Inequalities Notes

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Last Updated: May 5, 2025

1 Algebraic Inequalities

Theorem 1 (AM-GM). *Let $a_1, \dots, a_n$ be non-negative real numbers. Then:*

$$\frac{a_1 + \cdots + a_n}{2} \geq \sqrt[n]{a_1 \dots a_n}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 2 (Cauchy-Schwarz). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be real numbers. Then:*

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \dots a_n b_n)^2$$

Theorem 3 (Titu's Lemma). *Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ be positive real numbers. Then:*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

Theorem 4 (Young's Inequality). *Let a, b be nonnegative real numbers and if $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

Theorem 5 (Hölder's Inequality). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be positive real numbers. Suppose that $P > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \geq \sum_{i=1}^{n} a_i b_i$$

More generally, let $x_{ij} (i = 1, \dots, m, j = 1, \dots, n)$ be positive real numbers. Suppose that $w_1, w_2, \dots, w_n$ are positive real numbers satisfying $w_1 + w_2 + \cdots + w_n = 1$. Then:

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij}\right)^{w_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} x_{ij}^{w_j}\right)$$

Theorem 6 (Minkowski Inequality). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be positive real numbers. Suppose that $p > 1$. Then:*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}}$$

Theorem 7 (Generalized Minkowski Inequality). *Let* $a_{ij} \geq 0$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, n$ *and let* $p > 1$. *Then:*

$$\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)^{p}\right]^{\frac{1}{p}} \leq \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p}\right)^{\frac{1}{p}}$$

Theorem 8 (Chebyshev's Sum Inequality). *Let* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *be real numbers. Then:*

$$\frac{a_1 b_1 + \cdots + a_n b_n}{n} \geq \frac{(a_1 + \cdots + a_n)}{n} \frac{(b_1 + \cdots + b_n)}{n}$$
$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right)$$

Theorem 9 (Rearrangement Inequality). *Let* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *be real numbers. For any permutation* σ *of* $\{1, \ldots, n\}$, *we have:*

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\sigma(i)} \geq \sum_{i=1}^{n} a_i b_{n+1-i}$$