

Calculus Notes

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Contents

1	Limits	2
1.1	Precise Definition of a Limit	2
1.2	Precise Definition of One-Sided Limit	2
1.3	Precise Definition of Infinite Limit	3
1.4	Precise Definition of a Limit at Infinity	3
1.5	Precise Definition of Infinite Limit at Infinity	3
1.6	Limit Laws	3
1.7	Relationship between the Limit and One-Sided Limits	4
1.8	Comparison Theorem	4
1.9	Squeeze Theorem	4
1.10	Continuity	4
1.11	Properties of Continuous Functions	4
1.12	Types of Discontinuity	5
1.13	Limits of Continuous Functions	5
1.14	Intermediate Value Theorem	5
1.15	Asymptotes	5
1.16	Common Limits	6
2	Derivatives	6
2.1	Derivative at a Point	6
2.2	Derivative as a Function	6
2.3	Differentiability	7
2.4	Differentiability Implies Continuity	7
2.5	Properties of Derivatives	7
2.6	Table of Derivatives	8
2.7	Absolute and Local Extrema	8
2.8	Extreme Value Theorem	8
2.9	Critical and Stationary Points	8
2.10	Rolle's Theorem	9
2.11	Mean Value Theorem	9
2.12	Increasing/Decreasing Test	9
2.13	First Derivative Test	9
2.14	Concavity and Inflection Points	10
2.15	Concavity Test	10
2.16	Second Derivative Test	10
2.17	L'Hôpital's Rule	10
2.18	Indeterminate Forms	10

3	Integrals	11
3.1	Antiderivatives	11
3.2	Indefinite Integrals	11
3.3	Riemann Sum	11
3.4	Partitions of an Interval	11
3.5	Types of Riemann Sums	12
3.6	Definite Integral	13
3.7	Properties of Definite Integrals	14
3.8	Net and Total Area	14
3.9	Mean Value Theorem for Integrals	15
3.10	Fundamental Theorem of Calculus. Part 1	15
3.11	Fundamental Theorem of Calculus. Part 2	15
3.12	Table of Indefinite Integrals	15
4	Sequences and Series	16
4.1	Definition of a Sequence	16
4.2	Monotonic Sequence	16
4.3	Bounded Sequence	16
4.4	Limit of a Sequence	17
4.5	Limit of a Sequence Defined by a Function	17
4.6	Squeeze Theorem for Sequences	17
4.7	Limit of Absolute Value of a Sequence	17
4.8	Monotone Convergence Theorem	17
4.9	Series	17
4.10	Convergence and Divergence of Series	18
4.11	Types of Series	18

1 Limits

1.1 Precise Definition of a Limit

Standard Limit:

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.2 Precise Definition of One-Sided Limit

Right-Hand Limit:

$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Left-Hand Limit:

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.3 Precise Definition of Infinite Limit

Infinite Limit:

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow a} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -M.$$

1.4 Precise Definition of a Limit at Infinity

Limit at Infinity:

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x < -M \Rightarrow |f(x) - L| < \varepsilon.$$

1.5 Precise Definition of Infinite Limit at Infinity

Infinite Limit at Infinity:

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) < -M.$$

$$\lim_{x \rightarrow -\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) < -M.$$

1.6 Limit Laws

Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

1. $\lim_{x \rightarrow a} c = c$
2. $\lim_{x \rightarrow a} x = a$
3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
4. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
5. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
6. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$
7. $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$
8. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

1.7 Relationship between the Limit and One-Sided Limits

$$\begin{aligned}\lim_{x \rightarrow a} f(x) = L &\Leftrightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L. \\ \lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x) &\Rightarrow \lim_{x \rightarrow a} f(x) \text{ does not exist.}\end{aligned}$$

1.8 Comparison Theorem

If $f(x) \leq g(x)$ when x is near a , and $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

1.9 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a , and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L.$$

1.10 Continuity

A function $f(x)$ is **continuous at** $x = a$ if and only if it satisfies **all** the following:

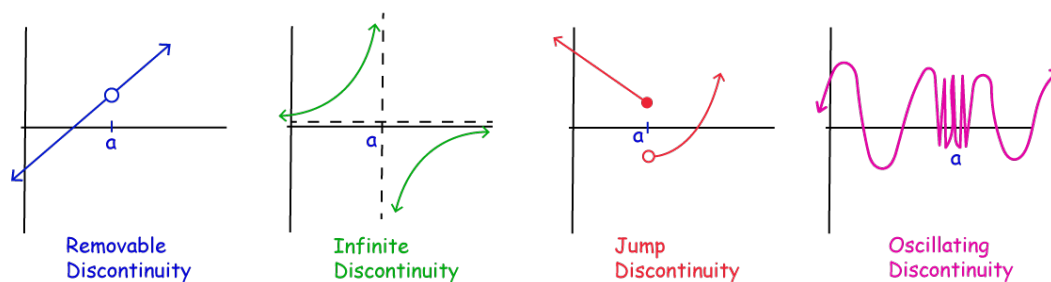
- (1) $f(a)$ exists
- (2) $\lim_{x \rightarrow a} f(x)$ exists
- (3) $\lim_{x \rightarrow a} f(x) = f(a)$

Otherwise, $f(x)$ is discontinuous at $x = a$.

1.11 Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$



Source: calcworkshop.com

1.12 Types of Discontinuity

1.13 Limits of Continuous Functions

If $f(x)$ is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(b).$$

If g is continuous at a and f is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a .

1.14 Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any N between $f(a)$ and $f(b)$,

$$\exists c \in [a, b] \text{ such that } f(c) = N$$

1.15 Asymptotes

Vertical Asymptote: $x = a$ is a vertical asymptote if

$$\lim_{x \rightarrow a^\pm} f(x) = \pm\infty.$$

Horizontal Asymptote: $y = L$ is a horizontal asymptote if

$$\lim_{x \rightarrow \pm\infty} f(x) = L.$$

For $f(x) = \frac{P(x)}{Q(x)}$, compare degrees of P and Q :

$$\deg P < \deg Q \Rightarrow y = 0.$$

$$\deg P = \deg Q \Rightarrow y = \frac{\text{leading coef. of } P}{\text{leading coef. of } Q}.$$

$$\deg P > \deg Q \Rightarrow \text{no horizontal asymptote.}$$

Oblique Asymptote: $y = mx + b$ is an oblique asymptote if

$$\lim_{x \rightarrow \pm\infty} (f(x) - (mx + b)) = 0.$$

For a rational function $f(x) = \frac{P(x)}{Q(x)}$, if $\deg P = \deg Q + 1$, then $f(x)$ has an oblique asymptote. Find it by polynomial long division:

$$f(x) = D(x) + \frac{R(x)}{Q(x)}, \quad \text{as } x \rightarrow \pm\infty, \quad f(x) \approx D(x).$$

Curvilinear Asymptote: $y = g(x)$ is a curvilinear asymptote if

$$\lim_{x \rightarrow \pm\infty} (f(x) - g(x)) = 0,$$

where $g(x)$ is any non-linear function.

1.16 Common Limits

Assume $a > 0$ in the following.

$$1. \lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

$$7. \lim_{x \rightarrow 0} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$$

$$2. \lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \frac{a}{b}$$

$$8. \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$3. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$9. \lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e$$

$$4. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$10. \lim_{x \rightarrow 0^+} x^x = 1$$

$$5. \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} = a$$

$$11. \lim_{x \rightarrow 0^+} x^a \ln x = 0$$

$$6. \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$12. \lim_{x \rightarrow +\infty} x^{-a} \ln x = 0$$

2 Derivatives

2.1 Derivative at a Point

The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** at that point:

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

2.2 Derivative as a Function

The derivative of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

2.3 Differentiability

A function $f(x)$ is **differentiable** at $x = a$ if its derivative $f'(x)$ exists. That is:

$$f(x) \text{ is differentiable at } x = a \iff \text{The limit } \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

2.4 Differentiability Implies Continuity

If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$:

$$f \text{ differentiable at } a \implies f \text{ continuous at } a.$$

However, the converse is false:

$$f \text{ continuous at } a \not\implies f \text{ differentiable at } a.$$

2.5 Properties of Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions. Then the following rules hold:

$$(1) \quad \frac{d}{dx}(c) = 0.$$

$$(2) \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

$$(3) \quad \frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx}f(x)\right].$$

$$(4) \quad \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

$$(5) \quad \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$(6) \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

$$(7) \quad \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

2.6 Table of Derivatives

$(\sin x)' = \cos x$	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(e^x)' = e^x$
$(\cos x)' = -\sin x$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$(a^x)' = a^x \ln a$
$(\tan x)' = \sec^2 x$	$(\arctan x)' = \frac{1}{1+x^2}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\cot x)' = -\csc^2 x$	$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$	$(\ln x)' = \frac{1}{x}$
$(\sec x)' = \sec x \tan x$	$(\operatorname{arcsec} x)' = \frac{1}{x\sqrt{x^2-1}}$	$(x)' = \frac{x}{ x }$
$(\csc x)' = -\csc x \cot x$	$(\operatorname{arccsc} x)' = -\frac{1}{x\sqrt{x^2-1}}$	$(x^x)' = x^x(1 + \ln x)$

2.7 Absolute and Local Extrema

Let f be defined on a domain D , and let $c \in D$.

- **Absolute Maximum:** $f(c)$ is an absolute maximum if $f(c) \geq f(x)$, $\forall x \in D$.
- **Absolute Minimum:** $f(c)$ is an absolute minimum if $f(c) \leq f(x)$, $\forall x \in D$.
- **Local Maximum:** $f(c)$ is a local maximum if $\exists \delta > 0$ such that $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.
- **Local Minimum:** $f(c)$ is a local minimum if $\exists \delta > 0$ such that $f(c) \leq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

2.8 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$:

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$

2.9 Critical and Stationary Points

Let f be defined on an interval I and $c \in I$.

- **Critical Point:** c is a critical point of f if either
 1. $f'(c) = 0$, or
 2. $f'(c)$ does not exist
- **Stationary Point:** c is a stationary point of f if $f'(c) = 0$.

Note: Every stationary point is a critical point, but not conversely.

2.10 Rolle's Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b) .
3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2.11 Mean Value Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b) .

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$.

2.12 Increasing/Decreasing Test

Let f be differentiable on an interval I . Then, for all $x \in I$:

1. If $f'(x) > 0$, then f is strictly increasing on I .
2. If $f'(x) < 0$, then f is strictly decreasing on I .
3. If $f'(x) = 0$, then f is constant on I .

2.13 First Derivative Test

Let c be a critical point of a differentiable function $f(x)$, meaning $f'(c) = 0$ or $f'(c)$ does not exist. Then:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is **neither** a local maximum nor a local minimum.

2.14 Concavity and Inflection Points

Concave up \iff Curve **lies above** all of its **tangent lines**.

Concave down \iff Curve **lies below** all of its **tangent lines**.

Inflection point \iff Point where **concavity changes**.

2.15 Concavity Test

Let $f(x)$ be twice differentiable on interval I . Then:

- If $f''(x) > 0$, $\forall x \in I \implies f(x)$ is **concave up** on I .
- If $f''(x) < 0$, $\forall x \in I \implies f(x)$ is **concave down** on I .

2.16 Second Derivative Test

Let c be a critical point of f where $f'(c) = 0$. If $f''(c)$ exists, then:

1. If $f''(c) > 0$, $f(x)$ is **concave up** at c , so $f(c)$ is a **local minimum**.

Local Maximum at $c \iff f'(c) = 0$ and $f''(c) < 0$

2. If $f''(c) < 0$, $f(x)$ is **concave down** at c , so $f(c)$ is a **local maximum**.

Local Minimum at $c \iff f'(c) = 0$ and $f''(c) > 0$

3. If $f''(c) = 0$ or **does not exist**, the test is **inconclusive**—use the First Derivative Test instead.

2.17 L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a (except possibly at a). If

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0 \text{ or } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \pm\infty$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

2.18 Indeterminate Forms

The following symbols are “indeterminate”:

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad 0 \cdot \infty \quad \infty - \infty \quad 1^\infty \quad 0^\infty \quad \infty^0$$

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \quad \frac{\infty}{0} \quad \frac{1}{\infty} \quad 1 \cdot \infty \quad \infty + \infty \quad 1 + \infty \quad 0^\infty$$

3 Integrals

3.1 Antiderivatives

A function $F(x)$ is an **antiderivative** (or primitive function) of $f(x)$ on an interval I if:

$$F'(x) = f(x), \quad \forall x \in I.$$

3.2 Indefinite Integrals

The **indefinite integral** (or general antiderivative) of a function $f(x)$ is the family of all its antiderivatives:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

3.3 Riemann Sum

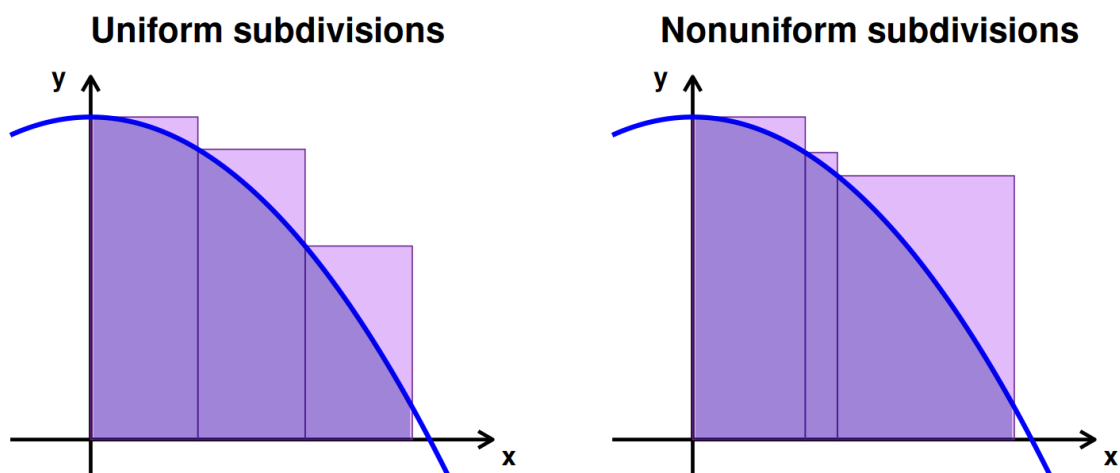
Let $f(x)$ be a function defined on a closed interval $[a, b]$, and let the interval be partitioned into n subintervals by inserting $n - 1$ points x_1, x_2, \dots, x_{n-1} such that:

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

where the subintervals $[x_{i-1}, x_i]$ have length $\Delta x_i = x_i - x_{i-1}$. For each subinterval $[x_{i-1}, x_i]$, let x_i^* be an **arbitrary sample point** within the interval. Then, a **Riemann sum**, which **approximates** the area under the curve $f(x)$ over $[a, b]$, is defined as:

$$S_n = \sum_{i=1}^n f(x_i^*) \Delta x_i, \quad x_i^* \in [x_{i-1}, x_i].$$

3.4 Partitions of an Interval



Comparison of uniform and non-uniform partitions of an interval (generated using R).

A partition of $[a, b]$ divides it into n subintervals:

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Each subinterval has width:

$$\Delta x_i = x_i - x_{i-1}.$$

- **Uniform Partition:** All subintervals have the same width:

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \quad \forall i.$$

- **Non-Uniform Partition:** Subintervals have different widths, and Δx_i varies for each i .

3.5 Types of Riemann Sums

The type of Riemann sum depends on how the sample points x_i^* are chosen within each subinterval $[x_{i-1}, x_i]$.

1. **Arbitrary-Point Rule:** $x_i^* \in [x_{i-1}, x_i] \rightarrow$ **General Riemann Sum**

$$S_n = \sum_{i=1}^n f(x_i^*) \Delta x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

Uniform Partition: If all subintervals have equal width $\Delta x = \frac{b-a}{n}$, then $x_i^* \in [x_{i-1}, x_i] = [a + (i-1)\Delta x, a + i\Delta x]$ or $x_i^* = a + (i-1+c)\Delta x$, where $c \in [0, 1]$ determines its position within the subinterval:

$$S_n = \sum_{i=1}^n f(a + (i-1+c)\Delta x) \Delta x = \sum_{i=1}^n f(a + (i-1+c)\frac{(b-a)}{n}) \frac{b-a}{n}.$$

- Uses arbitrary sample points x_i^* .

2. **Left Rule:** $x_i^* = x_{i-1} \rightarrow$ **Left Riemann Sum**

$$S_{\text{left}} = \sum_{i=1}^n f(x_{i-1}) \Delta x_i.$$

Uniform Partition: $c = 0$

$$S_{\text{left}} = \sum_{i=1}^n f(a + (i-1)\Delta x) \Delta x.$$

- Underestimates for increasing functions, overestimates for decreasing functions.

3. **Right Rule:** $x_i^* = x_i \rightarrow$ **Right Riemann Sum**

$$S_{\text{right}} = \sum_{i=1}^n f(x_i) \Delta x_i$$

Uniform Partition: $c = 1$

$$S_{\text{right}} = \sum_{i=1}^n f(a + i\Delta x) \Delta x.$$

- Overestimates for increasing functions, underestimates for decreasing functions.

4. **Midpoint Rule:** $x_i^* = \frac{x_{i-1} + x_i}{2} \rightarrow$ **Midpoint Riemann Sum**

$$S_{\text{mid}} = \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x_i.$$

Uniform Partition: $c = \frac{1}{2}$

$$S_{\text{mid}} = \sum_{i=1}^n f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right) \Delta x.$$

- Tends to give better approximations than left or right sums.

5. **Upper Rule:** $x_i^* = \arg \sup_{x \in [x_{i-1}, x_i]} f(x) \rightarrow$ **Upper Riemann Sum** (or **Upper Darboux Sum**)

$$S_{\text{upper}} = \sum_{i=1}^n \sup_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **overestimates** the integral.

6. **Lower Rule:** $x_i^* = \arg \inf_{x \in [x_{i-1}, x_i]} f(x) \rightarrow$ **Lower Riemann Sum** (or **Lower Darboux Sum**)

$$S_{\text{lower}} = \sum_{i=1}^n \inf_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **underestimates** the integral.

3.6 Definite Integral

The **definite integral** of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

where:

- $[a, b]$ is divided into n subintervals.
- $\Delta x_i = x_i - x_{i-1}$ is the width of the i -th subinterval.
- x_i^* is any sample point in the i -th subinterval.

3.7 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$ and c be a constant. Then:

- (1) $\int_a^b c \, dx = c(b - a).$
- (2) $\int_a^a f(x) \, dx = 0.$
- (3) $\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx.$
- (4) $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx.$
- (5) $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$
- (6) $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$
- (7) $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \quad \text{if } f(x) \leq g(x) \text{ on } [a, b].$
- (8) $\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$
- (9) $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a) \quad \text{if } m \leq f(x) \leq M \text{ on } [a, b].$

3.8 Net and Total Area

Let $f(x)$ be integrable on $[a, b]$. Define:

- A_1 = area of region where $f(x) > 0$ (area above x -axis).
- A_2 = area of region where $f(x) < 0$ (area below x -axis).

Then:

- **Net Area:**

$$\int_a^b f(x) \, dx = A_1 - A_2.$$

- **Total Area:**

$$\int_a^b |f(x)| \, dx = A_1 + A_2.$$

3.9 Mean Value Theorem for Integrals

Let f be continuous on $[a, b]$. Then:

$$\exists c \in (a, b) \text{ such that } \frac{1}{b-a} \int_a^b f(x) dx = f(c)$$

The expression on the left is the **average value** of the function $f(x)$ on the interval $[a, b]$.

3.10 Fundamental Theorem of Calculus. Part 1

Let f be a continuous on $[a, b]$. Define the function

$$F(x) = \int_a^x f(t) dt.$$

Then:

$$F'(x) = \frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x), \quad \forall x \in (a, b).$$

3.11 Fundamental Theorem of Calculus. Part 2

If f is continuous on $[a, b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b = F(x) \Big|_a^b.$$

3.12 Table of Indefinite Integrals

$$\int a dx = ax + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln |x| + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\int \csc x dx = \ln |\csc x - \cot x| + C$$

$$\int \log_a x dx = \frac{x}{\ln a} (\ln x - 1) + C$$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{c}{ax + b} dx = \frac{c}{a} \ln |ax + b| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C, \quad a > 0, a \neq 1$$

$$\int \tan x dx = \ln |\sec x| = -\ln |\cos x| + C$$

$$\int \cot x dx = -\ln |\csc x| = \ln |\sin x| + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\left. \begin{aligned} \int \sec x \tan x \, dx &= \sec x + C \\ \int \csc x \cot x \, dx &= -\csc x + C \end{aligned} \right| \begin{aligned} \int \frac{1}{\sqrt{1-x^2}} \, dx &= \arcsin x + C \\ \int \frac{1}{1+x^2} \, dx &= \arctan x + C \end{aligned}$$

4 Sequences and Series

4.1 Definition of a Sequence

A sequence is a function $a : \mathbb{N} \rightarrow \mathbb{R}$ that assigns to each $n \in \mathbb{N}$ a real number a_n .

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = (a_n) = \{a_1, a_2, a_3, \dots, a_n, \dots\}$$

4.2 Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if it is either monotonically increasing or monotonically decreasing.

- **Increasing:**

$$a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly increasing).}$$

$$a_n < a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly increasing).}$$

- **Decreasing:**

$$a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly decreasing).}$$

$$a_n > a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly decreasing).}$$

4.3 Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if and only if:

$$\exists M > 0 \text{ such that } |a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

or

$$\exists M_1, M_2 \in \mathbb{R} \text{ such that } M_1 \leq a_n \leq M_2, \quad \forall n \in \mathbb{N}.$$

Equivalently, $\{a_n\}$ is **bounded** if it is both **bounded above** and **bounded below**:

- **Bounded above:** $\exists M_2 \in \mathbb{R}$ such that $a_n \leq M_2, \quad \forall n \in \mathbb{N}.$

- **Bounded below:** $\exists M_1 \in \mathbb{R}$ such that $a_n \geq M_1, \quad \forall n \in \mathbb{N}.$

A sequence is bounded \iff It is both bounded above and bounded below

4.4 Limit of a Sequence

A sequence $\{a_n\}$ has a limit $L \in \mathbb{R}$ if:

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{if} \quad \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \varepsilon, \forall n \geq N.$$

If such an L exists, the sequence is **convergent**; otherwise, it is **divergent**.

4.5 Limit of a Sequence Defined by a Function

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $\{a_n\}$ be defined by $a_n = f(n)$. Then:

$$\lim_{x \rightarrow \infty} f(x) = L \implies \lim_{n \rightarrow \infty} a_n = L.$$

4.6 Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for all $n \geq N$, then:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L \implies \lim_{n \rightarrow \infty} b_n = L.$$

4.7 Limit of Absolute Value of a Sequence

$$\lim_{n \rightarrow \infty} |a_n| = 0 \implies \lim_{n \rightarrow \infty} a_n = 0.$$

4.8 Monotone Convergence Theorem

Every bounded and monotonic sequence is convergent.

- (1) Monotonic \wedge Bounded \implies Convergent.
- (2) Monotonically Increasing \wedge Bounded Above \implies Convergent.
- (3) Monotonically Decreasing \wedge Bounded Below \implies Convergent.

4.9 Series

An **infinite series** is the sum of the terms of a sequence $\{a_n\}$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The n^{th} partial sum of series:

$$S_n = \sum_{i=1}^n a_i.$$

4.10 Convergence and Divergence of Series

A infinite series $\sum a_n$ **converges** if the sequence of partial sums $\{S_n\}$ has a finite limit:

$$\sum_{i=1}^{\infty} a_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i \quad \text{or} \quad \lim_{n \rightarrow \infty} S_n = S.$$

Otherwise, the series **diverges**.

4.11 Types of Series

- | | | |
|--------------------------------|----------------------|-------------------|
| • Arithmetic Series | • Telescoping Series | • Taylor Series |
| • Geometric Series | • Alternating Series | • Laurent Series |
| • Arithmetico-Geometric Series | • Power Series | • Fourier Series |
| • Harmonic Series | • Taylor Series | • Binomial Series |
| • p-Series | • Maclaurin Series | • Mercator Series |