

Calculus Notes

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Contents

1	Limits	3
1.1	Precise Definition of a Limit	3
1.2	Precise Definition of One-Sided Limit	3
1.3	Precise Definition of Infinite Limit	4
1.4	Precise Definition of a Limit at Infinity	4
1.5	Precise Definition of Infinite Limit at Infinity	4
1.6	Limit Laws	4
1.7	Relationship between the Limit and One-Sided Limits	5
1.8	Comparison Theorem	5
1.9	Squeeze Theorem	5
1.10	Continuity	5
1.11	Properties of Continuous Functions	5
1.12	Types of Discontinuity	6
1.13	Limits of Continuous Functions	6
1.14	Intermediate Value Theorem	6
1.15	Asymptotes	6
1.16	Common Limits	7
2	Derivatives	7
2.1	Derivative at a Point	7
2.2	Derivative as a Function	7
2.3	Differentiability	8
2.4	Differentiability Implies Continuity	8
2.5	Properties of Derivatives	8
2.6	Table of Derivatives	9
2.7	Absolute and Local Extrema	9
2.8	Extreme Value Theorem	9
2.9	Critical and Stationary Points	9
2.10	Rolle's Theorem	10
2.11	Mean Value Theorem	10
2.12	Increasing/Decreasing Test	10
2.13	First Derivative Test	10
2.14	Concavity and Inflection Points	11
2.15	Concavity Test	11
2.16	Second Derivative Test	11
2.17	L'Hôpital's Rule	11
2.18	Indeterminate Forms	11

3	Integrals	12
3.1	Antiderivatives	12
3.2	Indefinite Integrals	12
3.3	Riemann Sum	12
3.4	Partitions of an Interval	12
3.5	Types of Riemann Sums	13
3.6	Definite Integral	14
3.7	Properties of Definite Integrals	15
3.8	Net and Total Area	15
3.9	Mean Value Theorem for Integrals	16
3.10	Fundamental Theorem of Calculus. Part 1	16
3.11	Fundamental Theorem of Calculus. Part 2	16
3.12	Table of Indefinite Integrals	16
4	Sequences and Series	17
4.1	Definition of a Sequence	17
4.2	Monotonic Sequence	17
4.3	Bounded Sequence	17
4.4	Limit of a Sequence	18
4.5	Limit of a Sequence Defined by a Function	18
4.6	Squeeze Theorem for Sequences	18
4.7	Limit of Absolute Value of a Sequence	18
4.8	Monotone Convergence Theorem	18
4.9	Series	18
4.10	Convergence and Divergence of Series	19
4.11	Types of Series	19
4.12	Properties of Series	19
4.13	Geometric Series	20
4.14	Telescoping Series	20
4.15	Harmonic Series	21
4.16	Limit of Terms in a Convergent Series	21
4.17	List of Convergence Tests	21
4.18	Divergence Test (n th-Term Test)	22
4.19	Direct Comparison Test	22
4.20	Limit Comparison Test	22
4.21	Integral Test (Maclaurin-Cauchy Test)	22
4.22	p -Series Test	22
4.23	Integral Remainder Estimate	23
4.24	Alternating Series Test (Leibniz Criterion)	23
4.25	Alternating Series Estimation Theorem	23
4.26	Ratio Test (d'Alembert's Criterion)	23
4.27	Root Test (Cauchy's Criterion)	24
4.28	Absolute Convergence Test	24
4.29	Riemann Series Theorem	24
4.30	Power Series	24
4.31	Radius of Convergence	25
4.32	Interval of Convergence	25
4.33	Differentiation and Integration of Power Series	26
4.34	Taylor Series	26
4.35	Taylor Polynomial	27
4.36	Taylor's Theorem	27

4.37	Taylor's Inequality	27
4.38	Binomial Series	27
5	Multivariable Calculus	28
5.1	Limit of a Function of Two Variables	28
5.2	Test for Nonexistence of a Limit	28
5.3	Partial Derivatives	28
5.4	Clairaut's Theorem	28
5.5	Equation of a Tangent Plane	28
5.6	Total Differential	28
5.7	Chain Rule For a Function of Two Variables	29
5.8	Implicit Differentiation of a Function of Two Variables	29
5.9	Gradient Vector	29
5.10	Directional Derivative	29
5.11	Tangent Plane to a Level Surface	30
5.12	Maximum and Minimum Values of a Function of Two Variables	30
5.13	Parametric Derivative	30
5.14	Area Under a Parametric Curve	30
5.15	Arc Length of a Parametric Curve	31
5.16	Surface Area of a Solid Revolution for a Parametric Curve	31
5.17	Polar Coordinates	31
5.18	Area in Polar Coordinates	31
5.19	Arc Length in Polar Coordinates	31
5.20	Tangents to Polar Curves	32
5.21	Double Integral	32
5.22	Iterated Integral	32
5.23	Average Value of a Function over a Region	32
5.24	Double Integral Over a General Region	32
5.25	Double Integral in Polar Coordinates	33
5.26	Double Integral in Polar Coordinates Over a General Region	33

1 Limits

1.1 Precise Definition of a Limit

Standard Limit:

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.2 Precise Definition of One-Sided Limit

Right-Hand Limit:

$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Left-Hand Limit:

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.3 Precise Definition of Infinite Limit

Infinite Limit:

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow a} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -M.$$

1.4 Precise Definition of a Limit at Infinity

Limit at Infinity:

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x < -M \Rightarrow |f(x) - L| < \varepsilon.$$

1.5 Precise Definition of Infinite Limit at Infinity

Infinite Limit at Infinity:

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) < -M.$$

$$\lim_{x \rightarrow -\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) < -M.$$

1.6 Limit Laws

Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

1. $\lim_{x \rightarrow a} c = c$
2. $\lim_{x \rightarrow a} x = a$
3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
4. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
5. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
6. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$
7. $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$
8. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

1.7 Relationship between the Limit and One-Sided Limits

$$\begin{aligned}\lim_{x \rightarrow a} f(x) = L &\iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L. \\ \lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x) &\implies \lim_{x \rightarrow a} f(x) \text{ does not exist.}\end{aligned}$$

1.8 Comparison Theorem

If $f(x) \leq g(x)$ when x is near a , and $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

1.9 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a , then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \implies \lim_{x \rightarrow a} g(x) = L.$$

1.10 Continuity

A function $f(x)$ is **continuous at** $x = a$ if and only if it satisfies **all** the following:

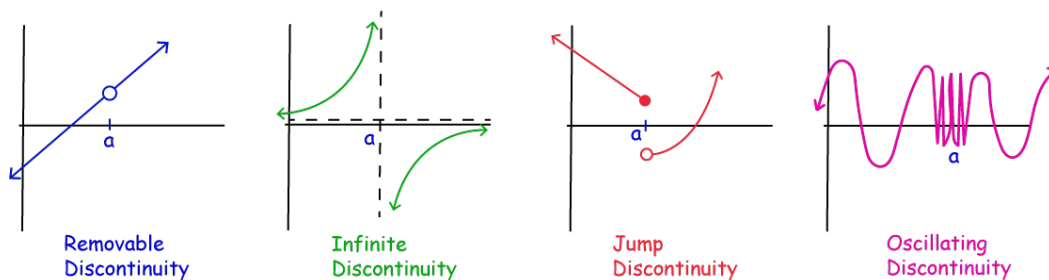
- (1) $f(a)$ exists
- (2) $\lim_{x \rightarrow a} f(x)$ exists
- (3) $\lim_{x \rightarrow a} f(x) = f(a)$

Otherwise, $f(x)$ is discontinuous at $x = a$.

1.11 Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$



Source: calcworkshop.com

1.12 Types of Discontinuity

1.13 Limits of Continuous Functions

If $f(x)$ is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(b).$$

If g is continuous at a and f is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a .

1.14 Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any N between $f(a)$ and $f(b)$,

$$\exists c \in [a, b] \text{ such that } f(c) = N$$

1.15 Asymptotes

Vertical Asymptote: $x = a$ is a vertical asymptote if

$$\lim_{x \rightarrow a^\pm} f(x) = \pm\infty.$$

Horizontal Asymptote: $y = L$ is a horizontal asymptote if

$$\lim_{x \rightarrow \pm\infty} f(x) = L.$$

For $f(x) = \frac{P(x)}{Q(x)}$, compare degrees of P and Q :

$$\deg P < \deg Q \implies y = 0.$$

$$\deg P = \deg Q \implies y = \frac{\text{leading coef. of } P}{\text{leading coef. of } Q}.$$

$$\deg P > \deg Q \implies \text{no horizontal asymptote.}$$

Oblique Asymptote: $y = mx + b$ is an oblique asymptote if

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + b)] = 0.$$

For a rational function $f(x) = \frac{P(x)}{Q(x)}$, if $\deg P = \deg Q + 1$, then $f(x)$ has an oblique asymptote. Find it by polynomial long division:

$$f(x) = D(x) + \frac{R(x)}{Q(x)}, \quad \text{as } x \rightarrow \pm\infty, \quad f(x) \approx D(x).$$

Curvilinear Asymptote: $y = g(x)$ is a curvilinear asymptote if

$$\lim_{x \rightarrow \pm\infty} [f(x) - g(x)] = 0,$$

where $g(x)$ is any non-linear function.

1.16 Common Limits

Assume $a > 0$ in the following.

$$1. \lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

$$8. \lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e$$

$$2. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$9. \lim_{x \rightarrow 0^+} x^x = 1$$

$$3. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$10. \lim_{x \rightarrow \infty} \sqrt[x]{x} = \lim_{x \rightarrow \infty} x^{\frac{1}{x}} = 1$$

$$4. \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} = a$$

$$11. \lim_{x \rightarrow 0^+} x^a \ln x = 0$$

$$5. \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$12. \lim_{x \rightarrow \infty} x^{-a} \ln x = 0$$

$$6. \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$$

$$13. \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

$$7. \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$14. \lim_{n \rightarrow \infty} \frac{n^n}{n!} = \infty$$

2 Derivatives

2.1 Derivative at a Point

The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** at that point:

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

2.2 Derivative as a Function

The derivative of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

2.3 Differentiability

A function $f(x)$ is **differentiable** at $x = a$ if its derivative $f'(x)$ exists. That is:

$$f(x) \text{ is differentiable at } x = a \iff \text{The limit } \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

2.4 Differentiability Implies Continuity

If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$:

$$f \text{ differentiable at } a \implies f \text{ continuous at } a.$$

However, the converse is false:

$$f \text{ continuous at } a \not\implies f \text{ differentiable at } a.$$

2.5 Properties of Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions. Then the following rules hold:

$$(1) \quad \frac{d}{dx}(c) = 0.$$

$$(2) \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

$$(3) \quad \frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx}f(x)\right].$$

$$(4) \quad \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

$$(5) \quad \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$(6) \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

$$(7) \quad \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

2.6 Table of Derivatives

$(\sin x)' = \cos x$	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(e^x)' = e^x$
$(\cos x)' = -\sin x$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$(a^x)' = a^x \ln a$
$(\tan x)' = \sec^2 x$	$(\arctan x)' = \frac{1}{1+x^2}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\cot x)' = -\csc^2 x$	$(x)' = -\frac{1}{1+x^2}$	$(\ln x)' = \frac{1}{x}$
$(\sec x)' = \sec x \tan x$	$(x)' = \frac{1}{x\sqrt{x^2-1}}$	$(x)' = \frac{x}{ x }$
$(\csc x)' = -\csc x \cot x$	$(x)' = -\frac{1}{x\sqrt{x^2-1}}$	$(x^x)' = x^x(1 + \ln x)$

2.7 Absolute and Local Extrema

Let f be defined on a domain D , and let $c \in D$.

- **Absolute Maximum:** $f(c)$ is an absolute maximum if $f(c) \geq f(x)$, $\forall x \in D$.
- **Absolute Minimum:** $f(c)$ is an absolute minimum if $f(c) \leq f(x)$, $\forall x \in D$.
- **Local Maximum:** $f(c)$ is a local maximum if $\exists \delta > 0$ such that $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.
- **Local Minimum:** $f(c)$ is a local minimum if $\exists \delta > 0$ such that $f(c) \leq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

2.8 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$:

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$

2.9 Critical and Stationary Points

Let f be defined on an interval I and $c \in I$.

- **Critical Point:** c is a critical point of f if either
 1. $f'(c) = 0$, or
 2. $f'(c)$ does not exist
- **Stationary Point:** c is a stationary point of f if $f'(c) = 0$.

Note: *Every stationary point is a critical point, but not conversely.*

2.10 Rolle's Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b) .
3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2.11 Mean Value Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b) .

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$.

2.12 Increasing/Decreasing Test

Let f be differentiable on an interval I . Then, for all $x \in I$:

1. If $f'(x) > 0$, then f is strictly increasing on I .
2. If $f'(x) < 0$, then f is strictly decreasing on I .
3. If $f'(x) = 0$, then f is constant on I .

2.13 First Derivative Test

Let c be a critical point of a differentiable function $f(x)$, meaning $f'(c) = 0$ or $f'(c)$ does not exist. Then:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is **neither** a local maximum nor a local minimum.

2.14 Concavity and Inflection Points

Concave up \iff Curve **lies above** all of its **tangent lines**.

Concave down \iff Curve **lies below** all of its **tangent lines**.

Inflection point \iff Point where **concavity changes**.

2.15 Concavity Test

Let $f(x)$ be twice differentiable on interval I . Then:

- If $f''(x) > 0$, $\forall x \in I \implies f(x)$ is **concave up** on I .
- If $f''(x) < 0$, $\forall x \in I \implies f(x)$ is **concave down** on I .

2.16 Second Derivative Test

Let c be a critical point of f where $f'(c) = 0$. If $f''(c)$ exists, then:

1. If $f''(c) > 0$, $f(x)$ is **concave up** at c , so $f(c)$ is a **local minimum**.

Local Maximum at $c \iff f'(c) = 0$ and $f''(c) < 0$

2. If $f''(c) < 0$, $f(x)$ is **concave down** at c , so $f(c)$ is a **local maximum**.

Local Minimum at $c \iff f'(c) = 0$ and $f''(c) > 0$

3. If $f''(c) = 0$ or **does not exist**, the test is **inconclusive**—use the First Derivative Test instead.

2.17 L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a (except possibly at a). If

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0 \text{ or } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \pm\infty$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

2.18 Indeterminate Forms

The following symbols are “indeterminate”:

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad 0 \cdot \infty \quad \infty - \infty \quad 1^\infty \quad 0^\infty \quad \infty^0$$

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \quad \frac{\infty}{0} \quad \frac{1}{\infty} \quad 1 \cdot \infty \quad \infty + \infty \quad 1 + \infty \quad 0^\infty$$

3 Integrals

3.1 Antiderivatives

A function $F(x)$ is an **antiderivative** (or primitive function) of $f(x)$ on an interval I if:

$$F'(x) = f(x), \quad \forall x \in I.$$

3.2 Indefinite Integrals

The **indefinite integral** (or general antiderivative) of a function $f(x)$ is the family of all its antiderivatives:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

3.3 Riemann Sum

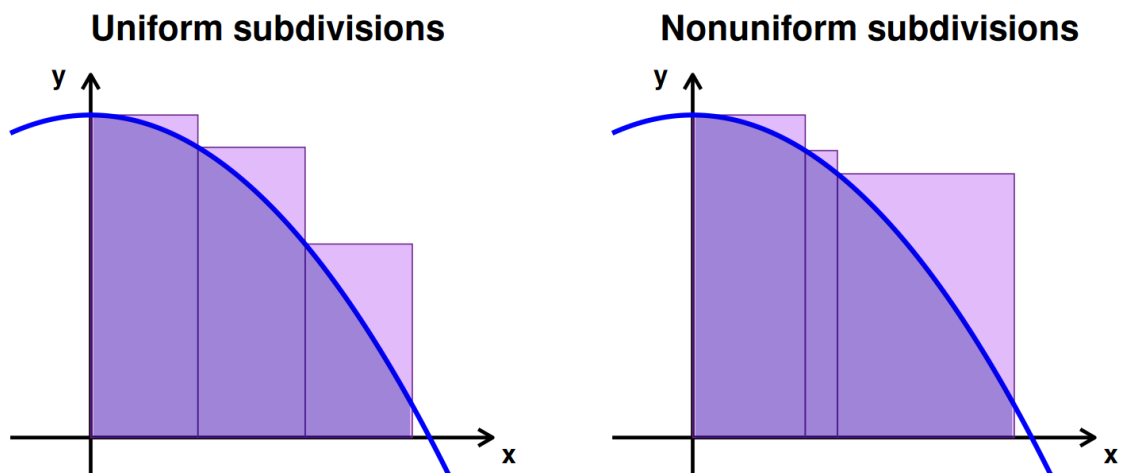
Let $f(x)$ be a function defined on a closed interval $[a, b]$, and let the interval be partitioned into n subintervals by inserting $n - 1$ points x_1, x_2, \dots, x_{n-1} such that:

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

where the subintervals $[x_{i-1}, x_i]$ have length $\Delta x_i = x_i - x_{i-1}$. For each subinterval $[x_{i-1}, x_i]$, let x_i^* be an **arbitrary sample point** within the interval. Then, a **Riemann sum**, which **approximates** the area under the curve $f(x)$ over $[a, b]$, is defined as:

$$S_n = \sum_{i=1}^n f(x_i^*)\Delta x_i, \quad x_i^* \in [x_{i-1}, x_i].$$

3.4 Partitions of an Interval



Comparison of uniform and non-uniform partitions of an interval (generated using R).

A partition of $[a, b]$ divides it into n subintervals:

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Each subinterval has width:

$$\Delta x_i = x_i - x_{i-1}.$$

- **Uniform Partition:** All subintervals have the same width:

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \quad \forall i.$$

- **Non-Uniform Partition:** Subintervals have different widths, and Δx_i varies for each i .

3.5 Types of Riemann Sums

The type of Riemann sum depends on how the sample points x_i^* are chosen within each subinterval $[x_{i-1}, x_i]$.

1. **Arbitrary-Point Rule:** $x_i^* \in [x_{i-1}, x_i] \rightarrow$ **General Riemann Sum**

$$S_n = \sum_{i=1}^n f(x_i^*) \Delta x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

Uniform Partition: If all subintervals have equal width $\Delta x = \frac{b-a}{n}$, then $x_i^* \in [x_{i-1}, x_i] = [a + (i-1)\Delta x, a + i\Delta x]$ or $x_i^* = a + (i-1+c)\Delta x$, where $c \in [0, 1]$ determines its position within the subinterval:

$$S_n = \sum_{i=1}^n f(a + (i-1+c)\Delta x) \Delta x = \sum_{i=1}^n f(a + (i-1+c)\frac{(b-a)}{n}) \frac{b-a}{n}.$$

- Uses arbitrary sample points x_i^* .

2. **Left Rule:** $x_i^* = x_{i-1} \rightarrow$ **Left Riemann Sum**

$$S_{\text{left}} = \sum_{i=1}^n f(x_{i-1}) \Delta x_i.$$

Uniform Partition: $c = 0$

$$S_{\text{left}} = \sum_{i=1}^n f(a + (i-1)\Delta x) \Delta x.$$

- Underestimates for increasing functions, overestimates for decreasing functions.

3. **Right Rule:** $x_i^* = x_i \rightarrow$ **Right Riemann Sum**

$$S_{\text{right}} = \sum_{i=1}^n f(x_i) \Delta x_i$$

Uniform Partition: $c = 1$

$$S_{\text{right}} = \sum_{i=1}^n f(a + i\Delta x) \Delta x.$$

- Overestimates for increasing functions, underestimates for decreasing functions.

4. **Midpoint Rule:** $x_i^* = \frac{x_{i-1} + x_i}{2} \rightarrow$ **Midpoint Riemann Sum**

$$S_{\text{mid}} = \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x_i.$$

Uniform Partition: $c = \frac{1}{2}$

$$S_{\text{mid}} = \sum_{i=1}^n f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right) \Delta x.$$

- Tends to give better approximations than left or right sums.

5. **Upper Rule:** $x_i^* = \arg \sup_{x \in [x_{i-1}, x_i]} f(x) \rightarrow$ **Upper Riemann Sum** (or **Upper Darboux Sum**)

$$S_{\text{upper}} = \sum_{i=1}^n \sup_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **overestimates** the integral.

6. **Lower Rule:** $x_i^* = \arg \inf_{x \in [x_{i-1}, x_i]} f(x) \rightarrow$ **Lower Riemann Sum** (or **Lower Darboux Sum**)

$$S_{\text{lower}} = \sum_{i=1}^n \inf_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **underestimates** the integral.

3.6 Definite Integral

The **definite integral** of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

where:

- $[a, b]$ is divided into n subintervals.
- $\Delta x_i = x_i - x_{i-1}$ is the width of the i -th subinterval.
- x_i^* is any sample point in the i -th subinterval.

3.7 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$ and c be a constant. Then:

$$(1) \quad \int_a^b c \, dx = c(b - a).$$

$$(2) \quad \int_a^a f(x) \, dx = 0.$$

$$(3) \quad \int_b^a f(x) \, dx = - \int_a^b f(x) \, dx.$$

$$(4) \quad \int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx.$$

$$(5) \quad \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$$

$$(6) \quad \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

$$(7) \quad \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \quad \text{if } f(x) \leq g(x) \text{ on } [a, b].$$

$$(8) \quad \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

$$(9) \quad m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a) \quad \text{if } m \leq f(x) \leq M \text{ on } [a, b].$$

3.8 Net and Total Area

Let $f(x)$ be integrable on $[a, b]$. Define:

- A_1 = area of region where $f(x) > 0$ (area above x -axis).
- A_2 = area of region where $f(x) < 0$ (area below x -axis).

Then:

- **Net Area:**

$$\int_a^b f(x) \, dx = A_1 - A_2.$$

- **Total Area:**

$$\int_a^b |f(x)| \, dx = A_1 + A_2.$$

3.9 Mean Value Theorem for Integrals

Let f be continuous on $[a, b]$. Then:

$$\exists c \in (a, b) \text{ such that } \frac{1}{b-a} \int_a^b f(x) dx = f(c)$$

The expression on the left is the **average value** of the function $f(x)$ on the interval $[a, b]$.

3.10 Fundamental Theorem of Calculus. Part 1

Let f be a continuous on $[a, b]$. Define the function

$$F(x) = \int_a^x f(t) dt.$$

Then:

$$F'(x) = \frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x), \quad \forall x \in (a, b).$$

3.11 Fundamental Theorem of Calculus. Part 2

If f is continuous on $[a, b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b = F(x) \Big|_a^b.$$

3.12 Table of Indefinite Integrals

$$\int a dx = ax + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln |x| + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\int \csc x dx = \ln |\csc x - \cot x| + C$$

$$\int \log_a x dx = \frac{x}{\ln a} (\ln x - 1) + C$$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{c}{ax + b} dx = \frac{c}{a} \ln |ax + b| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C, \quad a > 0, a \neq 1$$

$$\int \tan x dx = \ln |\sec x| = -\ln |\cos x| + C$$

$$\int \cot x dx = -\ln |\csc x| = \ln |\sin x| + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\left. \begin{aligned} \int \sec x \tan x \, dx &= \sec x + C \\ \int \csc x \cot x \, dx &= -\csc x + C \end{aligned} \right| \begin{aligned} \int \frac{1}{\sqrt{1-x^2}} \, dx &= \arcsin x + C \\ \int \frac{1}{1+x^2} \, dx &= \arctan x + C \end{aligned}$$

4 Sequences and Series

4.1 Definition of a Sequence

A sequence is a function $a : \mathbb{N} \rightarrow \mathbb{R}$ that assigns to each $n \in \mathbb{N}$ a real number a_n .

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = (a_n) = \{a_1, a_2, a_3, \dots, a_n, \dots\}$$

4.2 Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if it is either monotonically increasing or monotonically decreasing.

- **Increasing:**

$$a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly increasing).}$$

$$a_n < a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly increasing).}$$

- **Decreasing:**

$$a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly decreasing).}$$

$$a_n > a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly decreasing).}$$

4.3 Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if and only if:

$$\exists M > 0 \text{ such that } |a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

or

$$\exists M_1, M_2 \in \mathbb{R} \text{ such that } M_1 \leq a_n \leq M_2, \quad \forall n \in \mathbb{N}.$$

Equivalently, $\{a_n\}$ is **bounded** if it is both **bounded above** and **bounded below**:

- **Bounded above:** $\exists M_2 \in \mathbb{R}$ such that $a_n \leq M_2, \quad \forall n \in \mathbb{N}.$

- **Bounded below:** $\exists M_1 \in \mathbb{R}$ such that $a_n \geq M_1, \quad \forall n \in \mathbb{N}.$

A sequence is bounded \iff It is both bounded above and bounded below

4.4 Limit of a Sequence

A sequence $\{a_n\}$ has a limit $L \in \mathbb{R}$ if:

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{if} \quad \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \varepsilon, \forall n \geq N.$$

If such an L exists, the sequence is **convergent**; otherwise, it is **divergent**.

4.5 Limit of a Sequence Defined by a Function

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $\{a_n\}$ be defined by $a_n = f(n)$. Then:

$$\lim_{x \rightarrow \infty} f(x) = L \implies \lim_{n \rightarrow \infty} a_n = L.$$

4.6 Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for all $n \geq N$, then:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L \implies \lim_{n \rightarrow \infty} b_n = L.$$

4.7 Limit of Absolute Value of a Sequence

$$\lim_{n \rightarrow \infty} |a_n| = 0 \implies \lim_{n \rightarrow \infty} a_n = 0.$$

4.8 Monotone Convergence Theorem

Every bounded and monotonic sequence is convergent.

- (1) Monotonic \wedge Bounded \implies Convergent.
- (2) Monotonically Increasing \wedge Bounded Above \implies Convergent.
- (3) Monotonically Decreasing \wedge Bounded Below \implies Convergent.

4.9 Series

An **infinite series** is the sum of the terms of a sequence $\{a_n\}$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The n^{th} partial sum of series:

$$S_n = \sum_{k=1}^n a_k.$$

4.10 Convergence and Divergence of Series

A infinite series $\sum a_n$ **converges** if the sequence of partial sums $\{S_n\}$ has a finite limit:

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \quad \text{or} \quad \lim_{n \rightarrow \infty} S_n = S.$$

Otherwise, the series **diverges**.

4.11 Types of Series

- | | | |
|--------------------------------|----------------------|-------------------|
| • Arithmetic Series | • Telescoping Series | • Taylor Series |
| • Geometric Series | • Alternating Series | • Laurent Series |
| • Arithmetico-Geometric Series | • Power Series | • Fourier Series |
| • Harmonic Series | • Taylor Series | • Binomial Series |
| • p-Series | • Maclaurin Series | • Mercator Series |

4.12 Properties of Series

Let $\sum a_n$ and $\sum b_n$ be series, and let c, d be a constant. Then:

$$(1) \quad \sum_{k=a}^b c = c(b - a + 1).$$

$$(2) \quad \sum_{n=k}^{\infty} \pm c = \pm \infty, \quad c > 0.$$

$$(3) \quad \sum_{n=k}^m (ca_n \pm db_n) = c \sum_{n=k}^m a_n \pm d \sum_{n=k}^m b_n.$$

$$(4) \quad \sum_{n=k}^m a_n = \sum_{n=k}^p a_n + \sum_{n=p+1}^m a_n.$$

$$(5) \quad \sum_{n=k}^m a_n = \sum_{n=0}^{m-k} a_{m-n}.$$

$$(6) \quad \sum_{n=k}^m a_n = \sum_{j=k-h}^{m-h} a_{j+h}.$$

$$(7) \quad \sum_{k=1}^n k = \sum_{k=0}^{n-1} (k+1).$$

4.13 Geometric Series

$$\text{Geometric} \implies \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$$

- **Partial Sum:**

$$S_n = \sum_{k=0}^{n-1} ar^k = a \frac{1-r^n}{1-r}, \quad r \neq 1.$$

- **Convergence:**

$$|r| < 1 \implies \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

- **Divergence:**

$$|r| \geq 1 \implies \text{Series diverges.}$$

4.14 Telescoping Series

$$\text{Telescoping} \implies \sum_{n=1}^{\infty} (a_n - a_{n+1}) = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots$$

- **Partial Sum:**

$$S_n = \sum_{k=1}^n (a_k - a_{k+1}) = (a_1 - a_2) + (a_2 - a_3) + \dots + (a_n - a_{n+1}) = a_1 - a_{n+1}.$$

- **Convergence:**

$$\lim_{n \rightarrow \infty} a_{n+1} = L \implies \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - L.$$

- **Divergence:**

$$\lim_{n \rightarrow \infty} a_{n+1} \text{ does not exist} \implies \text{Series diverges.}$$

4.15 Harmonic Series

$$\text{Harmonic} \implies \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

- **Partial Sum and Approximation:**

$$H_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \dots + \frac{1}{n} \approx \ln(n) + \gamma$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant.

- **Divergence:**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ even though } \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

4.16 Limit of Terms in a Convergent Series

Let $\{a_n\}$ be a sequence and $\sum a_n$ be its series. Then:

$$\sum a_n \text{ convergent} \implies \lim_{n \rightarrow \infty} a_n = 0.$$

Note: The converse is false. ($\lim_{n \rightarrow \infty} a_n = 0 \not\Rightarrow \sum a_n \text{ convergent}$)

4.17 List of Convergence Tests

- **Tests for Positive Series:**

- Direct Comparison Test
- Integral Test
- Limit Comparison Test
- p-Series Test

- **Tests for Alternating Series:**

- Alternating Series Test
- Dirichlet's Test

- **General Tests:**

- Divergence Test
- Root Test
- Ratio Test
- Absolute Convergence Test

- **Advanced or Specialized Tests:**

- Cauchy Condensation Test
- Kummer's Test
- Abel's Test
- Gauss's Test
- Raabe's Test
- Bertrand's Test

4.18 Divergence Test (*n*th-Term Test)

$$\lim_{n \rightarrow \infty} a_n \neq 0 \vee \lim_{n \rightarrow \infty} a_n \text{ does not exist} \implies \sum a_n \text{ diverges.}$$

4.19 Direct Comparison Test

If $0 \leq a_n \leq b_n$ for all $n \geq N$:

- $\sum b_n \text{ converges} \implies \sum a_n \text{ converges.}$
- $\sum a_n \text{ diverges} \implies \sum b_n \text{ diverges.}$

4.20 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with $a_n > 0$, $b_n > 0$ for all $n \geq N$. Define the limit:

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}.$$

- $0 < L < \infty \implies \sum a_n \text{ and } \sum b_n \text{ both converge or both diverge.}$
- $L = 0 \wedge \sum b_n \text{ converges} \implies \sum a_n \text{ converges.}$
- $L = \infty \wedge \sum b_n \text{ diverges} \implies \sum a_n \text{ diverges.}$

Note: The order of division does not matter.

4.21 Integral Test (Maclaurin-Cauchy Test)

For a series $\sum a_n$ where $a_n = f(n)$, if $f(x)$ is a positive, continuous, and decreasing function for all $x \geq N$, then:

- $\sum_{n=N}^{\infty} a_n \text{ converges} \iff \int_N^{\infty} f(x) dx \text{ converges.}$
- $\sum_{n=N}^{\infty} a_n \text{ diverges} \iff \int_N^{\infty} f(x) dx \text{ diverges.}$

4.22 *p*-Series Test

For the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$:

- $p > 1 \iff \text{Series converges.}$
- $p \leq 1 \iff \text{Series diverges.}$

4.23 Integral Remainder Estimate

For a convergent series $\sum_{n=1}^{\infty} a_n$ with $a_n = f(n)$, where $f(x)$ is positive, continuous, and decreasing for all $x \geq N$, if the remainder $R_n = S - S_n$ then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$
$$S_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq S_n + \int_n^{\infty} f(x) dx$$

4.24 Alternating Series Test (Leibniz Criterion)

For an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where $a_n > 0$, the series converges if:

- $\lim_{n \rightarrow \infty} a_n = 0$. (terms approach zero)
- $a_{n+1} \leq a_n$ for all $n \geq N$. (monotonically decreasing)

4.25 Alternating Series Estimation Theorem

For an alternating series approximated by its n th partial sum, the absolute error (or remainder) is less than or equal to the absolute value of the next term in the series.

$$|R_n| = |S - S_n| \leq a_{n+1} = |S_{n+1} - S_n|.$$

4.26 Ratio Test (d'Alembert's Criterion)

Let $\sum a_n$ be an infinite series with $a_n \neq 0$. Define the limit:

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- $L < 1 \implies \sum a_n$ converges absolutely.
- $L > 1 \vee L = \infty \implies \sum a_n$ diverges.
- $L = 1 \implies$ Test is inconclusive.

4.27 Root Test (Cauchy's Criterion)

Let $\sum a_n$ be an infinite series. Define the limit:

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |a_n|^{1/n}.$$

- $L < 1 \implies \sum a_n$ converges absolutely.
- $L > 1 \vee L = \infty \implies \sum a_n$ diverges.
- $L = 1 \implies$ Test is inconclusive.

4.28 Absolute Convergence Test

- **Absolute Convergence:**

$$\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges absolutely.}$$

- **Conditional Convergence:**

$$\left(\sum a_n \text{ converges} \right) \wedge \left(\sum |a_n| \text{ diverges} \right) \implies \sum a_n \text{ converges conditionally.}$$

Note: All rearrangements of absolutely convergent series converge to the same sum.

4.29 Riemann Series Theorem

Conditionally convergent series $\sum_{n=1}^{\infty} a_n$ can be rearranged to:

- Converge to any real number: $\forall M \in \mathbb{R}, \exists \sigma$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)} = M$.
- Diverge to $\pm\infty$: $\exists \sigma$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)} = \pm\infty$.
- Fail to approach any limit: $\exists \sigma$ such that $\lim_{N \rightarrow \infty} \sum_{n=1}^N a_{\sigma(n)}$ does not exist.

4.30 Power Series

A power series centered at c is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

where $\{a_n\}$ is a sequence of coefficients, x is a variable, and c is the center of the series.

4.31 Radius of Convergence

For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, the radius of convergence R is a non-negative real number (possibly 0 or ∞) such that:

- If $R = 0$: The series converges only at $x = c$.
- If $R = \infty$: The series converges for all values of x .
- If $0 < R < \infty$:
 - The series converges absolutely for $|x - c| < R$.
 - The series diverges for $|x - c| > R$.
 - The series may or may not converge at $|x - c| = R$.

Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| < 1 \implies \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x-c| < 1$$

$$L \cdot |x-c| < 1 \implies |x-c| < \frac{1}{L} = \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

$$R = \frac{1}{L} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n(x-c)^n|} < 1 \implies \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \cdot |x-c| < 1$$

$$L \cdot |x-c| < 1 \implies |x-c| < \frac{1}{L} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

$$R = \frac{1}{L} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

4.32 Interval of Convergence

The interval of convergence of a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is the set of values of x for which the series converges absolutely.

- If $R = 0$, the interval is $\{c\}$.
- If $R = \infty$, the interval is $(-\infty, +\infty)$.
- If $0 < R < \infty$, the interval is one of the following depending on the convergence at the endpoints:
 - $(c-R, c+R)$ ◦ $[c-R, c+R)$ ◦ $(c-R, c+R]$ ◦ $[c-R, c+R]$

4.33 Differentiation and Integration of Power Series

For a power series centered at c

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

with radius of convergence R :

• **Differentiation:**

$$f'(x) = \frac{d}{dx} \left[\sum_{n=0}^{\infty} a_n(x-c)^n \right] = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x-c)^{n-1}$$

• **Integration:**

$$\int f(x) dx = \int \sum_{n=0}^{\infty} a_n(x-c)^n dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \dots$$

$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$$

Note: Both operations preserve the radius of convergence R , however, convergence at the endpoints $x = c \pm R$ may change and must be checked separately.

4.34 Taylor Series

Let f be infinitely differentiable at a point a . The Taylor Series of f centered at a is the infinite series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \end{aligned}$$

If there exists an interval where $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, then f is analytic at a . Note that analyticity implies infinite differentiability, but the converse is false.

4.35 Taylor Polynomial

The n th degree Taylor polynomial of a function $f(x)$ at point a is the finite sum:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n. \end{aligned}$$

4.36 Taylor's Theorem

Let f be a function that is $n+1$ times differentiable on an open interval I containing a point a . Then for any $x \in I$, there exists a point ξ between a and x such that:

$$\begin{aligned} f(x) &= P_n(x) + R_n(x) \\ &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}. \end{aligned}$$

Note: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$ is the Lagrange form of the remainder.

4.37 Taylor's Inequality

Taylor's Inequality provides an upper bound on the error when approximating a function with its Taylor polynomial. Let f be a function that is $n+1$ times differentiable on an open interval containing a and x . If there exists a constant M such that $f^{(n+1)}(t) \leq M$ for all t between a and x , then:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.$$

4.38 Binomial Series

For any real number $|x| < 1$, the binomial series is given by

$$\begin{aligned} (1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots \end{aligned}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

5 Multivariable Calculus

5.1 Limit of a Function of Two Variables

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall (x,y) \in D,$$
$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon.$$

5.2 Test for Nonexistence of a Limit

$$\bullet \lim_{(x,y) \rightarrow (a,b) \text{ along } C_1} f(x,y) = L_1 \qquad \bullet \lim_{(x,y) \rightarrow (a,b) \text{ along } C_2} f(x,y) = L_2$$

$$L_1 \neq L_2 \implies \lim_{(x,y) \rightarrow (a,b)} f(x,y) \text{ does not exist.}$$

5.3 Partial Derivatives

The **partial derivative** of a function $f(x,y)$ with respect to x is defined by:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

5.4 Clairaut's Theorem

If the mixed partial derivatives exist and are continuous then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

or equivalently,

$$f_{xy} = f_{yx}.$$

5.5 Equation of a Tangent Plane

The **tangent plane** to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is given by:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

5.6 Total Differential

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

5.7 Chain Rule For a Function of Two Variables

If $z = f(x, y)$ and $x = g(t), y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

5.8 Implicit Differentiation of a Function of Two Variables

If $F(x, y) = 0$ where y is implicitly defined as a function of x :

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

5.9 Gradient Vector

The **gradient vector** of a function points in the direction of steepest increase of the function at a given point, with its magnitude representing the rate of that increase. It collects all the partial derivatives of the function into a single vector.

The gradient vector of $f(x, y)$ at a point (x_0, y_0) is the vector whose components are the partial derivatives of f at that point.

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

or,

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right).$$

5.10 Directional Derivative

The **directional derivative** of a function $f(x, y)$ at a point (x_0, y_0) in the direction of a unit vector $\vec{u} = (u_1, u_2)$, where $\|\vec{u}\| = 1$, is the rate of change of f at (x_0, y_0) along the direction \vec{u} and is defined as:

$$D_{\vec{u}}f(x, y) = \lim_{h \rightarrow 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

If $f(x, y)$ is differentiable at a point (x_0, y_0) , the directional derivative of f in the direction of a unit vector $\vec{u} = (u_1, u_2)$ is defined as:

$$D_{\vec{u}}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)u_1 + \frac{\partial f}{\partial y}(x_0, y_0)u_2,$$

or equivalently,

$$D_{\vec{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}.$$

5.11 Tangent Plane to a Level Surface

The **tangent plane** to the level surface at point (x_0, y_0, z_0) of $F(x, y, z) = c$ is:

$$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

or written out fully,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

5.12 Maximum and Minimum Values of a Function of Two Variables

1. **Critical Points:** Solve the system of equations

$$\frac{\partial f}{\partial x}(x, y) = 0, \quad \frac{\partial f}{\partial y}(x, y) = 0.$$

2. **Second Derivative Test:** Compute the second-order partial derivatives f_{xx} , f_{yy} , f_{xy} and evaluate the discriminant

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2.$$

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$: Local minimum.
- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$: Local maximum.
- If $D < 0$: Saddle point (neither max nor min).
- If $D = 0$: Inconclusive.

3. **Boundary Analysis.**

5.13 Parametric Derivative

If $x = f(t)$ and $y = g(t)$, the derivative of y with respect to x is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

5.14 Area Under a Parametric Curve

Let a parametric curve be defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$. Then, the area is given by:

$$A = \int_{x=f(a)}^{x=f(b)} y \, dx = \int_{t=a}^{t=b} g(t) \cdot f'(t) \, dt.$$

5.15 Arc Length of a Parametric Curve

The **arc length** of a parametric curve defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$ is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

5.16 Surface Area of a Solid Revolution for a Parametric Curve

If the curve, defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$, is rotated about the x -axis, the surface area S is given by:

$$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

5.17 Polar Coordinates

- **From Polar to Cartesian:**

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$

- **From Cartesian to Polar:**

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

5.18 Area in Polar Coordinates

The area A of a region enclosed by a polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by:

$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

5.19 Arc Length in Polar Coordinates

The arc length L of a polar curve defined by $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

5.20 Tangents to Polar Curves

The slope of the tangent line to a polar curve $r = f(\theta)$ at a given point (r, θ) is given by:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta)}{\frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)}.$$

5.21 Double Integral

If R is a bounded region and f is integrable over R , then:

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}.$$

5.22 Iterated Integral

For a function $f(x, y)$ defined over a rectangular region $R = [a, b] \times [c, d]$, the double integral

$$\int_c^d \int_a^b f(x, y) dx dy.$$

5.23 Average Value of a Function over a Region

The average value of f the region R is defined by:

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA.$$

5.24 Double Integral Over a General Region

- **Type I Region** (vertical slices):

If $R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then:

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

- **Type II Region** (horizontal slices):

If $R = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then:

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

5.25 Double Integral in Polar Coordinates

The double integral in polar coordinates is given by:

$$\iint_R f(x, y) dA = \iint_{R'} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

5.26 Double Integral in Polar Coordinates Over a General Region

If $R = \{(r, \theta) : \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\}$, then:

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$