

Calculus Notes

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1 Limits

1.1 Precise Definition of a Limit

Standard Limit:

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.2 Precise Definition of One-Sided Limit

Right-Hand Limit:

$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Left-Hand Limit:

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

1.3 Precise Definition of Infinite Limit

Infinite Limit:

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow a} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -M.$$

1.4 Precise Definition of a Limit at Infinity

Limit at Infinity:

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists M > 0 \text{ such that } x < -M \Rightarrow |f(x) - L| < \varepsilon.$$

1.5 Precise Definition of Infinite Limit at Infinity

Infinite Limit at Infinity:

$$\lim_{x \rightarrow \infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow \infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x > N \Rightarrow f(x) < -M.$$

$$\lim_{x \rightarrow -\infty} f(x) = \infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) > M.$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \quad \text{if} \quad \forall M > 0, \exists N > 0 \text{ such that } x < -N \Rightarrow f(x) < -M.$$

1.6 Limit Laws

Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

1. $\lim_{x \rightarrow a} c = c$
2. $\lim_{x \rightarrow a} x = a$
3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
4. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
5. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
6. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$
7. $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$
8. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

1.7 Relationship between the Limit and One-Sided Limits

$$\begin{aligned}\lim_{x \rightarrow a} f(x) = L &\iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L. \\ \lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x) &\implies \lim_{x \rightarrow a} f(x) \text{ does not exist.}\end{aligned}$$

1.8 Comparison Theorem

If $f(x) \leq g(x)$ when x is near a , and $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

1.9 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a , then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \implies \lim_{x \rightarrow a} g(x) = L.$$

1.10 Continuity

A function $f(x)$ is **continuous at** $x = a$ if and only if it satisfies **all** the following:

- (1) $f(a)$ exists
- (2) $\lim_{x \rightarrow a} f(x)$ exists
- (3) $\lim_{x \rightarrow a} f(x) = f(a)$

Otherwise, $f(x)$ is discontinuous at $x = a$.

1.11 Properties of Continuous Functions

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$:

- | | | |
|------------|-----------------------------------|---------|
| 1. $f + g$ | 2. $f - g$ | 3. cf |
| 4. fg | 5. $\frac{f}{g}$ if $g(a) \neq 0$ | |



Source: calcworkshop.com

1.12 Types of Discontinuity

1.13 Limits of Continuous Functions

If $f(x)$ is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(b).$$

If g is continuous at a and f is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a .

1.14 Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any N between $f(a)$ and $f(b)$,

$$\exists c \in [a, b] \text{ such that } f(c) = N$$

1.15 Asymptotes

Vertical Asymptote: $x = a$ is a vertical asymptote if

$$\lim_{x \rightarrow a^\pm} f(x) = \pm\infty.$$

Horizontal Asymptote: $y = L$ is a horizontal asymptote if

$$\lim_{x \rightarrow \pm\infty} f(x) = L.$$

For $f(x) = \frac{P(x)}{Q(x)}$, compare degrees of P and Q :

$$\deg P < \deg Q \implies y = 0.$$

$$\deg P = \deg Q \implies y = \frac{\text{leading coef. of } P}{\text{leading coef. of } Q}.$$

$$\deg P > \deg Q \implies \text{no horizontal asymptote.}$$

Oblique Asymptote: $y = mx + b$ is an oblique asymptote if

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + b)] = 0.$$

For a rational function $f(x) = \frac{P(x)}{Q(x)}$, if $\deg P = \deg Q + 1$, then $f(x)$ has an oblique asymptote. Find it by polynomial long division:

$$f(x) = D(x) + \frac{R(x)}{Q(x)}, \quad \text{as } x \rightarrow \pm\infty, \quad f(x) \approx D(x).$$

Curvilinear Asymptote: $y = g(x)$ is a curvilinear asymptote if

$$\lim_{x \rightarrow \pm\infty} [f(x) - g(x)] = 0,$$

where $g(x)$ is any non-linear function.

1.16 Common Limits

Assume $a > 0$ in the following.

$$1. \lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

$$8. \lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e$$

$$2. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$9. \lim_{x \rightarrow 0^+} x^x = 1$$

$$3. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$10. \lim_{x \rightarrow \infty} \sqrt[x]{x} = \lim_{x \rightarrow \infty} x^{\frac{1}{x}} = 1$$

$$4. \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} = a$$

$$11. \lim_{x \rightarrow 0^+} x^a \ln x = 0$$

$$5. \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$12. \lim_{x \rightarrow \infty} x^{-a} \ln x = 0$$

$$6. \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$$

$$13. \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

$$7. \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$14. \lim_{n \rightarrow \infty} \frac{n^n}{n!} = \infty$$

2 Derivatives

2.1 Derivative at a Point

The **derivative** of $f(x)$ at $x = a$ is the **instantaneous rate of change** at that point:

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

2.2 Derivative as a Function

The derivative of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

2.3 Differentiability

A function $f(x)$ is **differentiable** at $x = a$ if its derivative $f'(x)$ exists. That is:

$$f(x) \text{ is differentiable at } x = a \iff \text{The limit } \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

2.4 Differentiability Implies Continuity

If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$:

$$f \text{ differentiable at } a \implies f \text{ continuous at } a.$$

However, the converse is false:

$$f \text{ continuous at } a \not\implies f \text{ differentiable at } a.$$

2.5 Properties of Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions. Then the following rules hold:

$$(1) \quad \frac{d}{dx}(c) = 0.$$

$$(2) \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

$$(3) \quad \frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx}f(x)\right].$$

$$(4) \quad \frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

$$(5) \quad \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$(6) \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

$$(7) \quad \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

2.6 Table of Derivatives

$(\sin x)' = \cos x$	$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$(e^x)' = e^x$
$(\cos x)' = -\sin x$	$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$(a^x)' = a^x \ln a$
$(\tan x)' = \sec^2 x$	$(\arctan x)' = \frac{1}{1+x^2}$	$(\log_a x)' = \frac{1}{x \ln a}$
$(\cot x)' = -\csc^2 x$	$(x)' = -\frac{1}{1+x^2}$	$(\ln x)' = \frac{1}{x}$
$(\sec x)' = \sec x \tan x$	$(x)' = \frac{1}{x\sqrt{x^2-1}}$	$(x)' = \frac{x}{ x }$
$(\csc x)' = -\csc x \cot x$	$(x)' = -\frac{1}{x\sqrt{x^2-1}}$	$(x^x)' = x^x(1 + \ln x)$

2.7 Absolute and Local Extrema

Let f be defined on a domain D , and let $c \in D$.

- **Absolute Maximum:** $f(c)$ is an absolute maximum if $f(c) \geq f(x)$, $\forall x \in D$.
- **Absolute Minimum:** $f(c)$ is an absolute minimum if $f(c) \leq f(x)$, $\forall x \in D$.
- **Local Maximum:** $f(c)$ is a local maximum if $\exists \delta > 0$ such that $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.
- **Local Minimum:** $f(c)$ is a local minimum if $\exists \delta > 0$ such that $f(c) \leq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

2.8 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$:

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$

2.9 Critical and Stationary Points

Let f be defined on an interval I and $c \in I$.

- **Critical Point:** c is a critical point of f if either
 1. $f'(c) = 0$, or
 2. $f'(c)$ does not exist
- **Stationary Point:** c is a stationary point of f if $f'(c) = 0$.

Note: *Every stationary point is a critical point, but not conversely.*

2.10 Rolle's Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b) .
3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

2.11 Mean Value Theorem

Let f satisfy all conditions:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b) .

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: Rolle's Theorem is a special case of the Mean Value Theorem where $f(a) = f(b)$.

2.12 Increasing/Decreasing Test

Let f be differentiable on an interval I . Then, for all $x \in I$:

1. If $f'(x) > 0$, then f is strictly increasing on I .
2. If $f'(x) < 0$, then f is strictly decreasing on I .
3. If $f'(x) = 0$, then f is constant on I .

2.13 First Derivative Test

Let c be a critical point of a differentiable function $f(x)$, meaning $f'(c) = 0$ or $f'(c)$ does not exist. Then:

1. If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a **local maximum**.
2. If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a **local minimum**.
3. If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is **neither** a local maximum nor a local minimum.

2.14 Concavity and Inflection Points

Concave up \iff Curve **lies above** all of its **tangent lines**.

Concave down \iff Curve **lies below** all of its **tangent lines**.

Inflection point \iff Point where **concavity changes**.

2.15 Concavity Test

Let $f(x)$ be twice differentiable on interval I . Then:

- If $f''(x) > 0$, $\forall x \in I \implies f(x)$ is **concave up** on I .
- If $f''(x) < 0$, $\forall x \in I \implies f(x)$ is **concave down** on I .

2.16 Second Derivative Test

Let c be a critical point of f where $f'(c) = 0$. If $f''(c)$ exists, then:

1. If $f''(c) > 0$, $f(x)$ is **concave up** at c , so $f(c)$ is a **local minimum**.

Local Maximum at $c \iff f'(c) = 0$ and $f''(c) < 0$

2. If $f''(c) < 0$, $f(x)$ is **concave down** at c , so $f(c)$ is a **local maximum**.

Local Minimum at $c \iff f'(c) = 0$ and $f''(c) > 0$

3. If $f''(c) = 0$ or **does not exist**, the test is **inconclusive**—use the First Derivative Test instead.

2.17 L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an open interval containing a (except possibly at a). If

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0 \text{ or } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \pm\infty$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

2.18 Indeterminate Forms

The following symbols are “indeterminate”:

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad 0 \cdot \infty \quad \infty - \infty \quad 1^\infty \quad 0^\infty \quad \infty^0$$

Warning: The following symbols are *not* indeterminate:

$$\frac{1}{0} \quad \frac{\infty}{0} \quad \frac{1}{\infty} \quad 1 \cdot \infty \quad \infty + \infty \quad 1 + \infty \quad 0^\infty$$

3 Integrals

3.1 Antiderivatives

A function $F(x)$ is an **antiderivative** (or primitive function) of $f(x)$ on an interval I if:

$$F'(x) = f(x), \quad \forall x \in I.$$

3.2 Indefinite Integrals

The **indefinite integral** (or general antiderivative) of a function $f(x)$ is the family of all its antiderivatives:

$$\int f(x)dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

3.3 Riemann Sum

Let $f(x)$ be a function defined on a closed interval $[a, b]$, and let the interval be partitioned into n subintervals by inserting $n - 1$ points x_1, x_2, \dots, x_{n-1} such that:

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

where the subintervals $[x_{i-1}, x_i]$ have length $\Delta x_i = x_i - x_{i-1}$. For each subinterval $[x_{i-1}, x_i]$, let x_i^* be an **arbitrary sample point** within the interval. Then, a **Riemann sum**, which **approximates** the area under the curve $f(x)$ over $[a, b]$, is defined as:

$$S_n = \sum_{i=1}^n f(x_i^*)\Delta x_i, \quad x_i^* \in [x_{i-1}, x_i].$$

3.4 Partitions of an Interval



Comparison of uniform and non-uniform partitions of an interval (generated using R).

A partition of $[a, b]$ divides it into n subintervals:

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Each subinterval has width:

$$\Delta x_i = x_i - x_{i-1}.$$

- **Uniform Partition:** All subintervals have the same width:

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \quad \forall i.$$

- **Non-Uniform Partition:** Subintervals have different widths, and Δx_i varies for each i .

3.5 Types of Riemann Sums

The type of Riemann sum depends on how the sample points x_i^* are chosen within each subinterval $[x_{i-1}, x_i]$.

1. **Arbitrary-Point Rule:** $x_i^* \in [x_{i-1}, x_i] \rightarrow$ **General Riemann Sum**

$$S_n = \sum_{i=1}^n f(x_i^*) \Delta x_i, \quad \Delta x_i = x_i - x_{i-1}.$$

Uniform Partition: If all subintervals have equal width $\Delta x = \frac{b-a}{n}$, then $x_i^* \in [x_{i-1}, x_i] = [a + (i-1)\Delta x, a + i\Delta x]$ or $x_i^* = a + (i-1+c)\Delta x$, where $c \in [0, 1]$ determines its position within the subinterval:

$$S_n = \sum_{i=1}^n f(a + (i-1+c)\Delta x) \Delta x = \sum_{i=1}^n f(a + (i-1+c)\frac{(b-a)}{n}) \frac{b-a}{n}.$$

- Uses arbitrary sample points x_i^* .

2. **Left Rule:** $x_i^* = x_{i-1} \rightarrow$ **Left Riemann Sum**

$$S_{\text{left}} = \sum_{i=1}^n f(x_{i-1}) \Delta x_i.$$

Uniform Partition: $c = 0$

$$S_{\text{left}} = \sum_{i=1}^n f(a + (i-1)\Delta x) \Delta x.$$

- Underestimates for increasing functions, overestimates for decreasing functions.

3. **Right Rule:** $x_i^* = x_i \rightarrow$ **Right Riemann Sum**

$$S_{\text{right}} = \sum_{i=1}^n f(x_i) \Delta x_i$$

Uniform Partition: $c = 1$

$$S_{\text{right}} = \sum_{i=1}^n f(a + i\Delta x) \Delta x.$$

- Overestimates for increasing functions, underestimates for decreasing functions.

4. **Midpoint Rule:** $x_i^* = \frac{x_{i-1} + x_i}{2} \rightarrow$ **Midpoint Riemann Sum**

$$S_{\text{mid}} = \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x_i.$$

Uniform Partition: $c = \frac{1}{2}$

$$S_{\text{mid}} = \sum_{i=1}^n f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right) \Delta x.$$

- Tends to give better approximations than left or right sums.

5. **Upper Rule:** $x_i^* = \arg \sup_{x \in [x_{i-1}, x_i]} f(x) \rightarrow$ **Upper Riemann Sum** (or **Upper Darboux Sum**)

$$S_{\text{upper}} = \sum_{i=1}^n \sup_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **overestimates** the integral.

6. **Lower Rule:** $x_i^* = \arg \inf_{x \in [x_{i-1}, x_i]} f(x) \rightarrow$ **Lower Riemann Sum** (or **Lower Darboux Sum**)

$$S_{\text{lower}} = \sum_{i=1}^n \inf_{x \in [x_{i-1}, x_i]} f(x) \Delta x_i.$$

- Always **underestimates** the integral.

3.6 Definite Integral

The **definite integral** of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

where:

- $[a, b]$ is divided into n subintervals.
- $\Delta x_i = x_i - x_{i-1}$ is the width of the i -th subinterval.
- x_i^* is any sample point in the i -th subinterval.

3.7 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$ and c be a constant. Then:

- (1) $\int_a^b c \, dx = c(b - a).$
- (2) $\int_a^a f(x) \, dx = 0.$
- (3) $\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx.$
- (4) $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx.$
- (5) $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$
- (6) $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$
- (7) $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \quad \text{if } f(x) \leq g(x) \text{ on } [a, b].$
- (8) $\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$
- (9) $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a) \quad \text{if } m \leq f(x) \leq M \text{ on } [a, b].$

3.8 Net and Total Area

Let $f(x)$ be integrable on $[a, b]$. Define:

- A_1 = area of region where $f(x) > 0$ (area above x -axis).
- A_2 = area of region where $f(x) < 0$ (area below x -axis).

Then:

- **Net Area:**

$$\int_a^b f(x) \, dx = A_1 - A_2.$$

- **Total Area:**

$$\int_a^b |f(x)| \, dx = A_1 + A_2.$$

3.9 Mean Value Theorem for Integrals

Let f be continuous on $[a, b]$. Then:

$$\exists c \in (a, b) \text{ such that } \frac{1}{b-a} \int_a^b f(x) dx = f(c)$$

The expression on the left is the **average value** of the function $f(x)$ on the interval $[a, b]$.

3.10 Fundamental Theorem of Calculus. Part 1

Let f be a continuous on $[a, b]$. Define the function

$$F(x) = \int_a^x f(t) dt.$$

Then:

$$F'(x) = \frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x), \quad \forall x \in (a, b).$$

3.11 Fundamental Theorem of Calculus. Part 2

If f is continuous on $[a, b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b = F(x) \Big|_a^b.$$

3.12 Table of Indefinite Integrals

$$\int a dx = ax + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln |x| + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\int \csc x dx = \ln |\csc x - \cot x| + C$$

$$\int \log_a x dx = \frac{x}{\ln a} (\ln x - 1) + C$$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C$$

$$\int \frac{c}{ax + b} dx = \frac{c}{a} \ln |ax + b| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C, \quad a > 0, a \neq 1$$

$$\int \tan x dx = \ln |\sec x| = -\ln |\cos x| + C$$

$$\int \cot x dx = -\ln |\csc x| = \ln |\sin x| + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\left. \begin{aligned} \int \sec x \tan x \, dx &= \sec x + C \\ \int \csc x \cot x \, dx &= -\csc x + C \end{aligned} \right| \begin{aligned} \int \frac{1}{\sqrt{1-x^2}} \, dx &= \arcsin x + C \\ \int \frac{1}{1+x^2} \, dx &= \arctan x + C \end{aligned}$$

4 Sequences and Series

4.1 Definition of a Sequence

A sequence is a function $a : \mathbb{N} \rightarrow \mathbb{R}$ that assigns to each $n \in \mathbb{N}$ a real number a_n .

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = (a_n) = \{a_1, a_2, a_3, \dots, a_n, \dots\}$$

4.2 Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if it is either monotonically increasing or monotonically decreasing.

- **Increasing:**

$$a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly increasing).}$$

$$a_n < a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly increasing).}$$

- **Decreasing:**

$$a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (weakly decreasing).}$$

$$a_n > a_{n+1}, \quad \forall n \in \mathbb{N} \text{ (strictly decreasing).}$$

4.3 Bounded Sequence

A sequence $\{a_n\}$ is **bounded** if and only if:

$$\exists M > 0 \text{ such that } |a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

or

$$\exists M_1, M_2 \in \mathbb{R} \text{ such that } M_1 \leq a_n \leq M_2, \quad \forall n \in \mathbb{N}.$$

Equivalently, $\{a_n\}$ is **bounded** if it is both **bounded above** and **bounded below**:

- **Bounded above:** $\exists M_2 \in \mathbb{R}$ such that $a_n \leq M_2, \quad \forall n \in \mathbb{N}.$

- **Bounded below:** $\exists M_1 \in \mathbb{R}$ such that $a_n \geq M_1, \quad \forall n \in \mathbb{N}.$

A sequence is bounded \iff It is both bounded above and bounded below

4.4 Limit of a Sequence

A sequence $\{a_n\}$ has a limit $L \in \mathbb{R}$ if:

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{if} \quad \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \varepsilon, \forall n \geq N.$$

If such an L exists, the sequence is **convergent**; otherwise, it is **divergent**.

4.5 Limit of a Sequence Defined by a Function

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $\{a_n\}$ be defined by $a_n = f(n)$. Then:

$$\lim_{x \rightarrow \infty} f(x) = L \implies \lim_{n \rightarrow \infty} a_n = L.$$

4.6 Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n$ for all $n \geq N$, then:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L \implies \lim_{n \rightarrow \infty} b_n = L.$$

4.7 Limit of Absolute Value of a Sequence

$$\lim_{n \rightarrow \infty} |a_n| = 0 \implies \lim_{n \rightarrow \infty} a_n = 0.$$

4.8 Monotone Convergence Theorem

Every bounded and monotonic sequence is convergent.

- (1) Monotonic \wedge Bounded \implies Convergent.
- (2) Monotonically Increasing \wedge Bounded Above \implies Convergent.
- (3) Monotonically Decreasing \wedge Bounded Below \implies Convergent.

4.9 Series

An **infinite series** is the sum of the terms of a sequence $\{a_n\}$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The n^{th} partial sum of series:

$$S_n = \sum_{k=1}^n a_k.$$

4.10 Convergence and Divergence of Series

A infinite series $\sum a_n$ **converges** if the sequence of partial sums $\{S_n\}$ has a finite limit:

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \quad \text{or} \quad \lim_{n \rightarrow \infty} S_n = S.$$

Otherwise, the series **diverges**.

4.11 Types of Series

- | | | |
|--------------------------------|----------------------|-------------------|
| • Arithmetic Series | • Telescoping Series | • Taylor Series |
| • Geometric Series | • Alternating Series | • Laurent Series |
| • Arithmetico-Geometric Series | • Power Series | • Fourier Series |
| • Harmonic Series | • Taylor Series | • Binomial Series |
| • p-Series | • Maclaurin Series | • Mercator Series |

4.12 Properties of Series

Let $\sum a_n$ and $\sum b_n$ be series, and let c, d be a constant. Then:

$$(1) \quad \sum_{k=a}^b c = c(b - a + 1).$$

$$(2) \quad \sum_{n=k}^{\infty} \pm c = \pm \infty, \quad c > 0.$$

$$(3) \quad \sum_{n=k}^m (ca_n \pm db_n) = c \sum_{n=k}^m a_n \pm d \sum_{n=k}^m b_n.$$

$$(4) \quad \sum_{n=k}^m a_n = \sum_{n=k}^p a_n + \sum_{n=p+1}^m a_n.$$

$$(5) \quad \sum_{n=k}^m a_n = \sum_{n=0}^{m-k} a_{m-n}.$$

$$(6) \quad \sum_{n=k}^m a_n = \sum_{j=k-h}^{m-h} a_{j+h}.$$

$$(7) \quad \sum_{k=1}^n k = \sum_{k=0}^{n-1} (k+1).$$

4.13 Geometric Series

$$\text{Geometric} \implies \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$$

- **Partial Sum:**

$$S_n = \sum_{k=0}^{n-1} ar^k = a \frac{1-r^n}{1-r}, \quad r \neq 1.$$

- **Convergence:**

$$|r| < 1 \implies \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

- **Divergence:**

$$|r| \geq 1 \implies \text{Series diverges.}$$

4.14 Telescoping Series

$$\text{Telescoping} \implies \sum_{n=1}^{\infty} (a_n - a_{n+1}) = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots$$

- **Partial Sum:**

$$S_n = \sum_{k=1}^n (a_k - a_{k+1}) = (a_1 - a_2) + (a_2 - a_3) + \dots + (a_n - a_{n+1}) = a_1 - a_{n+1}.$$

- **Convergence:**

$$\lim_{n \rightarrow \infty} a_{n+1} = L \implies \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - L.$$

- **Divergence:**

$$\lim_{n \rightarrow \infty} a_{n+1} \text{ does not exist} \implies \text{Series diverges.}$$

4.15 Harmonic Series

$$\text{Harmonic} \implies \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

- **Partial Sum and Approximation:**

$$H_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \dots + \frac{1}{n} \approx \ln(n) + \gamma$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant.

- **Divergence:**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ even though } \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

4.16 Limit of Terms in a Convergent Series

Let $\{a_n\}$ be a sequence and $\sum a_n$ be its series. Then:

$$\sum a_n \text{ convergent} \implies \lim_{n \rightarrow \infty} a_n = 0.$$

Note: The converse is false. ($\lim_{n \rightarrow \infty} a_n = 0 \not\Rightarrow \sum a_n \text{ convergent}$)

4.17 List of Convergence Tests

- **Tests for Positive Series:**

- Direct Comparison Test
- Integral Test
- Limit Comparison Test
- p-Series Test

- **Tests for Alternating Series:**

- Alternating Series Test
- Dirichlet's Test

- **General Tests:**

- Divergence Test
- Root Test
- Ratio Test
- Absolute Convergence Test

- **Advanced or Specialized Tests:**

- Cauchy Condensation Test
- Kummer's Test
- Abel's Test
- Gauss's Test
- Raabe's Test
- Bertrand's Test

4.18 Divergence Test (*n*th-Term Test)

$$\lim_{n \rightarrow \infty} a_n \neq 0 \vee \lim_{n \rightarrow \infty} a_n \text{ does not exist} \implies \sum a_n \text{ diverges.}$$

4.19 Direct Comparison Test

If $0 \leq a_n \leq b_n$ for all $n \geq N$:

- $\sum b_n \text{ converges} \implies \sum a_n \text{ converges.}$
- $\sum a_n \text{ diverges} \implies \sum b_n \text{ diverges.}$

4.20 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with $a_n > 0$, $b_n > 0$ for all $n \geq N$. Define the limit:

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}.$$

- $0 < L < \infty \implies \sum a_n \text{ and } \sum b_n \text{ both converge or both diverge.}$
- $L = 0 \wedge \sum b_n \text{ converges} \implies \sum a_n \text{ converges.}$
- $L = \infty \wedge \sum b_n \text{ diverges} \implies \sum a_n \text{ diverges.}$

Note: The order of division does not matter.

4.21 Integral Test (Maclaurin-Cauchy Test)

For a series $\sum a_n$ where $a_n = f(n)$, if $f(x)$ is a positive, continuous, and decreasing function for all $x \geq N$, then:

- $\sum_{n=N}^{\infty} a_n \text{ converges} \iff \int_N^{\infty} f(x) dx \text{ converges.}$
- $\sum_{n=N}^{\infty} a_n \text{ diverges} \iff \int_N^{\infty} f(x) dx \text{ diverges.}$

4.22 *p*-Series Test

For the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$:

- $p > 1 \iff \text{Series converges.}$
- $p \leq 1 \iff \text{Series diverges.}$

4.23 Integral Remainder Estimate

For a convergent series $\sum_{n=1}^{\infty} a_n$ with $a_n = f(n)$, where $f(x)$ is positive, continuous, and decreasing for all $x \geq N$, if the remainder $R_n = S - S_n$ then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$
$$S_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq S_n + \int_n^{\infty} f(x) dx$$

4.24 Alternating Series Test (Leibniz Criterion)

For an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where $a_n > 0$, the series converges if:

- $\lim_{n \rightarrow \infty} a_n = 0$. (terms approach zero)
- $a_{n+1} \leq a_n$ for all $n \geq N$. (monotonically decreasing)

4.25 Alternating Series Estimation Theorem

For an alternating series approximated by its n th partial sum, the absolute error (or remainder) is less than or equal to the absolute value of the next term in the series.

$$|R_n| = |S - S_n| \leq a_{n+1} = |S_{n+1} - S_n|.$$

4.26 Ratio Test (d'Alembert's Criterion)

Let $\sum a_n$ be an infinite series with $a_n \neq 0$. Define the limit:

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- $L < 1 \implies \sum a_n$ converges absolutely.
- $L > 1 \vee L = \infty \implies \sum a_n$ diverges.
- $L = 1 \implies$ Test is inconclusive.

4.27 Root Test (Cauchy's Criterion)

Let $\sum a_n$ be an infinite series. Define the limit:

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |a_n|^{1/n}.$$

- $L < 1 \implies \sum a_n$ converges absolutely.
- $L > 1 \vee L = \infty \implies \sum a_n$ diverges.
- $L = 1 \implies$ Test is inconclusive.

4.28 Absolute Convergence Test

- **Absolute Convergence:**

$$\sum |a_n| \text{ converges} \implies \sum a_n \text{ converges absolutely.}$$

- **Conditional Convergence:**

$$\left(\sum a_n \text{ converges} \right) \wedge \left(\sum |a_n| \text{ diverges} \right) \implies \sum a_n \text{ converges conditionally.}$$

Note: All rearrangements of absolutely convergent series converge to the same sum.

4.29 Riemann Series Theorem

Conditionally convergent series $\sum_{n=1}^{\infty} a_n$ can be rearranged to:

- Converge to any real number: $\forall M \in \mathbb{R}, \exists \sigma$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)} = M$.
- Diverge to $\pm\infty$: $\exists \sigma$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)} = \pm\infty$.
- Fail to approach any limit: $\exists \sigma$ such that $\lim_{N \rightarrow \infty} \sum_{n=1}^N a_{\sigma(n)}$ does not exist.

4.30 Power Series

A power series centered at c is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \dots$$

where $\{a_n\}$ is a sequence of coefficients, x is a variable, and c is the center of the series.

4.31 Radius of Convergence

For a power series $\sum_{n=0}^{\infty} a_n(x - c)^n$, the radius of convergence R is a non-negative real number (possibly 0 or ∞) such that:

- If $R = 0$: The series converges only at $x = c$.
- If $R = \infty$: The series converges for all values of x .
- If $0 < R < \infty$:
 - The series converges absolutely for $|x - c| < R$.
 - The series diverges for $|x - c| > R$.
 - The series may or may not converge at $|x - c| = R$.

Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x - c)^{n+1}}{a_n(x - c)^n} \right| < 1 \implies \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x - c| < 1$$

$$L \cdot |x - c| < 1 \implies |x - c| < \frac{1}{L} = \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

$$R = \frac{1}{L} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n(x - c)^n|} < 1 \implies \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \cdot |x - c| < 1$$

$$L \cdot |x - c| < 1 \implies |x - c| < \frac{1}{L} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

$$R = \frac{1}{L} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}}$$

4.32 Interval of Convergence

The interval of convergence of a power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ is the set of values of x for which the series converges absolutely.

- If $R = 0$, the interval is $\{c\}$.
- If $R = \infty$, the interval is $(-\infty, +\infty)$.
- If $0 < R < \infty$, the interval is one of the following depending on the convergence at the endpoints:
 - $(c - R, c + R)$
 - $[c - R, c + R)$
 - $(c - R, c + R]$
 - $[c - R, c + R]$

4.33 Differentiation and Integration of Power Series

For a power series centered at c

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

with radius of convergence R :

• **Differentiation:**

$$f'(x) = \frac{d}{dx} \left[\sum_{n=0}^{\infty} a_n(x-c)^n \right] = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x-c)^{n-1}$$

• **Integration:**

$$\int f(x) dx = \int \sum_{n=0}^{\infty} a_n(x-c)^n dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \dots$$

$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$$

Note: Both operations preserve the radius of convergence R , however, convergence at the endpoints $x = c \pm R$ may change and must be checked separately.

4.34 Taylor Series

Let f be infinitely differentiable at a point a . The Taylor Series of f centered at a is the infinite series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \end{aligned}$$

If there exists an interval where $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$, then f is analytic at a . Note that analyticity implies infinite differentiability, but the converse is false.

4.35 Taylor Polynomial

The n th degree Taylor polynomial of a function $f(x)$ at point a is the finite sum:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n. \end{aligned}$$

4.36 Taylor's Theorem

Let f be a function that is $n+1$ times differentiable on an open interval I containing a point a . Then for any $x \in I$, there exists a point ξ between a and x such that:

$$\begin{aligned} f(x) &= P_n(x) + R_n(x) \\ &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}. \end{aligned}$$

Note: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$ is the Lagrange form of the remainder.

4.37 Taylor's Inequality

Taylor's Inequality provides an upper bound on the error when approximating a function with its Taylor polynomial. Let f be a function that is $n+1$ times differentiable on an open interval containing a and x . If there exists a constant M such that $f^{(n+1)}(t) \leq M$ for all t between a and x , then:

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.$$

4.38 Binomial Series

For any real number $|x| < 1$, the binomial series is given by

$$\begin{aligned} (1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots \end{aligned}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

5 Multivariable Calculus

5.1 Limit of a Function of Two Variables

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L \quad \text{if} \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall (x,y) \in D,$$
$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon.$$

5.2 Test for Nonexistence of a Limit

$$\bullet \lim_{(x,y) \rightarrow (a,b) \text{ along } C_1} f(x,y) = L_1 \qquad \bullet \lim_{(x,y) \rightarrow (a,b) \text{ along } C_2} f(x,y) = L_2$$

$$L_1 \neq L_2 \implies \lim_{(x,y) \rightarrow (a,b)} f(x,y) \text{ does not exist.}$$

5.3 Partial Derivatives

The **partial derivative** of a function $f(x,y)$ with respect to x is defined by:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

5.4 Clairaut's Theorem

If the mixed partial derivatives exist and are continuous then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

or equivalently,

$$f_{xy} = f_{yx}.$$

5.5 Equation of a Tangent Plane

The **tangent plane** to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is given by:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

5.6 Total Differential

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

5.7 Chain Rule For a Function of Two Variables

If $z = f(x, y)$ and $x = g(t), y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

5.8 Implicit Differentiation of a Function of Two Variables

If $F(x, y) = 0$ where y is implicitly defined as a function of x :

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

5.9 Gradient Vector

The **gradient vector** of a function points in the direction of steepest increase of the function at a given point, with its magnitude representing the rate of that increase. It collects all the partial derivatives of the function into a single vector.

The gradient vector of $f(x, y)$ at a point (x_0, y_0) is the vector whose components are the partial derivatives of f at that point.

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

or,

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right).$$

5.10 Directional Derivative

The **directional derivative** of a function $f(x, y)$ at a point (x_0, y_0) in the direction of a unit vector $\vec{u} = (u_1, u_2)$, where $\|\vec{u}\| = 1$, is the rate of change of f at (x_0, y_0) along the direction \vec{u} and is defined as:

$$D_{\vec{u}}f(x, y) = \lim_{h \rightarrow 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

If $f(x, y)$ is differentiable at a point (x_0, y_0) , the directional derivative of f in the direction of a unit vector $\vec{u} = (u_1, u_2)$ is defined as:

$$D_{\vec{u}}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)u_1 + \frac{\partial f}{\partial y}(x_0, y_0)u_2,$$

or equivalently,

$$D_{\vec{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}.$$

5.11 Tangent Plane to a Level Surface

The **tangent plane** to the level surface at point (x_0, y_0, z_0) of $F(x, y, z) = c$ is:

$$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

or written out fully,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

5.12 Maximum and Minimum Values of a Function of Two Variables

1. **Critical Points:** Solve the system of equations

$$\frac{\partial f}{\partial x}(x, y) = 0, \quad \frac{\partial f}{\partial y}(x, y) = 0.$$

2. **Second Derivative Test:** Compute the second-order partial derivatives f_{xx} , f_{yy} , f_{xy} and evaluate the discriminant

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2.$$

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$: Local minimum.
- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$: Local maximum.
- If $D < 0$: Saddle point (neither max nor min).
- If $D = 0$: Inconclusive.

3. **Boundary Analysis.**

5.13 Parametric Derivative

If $x = f(t)$ and $y = g(t)$, the derivative of y with respect to x is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

5.14 Area Under a Parametric Curve

Let a parametric curve be defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$. Then, the area is given by:

$$A = \int_{x=f(a)}^{x=f(b)} y \, dx = \int_{t=a}^{t=b} g(t) \cdot f'(t) \, dt.$$

5.15 Arc Length of a Parametric Curve

The **arc length** of a parametric curve defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$ is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

5.16 Surface Area of a Solid Revolution for a Parametric Curve

If the curve, defined by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$, is rotated about the x -axis, the surface area S is given by:

$$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

5.17 Polar Coordinates

- **From Polar to Cartesian:**

$$x = r \cos(\theta), \quad y = r \sin(\theta).$$

- **From Cartesian to Polar:**

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

5.18 Area in Polar Coordinates

The area A of a region enclosed by a polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by:

$$A = \frac{1}{2} \int_a^b r^2 d\theta.$$

5.19 Arc Length in Polar Coordinates

The arc length L of a polar curve defined by $r = f(\theta)$ from $\theta = a$ to $\theta = b$ is given by:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

5.20 Tangents to Polar Curves

The slope of the tangent line to a polar curve $r = f(\theta)$ at a given point (r, θ) is given by:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta)}{\frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)}.$$

5.21 Double Integral

If R is a bounded region and f is integrable over R , then:

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}.$$

5.22 Iterated Integral

For a function $f(x, y)$ defined over a rectangular region $R = [a, b] \times [c, d]$, the double integral

$$\int_c^d \int_a^b f(x, y) dx dy.$$

5.23 Average Value of a Function over a Region

The average value of f the region R is defined by:

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA.$$

5.24 Double Integral Over a General Region

- **Type I Region** (vertical slices):

If $R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then:

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

- **Type II Region** (horizontal slices):

If $R = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then:

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

5.25 Double Integral in Polar Coordinates

The double integral in polar coordinates is given by:

$$\iint_R f(x, y) dA = \iint_{R'} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

5.26 Double Integral in Polar Coordinates Over a General Region

If $R = \{(r, \theta) : \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\}$, then:

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$