Inequalities Notes

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2.1 Majorization

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences of real numbers. Then x is said to *majorize* y, denoted $x \succ y$, if the following conditions are satisfied:

1. $x_1 \geq x_2 \geq \cdots \geq x_n$ *and* $y_1 \geq y_2 \geq \cdots \geq y_n$;

2. $\displaystyle\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$;

3. $\displaystyle\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n-1$.

Example: $(3, 1, 0) \succ (2, 1, 1)$, $(12, 0, 0) \succ (4, 4, 4)$.

2.2 Convex Function

A function $f : [a, b] \to \mathbb{R}$ is called *convex* (concave up) on $[a, b]$ if and only if for all $x, y \in [a, b]$ and all $\lambda \in [0, 1]$, the following inequality holds:

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y).$$

A function is called *concave* (concave down) if the inequality is flipped.

Additionally, convexity (concavity) can be determined by checking if $f''(x) \geq 0$ ($f''(x) \leq 0$) holds for all $x \in [a, b]$.

Note that f is convex if and only if $-f$ is concave.

Example (convex): x^2, e^x. Example (concave): $\ln x, \sqrt{x}$.

3 Inequalities

3.1 AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the following inequality holds:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.
More precisely,

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}.$$

Example: $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

3.2 Weighted AM-GM Inequality

Let $a_1, a_2, \ldots, a_n > 0$ and $w_1, w_2, \ldots, w_n$ be positive integers. Then, by AM-GM we have:

$$\frac{\underbrace{a_1 + a_1 + \cdots + a_1}_{w_1} + \underbrace{a_2 + a_2 + \cdots + a_2}_{w_2} + \cdots + \underbrace{a_n + a_n + \cdots + a_n}_{w_n}}{w_1 + w_2 + \cdots + w_n}$$

$$\geq \left(\underbrace{a_1 a_1 \ldots a_1}_{w_1} \underbrace{a_2 a_2 \ldots a_2}_{w_2} \ldots \underbrace{a_n a_n \ldots a_n}_{w_n} \right)^{\frac{1}{w_1 + w_2 + \cdots + w_n}}.$$

The above is equivalent to the following

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n} \geq \left(a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} \right)^{\frac{1}{w_1 + w_2 + \ldots w_n}}.$$

More precisely,

$$\frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i} \geq \left(\prod_{i=1}^{n} a_i^{w_i} \right)^{\frac{1}{\sum_{i=1}^{n} w_i}}$$

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 a_1 + w_2 a_2 + \cdots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n},$$

or, more precisely,

$$\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i}.$$

Example: $\frac{3a+2b+c}{6} \geq \sqrt[6]{a^3 b^2 c}$.

3.3 Power Mean Inequality

Let $a_1, a_2, \ldots, a_n > 0$. Then, the r-th power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(\dfrac{a_1^r + \cdots + a_n^r}{n} \right)^{1/r} & r \neq 0, \\[2em] \sqrt[n]{a_1 a_2 \ldots a_n} & r = 0. \end{cases}$$

Example:

- $r = -1:$ $\qquad \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = \dfrac{n}{\sum_{i=1}^n \frac{1}{a_i}}$ (Harmonic Mean)

- $r = 0:$ $\qquad \sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}}$ (Geomteric Mean)

- $r = 1:$ $\qquad \dfrac{a_1 + a_2 + \cdots + a_n}{n} = \dfrac{1}{n} \sum_{i=1}^n a_i$ (Arithmetic Mean)

- $r = 2:$ $\qquad \sqrt{\dfrac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \sqrt{\dfrac{\sum_{i=1}^n a_i^2}{n}}$ (Quadratic Mean)

If $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $\mathcal{P}(2) \geq \mathcal{P}(1) \iff \sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a+b}{2}$.

3.4 Weighted Power Mean Inequality

Let $a_1, a_2, \ldots a_n > 0$ and $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$. Then, the r-th weighted power mean is defined as:

$$\mathcal{P}(r) = \begin{cases} \left(w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r \right)^{1/r} & r \neq 0, \\[1em] a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} & r = 0. \end{cases}$$

Similarly, if $r > s$, then

$$\mathcal{P}(r) \geq \mathcal{P}(s)$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Example: $\left(\frac{1}{6} a^3 + \frac{1}{3} b^3 + \frac{1}{2} c^3 \right)^{1/3} \geq a^{1/6} b^{1/3} c^{1/2}$.

3.5 HM-GM-AM-QM Inequalities

Let $a_1, a_2, \ldots, a_n > 0$. Then:

$$0 < \text{HM} \leq \text{GM} \leq \text{AM} \leq \text{QM}$$

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}.$$

More precisely,

$$0 < \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \sqrt[n]{\prod_{i=1}^{n} a_i} \leq \frac{1}{n} \sum_{i=1}^{n} a_i \leq \sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}.$$

Example: $\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$.

3.6 Bernoulli's Inequality

For all $x \geq -1$ and $r \geq 1$:
$$(1 + x)^r \geq 1 + rx.$$

Example: $(1 + x)^5 \geq 1 + 5x$.

3.7 Jensen's Inequality

If f is *convex*, then:

$$\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

with equality if and only if f is *linear* or $a_1 = a_2 = \cdots = a_n$.

If we let $w_1, w_2, \ldots, w_n \geq 0$ with $w_1 + w_2 + \cdots + w_n = 1$, we have:

$$w_1 f(a_1) + w_2 f(a_2) + \cdots + w_n f(a_n) \geq f(w_1 a_1 + w_2 a_2 + \cdots + w_n a_n),$$

or, more precisely,

$$\sum_{i=1}^{n} w_i f(a_i) \geq f\left(\sum_{i=1}^{n} w_i a_i\right).$$

The inequality is reversed if f is concave.

Example: $\sqrt{\frac{x+y}{2}} \geq \frac{\sqrt{x}+\sqrt{y}}{2}$.

3.8 Karamata's Inequality

If f is *convex*, and (a_i) *majorizes* (b_i), then:

$$f(a_1) + f(a_2) + \ldots f(a_n) \geq f(b_1) + f(b_2) + \ldots f(b_n),$$

or, more precisely,

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

The inequality is reversed if f is concave.

Example: $f(x) = x^2 \implies (4)^2 + (1)^2 \geq (2.5)^2 + (2.5)^2 \implies 17 \geq 12.5$.

3.9 Popoviciu's Inequality

If f is *convex*, and $a, b, c > 0$, then:

$$af(x) + bf(y) + cf(z) + (a + b + c)f\left(\frac{ax + by + cz}{a + b + c}\right) \geq$$

$$(a + b)f\left(\frac{ax + by}{a + b}\right) + (b + c)f\left(\frac{by + cz}{b + c}\right) + (c + a)f\left(\frac{cz + ax}{c + a}\right)$$

Particularly, if $a = b = c = 1$, we have:

$$\frac{f(x) + f(y) + f(c)}{3} + f\left(\frac{x + y + z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right)\right].$$

Equality holds if and only if f is *linear* or $x = y = z$.

Example: $f(x) = x^2 \implies \frac{(1)^2 + (2)^2 + (3)^2}{3} + \left(\frac{1 + 2 + 3}{3}\right)^2 \geq \frac{2}{3}\left[\left(\frac{1 + 2}{2}\right)^2 + \left(\frac{2 + 3}{2}\right)^2 + \left(\frac{3 + 1}{2}\right)^2\right] \implies \frac{26}{3} \geq \frac{25}{3}$.

4   Selected Inequalities

5   Proofs

5.1   Proof of AM-GM Inequality using Induction

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

  i. Base case is true $(n = 2)$.

 ii. n is true $\implies n + 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n} \implies$$

$$\frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} = \frac{n \frac{a_1 + \cdots + a_n}{n} + a_{n+1}}{n+1}$$

$$\geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}}$$

$$\geq \left(\sqrt[n]{a_1 \ldots a_n}\right)^{\frac{n}{n+1}} (a_{n+1})^{\frac{1}{n+1}}$$

$$= \sqrt[n+1]{a_1 \ldots a_n a_{n+1}}$$

$\square$

5.2 Proof of AM-GM Inequality using Forward-Backward Induction (a.k.a. Cauchy Induction)

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

i. Base case is true ($n = 2$).

ii. n is true $\implies 2n$ is true.

iii. n is true $\implies n - 1$ is true.

Proof:

Step 1:

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \implies (\sqrt{a_1})^2 - 2\sqrt{a_1 a_2} + (\sqrt{a_2})^2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0.$$

Step 2:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n} \implies$$

$$\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = \frac{1}{2}\left(\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}\right)$$

$$\geq \frac{\sqrt[n]{a_1 a_2 \ldots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}}{2}$$

$$\geq \sqrt[2]{\sqrt[n]{a_1 a_2 \ldots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2n}}}$$

$$= \sqrt[2n]{a_1 a_2 \ldots a_{2n}}$$

Step 3:

$$\frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} a_n} \implies$$

$$\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + \cdots + a_{n-1}}{n-1}}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}}$$

$$\frac{(n-1)(a_1 + a_2 + \cdots + a_{n-1}) + (a_1 + \cdots + a_{n-1})}{n \cdot (n-1)} = \frac{(n-1+1)(a_1 + a_2 + \cdots + a_{n-1})}{n \cdot (n-1)}$$

$$\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n]{a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}}$$

$$\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \right)^n \geq a_1 a_2 \ldots a_{n-1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1}$$

$$\left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \right)^{n-1} \geq a_1 a_2 \ldots a_{n-1}$$

$$\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}}$$

$\square$

5.3 Proof of AM-GM Inequality using Jensen's Method

Let $a_1, a_2, \ldots, a_n > 0$ and $f(x) = \ln x$ be a *concave* function on $(0, \infty)$. By Jensen's Inequality we have:

$$f\left(\frac{1}{n} \sum_{i=1}^{n} a_i \right) \geq \frac{1}{n} \sum_{i=1}^{n} f(a_i)$$

$$\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right) \geq \frac{\ln(a_1) + \ln(a_2) + \cdots + \ln(a_n)}{n}$$

$$= \frac{\ln(a_1 a_2 \ldots a_n)}{n}$$

$$= \ln\left(\sqrt[n]{a_1 a_2 \ldots a_n} \right)$$

$$e^{\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)} \geq e^{\ln\left(\sqrt[n]{a_1 a_2 \ldots a_n} \right)}$$

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

$\square$

6 Selected Problems

1. –