Inequalities Notes

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Contents

1 Introduction

The purpose of these notes is to expose learners to all main inequalities that one may encounter during his or her mathematical journey, while keeping everything very concise and well-structured. Please note that the notes are not finished yet. For any suggestions, contact me on Telegram at abrormaksudov.t.me.

2 Definitions

2.1 Majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences of real numbers. Then x is said to *majorize* y, denoted $x \succ y$, if the following conditions are satisfied:

1. $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n$;
2. $\displaystyle\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$;
3. $\displaystyle\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n-1$.

Example: $(3, 1, 0) \succ (2, 1, 1), \quad (12, 0, 0) \succ (4, 4, 4)$.

2.2 Convex Function

A function $f : [a, b] \to \mathbb{R}$ is called *convex* (concave up) on $[a, b]$ if and only if for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$, the following inequality holds:

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if $f''(x) \geq 0$ ($f''(x) \leq 0$) holds for all $x \in [a, b]$.

Note that f is convex if and only if $-f$ is concave.

Example (convex): x^2, e^x. Example (concave): $\ln x, \sqrt{x}$.

2.3 Cyclic Sum

The *cyclic sum* of a function f over n variables is the sum over *all cyclic permutations* of its arguments:

$$\sum_{\text{cyc}} f(a_1, a_2, \ldots, a_n) = f(a_1, a_2, \ldots, a_n) + f(a_2, a_3, \ldots, a_1) + \cdots + f(a_n, a_1, \ldots, a_{n-1}).$$

The number of terms is equal to the number of variables: n.

Example: $f(a, b, c) = a^2b \implies \sum_{\text{cyc}} = a^2b + b^2c + c^2a.$

2.4 Symmetric Sum

The *symmetric sum* of a function f over n variables is the sum over *all possible permutations* of its arguments:

$$\sum_{\text{sym}} f(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in S_n} f(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}),$$

where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$.

The number of terms equals to the number of all permutations of the variables $= n!$.

Example: $f(a, b, c) = a^2b \implies \sum_{\text{sym}} = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$

2.5 Elementary Symmetric Polynomials

Let t be a variable and $x_1, x_2, \ldots, x_n$ be real numbers. Define:

$$\begin{aligned}
P(x) &= \prod_{i=1}^{n}(t + x_i) = (t + x_1)(t + x_2)\ldots(t + x_n) \\
&= t^n + (x_1 + \cdots + x_n)t^{n-1} + (x_1x_2 + x_1x_3 + \ldots)t^{n-2} + \ldots \\
&+ (x_2x_3\ldots x_n + x_1x_3\ldots x_n + \ldots)t + x_1x_2x_3\ldots x_n \\
&= 1 \cdot t^n + \left(\sum_{1 \le i \le n} x_i\right) t^{n-1} + \left(\sum_{1 \le i < j \le n} x_ix_j\right) t^{n-2} + \ldots \\
&+ \left(\sum_{1 \le i_1 < \cdots < i_{n-1} \le n} x_{i_1}x_{i_2}\ldots x_{i_{n-1}}\right) t + \prod_{i=1}^{n} x_i.
\end{aligned}$$

In other words,

$$P(x) = \prod_{i=1}^{n}(t + x_i) = c_0t^n + c_1t^{n-1} + c_2t^{n-2} + \cdots + c_{n-1}t + c_n,$$

where the coefficient c_k is the k-th elementary symmetric sum:

$$c_0 = 1, \qquad c_1 = \sum_{1 \le i \le n} x_i, \qquad c_2 = \sum_{1 \le i < j \le n} x_ix_j,$$

$$c_3 = \sum_{1 \le i < j < k \le n} x_ix_jx_k, \qquad \ldots, \qquad c_n = \prod_{i=1}^{n} x_i.$$

In general, for $0 \leq k \leq n$

$$c_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies (x+1)(x+2)(x+3) = x^3 + (1+2+3)x^2 + (1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1)x + 1 \cdot 2 \cdot 3 = x^3 + 6x^2 + 11x + 6.$

2.6 Elementary Symmetric Mean

Let $x_1, x_2, \dots, x_n$ be real numbers. The k-th elementary symmetric mean is defined as:

$$d_k = \frac{c_k}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies d_2 = \frac{c_2}{\binom{3}{2}} = \frac{11}{3}$.

3 Inequalities

3.1 AM-GM Inequality

Let $a_1, a_2, \dots, a_n > 0$. Then, the following inequality holds:

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n},$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.
More precisely,

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}.$$

Example: $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

3.2 Weighted AM-GM Inequality

Let $a_1, a_2, \dots, a_n > 0$ and $w_1, w_2, \dots, w_n$ be positive integers. Then, by AM-GM we have:

$$\frac{\underbrace{a_1 + a_1 + \dots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \dots + a_2}_{w_2} + \dots + \underbrace{a_n + a_n + \dots + a_n}_{w_n}}{w_1 + w_2 + \dots + w_n}$$
$$\geq \left(\underbrace{a_1 a_1 \dots a_1}_{w_1} \underbrace{a_2 a_2 \dots a_2}_{w_2} \dots \underbrace{a_n a_n \dots a_n}_{w_n} \right)^{\frac{1}{w_1 + w_2 + \dots + w_n}}.$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \dots + w_n a_n}{w_1 + w_2 + \dots + w_n} \geq (a_1^{w_1} a_2^{w_2} \dots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \dots w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} \geq \left(\prod_{i=1}^{n} a_i^{w_i}\right)^{\frac{1}{\sum_{i=1}^{n} w_i}}$$

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}.$$

Example: $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$.

3.3 Power Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the r-th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(\dfrac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r} & r \neq 0, \\ \sqrt[n]{a_1 a_2 \ldots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1:$ $$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$$ (Harmonic Mean)
- $r = 0:$ $$\sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$ (Geomteric Mean)
- $r = 1:$ $$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n} \sum_{i=1}^{n} a_i$$ (Arithmetic Mean)
- $r = 2:$ $$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}$$ (Quadratic Mean)

If $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$.

3.4 Weighted Power Mean Inequality

Let $a_1, a_2, \ldots a_n > 0$ and $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$. Then, the r-th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r)^{1/r} & r \neq 0, \\ a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $(\frac{1}{6}a^3 + \frac{1}{3}b^3 + \frac{1}{2}c^3)^{1/3} \geq a^{1/6}b^{1/3}c^{1/2}$.

3.5 HM-GM-AM-QM Inequalities

Let $a_1, a_2, \ldots, a_n > 0$. Then:

$$0 < \text{HM} \leq \text{GM} \leq \text{AM} \leq \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^n \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^n a_i} \leq \frac{1}{n}\sum_{i=1}^n a_i \leq \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}}.$$

Example: $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$.

3.6 Bernoulli's Inequality

For all $x \geq -1$ and $r \geq 1$:

$$(1+x)^r \geq 1 + rx.$$

Example: $(1+x)^5 \geq 1 + 5x$.

3.7 Jensen's Inequality

If f is *convex*, then:

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

with equality if and only if f is *linear* or $a_1 = a_2 = \cdots = a_n$.

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^n w_i f(a_i) \geq f\left(\sum_{i=1}^n w_i a_i\right).$$

The inequality is reversed if f is concave.

Example: $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x}+\sqrt{y}}{2}$.

3.8 Karamata's Inequality

If f is *convex*, and (a_i) *majorizes* (b_i), then:

$$f(a_1) + f(a_2) + \dots f(a_n) \geq f(b_1) + f(b_2) + \dots f(b_n),$$

or, more precisely,

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

The inequality is reversed if f is concave.

Example: $f(x) = x^2 \implies (4)^2 + (1)^2 \geq (2.5)^2 + (2.5)^2 \implies 17 \geq 12.5.$

3.9 Popoviciu's Inequality

If f is *convex*, and $a, b, c > 0$, then:

$$af(x) + bf(y) + cf(z) + (a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right) \geq$$
$$(a+b)f\left(\frac{ax+by}{a+b}\right) + (b+c)f\left(\frac{by+cz}{b+c}\right) + (c+a)f\left(\frac{cz+ax}{c+a}\right)$$

Particularly, if $a = b = c = 1$, we have:

$$\frac{f(x) + f(y) + f(c)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right].$$

Equality holds if and only if f is *linear* or $x = y = z$.

Example: $f(x) = x^2 \implies \frac{(1)^2+(2)^2+(3)^2}{3} + \left(\frac{1+2+3}{3}\right)^2 \geq \frac{2}{3}\left[\left(\frac{1+2}{2}\right)^2 + \left(\frac{2+3}{2}\right)^2 + \left(\frac{3+1}{2}\right)^2\right] \implies \frac{26}{3} \geq \frac{25}{3}$.

3.10 Newton's Inequality

For $x_1, x_2, \dots, x_n > 0$ and $k = 1, 2, \dots, n-1$, we have:

$$d_i^2 \geq d_{i-1}d_{i+1},$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Example: $x = 1, y = 2, z = 3 \implies (\frac{xy+yz+zx}{3})^2 \geq (\frac{x+y+z}{3}) \cdot xyz$

$\implies \left(\frac{1\cdot2+2\cdot3+3\cdot1}{3}\right)^2 \geq \frac{1+2+3}{3}(1\cdot2\cdot3) \implies \left(\frac{11}{3}\right)^2 \geq 2\cdot6 \implies 13.444 \geq 12.$

3.11 Maclaurin's Inequality

For $x_1, x_2, \dots, x_n > 0$, we have:

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \dots \geq \sqrt[n]{d_n}$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Equivalently, it can be written as:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt{\frac{\sum_{1\leq i<j\leq n} x_i x_j}{\binom{n}{2}}} \geq \sqrt[3]{\frac{\sum_{1\leq i<j<k\leq n} x_i x_j x_k}{\binom{n}{3}}} \geq \cdots \geq \sqrt[n]{x_1 x_2 \dots x_n}.$$

Example: $x = 1, y = 2, z = 3 \implies \frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz}$

$\implies \frac{1+2+3}{3} \geq \sqrt{\frac{1\cdot 2+2\cdot 3+3\cdot 1}{3}} \geq \sqrt[3]{1\cdot 2\cdot 3} \implies 2 \geq \frac{11}{3} \geq \sqrt[3]{6} \implies 2 \geq 1.915 \geq 1.817.$

3.12 Cauchy–Schwarz Inequality

Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ be real numbers. Then:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \dots a_n b_n)^2,$$

with equality if and only if there is a contant $\lambda \in \mathbb{R}$ such that $a_i = \lambda b_i$ for all $1 \leq i \leq n$. That is, if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = \lambda$.

More precisely,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Example: $(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2 \implies (2^2 + 3^2)(4^2 + 5^2) \geq (2\cdot 4 + 3\cdot 5)^2$

$\implies 13\cdot 41 \geq 23^2 \implies 533 \geq 529.$

3.13 Titu's Lemma/Sedrakyan's Inequality/Engel's Form

Let $a_1, a_2, \dots, a_n \geq 0$ and $b_1, b_2, \dots, b_n > 0$. Then:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

Example: $\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$.

3.14 Hölder's Inequality

Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$, $\dots$, $z_1, z_2, \dots, z_n$ be positive real numbers, and let $\lambda_a, \lambda_b, \dots, \lambda_z$ be positive reals with $\lambda_a + \lambda_b + \cdots + \lambda_z = 1$. Then:

$$(a_1 + \cdots + a_n)^{\lambda_a}(b_1 + \cdots + b_n)^{\lambda_b} \dots (z_1 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \dots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \dots z_n^{\lambda_z},$$

or, more precisely,

$$\underbrace{\left(\sum_{i=1}^{n} a_i\right)^{\lambda_a}\left(\sum_{i=1}^{n} b_i\right)^{\lambda_b}\cdots\left(\sum_{i=1}^{n} z_i\right)^{\lambda_z}}_{m\text{ factors}} \geq \sum_{i=1}^{n}\left(\underbrace{a_i^{\lambda_a} b_i^{\lambda_b}\cdots z_i^{\lambda_z}}_{m\text{ variables}}\right)$$

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right)^{\lambda_j} \geq \sum_{i=1}^{n}\left(\prod_{j=1}^{m} a_{ij}^{\lambda_j}\right).$$

Example: $m = 3, n = 2, \lambda_a = 0.5, \lambda_b = 0.3, \lambda_c = 0.2, (a) = (1,3), (b) = (2,4), (c) = (5,6)$

$\implies (a_1 + a_2)^{0.5}(b_1 + b_2)^{0.3}(c_1 + c_2)^{0.2} \geq a_1^{0.5}b_1^{0.3}c_1^{0.2} + a_2^{0.5}b_2^{0.3}c_2^{0.2}$

$\implies (1+3)^{0.5}(2+4)^{0.3}(5+6)^{0.2} \geq 1^{0.5}2^{0.3}5^{0.2} + 3^{0.5}4^{0.3}6^{0.2} \implies 5.53 \geq 5.45.$

3.15 Minkowski Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be positive real numbers and $p > 1$. Then:

$$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} + (b_1^p + b_2^p + \cdots + b_n^p)^{\frac{1}{p}} \geq ((a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p)^{\frac{1}{p}},$$

or, more precisely,

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}}$$

Example: $p = 2, a = (3,4), b = (6,8)$

$\implies \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$

$\implies \sqrt{3^2 + 4^2} + \sqrt{6^2 + 8^2} \geq \sqrt{(3+6)^2 + (4+8)^2}$

$\implies \sqrt{25} + \sqrt{100} \geq \sqrt{225} \implies 15 \geq 15.$ (equal. why?)

3.16 Generalized Minkowski Inequality

Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, $\ldots$, $z_1, z_2, \ldots, z_n$ be positive real numbers, and $p > 1$. Then:

$$\underbrace{\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} + \cdots + \left(\sum_{i=1}^{n} z_i^p\right)^{\frac{1}{p}}}_{m\text{ terms}} \geq \left(\sum_{i=1}^{n}\underbrace{(a_i + b_i + \cdots + z_i)^p}_{m\text{ terms}}\right)^{\frac{1}{p}}.$$

More precisely,

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^p\right)^{\frac{1}{p}} \geq \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)^p\right]^{\frac{1}{p}}$$

3.17 Young's Inequality

Let $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

with equality if and only if $a^p = b^q$.

Moreover, for increasing functions

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab$$

with equality if and only if $f(a) = b$.

Example: $a = 2, b = 3, p = 3, q = \frac{3}{2} \implies \frac{2^3}{3} + \frac{3^{3/2}}{3/2} \geq 2 \cdot 3 \implies 6.13 \geq 6$.

3.18 Rearrangement Inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$ be two sequences that are both increasing (or both decreasing). Then:

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{\sigma(1)} + a_1b_{\sigma(2)} + \cdots + a_1b_{\sigma(n)} \geq a_1b_n + a_2b_{n-1} + \ldots a_nb_1,$$

where σ is a permutation function, which sends each of $1, 2, \ldots, n$ to a different value in $\{1, 2, \ldots, n\}$.

More precisely,

$$\sum_{i=1}^{n} a_ib_i \geq \sum_{i=1}^{n} a_ib_{\sigma(i)} \geq \sum_{i=1}^{n} a_ib_{n+1-i}.$$

In other words, the sum is *maximized* when both sequences are ordered *similarly* (both increasing or both decreasing), and is *minimazied* when both sequences are ordered *oppositely* (one increasing, the other decreasing).

Example: $a^2 + b^2 + c^2 \geq ab + bc + ca$;

$a = (1, 3, 5), \quad b = (2, 4, 6) \implies 1\cdot2+3\cdot4+5\cdot6 \geq 1\cdot4+3\cdot6+5\cdot2 \geq 1\cdot6+3\cdot4+5\cdot2 \implies 44 \geq 32 \geq 28$.

3.19 Chebyshev's Sum Inequality

Let $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$ be two sequences that are both increasing (or both decreasing). Then:

$$\frac{a_1b_1 + \cdots + a_nb_n}{n} \geq \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} \geq \frac{a_1b_n + \cdots + a_nb_1}{n},$$

or, more precisely,

$$\frac{\sum_{i=1}^n a_ib_i}{n} \geq \frac{\sum_{i=1}^n a_i}{n} \times \frac{\sum_{i=1}^n b_i}{n} \geq \frac{\sum_{i=1}^n a_ib_{n+1-i}}{n}.$$

Example: $a_1 \leq a_2 \leq a_3,\ b_1 \leq b_2 \leq b_3 \implies a_1b_1{+}a_2b_2{+}a_3b_3 \geq \frac{1}{3}(a_1{+}a_2{+}a_3)(b_1{+}b_2{+}b_3)$.

3.20 Schur's Inequality

Let $a, b, c \geq 0$ and $r > 0$. Then:

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or two of them are equal and the other is zero.

Example: $r = 1 \implies a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(c+a) + c^2(a+b)$.

3.21 Muirhead's Inequality

Let $a_1, a_2, \ldots, a_n \geq 0$ and suppose that (x_n) *majorizes* (y_n), $x \succ y$. Then:

$$\sum_{\text{sym}} a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n} \geq \sum_{\text{sym}} a_1^{y_1} a_2^{y_2} \ldots a_n^{y_n}.$$

Example: $(5,0) \succ (3,2) \implies x^5 + y^5 \geq x^3y^2 + x^2y^3$.

3.22 Nesbitt's Inequality

Let $a, b, c > 0$. Then:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

with equality if and only if $a = b = c$.

Example: $a = 1, b = 2, c = 3 \implies \frac{1}{2+3} + \frac{2}{3+1} + \frac{3}{1+2} \geq \frac{3}{2} \implies \frac{1}{5} + \frac{2}{4} + \frac{3}{3} \geq \frac{3}{2} \implies 1.7 \geq 1.5$.

3.23 Aczel's Inequality

Let $a_1, \ldots, a_n,\ b_1, \ldots, b_n$ be positive real numbers. If $a_1^2 \geq a_2^2 + \cdots + a_n^2$ and $b_1^2 \geq b_2^2 + \cdots + b_n^2$, then:

$$(a_1b_1 - a_2b_2 - \cdots - a_nb_n)^2 \geq (a_1^2 - a_2^2 - \ldots\ a_n^2)(b_1^2 - b_2^2 - \ldots\ b_n^2)$$

with equality if and only if the sequences are proportional.

More precisely,

$$\left(a_1b_1 - \sum_{i=2}^{n} a_ib_i\right)^2 \geq \left(a_1^2 - \sum_{i=2}^{n} a_i^2\right)\left(b_1^2 - \sum_{i=2}^{n} b_i^2\right).$$

Example: $(a) = (6, 3, 2), (b) = (5, 4, 1)$

$\implies (6 \cdot 5 - 3 \cdot 4 - 2 \cdot 1)^2 \geq (6^2 - 3^2 - 2^2)(5^2 - 4^2 - 1^2) \implies 16^2 \geq 23 \cdot 8 \implies 256 \geq 184.$

3.24 Huygens Inequality

Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $w_1, \ldots, w_n$ be positive real numbers with $w_1 + \cdots + w_n = 1$. Then:

$$\prod_{i=1}^{n}(a_i + b_i)^{w_i} \geq \prod_{i=1}^{n} a_i^{w_i} + \prod_{i=1}^{n} b_i^{w_i}.$$

Example: $(a_i) = (6, 11), (b_i) = (13, 2), (w_i) = (3/4, 1/4)$

$\implies (6 + 13)^{3/4}(11 + 2)^{1/4} \geq 6^{3/4}11^{1/4} + 13^{3/4}2^{1/4} \implies 17.280 \geq 15.123.$

3.25 Heinz Mean Inequality

Let $a, b > 0$ and $0 \leq \nu \leq 1$. Then:

$$\sqrt{ab} \leq \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2} \leq \frac{a + b}{2}.$$

Example: $a = 9, b = 7, \nu = 0.3$

$\implies \sqrt{4 \cdot 17} \leq \frac{4^{0.3}17^{0.7}+4^{0.7}17^{0.3}}{2} \leq \frac{4+17}{2} \implies 8.246 \leq 8.593 \leq 10.5.$

3.26 Mildorf's Inequality

Let $k \geq -1$ be an integer and $a, b > 0$. Then:

$$\frac{(1 + k)(a - b)^2 + 8ab}{4(a + b)} \geq \sqrt[k]{\frac{a^k + b^k}{2}}$$

with equality if and only if $a = b$ or $k \in \{-1, 1\}$, where the power mean $k = 0$ is interpreted as the geometric mean $\sqrt{ab}$. Moreover, the inequality is flipped if $k < -1$.

Example: $a = 22, b = 13, k = 5$

$\implies \frac{(1+5)(22-13)^2+8 \cdot 22 \cdot 13}{4(22+13)} \geq \sqrt[5]{\frac{22^5+13^5}{2}} \implies \frac{6 \cdot 9^2+2{,}288}{4 \cdot 35} \geq \sqrt[5]{\frac{5{,}153{,}632+371{,}293}{2}}$

$\implies \frac{2774}{140} \geq \sqrt[5]{2, 762, 462.5} \implies 19.814 \geq 19.420.$

4 Selected Inequalities

$$(a + b)(b + c)(c + a) \geq 8abc \tag{1}$$

$$\sqrt{1+\sqrt{a}}+\sqrt{1+\sqrt{a+\sqrt{a^2}}}+\ldots+\sqrt{1+\sqrt{a+\ldots+\sqrt{a^n}}}<na, \quad n\geq 2, a\geq 2, n\in\mathbb{N} \tag{2}$$

$$(n!)^2 \geq n^n, \quad n\in\mathbb{N} \tag{3}$$

$$\begin{gathered}\frac{1}{3}+\frac{2}{3\cdot 5}+\frac{3}{3\cdot 5\cdot 7}+\ldots+\frac{n}{3\cdot 5\ldots(2n+1)}<\frac{1}{2}, \quad n\in\mathbb{N}\\ \sum_{k=1}^{n}\frac{k}{\prod_{j=1}^{k}(2j+1)}<\frac{1}{2}, \quad n\in\mathbb{N}\end{gathered} \tag{4}$$

$$\begin{gathered}\frac{2^3+1}{2^3-1}\cdot\frac{3^3+1}{3^3-1}\cdot\ldots\cdot\frac{n^3+1}{n^3-1}<\frac{3}{2}, \quad n\geq 2, n\in\mathbb{N}\\ \prod_{k=2}^{n}\frac{k^3+1}{k^3-1}<\frac{3}{2}, \quad n\geq 2, n\in\mathbb{N}\end{gathered} \tag{5}$$

$$\begin{gathered}1\cdot 1!+2\cdot 2!+3\cdot 3!+\ldots+n\cdot n!<(n+1)!, \quad n\in\mathbb{N}\\ \sum_{k=1}^{n}k\cdot k!<(n+1)!, \quad n\in\mathbb{N}\end{gathered} \tag{6}$$

$$\begin{gathered}\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\ldots\left(1+\frac{1}{n^2}\right)<2, \quad n\geq 2, n\in\mathbb{N}\\ \prod_{k=2}^{n}\left(1+\frac{1}{k^2}\right)<2, \quad n\geq 2, n\in\mathbb{N}\end{gathered} \tag{7}$$

$$x^8+y^8\geq\frac{1}{128}, \quad x+y=1 \tag{8}$$

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2\geq 12.5, \quad x,y>0,\, x+y=1 \tag{9}$$

$$\begin{gathered}\left(x_1+\frac{1}{x_1}\right)^2+\left(x_2+\frac{1}{x_2}\right)^2+\cdots+\left(x_n+\frac{1}{x_n}\right)^2\geq n\left(n+\frac{1}{n}\right)^2, \quad x_k>0,\, \sum x_k=1\\ \sum_{k=1}^{n}\left(x_k+\frac{1}{x_k}\right)^2\geq n\left(n+\frac{1}{n}\right)^2, \quad x_k>0,\, \sum x_k=1\end{gathered} \tag{10}$$

$$n!\leq\left(\frac{n+1}{2}\right)^n, \quad n\in\mathbb{N} \tag{11}$$

$$\begin{gathered}\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}\geq n, \quad a_k>0\\ \sum_{k=1}^{n}\frac{a_k}{a_{k+1}}\geq n, \quad a_k>0,\, a_{n+1}:=a_1\end{gathered} \tag{12}$$

$$\sqrt{a_1b_1}+\sqrt{a_2b_2}+\cdots+\sqrt{a_nb_n}\le\sqrt{a_1+a_2+\cdots+a_n}\cdot\sqrt{b_1+b_2+\cdots+b_n},\quad a_k,b_k>0$$
$$\sum_{k=1}^{n}\sqrt{a_kb_k}\le\sqrt{\sum_{k=1}^{n}a_k}\cdot\sqrt{\sum_{k=1}^{n}b_k},\quad a_k,b_k>0 \tag{13}$$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\ge n\sqrt{\frac{2}{n+1}},\quad n\in\mathbb{N}$$
$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\ge n\sqrt{\frac{2}{n+1}},\quad n\in\mathbb{N} \tag{14}$$

$$\sqrt{a+\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}\le\frac{1+\sqrt{1+4a}}{2},\quad a\ge 0 \tag{15}$$

$$\sqrt[n]{n}>\sqrt[n+1]{n+1},\quad n\ge 3 \tag{16}$$

$$1^1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n>(2n)!,\quad n\ge 5,n\in\mathbb{N}$$
$$\prod_{k=1}^{n}k^k>(2n)!,\quad n\ge 5,n\in\mathbb{N} \tag{17}$$

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}>\frac{2}{3}n\sqrt{n},\quad n\in\mathbb{N}$$
$$\sum_{k=1}^{n}\sqrt{k}>\frac{2}{3}n\sqrt{n},\quad n\in\mathbb{N} \tag{18}$$

$$e^x\ge x^e,\quad x\ge e \tag{19}$$

$$\sqrt{2\sqrt[3]{3\sqrt[4]{4\sqrt[5]{5\ldots\sqrt[n]{n}}}}}<2,\quad n\ge 2,n\in\mathbb{N}$$
$$2^{\frac{1}{2}}\cdot 3^{\frac{1}{2\cdot 3}}\cdot 4^{\frac{1}{2\cdot 3\cdot 4}}\cdot\ldots\cdot n^{\frac{1}{2\cdot 3\cdot 4\cdot\ldots\cdot n}}=\prod_{k=2}^{n}k^{1/k!}<2,\quad n\ge 2,n\in\mathbb{N} \tag{20}$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\cdots+\sqrt{n}}}}}<2,\quad n\in\mathbb{N} \tag{21}$$

5 Selected Problems

1. If x is a positive real number, show that
$$x + \frac{1}{x} \geq 2.$$

2. If x, y and z are non-negative real numbers, show that
$$x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy} \leq xy + yz + zx.$$

3. If a, b, c, d are positive real numbers, show that
$$\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}.$$

4. If $x_1, x_2, \ldots, x_n$ are n positive real numbers with $x_1 \cdot x_2 \cdot \ldots \cdot x_n = 1$, show that
$$(1+x_1)(1+x_2)\ldots(1+x_n) \geq 2^n.$$

5. For n non-negative numbers $a_1, a_2, \ldots, a_n$, show that
$$\sqrt{a_1a_2} + \sqrt{a_2a_3} + \ldots + \sqrt{a_{n-1}a_n} + \sqrt{a_na_1} \leq a_1 + a_2 + \ldots + a_n.$$

6 Proofs

6.1 Proof of AM-GM Inequality using Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1a_2\ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies$ $n+1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1\ldots a_n} \implies \\
\frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} &= \frac{n\frac{a_1+\cdots+a_n}{n} + a_{n+1}}{n+1} \\
&\geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&\geq (\sqrt[n]{a_1\ldots a_n})^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\
&= \sqrt[n+1]{a_1\ldots a_na_{n+1}}
\end{aligned}$$

□

6.2 Proof of AM-GM Inequality using Cauchy Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies 2n$ is true.

iii. n is true $\implies n - 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned}
\frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \ldots a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} &= \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}\right) \\
&\geq \frac{\sqrt[n]{a_1 a_2 \ldots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}}{2} \\
&\geq \sqrt[2]{\sqrt[n]{a_1 a_2 \ldots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}} \\
&= \sqrt[2n]{a_1 a_2 \ldots a_{2n}}
\end{aligned}$$

Step 3:

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + \cdots + a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\frac{(n-1)(a_1 + a_2 + \cdots + a_{n-1}) + (a_1 + \cdots + a_{n-1})}{n \cdot (n-1)} &= \frac{(n - 1 + 1)(a_1 + a_2 + \cdots + a_{n-1})}{n \cdot (n-1)} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^n &\geq a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^{n-1} &\geq a_1 a_2 \ldots a_{n-1} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}}
\end{aligned}$$

□

6.3 Proof of AM-GM Inequality using Jensen's Method

Let $a_1, a_2, \ldots, a_n > 0$ and $f(x) = \ln x$ be a *concave* function on $(0, \infty)$. By Jensen's

Inequality we have:

$$
\begin{aligned}
f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) &\geq \frac{1}{n}\sum_{i=1}^{n} f(a_i) \\
\ln\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \ln(a_2) + \dots + \ln(a_n)}{n} \\
&= \frac{\ln(a_1 a_2 \dots a_n)}{n} \\
&= \ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right) \\
e^{\ln\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)} &\geq e^{\ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right)} \\
\frac{a_1 + a_2 + \dots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_n}
\end{aligned}
$$

$\square$