Inequalities Notes

Abror Maksudov

Last Updated: May 5, 2025

1 Algebraic Inequalities

Theorem 1 (AM-GM). *Let $a_1, \dots, a_n$ be non-negative real numbers. Then:*

$$\frac{a_1 + \dots + a_n}{2} \geq \sqrt[n]{a_1 \dots a_n}$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

Theorem 2 (Cauchy-Schwarz). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be real numbers. Then:*

$$(a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) \geq (a_1 b_1 + \dots a_n b_n)^2$$

Theorem 3 (Titu's Lemma). *Let $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$ be positive real numbers. Then:*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}$$

Theorem 4 (Young's Inequality). *Let a, b be nonnegative real numbers and if $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

Theorem 5 (Hölder's Inequality). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be positive real numbers. Suppose that $P > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \geq \sum_{i=1}^n a_i b_i$$

More generally, let $x_{ij}(i = 1, \dots, m, j = 1, \dots, n)$ be positive real numbers. Suppose that $w_1, w_2, \dots, w_n$ are positive real numbers satisfying $w_1 + w_2 + \dots + w_n = 1$. Then:

$$\prod_{j=1}^n \left(\sum_{i=1}^m x_{ij}\right)^{w_j} \geq \sum_{i=1}^m \left(\prod_{j=1}^n x_{ij}^{w_j}\right)$$

Theorem 6 (Minkowski Inequality). *Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be positive real numbers. Suppose that $p > 1$. Then:*

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{\frac{1}{p}}$$

Theorem 7 (Generalized Minkowski Inequality). *Let $a_{ij} \geq 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$ and let $p > 1$. Then:*

$$\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)^{p}\right]^{\frac{1}{p}} \leq \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p}\right)^{\frac{1}{p}}$$

Theorem 8 (Chebyshev's Sum Inequality). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Then:*

$$\frac{a_1b_1 + \cdots + a_nb_n}{n} \geq \frac{(a_1 + \cdots + a_n)}{n}\frac{(b_1 + \cdots + b_n)}{n}$$
$$\frac{1}{n}\sum_{i=1}^{n} a_ib_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right)$$

Theorem 9 (Rearrangement Inequality). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. For any permutation σ of $\{1, \ldots, n\}$, we have:*

$$\sum_{i=1}^{n} a_ib_i \geq \sum_{i=1}^{n} a_ib_{\sigma(i)} \geq \sum_{i=1}^{n} a_ib_{n+1-i}$$

Definition 1 (Convex Function). *Suppose that f is a one-variable function defined on $[a, b] \subset \mathbb{R}$. f is called a convex function on $[a, b]$ if and only if for all $x, y \in [a, b]$ and for all $0 \leq t \leq 1$, we have:*

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y)$$

Theorem 10 (Jensen's Inequality). *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then for any $x_1, \ldots, x_n \in [a, b]$ and non-negative real numbers $w_1, \ldots, w_n$ with $w_1 + \cdots + w_n = 1$, we have:*

$$\sum_{i=1}^{n} w_if(x_i) \geq f(\sum_{i=1}^{n} w_ix_i)$$

Theorem 11 (Popoviciu's Inequality). *Let $f : I \to \mathbb{R}$. If f is convex, then for any three points x, y, z in I:*

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

Definition 2 (Majorization). *Given two sequences $(a) = (a_1, a_2, \ldots, a_n)$ and $(b) = (b_1, b_2, \ldots, b_n)$ (where $a_i, b_i \in \mathbb{R} \quad \forall i \in \{1, 2, \ldots, n\}$). We say that the sequence (a) majorizes the sequence (b), and write $(a) \succ (b)$, if the following conditions are fulfilled:*

$$\begin{aligned}
&a_1 \geq a_2 \geq \cdots \geq a_n;\\
&b_1 \geq b_2 \geq \cdots \geq b_n;\\
&a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n;\\
&a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_k \quad \forall k \in \{1, 2, \ldots, n-1\}
\end{aligned}$$

Theorem 12 (Karamata's Inequality). *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Suppose that $(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n)$ where $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in [a, b]$. Then:*

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i)$$

Theorem 13 (Weighted AM-GM Inequality). *Let $w_1, \ldots, w_n \geq 0$ such that $w_1 + \ldots w_n = 1$. For all $x_1, \ldots, x_n \geq 0$, we have:*

$$\sum_{i=1}^{n} w_i x_i \geq \prod_{i=1}^{n} x_i^{w_i}$$

Theorem 14 (Schur's Inequality). *Let x, y, z be non-negative real numbers. For any $r > 0$, we have:*

$$\sum_{cyc} x^r (x-y)(x-z) \geq 0$$

Theorem 15 (Generalized Schur's Inequality). *Let a, b, c, x, y, z be six non-negative real numbers such that the sequences (a, b, c) and (x, y, z) are similarly sorted. Then:*

$$x(a-b)(a-c) + y(b-c)(b-a) + z(c-a)(c-b) \geq 0$$

Theorem 16 (Newton's Inequality). *Let $x_1, \ldots, x_n$ be non-negative real numbers. Define the symmetric polynomials $s_0, s_1, \ldots, s_n$ by $(x + x_1)(x + x_2) \ldots (x + x_n) = s_n x^n + \cdots + s_1 + s_0$, and define the symmetric averages by $d_i = \frac{s_i}{\binom{n}{i}}$. Then:*

$$d_i^2 \geq d_{i+1} d_{i-1}$$

Theorem 17 (Maclaurin's Inequality). *Let $x_1, \ldots, x_n$ be non-negative real numbers. Define the symmetric polynomials $s_0, s_1, \ldots, s_n$ by $(x + x_1)(x + x_2) \ldots (x + x_n) = s_n x^n + \cdots + s_1 + s_0$, and define the symmetric averages by $d_i = \frac{s_i}{\binom{n}{i}}$. Then:*

$$d_1 \geq \sqrt[2]{d_2} \geq \sqrt[3]{d_3} \geq \cdots \geq \sqrt[n]{d_n}$$

Theorem 18 (Muirhead's Inequality). *Suppose that $(a_1, \ldots, a_n) \succ (b_1, \ldots, b_n)$, and $x_1, \ldots, x_n$ are positive real numbers. Then:*

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$$

where the symmetric sum is taken over all $n!$ permutations of $(x_1, x_2, \ldots, x_n)$.