# Inequalities Notes

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## 1 Algebraic Inequalities

**Theorem 1 (AM-GM).** *Let $a_1, \ldots, a_n$ be non-negative real numbers. Then:*

$$\frac{a_1 + \cdots + a_n}{2} \geq \sqrt[n]{a_1 \ldots a_n}$$

*with equality if and only if $a_1 = a_2 = \cdots = a_n$.*

**Theorem 2 (Cauchy-Schwarz).** *Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be real numbers. Then:*

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \ldots a_n b_n)^2$$

**Theorem 3 (Titu's Lemma).** *Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be positive real numbers. Then:*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

**Theorem 4 (Young's Inequality).** *Let $a, b$ be nonnegative real numbers and if $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

*with equality if and only if $a^p = b^q$.*

**Theorem 5 (Hölder's Inequality).** *Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be positive real numbers. Suppose that $P > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \geq \sum_{i=1}^n a_i b_i$$

*More generally, let $x_{ij} (i = 1, \ldots, m, j = 1, \ldots, n)$ be positive real numbers. Suppose that $w_1, w_2, \ldots, w_n$ are positive real numbers satisfying $w_1 + w_2 + \cdots + w_n = 1$. Then:*

$$\prod_{j=1}^n \left(\sum_{i=1}^m x_{ij}\right)^{w_j} \geq \sum_{i=1}^m \left(\prod_{j=1}^n x_{ij}^{w_j}\right)$$

**Theorem 6 (Minkowski Inequality).** *Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be positive real numbers. Suppose that $p > 1$. Then:*

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{\frac{1}{p}}$$

**Theorem 7 (Generalized Minkowski Inequality).** *Let $a_{ij} \geq 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$ and let $p > 1$. Then:*

$$\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)^{p}\right]^{\frac{1}{p}} \leq \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p}\right)^{\frac{1}{p}}$$

**Theorem 8 (Chebyshev's Sum Inequality).** *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Then:*

$$\frac{a_1 b_1 + \cdots + a_n b_n}{n} \geq \frac{(a_1 + \cdots + a_n)}{n} \frac{(b_1 + \cdots + b_n)}{n}$$
$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right)$$

**Theorem 9 (Rearrangement Inequality).** *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. For any permutation $\sigma$ of $\{1, \ldots, n\}$, we have:*

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\sigma(i)} \geq \sum_{i=1}^{n} a_i b_{n+1-i}$$

**Definition 1 (Convex Function).** *Suppose that $f$ is a one-variable function defined on $[a, b] \subset \mathbb{R}$. $f$ is called a convex function on $[a, b]$ if and only if for all $x, y \in [a, b]$ and for all $0 \leq t \leq 1$, we have:*

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y)$$

**Theorem 10 (Jensen's Inequality).** *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then for any $x_1, \ldots, x_n \in [a, b]$ and non-negative real numbers $w_1, \ldots, w_n$ with $w_1 + \cdots + w_n = 1$, we have:*

$$\sum_{i=1}^{n} w_i f(x_i) \geq f(\sum_{i=1}^{n} w_i x_i)$$

**Theorem 11 (Popoviciu's Inequality).** *Let $f : I \to \mathbb{R}$. If $f$ is convex, then for any three points $x, y, z$ in $I$:*

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

**Definition 2 (Majorization).** *Given two sequences $(a) = (a_1, a_2, \ldots, a_n)$ and $(b) = (b_1, b_2, \ldots, b_n)$ (where $a_i, b_i \in \mathbb{R} \quad \forall i \in \{1, 2, \ldots, n\}$). We say that the sequence $(a)$ majorizes the sequence $(b)$, and write $(a) \succ (b)$, if the following conditions are fulfilled:*

$$\begin{aligned}
&a_1 \geq a_2 \geq \cdots \geq a_n; \\
&b_1 \geq b_2 \geq \cdots \geq b_n; \\
&a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n; \\
&a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_k \quad \forall k \in \{1, 2, \ldots, n-1\}
\end{aligned}$$

**Theorem 12 (Karamata's Inequality).** *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Suppose that $(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n)$ where $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in [a, b]$. Then:*

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i)$$

**Theorem 13 (Weighted AM-GM Inequality).** *Let $w_1, \ldots, w_n \geq 0$ such that $w_1 + \ldots w_n = 1$. For all $x_1, \ldots, x_n \geq 0$, we have:*

$$\sum_{i=1}^{n} w_i x_i \geq \prod_{i=1}^{n} x_i^{w_i}$$

**Theorem 14 (Schur's Inequality).** *Let $x, y, z$ be non-negative real numbers. For any $r > 0$, we have:*

$$\sum_{cyc} x^r (x-y)(x-z) \geq 0$$

**Theorem 15 (Generalized Schur's Inequality).** *Let $a, b, c, x, y, z$ be six non-negative real numbers such that the sequences $(a, b, c)$ and $(x, y, z)$ are similarly sorted. Then:*

$$x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \geq 0$$

**Theorem 16 (Newton's Inequality).** *Let $x_1, \ldots, x_n$ be non-negative real numbers. Define the symmetric polynomials $s_0, s_1, \ldots, s_n$ by $(x+x_1)(x+x_2)\ldots(x+x_n) = s_n x^n + \cdots + s_1 + s_0$, and define the symmetric averages by $d_i = \frac{s_i}{\binom{n}{i}}$. Then:*

$$d_i^2 \geq d_{i+1} d_{i-1}$$

**Theorem 17 (Maclaurin's Inequality).** *Let $x_1, \ldots, x_n$ be non-negative real numbers. Define the symmetric polynomials $s_0, s_1, \ldots, s_n$ by $(x+x_1)(x+x_2)\ldots(x+x_n) = s_n x^n + \cdots + s_1 + s_0$, and define the symmetric averages by $d_i = \frac{s_i}{\binom{n}{i}}$. Then:*

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \cdots \geq \sqrt[n]{d_n}$$

**Theorem 18 (Muirhead's Inequality).** *Suppose that $(a_1, \ldots, a_n) \succ (b_1, \ldots, b_n)$, and $x_1, \ldots, x_n$ are positive real numbers. Then:*

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$$

*where the symmetric sum is taken over all $n!$ permutations of $(x_1, x_2, \ldots, x_n)$.*

**Theorem 19 (Power Mean Inequality).** *Let $x_1, \ldots, x_n > 0$. The power mean of order $r$ is defined by:*

$$\begin{cases} M_{(x_1,\ldots,x_n)}(0) = \sqrt[n]{x_1 \ldots x_n} \\ M_{(x_1,\ldots,x_n)}(r) = \left(\dfrac{x_1^r + \cdots + x_n^r}{n}\right)^{\frac{1}{r}}, \quad r \neq 0 \end{cases}$$

*Then, $M_{(x_1,\ldots,x_n)} : \mathbb{R} \to \mathbb{R}$ is continuous and monotone increasing.* **Theorem (Bernoulli's Inequality).** *For every real number $r \geq 1$ and real number $x \geq -1$, we have:*

$$(1+x)^r \geq 1 + rx$$

*while for $0 \leq r \leq 1$ and real number $x \geq -1$, we have:*

$$(1+x)^r \leq 1 + rx$$

**Theorem 20 (Aczel's Inequality).** *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be non-negative real numbers satisfying $a_1^2 \geq a_2^2 + \cdots + a_n^2$ and $b_1^2 \geq b_2^2 + \cdots + b_n^2$. Then:*

$$a_1 b_1 - (a_2 b_2 + \cdots + a_n b_n) \geq \sqrt{(a_1^2 - (a_2^2 + \cdots + a_n^2))(b_1^2 - (b_2^2 + \cdots + b_n^2))}$$

**Theorem 21 (Huygens Inequality).** *Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, w_1, w_2, \ldots, w_n$ be positive real numbers such that $w_1 + w_2 + \cdots + w_n = 1$. Then:*

$$\prod_{i=1}^{n}(a_i + b_i)^{w_i} \geq \prod_{i=1}^{n} a_i^{w_i} + \prod_{i=1}^{n} b_i^{w_i}$$

**Theorem 22 (Heinz Inequality).** *For $a, b > 0$ and $\alpha \in [0,1], wehave:$*

$$\sqrt{ab} \leq \frac{a^{\alpha}b^{1-\alpha} + a^{1-\alpha}b^{\alpha}}{2} \leq \frac{a+b}{2}$$