Calculus Notes

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Contents

1 Limits

1.1 Precise Definition of a Limit

Standard Limit:

$$\lim_{x\to a} f(x) = L \quad \text{if} \quad \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.2 Precise Definition of One-Sided Limit

Right-Hand Limit:

$$\lim_{x\to a^+} f(x) = L \quad \text{if} \quad \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } 0 < x-a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Left-Hand Limit:

$$\lim_{x\to a^-} f(x) = L \quad \text{if} \quad \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } 0 < a-x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.3 Precise Definition of Infinite Limit

Infinite Limit:

$$\lim_{x \to a} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

$$\lim_{x \to a} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -M.$$

1.4 Precise Definition of a Limit at Infinity

Limit at Infinity:

$$\lim_{x \to \infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \to -\infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x < -M \Rightarrow |f(x) - L| < \varepsilon.$$

1.5 Precise Definition of Infinite Limit at Infinity

Infinite Limit at Infinity:

$$\lim_{x \to \infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$

$$\lim_{x \to \infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) < -M.$$

$$\lim_{x \to -\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) > M.$$

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) < -M.$$

1.6 Limit Laws

Suppose that c is a constant and the limits $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. Then

1. $\lim\limits_{x \to a} c = c$
2. $\lim\limits_{x \to a} x = a$
3. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
4. $\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} f(x)$
5. $\lim\limits_{x \to a} [f(x)g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
6. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, if $\lim\limits_{x \to a} g(x) \neq 0$
7. $\lim\limits_{x \to a} [f(x)]^n = [\lim\limits_{x \to a} f(x)]^n$
8. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$

1.7 Relationship between the Limit and One-Sided Limits

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \implies \lim_{x\to a} f(x) \text{ does not exist.}$$

1.8 Comparison Theorem

If $f(x) \leq g(x)$ when x is near a, and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

1.9 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \implies \lim_{x\to a} g(x) = L.$$

1.10 Continuity

A function $f(x)$ is **continuous at** $x = a$ if and only if it satisfies **all** the following:

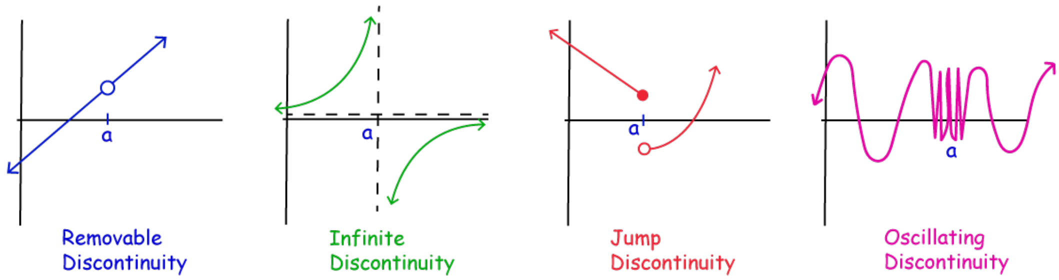
$$\begin{aligned}
&(1) \quad f(a) \text{ exists} \\
&(2) \quad \lim_{x\to a} f(x) \text{ exists} \\
&(3) \quad \lim_{x\to a} f(x) = f(a)
\end{aligned}$$

Otherwise, $f(x)$ is discontinuous at $x = a$.

1.11 Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$



Source: calcworkshop.com

1.12 Types of Discontinuity

1.13 Limits of Continuous Functions

If $f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

If g is continuous at a and a is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a.

1.14 Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any N between $f(a)$ and $f(b)$,

$$\exists c \in [a, b] \text{ such that } f(c) = N$$

1.15 Asymptotes

Vertical Asymptote: $x = a$ is a vertical asymptote if

$$\lim_{x\to a^{\pm}} f(x) = \pm\infty.$$

Horizontal Asymptote: $y = L$ is a horizontal asymptote if

$$\lim_{x\to\pm\infty} f(x) = L.$$

For $f(x) = \dfrac{P(x)}{Q(x)}$, compare degrees of P and Q:

$$\begin{aligned}
\deg P < \deg Q \quad &\Longrightarrow \quad y = 0. \\
\deg P = \deg Q \quad &\Longrightarrow \quad y = \frac{\text{leading coef. of } P}{\text{leading coef. of } Q}. \\
\deg P > \deg Q \quad &\Longrightarrow \quad \text{no horizontal asymptote.}
\end{aligned}$$

Oblique Asymptote: $y = mx + b$ is an oblique asymptote if

$$\lim_{x\to\pm\infty} [f(x) - (mx + b)] = 0.$$

For a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, if $\deg P = \deg Q + 1$, then $f(x)$ has an oblique asymptote. Find it by polynomial long division:

$$f(x) = D(x) + \frac{R(x)}{Q(x)}, \quad \text{as } x \to \pm\infty, \quad f(x) \approx D(x).$$

Curvilinear Asymptote: $y = g(x)$ is a curvilinear asymptote if

$$\lim_{x\to\pm\infty} [f(x) - g(x)] = 0,$$

where $g(x)$ is any non-linear function.

1.16 Common Limits

Assume $a > 0$ in the following.

1. $\displaystyle\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$
2. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x} = 0$
3. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
4. $\displaystyle\lim_{x\to 0} \frac{e^{ax}-1}{x} = a$
5. $\displaystyle\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$
6. $\displaystyle\lim_{x\to \infty} \left(1+\frac{k}{x}\right)^{mx} = e^{mk}$
7. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
8. $\displaystyle\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$
9. $\displaystyle\lim_{x\to 0^+} x^x = 1$
10. $\displaystyle\lim_{x\to \infty} \sqrt[x]{x} = \lim_{x\to\infty} x^{\frac{1}{x}} = 1$
11. $\displaystyle\lim_{x\to 0^+} x^a \ln x = 0$
12. $\displaystyle\lim_{x\to \infty} x^{-a} \ln x = 0$
13. $\displaystyle\lim_{n\to \infty} \frac{x^n}{n!} = 0 \lim$
14. $\displaystyle\lim_{n\to \infty} \frac{n^n}{n!} = \infty$

2 Derivatives

2.1 Derivative at a Point

The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** at that point:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

2.2 Derivative as a Function

The derivative of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

2.3 Differentiability

A function $f(x)$ is **differentiable** at $x = a$ if its derivative $f'(x)$ exists. That is:

$$f(x) \text{ is differentiable at } x = a \iff \text{The limit } \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

2.4 Differentiability Implies Continuity

If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$:

$$f \text{ differentiable at } a \implies f \text{ continuous at } a.$$

However, the converse is false:

$$f \text{ continuous at } a \not\Longrightarrow f \text{ differentiable at } a.$$

2.5 Properties of Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions. Then the following rules hold:

$$(1) \quad \frac{d}{dx}(c) = 0.$$

$$(2) \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

$$(3) \quad \frac{d}{dx}[cf(x)] = c[\frac{d}{dx}f(x)].$$

$$(4) \quad \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

$$(5) \quad \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$(6) \quad \frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

$$(7) \quad \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

2.6 Table of Derivatives

$(\sin x)' = \cos x$	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(e^x)' = e^x$
$(\cos x)' = -\sin x$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$(a^x)' = a^x \ln a$
$(\tan x)' = \sec^2 x$	$(\arctan x)' = \frac{1}{1+x^2}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\cot x)' = -\csc^2 x$	$(x)' = -\frac{1}{1+x^2}$	$(\ln x)' = \frac{1}{x}$
$(\sec x)' = \sec x \tan x$	$(x)' = \frac{1}{x\sqrt{x^2-1}}$	$(\lvert x\rvert)' = \frac{x}{\lvert x\rvert}$
$(\csc x)' = -\csc x \cot x$	$(x)' = -\frac{1}{x\sqrt{x^2-1}}$	$(x^x)' = x^x(1 + \ln x)$

2.7 Absolute and Local Extrema

Let f be defined on a domain D, and let $c \in D$.

- **Absolute Maximum:** $f(c)$ is an absolute maximum if $f(c) \geq f(x),\ \forall x \in D$.
- **Absolute Minimum:** $f(c)$ is an absolute minimum if $f(c) \leq f(x),\ \forall x \in D$.
- **Local Maximum:** $f(c)$ is a local maximum if $\exists \delta > 0$ such that $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.
- **Local Minimum:** $f(c)$ is a local minimum if $\exists \delta > 0$ such that $f(c) \leq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

2.8 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$:

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$

2.9 Critical and Stationary Points

Let f be defined on an interval I and $c \in I$.

- **Critical Point:** c is a critical point of f if either
  1. $f'(c) = 0$, or
  2. $f'(c)$ does not exist
- **Stationary Point:** c is a stationary point of f if $f'(c) = 0$.

Note: *Every stationary point is a critical point, but not conversely.*

2.10 Rolle's Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).
3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2.11 Mean Value Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$.

2.12 Increasing/Decreasing Test

Let f be differentiable on an interval I. Then, for all $x \in I$:

1. If $f'(x) > 0$, then f is strictly increasing on I.
2. If $f'(x) < 0$, then f is strictly decreasing on I.
3. If $f'(x) = 0$, then f is constant on I.

2.13 First Derivative Test

Let c be a critical point of a differentiable function $f(x)$, meaning $f'(c) = 0$ or $f'(c)$ does not exist. Then:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is **neither** a local maximum nor a local minimum.

2.14 Concavity and Inflection Points

$$\text{Concave up} \iff \text{Curve } \textbf{lies above} \text{ all of its } \textbf{tangent lines}.$$
$$\text{Concave down} \iff \text{Curve } \textbf{lies below} \text{ all of its } \textbf{tangent lines}.$$
$$\text{Inflection point} \iff \text{Point where } \textbf{concavity changes}.$$

2.15 Concavity Test

Let $f(x)$ be twice differentiable on interval I. Then:

- If $f''(x) > 0, \quad \forall x \in I \implies f(x)$ is **concave up** on I.
- If $f''(x) < 0, \quad \forall x \in I \implies f(x)$ is **concave down** on I.

2.16 Second Derivative Test

Let c be a critical point of f where $f(c) = 0$. If $f''(c)$ exists, then:

1. If $f''(c) > 0$, $f(x)$ is **concave up** at c, so $f(c)$ is a **local minimum**.

$$\text{Local Maximum at } c \iff f'(c) = 0 \text{ and } f''(c) < 0$$

2. If $f''(c) < 0$, $f(x)$ is **concave down** at c, so $f(c)$ is a **local maximum**.

$$\text{Local Minimum at } c \iff f'(c) = 0 \text{ and } f''(c) > 0$$

3. If $f''(c) = 0$ or **does not exist**, the test is **inconclusive**—use the First Derivative Test instead.

2.17 L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a (except possibly at a). If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2.18 Indeterminate Forms

The following symbols are "indeterminate":

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 1^{\infty} \qquad 0^{\infty} \qquad \infty^{0}$$

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \qquad \frac{\infty}{0} \qquad \frac{1}{\infty} \qquad 1\cdot\infty \qquad \infty+\infty \qquad 1+\infty \qquad 0^{\infty}$$

3 Integrals

3.1 Antiderivatives

A function $F(x)$ is an **antiderivative** (or primitive function) of $f(x)$ on an interval I if:

$$F'(x) = f(x), \quad \forall x \in I.$$

3.2 Indefinite Integrals

The **indefinite integral** (or general antiderivative) of a function $f(x)$ is the family of all its antiderivatives:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

3.3 Riemann Sum

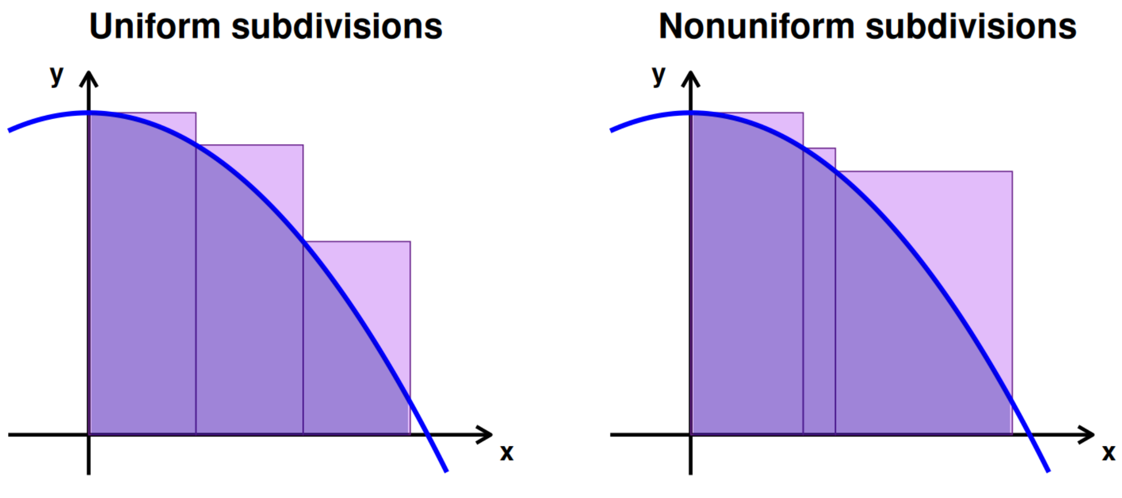
Let $f(x)$ be a function defined on a closed interval $[a, b]$, and let the interval be partitioned into n subintervals by inserting $n-1$ points $x_1, x_2, \ldots, x_{n-1}$ such that:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

where the subintervals $[x_{i-1}, x_i]$ have length $\Delta x_i = x_i - x_{i-1}$. For each subinterval $[x_{i-1}, x_i]$, let x_i^* be an **arbitrary sample point** within the interval. Then, a **Riemann sum**, which **approximates** the area under the curve $f(x)$ over $[a, b]$, is defined as:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i, \quad x_i^* \in [x_{i-1}, x_i].$$

3.4 Partitions of an Interval



Comparison of uniform and non-uniform partitions of an interval (generated using R).

A partition of $[a, b]$ divides it into n subintervals:

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Each subinterval has width:

$$\Delta x_i = x_i - x_{i-1}.$$

- **Uniform Partition:** All subintervals have the same width:

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \quad \forall i.$$

- **Non-Uniform Partition:** Subintervals have different widths, and Δx_i varies for each i.

3.5 Types of Riemann Sums

The type of Riemann sum depends on how the sample points x_i^* are chosen within each subinterval $[x_{i-1}, x_i]$.

1. **Arbitrary-Point Rule:** $x_i^* \in [x_{i-1}, x_i] \to$ **General Riemann Sum**

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

Uniform Partition: If all subintervals have equal width $\Delta x = \dfrac{b-a}{n}$, then $x_i^* \in [x_{i-1}, x_i] = [a + (i-1)\Delta x, a + i\Delta x]$ or $x_i^* = a + (i - 1 + c)\Delta x$, where $c \in [0, 1]$ determines its position within the subinterval:

$$S_n = \sum_{i=1}^{n} f(a + (i-1+c)\Delta x)\Delta x = \sum_{i=1}^{n} f(a + (i-1+c)\frac{(b-a)}{n})\frac{b-a}{n}.$$

- Uses arbitrary sample points x_i^*.

2. **Left Rule:** $x_i^* = x_{i-1} \to$ **Left Riemann Sum**

$$S_{\text{left}} = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i.$$

Uniform Partition: $c = 0$

$$S_{\text{left}} = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\Delta x.$$

- Underestimates for increasing functions, overestimates for decreasing functions.

3. **Right Rule:** $x_i^* = x_i \to$ **Right Riemann Sum**

$$S_{\text{right}} = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Uniform Partition: $c = 1$

$$S_{\text{right}} = \sum_{i=1}^{n} f(a + i\Delta x)\Delta x.$$

- Overestimates for increasing functions, underestimates for decreasing functions.

4. **Midpoint Rule:** $x_i^* = \dfrac{x_{i-1} + x_i}{2}$ → **Midpoint Riemann Sum**

$$S_{\text{mid}} = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x_i.$$

Uniform Partition: $c = \dfrac{1}{2}$

$$S_{\text{mid}} = \sum_{i=1}^{n} f(a + (i - \frac{1}{2})\Delta x)\Delta x.$$

- Tends to give better approximations than left or right sums.

5. **Upper Rule:** $x_i^* = \arg \sup_{x \in [x_{i-1}, x_i]} f(x)$ → **Upper Riemann Sum** (or **Upper Darboux Sum**)

$$S_{\text{upper}} = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)\Delta x_i.$$

- Always **overestimates** the integral.

6. **Lower Rule:** $x_i^* = \arg \inf_{x \in [x_{i-1}, x_i]} f(x)$ → **Lower Riemann Sum** (or **Lower Darboux Sum**)

$$S_{\text{lower}} = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)\Delta x_i.$$

- Always **underestimates** the integral.

3.6 Definite Integral

The **definite integral** of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where:

- $[a, b]$ is divided into n subintervals.
- $\Delta x_i = x_i - x_{i-1}$ is the width of the i-th subinterval.
- x_i^* is any sample point in the i-th subinterval.

3.7 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$ and c be a constant. Then:

(1) $\displaystyle\int_a^b c\,dx = c(b-a).$

(2) $\displaystyle\int_a^a f(x)\,dx = 0.$

(3) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

(4) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

(5) $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

(6) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

(7) $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \quad \text{if } f(x) \le g(x) \text{ on } [a, b].$

(8) $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

(9) $\displaystyle m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \quad \text{if } m \le f(x) \le M \text{ on } [a, b].$

3.8 Net and Total Area

Let $f(x)$ be integrable on $[a, b]$. Define:

- A_1 = area of region where $f(x) > 0$ (area above x-axis).
- A_2 = area of region where $f(x) < 0$ (area below x-axis).

Then:

- **Net Area:**
$$\int_a^b f(x)\,dx = A_1 - A_2.$$
- **Total Area:**
$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

3.9 Mean Value Theorem for Integrals

Let f be continuous on [a,b]. Then:

$$\exists c \in (a,b) \text{ such that } \frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

The expression on the left is the **average value** of the function $f(x)$ on the interval $[a,b]$.

3.10 Fundamental Theorem of Calculus. Part 1

Let f be a continuous on $[a,b]$. Define the function

$$F(x) = \int_a^x f(t)\,dt.$$

Then:

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x), \quad \forall x \in (a,b).$$

3.11 Fundamental Theorem of Calculus. Part 2

If f is continuous on $[a,b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b = F(x)\Big|_a^b.$$

3.12 Table of Indefinite Integrals

$$\int a\,dx = ax + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \log_a x\,dx = \frac{x}{\ln a}(\ln x - 1) + C$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{c}{ax+b}\,dx = \frac{c}{a}\ln|ax+b| + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \quad a > 0,\ a \neq 1$$

$$\int \tan x\,dx = \ln|\sec x| = -\ln|\cos x| + C$$

$$\int \cot x\,dx = -\ln|\csc x| = \ln|\sin x| + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C \qquad \int \frac{1}{1+x^2} \, dx = \arctan x + C$$

4 Sequences and Series

4.1 Definition of a Sequence

A sequence is a function $a : \mathbb{N} \to \mathbb{R}$ that assigns to each $n \in N$ a real number a_n.

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = (a_n) = \{a_1, a_2, a_3, \dots, a_n, \dots\}$$

4.2 Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if it is either monotonically increasing or monotonically decreasing.

- **Increasing:**

$$a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly increasing).}$$

$$a_n < a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly increasing).}$$

- **Decreasing:**

$$a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly decreasing).}$$

$$a_n > a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly decreasing).}$$

4.3 Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if and only if:

$$\exists M > 0 \text{ such that } |a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

or

$$\exists M_1, M_2 \in \mathbb{R} \text{ such that } M_1 \leq a_n \leq M_2, \quad \forall n \in \mathbb{N}.$$

Equivalently, $\{a_n\}$ is **bounded** if it is both **bounded above** and **bounded below**:

- **Bounded above:** $\exists M_2 \in \mathbb{R}$ such that $a_n \leq M_2, \quad \forall n \in \mathbb{N}$.
- **Bounded below:** $\exists M_1 \in \mathbb{R}$ such that $a_n \geq M_1, \quad \forall n \in \mathbb{N}$.

$$\text{A sequence is bounded} \iff \text{It is both bounded above and bounded below}$$

4.4 Limit of a Sequence

A sequence $\{a_n\}$ has a limit $L \in \mathbb{R}$ if:

$$\lim_{n\to\infty} a_n = L \quad \text{if} \quad \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \varepsilon, \forall n \geq N.$$

If such an L exists, the sequence is **convergent**; otherwise, it is **divergent**.

4.5 Limit of a Sequence Defined by a Function

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $\{a_n\}$ be defined by $a_n = f(n)$. Then:

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L.$$

4.6 Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for all $n \geq N$, then:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \implies \lim_{n\to\infty} b_n = L.$$

4.7 Limit of Absolute Value of a Sequence

$$\lim_{n\to\infty} |a_n| = 0 \implies \lim_{n\to\infty} a_n = 0.$$

4.8 Monotone Convergence Theorem

Every bounded and monotonic sequence is convergent.

$$\text{(1)} \quad \text{Monotonic} \ \wedge \ \text{Bounded} \implies \text{Convergent.}$$

$$\text{(2)} \quad \text{Monotonically Increasing} \ \wedge \ \text{Bounded Above} \implies \text{Convergent.}$$

$$\text{(3)} \quad \text{Monotonically Decreasing} \ \wedge \ \text{Bounded Below} \implies \text{Convergent.}$$

4.9 Series

An **infinite series** is the sum of the terms of a sequence $\{a_n\}$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The $\boldsymbol{n^{\text{th}}}$ **partial sum** of series:

$$S_n = \sum_{k=1}^{n} a_k.$$

4.10 Convergence and Divergence of Series

A infinite series $\sum a_n$ **converges** if the sequence of partial sums $\{S_n\}$ has a finite limit:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \quad \text{or} \quad \lim_{n\to\infty} S_n = S.$$

Otherwise, the series **diverges**.

4.11 Types of Series

- Arithmetic Series
- Geometric Series
- Arithmetico-Geometric Series
- Harmonic Series
- p-Series
- Telescoping Series
- Alternating Series
- Power Series
- Taylor Series
- Maclaurin Series
- Taylor Series
- Laurent Series
- Fourier Series
- Binomial Series
- Mercator Series

4.12 Properties of Series

Let $\sum a_n$ and $\sum b_n$ be series, and let c, d be a constant. Then:

$$(1) \quad \sum_{k=a}^{b} c = c(b-a+1).$$

$$(2) \quad \sum_{n=k}^{\infty} \pm c = \pm\infty, \quad c > 0.$$

$$(3) \quad \sum_{n=k}^{m} (ca_n \pm db_n) = c\sum_{n=k}^{m} a_n \pm d\sum_{n=k}^{m} b_n.$$

$$(4) \quad \sum_{n=k}^{m} a_n = \sum_{n=k}^{p} a_n + \sum_{n=p+1}^{m} a_n.$$

$$(5) \quad \sum_{n=k}^{m} a_n = \sum_{n=0}^{m-k} a_{m-n}.$$

$$(6) \quad \sum_{n=k}^{m} a_n = \sum_{j=k-h}^{m-h} a_{j+h}.$$

$$(7) \quad \sum_{k=1}^{n} k = \sum_{k=0}^{n-1} (k+1).$$

4.13 Geometric Series

$$\text{Geometric} \implies \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$$

- **Partial Sum:**

$$S_n = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}, \quad r \neq 1.$$

- **Convergence:**

$$|r| < 1 \implies \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

- **Divergence:**

$$|r| \geq 1 \implies \text{Series diverges.}$$

4.14 Telescoping Series

$$\text{Telescoping} \implies \sum_{n=1}^{\infty} (a_n - a_{n+1}) = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots$$

- **Partial Sum:**

$$S_n = \sum_{k=1}^{n} (a_k - a_{k+1}) = (a_1 - a_2) + (a_2 - a_3) + \dots + (a_n - a_{n+1}) = a_1 - a_{n+1}.$$

- **Convergence:**

$$\lim_{n\to\infty} a_{n+1} = L \implies \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - L.$$

- **Divergence:**

$$\lim_{n\to\infty} a_{n+1} \text{ does not exist} \implies \text{Series diverges.}$$

4.15 Harmonic Series

$$\text{Harmonic} \implies \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

- **Partial Sum and Approximation:**

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln(n) + \gamma$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant.

- **Divergence:**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ even though } \lim_{n\to\infty} \frac{1}{n} = 0.$$

4.16 Limit of Terms in a Convergent Series

Let $\{a_n\}$ be a sequence and $\sum a_n$ be its series. Then:

$$\sum a_n \text{ convergent} \implies \lim_{n\to\infty} a_n = 0.$$

Note: The converse is false. ($\lim_{n\to\infty} a_n = 0 \not\Longrightarrow \sum a_n$ convergent)

4.17 List of Convergence Tests

- **Tests for Positive Series:**
  - Direct Comparison Test
  - Limit Comparison Test
  - Integral Test
  - p-Series Test
- **Tests for Alternating Series:**
  - Alternating Series Test
  - Dirichlet's Test
- **General Tests:**
  - Divergence Test
  - Ratio Test
  - Root Test
  - Absolute Convergence Test
- **Advanced or Specialized Tests:**
  - Cauchy Condensation Test
  - Abel's Test
  - Raabe's Test
  - Kummer's Test
  - Gauss's Test
  - Bertrand's Test

4.18 Divergence Test (nth-Term Test)

$$\lim_{n\to\infty} a_n \neq 0 \ \vee \ \lim_{n\to\infty} a_n \text{ does not exist} \implies \sum a_n \text{ diverges.}$$

4.19 Direct Comparison Test

If $0 \le a_n \le b_n$ for all $n \ge N$:

- $\sum b_n$ converges $\implies \sum a_n$ converges.
- $\sum a_n$ diverges $\implies \sum b_n$ diverges.

4.20 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with $a_n > 0$, $b_n > 0$ for all $n \ge N$. Define the limit:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

- $0 < L < \infty \implies \sum a_n$ and $\sum b_n$ both converge or both diverge.
- $L = 0 \ \wedge \ \sum b_n$ converges $\implies \sum a_n$ converges.
- $L = \infty \ \wedge \ \sum b_n$ diverges $\implies \sum a_n$ diverges.

Note: The order of division does not matter.

4.21 Integral Test (Maclaurin-Cauchy Test)

For a series $\sum a_n$ where $a_n = f(n)$, if $f(x)$ is a positive, continuous, and decreasing function for all $x \ge N$, then:

- $\displaystyle\sum_{n=N}^{\infty} a_n$ converges $\iff \displaystyle\int_N^{\infty} f(x)\,dx$ converges.
- $\displaystyle\sum_{n=N}^{\infty} a_n$ diverges $\iff \displaystyle\int_N^{\infty} f(x)\,dx$ diverges.

4.22 p-Series Test

For the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$:

- $p > 1 \iff$ Series converges.
- $p \le 1 \iff$ Series diverges.

4.23 Integral Remainder Estimate

For a convergent series $\sum_{n=1}^{\infty} a_n$ with $a_n = f(n)$, where $f(x)$ is positive, continuous, and decreasing for all $x \geq N$, if the remainder $R_n = S - S_n$ then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq S_n + \int_{n}^{\infty} f(x)\,dx$$

4.24 Alternating Series Test (Leibniz Criterion)

For an alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where $a_n > 0$, the series converges if:

- $\lim\limits_{n\to\infty} a_n = 0$. (terms approach zero)
- $a_{n+1} \leq a_n$ for all $n \geq N$. (monotonically decreasing)

4.25 Alternating Series Estimation Theorem

For an alternating series approximated by its nth partial sum, the absolute error (or remainder) is less than or equal to the absolute value of the next term in the series.

$$|R_n| = |S - S_n| \ \leq \ a_{n+1} = |S_{n+1} - S_n|.$$

4.26 Ratio Test (d'Alembert's Criterion)

Let $\sum a_n$ be an infinite series with $a_n \neq 0$. Define the limit:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- $L < 1 \implies \sum a_n$ converges absolutely.
- $L > 1 \ \vee \ L = \infty \implies \sum a_n$ diverges.
- $L = 1 \implies$ Test is inconclusive.

4.27 Root Test (Cauchy's Criterion)

Let $\sum a_n$ be an infinite series. Define the limit:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}.$$

- $L < 1 \implies \sum a_n$ converges absolutely.
- $L > 1 \lor L = \infty \implies \sum a_n$ diverges.
- $L = 1 \implies$ Test is inconclusive.

4.28 Absolute Convergence Test

- **Absolute Convergence:**

$$\sum |a_n| \text{ converges } \implies \sum a_n \text{ converges absolutely.}$$

- **Conditional Convergence:**

$$\left(\sum a_n \text{ converges}\right) \land \left(\sum |a_n| \text{ diverges}\right) \implies \sum a_n \text{ converges conditionally.}$$

Note: All rearrangements of absolutely convergent series converge to the same sum.

4.29 Riemann Series Theorem

Conditionally convergent series $\sum_{n=1}^{\infty} a_n$ can be rearranged to:

- Converge to any real number: $\forall M \in \mathbb{R}, \exists \sigma$ such that $\displaystyle\sum_{n=1}^{\infty} a_{\sigma(n)} = M$.
- Diverge to $\pm\infty$: $\exists \sigma$ such that $\displaystyle\sum_{n=1}^{\infty} a_{\sigma(n)} = \pm\infty$.
- Fail to approach any limit: $\exists \sigma$ such that $\displaystyle\lim_{N\to\infty} \sum_{n=1}^{N} a_{\sigma(n)}$ does not exist.

4.30 Power Series

A power series centered at c is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

where $\{a_n\}$ is a sequence of coefficients, x is a variable, and c is the center of the series.

4.31 Radius of Convergence

For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, the radius of convergence R is a non-negative real number (possibly 0 or ∞) such that:

- If $R = 0$: The series converges only at $x = c$.
- If $R = \infty$: The series converges for all values of x.
- If $0 < R < \infty$:
  - The series converges absolutely for $|x - c| < R$.
  - The series diverges for $|x - c| > R$.
  - The series may or may not converge at $|x - c| = R$.

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| < 1 \implies \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x-c| < 1$$

$$L \cdot |x-c| < 1 \implies |x-c| < \frac{1}{L} = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

$$\boxed{R = \frac{1}{L} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|}$$

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n(x-c)^n|} < 1 \implies \lim_{n\to\infty} \sqrt[n]{|a_n|} \cdot |x-c| < 1$$

$$L \cdot |x-c| < 1 \implies |x-c| < \frac{1}{L} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

$$\boxed{R = \frac{1}{L} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}}$$

4.32 Interval of Convergence

The interval of convergence of a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is the set of values of x for which the series converges absolutely.

- If $R = 0$, the interval is $\{c\}$.
- If $R = \infty$, the interval is $(-\infty, +\infty)$.
- If $0 < R < \infty$, the interval is one of the following depending on the convergence at the endpoints:
  - $(c-R, c+R)$
  - $[c-R, c+R)$
  - $(c-R, c+R]$
  - $[c-R, c+R]$

4.33 Differentiation and Integration of Power Series

For a power series centered at c

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

with radius of convergence R:

- **Differentiation:**

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n (x-c)^n\right] = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

$$\boxed{f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}}$$

- **Integration:**

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} a_n (x-c)^n\,dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \dots$$

$$\boxed{\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}}$$

Note: Both operations preserve the radius of convergence R, however, convergence at the endpoints $x = c \pm R$ may change and must be checked separately.

4.34 Taylor Series

Let f be infinitely differentiable at a point a. The Taylor Series of f centered at a is the infinite series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \end{aligned}$$

If there exists an interval where $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, then f is analytic at a. Note that analyticity implies infinite differentiability, but the converse is false.

4.35 Taylor Polynomial

The nth degree Taylor polynomial of a function $f(x)$ at point a is the finite sum:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

4.36 Taylor's Theorem

Let f be a function that is $n+1$ times differentiable on an open interval I containing a point a. Then for any $x \in I$, there exists a point ξ between a and x such that:

$$\begin{aligned} f(x) &= P_n(x) + R_n(x) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

Note: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$ is the Lagrange form of the remainder.

4.37 Taylor's Inequality

Taylor's Inequality provides an upper bound on the error when approximating a function with its Taylor polynomial. Let f be a function that is $n+1$ times differentiable on an open interval containing a and x. If there exists a constant M such that $f^{(n+1)}(t) \leq M$ for all t between a and x, then:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.$$

4.38 Binomial Series

For any real number $|x| < 1$, the binomial series is given by

$$\begin{aligned} (1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \end{aligned}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}.$$

5 Multivariable Calculus

5.1 Limit of a Function of Two Variables

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{if} \quad \forall\varepsilon > 0,\ \exists\delta > 0 \text{ such that } \forall(x,y) \in D,$$

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon.$$

5.2 Test for Nonexistence of a Limit

- $\displaystyle\lim_{(x,y)\to(a,b) \text{ along } C_1} f(x,y) = L_1$
- $\displaystyle\lim_{(x,y)\to(a,b) \text{ along } C_2} f(x,y) = L_2$

$$L_1 \neq L_2 \implies \lim_{(x,y)\to(a,b)} f(x,y) \text{ does not exist.}$$

5.3 Partial Derivatives

The **partial derivative** of a function $f(x,y)$ with respect to x is defined by:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

5.4 Clairaut's Theorem

If the mixed partial derivatives exist and are continuous then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

or equivalently,

$$f_{xy} = f_{yx}.$$

5.5 Equation of a Tangent Plane

The **tangent plane** to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$ is given by:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(x-y).$$

5.6 Total Differential

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

5.7 Chain Rule For a Function of Two Variables

If $z = f(x, y)$ and $x = g(t), y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

5.8 Implicit Differentiation of a Function of Two Variables

If $F(x, y) = 0$ where y is implicitly defined as a function of x:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

5.9 Gradient Vector

The **gradient vector** of a function points in the direction of steepest increase of the function at a given point, with its magnitude representing the rate of that increase. It collects all the partial derivatives of the function into a single vector.

The gradient vector of $f(x, y)$ at a point (x_0, y_0) is the vector whose components are the partial derivatives of f at that point.

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

or,

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right).$$

5.10 Directional Derivative

The **directional derivative** of a function $f(x, y)$ at a point (x_0, y_0) in the direction of a unit vector $\vec{u} = (u_1, u_2)$, where $\|\vec{u}\| = 1$, is the rate of change of f at (x_0, y_0) along the direction $\vec{u}$ and is defined as:

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

If $f(x, y)$ is differentiable at a point (x_0, y_0), the directional derivative of f in the direction of a unit vector $\vec{u} = (u_1, u_2)$ is defined as:

$$D_u f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)u_1 + \frac{\partial f}{\partial y}(x_0, y_0)u_2,$$

or equivalently,

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}.$$

5.11 Tangent Plane to a Level Surface

The **tangent plane** to the level surfact at point (x_0, y_0, z_0) of $F(x, y, z) = c$ is:

$$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

or written out fully,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

5.12 Maximum and Minimum Values of a Function of Two Variables

1. **Critical Points:** Solve the system of equations

$$\frac{\partial f}{\partial x}(x, y) = 0, \quad \frac{\partial f}{\partial y}(x, y) = 0.$$

2. **Second Derivative Test:** Compute the second-order partial derivatives f_{xx}, f_{yy}, f_{xy} and evaluate the discriminant

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2.$$

   - If $D > 0$ and $f_{xx}(x_0, y_0) > 0$: Local minimum.
   - If $D > 0$ and $f_{xx}(x_0, y_0) < 0$: Local maximum.
   - If $D < 0$: Saddle point (neither max nor min).
   - If $D = 0$: Inconclusive.

3. **Boundary Analysis.**

5.13 Parametric Derivative

If $x = f(t)$ and $y = g(t)$, the derivative of y with respect to x is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

5.14 Area Under a Parametric Curve

Let a parametric curve be defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$. Then, the area is given by:

$$A = \int_{x=f(a)}^{x=f(b)} y\,dx = \int_{t=a}^{t=b} g(t) \cdot f'(t)\,dt.$$

5.15 Arc Length of a Parametric Curve

The **arc length** of a parametric curve defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$ is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

5.16 Surface Area of a Solid Revolution for a Parametric Curve

If the curve, defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$, is rotated about the x-axis, the surface area S is given by:

$$S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

5.17 Polar Coordinates

- **From Polar to Cartesian:**

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

- **From Cartesian to Polar:**

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

5.18 Area in Polar Coordinates

The area A of a region enclosed by a polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by:

$$A = \frac{1}{2}\int_a^b r^2\, d\theta.$$

5.19 Arc Length in Polar Coordinates

The arc length L of a polar curve defined by $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

5.20 Tangents to Polar Curves

The slope of the tangent line to a polar curve $r = f(\theta)$ at a given point (r, θ) is given by:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}.$$

5.21 Double Integral

If R is a bounded region and f is integrable over R, then:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}.$$

5.22 Iterated Integral

For a function $f(x, y)$ defined over a rectangular region $R = [a, b] \times [c, d]$, the double integral

$$\int_c^d \int_a^b f(x,y)\,dx\,dy.$$

5.23 Average Value of a Function over a Region

The average value of f the region R is defined by:

$$f_{avg} = \frac{1}{A(R)}\iint_R f(x,y)\,dA.$$

5.24 Double Integral Over a General Region

- **Type I Region** (vertical slices):
  If $R = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

- **Type II Region** (horizontal slices):
  If $R = \{(x, y) : c \le y \le d, h_1(x) \le x \le h_2(x)\}$, then:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

5.25 Double Integral in Polar Coordinates

The double integral in polar coordinates is given by:

$$\iint_R f(x,y)\,dA = \iint_{R'} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

5.26 Double Integral in Polar Coordinates Over a General Region

If $R = \{(r,\theta) : \alpha \le \theta \le \beta, r_1(\theta) \le r \le r_2(\theta)\}$, then:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$