Let $P_0, \cdots, P_3$ be the control points of curve p and P_0, Q_1, Q_2, P_3 the control points of curve q. We have $\|q(t) - p(t)\|$ over-estimates the true distance between the point $p(t)$ and the curve q, since the distance $\operatorname{dist}(q(t), p)$ is defined by $\min_{t'} \|p(t) - q(t')\|$. Therefore $\max_t \|p(t) - q(t)\|$ over-estimates $\max_t \operatorname{dist}(p(t), q)$.

$$\min_{t'} \|p(t) - q(t')\| \leq \|p(t) - q(t)\| \tag{0.1}$$

$$\Rightarrow \operatorname{dist}(p(t), q) \leq \|p(t) - q(t)\| \tag{0.2}$$

$$\Rightarrow \max_t \operatorname{dist}(p(t), q) \leq \max_t \|p(t) - q(t)\| \tag{0.3}$$

0.1 Calculating $\operatorname{argmax}_t \|p(t) - q(t)\|_2^2$

We calculate $\operatorname{argmax}_t \|p(t) - q(t)\|_2^2$:

$$\begin{aligned}
\|p(t) - q(t)\|_2^2 &= \|(3(1-t)^2 t P_1 + 3(1-t)t^2 P_2) - (3(1-t)^2 t Q_1 + 3(1-t)t^2 Q_2)\|_2^2 && (0.4)\\
&= \|3(1-t)^2 t(P_1 - Q_1) + 3(1-t)t^2(P_2 - Q_2)\|_2^2 && (0.5)\\
&= 9t^2(1-t)^2 \|(1-t)(P_1 - Q_1) + t(P_2 - Q_2)\|_2^2 && (0.6)\\
&= 9t^2(1-t)^2 \left\| \begin{pmatrix} (1-t)(P_{1x} - Q_{1x}) + t(P_{2x} - Q_{2x}) \\ (1-t)(P_{1y} - Q_{1y}) + t(P_{2y} - Q_{2y}) \end{pmatrix} \right\|_2^2 && (0.7)
\end{aligned}$$

For ease of calculation we introduce variable substitutions:

$$\begin{aligned}
\alpha_{1x} &:= P_{1x} - Q_{1x} && (0.8)\\
\alpha_{2x} &:= P_{2x} - Q_{2x} && (0.9)\\
\alpha_{1y} &:= P_{1y} - Q_{1y} && (0.10)\\
\alpha_{2y} &:= P_{2y} - Q_{2y} && (0.11)\\
\Rightarrow \|p(t) - q(t)\|_2^2 &= 9t^2(1-t)^2[((1-t)\alpha_{1x} + t\alpha_{2x})^2 + ((1-t)\alpha_{1y} + t\alpha_{2y})^2] && (0.12)
\end{aligned}$$

Let's calculate the first derivative wrt. t and see what we get: Well, we obviously get a polynomial of degree 3 and we need to find the roots which is no fun, so lets's do something else: Triangle inequality.

0.2 Using the triangle inequality

We have $\|a + b\| \leq \|a\| + \|b\|$ for any two vectors a, b and any norm $\|\cdot\|$. Therefore:

$$\begin{aligned}
\|p(t) - q(t)\| &= \|(3(1-t)^2 t P_1 + 3(1-t)t^2 P_2) - (3(1-t)^2 t Q_1 + 3(1-t)t^2 Q_2)\| && (0.13)\\
&\leq \|3(1-t)^2 t(P_1 - Q_1)\| + \|3(1-t)t^2(P_2 - Q_2)\| =: \psi(t) && (0.14)\\
&= 3(1-t)^2 t \underbrace{\|P_1 - Q_1\|}_{:=\alpha_1} + 3(1-t)t^2 \underbrace{\|P_2 - Q_2\|}_{\alpha_2} && (0.15)\\
&= 3(1-t)^2 t\alpha_1 + 3(1-t)t^2\alpha_2 && (0.16)\\
& && (0.17)
\end{aligned}$$

Now we have a upper bound $\psi(t)$ for each $t \in [0, 1]$. We maximize it wrt. t and obtain an upper bound for $\mathrm{dist}(p, q)$.

$$
\begin{align}
0 &= \partial_t[3(1-t)^2 t\alpha_1 + 3(1-t)t^2\alpha_2] \tag{0.18}\\
\Leftrightarrow 0 &= \partial_t[(1-2t+t^2)t\alpha_1 + (t^2-t^3)\alpha_2] \tag{0.19}\\
\Leftrightarrow 0 &= \partial_t[(t-2t^2+t^3)\alpha_1 + (t^2-t^3)\alpha_2] \tag{0.20}\\
\Leftrightarrow 0 &= (1-4t+3t^2)\alpha_1 + (2t-3t^2)\alpha_2 \tag{0.21}\\
\Leftrightarrow 0 &= \alpha_1 + (2\alpha_2 - 4\alpha_1)t + (3\alpha_1 - 3\alpha_2)t^2 \tag{0.22}\\
& \tag{0.23}
\end{align}
$$

And that's just implementation stuff, calculate both roots t_1, t_2, the upper bound then is $\max\{\psi(t_1), \psi(t_2)\}$. t_1 and t_2 are probably good candidates for the actual time values maximizing the distance between p and q, therefore one could calculate:

$$
\begin{align}
l_1 &:= \min_{t'} \|p(t_1) - q(t')\| \tag{0.24}\\
l_2 &:= \min_{t'} \|p(t_2) - q(t')\| \tag{0.25}\\
l_3 &:= \min_{t'} \|p(t') - q(t_1)\| \tag{0.26}\\
l_4 &:= \min_{t'} \|p(t') - q(t_2)\| \tag{0.27}\\
& \tag{0.28}
\end{align}
$$

Then $\max\{l_1, l_2, l_3, l_4\}$ is a lower bound for the maximum distance of p and q.

0.3 Estimation without finding roots of polynomials

We want to do something even simpler. We know now that:

$$\mathrm{dist}(p, q) \leq \max_t [3(1-t)^2 t\alpha_1 + 3(1-t)t^2\alpha_2]$$

We want to find the maximum value of both summands.

$$
\begin{aligned}
f_1(t) &:= (1-t)^2 t = (1-2t+t^2)t = t - 2t^2 + t^3\\
f_1(1) &= f_1(0) = 0\\
f_1'(t) &= 3t^2 - 4t + 1\\
t_{1,2} &:= \frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6} = \{1, 1/3\}\\
f_1(1/3) &= (2/3)^2 \frac{1}{3} = \frac{4}{9}\frac{1}{3} = \frac{4}{27}\\
f_2(t) &:= (1-t)t^2 = f_1(1-t)\\
\Rightarrow f_2(2/3) &= \frac{4}{27}
\end{aligned}
$$

Now we can derive a simper, but coarser estimation:

$$\begin{aligned}
\operatorname{dist}(p,q) &\leq \max_t [3(1-t)^2 t\alpha_1 + 3(1-t)t^2\alpha_2] \\
&\leq \max_t 3(1-t)^2 t\alpha_1 + \max_t 3(1-t)t^2\alpha_2 \\
&= \frac{4}{9}\alpha_1 + \frac{4}{9}\alpha_2 = \frac{4}{9}(\alpha_1+\alpha_2) \\
&= \frac{4}{9}(\|P_1-Q_1\| + \|P_2-Q_2\|)
\end{aligned}$$

0.4 Estimation using matrix norms

For a matrix A and vector b we have $\|Ab\|_2 \leq \|A\|_2\|b\|_2$ where $\|A\|_2$ denotes the spectral norm of A.

We have:

$$\begin{aligned}
\|p(t)-q(t)\|_2 &= \|3(1-t)^2 t(P_1-Q_1) + 3(1-t)t^2(P_2-Q_2)\|_2 \\
&= \left\| \begin{pmatrix} 3(1-t)^2 t & 3(1-t)t^2 \end{pmatrix} \begin{pmatrix} P_1-Q_1 \\ P_2-Q_2 \end{pmatrix} \right\|_2 \\
&= 3 \left\| \underbrace{\begin{pmatrix} (1-t)^2 t & (1-t)t^2 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} P_1-Q_1 \\ P_2-Q_2 \end{pmatrix}}_{=:b} \right\|_2 \\
&\leq 3 \left\| \begin{pmatrix} (1-t)^2 t & (1-t)t^2 \end{pmatrix} \right\|_2 \left\| \begin{pmatrix} P_1-Q_1 \\ P_2-Q_2 \end{pmatrix} \right\|_2
\end{aligned}$$

For this special matrix the spectral norm is equal to the Euklidean norm of the row A, so we need to maximize:

$$f(t) := ((1-t)^2 t)^2 + ((1-t)t^2)^2$$

We used a CAS to find that f has only one local maximum in the interval $[0,1]$ at $t = 1/2$ $f(1/2) = 1/32$, so $\|A\|_2 = \sqrt{1/32} = \sqrt{1/2}\sqrt{1/16} = 1/(4\sqrt{2})$. Therefore:

$$\operatorname{dist}(p,q) \leq \frac{3}{4\sqrt{2}} \left\| \underbrace{\begin{pmatrix} P_1-Q_1 \\ P_2-Q_2 \end{pmatrix}}_{=b} \right\|_2$$

Note that b is in this case a 2×2-matrix, so we need a estimation of its spectral norm. Standard estimations are $\sqrt{\|b\|_\infty \cdot \|b\|_1}$ and $\|b\|_F$.