

Inequalities Notes

Abror Maksudov

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1 Introduction

2 Definitions

2.1 Majorization

Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be non-increasing sequences of real numbers. Then x is said to *majorize* y , denoted $x \succ y$, if the following conditions are satisfied:

1. $x_1 \geq x_2 \geq \dots \geq x_n$ and $y_1 \geq y_2 \geq \dots \geq y_n$;
2. $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$;
3. $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$ for all $k = 1, 2, \dots, n-1$.

Example: $(3, 1, 0) \succ (2, 1, 1)$, $(12, 0, 0) \succ (4, 4, 4)$.

2.2 Convex Function

A function $f : [a, b] \rightarrow \mathbb{R}$ is called *convex* (concave up) on $[a, b]$ if and only if for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$, the following inequality holds:

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if $f''(x) \geq 0$ ($f''(x) \leq 0$) holds for all $x \in [a, b]$.

Note that f is convex if and only if $-f$ is concave.

Example (convex): x^2, e^x . Example (concave): $\ln x, \sqrt{x}$.

2.3 Elementary Symmetric Polynomials

Let t be a variable and x_1, x_2, \dots, x_n be real numbers. Define:

$$\begin{aligned}
 P(x) &= \prod_{i=1}^n (t + x_i) = (t + x_1)(t + x_2) \dots (t + x_n) \\
 &= t^n + (x_1 + \dots + x_n)t^{n-1} + (x_1x_2 + x_1x_3 + \dots)t^{n-2} + \dots \\
 &\quad + (x_2x_3 \dots x_n + x_1x_3 \dots x_n + \dots)t + x_1x_2x_3 \dots x_n \\
 &= 1 \cdot t^n + \left(\sum_{1 \leq i \leq n} x_i \right) t^{n-1} + \left(\sum_{1 \leq i < j \leq n} x_i x_j \right) t^{n-2} + \dots \\
 &\quad + \left(\sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} x_{i_1} x_{i_2} \dots x_{i_{n-1}} \right) t + \prod_{i=1}^n x_i.
 \end{aligned}$$

In other words,

$$P(x) = \prod_{i=1}^n (t + x_i) = c_0 t^n + c_1 t^{n-1} + c_2 t^{n-2} + \dots + c_{n-1} t + c_n,$$

where the coefficient c_k is the k -th elementary symmetric sum:

$$\begin{aligned}
 c_0 &= 1, & c_1 &= \sum_{1 \leq i \leq n} x_i, & c_2 &= \sum_{1 \leq i < j \leq n} x_i x_j, \\
 c_3 &= \sum_{1 \leq i < j < k \leq n} x_i x_j x_k, & \dots, & & c_n &= \prod_{i=1}^n x_i.
 \end{aligned}$$

In general, for $0 \leq k \leq n$

$$c_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies (x+1)(x+2)(x+3) = x^3 + (1+2+3)x^2 + (1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3)x + 1 \cdot 2 \cdot 3 = x^3 + 6x^2 + 11x + 6$.

2.4 Elementary Symmetric Mean

Let x_1, x_2, \dots, x_n be real numbers. The k -th elementary symmetric mean is defined as:

$$d_k = \frac{c_k}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Example: $x_1 = 1, x_2 = 2, x_3 = 3 \implies d_2 = \frac{c_2}{\binom{3}{2}} = \frac{11}{3}$.

3 Inequalities

3.1 AM-GM Inequality

Let $a_1, a_2, \dots, a_n > 0$. Then, the following inequality holds:

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n},$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

More precisely,

$$\frac{1}{n} \sum_{i=1}^n a_i \geq \sqrt[n]{\prod_{i=1}^n a_i}.$$

Example: $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

3.2 Weighted AM-GM Inequality

Let $a_1, a_2, \dots, a_n > 0$ and w_1, w_2, \dots, w_n be positive integers. Then, by AM-GM we have:

$$\begin{aligned} & \frac{\underbrace{a_1 + a_1 + \dots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \dots + a_2}_{w_2} + \dots + \underbrace{a_n + a_n + \dots + a_n}_{w_n}}{w_1 + w_2 + \dots + w_n} \\ & \geq \left(\underbrace{a_1 a_1 \dots a_1}_{w_1} \underbrace{a_2 a_2 \dots a_2}_{w_2} \dots \underbrace{a_n a_n \dots a_n}_{w_n} \right)^{\frac{1}{w_1 + w_2 + \dots + w_n}}. \end{aligned}$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \dots + w_n a_n}{w_1 + w_2 + \dots + w_n} \geq (a_1^{w_1} a_2^{w_2} \dots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \dots + w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^n w_i a_i}{\sum_{i=1}^n w_i} \geq \left(\prod_{i=1}^n a_i^{w_i} \right)^{\frac{1}{\sum_{i=1}^n w_i}}$$

If we let $w_1, w_2, \dots, w_n \geq 0$ with $w_1 + w_2 + \dots + w_n = 1$, we have:

$$w_1 a_1 + w_2 a_2 + \dots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \dots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^n w_i a_i \geq \prod_{i=1}^n a_i^{w_i}.$$

Example: $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$.

3.3 Power Mean Inequality

Let $a_1, a_2, \dots, a_n > 0$. Then, the r -th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(\frac{a_1^r + \dots + a_n^r}{n} \right)^{1/r} & r \neq 0, \\ \sqrt[n]{a_1 a_2 \dots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1$: $\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} = \frac{n}{\sum_{i=1}^n \frac{1}{a_i}}$ (Harmonic Mean)
- $r = 0$: $\sqrt[n]{a_1 a_2 \dots a_n} = \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}}$ (Geometric Mean)
- $r = 1$: $\frac{a_1 + a_2 + \dots + a_n}{n} = \frac{1}{n} \sum_{i=1}^n a_i$ (Arithmetic Mean)
- $r = 2$: $\sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}}$ (Quadratic Mean)

If $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

Example: $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$.

3.4 Weighted Power Mean Inequality

Let $a_1, a_2, \dots, a_n > 0$ and $w_1, w_2, \dots, w_n \geq 0$ with $w_1 + w_2 + \dots + w_n = 1$. Then, the r -th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \dots + w_n a_n^r)^{1/r} & r \neq 0, \\ a_1^{w_1} a_2^{w_2} \dots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

Example: $(\frac{1}{6}a^3 + \frac{1}{3}b^3 + \frac{1}{2}c^3)^{1/3} \geq a^{1/6}b^{1/3}c^{1/2}$.

3.5 HM-GM-AM-QM Inequalities

Let $a_1, a_2, \dots, a_n > 0$. Then:

$$0 < \text{HM} \leq \text{GM} \leq \text{AM} \leq \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^n \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^n a_i} \leq \frac{1}{n} \sum_{i=1}^n a_i \leq \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}}.$$

Example: $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}.$

3.6 Bernoulli's Inequality

For all $x \geq -1$ and $r \geq 1$:

$$(1+x)^r \geq 1+rx.$$

Example: $(1+x)^5 \geq 1+5x.$

3.7 Jensen's Inequality

If f is *convex*, then:

$$\frac{f(a_1) + f(a_2) + \dots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

with equality if and only if f is *linear* or $a_1 = a_2 = \dots = a_n$.

If we let $w_1, w_2, \dots, w_n \geq 0$ with $w_1 + w_2 + \dots + w_n = 1$, we have:

$$w_1 f(a_1) + w_2 f(a_2) + \dots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \dots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^n w_i f(a_i) \geq f\left(\sum_{i=1}^n w_i a_i\right).$$

The inequality is reversed if f is concave.

Example: $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x}+\sqrt{y}}{2}.$

3.8 Karamata's Inequality

If f is *convex*, and (a_i) *majorizes* (b_i) , then:

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n),$$

or, more precisely,

$$\sum_{i=1}^n f(a_i) \geq \sum_{i=1}^n f(b_i).$$

The inequality is reversed if f is *concave*.

Example: $f(x) = x^2 \implies (4)^2 + (1)^2 \geq (2.5)^2 + (2.5)^2 \implies 17 \geq 12.5$.

3.9 Popoviciu's Inequality

If f is *convex*, and $a, b, c > 0$, then:

$$\begin{aligned} &af(x) + bf(y) + cf(z) + (a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right) \geq \\ &(a+b)f\left(\frac{ax+by}{a+b}\right) + (b+c)f\left(\frac{by+cz}{b+c}\right) + (c+a)f\left(\frac{cz+ax}{c+a}\right) \end{aligned}$$

Particularly, if $a = b = c = 1$, we have:

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3} \left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right].$$

Equality holds if and only if f is *linear* or $x = y = z$.

Example: $f(x) = x^2 \implies \frac{(1)^2 + (2)^2 + (3)^2}{3} + \left(\frac{1+2+3}{3}\right)^2 \geq \frac{2}{3} \left[\left(\frac{1+2}{2}\right)^2 + \left(\frac{2+3}{2}\right)^2 + \left(\frac{3+1}{2}\right)^2 \right] \implies \frac{26}{3} \geq \frac{25}{3}$.

3.10 Newton's Inequality

For $x_1, x_2, \dots, x_n > 0$ and $k = 1, 2, \dots, n-1$, we have:

$$d_k^2 \geq d_{k-1}d_{k+1},$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Example: $a = 1, b = 2, c = 3 \implies \left(\frac{ab+bc+ca}{3}\right)^2 \geq \left(\frac{a+b+c}{3}\right)abc \implies \left(\frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1}{3}\right)^2 \geq \frac{1+2+3}{3}(1 \cdot 2 \cdot 3) \implies \left(\frac{11}{3}\right)^2 \geq 2 \cdot 6 \implies 13.444 \geq 12$.

3.11 Maclaurin's Inequality

For $x_1, x_2, \dots, x_n > 0$, we have:

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \dots \geq \sqrt[n]{d_n}$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Example: $a = 1, b = 2, c = 3 \implies \frac{a+b+c}{3} \geq \sqrt{\frac{ab+bc+ca}{3}} \geq \sqrt[3]{abc} \implies \frac{1+2+3}{3} \geq \sqrt{\frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1}{3}} \geq \sqrt[3]{1 \cdot 2 \cdot 3} \implies 2 \geq \frac{11}{3} \geq \sqrt[3]{6} \implies 2 \geq 1.915 \geq 1.817.$

3.12 Young's Inequality

3.13 Cauchy–Schwarz Inequality

3.14 Muirhead's Inequality

3.15 Hölder's Inequality

3.16 Minkowski Inequality

3.17 Generalized Minkowski Inequality

3.18 Chebyshev's Sum Inequality

3.19 Rearrangement Inequality

3.20 Schur's Inequality

3.21 Generalized Schur's Inequality

3.22 Aczel's Inequality

3.23 Huygens Inequality

3.24 Heinz Inequality

3.25 Nesbitt's Inequality

3.26 Cesàro's Inequality

3.27 Mildorf's Inequality

4 Selected Inequalities

5 Proofs

5.1 Proof of AM-GM Inequality using Induction

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies n + 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned} \frac{a_1 + \dots + a_n}{n} &\geq \sqrt[n]{a_1 \dots a_n} \implies \\ \frac{a_1 + \dots + a_n + a_{n+1}}{n+1} &= \frac{n \frac{a_1 + \dots + a_n}{n} + a_{n+1}}{n+1} \\ &\geq \left(\frac{a_1 + \dots + a_n}{n} \right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\ &\geq \left(\sqrt[n]{a_1 \dots a_n} \right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\ &= \sqrt[n+1]{a_1 \dots a_n a_{n+1}} \end{aligned}$$

□

5.2 Proof of AM-GM Inequality using Forward-Backward Induction (a.k.a. Cauchy Induction)

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}$$

- i. Base case is true ($n = 2$).
- ii. n is true $\implies 2n$ is true.
- iii. n is true $\implies n - 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned} \frac{a_1 + \dots + a_n}{n} &\geq \sqrt[n]{a_1 \dots a_n} \implies \\ \frac{a_1 + a_2 + \dots + a_{2n}}{2n} &= \frac{1}{2} \left(\frac{a_1 + a_2 + \dots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \dots + a_{2n}}{n} \right) \\ &\geq \frac{\sqrt[n]{a_1 a_2 \dots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}}{2} \\ &\geq \sqrt[2]{\sqrt[n]{a_1 a_2 \dots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}} \\ &= \sqrt[2n]{a_1 a_2 \dots a_{2n}} \end{aligned}$$

Step 3:

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} a_n} \implies \\
\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + \cdots + a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\frac{(n-1)(a_1 + a_2 + \cdots + a_{n-1}) + (a_1 + \cdots + a_{n-1})}{n \cdot (n-1)} &= \frac{(n-1+1)(a_1 + a_2 + \cdots + a_{n-1})}{n \cdot (n-1)} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \right)^n &\geq a_1 a_2 \cdots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \\
\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \right)^{n-1} &\geq a_1 a_2 \cdots a_{n-1} \\
\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}
\end{aligned}$$

□

5.3 Proof of AM-GM Inequality using Jensen's Method

Let $a_1, a_2, \dots, a_n > 0$ and $f(x) = \ln x$ be a *concave* function on $(0, \infty)$. By Jensen's Inequality we have:

$$\begin{aligned}
f\left(\frac{1}{n} \sum_{i=1}^n a_i\right) &\geq \frac{1}{n} \sum_{i=1}^n f(a_i) \\
\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \ln(a_2) + \cdots + \ln(a_n)}{n} \\
&= \frac{\ln(a_1 a_2 \cdots a_n)}{n} \\
&= \ln\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right) \\
e^{\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)} &\geq e^{\ln\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right)} \\
\frac{a_1 + a_2 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \cdots a_n}
\end{aligned}$$

□

6 Selected Problems

1. –