

## 0.1 Calculating $\operatorname{argmax}_t \|p(t) - q(t)\|_2^2$

Let  $P_0, \dots, P_3$  be the control points of curve  $p$  and  $P_0, Q_1, Q_2, P_3$  the control points of curve  $q$ . We have  $\|q(t) - p(t)\|$  over-estimates the true distance between the point  $p(t)$  and the curve  $q$ , since the distance  $\operatorname{dist}(q(t), p)$  is defined by  $\min_{t'} \|p(t) - q(t')\|$ . Therefore  $\max_t \|p(t) - q(t)\|$  over-estimates  $\max_t \operatorname{dist}(p(t), q)$ .

$$\min_{t'} \|p(t) - q(t')\| \leq \|p(t) - q(t)\| \quad (0.1)$$

$$\Rightarrow \operatorname{dist}(p(t), q) \leq \|p(t) - q(t)\| \quad (0.2)$$

$$\Rightarrow \max_t \operatorname{dist}(p(t), q) \leq \max_t \|p(t) - q(t)\| \quad (0.3)$$

## 0.1 Calculating $\operatorname{argmax}_t \|p(t) - q(t)\|_2^2$

We calculate  $\operatorname{argmax}_t \|p(t) - q(t)\|_2^2$ :

$$\|p(t) - q(t)\|_2^2 = \|(3(1-t)^2tP_1 + 3(1-t)t^2P_2) - (3(1-t)^2tQ_1 + 3(1-t)t^2Q_2)\|_2^2 \quad (0.4)$$

$$= \|3(1-t)^2t(P_1 - Q_1) + 3(1-t)t^2(P_2 - Q_2)\|_2^2 \quad (0.5)$$

$$= 9t^2(1-t)^2\|(1-t)(P_1 - Q_1) + t(P_2 - Q_2)\|_2^2 \quad (0.6)$$

$$= 9t^2(1-t)^2 \left\| \begin{pmatrix} (1-t)(P_{1x} - Q_{1x}) + t(P_{2x} - Q_{2x}) \\ (1-t)(P_{1y} - Q_{1y}) + t(P_{2y} - Q_{2y}) \end{pmatrix} \right\|_2^2 \quad (0.7)$$

For ease of calculation we introduce variable substitutions:

$$\alpha_{1x} := P_{1x} - Q_{1x} \quad (0.8)$$

$$\alpha_{2x} := P_{2x} - Q_{2x} \quad (0.9)$$

$$\alpha_{1y} := P_{1y} - Q_{1y} \quad (0.10)$$

$$\alpha_{2y} := P_{2y} - Q_{2y} \quad (0.11)$$

$$\Rightarrow \|p(t) - q(t)\|_2^2 = 9t^2(1-t)^2 [((1-t)\alpha_{1x} + t\alpha_{2x})^2 + ((1-t)\alpha_{1y} + t\alpha_{2y})^2] \quad (0.12)$$

Let's calculate the first derivative wrt.  $t$  and see what we get: Well, we obviously get a polynomial of degree 3 and we need to find the roots which is no fun, so let's do something else: Triangle inequality.

## 0.2 Using the triangle inequality

We have  $\|a + b\| \leq \|a\| + \|b\|$  for any two vectors  $a, b$  and any norm  $\|\cdot\|$ . Therefore:

$$\|p(t) - q(t)\| = \|3(1-t)^2tP_1 + 3(1-t)t^2P_2 - (3(1-t)^2tQ_1 + 3(1-t)t^2Q_2)\| \quad (0.13)$$

$$\leq \|3(1-t)^2t(P_1 - Q_1)\| + \|3(1-t)t^2(P_2 - Q_2)\| =: \psi(t) \quad (0.14)$$

$$= 3(1-t)^2t \underbrace{\|P_1 - Q_1\|}_{:=\alpha_1} + 3(1-t)t^2 \underbrace{\|P_2 - Q_2\|}_{\alpha_2} \quad (0.15)$$

$$= 3(1-t)^2t\alpha_1 + 3(1-t)t^2\alpha_2 \quad (0.16)$$

$$(0.17)$$

Now we have a upper bound  $\psi(t)$  for each  $t \in [0, 1]$ . We maximize it wrt.  $t$  and obtain an upper bound for  $\text{dist}(p, q)$ .

$$0 = \partial_t [3(1-t)^2 t \alpha_1 + 3(1-t) t^2 \alpha_2] \quad (0.18)$$

$$\Leftrightarrow 0 = \partial_t [(1-2t+t^2) t \alpha_1 + (t^2-t^3) \alpha_2] \quad (0.19)$$

$$\Leftrightarrow 0 = \partial_t [(t-2t^2+t^3) \alpha_1 + (t^2-t^3) \alpha_2] \quad (0.20)$$

$$\Leftrightarrow 0 = (1-4t+3t^2) \alpha_1 + (2t-3t^2) \alpha_2 \quad (0.21)$$

$$\Leftrightarrow 0 = \alpha_1 + (2\alpha_2 - 4\alpha_1)t + (3\alpha_1 - 3\alpha_2)t^2 \quad (0.22)$$

$$(0.23)$$

And that's just implementation stuff, calculate both roots  $t_1, t_2$ , the upper bound then is  $\max\{\psi(t_1), \psi(t_2)\}$ .  $t_1$  and  $t_2$  are probably good candidates for the actual time values maximizing the distance between  $p$  and  $q$ , therefore one could calculate:

$$l_1 := \min_{t'} \|p(t_1) - q(t')\| \quad (0.24)$$

$$l_2 := \min_{t'} \|p(t_2) - q(t')\| \quad (0.25)$$

$$l_3 := \min_{t'} \|p(t') - q(t_1)\| \quad (0.26)$$

$$l_4 := \min_{t'} \|p(t') - q(t_2)\| \quad (0.27)$$

$$(0.28)$$

Then  $\max\{l_1, l_2, l_3, l_4\}$  is a lower bound for the maximum distance of  $p$  and  $q$ .

### 0.3 Estimation without finding roots of polynomials

We want to do something even simpler. We know now that:

$$\text{dist}(p, q) \leq \max_t [3(1-t)^2 t \alpha_1 + 3(1-t) t^2 \alpha_2]$$

We want to find the maximum value of both summands.

$$f_1(t) := (1-t)^2 t = (1-2t+t^2)t = t - 2t^2 + t^3$$

$$f_1(1) = f_1(0) = 0$$

$$f_1'(t) = 3t^2 - 4t + 1$$

$$t_{1,2} := \frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6} = \{1, 1/3\}$$

$$f_1(1/3) = (2/3)^2 \frac{1}{3} = \frac{4}{9} \frac{1}{3} = \frac{4}{27}$$

$$f_2(t) := (1-t)t^2 = f_1(1-t)$$

$$\Rightarrow f_2(2/3) = \frac{4}{27}$$

### 0.3 Estimation without finding roots of polynomials

Now we can derive a simpler, but coarser estimation:

$$\begin{aligned}\text{dist}(p, q) &\leq \max_t [3(1-t)^2 t \alpha_1 + 3(1-t) t^2 \alpha_2] \\ &\leq \max_t 3(1-t)^2 t \alpha_1 + \max_t 3(1-t) t^2 \alpha_2 \\ &= \frac{4}{9} \alpha_1 + \frac{4}{9} \alpha_2 = \frac{4}{9} (\alpha_1 + \alpha_2) \\ &= \frac{4}{9} (\|P_1 - Q_1\| + \|P_2 - Q_2\|)\end{aligned}$$