

# Inequalities Notes

Abror Maksudov

Last Updated: May 20, 2025

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Definitions</b>	<b>2</b>
2.1	Majorization . . . . .	2
2.2	Convex Function . . . . .	2
2.3	Elementary Symmetric Polynomials . . . . .	3
2.4	Elementary Symmetric Mean . . . . .	3
<b>3</b>	<b>Inequalities</b>	<b>4</b>
3.1	AM-GM Inequality . . . . .	4
3.2	Weighted AM-GM Inequality . . . . .	4
3.3	Power Mean Inequality . . . . .	5
3.4	Weighted Power Mean Inequality . . . . .	5
3.5	HM-GM-AM-QM Inequalities . . . . .	6
3.6	Bernoulli's Inequality . . . . .	6
3.7	Jensen's Inequality . . . . .	6
3.8	Karamata's Inequality . . . . .	7
3.9	Popoviciu's Inequality . . . . .	7
3.10	Newton's Inequality . . . . .	7
3.11	Maclaurin's Inequality . . . . .	8
3.12	Cauchy–Schwarz Inequality . . . . .	8
3.13	Hölder's Inequality . . . . .	8
3.14	Minkowski Inequality . . . . .	9
3.15	Generalized Minkowski Inequality . . . . .	9
3.16	Young's Inequality . . . . .	10
3.17	Rearrangement Inequality . . . . .	10
3.18	Chebyshev's Sum Inequality . . . . .	10
3.19	Schur's Inequality . . . . .	10
3.20	Generalized Schur's Inequality . . . . .	10
3.21	Muirhead's Inequality . . . . .	11
3.22	Aczel's Inequality . . . . .	11
3.23	Huygens Inequality . . . . .	11
3.24	Heinz Inequality . . . . .	11
3.25	Nesbitt's Inequality . . . . .	11
3.26	Cesàro's Inequality . . . . .	11
3.27	Mildorf's Inequality . . . . .	11
<b>4</b>	<b>Selected Inequalities</b>	<b>11</b>

<b>5</b>	<b>Proofs</b>	<b>11</b>
5.1	Proof of AM-GM Inequality using Induction . . . . .	11
5.2	Proof of AM-GM Inequality using Cauchy Induction . . . . .	11
5.3	Proof of AM-GM Inequality using Jensen's Method . . . . .	13
<b>6</b>	<b>Selected Problems</b>	<b>13</b>

# 1 Introduction

## 2 Definitions

### 2.1 Majorization

Let  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  be non-increasing sequences of real numbers. Then  $x$  is said to *majorize*  $y$ , denoted  $x \succ y$ , if the following conditions are satisfied:

1.  $x_1 \geq x_2 \geq \dots \geq x_n$  and  $y_1 \geq y_2 \geq \dots \geq y_n$ ;

2.  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ ;

3.  $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$  for all  $k = 1, 2, \dots, n-1$ .

Example:  $(3, 1, 0) \succ (2, 1, 1)$ ,  $(12, 0, 0) \succ (4, 4, 4)$ .

### 2.2 Convex Function

A function  $f : [a, b] \rightarrow \mathbb{R}$  is called *convex* (concave up) on  $[a, b]$  if and only if for all  $x, y \in [a, b]$  and all  $\lambda \in [0, 1]$ , the following inequality holds:

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if  $f''(x) \geq 0$  ( $f''(x) \leq 0$ ) holds for all  $x \in [a, b]$ .

Note that  $f$  is convex if and only if  $-f$  is concave.

Example (convex):  $x^2, e^x$ . Example (concave):  $\ln x, \sqrt{x}$ .

## 2.3 Elementary Symmetric Polynomials

Let  $t$  be a variable and  $x_1, x_2, \dots, x_n$  be real numbers. Define:

$$\begin{aligned}
 P(x) &= \prod_{i=1}^n (t + x_i) = (t + x_1)(t + x_2) \dots (t + x_n) \\
 &= t^n + (x_1 + \dots + x_n)t^{n-1} + (x_1x_2 + x_1x_3 + \dots)t^{n-2} + \dots \\
 &\quad + (x_2x_3 \dots x_n + x_1x_3 \dots x_n + \dots)t + x_1x_2x_3 \dots x_n \\
 &= 1 \cdot t^n + \left( \sum_{1 \leq i \leq n} x_i \right) t^{n-1} + \left( \sum_{1 \leq i < j \leq n} x_i x_j \right) t^{n-2} + \dots \\
 &\quad + \left( \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} x_{i_1} x_{i_2} \dots x_{i_{n-1}} \right) t + \prod_{i=1}^n x_i.
 \end{aligned}$$

In other words,

$$P(x) = \prod_{i=1}^n (t + x_i) = c_0 t^n + c_1 t^{n-1} + c_2 t^{n-2} + \dots + c_{n-1} t + c_n,$$

where the coefficient  $c_k$  is the  $k$ -th elementary symmetric sum:

$$\begin{aligned}
 c_0 &= 1, & c_1 &= \sum_{1 \leq i \leq n} x_i, & c_2 &= \sum_{1 \leq i < j \leq n} x_i x_j, \\
 c_3 &= \sum_{1 \leq i < j < k \leq n} x_i x_j x_k, & \dots, & & c_n &= \prod_{i=1}^n x_i.
 \end{aligned}$$

In general, for  $0 \leq k \leq n$

$$c_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Example:  $x_1 = 1, x_2 = 2, x_3 = 3 \implies (x+1)(x+2)(x+3) = x^3 + (1+2+3)x^2 + (1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3)x + 1 \cdot 2 \cdot 3 = x^3 + 6x^2 + 11x + 6$ .

## 2.4 Elementary Symmetric Mean

Let  $x_1, x_2, \dots, x_n$  be real numbers. The  $k$ -th elementary symmetric mean is defined as:

$$d_k = \frac{c_k}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

Example:  $x_1 = 1, x_2 = 2, x_3 = 3 \implies d_2 = \frac{c_2}{\binom{3}{2}} = \frac{11}{3}$ .

### 3 Inequalities

#### 3.1 AM-GM Inequality

Let  $a_1, a_2, \dots, a_n > 0$ . Then, the following inequality holds:

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n},$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

More precisely,

$$\frac{1}{n} \sum_{i=1}^n a_i \geq \sqrt[n]{\prod_{i=1}^n a_i}.$$

Example:  $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ .

#### 3.2 Weighted AM-GM Inequality

Let  $a_1, a_2, \dots, a_n > 0$  and  $w_1, w_2, \dots, w_n$  be positive integers. Then, by AM-GM we have:

$$\begin{aligned} & \frac{\underbrace{a_1 + a_1 + \dots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \dots + a_2}_{w_2} + \dots + \underbrace{a_n + a_n + \dots + a_n}_{w_n}}{w_1 + w_2 + \dots + w_n} \\ & \geq \left( \underbrace{a_1 a_1 \dots a_1}_{w_1} \underbrace{a_2 a_2 \dots a_2}_{w_2} \dots \underbrace{a_n a_n \dots a_n}_{w_n} \right)^{\frac{1}{w_1 + w_2 + \dots + w_n}}. \end{aligned}$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \dots + w_n a_n}{w_1 + w_2 + \dots + w_n} \geq (a_1^{w_1} a_2^{w_2} \dots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \dots + w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^n w_i a_i}{\sum_{i=1}^n w_i} \geq \left( \prod_{i=1}^n a_i^{w_i} \right)^{\frac{1}{\sum_{i=1}^n w_i}}$$

If we let  $w_1, w_2, \dots, w_n \geq 0$  with  $w_1 + w_2 + \dots + w_n = 1$ , we have:

$$w_1 a_1 + w_2 a_2 + \dots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \dots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^n w_i a_i \geq \prod_{i=1}^n a_i^{w_i}.$$

Example:  $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$ .

### 3.3 Power Mean Inequality

Let  $a_1, a_2, \dots, a_n > 0$ . Then, the  $r$ -th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left( \frac{a_1^r + \dots + a_n^r}{n} \right)^{1/r} & r \neq 0, \\ \sqrt[n]{a_1 a_2 \dots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1$  :  $\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} = \frac{n}{\sum_{i=1}^n \frac{1}{a_i}}$  (Harmonic Mean)
- $r = 0$  :  $\sqrt[n]{a_1 a_2 \dots a_n} = \left( \prod_{i=1}^n a_i \right)^{\frac{1}{n}}$  (Geometric Mean)
- $r = 1$  :  $\frac{a_1 + a_2 + \dots + a_n}{n} = \frac{1}{n} \sum_{i=1}^n a_i$  (Arithmetic Mean)
- $r = 2$  :  $\sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}}$  (Quadratic Mean)

If  $r > s$ , then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

Example:  $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}.$

### 3.4 Weighted Power Mean Inequality

Let  $a_1, a_2, \dots, a_n > 0$  and  $w_1, w_2, \dots, w_n \geq 0$  with  $w_1 + w_2 + \dots + w_n = 1$ . Then, the  $r$ -th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r + \dots + w_n a_n^r)^{1/r} & r \neq 0, \\ a_1^{w_1} a_2^{w_2} \dots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if  $r > s$ , then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

Example:  $(\frac{1}{6}a^3 + \frac{1}{3}b^3 + \frac{1}{2}c^3)^{1/3} \geq a^{1/6}b^{1/3}c^{1/2}.$

### 3.5 HM-GM-AM-QM Inequalities

Let  $a_1, a_2, \dots, a_n > 0$ . Then:

$$0 < \text{HM} \leq \text{GM} \leq \text{AM} \leq \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^n \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^n a_i} \leq \frac{1}{n} \sum_{i=1}^n a_i \leq \sqrt{\frac{\sum_{i=1}^n a_i^2}{n}}.$$

Example:  $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}.$

### 3.6 Bernoulli's Inequality

For all  $x \geq -1$  and  $r \geq 1$ :

$$(1+x)^r \geq 1+rx.$$

Example:  $(1+x)^5 \geq 1+5x.$

### 3.7 Jensen's Inequality

If  $f$  is *convex*, then:

$$\frac{f(a_1) + f(a_2) + \dots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)$$

with equality if and only if  $f$  is *linear* or  $a_1 = a_2 = \dots = a_n$ .

If we let  $w_1, w_2, \dots, w_n \geq 0$  with  $w_1 + w_2 + \dots + w_n = 1$ , we have:

$$w_1 f(a_1) + w_2 f(a_2) + \dots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \dots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^n w_i f(a_i) \geq f\left(\sum_{i=1}^n w_i a_i\right).$$

The inequality is reversed if  $f$  is *concave*.

Example:  $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x} + \sqrt{y}}{2}.$

### 3.8 Karamata's Inequality

If  $f$  is *convex*, and  $(a_i)$  *majorizes*  $(b_i)$ , then:

$$f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n),$$

or, more precisely,

$$\sum_{i=1}^n f(a_i) \geq \sum_{i=1}^n f(b_i).$$

The inequality is reversed if  $f$  is *concave*.

Example:  $f(x) = x^2 \implies (4)^2 + (1)^2 \geq (2.5)^2 + (2.5)^2 \implies 17 \geq 12.5$ .

### 3.9 Popoviciu's Inequality

If  $f$  is *convex*, and  $a, b, c > 0$ , then:

$$\begin{aligned} &af(x) + bf(y) + cf(z) + (a+b+c)f\left(\frac{ax+by+cz}{a+b+c}\right) \geq \\ &(a+b)f\left(\frac{ax+by}{a+b}\right) + (b+c)f\left(\frac{by+cz}{b+c}\right) + (c+a)f\left(\frac{cz+ax}{c+a}\right) \end{aligned}$$

Particularly, if  $a = b = c = 1$ , we have:

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3} \left[ f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right].$$

Equality holds if and only if  $f$  is *linear* or  $x = y = z$ .

Example:  $f(x) = x^2 \implies \frac{(1)^2 + (2)^2 + (3)^2}{3} + \left(\frac{1+2+3}{3}\right)^2 \geq \frac{2}{3} \left[ \left(\frac{1+2}{2}\right)^2 + \left(\frac{2+3}{2}\right)^2 + \left(\frac{3+1}{2}\right)^2 \right] \implies \frac{26}{3} \geq \frac{25}{3}$ .

### 3.10 Newton's Inequality

For  $x_1, x_2, \dots, x_n > 0$  and  $k = 1, 2, \dots, n-1$ , we have:

$$d_k^2 \geq d_{k-1}d_{k+1},$$

with equality if and only if  $x_1 = x_2 = \dots = x_n$ .

Example:  $x = 1, y = 2, z = 3 \implies \left(\frac{xy+yz+zx}{3}\right)^2 \geq \left(\frac{x+y+z}{3}\right) \cdot xyz$

$\implies \left(\frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1}{3}\right)^2 \geq \frac{1+2+3}{3}(1 \cdot 2 \cdot 3) \implies \left(\frac{11}{3}\right)^2 \geq 2 \cdot 6 \implies 13.444 \geq 12$ .

### 3.11 Maclaurin's Inequality

For  $x_1, x_2, \dots, x_n > 0$ , we have:

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \dots \geq \sqrt[n]{d_n}$$

with equality if and only if  $x_1 = x_2 = \dots = x_n$ .

Equivalently, it can be written as:

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt{\frac{\sum_{1 \leq i < j \leq n} x_i x_j}{\binom{n}{2}}} \geq \sqrt[3]{\frac{\sum_{1 \leq i < j < k \leq n} x_i x_j x_k}{\binom{n}{3}}} \geq \dots \geq \sqrt[n]{x_1 x_2 \dots x_n}.$$

Example:  $x = 1, y = 2, z = 3 \implies \frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz}$

$$\implies \frac{1+2+3}{3} \geq \sqrt{\frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1}{3}} \geq \sqrt[3]{1 \cdot 2 \cdot 3} \implies 2 \geq \frac{11}{3} \geq \sqrt[3]{6} \implies 2 \geq 1.915 \geq 1.817.$$

### 3.12 Cauchy–Schwarz Inequality

Let  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  be real numbers. Then:

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2,$$

with equality if and only if there is a constant  $\lambda \in \mathbb{R}$  such that  $a_i = \lambda b_i$  for all  $1 \leq i \leq n$ . That is, if  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n} = \lambda$ .

More precisely,

$$\left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \geq \left( \sum_{i=1}^n a_i b_i \right)^2.$$

Example:  $(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2 \implies (2^2 + 3^2)(4^2 + 5^2) \geq (2 \cdot 4 + 3 \cdot 5)^2$

$$\implies 13 \cdot 41 \geq 23^2 \implies 533 \geq 529.$$

### 3.13 Hölder's Inequality

Let  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, \dots, z_1, z_2, \dots, z_n$  be positive real numbers, and let  $\lambda_a, \lambda_b, \dots, \lambda_z$  be positive reals with  $\lambda_a + \lambda_b + \dots + \lambda_z = 1$ . Then:

$$(a_1 + \dots + a_n)^{\lambda_a} (b_1 + \dots + b_n)^{\lambda_b} \dots (z_1 + \dots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \dots z_1^{\lambda_z} + \dots + a_n^{\lambda_a} b_n^{\lambda_b} \dots z_n^{\lambda_z},$$



or, more precisely,

$$\underbrace{\left(\sum_{i=1}^n a_i\right)^{\lambda_a} \left(\sum_{i=1}^n b_i\right)^{\lambda_b} \cdots \left(\sum_{i=1}^n z_i\right)^{\lambda_z}}_{m \text{ factors}} \geq \sum_{i=1}^n \underbrace{\left(a_i^{\lambda_a} b_i^{\lambda_b} \cdots z_i^{\lambda_z}\right)}_{m \text{ variables}}$$

$$\prod_{j=1}^m \left(\sum_{i=1}^n a_{ij}\right)^{\lambda_j} \geq \sum_{i=1}^n \left(\prod_{j=1}^m a_{ij}^{\lambda_j}\right).$$

Example:  $m = 3, n = 2, \lambda_a = 0.5, \lambda_b = 0.3, \lambda_c = 0.2, (a) = (1, 3), (b) = (2, 4), (c) = (5, 6)$

$$\implies (a_1 + a_2)^{0.5} (b_1 + b_2)^{0.3} (c_1 + c_2)^{0.2} \geq a_1^{0.5} b_1^{0.3} c_1^{0.2} + a_2^{0.5} b_2^{0.3} c_2^{0.2}$$

$$\implies (1 + 3)^{0.5} (2 + 4)^{0.3} (5 + 6)^{0.2} \geq 1^{0.5} 2^{0.3} 5^{0.2} + 3^{0.5} 4^{0.3} 6^{0.2} \implies 5.53 \geq 5.45.$$

### 3.14 Minkowski Inequality

Let  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  be positive real numbers and  $p > 1$ . Then:

$$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} + (b_1^p + b_2^p + \cdots + b_n^p)^{\frac{1}{p}} \geq ((a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p)^{\frac{1}{p}},$$

or, more precisely,

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{\frac{1}{p}}$$

Example:  $p = 2, a = (3, 4), b = (6, 8)$

$$\implies \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$$

$$\implies \sqrt{3^2 + 4^2} + \sqrt{6^2 + 8^2} \geq \sqrt{(3 + 6)^2 + (4 + 8)^2}$$

$$\implies \sqrt{25} + \sqrt{100} \geq \sqrt{225} \implies 15 \geq 15. \text{ (equal. why?)}$$

### 3.15 Generalized Minkowski Inequality

Let  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, \dots, z_1, z_2, \dots, z_n$  be positive real numbers, and  $p > 1$ . Then:

$$\underbrace{\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} + \cdots + \left(\sum_{i=1}^n z_i^p\right)^{\frac{1}{p}}}_{m \text{ terms}} \geq \left(\sum_{i=1}^n \underbrace{(a_i + b_i + \cdots + z_i)^p}_{m \text{ terms}}\right)^{\frac{1}{p}}.$$

More precisely,

$$\sum_{j=1}^m \left(\sum_{i=1}^n a_{ij}^p\right)^{\frac{1}{p}} \geq \left[\sum_{i=1}^n \left(\sum_{j=1}^m a_{ij}\right)^p\right]^{\frac{1}{p}}$$

### 3.16 Young's Inequality

Let  $a, b \geq 0$  and  $p, q > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then:

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

with equality if and only if  $a^p = b^q$ .

Moreover, for increasing functions

$$\int_0^a f(x) dx + \int_0^b f^{-1}(x) dx \geq ab$$

with equality if and only if  $f(a) = b$ .

Example:  $a = 2, b = 3, p = 3, q = \frac{3}{2} \implies \frac{2^3}{3} + \frac{3^{3/2}}{3/2} \geq 2 \cdot 3 \implies 6.13 \geq 6$ .

### 3.17 Rearrangement Inequality

### 3.18 Chebyshev's Sum Inequality

### 3.19 Schur's Inequality

### 3.20 Generalized Schur's Inequality

### 3.21 Muirhead's Inequality

### 3.22 Aczel's Inequality

### 3.23 Huygens Inequality

### 3.24 Heinz Inequality

### 3.25 Nesbitt's Inequality

### 3.26 Cesàro's Inequality

### 3.27 Mildorf's Inequality

## 4 Selected Inequalities

## 5 Proofs

### 5.1 Proof of AM-GM Inequality using Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

- i. Base case is true ( $n = 2$ ).
- ii.  $n$  is true  $\implies n + 1$  is true.

*Proof:*

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned} \frac{a_1 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 \cdots a_n} \implies \\ \frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} &= \frac{n \frac{a_1 + \cdots + a_n}{n} + a_{n+1}}{n+1} \\ &\geq \left( \frac{a_1 + \cdots + a_n}{n} \right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\ &\geq \left( \sqrt[n]{a_1 \cdots a_n} \right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}} \\ &= \sqrt[n+1]{a_1 \cdots a_n a_{n+1}} \end{aligned}$$

□

### 5.2 Proof of AM-GM Inequality using Cauchy Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

- i. Base case is true ( $n = 2$ ).
- ii.  $n$  is true  $\implies 2n$  is true.
- iii.  $n$  is true  $\implies n - 1$  is true.

*Proof:*

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\begin{aligned} \frac{a_1 + \dots + a_n}{n} &\geq \sqrt[n]{a_1 \dots a_n} \implies \\ \frac{a_1 + a_2 + \dots + a_{2n}}{2n} &= \frac{1}{2} \left( \frac{a_1 + a_2 + \dots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \dots + a_{2n}}{n} \right) \\ &\geq \frac{\sqrt[n]{a_1 a_2 \dots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}}{2} \\ &\geq \sqrt[2]{\sqrt[n]{a_1 a_2 \dots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}} \\ &= \sqrt[2n]{a_1 a_2 \dots a_{2n}} \end{aligned}$$

Step 3:

$$\begin{aligned} \frac{a_1 + a_2 + \dots + a_{n-1} + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} a_n} \implies \\ \frac{a_1 + a_2 + \dots + a_{n-1} + \frac{a_1 + \dots + a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \dots + a_{n-1}}{n-1}} \\ \frac{(n-1)(a_1 + a_2 + \dots + a_{n-1}) + (a_1 + \dots + a_{n-1})}{n \cdot (n-1)} &= \frac{(n-1+1)(a_1 + a_2 + \dots + a_{n-1})}{n \cdot (n-1)} \\ \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \dots + a_{n-1}}{n-1}} \\ \left( \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} \right)^n &\geq a_1 a_2 \dots a_{n-1} \cdot \frac{a_1 + \dots + a_{n-1}}{n-1} \\ \left( \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} \right)^{n-1} &\geq a_1 a_2 \dots a_{n-1} \\ \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1 a_2 \dots a_{n-1}} \end{aligned}$$

□

### 5.3 Proof of AM-GM Inequality using Jensen's Method

Let  $a_1, a_2, \dots, a_n > 0$  and  $f(x) = \ln x$  be a *concave* function on  $(0, \infty)$ . By Jensen's Inequality we have:

$$\begin{aligned} f\left(\frac{1}{n} \sum_{i=1}^n a_i\right) &\geq \frac{1}{n} \sum_{i=1}^n f(a_i) \\ \ln\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) &\geq \frac{\ln(a_1) + \ln(a_2) + \dots + \ln(a_n)}{n} \\ &= \frac{\ln(a_1 a_2 \dots a_n)}{n} \\ &= \ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right) \\ e^{\ln\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)} &\geq e^{\ln\left(\sqrt[n]{a_1 a_2 \dots a_n}\right)} \\ \frac{a_1 + a_2 + \dots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_n} \end{aligned}$$

□

## 6 Selected Problems

1. –