Calculus Notes

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Contents

1 Limits

1.1 Precise Definition of a Limit

Standard Limit:

$$\lim_{x\to a} f(x) = L \quad \text{if} \quad \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } 0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon.$$

1.2 Precise Definition of One-Sided Limit

Right-Hand Limit:

$$\lim_{x \to a^+} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Left-Hand Limit:

$$\lim_{x \to a^-} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.3 Precise Definition of Infinite Limit

Infinite Limit:

$$\lim_{x \to a} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

$$\lim_{x \to a} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -M.$$

1.4 Precise Definition of a Limit at Infinity

Limit at Infinity:

$$\lim_{x \to \infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \to -\infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x < -M \Rightarrow |f(x) - L| < \varepsilon.$$

1.5 Precise Definition of Infinite Limit at Infinity

Infinite Limit at Infinity:

$$\lim_{x \to \infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$

$$\lim_{x \to \infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) < -M.$$

$$\lim_{x \to -\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) > M.$$

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) < -M.$$

1.6 Limit Laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
4. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
5. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
6. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$
7. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$
8. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

1.7 Relationship between the Limit and One-Sided Limits

$$\lim_{x\to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \quad \Rightarrow \quad \lim_{x\to a} f(x) \text{ does not exist.}$$

1.8 Comparison Theorem

If $f(x) \leq g(x)$ when x is near a, and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

1.9 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

1.10 Continuity

A function $f(x)$ is **continuous at** $x = a$ if and only if it satisfies **all** the following:

$$\begin{aligned} &(1) \quad f(a) \text{ exists} \\ &(2) \quad \lim_{x\to a} f(x) \text{ exists} \\ &(3) \quad \lim_{x\to a} f(x) = f(a) \end{aligned}$$

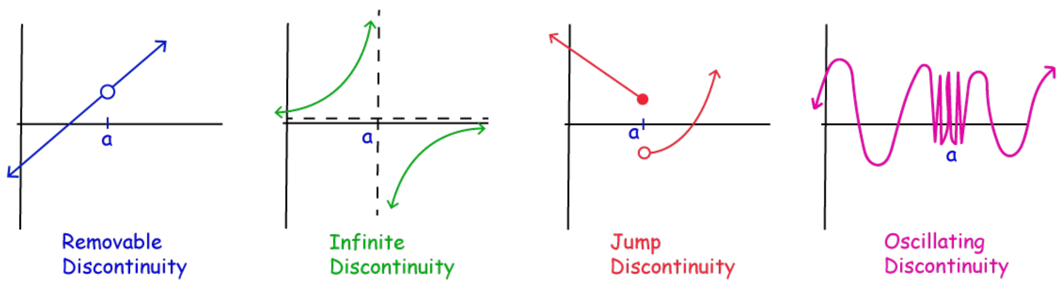
Otherwise, $f(x)$ is discontinuous at $x = a$.

1.11 Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

1.12 Types of Discontinuity



Source: calcworkshop.com

1.13 Limits of Continuous Functions

If $f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

If g is continuous at a and f is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a.

1.14 Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any N between $f(a)$ and $f(b)$,

$$\exists\, c \in [a, b] \text{ such that } f(c) = N$$

1.15 Asymptotes

Vertical Asymptote: $x = a$ is a vertical asymptote if

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty.$$

Horizontal Asymptote: $y = L$ is a horizontal asymptote if

$$\lim_{x \to \pm\infty} f(x) = L.$$

For $f(x) = \dfrac{P(x)}{Q(x)}$, compare degrees of P and Q:

$$\begin{aligned}
\deg P < \deg Q \quad &\Rightarrow \quad y = 0. \\
\deg P = \deg Q \quad &\Rightarrow \quad y = \frac{\text{leading coef. of } P}{\text{leading coef. of } Q}. \\
\deg P > \deg Q \quad &\Rightarrow \quad \text{no horizontal asymptote.}
\end{aligned}$$

Oblique Asymptote: $y = mx + b$ is an oblique asymptote if

$$\lim_{x \to \pm\infty} \big(f(x) - (mx + b)\big) = 0.$$

For a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, if $\deg P = \deg Q + 1$, then $f(x)$ has an oblique asymptote. Find it by polynomial long division:

$$f(x) = D(x) + \frac{R(x)}{Q(x)}, \quad \text{as } x \to \pm\infty, \quad f(x) \approx D(x).$$

Curvilinear Asymptote: $y = g(x)$ is a curvilinear asymptote if

$$\lim_{x \to \pm\infty} \big(f(x) - g(x)\big) = 0,$$

where $g(x)$ is any non-linear function.

1.16 Common Limits

Assume $a > 0$ in the following.

1. $\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$
2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$
3. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
4. $\lim_{x\to 0} \frac{e^{ax}-1}{x} = a$
5. $\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$
6. $\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^{mx} = e^{mk}$
7. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
8. $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$
9. $\lim_{x\to 0^+} x^x = 1$
10. $\lim_{x\to\infty} \sqrt[x]{x} = \lim_{x\to\infty} x^{\frac{1}{x}} = 1$
11. $\lim_{x\to 0^+} x^a \ln x = 0$
12. $\lim_{x\to\infty} x^{-a} \ln x = 0$
13. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \lim$
14. $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$

2 Derivatives

2.1 Derivative at a Point

The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** at that point:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

2.2 Derivative as a Function

The derivative of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

2.3 Differentiability

A function $f(x)$ is **differentiable** at $x = a$ if its derivative $f'(x)$ exists. That is:

$$f(x) \text{ is differentiable at } x = a \iff \text{The limit } \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ exists.}$$

2.4 Differentiability Implies Continuity

If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$:

$$f \text{ differentiable at } a \implies f \text{ continuous at } a.$$

However, the converse is false:

$$f \text{ continuous at } a \not\Longrightarrow f \text{ differentiable at } a.$$

2.5 Properties of Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions. Then the following rules hold:

$$(1) \quad \frac{d}{dx}(c) = 0.$$

$$(2) \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

$$(3) \quad \frac{d}{dx}[cf(x)] = c[\frac{d}{dx}f(x)].$$

$$(4) \quad \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

$$(5) \quad \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$(6) \quad \frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

$$(7) \quad \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

2.6 Table of Derivatives

$(\sin x)' = \cos x$	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(e^x)' = e^x$
$(\cos x)' = -\sin x$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$(a^x)' = a^x \ln a$
$(\tan x)' = \sec^2 x$	$(\arctan x)' = \frac{1}{1+x^2}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\cot x)' = -\csc^2 x$	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$	$(\ln x)' = \frac{1}{x}$
$(\sec x)' = \sec x \tan x$	$(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$	$(\lvert x\rvert)' = \frac{x}{\lvert x\rvert}$
$(\csc x)' = -\csc x \cot x$	$(\operatorname{arccsc} x)' = -\frac{1}{x\sqrt{x^2-1}}$	$(x^x)' = x^x(1 + \ln x)$

2.7 Absolute and Local Extrema

Let f be defined on a domain D, and let $c \in D$.

- **Absolute Maximum:** $f(c)$ is an absolute maximum if $f(c) \geq f(x),\ \forall x \in D$.
- **Absolute Minimum:** $f(c)$ is an absolute minimum if $f(c) \leq f(x),\ \forall x \in D$.
- **Local Maximum:** $f(c)$ is a local maximum if $\exists \delta > 0$ such that $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.
- **Local Minimum:** $f(c)$ is a local minimum if $\exists \delta > 0$ such that $f(c) \leq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

2.8 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$:

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$

2.9 Critical and Stationary Points

Let f be defined on an interval I and $c \in I$.

- **Critical Point:** c is a critical point of f if either
  1. $f'(c) = 0$, or
  2. $f'(c)$ does not exist
- **Stationary Point:** c is a stationary point of f if $f'(c) = 0$.

Note: *Every stationary point is a critical point, but not conversely.*

2.10 Rolle's Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).
3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2.11 Mean Value Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$.

2.12 Increasing/Decreasing Test

Let f be differentiable on an interval I. Then, for all $x \in I$:

1. If $f'(x) > 0$, then f is strictly increasing on I.
2. If $f'(x) < 0$, then f is strictly decreasing on I.
3. If $f'(x) = 0$, then f is constant on I.

2.13 First Derivative Test

Let c be a critical point of a differentiable function $f(x)$, meaning $f'(c) = 0$ or $f'(c)$ does not exist. Then:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is **neither** a local maximum nor a local minimum.

2.14 Concavity and Inflection Points

Concave up $\iff$ Curve **lies above** all of its **tangent lines**.

Concave down $\iff$ Curve **lies below** all of its **tangent lines**.

Inflection point $\iff$ Point where **concavity changes**.

2.15 Concavity Test

Let $f(x)$ be twice differentiable on interval I. Then:

- If $f''(x) > 0, \quad \forall x \in I \implies f(x)$ is **concave up** on I.
- If $f''(x) < 0, \quad \forall x \in I \implies f(x)$ is **concave down** on I.

2.16 Second Derivative Test

Let c be a critical point of f where $f(c) = 0$. If $f''(c)$ exists, then:

1. If $f''(c) > 0$, $f(x)$ is **concave up** at c, so $f(c)$ is a **local minimum**.

$$\text{Local Maximum at } c \iff f'(c) = 0 \text{ and } f''(c) < 0$$

2. If $f''(c) < 0$, $f(x)$ is **concave down** at c, so $f(c)$ is a **local maximum**.

$$\text{Local Minimum at } c \iff f'(c) = 0 \text{ and } f''(c) > 0$$

3. If $f''(c) = 0$ or **does not exist**, the test is **inconclusive**—use the First Derivative Test instead.

2.17 L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a (except possibly at a). If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2.18 Indeterminate Forms

The following symbols are "indeterminate":

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 1^{\infty} \qquad 0^{\infty} \qquad \infty^{0}$$

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \qquad \frac{\infty}{0} \qquad \frac{1}{\infty} \qquad 1\cdot\infty \qquad \infty+\infty \qquad 1+\infty \qquad 0^{\infty}$$

3 Integrals

3.1 Antiderivatives

A function $F(x)$ is an **antiderivative** (or primitive function) of $f(x)$ on an interval I if:

$$F'(x) = f(x), \quad \forall x \in I.$$

3.2 Indefinite Integrals

The **indefinite integral** (or general antiderivative) of a function $f(x)$ is the family of all its antiderivatives:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

3.3 Riemann Sum

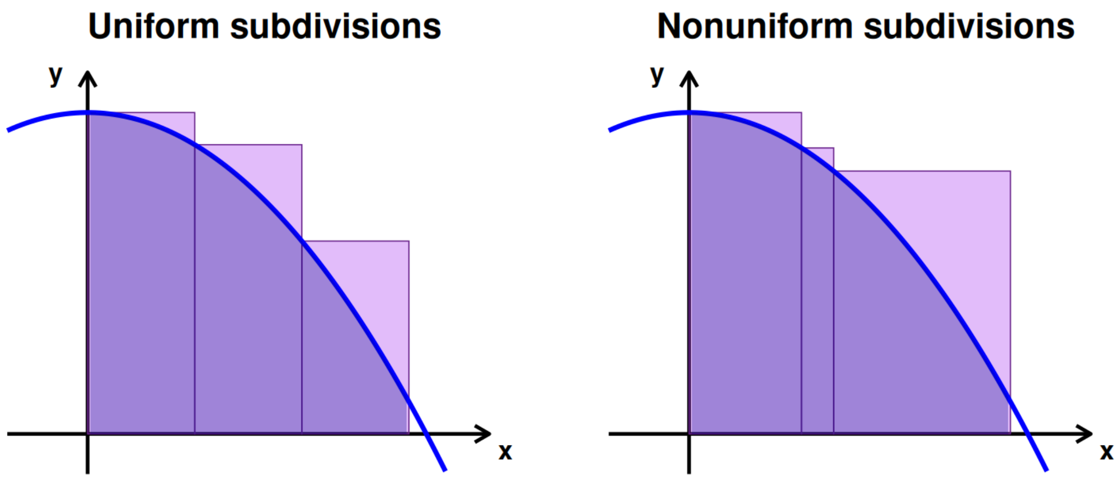
Let $f(x)$ be a function defined on a closed interval $[a, b]$, and let the interval be partitioned into n subintervals by inserting $n - 1$ points $x_1, x_2, \ldots, x_{n-1}$ such that:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

where the subintervals $[x_{i-1}, x_i]$ have length $\Delta x_i = x_i - x_{i-1}$. For each subinterval $[x_{i-1}, x_i]$, let x_i^* be an **arbitrary sample point** within the interval. Then, a **Riemann sum**, which **approximates** the area under the curve $f(x)$ over $[a, b]$, is defined as:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i, \quad x_i^* \in [x_{i-1}, x_i].$$

3.4 Partitions of an Interval



Comparison of uniform and non-uniform partitions of an interval (generated using R).

A partition of $[a, b]$ divides it into n subintervals:

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Each subinterval has width:

$$\Delta x_i = x_i - x_{i-1}.$$

- **Uniform Partition:** All subintervals have the same width:

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \quad \forall i.$$

- **Non-Uniform Partition:** Subintervals have different widths, and Δx_i varies for each i.

3.5 Types of Riemann Sums

The type of Riemann sum depends on how the sample points x_i^* are chosen within each subinterval $[x_{i-1}, x_i]$.

1. **Arbitrary-Point Rule:** $x_i^* \in [x_{i-1}, x_i] \rightarrow$ **General Riemann Sum**

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

   Uniform Partition: If all subintervals have equal width $\Delta x = \dfrac{b-a}{n}$, then $x_i^* \in [x_{i-1}, x_i] = [a + (i-1)\Delta x, a + i\Delta x]$ or $x_i^* = a + (i - 1 + c)\Delta x$, where $c \in [0, 1]$ determines its position within the subinterval:

$$S_n = \sum_{i=1}^{n} f(a + (i-1+c)\Delta x)\Delta x = \sum_{i=1}^{n} f(a + (i-1+c)\frac{(b-a)}{n})\frac{b-a}{n}.$$

   - Uses arbitrary sample points x_i^*.

2. **Left Rule:** $x_i^* = x_{i-1} \rightarrow$ **Left Riemann Sum**

$$S_{\text{left}} = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i.$$

   Uniform Partition: $c = 0$

$$S_{\text{left}} = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\Delta x.$$

   - Underestimates for increasing functions, overestimates for decreasing functions.

3. **Right Rule:** $x_i^* = x_i \rightarrow$ **Right Riemann Sum**

$$S_{\text{right}} = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

   Uniform Partition: $c = 1$

$$S_{\text{right}} = \sum_{i=1}^{n} f(a + i\Delta x)\Delta x.$$

   - Overestimates for increasing functions, underestimates for decreasing functions.

4. **Midpoint Rule:** $x_i^* = \dfrac{x_{i-1} + x_i}{2}$ → **Midpoint Riemann Sum**

$$S_{\text{mid}} = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x_i.$$

Uniform Partition: $c = \dfrac{1}{2}$

$$S_{\text{mid}} = \sum_{i=1}^{n} f(a + (i - \frac{1}{2})\Delta x)\Delta x.$$

- Tends to give better approximations than left or right sums.

5. **Upper Rule:** $x_i^* = \arg \sup_{x \in [x_{i-1}, x_i]} f(x)$ → **Upper Riemann Sum** (or **Upper Darboux Sum**)

$$S_{\text{upper}} = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **overestimates** the integral.

6. **Lower Rule:** $x_i^* = \arg \inf_{x \in [x_{i-1}, x_i]} f(x)$ → **Lower Riemann Sum** (or **Lower Darboux Sum**)

$$S_{\text{lower}} = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **underestimates** the integral.

3.6 Definite Integral

The **definite integral** of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

where:

- $[a, b]$ is divided into n subintervals.
- $\Delta x_i = x_i - x_{i-1}$ is the width of the i-th subinterval.
- x_i^* is any sample point in the i-th subinterval.

3.7 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$ and c be a constant. Then:

(1) $\displaystyle\int_a^b c\,dx = c(b-a).$

(2) $\displaystyle\int_a^a f(x)\,dx = 0.$

(3) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

(4) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

(5) $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

(6) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

(7) $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \quad \text{if } f(x) \le g(x) \text{ on } [a, b].$

(8) $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

(9) $\displaystyle m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \quad \text{if } m \le f(x) \le M \text{ on } [a, b].$

3.8 Net and Total Area

Let $f(x)$ be integrable on $[a, b]$. Define:

- A_1 = area of region where $f(x) > 0$ (area above x-axis).
- A_2 = area of region where $f(x) < 0$ (area below x-axis).

Then:

- **Net Area:**
$$\int_a^b f(x)\,dx = A_1 - A_2.$$
- **Total Area:**
$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

3.9 Mean Value Theorem for Integrals

Let f be continuous on [a,b]. Then:

$$\exists c \in (a,b) \text{ such that } \frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

The expression on the left is the **average value** of the function $f(x)$ on the interval $[a,b]$.

3.10 Fundamental Theorem of Calculus. Part 1

Let f be a continuous on $[a,b]$. Define the function

$$F(x) = \int_a^x f(t)\,dt.$$

Then:

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x), \quad \forall x \in (a,b).$$

3.11 Fundamental Theorem of Calculus. Part 2

If f is continuous on $[a,b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b = F(x)\Big|_a^b.$$

3.12 Table of Indefinite Integrals

$$\int a\,dx = ax + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \log_a x\,dx = \frac{x}{\ln a}(\ln x - 1) + C$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{c}{ax+b}\,dx = \frac{c}{a}\ln|ax+b| + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \quad a > 0,\ a \neq 1$$

$$\int \tan x\,dx = \ln|\sec x| = -\ln|\cos x| + C$$

$$\int \cot x\,dx = -\ln|\csc x| = \ln|\sin x| + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C \qquad \int \frac{1}{1+x^2} \, dx = \arctan x + C$$

4 Sequences and Series

4.1 Definition of a Sequence

A sequence is a function $a : \mathbb{N} \to \mathbb{R}$ that assigns to each $n \in N$ a real number a_n.

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = (a_n) = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

4.2 Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if it is either monotonically increasing or monotonically decreasing.

- **Increasing:**

$$a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly increasing)}.$$

$$a_n < a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly increasing)}.$$

- **Decreasing:**

$$a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly decreasing)}.$$

$$a_n > a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly decreasing)}.$$

4.3 Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if and only if:

$$\exists M > 0 \text{ such that } |a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

or

$$\exists M_1, M_2 \in \mathbb{R} \text{ such that } M_1 \leq a_n \leq M_2, \quad \forall n \in \mathbb{N}.$$

Equivalently, $\{a_n\}$ is **bounded** if it is both **bounded above** and **bounded below**:

- **Bounded above:** $\exists M_2 \in \mathbb{R}$ such that $a_n \leq M_2, \quad \forall n \in \mathbb{N}$.
- **Bounded below:** $\exists M_1 \in \mathbb{R}$ such that $a_n \geq M_1, \quad \forall n \in \mathbb{N}$.

$$\text{A sequence is bounded} \iff \text{It is both bounded above and bounded below}$$

4.4 Limit of a Sequence

A sequence $\{a_n\}$ has a limit $L \in \mathbb{R}$ if:

$$\lim_{n\to\infty} a_n = L \quad \text{if} \quad \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \varepsilon, \forall n \geq N.$$

If such an L exists, the sequence is **convergent**; otherwise, it is **divergent**.

4.5 Limit of a Sequence Defined by a Function

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $\{a_n\}$ be defined by $a_n = f(n)$. Then:

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L.$$

4.6 Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for all $n \geq N$, then:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \implies \lim_{n\to\infty} b_n = L.$$

4.7 Limit of Absolute Value of a Sequence

$$\lim_{n\to\infty} |a_n| = 0 \implies \lim_{n\to\infty} a_n = 0.$$

4.8 Monotone Convergence Theorem

Every bounded and monotonic sequence is convergent.

$$\text{(1)}\quad \text{Monotonic} \ \wedge\ \text{Bounded} \implies \text{Convergent.}$$

$$\text{(2)}\quad \text{Monotonically Increasing} \ \wedge\ \text{Bounded Above} \implies \text{Convergent.}$$

$$\text{(3)}\quad \text{Monotonically Decreasing} \ \wedge\ \text{Bounded Below} \implies \text{Convergent.}$$

4.9 Series

An **infinite series** is the sum of the terms of a sequence $\{a_n\}$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The $\boldsymbol{n^{\text{th}}}$ **partial sum** of series:

$$S_n = \sum_{k=1}^{n} a_k.$$

4.10 Convergence and Divergence of Series

A infinite series $\sum a_n$ **converges** if the sequence of partial sums $\{S_n\}$ has a finite limit:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \quad \text{or} \quad \lim_{n\to\infty} S_n = S.$$

Otherwise, the series **diverges**.

4.11 Types of Series

- Arithmetic Series
- Geometric Series
- Arithmetico-Geometric Series
- Harmonic Series
- p-Series
- Telescoping Series
- Alternating Series
- Power Series
- Taylor Series
- Maclaurin Series
- Taylor Series
- Laurent Series
- Fourier Series
- Binomial Series
- Mercator Series

4.12 Properties of Series

Let $\sum a_n$ and $\sum b_n$ be series, and let c, d be a constant. Then:

$$(1) \quad \sum_{k=a}^{b} c = c(b-a+1).$$

$$(2) \quad \sum_{n=k}^{\infty} \pm c = \pm\infty, \quad c > 0.$$

$$(3) \quad \sum_{n=k}^{m} (ca_n \pm db_n) = c\sum_{n=k}^{m} a_n \pm d\sum_{n=k}^{m} b_n.$$

$$(4) \quad \sum_{n=k}^{m} a_n = \sum_{n=k}^{p} a_n + \sum_{n=p+1}^{m} a_n.$$

$$(5) \quad \sum_{n=k}^{m} a_n = \sum_{n=0}^{m-k} a_{m-n}.$$

$$(6) \quad \sum_{n=k}^{m} a_n = \sum_{j=k-h}^{m-h} a_{j+h}.$$

$$(7) \quad \sum_{k=1}^{n} k = \sum_{k=0}^{n-1} (k+1).$$

4.13 Geometric Series

$$\text{Geometric} \implies \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$$

- **Partial Sum:**
$$S_n = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}, \quad r \neq 1.$$
- **Convergence:**
$$|r| < 1 \implies \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$
- **Divergence:**
$$|r| \geq 1 \implies \text{Series diverges.}$$

4.14 Limit of Terms in a Convergent Series

Let $\{a_n\}$ be a sequence and $\sum a_n$ be its series. Then:

$$\sum a_n \text{ convergent} \implies \lim_{n\to\infty} a_n = 0.$$

Note: The converse is false. ($\lim_{n\to\infty} a_n = 0 \;\not\Longrightarrow\; \sum a_n$ convergent)

4.15 List of Convergence Tests

- **Tests for Positive Series:**
  - Direct Comparison Test
  - Limit Comparison Test
  - Integral Test
  - p-Series Test
- **Tests for Alternating Series:**
  - Alternating Series Test
  - Dirichlet's Test
- **General Tests:**
  - Divergence Test
  - Ratio Test
  - Root Test
  - Absolute Convergence Test
- **Advanced or Specialized Tests:**
  - Cauchy Condensation Test
  - Abel's Test
  - Raabe's Test
  - Kummer's Test
  - Gauss's Test
  - Bertrand's Test

4.16 Divergence Test (nth-Term Test)

$$\lim_{n\to\infty} a_n \neq 0 \ \lor \ \lim_{n\to\infty} a_n \text{ does not exist } \implies \sum a_n \text{ diverges.}$$

4.17 Direct Comparison Test

If $0 \leq a_n \leq b_n$ for all $n \geq N$:

- $\sum b_n$ converges $\implies \sum a_n$ converges.
- $\sum a_n$ diverges $\implies \sum b_n$ diverges.

4.18 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with $a_n > 0$, $b_n > 0$ for all $n \geq N$. Define the limit:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

- $0 < L < \infty \implies \sum a_n$ and $\sum b_n$ both converge or both diverge.
- $L = 0 \ \land \ \sum b_n$ converges $\implies \sum a_n$ converges.
- $L = \infty \ \land \ \sum b_n$ diverges $\implies \sum a_n$ diverges.

Note: The order of division does not matter.

4.19 Integral Test (Maclaurin-Cauchy Test)

For a series $\sum a_n$ where $a_n = f(n)$, if $f(x)$ is a positive, continuous, and decreasing function for all $x \geq N$, then:

- $\displaystyle\sum_{n=N}^{\infty} a_n$ converges $\iff \displaystyle\int_N^{\infty} f(x)\,dx$ converges.
- $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges $\iff \displaystyle\int_N^{\infty} f(x)\,dx$ diverges.

4.20 p-Series Test

For the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$:

- $p > 1 \iff$ Series converges.
- $p \leq 1 \iff$ Series diverges.