Inequalities Notes

Abror Maksudov

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2.1 Majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences of real numbers. Then x is said to *majorize* y, denoted $x \succ y$, if the following conditions are satisfied:

1. $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n$;
2. $\displaystyle\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$;
3. $\displaystyle\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n-1$.

Example: $(3,1,0) \succ (2,1,1)$, $\quad (12,0,0) \succ (4,4,4)$.

2.2 Convex Function

A function $f : [a,b] \to \mathbb{R}$ is called *convex* (concave up) on $[a,b]$ if and only if for all $x, y \in [a,b]$ and all $\lambda \in [0,1]$, the following inequality holds:

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if $f''(x) \geq 0$ ($f''(x) \leq 0$) holds for all $x \in [a,b]$.

Note that f is convex if and only if $-f$ is concave.

Example (convex): x^2, e^x. Example (concave): $\ln x, \sqrt{x}$.

2.3 Cyclic Sum

The *cyclic sum* of a function f over n variables is the sum over *all cyclic permutations* of its arguments:

$$\sum_{\text{cyc}} f(a_1, a_2, \ldots, a_n) = f(a_1, a_2, \ldots, a_n) + f(a_2, a_3, \ldots, a_1) + \cdots + f(a_n, a_1, \ldots, a_{n-1}).$$

The number of terms is equal to the number of variables: n.

Example: $f(a,b,c) = a^2b \implies \sum_{\text{cyc}} = a^2b + b^2c + c^2a.$

2.4 Symmetric Sum

The *symmetric sum* of a function f over n variables is the sum over *all possible permutations* of its arguments:

$$\sum_{\text{sym}} f(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in S_n} f(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}),$$

where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$.

The number of terms equals to the number of all permutations of the variables $= n!$.

Example: $f(a,b,c) = a^2b \implies \sum_{\text{sym}} = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$

2.5 Elementary Symmetric Polynomials

Let t be a variable and $x_1, x_2, \ldots, x_n$ be real numbers. Define:

$$\begin{aligned}
P(x) = \prod_{i=1}^{n}(t + x_i) &= (t + x_1)(t + x_2)\ldots(t + x_n) \\
&= t^n + (x_1 + \cdots + x_n)t^{n-1} + (x_1x_2 + x_1x_3 + \ldots)t^{n-2} + \ldots \\
&+ (x_2x_3\ldots x_n + x_1x_3\ldots x_n + \ldots)t + x_1x_2x_3\ldots x_n \\
&= 1 \cdot t^n + \left(\sum_{1 \le i \le n} x_i\right) t^{n-1} + \left(\sum_{1 \le i < j \le n} x_ix_j\right) t^{n-2} + \ldots \\
&+ \left(\sum_{1 \le i_1 < \cdots < i_{n-1} \le n} x_{i_1}x_{i_2}\ldots x_{i_{n-1}}\right) t + \prod_{i=1}^{n} x_i.
\end{aligned}$$

In other words,

$$P(x) = \prod_{i=1}^{n}(t + x_i) = c_0t^n + c_1t^{n-1} + c_2t^{n-2} + \cdots + c_{n-1}t + c_n,$$

where the coefficient c_k is the k-th elementary symmetric sum:

$$\begin{aligned}
&c_0 = 1, && c_1 = \sum_{1 \le i \le n} x_i, && c_2 = \sum_{1 \le i < j \le n} x_ix_j, \\
&c_3 = \sum_{1 \le i < j < k \le n} x_ix_jx_k, && \ldots, && c_n = \prod_{i=1}^{n} x_i.
\end{aligned}$$

In general, for $0 \le k \le n$

$$c_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies (x+1)(x+2)(x+3) = x^3 + (1+2+3)x^2 + (1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1)x + 1 \cdot 2 \cdot 3 = x^3 + 6x^2 + 11x + 6$.

2.6 Elementary Symmetric Mean

Let $x_1, x_2, \ldots, x_n$ be real numbers. The k-th elementary symmetric mean is defined as:

$$d_k = \frac{c_k}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies d_2 = \frac{c_2}{\binom{3}{2}} = \frac{11}{3}$.

3 Inequalities

3.1 AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the following inequality holds:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \ldots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.
More precisely,

$$\frac{1}{n} \sum_{i=1}^{n} a_i \ge \sqrt[n]{\prod_{i=1}^{n} a_i}.$$

Example: $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$.

3.2 Weighted AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$ and $w_1, w_2, \ldots, w_n$ be positive integers. Then, by AM-GM we have:

$$\frac{\underbrace{a_1 + a_1 + \cdots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \cdots + a_2}_{w_2} + \cdots + \underbrace{a_n + a_n + \cdots + a_n}_{w_n}}{w_1 + w_2 + \cdots + w_n}$$
$$\ge \left(\underbrace{a_1 a_1 \ldots a_1}_{w_1} \underbrace{a_2 a_2 \ldots a_2}_{w_2} \ldots \underbrace{a_n a_n \ldots a_n}_{w_n} \right)^{\frac{1}{w_1 + w_2 + \cdots + w_n}}.$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n} \ge (a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \ldots w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} \geq \left(\prod_{i=1}^{n} a_i^{w_i}\right)^{\frac{1}{\sum_{i=1}^{n} w_i}}$$

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}.$$

Example: $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$.

3.3 Power Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the r-th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(\dfrac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} & r \neq 0, \\ \sqrt[n]{a_1 a_2 \ldots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1:$ $$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$$ (Harmonic Mean)
- $r = 0:$ $$\sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$ (Geomteric Mean)
- $r = 1:$ $$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n} \sum_{i=1}^{n} a_i$$ (Arithmetic Mean)
- $r = 2:$ $$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}$$ (Quadratic Mean)

If $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$.

3.4 Weighted Power Mean Inequality

Let $a_1, a_2, \ldots a_n > 0$ and $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$. Then, the r-th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r)^{1/r} & r \neq 0, \\ a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $(\frac{1}{6}a^3 + \frac{1}{3}b^3 + \frac{1}{2}c^3)^{1/3} \geq a^{1/6}b^{1/3}c^{1/2}$.

3.5 HM-GM-AM-QM Inequalities

Let $a_1, a_2, \ldots, a_n > 0$. Then:

$$0 < \text{HM} \leq \text{GM} \leq \text{AM} \leq \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^{n} a_i} \leq \frac{1}{n}\sum_{i=1}^{n} a_i \leq \sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}.$$

Example: $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$.

3.6 Bernoulli's Inequality

For all $x \geq -1$ and $r \geq 1$:

$$(1+x)^r \geq 1 + rx.$$

Example: $(1+x)^5 \geq 1 + 5x$.

3.7 Jensen's Inequality

If f is *convex*, then:

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

with equality if and only if f is *linear* or $a_1 = a_2 = \cdots = a_n$.

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^{n} w_i f(a_i) \geq f\left(\sum_{i=1}^{n} w_i a_i\right).$$

The inequality is reversed if f is concave.

Example: $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x}+\sqrt{y}}{2}$.

3.8 Karamata's Inequality

If f is *convex*, and (a_i) *majorizes* (b_i), then:

$$f(a_1)+f(a_2)+\dots f(a_n)\geq f(b_1)+f(b_2)+\dots f(b_n),$$

or, more precisely,

$$\sum_{i=1}^{n} f(a_i)\geq \sum_{i=1}^{n} f(b_i).$$

The inequality is reversed if f is concave.

Example: $f(x)=x^2 \implies (4)^2+(1)^2\geq (2.5)^2+(2.5)^2 \implies 17\geq 12.5.$

3.9 Popoviciu's Inequality

If f is *convex*, and $a,b,c>0$, then:

$$af(x)+bf(y)+cf(z)+(a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right)\geq$$
$$(a+b)f\left(\frac{ax+by}{a+b}\right)+(b+c)f\left(\frac{by+cz}{b+c}\right)+(c+a)f\left(\frac{cz+ax}{c+a}\right)$$

Particularly, if $a=b=c=1$, we have:

$$\frac{f(x)+f(y)+f(c)}{3}+f\left(\frac{x+y+z}{3}\right)\geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right].$$

Equality holds if and only if f is *linear* or $x=y=z$.

Example: $f(x)=x^2 \implies \frac{(1)^2+(2)^2+(3)^2}{3}+\left(\frac{1+2+3}{3}\right)^2\geq \frac{2}{3}\left[\left(\frac{1+2}{2}\right)^2+\left(\frac{2+3}{2}\right)^2+\left(\frac{3+1}{2}\right)^2\right] \implies \frac{26}{3}\geq\frac{25}{3}$.

3.10 Newton's Inequality

For $x_1,x_2,\dots,x_n>0$ and $k=1,2,\dots,n-1$, we have:

$$d_i^2\geq d_{i-1}d_{i+1},$$

with equality if and only if $x_1=x_2=\dots=x_n$.

Example: $x=1,y=2,z=3 \implies (\frac{xy+yz+zx}{3})^2\geq(\frac{x+y+z}{3})\cdot xyz$

$\implies \left(\frac{1\cdot2+2\cdot3+3\cdot1}{3}\right)^2\geq\frac{1+2+3}{3}(1\cdot2\cdot3) \implies \left(\frac{11}{3}\right)^2\geq 2\cdot 6 \implies 13.444\geq 12.$

3.11 Maclaurin's Inequality

For $x_1,x_2,\dots,x_n>0$, we have:

$$d_1\geq\sqrt[2]{d_2}\geq\sqrt[3]{d_3}\geq\dots\geq\sqrt[n]{d_n}$$

with equality if and only if $x_1=x_2=\dots=x_n$.

Equivalently, it can be written as:

$$\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt{\frac{\sum_{1\leq i<j\leq n} x_i x_j}{\binom{n}{2}}} \geq \sqrt[3]{\frac{\sum_{1\leq i<j<k\leq n} x_i x_j x_k}{\binom{n}{3}}} \geq \cdots \geq \sqrt[n]{x_1 x_2 \dots x_n}.$$

Example: $x=1, y=2, z=3 \implies \frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz}$

$\implies \frac{1+2+3}{3} \geq \sqrt{\frac{1\cdot2+2\cdot3+3\cdot1}{3}} \geq \sqrt[3]{1\cdot2\cdot3} \implies 2 \geq \frac{11}{3} \geq \sqrt[3]{6} \implies 2 \geq 1.915 \geq 1.817.$

3.12 Cauchy–Schwarz Inequality

Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ be real numbers. Then:

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \geq (a_1b_1+a_2b_2+\dots a_nb_n)^2,$$

with equality if and only if there is a contant $\lambda \in \mathbb{R}$ such that $a_i = \lambda b_i$ for all $1 \leq i \leq n$. That is, if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = \lambda$.

More precisely,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Example: $(a^2+b^2)(x^2+y^2) \geq (ax+by)^2 \implies (2^2+3^2)(4^2+5^2) \geq (2\cdot4+3\cdot5)^2$

$\implies 13\cdot41 \geq 23^2 \implies 533 \geq 529.$

3.13 Titu's Lemma/Sedrakyan's Inequality/Engel's Form

Let $a_1, a_2, \dots, a_n \geq 0$ and $b_1, b_2, \dots, b_n > 0$. Then:

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}.$$

Example: $\frac{a^2}{x}+\frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$.

3.14 Hölder's Inequality

Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$, $\dots$, $z_1, z_2, \dots, z_n$ be positive real numbers, and let $\lambda_a, \lambda_b, \dots, \lambda_z$ be positive reals with $\lambda_a+\lambda_b+\cdots+\lambda_z = 1$. Then:

$$(a_1+\cdots+a_n)^{\lambda_a}(b_1+\cdots+b_n)^{\lambda_b}\dots(z_1+\cdots+z_n)^{\lambda_z} \geq a_1^{\lambda_a}b_1^{\lambda_b}\dots z_1^{\lambda_z}+\cdots+a_n^{\lambda_a}b_n^{\lambda_b}\dots z_n^{\lambda_z},$$

or, more precisely,

$$\underbrace{\left(\sum_{i=1}^{n} a_i\right)^{\lambda_a} \left(\sum_{i=1}^{n} b_i\right)^{\lambda_b} \cdots \left(\sum_{i=1}^{n} z_i\right)^{\lambda_z}}_{m \text{ factors}} \geq \sum_{i=1}^{n} \left(\underbrace{a_i^{\lambda_a} b_i^{\lambda_b} \ldots z_i^{\lambda_z}}_{m \text{ variables}}\right)$$

$$\prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}\right)^{\lambda_j} \geq \sum_{i=1}^{n} \left(\prod_{j=1}^{m} a_{ij}^{\lambda_j}\right).$$

Example: $m = 3, n = 2, \lambda_a = 0.5, \lambda_b = 0.3, \lambda_c = 0.2, (a) = (1, 3), (b) = (2, 4), (c) = (5, 6)$

$\implies (a_1 + a_2)^{0.5}(b_1 + b_2)^{0.3}(c_1 + c_2)^{0.2} \geq a_1^{0.5} b_1^{0.3} c_1^{0.2} + a_2^{0.5} b_2^{0.3} c_2^{0.2}$

$\implies (1 + 3)^{0.5}(2 + 4)^{0.3}(5 + 6)^{0.2} \geq 1^{0.5} 2^{0.3} 5^{0.2} + 3^{0.5} 4^{0.3} 6^{0.2} \implies 5.53 \geq 5.45.$

3.15 Minkowski Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be positive real numbers and $p > 1$. Then:

$$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} + (b_1^p + b_2^p + \cdots + b_n^p)^{\frac{1}{p}} \geq ((a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p)^{\frac{1}{p}},$$

or, more precisely,

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}}$$

Example: $p = 2, a = (3, 4), b = (6, 8)$

$\implies \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$

$\implies \sqrt{3^2 + 4^2} + \sqrt{6^2 + 8^2} \geq \sqrt{(3 + 6)^2 + (4 + 8)^2}$

$\implies \sqrt{25} + \sqrt{100} \geq \sqrt{225} \implies 15 \geq 15.$ (equal. why?)

3.16 Generalized Minkowski Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, $\ldots$, $z_1, z_2, \ldots, z_n$ be positive real numbers, and $p > 1$. Then:

$$\underbrace{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} + \cdots + \left(\sum_{i=1}^{n} z_i^p\right)^{\frac{1}{p}}}_{m \text{ terms}} \geq \left(\sum_{i=1}^{n} \underbrace{(a_i + b_i + \cdots + z_i)^p}_{m \text{ terms}}\right)^{\frac{1}{p}}.$$

More precisely,

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^p\right)^{\frac{1}{p}} \geq \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij}\right)^p\right]^{\frac{1}{p}}$$

3.17 Young's Inequality

Let $a, b \geq 0$ and $p, q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then:

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

with equality if and only if $a^p = b^q$.

Moreover, for increasing functions

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab$$

with equality if and only if $f(a) = b$.

Example: $a = 2, b = 3, p = 3, q = \frac{3}{2} \implies \frac{2^3}{3} + \frac{3^{3/2}}{3/2} \geq 2 \cdot 3 \implies 6.13 \geq 6.$

3.18 Rearrangement Inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$ be two sequences that are both increasing (or both decreasing). Then:

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{\sigma(1)} + a_1b_{\sigma(2)} + \cdots + a_1b_{\sigma(n)} \geq a_1b_n + a_2b_{n-1} + \ldots a_nb_1,$$

where σ is a permutation function, which sends each of $1, 2, \ldots, n$ to a different value in $\{1, 2, \ldots, n\}$.

More precisely,

$$\sum_{i=1}^n a_ib_i \geq \sum_{i=1}^n a_ib_{\sigma(i)} \geq \sum_{i=1}^n a_ib_{n+1-i}.$$

In other words, the sum is *maximized* when both sequences are ordered *similarly* (both increasing or both decreasing), and is *minimazied* when both sequences are ordered *oppositely* (one increasing, the other decreasing).

Example: $a^2 + b^2 + c^2 \geq ab + bc + ca$;

$a = (1, 3, 5), \quad b = (2, 4, 6) \implies 1\cdot2+3\cdot4+5\cdot6 \geq 1\cdot4+3\cdot6+5\cdot2 \geq 1\cdot6+3\cdot4+5\cdot2 \implies 44 \geq 32 \geq 28.$

3.19 Chebyshev's Sum Inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$ be two sequences that are both increasing (or both decreasing). Then:

$$\frac{a_1b_1 + \cdots + a_nb_n}{n} \geq \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} \geq \frac{a_1b_n + \cdots + a_nb_1}{n},$$

or, more precisely,

$$\frac{\sum_{i=1}^n a_ib_i}{n} \geq \frac{\sum_{i=1}^n a_i}{n} \times \frac{\sum_{i=1}^n b_i}{n} \geq \frac{\sum_{i=1}^n a_ib_{n+1-i}}{n}.$$

Example: $a_1 \leq a_2 \leq a_3,\ b_1 \leq b_2 \leq b_3 \implies a_1b_1+a_2b_2+a_3b_3 \geq \frac{1}{3}(a_1+a_2+a_3)(b_1+b_2+b_3)$.

3.20 Schur's Inequality

Let $a, b, c \geq 0$ and $r > 0$. Then:

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or two of them are equal and the other is zero.

Example: $r = 1 \implies a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(c+a) + c^2(a+b)$.

3.21 Muirhead's Inequality

Let $a_1, a_2, \ldots, a_n \geq 0$ and suppose that (x_n) *majorizes* (y_n), $x \succ y$. Then:

$$\sum_{\text{sym}} a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n} \geq \sum_{\text{sym}} a_1^{y_1} a_2^{y_2} \ldots a_n^{y_n}.$$

Example: $(5,0) \succ (3,2) \implies x^5 + y^5 \geq x^3y^2 + x^2y^3$.

3.22 Nesbitt's Inequality

Let $a, b, c > 0$. Then:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

with equality if and only if $a = b = c$.

Example: $a = 1, b = 2, c = 3 \implies \frac{1}{2+3} + \frac{2}{3+1} + \frac{3}{1+2} \geq \frac{3}{2} \implies \frac{1}{5} + \frac{2}{4} + \frac{3}{3} \geq \frac{3}{2} \implies 1.7 \geq 1.5$.

3.23 Aczel's Inequality

Let $a_1, \ldots, a_n,\ b_1, \ldots, b_n$ be positive real numbers. If $a_1^2 \geq a_2^2 + \cdots + a_n^2$ and $b_1^2 \geq b_2^2 + \cdots + b_n^2$, then:

$$(a_1b_1 - a_2b_2 - \cdots - a_nb_n)^2 \geq (a_1^2 - a_2^2 - \ldots\ a_n^2)(b_1^2 - b_2^2 - \ldots\ b_n^2)$$

with equality if and only if the sequences are proportional.

More precisely,

$$\left(a_1b_1 - \sum_{i=2}^{n} a_ib_i\right)^2 \geq \left(a_1^2 - \sum_{i=2}^{n} a_i^2\right)\left(b_1^2 - \sum_{i=2}^{n} b_i^2\right).$$

Example: $(a) = (6,3,2), (b) = (5,4,1)$

$\implies (6\cdot 5 - 3\cdot 4 - 2\cdot 1)^2 \geq (6^2 - 3^2 - 2^2)(5^2 - 4^2 - 1^2) \implies 16^2 \geq 23\cdot 8 \implies 256 \geq 184.$

3.24 Huygens Inequality

Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $w_1, \ldots, w_n$ be positive real numbers with $w_1 + \cdots + w_n = 1$. Then:

$$\prod_{i=1}^{n}(a_i + b_i)^{w_i} \geq \prod_{i=1}^{n} a_i^{w_i} + \prod_{i=1}^{n} b_i^{w_i}.$$

Example: $(a_i) = (6,11), (b_i) = (13,2), (w_i) = (3/4, 1/4)$

$\implies (6+13)^{3/4}(11+2)^{1/4} \geq 6^{3/4}11^{1/4} + 13^{3/4}2^{1/4} \implies 17.280 \geq 15.123.$

3.25 Heinz Mean Inequality

Let $a, b > 0$ and $0 \leq \nu \leq 1$. Then:

$$\sqrt{ab} \leq \frac{a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}}{2} \leq \frac{a+b}{2}.$$

Example: $a = 9, b = 7, \nu = 0.3$

$\implies \sqrt{4\cdot 17} \leq \frac{4^{0.3}17^{0.7} + 4^{0.7}17^{0.3}}{2} \leq \frac{4+17}{2} \implies 8.246 \leq 8.593 \leq 10.5.$

3.26 Mildorf's Inequality

Let $k \geq -1$ be an integer and $a, b > 0$. Then:

$$\frac{(1+k)(a-b)^2 + 8ab}{4(a+b)} \geq \sqrt[k]{\frac{a^k + b^k}{2}}$$

with equality if and only if $a = b$ or $k \in \{-1, 1\}$, where the power mean $k = 0$ is interpreted as the geometric mean $\sqrt{ab}$. Moreover, the inequality is flipped if $k < -1$.

Example: $a = 22, b = 13, k = 5$

$\implies \frac{(1+5)(22-13)^2 + 8\cdot 22\cdot 13}{4(22+13)} \geq \sqrt[5]{\frac{22^5 + 13^5}{2}} \implies \frac{6\cdot 9^2 + 2{,}288}{4\cdot 35} \geq \sqrt[5]{\frac{5{,}153{,}632 + 371{,}293}{2}}$

$\implies \frac{2774}{140} \geq \sqrt[5]{2{,}762{,}462.5} \implies 19.814 \geq 19.420.$

4 Selected Inequalities

$$(a+b)(b+c)(c+a) \geq 8abc \tag{1}$$

$$\sqrt{1+\sqrt{a}}+\sqrt{1+\sqrt{a+\sqrt{a^2}}}+\ldots+\sqrt{1+\sqrt{a+\ldots+\sqrt{a^n}}}<na, \quad n\geq 2, a\geq 2, n\in\mathbb{N} \tag{2}$$

$$(n!)^2\geq n^n, \quad n\in\mathbb{N} \tag{3}$$

$$\begin{aligned}\frac{1}{3}+\frac{2}{3\cdot 5}+\frac{3}{3\cdot 5\cdot 7}+\ldots+\frac{n}{3\cdot 5\ldots(2n+1)}&<\frac{1}{2}, \quad n\in\mathbb{N}\\ \sum_{k=1}^{n}\frac{k}{\prod_{j=1}^{k}(2j+1)}&<\frac{1}{2}, \quad n\in\mathbb{N}\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{2^3+1}{2^3-1}\cdot\frac{3^3+1}{3^3-1}\cdot\ldots\cdot\frac{n^3+1}{n^3-1}&<\frac{3}{2}, \quad n\geq 2, n\in\mathbb{N}\\ \prod_{k=2}^{n}\frac{k^3+1}{k^3-1}&<\frac{3}{2}, \quad n\geq 2, n\in\mathbb{N}\end{aligned} \tag{5}$$

$$\begin{aligned}1\cdot 1!+2\cdot 2!+3\cdot 3!+\ldots+n\cdot n!&<(n+1)!, \quad n\in\mathbb{N}\\ \sum_{k=1}^{n}k\cdot k!&<(n+1)!, \quad n\in\mathbb{N}\end{aligned} \tag{6}$$

5 Selected Problems

6 Proofs

6.1 Proof of AM-GM Inequality using Induction

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1a_2\ldots a_n}$$

i. Base case is true ($n=2$).

ii. n is true $\implies$ $n+1$ is true.

Proof:

Step 1:

$$\frac{a_1+a_2}{2}\geq\sqrt{a_1a_2} \implies (\sqrt{a_1})^2-2\sqrt{a_1a_2}+(\sqrt{a_2})^2=(\sqrt{a_1}-\sqrt{a_2})^2\geq 0.$$

Step 2:

$$\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \ldots a_n} \implies \\
\frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} &= \frac{n\frac{a_1+\cdots+a_n}{n} + a_{n+1}}{n+1} \\
&\geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&\geq (\sqrt[n]{a_1 \ldots a_n})^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&= \sqrt[n+1]{a_1 \ldots a_n a_{n+1}}
\end{aligned}$$

□

6.2 Proof of AM-GM Inequality using Cauchy Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $2n$ is true.

iii. n is true $\implies$ $n - 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \ldots a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} &= \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}\right) \\
&\geq \frac{\sqrt[n]{a_1 a_2 \ldots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}}{2} \\
&\geq \sqrt[2]{\sqrt[n]{a_1 a_2 \ldots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}} \\
&= \sqrt[2n]{a_1 a_2 \ldots a_{2n}}
\end{aligned}$$

Step 3:

$$\begin{aligned}
\frac{a_1 + a_2 + \dots + a_{n-1} + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} a_n} \implies \\
\frac{a_1 + a_2 + \dots + a_{n-1} + \frac{a_1 + \dots + a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \dots + a_{n-1}}{n-1}} \\
\frac{(n-1)(a_1 + a_2 + \dots + a_{n-1}) + (a_1 + \dots + a_{n-1})}{n \cdot (n-1)} &= \frac{(n-1+1)(a_1 + a_2 + \dots + a_{n-1})}{n \cdot (n-1)} \\
\frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \dots + a_{n-1}}{n-1}} \\
\left(\frac{a_1 + a_2 + \dots + a_{n-1}}{n-1}\right)^n &\geq a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \dots + a_{n-1}}{n-1} \\
\left(\frac{a_1 + a_2 + \dots + a_{n-1}}{n-1}\right)^{n-1} &\geq a_1 a_2 \dots a_{n-1} \\
\frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1 a_2 \dots a_{n-1}}
\end{aligned}$$

$\square$

6.3 Proof of AM-GM Inequality using Jensen's Method

Let $a_1, a_2, \dots, a_n > 0$ and $f(x) = \ln x$ be a *concave* function on $(0, \infty)$. By Jensen's Inequality we have:

$$\begin{aligned}
f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) &\geq \frac{1}{n}\sum_{i=1}^{n} f(a_i) \\
\ln\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \ln(a_2) + \dots + \ln(a_n)}{n} \\
&= \frac{\ln(a_1 a_2 \dots a_n)}{n} \\
&= \ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right) \\
e^{\ln\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)} &\geq e^{\ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right)} \\
\frac{a_1 + a_2 + \dots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_n}
\end{aligned}$$

$\square$