Inequalities Notes

Abror Maksudov

Last Updated: May 21, 2025

Contents

1 Introduction **2**

2 Definitions **2**
2.1 Majorization . . . 2
2.2 Convex Function . . . 2
2.3 Elementary Symmetric Polynomials . . . 3
2.4 Elementary Symmetric Mean . . . 3

3 Inequalities **4**
3.1 AM-GM Inequality . . . 4
3.2 Weighted AM-GM Inequality . . . 4
3.3 Power Mean Inequality . . . 5
3.4 Weighted Power Mean Inequality . . . 5
3.5 HM-GM-AM-QM Inequalities . . . 6
3.6 Bernoulli's Inequality . . . 6
3.7 Jensen's Inequality . . . 6
3.8 Karamata's Inequality . . . 7
3.9 Popoviciu's Inequality . . . 7
3.10 Newton's Inequality . . . 7
3.11 Maclaurin's Inequality . . . 8
3.12 Cauchy–Schwarz Inequality . . . 8
3.13 Titu's Lemma/Sedrakyan's Inequality/Engel's Form . . . 8
3.14 Hölder's Inequality . . . 8
3.15 Minkowski Inequality . . . 9
3.16 Generalized Minkowski Inequality . . . 9
3.17 Young's Inequality . . . 10
3.18 Rearrangement Inequality . . . 10
3.19 Chebyshev's Sum Inequality . . . 11
3.20 Schur's Inequality . . . 11
3.21 Muirhead's Inequality . . . 11
3.22 Aczel's Inequality . . . 12
3.23 Huygens Inequality . . . 12
3.24 Heinz Inequality . . . 12
3.25 Nesbitt's Inequality . . . 12
3.26 Cesàro's Inequality . . . 12
3.27 Mildorf's Inequality . . . 12

4 Selected Inequalities **12**

5 Proofs **12**
5.1 Proof of AM-GM Inequality using Induction 12
5.2 Proof of AM-GM Inequality using Cauchy Induction 12
5.3 Proof of AM-GM Inequality using Jensen's Method 14

6 Selected Problems **14**

1 Introduction

2 Definitions

2.1 Majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences of real numbers. Then x is said to *majorize* y, denoted $x \succ y$, if the following conditions are satisfied:

1. $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n$;
2. $\displaystyle\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$;
3. $\displaystyle\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n-1$.

Example: $(3, 1, 0) \succ (2, 1, 1), \quad (12, 0, 0) \succ (4, 4, 4)$.

2.2 Convex Function

A function $f : [a, b] \to \mathbb{R}$ is called *convex* (concave up) on $[a, b]$ if and only if for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$, the following inequality holds:

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if $f''(x) \geq 0$ ($f''(x) \leq 0$) holds for all $x \in [a, b]$.

Note that f is convex if and only if $-f$ is concave.

Example (convex): x^2, e^x. Example (concave): $\ln x, \sqrt{x}$.

2.3 Elementary Symmetric Polynomials

Let t be a variable and $x_1, x_2, \dots, x_n$ be real numbers. Define:

$$
\begin{aligned}
P(x) = \prod_{i=1}^{n}(t+x_i) &= (t+x_1)(t+x_2)\dots(t+x_n) \\
&= t^n + (x_1 + \dots + x_n)t^{n-1} + (x_1x_2 + x_1x_3 + \dots)t^{n-2} + \dots \\
&+ (x_2x_3\dots x_n + x_1x_3\dots x_n + \dots)t + x_1x_2x_3\dots x_n \\
&= 1 \cdot t^n + \left(\sum_{1\le i\le n} x_i\right)t^{n-1} + \left(\sum_{1\le i<j\le n} x_ix_j\right)t^{n-2} + \dots \\
&+ \left(\sum_{1\le i_1<\dots<i_{n-1}\le n} x_{i_1}x_{i_2}\dots x_{i_{n-1}}\right)t + \prod_{i=1}^{n} x_i.
\end{aligned}
$$

In other words,

$$
P(x) = \prod_{i=1}^{n}(t+x_i) = c_0t^n + c_1t^{n-1} + c_2t^{n-2} + \dots + c_{n-1}t + c_n,
$$

where the coefficient c_k is the k-th elementary symmetric sum:

$$
c_0 = 1, \qquad c_1 = \sum_{1\le i\le n} x_i, \qquad c_2 = \sum_{1\le i<j\le n} x_ix_j,
$$

$$
c_3 = \sum_{1\le i<j<k\le n} x_ix_jx_k, \qquad \dots, \qquad c_n = \prod_{i=1}^{n} x_i.
$$

In general, for $0 \le k \le n$

$$
c_k = \sum_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}.
$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies (x+1)(x+2)(x+3) = x^3 + (1+2+3)x^2 + (1\cdot 2 + 2\cdot 3 + 3\cdot 1)x + 1\cdot 2\cdot 3 = x^3 + 6x^2 + 11x + 6$.

2.4 Elementary Symmetric Mean

Let $x_1, x_2, \dots, x_n$ be real numbers. The k-th elementary symmetric mean is defined as:

$$
d_k = \frac{c_k}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}.
$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies d_2 = \frac{c_2}{\binom{3}{2}} = \frac{11}{3}$.

3 Inequalities

3.1 AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the following inequality holds:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.
More precisely,

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}.$$

Example: $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

3.2 Weighted AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$ and $w_1, w_2, \ldots, w_n$ be positive integers. Then, by AM-GM we have:

$$\frac{\underbrace{a_1 + a_1 + \cdots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \cdots + a_2}_{w_2} + \cdots + \underbrace{a_n + a_n + \cdots + a_n}_{w_n}}{w_1 + w_2 + \cdots + w_n}$$
$$\geq \left(\underbrace{a_1 a_1 \ldots a_1}_{w_1} \underbrace{a_2 a_2 \ldots a_2}_{w_2} \ldots \underbrace{a_n a_n \ldots a_n}_{w_n}\right)^{\frac{1}{w_1 + w_2 + \cdots + w_n}}.$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n} \geq (a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \ldots w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} \geq \left(\prod_{i=1}^{n} a_i^{w_i}\right)^{\frac{1}{\sum_{i=1}^{n} w_i}}$$

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}.$$

Example: $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$.

3.3 Power Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the r-th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(\dfrac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} & r \neq 0, \\ \sqrt[n]{a_1 a_2 \ldots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1:$ $$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \qquad \text{(Harmonic Mean)}$$
- $r = 0:$ $$\sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \qquad \text{(Geomteric Mean)}$$
- $r = 1:$ $$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i \qquad \text{(Arithmetic Mean)}$$
- $r = 2:$ $$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}} \qquad \text{(Quadratic Mean)}$$

If $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$.

3.4 Weighted Power Mean Inequality

Let $a_1, a_2, \ldots a_n > 0$ and $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$. Then, the r-th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r)^{1/r} & r \neq 0, \\ a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $(\frac{1}{6}a^3 + \frac{1}{3}b^3 + \frac{1}{2}c^3)^{1/3} \geq a^{1/6} b^{1/3} c^{1/2}$.

3.5 HM-GM-AM-QM Inequalities

Let $a_1, a_2, \ldots, a_n > 0$. Then:

$$0 < \text{HM} \leq \text{GM} \leq \text{AM} \leq \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^{n} a_i} \leq \frac{1}{n} \sum_{i=1}^{n} a_i \leq \sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}.$$

Example: $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$.

3.6 Bernoulli's Inequality

For all $x \geq -1$ and $r \geq 1$:

$$(1+x)^r \geq 1 + rx.$$

Example: $(1+x)^5 \geq 1 + 5x$.

3.7 Jensen's Inequality

If f is *convex*, then:

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

with equality if and only if f is *linear* or $a_1 = a_2 = \cdots = a_n$.

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^{n} w_i f(a_i) \geq f\left(\sum_{i=1}^{n} w_i a_i\right).$$

The inequality is reversed if f is concave.

Example: $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x}+\sqrt{y}}{2}$.

3.8 Karamata's Inequality

If f is *convex*, and (a_i) *majorizes* (b_i), then:

$$f(a_1) + f(a_2) + \dots f(a_n) \geq f(b_1) + f(b_2) + \dots f(b_n),$$

or, more precisely,

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

The inequality is reversed if f is concave.

Example: $f(x) = x^2 \implies (4)^2 + (1)^2 \geq (2.5)^2 + (2.5)^2 \implies 17 \geq 12.5$.

3.9 Popoviciu's Inequality

If f is *convex*, and $a, b, c > 0$, then:

$$af(x) + bf(y) + cf(z) + (a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right) \geq$$

$$(a+b)f\left(\frac{ax+by}{a+b}\right) + (b+c)f\left(\frac{by+cz}{b+c}\right) + (c+a)f\left(\frac{cz+ax}{c+a}\right)$$

Particularly, if $a = b = c = 1$, we have:

$$\frac{f(x)+f(y)+f(c)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right].$$

Equality holds if and only if f is *linear* or $x = y = z$.

Example: $f(x) = x^2 \implies \frac{(1)^2+(2)^2+(3)^2}{3} + \left(\frac{1+2+3}{3}\right)^2 \geq \frac{2}{3}\left[\left(\frac{1+2}{2}\right)^2 + \left(\frac{2+3}{2}\right)^2 + \left(\frac{3+1}{2}\right)^2\right] \implies \frac{26}{3} \geq \frac{25}{3}$.

3.10 Newton's Inequality

For $x_1, x_2, \dots, x_n > 0$ and $k = 1, 2, \dots, n-1$, we have:

$$d_i^2 \geq d_{i-1}d_{i+1},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Example: $x = 1, y = 2, z = 3 \implies (\frac{xy+yz+zx}{3})^2 \geq (\frac{x+y+z}{3}) \cdot xyz$

$\implies \left(\frac{1\cdot2+2\cdot3+3\cdot1}{3}\right)^2 \geq \frac{1+2+3}{3}(1\cdot2\cdot3) \implies \left(\frac{11}{3}\right)^2 \geq 2\cdot6 \implies 13.444 \geq 12$.

3.11 Maclaurin's Inequality

For $x_1, x_2, \ldots, x_n > 0$, we have:

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \cdots \geq \sqrt[n]{d_n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Equivalently, it can be written as:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt{\frac{\sum_{1 \leq i < j \leq n} x_i x_j}{\binom{n}{2}}} \geq \sqrt[3]{\frac{\sum_{1 \leq i < j < k \leq n} x_i x_j x_k}{\binom{n}{3}}} \geq \cdots \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Example: $x = 1, y = 2, z = 3 \implies \frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz}$

$\implies \frac{1+2+3}{3} \geq \sqrt{\frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1}{3}} \geq \sqrt[3]{1 \cdot 2 \cdot 3} \implies 2 \geq \frac{11}{3} \geq \sqrt[3]{6} \implies 2 \geq 1.915 \geq 1.817.$

3.12 Cauchy–Schwarz Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers. Then:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \ldots a_n b_n)^2,$$

with equality if and only if there is a contant $\lambda \in \mathbb{R}$ such that $a_i = \lambda b_i$ for all $1 \leq i \leq n$. That is, if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = \lambda$.

More precisely,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Example: $(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2 \implies (2^2 + 3^2)(4^2 + 5^2) \geq (2 \cdot 4 + 3 \cdot 5)^2$

$\implies 13 \cdot 41 \geq 23^2 \implies 533 \geq 529.$

3.13 Titu's Lemma/Sedrakyan's Inequality/Engel's Form

Let $a_1, a_2, \ldots, a_n \geq 0$ and $b_1, b_2, \ldots, b_n > 0$. Then:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

Example: $\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$.

3.14 Hölder's Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, $\ldots$, $z_1, z_2, \ldots, z_n$ be positive real numbers, and let $\lambda_a, \lambda_b, \ldots, \lambda_z$ be positive reals with $\lambda_a + \lambda_b + \cdots + \lambda_z = 1$. Then:

$$(a_1 + \cdots + a_n)^{\lambda_a}(b_1 + \cdots + b_n)^{\lambda_b} \ldots (z_1 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \ldots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \ldots z_n^{\lambda_z},$$

or, more precisely,

$$\underbrace{\left(\sum_{i=1}^{n} a_i\right)^{\lambda_a}\left(\sum_{i=1}^{n} b_i\right)^{\lambda_b}\cdots\left(\sum_{i=1}^{n} z_i\right)^{\lambda_z}}_{m \text{ factors}} \geq \sum_{i=1}^{n}\left(\underbrace{a_i^{\lambda_a} b_i^{\lambda_b}\ldots z_i^{\lambda_z}}_{m \text{ variables}}\right)$$

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right)^{\lambda_j} \geq \sum_{i=1}^{n}\left(\prod_{j=1}^{m} a_{ij}^{\lambda_j}\right).$$

Example: $m = 3, n = 2, \lambda_a = 0.5, \lambda_b = 0.3, \lambda_c = 0.2, (a) = (1, 3), (b) = (2, 4), (c) = (5, 6)$

$\implies (a_1 + a_2)^{0.5}(b_1 + b_2)^{0.3}(c_1 + c_2)^{0.2} \geq a_1^{0.5}b_1^{0.3}c_1^{0.2} + a_2^{0.5}b_2^{0.3}c_2^{0.2}$

$\implies (1 + 3)^{0.5}(2 + 4)^{0.3}(5 + 6)^{0.2} \geq 1^{0.5}2^{0.3}5^{0.2} + 3^{0.5}4^{0.3}6^{0.2} \implies 5.53 \geq 5.45.$

3.15 Minkowski Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be positive real numbers and $p > 1$. Then:

$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} + (b_1^p + b_2^p + \cdots + b_n^p)^{\frac{1}{p}} \geq ((a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p)^{\frac{1}{p}}$,

or, more precisely,

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}}$$

Example: $p = 2, a = (3, 4), b = (6, 8)$

$\implies \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$

$\implies \sqrt{3^2 + 4^2} + \sqrt{6^2 + 8^2} \geq \sqrt{(3 + 6)^2 + (4 + 8)^2}$

$\implies \sqrt{25} + \sqrt{100} \geq \sqrt{225} \implies 15 \geq 15.$ (equal. why?)

3.16 Generalized Minkowski Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, $\ldots$, $z_1, z_2, \ldots, z_n$ be positive real numbers, and $p > 1$. Then:

$$\underbrace{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} + \cdots + \left(\sum_{i=1}^{n} z_i^p\right)^{\frac{1}{p}}}_{m \text{ terms}} \geq \left(\sum_{i=1}^{n}\underbrace{(a_i + b_i + \cdots + z_i)^p}_{m \text{ terms}}\right)^{\frac{1}{p}}.$$

More precisely,

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^p\right)^{\frac{1}{p}} \geq \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)^p\right]^{\frac{1}{p}}$$

3.17 Young's Inequality

Let $a, b \geq 0$ and $p, q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then:

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

with equality if and only if $a^p = b^q$.

Moreover, for increasing functions

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab$$

with equality if and only if $f(a) = b$.

Example: $a = 2, b = 3, p = 3, q = \frac{3}{2} \implies \frac{2^3}{3} + \frac{3^{3/2}}{3/2} \geq 2 \cdot 3 \implies 6.13 \geq 6.$

3.18 Rearrangement Inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$ be two sequences that are both increasing (or both decreasing). Then:

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{\sigma(1)} + a_1b_{\sigma(2)} + \cdots + a_1b_{\sigma(n)} \geq a_1b_n + a_2b_{n-1} + \ldots a_nb_1,$$

where σ is a permutation function, which sends each of $1, 2, \ldots, n$ to a different value in $\{1, 2, \ldots, n\}$.

More precisely,

$$\sum_{i=1}^n a_ib_i \geq \sum_{i=1}^n a_ib_{\sigma(i)} \geq \sum_{i=1}^n a_ib_{n+1-i}.$$

In other words, the sum is *maximized* when both sequences are ordered *similarly* (both increasing or both decreasing), and is *minimazied* when both sequences are ordered *oppositely* (one increasing, the other decreasing).

Example: $a^2 + b^2 + c^2 \geq ab + bc + ca$;

$a = (1, 3, 5), \quad b = (2, 4, 6) \implies 1\cdot2+3\cdot4+5\cdot6 \geq 1\cdot4+3\cdot6+5\cdot2 \geq 1\cdot6+3\cdot4+5\cdot2 \implies 44 \geq 32 \geq 28.$

3.19 Chebyshev's Sum Inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$ be two sequences that are both increasing (or both decreasing). Then:

$$\frac{a_1b_1 + \cdots + a_nb_n}{n} \geq \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} \geq \frac{a_1b_n + \cdots + a_nb_1}{n},$$

or, more precisely,

$$\frac{\sum_{i=1}^{n} a_i b_i}{n} \geq \frac{\sum_{i=1}^{n} a_i}{n} \times \frac{\sum_{i=1}^{n} b_i}{n} \geq \frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n}.$$

Example: $a_1 \leq a_2 \leq a_3,\ b_1 \leq b_2 \leq b_3 \implies a_1b_1+a_2b_2+a_3b_3 \geq \frac{1}{3}(a_1+a_2+a_3)(b_1+b_2+b_3)$.

3.20 Schur's Inequality

Let $a, b, c \geq 0$ and $r > 0$. Then:

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or two of them are equal and the other is zero.

Example: $r = 1 \implies a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(c+a) + c^2(a+b)$.

3.21 Muirhead's Inequality

Let $a_1, a_2, \ldots, a_n \geq 0$ and suppose that (x_n) *majorizes* (y_n), $x \succ y$. Then:

$$\sum_{\text{sym}} a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n} \geq \sum_{\text{sym}} a_1^{y_1} a_2^{y_2} \ldots a_n^{y_n}.$$

Example: $(5,0) \succ (3,2) \implies x^5 + y^5 \geq x^3y^2 + x^2y^3$.

3.22 Aczel's Inequality

3.23 Huygens Inequality

3.24 Heinz Inequality

3.25 Nesbitt's Inequality

3.26 Cesàro's Inequality

3.27 Mildorf's Inequality

4 Selected Inequalities

5 Proofs

5.1 Proof of AM-GM Inequality using Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $n+1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n} \implies$$

$$\begin{aligned} \frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} &= \frac{n\frac{a_1 + \cdots + a_n}{n} + a_{n+1}}{n+1} \\ &\geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\ &\geq (\sqrt[n]{a_1 \ldots a_n})^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\ &= \sqrt[n+1]{a_1 \ldots a_n a_{n+1}} \end{aligned}$$

□

5.2 Proof of AM-GM Inequality using Cauchy Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $2n$ is true.

iii. n is true $\implies$ $n-1$ is true.

Proof:

Step 1:

$$\frac{a_1+a_2}{2} \geq \sqrt{a_1a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1}-\sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned}
\frac{a_1+\cdots+a_n}{n} &\geq \sqrt[n]{a_1\ldots a_n} \implies \\
\frac{a_1+a_2+\cdots+a_{2n}}{2n} &= \frac{1}{2}\left(\frac{a_1+a_2+\cdots+a_n}{n} + \frac{a_{n+1}+a_{n+2}+\cdots+a_{2n}}{n}\right) \\
&\geq \frac{\sqrt[n]{a_1a_2\ldots a_n} + \sqrt[n]{a_{n+1}a_{n+2}\ldots a_{2n}}}{2} \\
&\geq \sqrt[2]{\sqrt[n]{a_1a_2\ldots a_n}\cdot\sqrt[n]{a_{n+1}a_{n+2}\ldots a_{2n}}} \\
&= \sqrt[2n]{a_1a_2\ldots a_{2n}}
\end{aligned}$$

Step 3:

$$\begin{aligned}
\frac{a_1+a_2+\cdots+a_{n-1}+a_n}{n} &\geq \sqrt[n]{a_1a_2\ldots a_{n-1}a_n} \implies \\
\frac{a_1+a_2+\cdots+a_{n-1}+\frac{a_1+\cdots+a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1a_2\ldots a_{n-1}\cdot\frac{a_1+\cdots+a_{n-1}}{n-1}} \\
\frac{(n-1)(a_1+a_2+\cdots+a_{n-1})+(a_1+\cdots+a_{n-1})}{n\cdot(n-1)} &= \frac{(n-1+1)(a_1+a_2+\cdots+a_{n-1})}{n\cdot(n-1)} \\
\frac{a_1+a_2+\cdots+a_{n-1}}{n-1} &\geq \sqrt[n]{a_1a_2\ldots a_{n-1}\cdot\frac{a_1+\cdots+a_{n-1}}{n-1}} \\
\left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}\right)^n &\geq a_1a_2\ldots a_{n-1}\cdot\frac{a_1+\cdots+a_{n-1}}{n-1} \\
\left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}\right)^{n-1} &\geq a_1a_2\ldots a_{n-1} \\
\frac{a_1+a_2+\cdots+a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1a_2\ldots a_{n-1}}
\end{aligned}$$

$\square$

5.3 Proof of AM-GM Inequality using Jensen's Method

Let $a_1, a_2, \ldots, a_n > 0$ and $f(x) = \ln x$ be a *concave* function on $(0, \infty)$. By Jensen's Inequality we have:

$$\begin{aligned}
f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) &\geq \frac{1}{n}\sum_{i=1}^{n} f(a_i) \\
\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \ln(a_2) + \cdots + \ln(a_n)}{n} \\
&= \frac{\ln(a_1 a_2 \ldots a_n)}{n} \\
&= \ln\left(\sqrt[n]{a_1 a_2 \ldots a_n}\right) \\
e^{\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)} &\geq e^{\ln\left(\sqrt[n]{a_1 a_2 \ldots a_n}\right)} \\
\frac{a_1 + a_2 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_n}
\end{aligned}$$

□

6 Selected Problems

1. –