Inequalities Notes

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Last Updated: May 19, 2025

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2.1 Majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences of real numbers. Then x is said to *majorize* y, denoted $x \succ y$, if the following conditions are satisfied:

1. $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n$;
2. $\displaystyle\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$;
3. $\displaystyle\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n-1$.

Example: $(3, 1, 0) \succ (2, 1, 1)$, $\quad (12, 0, 0) \succ (4, 4, 4)$.

2.2 Convex Function

A function $f : [a, b] \to \mathbb{R}$ is called *convex* (concave up) on $[a, b]$ if and only if for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$, the following inequality holds:

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if $f''(x) \geq 0$ ($f''(x) \leq 0$) holds for all $x \in [a, b]$.

Note that f is convex if and only if $-f$ is concave.

Example (convex): x^2, e^x. Example (concave): $\ln x, \sqrt{x}$.

3 Inequalities

3.1 AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the following inequality holds:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.
More precisely,

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}.$$

Example: $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

3.2 Weighted AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$ and $w_1, w_2, \ldots, w_n$ be positive integers. Then, by AM-GM we have:

$$\frac{\underbrace{a_1 + a_1 + \cdots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \cdots + a_2}_{w_2} + \cdots + \underbrace{a_n + a_n + \cdots + a_n}_{w_n}}{w_1 + w_2 + \cdots + w_n}$$
$$\geq \left(\underbrace{a_1 a_1 \ldots a_1}_{w_1} \underbrace{a_2 a_2 \ldots a_2}_{w_2} \ldots \underbrace{a_n a_n \ldots a_n}_{w_n} \right)^{\frac{1}{w_1 + w_2 + \cdots + w_n}}.$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n} \geq (a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \ldots w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} \geq \left(\prod_{i=1}^{n} a_i^{w_i} \right)^{\frac{1}{\sum_{i=1}^{n} w_i}}$$

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}.$$

Example: $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$.

3.3 Power Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the r-th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(\dfrac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} & r \neq 0, \\ \sqrt[n]{a_1 a_2 \ldots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1:$ $$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \frac{n}{\sum_{i=1}^n \frac{1}{a_i}}$$ (Harmonic Mean)
- $r = 0:$ $$\sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}}$$ (Geomteric Mean)
- $r = 1:$ $$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^n a_i$$ (Arithmetic Mean)
- $r = 2:$ $$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}}$$ (Quadratic Mean)

If $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$.

3.4 Weighted Power Mean Inequality

Let $a_1, a_2, \ldots a_n > 0$ and $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$. Then, the r-th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r)^{1/r} & r \neq 0, \\ a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $(\frac{1}{6}a^3 + \frac{1}{3}b^3 + \frac{1}{2}c^3)^{1/3} \geq a^{1/6}b^{1/3}c^{1/2}$.

3.5 HM-GM-AM-QM Inequalities

Let $a_1, a_2, \ldots, a_n > 0$. Then:

$$0 < \text{HM} \leq \text{GM} \leq \text{AM} \leq \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^n \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^n a_i} \leq \frac{1}{n}\sum_{i=1}^n a_i \leq \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}}.$$

Example: $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$.

3.6 Bernoulli's Inequality

For all $x \geq -1$ and $r \geq 1$:

$$(1+x)^r \geq 1 + rx.$$

Example: $(1+x)^5 \geq 1 + 5x$.

3.7 Jensen's Inequality

If f is convex, then:

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

with equality if and only if f is *linear* or $a_1 = a_2 = \cdots = a_n$.

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^n w_i f(a_i) \geq f\left(\sum_{i=1}^n w_i a_i\right).$$

The inequality is reversed if f is concave.

Example: $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x}+\sqrt{y}}{2}$.

3.8 Young's Inequality

3.9 Cauchy–Schwarz Inequality

3.10 Muirhead's Inequality

3.11 Popoviciu's Inequality

3.12 Karamata's Inequality

3.13 Hölder's Inequality

3.14 Minkowski Inequality

3.15 Generalized Minkowski Inequality

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3.23 Huygens Inequality

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3.25 Nesbitt's Inequality

3.26 Cesàro's Inequality

3.27 Mildorf's Inequality

4 Selected Inequalities

5 Proofs

5.1 Proof of AM-GM Inequality using Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $n + 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$
\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \dots a_n} \implies \\
\frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} &= \frac{n\frac{a_1+\cdots+a_n}{n} + a_{n+1}}{n+1} \\
&\geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&\geq (\sqrt[n]{a_1 \dots a_n})^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&= \sqrt[n+1]{a_1 \dots a_n a_{n+1}}
\end{aligned}
$$

□

5.2 Proof of AM-GM Inequality using Forward-Backward Induction (a.k.a. Cauchy Induction)

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $2n$ is true.

iii. n is true $\implies$ $n - 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$
\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \dots a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} &= \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}\right) \\
&\geq \frac{\sqrt[n]{a_1 a_2 \dots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}}{2} \\
&\geq \sqrt[2]{\sqrt[n]{a_1 a_2 \dots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}} \\
&= \sqrt[2n]{a_1 a_2 \dots a_{2n}}
\end{aligned}
$$

Step 3:

$$
\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + \cdots + a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\frac{(n-1)(a_1 + a_2 + \cdots + a_{n-1}) + (a_1 + \cdots + a_{n-1})}{n \cdot (n-1)} &= \frac{(n-1+1)(a_1 + a_2 + \cdots + a_{n-1})}{n \cdot (n-1)} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^n &\geq a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^{n-1} &\geq a_1 a_2 \dots a_{n-1} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1 a_2 \dots a_{n-1}}
\end{aligned}
$$

$\square$

5.3 Proof of AM-GM Inequality using Jensen's Method

Let $a_1, a_2, \dots, a_n > 0$ and $f(x) = \ln x$ be a *concave* function on $(0, \infty)$. By Jensen's Inequality we have:

$$
\begin{aligned}
f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) &\geq \frac{1}{n}\sum_{i=1}^{n} f(a_i) \\
\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \ln(a_2) + \cdots + \ln(a_n)}{n} \\
&= \frac{\ln(a_1 a_2 \dots a_n)}{n} \\
&= \ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right) \\
e^{\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)} &\geq e^{\ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right)} \\
\frac{a_1 + a_2 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_n}
\end{aligned}
$$

$\square$

6 Selected Problems

1. –