

Inequalities Notes

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1 Algebraic Inequalities

Theorem 1 (AM-GM). *Let a_1, \dots, a_n be non-negative real numbers. Then:*

$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n}$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

Theorem 2 (Cauchy-Schwarz). *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be real numbers. Then:*

$$(a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) \geq (a_1 b_1 + \dots + a_n b_n)^2$$

Theorem 3 (Titu's Lemma). *Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be positive real numbers. Then:*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}$$

Theorem 4 (Young's Inequality). *Let a, b be nonnegative real numbers and if $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

Theorem 5 (Hölder's Inequality). *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} \geq \sum_{i=1}^n a_i b_i$$

More generally, let x_{ij} ($i = 1, \dots, m, j = 1, \dots, n$) be positive real numbers. Suppose that w_1, w_2, \dots, w_n are positive real numbers satisfying $w_1 + w_2 + \dots + w_n = 1$. Then:

$$\prod_{j=1}^n \left(\sum_{i=1}^m x_{ij} \right)^{w_j} \geq \sum_{i=1}^m \left(\prod_{j=1}^n x_{ij}^{w_j} \right)$$

Theorem 6 (Minkowski Inequality). *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be positive real numbers. Suppose that $p > 1$. Then:*

$$\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n (a_i + b_i)^p \right)^{\frac{1}{p}}$$

Theorem 7 (Generalized Minkowski Inequality). Let $a_{ij} \geq 0$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ and let $p > 1$. Then:

$$\left[\sum_{i=1}^n \left(\sum_{j=1}^m a_{ij} \right)^p \right]^{\frac{1}{p}} \leq \sum_{j=1}^m \left(\sum_{i=1}^n a_{ij}^p \right)^{\frac{1}{p}}$$

Theorem 8 (Chebyshev's Sum Inequality). Let a_1, \dots, a_n and b_1, \dots, b_n be real numbers. Then:

$$\frac{a_1 b_1 + \dots + a_n b_n}{n} \geq \frac{(a_1 + \dots + a_n)}{n} \frac{(b_1 + \dots + b_n)}{n}$$

$$\frac{1}{n} \sum_{i=1}^n a_i b_i \geq \left(\frac{1}{n} \sum_{i=1}^n a_i \right) \left(\frac{1}{n} \sum_{i=1}^n b_i \right)$$

Theorem 9 (Rearrangement Inequality). Let a_1, \dots, a_n and b_1, \dots, b_n be real numbers. For any permutation σ of $\{1, \dots, n\}$, we have:

$$\sum_{i=1}^n a_i b_i \geq \sum_{i=1}^n a_i b_{\sigma(i)} \geq \sum_{i=1}^n a_i b_{n+1-i}$$

Definition 1 (Convex Function). Suppose that f is a one-variable function defined on $[a, b] \subset \mathbb{R}$. f is called a convex function on $[a, b]$ if and only if for all $x, y \in [a, b]$ and for all $0 \leq t \leq 1$, we have:

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y)$$

Theorem 10 (Jensen's Inequality). Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. Then for any $x_1, \dots, x_n \in [a, b]$ and non-negative real numbers w_1, \dots, w_n with $w_1 + \dots + w_n = 1$, we have:

$$\sum_{i=1}^n w_i f(x_i) \geq f\left(\sum_{i=1}^n w_i x_i\right)$$

Theorem 11 (Popoviciu's Inequality). Let $f : I \rightarrow \mathbb{R}$. If f is convex, then for any three points x, y, z in I :

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3} \left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right]$$

Definition 2 (Majorization). Given two sequences $(a) = (a_1, a_2, \dots, a_n)$ and $(b) = (b_1, b_2, \dots, b_n)$ (where $a_i, b_i \in \mathbb{R} \quad \forall i \in \{1, 2, \dots, n\}$). We say that the sequence (a) majorizes the sequence (b) , and write $(a) \succ (b)$, if the following conditions are fulfilled:

$$\begin{aligned} a_1 &\geq a_2 \geq \dots \geq a_n; \\ b_1 &\geq b_2 \geq \dots \geq b_n; \\ a_1 + a_2 + \dots + a_n &= b_1 + b_2 + \dots + b_n; \\ a_1 + a_2 + \dots + a_k &= b_1 + b_2 + \dots + b_k \quad \forall k \in \{1, 2, \dots, n-1\} \end{aligned}$$