Calculus Notes

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Contents

1 Limits

1.1 Precise Definition of a Limit

Standard Limit:

$$\lim_{x \to a} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.2 Precise Definition of One-Sided Limit

Right-Hand Limit:

$$\lim_{x \to a^+} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\, \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Left-Hand Limit:

$$\lim_{x \to a^-} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\, \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.3 Precise Definition of Infinite Limit

Infinite Limit:

$$\lim_{x \to a} f(x) = \infty \quad \text{if} \quad \forall M > 0,\, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

$$\lim_{x \to a} f(x) = -\infty \quad \text{if} \quad \forall M > 0,\, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -M.$$

1.4 Precise Definition of a Limit at Infinity

Limit at Infinity:

$$\lim_{x \to \infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\, \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \to -\infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\, \exists M > 0 \text{ such that } x < -M \Rightarrow |f(x) - L| < \varepsilon.$$

1.5 Precise Definition of Infinite Limit at Infinity

Infinite Limit at Infinity:

$$\lim_{x\to\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$

$$\lim_{x\to\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) < -M.$$

$$\lim_{x\to-\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) > M.$$

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) < -M.$$

1.6 Limit Laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
5. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
6. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$
7. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$
8. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

1.7 Relationship between the Limit and One-Sided Limits

$$\lim_{x\to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \quad \Rightarrow \quad \lim_{x\to a} f(x) \text{ does not exist.}$$

1.8 Comparison Theorem

If $f(x) \leq g(x)$ when x is near a, and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

1.9 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L.$$

1.10 Continuity

A function $f(x)$ is **continuous at** $x = a$ if and only if it satisfies **all** the following:

$$\begin{aligned} &(1) \quad f(a) \text{ exists} \\ &(2) \quad \lim_{x \to a} f(x) \text{ exists} \\ &(3) \quad \lim_{x \to a} f(x) = f(a) \end{aligned}$$

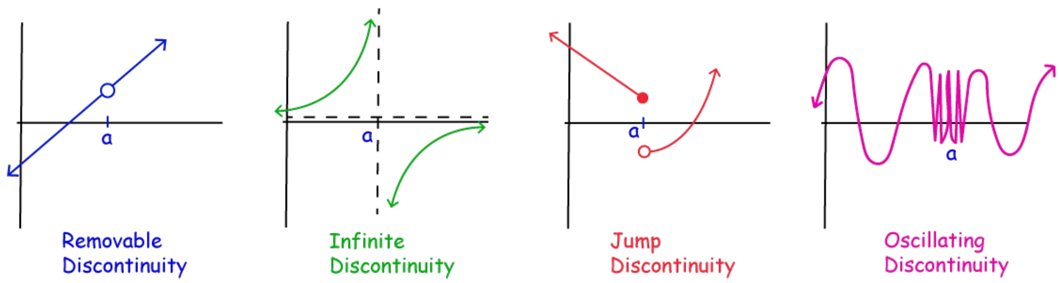
Otherwise, $f(x)$ is discontinuous at $x = a$.

1.11 Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

1.12 Types of Discontinuity



Source: calcworkshop.com

1.13 Limits of Continuous Functions

If $f(x)$ is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b).$$

If g is continuous at a and f is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a.

1.14 Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any N between $f(a)$ and $f(b)$,

$$\exists\, c \in [a, b] \text{ such that } f(c) = N$$

1.15 Asymptotes

Vertical Asymptote: $x = a$ is a vertical asymptote if

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty.$$

Horizontal Asymptote: $y = L$ is a horizontal asymptote if

$$\lim_{x \to \pm\infty} f(x) = L.$$

For $f(x) = \dfrac{P(x)}{Q(x)}$, compare degrees of P and Q:

$$\begin{aligned} \deg P < \deg Q &\quad \Rightarrow \quad y = 0. \\ \deg P = \deg Q &\quad \Rightarrow \quad y = \frac{\text{leading coef. of } P}{\text{leading coef. of } Q}. \\ \deg P > \deg Q &\quad \Rightarrow \quad \text{no horizontal asymptote.} \end{aligned}$$

Oblique Asymptote: $y = mx + b$ is an oblique asymptote if

$$\lim_{x \to \pm\infty} \big(f(x) - (mx + b)\big) = 0.$$

For a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, if $\deg P = \deg Q + 1$, then $f(x)$ has an oblique asymptote. Find it by polynomial long division:

$$f(x) = D(x) + \frac{R(x)}{Q(x)}, \quad \text{as } x \to \pm\infty, \quad f(x) \approx D(x).$$

Curvilinear Asymptote: $y = g(x)$ is a curvilinear asymptote if

$$\lim_{x \to \pm\infty} \big(f(x) - g(x)\big) = 0,$$

where $g(x)$ is any non-linear function.

1.16 Common Limits

Assume $a > 0$ in the following.

1. $\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$
2. $\lim_{x \to 0} \frac{\tan ax}{bx} = \frac{a}{b}$
3. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
4. $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
5. $\lim_{x \to 0} \frac{e^{ax} - 1}{x} = a$
6. $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$
7. $\lim_{x \to 0} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$
8. $\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$
9. $\lim_{x \to 0} \frac{\log_a(1 + x)}{x} = \log_a e$
10. $\lim_{x \to 0^+} x^x = 1$
11. $\lim_{x \to 0^+} x^a \ln x = 0$
12. $\lim_{x \to +\infty} x^{-a} \ln x = 0$

2 Derivatives

2.1 Derivative at a Point

The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** at that point:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

2.2 Derivative as a Function

The derivative of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

2.3 Differentiability

A function $f(x)$ is **differentiable** at $x = a$ if its derivative $f'(x)$ exists. That is:

$$f(x) \text{ is differentiable at } x = a \iff \text{The limit } \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \text{ exists.}$$

2.4 Differentiability Implies Continuity

If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$:

$$f \text{ differentiable at } a \implies f \text{ continuous at } a.$$

However, the converse is false:

$$f \text{ continuous at } a \not\Longrightarrow f \text{ differentiable at } a.$$

2.5 Basic Differentiation Rules

Let f and g be differentiable functions. Then the following rules hold:

$$\begin{aligned}
&(1) \quad \frac{d}{dx}(c) = 0 \\
&(2) \quad \frac{d}{dx}(x^n) = nx^{n-1} \\
&(3) \quad \frac{d}{dx}[cf(x)] = c[\frac{d}{dx}f(x)] \\
&(4) \quad \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \\
&(5) \quad \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \\
&(6) \quad \frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\
&(7) \quad \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).
\end{aligned}$$

2.6 Table of Derivatives

$(\sin x)' = \cos x$	$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$	$(e^x)' = e^x$
$(\cos x)' = -\sin x$	$(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$	$(a^x)' = a^x \ln a$
$(\tan x)' = \sec^2 x$	$(\arctan x)' = \dfrac{1}{1+x^2}$	$(\log_a x)' = \dfrac{1}{x \ln a}$
$(\cot x)' = -\csc^2 x$	$(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$	$(\ln x)' = \dfrac{1}{x}$
$(\sec x)' = \sec x \tan x$	$(\operatorname{arcsec} x)' = \dfrac{1}{x\sqrt{x^2-1}}$	$(\lvert x\rvert)' = \dfrac{x}{\lvert x\rvert}$
$(\csc x)' = -\csc x \cot x$	$(\operatorname{arccsc} x)' = -\dfrac{1}{x\sqrt{x^2-1}}$	$(x^x)' = x^x(1 + \ln x)$

2.7 Absolute and Local Extrema

Let f be defined on a domain D, and let $c \in D$.

- **Absolute Maximum:** $f(c)$ is an absolute maximum if $f(c) \geq f(x),\ \forall x \in D$.
- **Absolute Minimum:** $f(c)$ is an absolute minimum if $f(c) \leq f(x),\ \forall x \in D$.
- **Local Maximum:** $f(c)$ is a local maximum if $\exists \delta > 0$ such that $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.
- **Local Minimum:** $f(c)$ is a local minimum if $\exists \delta > 0$ such that $f(c) \leq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

2.8 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$:

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d),\ \forall x \in [a, b].$$

2.9 Critical and Stationary Points

Let f be defined on an interval I and $c \in I$.

- **Critical Point:** c is a critical point of f if either
  1. $f'(c) = 0$, or
  2. $f'(c)$ does not exist
- **Stationary Point:** c is a stationary point of f if $f'(c) = 0$.

Note: *Every stationary point is a critical point, but not conversely.*

2.10 Rolle's Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).
3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2.11 Mean Value Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$.

2.12 Increasing/Decreasing Test

Let f be differentiable on an interval I. Then, for all $x \in I$:

1. If $f'(x) > 0$, then f is strictly increasing on I.
2. If $f'(x) < 0$, then f is strictly decreasing on I.
3. If $f'(x) = 0$, then f is constant on I.

2.13 First Derivative Test

Let c be a critical point of a differentiable function $f(x)$, meaning $f'(c) = 0$ or $f'(c)$ does not exist. Then:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is **neither** a local maximum nor a local minimum.

2.14 Concavity and Inflection Points

Concave up $\iff$ Curve **lies above** all of its **tangent lines**.

Concave down $\iff$ Curve **lies below** all of its **tangent lines**.

Inflection point $\iff$ Point where **concavity changes**.

2.15 Concavity Test

Let $f(x)$ be twice differentiable on interval I. Then:

- If $f''(x) > 0 \ \forall x \in I \implies f(x)$ is **concave up** on I.
- If $f''(x) < 0 \ \forall x \in I \implies f(x)$ is **concave down** on I.

2.16 Second Derivative Test

Let c be a critical point of f where $f(c) = 0$. If $f''(c)$ exists, then:

1. If $f''(c) > 0$, $f(x)$ is **concave up** at c, so $f(c)$ is a **local minimum**.

$$\text{Local Maximum at } c \iff f'(c) = 0 \text{ and } f''(c) < 0$$

2. If $f''(c) < 0$, $f(x)$ is **concave down** at c, so $f(c)$ is a **local maximum**.

$$\text{Local Minimum at } c \iff f'(c) = 0 \text{ and } f''(c) > 0$$

3. If $f''(c) = 0$ or **does not exist**, the test is **inconclusive**—use the First Derivative Test instead.

2.17 L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a (except possibly at a). If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2.18 Indeterminate Forms

The following symbols are "indeterminate":

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 1^{\infty} \qquad 0^{\infty} \qquad \infty^{0}$$

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \qquad \frac{\infty}{0} \qquad \frac{1}{\infty} \qquad 1\cdot\infty \qquad \infty+\infty \qquad 1+\infty \qquad 0^{\infty}$$

3 Integrals

3.1 Antiderivatives

A function $F(x)$ is an **antiderivative** (or primitive function) of $f(x)$ on an interval I if:

$$F'(x) = f(x), \ \forall x \in I.$$

3.2 Indefinite Integrals

The **indefinite integral** (or general antiderivative) of a function $f(x)$ is the family of all its antiderivatives:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

3.3 Riemann Sum

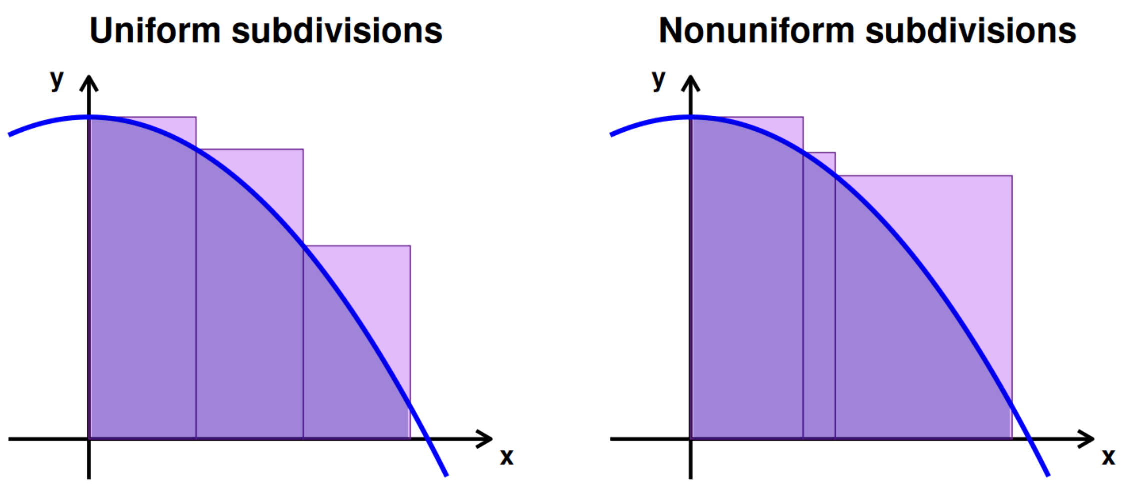
Let $f(x)$ be a function defined on a closed interval $[a, b]$, and let the interval be partitioned into n subintervals by inserting $n - 1$ points $x_1, x_2, \dots, x_{n-1}$ such that:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

where the subintervals $[x_{i-1}, x_i]$ have length $\Delta x_i = x_i - x_{i-1}$. For each subinterval $[x_{i-1}, x_i]$, let x_i^* be an **arbitrary sample point** within the interval, $x_i^* \in [x_{i-1}, x_i]$. Then, a **Riemann sum**, which **approximates** the area under the curve $f(x)$ over $[a, b]$, is defined as:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

3.4 Partitions of an Interval



Comparison of uniform and non-uniform partitions of an interval (generated using R).

A partition of $[a, b]$ divides it into n subintervals:

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Each subinterval has width:

$$\Delta x_i = x_i - x_{i-1}.$$

- **Uniform Partition:** All subintervals have the same width:

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \quad \forall i.$$

- **Non-Uniform Partition:** Subintervals have different widths, and Δx_i varies for each i.

3.5 Types of Riemann Sums

The type of Riemann sum depends on how the sample points x_i^* are chosen within each subinterval $[x_{i-1}, x_i]$.

1. **Left Rule:** $x_i^* = x_{i-1} \rightarrow$ **Left Riemann Sum**

$$S_{\text{left}} = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

   - Underestimates for increasing functions, overestimates for decreasing functions.

2. **Right Rule:** $x_i^* = x_i \rightarrow$ **Right Riemann Sum**

$$S_{\text{right}} = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

   - Overestimates for increasing functions, underestimates for decreasing functions.

3. **Midpoint Rule:** $x_i^* = \dfrac{x_{i-1}+x_i}{2} \rightarrow$ **Midpoint Riemann Sum**

$$S_{\text{mid}} = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i$$

   - Tends to give better approximations than left or right sums.

4. **Upper Rule:** $x_i^* = \arg \sup\limits_{x\in[x_{i-1},x_i]} f(x) \rightarrow$ **Upper Riemann Sum** (or **Upper Darboux Sum**)

$$S_{\text{upper}} = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x)\Delta x_i$$

   - Always **overestimates** the integral.

5. **Lower Rule:** $x_i^* = \arg \inf\limits_{x\in[x_{i-1},x_i]} f(x) \rightarrow$ **Lower Riemann Sum** (or **Lower Darboux Sum**)

$$S_{\text{lower}} = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)\Delta x_i$$

   - Always **underestimates** the integral.