Calculus Notes

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Contents

1 Limits

1.1 Precise Definition of a Limit

Standard Limit:

$$\lim_{x \to a} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.2 Precise Definition of One-Sided Limit

Right-Hand Limit:

$$\lim_{x \to a^+} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Left-Hand Limit:

$$\lim_{x \to a^-} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.3 Precise Definition of Infinite Limit

Infinite Limit:

$$\lim_{x \to a} f(x) = \infty \quad \text{if} \quad \forall M > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

$$\lim_{x \to a} f(x) = -\infty \quad \text{if} \quad \forall M > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -M.$$

1.4 Precise Definition of a Limit at Infinity

Limit at Infinity:

$$\lim_{x \to \infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\ \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \to -\infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0,\ \exists M > 0 \text{ such that } x < -M \Rightarrow |f(x) - L| < \varepsilon.$$

1.5 Precise Definition of Infinite Limit at Infinity

Infinite Limit at Infinity:

$$\lim_{x \to \infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \ \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$

$$\lim_{x \to \infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \ \exists N > 0 \text{ such that } x > N \Rightarrow f(x) < -M.$$

$$\lim_{x \to -\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \ \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) > M.$$

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \ \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) < -M.$$

1.6 Limit Laws

Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $\lim_{x \to a} c = c$

2. $\lim_{x \to a} x = a$

3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

4. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

5. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \text{if} \ \lim_{x \to a} g(x) \neq 0$

7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

8. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

1.7 Relationship between the Limit and One-Sided Limits

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \quad \Rightarrow \quad \lim_{x \to a} f(x) \text{ does not exist.}$$

1.8 Comparison Theorem

If $f(x) \leq g(x)$ when x is near a, and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

1.9 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L.$$

1.10 Continuity

A function $f(x)$ is **continuous at** $x = a$ if and only if it satisfies **all** the following:

(1) $f(a)$ exists

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$

Otherwise, $f(x)$ is discontinuous at $x = a$.

1.11 Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

1.12 Types of Discontinuity

Removable Discontinuity   Infinite Discontinuity   Jump Discontinuity   Oscillating Discontinuity
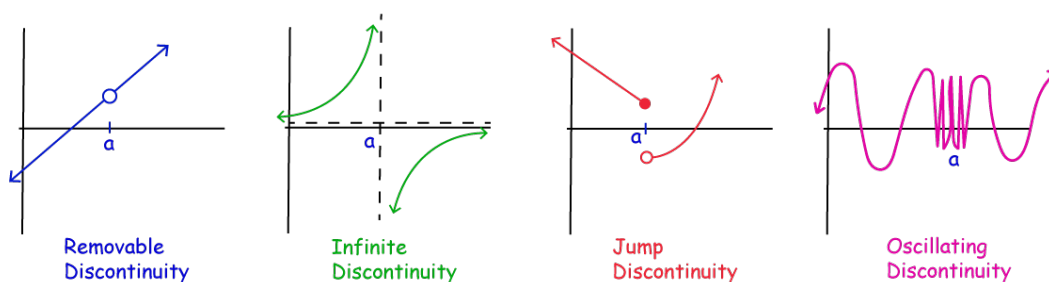
Source: calcworkshop.com

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1.13   Limits of Continuous Functions

If $f(x)$ is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b).$$

If g is continuous at a and f is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a.

1.14   Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any N between $f(a)$ and $f(b)$,

$$\exists \, c \in [a, b] \text{ such that } f(c) = N$$

1.15   Asymptotes

Vertical Asymptote: $x = a$ is a vertical asymptote if

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty.$$

Horizontal Asymptote: $y = L$ is a horizontal asymptote if

$$\lim_{x \to \pm\infty} f(x) = L.$$

For $f(x) = \dfrac{P(x)}{Q(x)}$, compare degrees of P and Q:

$$\deg P < \deg Q \quad \Rightarrow \quad y = 0.$$
$$\deg P = \deg Q \quad \Rightarrow \quad y = \frac{\text{leading coef. of } P}{\text{leading coef. of } Q}.$$
$$\deg P > \deg Q \quad \Rightarrow \quad \text{no horizontal asymptote.}$$

Oblique Asymptote: $y = mx + b$ is an oblique asymptote if

$$\lim_{x \to \pm\infty} \big(f(x) - (mx + b)\big) = 0.$$

For a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, if $\deg P = \deg Q + 1$, then $f(x)$ has an oblique asymptote. Find it by polynomial long division:

$$f(x) = D(x) + \frac{R(x)}{Q(x)}, \quad \text{as } x \to \pm\infty, \quad f(x) \approx D(x).$$

Curvilinear Asymptote: $y = g(x)$ is a curvilinear asymptote if

$$\lim_{x \to \pm\infty} \big(f(x) - g(x)\big) = 0,$$

where $g(x)$ is any non-linear function.

1.16   Common Limits

Assume $a > 0$ in the following.

1. $\displaystyle\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

2. $\displaystyle\lim_{x \to 0} \frac{\tan ax}{bx} = \frac{a}{b}$

3. $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

4. $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$

5. $\displaystyle\lim_{x \to 0} \frac{e^{ax} - 1}{x} = a$

6. $\displaystyle\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$

7. $\displaystyle\lim_{x \to 0} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$

8. $\displaystyle\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$

9. $\displaystyle\lim_{x \to 0} \frac{\log_a(1 + x)}{x} = \log_a e$

10. $\displaystyle\lim_{x \to 0^+} x^x = 1$

11. $\displaystyle\lim_{x \to 0^+} x^a \ln x = 0$

12. $\displaystyle\lim_{x \to +\infty} x^{-a} \ln x = 0$

2   Derivatives

2.1   Derivative at a Point

The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** at that point:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

2.2   Derivative as a Function

The derivative of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

2.3   Differentiability

A function $f(x)$ is **differentiable** at $x = a$ if its derivative $f'(x)$ exists. That is:

$f(x)$ is differentiable at $x = a$ $\iff$ The limit $\displaystyle\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ exists.

2.4 Differentiability Implies Continuity

If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$:

$$f \text{ differentiable at } a \implies f \text{ continuous at } a.$$

However, the converse is false:

$$f \text{ continuous at } a \nimplies f \text{ differentiable at } a.$$

2.5 Properties of Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions. Then the following rules hold:

(1) $\quad \dfrac{d}{dx}(c) = 0.$

(2) $\quad \dfrac{d}{dx}(x^n) = nx^{n-1}.$

(3) $\quad \dfrac{d}{dx}[cf(x)] = c[\dfrac{d}{dx}f(x)].$

(4) $\quad \dfrac{d}{dx}[f(x) \pm g(x)] = \dfrac{d}{dx}f(x) \pm \dfrac{d}{dx}g(x).$

(5) $\quad \dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$

(6) $\quad \dfrac{d}{dx}[\dfrac{f(x)}{g(x)}] = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$

(7) $\quad \dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$

2.6 Table of Derivatives

$(\sin x)' = \cos x$	$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$	$(e^x)' = e^x$
$(\cos x)' = -\sin x$	$(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$	$(a^x)' = a^x \ln a$
$(\tan x)' = \sec^2 x$	$(\arctan x)' = \dfrac{1}{1+x^2}$	$(\log_a x)' = \dfrac{1}{x \ln a}$
$(\cot x)' = -\csc^2 x$	$(\arccot x)' = -\dfrac{1}{1+x^2}$	$(\ln x)' = \dfrac{1}{x}$
$(\sec x)' = \sec x \tan x$	$(\arcsec x)' = \dfrac{1}{x\sqrt{x^2-1}}$	$(|x|)' = \dfrac{x}{|x|}$
$(\csc x)' = -\csc x \cot x$	$(\arccsc x)' = -\dfrac{1}{x\sqrt{x^2-1}}$	$(x^x)' = x^x(1+\ln x)$

2.7 Absolute and Local Extrema

Let f be defined on a domain D, and let $c \in D$.

- **Absolute Maximum:** $f(c)$ is an absolute maximum if $f(c) \geq f(x)$, $\forall x \in D$.

- **Absolute Minimum:** $f(c)$ is an absolute minimum if $f(c) \leq f(x)$, $\forall x \in D$.

- **Local Maximum:** $f(c)$ is a local maximum if $\exists \delta > 0$ such that $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

- **Local Minimum:** $f(c)$ is a local minimum if $\exists \delta > 0$ such that $f(c) \leq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

2.8 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$:

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d), \ \forall x \in [a, b].$$

2.9 Critical and Stationary Points

Let f be defined on an interval I and $c \in I$.

- **Critical Point:** c is a critical point of f if either

    1. $f'(c) = 0$, or
    2. $f'(c)$ does not exist

- **Stationary Point:** c is a stationary point of f if $f'(c) = 0$.

Note: *Every stationary point is a critical point, but not conversely.*

2.10 Rolle's Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.

2. f is differentiable on (a, b).

3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2.11 Mean Value Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.

2. f is differentiable on (a, b).

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$.

2.12 Increasing/Decreasing Test

Let f be differentiable on an interval I. Then, for all $x \in I$:

1. If $f'(x) > 0$, then f is strictly increasing on I.

2. If $f'(x) < 0$, then f is strictly decreasing on I.

3. If $f'(x) = 0$, then f is constant on I.

2.13 First Derivative Test

Let c be a critical point of a differentiable function $f(x)$, meaning $f'(c) = 0$ or $f'(c)$ does not exist. Then:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum**.

2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum**.

3. If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is **neither** a local maximum nor a local minimum.

2.14 Concavity and Inflection Points

Concave up $\iff$ Curve **lies above** all of its **tangent lines**.

Concave down $\iff$ Curve **lies below** all of its **tangent lines**.

Inflection point $\iff$ Point where **concavity changes**.

2.15 Concavity Test

Let $f(x)$ be twice differentiable on interval I. Then:

- If $f''(x) > 0 \ \forall x \in I \implies f(x)$ is **concave up** on I.

- If $f''(x) < 0 \ \forall x \in I \implies f(x)$ is **concave down** on I.

2.16 Second Derivative Test

Let c be a critical point of f where $f(c) = 0$. If $f''(c)$ exists, then:

1. If $f''(c) > 0$, $f(x)$ is **concave up** at c, so $f(c)$ is a **local minimum**.

$$\text{Local Maximum at } c \iff f'(c) = 0 \text{ and } f''(c) < 0$$

2. If $f''(c) < 0$, $f(x)$ is **concave down** at c, so $f(c)$ is a **local maximum**.

$$\text{Local Minimum at } c \iff f'(c) = 0 \text{ and } f''(c) > 0$$

3. If $f''(c) = 0$ or **does not exist**, the test is **inconclusive**—use the First Derivative Test instead.

2.17 L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a (except possibly at a). If
$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \text{ or } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$
then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

2.18 Indeterminate Forms

The following symbols are "indeterminate":

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 1^{\infty} \qquad 0^{\infty} \qquad \infty^{0}$$

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \qquad \frac{\infty}{0} \qquad \frac{1}{\infty} \qquad 1 \cdot \infty \qquad \infty + \infty \qquad 1 + \infty \qquad 0^{\infty}$$

3 Integrals

3.1 Antiderivatives

A function $F(x)$ is an **antiderivative** (or primitive function) of $f(x)$ on an interval I if:

$$F'(x) = f(x), \ \forall x \in I.$$

3.2 Indefinite Integrals

The **indefinite integral** (or general antiderivative) of a function $f(x)$ is the family of all its antiderivatives:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

3.3 Riemann Sum

Let $f(x)$ be a function defined on a closed interval $[a, b]$, and let the interval be partitioned into n subintervals by inserting $n - 1$ points $x_1, x_2, \ldots, x_{n-1}$ such that:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

where the subintervals $[x_{i-1}, x_i]$ have length $\Delta x_i = x_i - x_{i-1}$. For each subinterval $[x_{i-1}, x_i]$, let x_i^* be an **arbitrary sample point** within the interval. Then, a **Riemann sum**, which **approximates** the area under the curve $f(x)$ over $[a, b]$, is defined as:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i, \quad x_i^* \in [x_{i-1}, x_i].$$

3.4 Partitions of an Interval
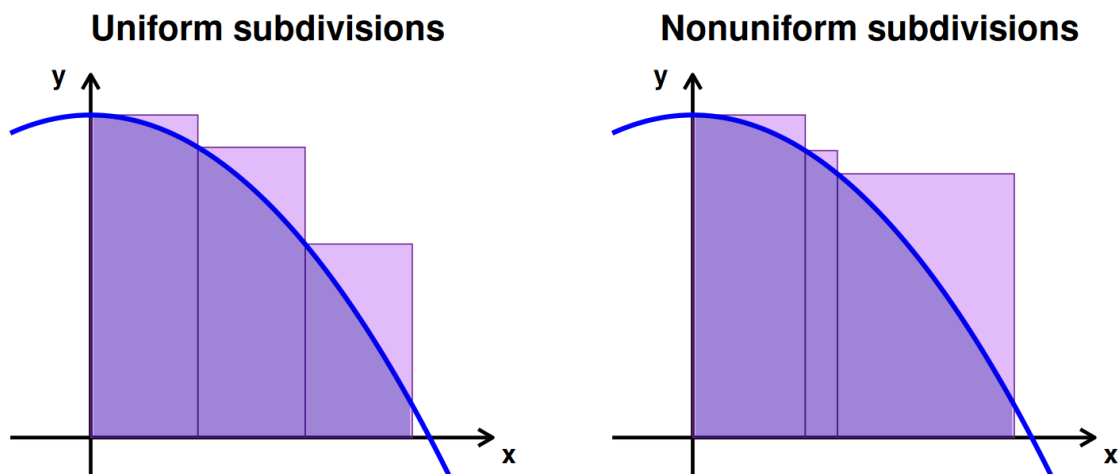
Comparison of uniform and non-uniform partitions of an interval (generated using R).

A partition of $[a, b]$ divides it into n subintervals:

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Each subinterval has width:

$$\Delta x_i = x_i - x_{i-1}.$$

- **Uniform Partition:** All subintervals have the same width:

$$\Delta x_i = \Delta x = \frac{b - a}{n}, \quad \forall i.$$

- **Non-Uniform Partition:** Subintervals have different widths, and Δx_i varies for each i.

3.5 Types of Riemann Sums

The type of Riemann sum depends on how the sample points x_i^* are chosen within each subinterval $[x_{i-1}, x_i]$.

1. **Arbitrary-Point Rule:** $x_i^* \in [x_{i-1}, x_i] \rightarrow$ **General Riemann Sum**

$$S_n = \sum_{i=1}^{n} f(x_i^*) \Delta x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

   Uniform Partition: If all subintervals have equal width $\Delta x = \dfrac{b - a}{n}$, then $x_i^* \in [x_{i-1}, x_i] = [a + (i-1)\Delta x, a + i\Delta x]$ or $x_i^* = a + (i - 1 + c)\Delta x$, where $c \in [0, 1]$ determines its position within the subinterval:

$$S_n = \sum_{i=1}^{n} f(a + (i - 1 + c)\Delta x)\Delta x = \sum_{i=1}^{n} f(a + (i - 1 + c)\frac{(b-a)}{n})\frac{b - a}{n}.$$

   - Uses arbitrary sample points x_i^*.

2. **Left Rule:** $x_i^* = x_{i-1} \rightarrow$ **Left Riemann Sum**

$$S_{\text{left}} = \sum_{i=1}^{n} f(x_{i-1}) \Delta x_i.$$

   Uniform Partition: $c = 0$

$$S_{\text{left}} = \sum_{i=1}^{n} f(a + (i - 1)\Delta x)\Delta x.$$

   - Underestimates for increasing functions, overestimates for decreasing functions.

3. **Right Rule:** $x_i^* = x_i \rightarrow$ **Right Riemann Sum**

$$S_{\text{right}} = \sum_{i=1}^{n} f(x_i) \Delta x_i$$

   Uniform Partition: $c = 1$

$$S_{\text{right}} = \sum_{i=1}^{n} f(a + i\Delta x)\Delta x.$$

   - Overestimates for increasing functions, underestimates for decreasing functions.

4. **Midpoint Rule:** $x_i^* = \dfrac{x_{i-1} + x_i}{2} \to$ **Midpoint Riemann Sum**

$$S_{\text{mid}} = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x_i.$$

Uniform Partition: $c = \dfrac{1}{2}$

$$S_{\text{mid}} = \sum_{i=1}^{n} f(a + (i - \frac{1}{2})\Delta x)\Delta x.$$

- Tends to give better approximations than left or right sums.

5. **Upper Rule:** $x_i^* = \arg \sup\limits_{x \in [x_{i-1}, x_i]} f(x) \to$ **Upper Riemann Sum** (or **Upper Darboux Sum**)

$$S_{\text{upper}} = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)\Delta x_i.$$

- Always **overestimates** the integral.

6. **Lower Rule:** $x_i^* = \arg \inf\limits_{x \in [x_{i-1}, x_i]} f(x) \to$ **Lower Riemann Sum** (or **Lower Darboux Sum**)

$$S_{\text{lower}} = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)\Delta x_i.$$

- Always **underestimates** the integral.

3.6   Definite Integral

The **definite integral** of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum:

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where:

- $[a, b]$ is divided into n subintervals.

- $\Delta x_i = x_i - x_{i-1}$ is the width of the i-th subinterval.

- x_i^* is any sample point in the i-th subinterval.

3.7 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$ and c be a constant. Then:

(1) $\quad \displaystyle\int_a^b c\, dx = c(b - a).$

(2) $\quad \displaystyle\int_a^a f(x)\, dx = 0.$

(3) $\quad \displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx.$

(4) $\quad \displaystyle\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx.$

(5) $\quad \displaystyle\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$

(6) $\quad \displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$

(7) $\quad \displaystyle\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx \quad$ if $f(x) \le g(x)$ on $[a, b]$.

(8) $\quad \left| \displaystyle\int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx.$

(9) $\quad m(b - a) \le \displaystyle\int_a^b f(x)\, dx \le M(b - a) \quad$ if $m \le f(x) \le M$ on $[a, b]$.