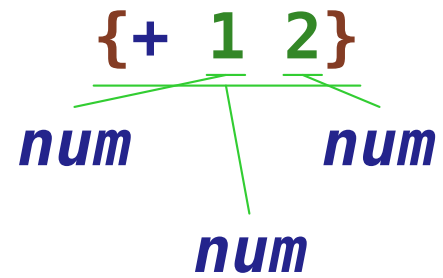
{+ 1 2}
 num *num*



Types

1 : *num*

true : *bool*



Types

1 : *num*

true : *bool*

{+ 1 2}
num *num*
num

{+ 1 false}

Types

1 : *num*

true : *bool*

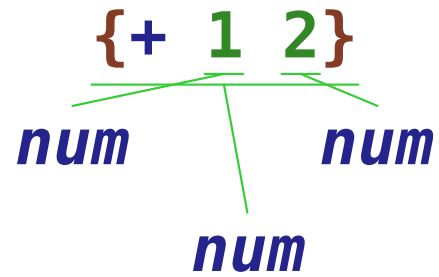
{+ 1 2}
num *num*
num

{+ 1 false}
num

Types

1 : *num*

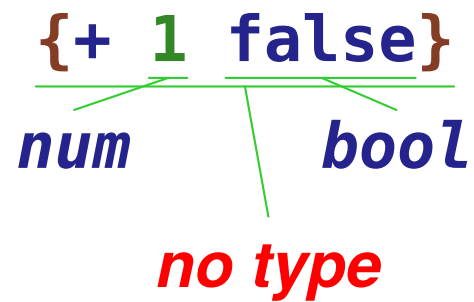
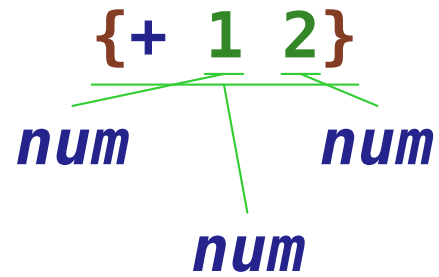
true : *bool*



Types

1 : *num*

true : *bool*



Type Rules

<num> : *num*

true : *bool*

false : *bool*

<MFAE>₁ : *num*

<MFAE>₂ : *num*

{+ <MFAE>₁ <MFAE>₂} : *num*

Type Rules

<num> : num

true : bool

false : bool

<MFAE>₁ : num

<MFAE>₂ : num

{+ <MFAE>₁ <MFAE>₂} : num

1 : num

true : bool

Type Rules

<num> : num

true : bool

false : bool

<MFAE>₁ : num

<MFAE>₂ : num

{+ <MFAE>₁ <MFAE>₂} : num

1 : num

true : bool

1 : num

2 : num

{+ 1 2} : num

Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

$\{+ \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2\} : \text{num}$

$1 : \text{num}$

$\text{true} : \text{bool}$

$1 : \text{num}$

$2 : \text{num}$

$\{+ 1 2\} : \text{num}$

$1 : \text{num}$

$\text{false} : \text{bool}$

$\{+ 1 \text{false}\} : \text{no type}$

Type Rules

$\langle \text{num} \rangle : \text{num}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

$\langle \text{MFAE} \rangle_1 : \text{num}$

$\langle \text{MFAE} \rangle_2 : \text{num}$

$\{+ \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2\} : \text{num}$

$1 : \text{num}$

$2 : \text{num}$

$\{+ 1 2\} : \text{num}$

$3 : \text{num}$

$\{+ \{+ 1 2\} 3\} : \text{num}$


Types: Conditionals

```
{if true 1 2}
```

Types: Conditionals

```
{if true 1 2}
```

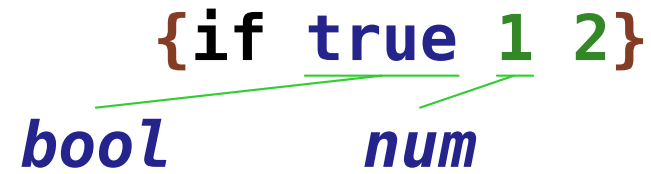
bool



Types: Conditionals

{if true 1 2}

bool *num*

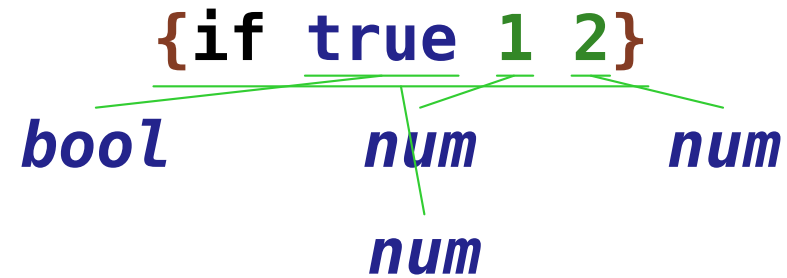


Types: Conditionals

{if true 1 2}

bool *num* *num*

Types: Conditionals



Types: Conditionals

`{if true 1 2}`

bool *num* *num*

`{if {+ 1 2} 1 2}`

Types: Conditionals

`{if true 1 2}`

bool *num* *num*

`{if {+ 1 2} 1 2}`

num

Types: Conditionals

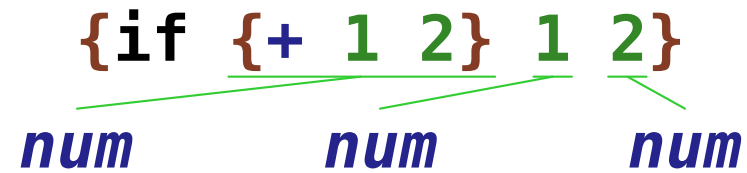
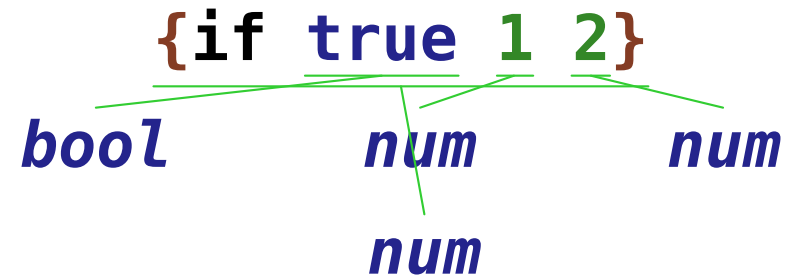
{if true 1 2}

bool *num* *num*

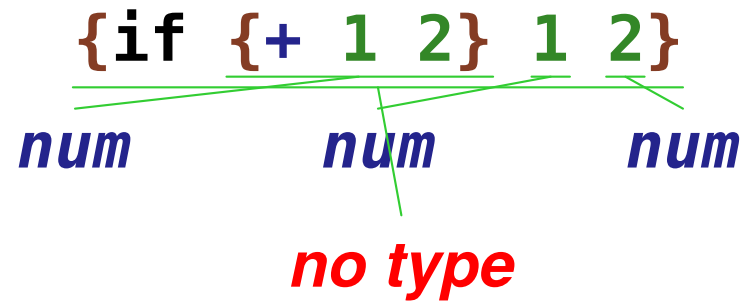
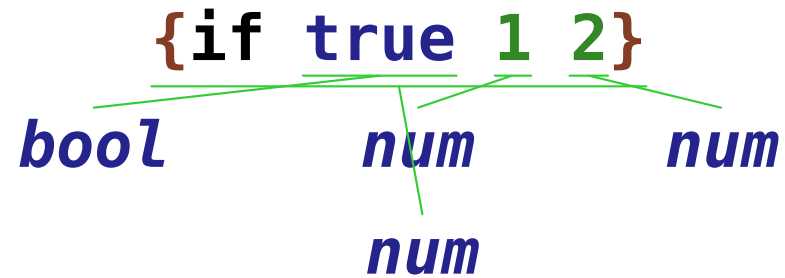
{if {+ 1 2} 1 2}

num *num*

Types: Conditionals



Types: Conditionals



Types: Conditionals

`{if true 1 2}`

bool *num* *num*

num

`{if {+ 1 2} 1 2}`

num *num* *num*

no type

`{if false 1 false}`

Types: Conditionals

`{if true 1 2}`

bool *num* *num*

num

`{if {+ 1 2} 1 2}`

num *num* *num*

no type

`{if false 1 false}`

bool

Types: Conditionals

{if true 1 2}

bool *num* *num*

num

{if {+ 1 2} 1 2}

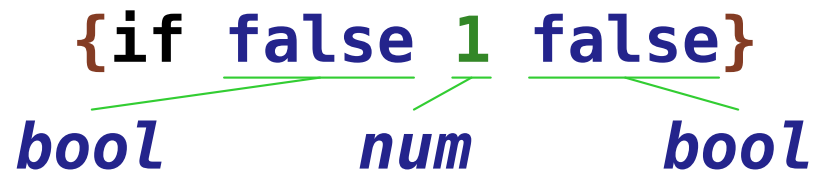
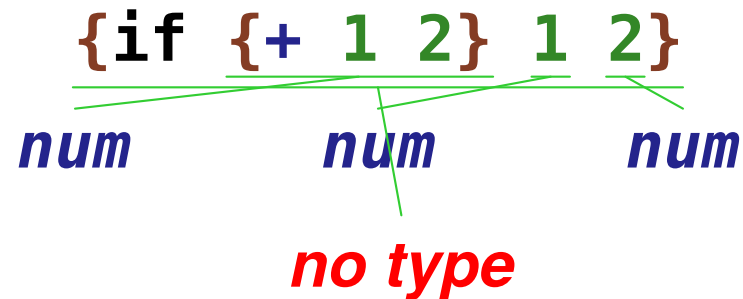
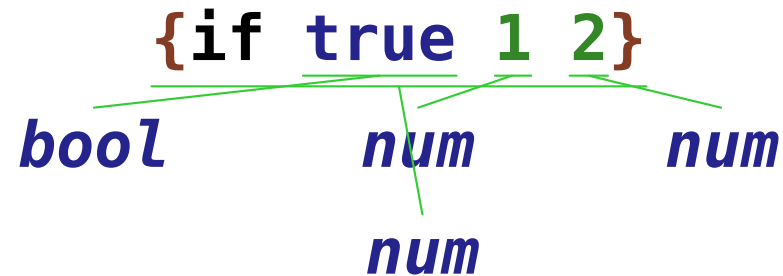
num *num* *num*

no type

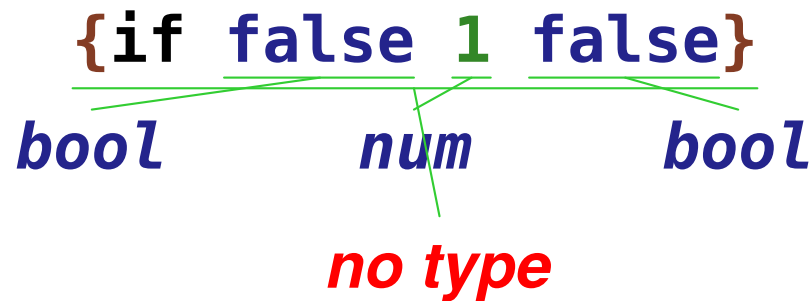
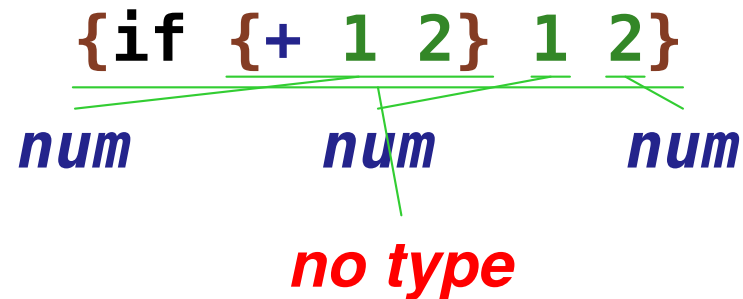
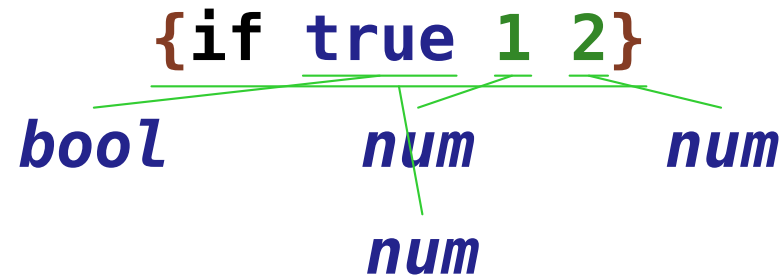
{if false 1 false}

bool *num*

Types: Conditionals



Types: Conditionals



Conditional Type Rules

$$\frac{\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0}{\{\text{if } \langle \text{MFAE} \rangle_1 \ \langle \text{MFAE} \rangle_2 \ \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0}$$

Conditional Type Rules

$$\frac{\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0}{\{\text{if } \langle \text{MFAE} \rangle_1 \ \langle \text{MFAE} \rangle_2 \ \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0}$$
$$\frac{\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}}{\{\text{if true } 1 \ 2\} : \text{num}}$$

Conditional Type Rules

$$\frac{\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0}{\{\text{if } \langle \text{MFAE} \rangle_1 \ \langle \text{MFAE} \rangle_2 \ \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0}$$
$$\frac{\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}}{\{\text{if true } 1 \ 2\} : \text{num}}$$
$$\frac{\{+ \ 1 \ 2\} : \text{num} \quad 1 : \text{num} \quad 2 : \text{num}}{\{\text{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \text{no type}}$$

Conditional Type Rules

$$\frac{\langle \text{MFAE} \rangle_1 : \text{bool} \quad \langle \text{MFAE} \rangle_2 : \langle \text{type} \rangle_0 \quad \langle \text{MFAE} \rangle_3 : \langle \text{type} \rangle_0}{\{\text{if } \langle \text{MFAE} \rangle_1 \langle \text{MFAE} \rangle_2 \langle \text{MFAE} \rangle_3\} : \langle \text{type} \rangle_0}$$
$$\frac{\text{true} : \text{bool} \quad 1 : \text{num} \quad 2 : \text{num}}{\{\text{if true } 1 \ 2\} : \text{num}}$$
$$\frac{\{+ \ 1 \ 2\} : \text{num} \quad 1 : \text{num} \quad 2 : \text{num}}{\{\text{if } \{+ \ 1 \ 2\} \ 1 \ 2\} : \text{no type}}$$
$$\frac{\text{false} : \text{bool} \quad 1 : \text{num} \quad \text{false} : \text{bool}}{\{\text{if false } 1 \ \text{false}\} : \text{no type}}$$

Types: Variables and Functions

x : *no type*

Types: Variables and Functions


`x : no type`

`{fun {x : bool} x}`

Types: Variables and Functions

x : *no type*

{fun {x : bool} x}
bool



Types: Variables and Functions

x : *no type*

{fun {x : bool} x}

bool

(bool → bool)

Types: Variables and Functions

x : *no type*

{fun {x : bool} x}

bool

(*bool* → *bool*)

{fun {x : bool} {if x 1 2}}

Types: Variables and Functions

x : *no type*

{fun {x : bool} x}

bool

(bool → bool)

{fun {x : bool} {if x 1 2}}

bool

Types: Variables and Functions

x : *no type*

{fun {x : bool} x}

bool

(*bool* → *bool*)

{fun {x : bool} {if x 1 2}}

bool

num

Types: Variables and Functions

x : *no type*

{fun {x : bool} x}

bool

(*bool* → *bool*)

{fun {x : bool} {if x 1 2}}

bool

num

num

Types: Variables and Functions

x : *no type*

{fun {x : bool} x}

bool

(*bool* → *bool*)

{fun {x : bool} {if x 1 2}}

bool

num

num

num

Types: Variables and Functions

x : no type

{fun {x : bool} x}

bool

(bool → bool)

{fun {x : bool} {if x 1 2}}

bool

num

num

num

(bool → num)

Variable and Function Type Rules

$$[\dots \textcolor{blue}{<id>} \leftarrow \textcolor{blue}{\tau} \dots] \vdash \textcolor{blue}{<id>} : \tau$$

$$\Gamma[\textcolor{blue}{<id>} \leftarrow \tau_1] \vdash \textcolor{blue}{e} : \tau_2$$

$$\Gamma \vdash \textcolor{brown}{\{fun \{<id> : \tau_1\} e\}} : (\tau_1 \rightarrow \tau_2)$$

Abbreviations: $\tau = \textcolor{blue}{<type>}$
 $e = \textcolor{blue}{<MFAE>}$
 $\Gamma = \textcolor{blue}{<env>}$

Variable and Function Type Rules

$$[\dots \textcolor{blue}{<id>} \leftarrow \tau \dots] \vdash \textcolor{blue}{<id>} : \tau$$

$$\Gamma[\textcolor{blue}{<id>} \leftarrow \tau_1] \vdash \textcolor{blue}{e} : \tau_2$$

$$\Gamma \vdash \textcolor{brown}{\{fun \{<id> : \tau_1\} e\}} : (\tau_1 \rightarrow \tau_2)$$

$$\emptyset \vdash \textcolor{blue}{x} : \textcolor{red}{no type}$$

Variable and Function Type Rules

$$[\dots \textcolor{blue}{<id>} \leftarrow \tau \dots] \vdash \textcolor{blue}{<id>} : \tau$$

$$\Gamma[\textcolor{blue}{<id>} \leftarrow \tau_1] \vdash \textcolor{blue}{e} : \tau_2$$

$$\Gamma \vdash \{\textcolor{brown}{fun} \{ \textcolor{blue}{<id>} : \tau_1 \} \textcolor{brown}{e} \} : (\tau_1 \rightarrow \tau_2)$$

$$\emptyset \vdash \textcolor{blue}{x} : \textcolor{red}{no type}$$

$$[\textcolor{blue}{x} \leftarrow \textcolor{blue}{bool}] \vdash \textcolor{blue}{x} : \textcolor{blue}{bool}$$

$$\emptyset \vdash \{\textcolor{brown}{fun} \{ \textcolor{blue}{x} : \textcolor{blue}{bool} \} \textcolor{brown}{x} \} : (\textcolor{blue}{bool} \rightarrow \textcolor{blue}{bool})$$

Variable and Function Type Rules

$$[\dots \textcolor{blue}{<id>} \leftarrow \tau \dots] \vdash \textcolor{blue}{<id>} : \tau$$

$$\Gamma[\textcolor{blue}{<id>} \leftarrow \tau_1] \vdash \textcolor{blue}{e} : \tau_2$$

$$\Gamma \vdash \{\textcolor{blue}{fun} \{ \textcolor{blue}{<id>} : \tau_1 \} \textcolor{brown}{e} \} : (\tau_1 \rightarrow \tau_2)$$

$$\emptyset \vdash \textcolor{blue}{x} : \textcolor{red}{no type}$$

$$[\textcolor{blue}{x} \leftarrow \textcolor{blue}{bool}] \vdash \textcolor{blue}{x} : \textcolor{blue}{bool}$$

$$\emptyset \vdash \{\textcolor{brown}{fun} \{ \textcolor{blue}{x} : \textcolor{blue}{bool} \} \textcolor{brown}{x} \} : (\textcolor{blue}{bool} \rightarrow \textcolor{blue}{bool})$$

$$[\textcolor{blue}{x} \leftarrow \textcolor{blue}{bool}] \vdash \textcolor{blue}{x} : \textcolor{blue}{bool} \quad [\textcolor{blue}{x} \leftarrow \textcolor{blue}{bool}] \vdash \textcolor{green}{1} : \textcolor{blue}{num} \quad [\textcolor{blue}{x} \leftarrow \textcolor{blue}{bool}] \vdash \textcolor{green}{2} : \textcolor{blue}{num}$$

$$[\textcolor{blue}{x} \leftarrow \textcolor{blue}{bool}] \vdash \{\textcolor{brown}{if} \textcolor{blue}{x} \textcolor{green}{1} \textcolor{green}{2} \} : \textcolor{blue}{num}$$

$$\emptyset \vdash \{\textcolor{brown}{fun} \{ \textcolor{blue}{x} : \textcolor{blue}{bool} \} \{\textcolor{brown}{if} \textcolor{blue}{x} \textcolor{green}{1} \textcolor{green}{2} \} \} : (\textcolor{blue}{bool} \rightarrow \textcolor{blue}{num})$$

Revised Rules

$$\Gamma \vdash \text{<num>} : \textit{num}$$
$$\Gamma \vdash \text{true} : \textit{bool}$$
$$\Gamma \vdash \text{false} : \textit{bool}$$
$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{num} \quad \Gamma \vdash \mathbf{e}_2 : \textit{num}}{\Gamma \vdash \{+ \mathbf{e}_1 \mathbf{e}_2\} : \textit{num}}$$
$$\frac{\Gamma \vdash \mathbf{e}_1 : \textit{bool} \quad \Gamma \vdash \mathbf{e}_2 : \tau_0 \quad \Gamma \vdash \mathbf{e}_3 : \tau_0}{\Gamma \vdash \{\mathbf{if} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3\} : \tau_0}$$

Types: Function Calls

```
{{fun {x : bool} {if x 1 2}}} true}
```

Types: Function Calls

{{fun {x : bool} {if x 1 2}}} true}

(*bool* → *num*)

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}
 (*bool* → *num*) *bool*

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool} \quad \text{num}}$

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{\text{num}}$

$(\text{bool} \rightarrow \text{num})$ bool

$\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}$

Types: Function Calls

$\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \text{true}\}$

$(\text{bool} \rightarrow \text{num})$ bool

num

$\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}$

$(\text{bool} \rightarrow \text{num})$

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool} \quad \text{num}}$

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num}}$

Types: Function Calls

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ \text{true}\}}{(\text{bool} \rightarrow \text{num}) \quad \text{bool} \quad \text{num}}$

$\frac{\{\{\text{fun } \{x : \text{bool}\} \{\text{if } x \ 1 \ 2\}\} \ 5\}}{(\text{bool} \rightarrow \text{num}) \quad \text{num} \quad \text{no type}}$

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}

$(bool \rightarrow num)$ $bool$

num

{{fun {x : bool} {if x 1 2}} 5}

$(bool \rightarrow num)$ num

no type

$\{7\ 5\}$

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}

$(bool \rightarrow num)$ *bool*

num

{{fun {x : bool} {if x 1 2}} 5}

$(bool \rightarrow num)$ *num*

no type

{7 5}

num

Types: Function Calls

{{fun {x : bool} {if x 1 2}} true}

$(bool \rightarrow num)$ $bool$

num

{{fun {x : bool} {if x 1 2}} 5}

$(bool \rightarrow num)$ num

no type

{7 5}

num num

Types: Function Calls

`{{fun {x : bool} {if x 1 2}} true}`

$(bool \rightarrow num)$ $bool$

num

`{{fun {x : bool} {if x 1 2}} 5}`

$(bool \rightarrow num)$ num

no type

`{7 5}`

num num

no type

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

Function Call Type Rule

$$\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \qquad \Gamma \vdash \mathbf{e}_2 : \tau_2$$

$$\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3$$

$$\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \qquad \emptyset \vdash \text{true} : \text{bool}$$

$$\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{num}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

Function Call Type Rule

$$\frac{\Gamma \vdash \mathbf{e}_1 : (\tau_2 \rightarrow \tau_3) \quad \Gamma \vdash \mathbf{e}_2 : \tau_2}{\Gamma \vdash \{\mathbf{e}_1 \ \mathbf{e}_2\} : \tau_3}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash \text{true} : \text{bool}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} \ \text{true}\} : \text{num}}$$

$$\frac{\emptyset \vdash \{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} : (\text{bool} \rightarrow \text{num}) \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{\{\text{fun } \{x : \text{bool}\} \ \{\text{if } x \ 1 \ 2\}\} \ 5\} : \text{no type}}$$

$$\frac{\emptyset \vdash 7 : \text{num} \quad \emptyset \vdash 5 : \text{num}}{\emptyset \vdash \{7 \ 5\} : \text{no type}}$$

Types: Multiple Arguments

```
{fun {x : num y : num} {+ x y}}
```

Types: Multiple Arguments

```
{fun {x : num y : num} {+ x y}}
```



num

Types: Multiple Arguments

{fun {x : num y : num} {+ x y}}



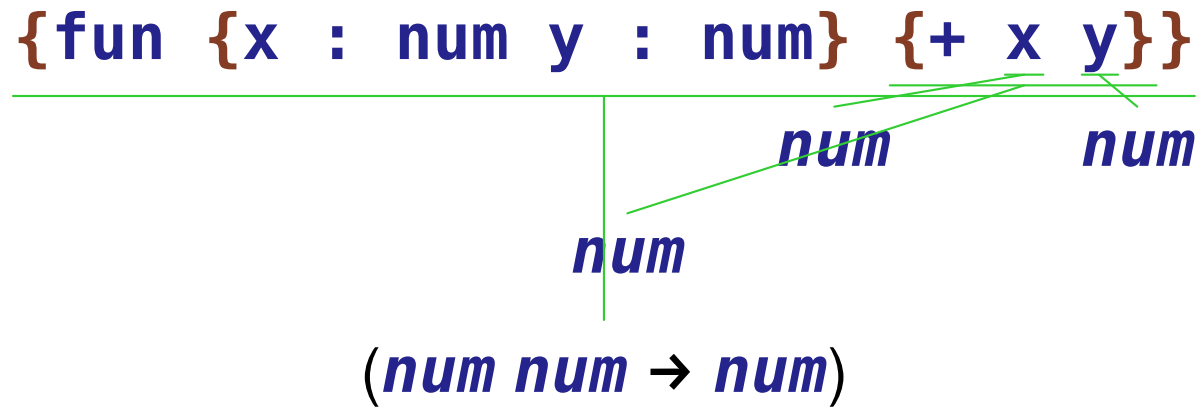
The diagram illustrates the type annotations for the arguments *x* and *y* in the function definition. Two green lines point from the *num* type annotations below to the variables *x* and *y* in the function body. The first *num* is positioned under the *x* in the argument list, and the second *num* is positioned under the *y* in the function body.

Types: Multiple Arguments

{fun {x : num y : num} {+ x y}}

The diagram illustrates the type annotations for the function `{fun {x : num y : num} {+ x y}}`. Three green lines point from the word `num` to the arguments `x`, `y`, and the expression `+ x y`. The first `num` is positioned below `x`, the second `num` is positioned below `y`, and the third `num` is positioned below the `+` operator.

Types: Multiple Arguments



Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

num num num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

num num num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$

Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

num num

num

$(\text{num num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num num} \rightarrow \text{num})$ num

Types: Multiple Arguments

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num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num num

Types: Multiple Arguments

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num num num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num num

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Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

num num num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num num

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$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}$

Types: Multiple Arguments

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num num num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num num

num

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}$

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Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

num num num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num num

num

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num

Types: Multiple Arguments

$\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\}$

num num num

$(\text{num } \text{num} \rightarrow \text{num})$

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5 6\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num num

num

$\{\{\text{fun } \{x : \text{num } y : \text{num}\} \{+ x y\}\} 5\}$

$(\text{num } \text{num} \rightarrow \text{num})$ num

no type

Revised Function and Call Rules

$$\frac{\Gamma[\textcolor{blue}{<id>}_1 \leftarrow \tau_1 \dots \textcolor{blue}{<id>}_n \leftarrow \tau_n] \vdash \textcolor{blue}{e} : \tau_0}{\Gamma \vdash \{\textcolor{brown}{fun} \{\textcolor{blue}{<id>}_1 : \tau_1 \dots \textcolor{blue}{<id>}_n : \tau_n\} \textcolor{brown}{e}\} : (\tau_1 \dots \tau_n \rightarrow \tau_0)}$$

$$\frac{\Gamma \vdash \textcolor{blue}{e}_0 : (\tau_1 \dots \tau_n \rightarrow \tau_0) \quad \Gamma \vdash \textcolor{blue}{e}_1 : \tau_1 \quad \dots \quad \Gamma \vdash \textcolor{blue}{e}_n : \tau_n}{\Gamma \vdash \{\textcolor{brown}{e}_0 \textcolor{blue}{e}_1 \dots \textcolor{blue}{e}_n\} : \tau_0}$$