Inequalities Notes

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2.1 Majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences of real numbers. Then x is said to *majorize* y, denoted $x \succ y$, if the following conditions are satisfied:

1. $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n$;
2. $\displaystyle\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$;
3. $\displaystyle\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n-1$.

Example: $(3, 1, 0) \succ (2, 1, 1)$, $\quad (12, 0, 0) \succ (4, 4, 4)$.

2.2 Convex Function

A function $f : [a, b] \to \mathbb{R}$ is called *convex* (concave up) on $[a, b]$ if and only if for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$, the following inequality holds:

$$\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if $f''(x) \geq 0$ ($f''(x) \leq 0$) holds for all $x \in [a, b]$.

Note that f is convex if and only if $-f$ is concave.

Example (convex): x^2, e^x. Example (concave): $\ln x, \sqrt{x}$.

2.3 Elementary Symmetric Polynomials

Let t be a variable and $x_1, x_2, \dots, x_n$ be real numbers. Define:

$$\begin{aligned} P(x) = \prod_{i=1}^{n}(t+x_i) &= (t+x_1)(t+x_2)\dots(t+x_n) \\ &= t^n + (x_1+\dots+x_n)t^{n-1} + (x_1x_2 + x_1x_3 + \dots)t^{n-2} + \dots \\ &+ (x_2x_3\dots x_n + x_1x_3\dots x_n + \dots)t + x_1x_2x_3\dots x_n \\ &= 1\cdot t^n + \left(\sum_{1\le i\le n} x_i\right)t^{n-1} + \left(\sum_{1\le i<j\le n} x_ix_j\right)t^{n-2} + \dots \\ &+ \left(\sum_{1\le i_1<\dots<i_{n-1}\le n} x_{i_1}x_{i_2}\dots x_{i_{n-1}}\right)t + \prod_{i=1}^{n} x_i. \end{aligned}$$

In other words,

$$P(x) = \prod_{i=1}^{n}(t+x_i) = c_0t^n + c_1t^{n-1} + c_2t^{n-2} + \dots + c_{n-1}t + c_n,$$

where the coefficient c_k is the k-th elementary symmetric sum:

$$c_0 = 1, \qquad c_1 = \sum_{1\le i\le n} x_i, \qquad c_2 = \sum_{1\le i<j\le n} x_ix_j,$$

$$c_3 = \sum_{1\le i<j<k\le n} x_ix_jx_k, \qquad \dots, \qquad c_n = \prod_{i=1}^{n} x_i.$$

In general, for $0 \le k \le n$

$$c_k = \sum_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies (x+1)(x+2)(x+3) = x^3 + (1+2+3)x^2 + (1\cdot 2 + 2\cdot 3 + 3\cdot 1)x + 1\cdot 2\cdot 3 = x^3 + 6x^2 + 11x + 6$.

2.4 Elementary Symmetric Mean

Let $x_1, x_2, \dots, x_n$ be real numbers. The k-th elementary symmetric mean is defined as:

$$d_k = \frac{c_k}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies d_2 = \frac{c_2}{\binom{3}{2}} = \frac{11}{3}$.

3 Inequalities

3.1 AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the following inequality holds:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.
More precisely,

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}.$$

Example: $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

3.2 Weighted AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$ and $w_1, w_2, \ldots, w_n$ be positive integers. Then, by AM-GM we have:

$$\frac{\underbrace{a_1 + a_1 + \cdots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \cdots + a_2}_{w_2} + \cdots + \underbrace{a_n + a_n + \cdots + a_n}_{w_n}}{w_1 + w_2 + \cdots + w_n}$$
$$\geq \left(\underbrace{a_1 a_1 \ldots a_1}_{w_1} \underbrace{a_2 a_2 \ldots a_2}_{w_2} \ldots \underbrace{a_n a_n \ldots a_n}_{w_n} \right)^{\frac{1}{w_1 + w_2 + \cdots + w_n}}.$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n} \geq (a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \ldots w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} \geq \left(\prod_{i=1}^{n} a_i^{w_i} \right)^{\frac{1}{\sum_{i=1}^{n} w_i}}$$

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}.$$

Example: $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$.

3.3 Power Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the r-th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(\dfrac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} & r \neq 0, \\ \sqrt[n]{a_1 a_2 \ldots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1:$ $$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \frac{n}{\sum_{i=1}^n \frac{1}{a_i}} \qquad \text{(Harmonic Mean)}$$
- $r = 0:$ $$\sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}} \qquad \text{(Geomteric Mean)}$$
- $r = 1:$ $$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^n a_i \qquad \text{(Arithmetic Mean)}$$
- $r = 2:$ $$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}} \qquad \text{(Quadratic Mean)}$$

If $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$.

3.4 Weighted Power Mean Inequality

Let $a_1, a_2, \ldots a_n > 0$ and $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$. Then, the r-th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r)^{1/r} & r \neq 0, \\ a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $(\frac{1}{6}a^3 + \frac{1}{3}b^3 + \frac{1}{2}c^3)^{1/3} \geq a^{1/6}b^{1/3}c^{1/2}$.

3.5 HM-GM-AM-QM Inequalities

Let $a_1, a_2, \ldots, a_n > 0$. Then:

$$0 < \text{HM} \le \text{GM} \le \text{AM} \le \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \le \sqrt[n]{a_1 a_2 \ldots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n} \le \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \le \sqrt[n]{\prod_{i=1}^{n} a_i} \le \frac{1}{n} \sum_{i=1}^{n} a_i \le \sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}.$$

Example: $\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$.

3.6 Bernoulli's Inequality

For all $x \ge -1$ and $r \ge 1$:

$$(1+x)^r \ge 1 + rx.$$

Example: $(1+x)^5 \ge 1 + 5x$.

3.7 Jensen's Inequality

If f is *convex*, then:

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \ge f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

with equality if and only if f is *linear* or $a_1 = a_2 = \cdots = a_n$.

If we let $w_1, w_2, \ldots, w_n \ge 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n) \ge f(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^{n} w_i f(a_i) \ge f\left(\sum_{i=1}^{n} w_i a_i\right).$$

The inequality is reversed if f is concave.

Example: $\sqrt{\frac{x+y}{2}} \ge \frac{\sqrt{x}+\sqrt{y}}{2}$.

3.8 Karamata's Inequality

If f is *convex*, and (a_i) *majorizes* (b_i), then:

$$f(a_1) + f(a_2) + \dots f(a_n) \geq f(b_1) + f(b_2) + \dots f(b_n),$$

or, more precisely,

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

The inequality is reversed if f is concave.

Example: $f(x) = x^2 \implies (4)^2 + (1)^2 \geq (2.5)^2 + (2.5)^2 \implies 17 \geq 12.5$.

3.9 Popoviciu's Inequality

If f is *convex*, and $a, b, c > 0$, then:

$$af(x) + bf(y) + cf(z) + (a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right) \geq$$

$$(a+b)f\left(\frac{ax+by}{a+b}\right) + (b+c)f\left(\frac{by+cz}{b+c}\right) + (c+a)f\left(\frac{cz+ax}{c+a}\right)$$

Particularly, if $a = b = c = 1$, we have:

$$\frac{f(x)+f(y)+f(c)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right].$$

Equality holds if and only if f is *linear* or $x = y = z$.

Example: $f(x) = x^2 \implies \frac{(1)^2+(2)^2+(3)^2}{3} + \left(\frac{1+2+3}{3}\right)^2 \geq \frac{2}{3}\left[\left(\frac{1+2}{2}\right)^2 + \left(\frac{2+3}{2}\right)^2 + \left(\frac{3+1}{2}\right)^2\right] \implies \frac{26}{3} \geq \frac{25}{3}$.

3.10 Newton's Inequality

For $x_1, x_2, \dots, x_n > 0$ and $k = 1, 2, \dots, n-1$, we have:

$$d_i^2 \geq d_{i-1}d_{i+1},$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Example: $x = 1, y = 2, z = 3 \implies (\frac{xy+yz+zx}{3})^2 \geq (\frac{x+y+z}{3}) \cdot xyz \implies \left(\frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1}{3}\right)^2 \geq \frac{1+2+3}{3}(1 \cdot 2 \cdot 3) \implies \left(\frac{11}{3}\right)^2 \geq 2 \cdot 6 \implies 13.444 \geq 12$.

3.11 Maclaurin's Inequality

For $x_1, x_2, \ldots, x_n > 0$, we have:

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \cdots \geq \sqrt[n]{d_n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Equivalently, it can be written as:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt{\frac{\sum_{1 \leq i < j \leq n} x_i x_j}{\binom{n}{2}}} \geq \sqrt[3]{\frac{\sum_{1 \leq i < j < k \leq n} x_i x_j x_k}{\binom{n}{3}}} \geq \cdots \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Example: $x = 1, y = 2, z = 3 \implies \frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz} \implies \frac{1+2+3}{3} \geq \sqrt{\frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1}{3}} \geq \sqrt[3]{1 \cdot 2 \cdot 3} \implies 2 \geq \frac{11}{3} \geq \sqrt[3]{6} \implies 2 \geq 1.915 \geq 1.817.$

3.12 Cauchy–Schwarz Inequality

3.13 Hölder's Inequality

3.14 Minkowski Inequality

3.15 Generalized Minkowski Inequality

3.16 Young's Inequality

3.17 Rearrangement Inequality

3.18 Chebyshev's Sum Inequality

3.19 Schur's Inequality

3.20 Generalized Schur's Inequality

3.21 Muirhead's Inequality

3.22 Aczel's Inequality

3.23 Huygens Inequality

3.24 Heinz Inequality

3.25 Nesbitt's Inequality

3.26 Cesàro's Inequality

3.27 Mildorf's Inequality

4 Selected Inequalities

5 Proofs

5.1 Proof of AM-GM Inequality using Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $n+1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \ldots a_n} \implies \\
\frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} &= \frac{n\frac{a_1 + \cdots + a_n}{n} + a_{n+1}}{n+1} \\
&\geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&\geq (\sqrt[n]{a_1 \ldots a_n})^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&= \sqrt[n+1]{a_1 \ldots a_n a_{n+1}}
\end{aligned}$$

□

5.2 Proof of AM-GM Inequality using Forward-Backward Induction (a.k.a. Cauchy Induction)

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $2n$ is true.

iii. n is true $\implies$ $n - 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \ldots a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} &= \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}\right) \\
&\geq \frac{\sqrt[n]{a_1 a_2 \ldots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}}{2} \\
&\geq \sqrt[2]{\sqrt[n]{a_1 a_2 \ldots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}} \\
&= \sqrt[2n]{a_1 a_2 \ldots a_{2n}}
\end{aligned}$$

Step 3:

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + \cdots + a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\frac{(n-1)(a_1 + a_2 + \cdots + a_{n-1}) + (a_1 + \cdots + a_{n-1})}{n \cdot (n-1)} &= \frac{(n-1+1)(a_1 + a_2 + \cdots + a_{n-1})}{n \cdot (n-1)} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^n &\geq a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^{n-1} &\geq a_1 a_2 \ldots a_{n-1} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}}
\end{aligned}$$

$\square$

5.3 Proof of AM-GM Inequality using Jensen's Method

Let $a_1, a_2, \ldots, a_n > 0$ and $f(x) = \ln x$ be a *concave* function on $(0, \infty)$. By Jensen's Inequality we have:

$$\begin{aligned}
f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) &\geq \frac{1}{n}\sum_{i=1}^{n} f(a_i) \\
\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \ln(a_2) + \cdots + \ln(a_n)}{n} \\
&= \frac{\ln(a_1 a_2 \ldots a_n)}{n} \\
&= \ln\left(\sqrt[n]{a_1 a_2 \ldots a_n}\right) \\
e^{\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)} &\geq e^{\ln\left(\sqrt[n]{a_1 a_2 \ldots a_n}\right)} \\
\frac{a_1 + a_2 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_n}
\end{aligned}$$

□

6 Selected Problems

1. –