AMA3020 Short Investigations

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February 4, 2025

Guidance will be provided by Drs Brown and Grüning. Topics chosen for the pair investigations will generally not be available for the solo investigations. In a large class, some projects will probably be chosen by more than one pair or more than one student. However, we would like the class to do as many *different* projects as possible. Although your preferences will be taken into account, there is no guarantee that you will always be assigned a project of your first choice.

Contents

1 The simple pendulum
(Pre-requisite: Classical Mechanics)

The bob of a simple pendulum of length l moves in accordance with the differential equation

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0,$$

where θ is the angle the pendulum makes with the vertical and g is the acceleration due to gravity. The most elementary approach in classical mechanics books considers only small oscillations for which $|\theta| \ll 1$ (in radians, of course!). In this case $\sin\theta \simeq \theta$ and the equation describes simple harmonic motion,

$$\theta(t) = a\cos(\omega t + \alpha),$$

with the amplitude a, initial phase α, angular frequency $\omega = \sqrt{g/l}$ and period $T = 2\pi/\omega$.

What happens if θ is not small?

2 No one should miss a penalty!
(Pre-requisite: None)

By tradition, the penalty spot in football is 12 yards (36 feet) from the centre of the goal which measures 24 feet wide by 8 feet high.

Is this the best position to maximise the chance that a goal is scored?

3 Stiff differential equations
(Pre-requisite: None, but familiarity with coding would help)

Many differential equations that are of importance in mathematics and its applications can only be solved numerically. In this case one must ensure that the solution obtained is *accurate*. In particular, errors can occur when solutions contain terms that decrease rapidly. This problem can be illustrated by the following equation

$$\frac{dx}{dt} = -30x,$$

that needs to be solved with the initial conditions $x(0) = 1$.

- Calculate the estimated value of $x(h)$ using a single step of the Runge-Kutta method of order 4 for various step sizes h and compare with the exact result.
- Use the scipy differential equation solver `vode` with integrator option `'bdf'` (stiff problems) and `'adams'` (non-stiff problems) to find numerical solutions and compare them with the exact result.
- Repeat the last two steps for

  $$\dot{x} = -20x + 20\sin\omega t + \omega\cos\omega t$$

  What happens for $\omega = 0.02$? What happens for other values of ω?

See also *Numerical Analysis* by R. L. Burden and J. D. Faires.

4 Fibonacci and related numbers
(Pre-requisite: None)

Fibonacci numbers $\{F_n\}$ satisfy the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \qquad (n \geq 1),$$

with $F_1 = F_2 = 1$. They are given explicitly by Binet's formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \tag{1}$$

where α, β are the roots of $x^2 - x - 1$ = 0.

Lucas numbers $\{L_n\}$ satisfy the same recurrence relation

$$L_{n+2} = L_{n+1} + L_n \qquad (n \geq 1),$$

but have $L_1 = 1$, $L_2 = 3$.

- Find an expression for L_n similar to that for F_n, given by Eq. (1).
- Show that $F_{n+2} + F_n = L_{n+1}$.
- Consider other relations involving these number sequences.
- Consider other related sequences.

A related problem (suggested by Dr Alex Schuchinsky) is to consider a sequence of "words" constructed from two symbols, say a and b, in which the next member of the sequence is constructed by concatenating the two previous "words". Starting from the two smallest "words", i.e., a and b, one obtains

$$a,\ b,\ ab,\ bab,\ abbab,\ bababbab,\ \text{etc.}$$

As one can see, the length of the nth word in the sequence is F_n. One can also notice that starting from the fifth member, the "words" contain at least one "double-b", i.e., bb. Is it possible to determine, how many bb's are in the nth "word"?

5 Circles on a lattice

(Pre-requisite: None)

Consider all points with integer coordinates on the xy plane, such as $(0, 0)$, $(1, 0)$, $(-2, 5)$, etc. These points form a square lattice. A circle of radius R centered at the origin may pass through a number of lattice points for some values of R, while for others it will pass through none. For example, for $R = \sqrt{2}$ the circle contains four lattice points: $(\pm 1, \pm 1)$, but for $R = 3/2$ no lattice points lie on the circle.

It is easy to see that for a circle to contain any lattice points, its radius must be the square root of a natural number, i.e., $R = \sqrt{n}$, where $n \in \mathbb{N}$. However, this condition is not sufficient, e.g., the circle with $R = \sqrt{3}$ contains no lattice points, while that with $R = \sqrt{5}$ contains a few (how many?).

Investigate this problem. The aim is to find out as much as possible about the number of lattice points that a circle may pass through, as a function of n. In particular, can we say anything about the *average* number of lattice points per circle?

6 Cycling into the wind

(Pre-requisite: Classical Mechanics)

It often seems that you are more likely to cycle into the wind than to have the wind in your back. While cynics may say that this is a purely psychological effect of the effort you make to overcome the wind, good modelling suggest that the effect is due to air resistance.

- Assume that the air is a collection of particles which move in the same direction and the same (wind) speed. How does the air resistance depend on the wind speed, the cyclist's speed and the angle between the direction of the wind and that of the cyclist?

- If a cyclist can reach a speed of 20 mph in the absence of any wind, and 25 mph with wind fully in the back, what maximum speed will the cyclist reach in full headwind? What speed can be reached for other wind directions?
- What effect does resistance due to the road surface have?

7 Bungee jumping

(Pre-requisite: Classical Mechanics)

In bungee jumping, a person jumps from a tall structure, e.g., a bridge, while attached to a long, heavy elastic band. The elastic band ensures that the person does not hit the surface below. The aim of this project is to find the acceleration the person experiences while falling.

To simplify the problem, you can replace the elastic band with a (non-elastic) rope. Assume also that the rope initially hangs down from the structure and the person.

- How does the speed of the person depend on the distance travelled? Determine the acceleration of the person by taking the derivative of the velocity with respect to time. How does it compare to g? [Hint: use energy conservation.]
- What happens for other initial positions of the rope?
- Extend, for example, by numerically determining the full motion, or by replacing the rope by an elastic band.

8 The error function

(Pre-requisite: Familiarity with integration)

The function

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

occurs in discussions of many probabilistic phenomena, e.g., those involving the normal distribution or diffusion processes. Most elementary statistics texts will provide a set of values of $\Phi(x)$ tabulated at closely spaced values of x.

Investigate ways of evaluating this function to whatever accuracy you care to specify, without recourse to such tables.

9 Square wheels

(Pre-requisite: None)

The invention of the wheel made for smooth transport, as long as the wheels of a vehicle rotated about axles passing through the centres of the wheels. The key point is that the axles move in a straight line when the vehicle is travelling on a flat road.

Square wheels will give you a bumpy ride on a flat road. What shape should the road surface be to give a smooth ride for a vehicle with square wheels? What about other shapes of the wheels?

10 Flat green bowling

(Pre-requisite: Classical Mechanics)

This is a game so called because it involves rolling balls on a flat horizontal surface, either made of grass or (when played indoors) consisting of long mats. The aim of the game is for one's bowl(s) to reach as near as possible to a specified position (the position of the 'jack').

It is made more interesting by the bowls having their mass distributed non-spherically – they have one axis of rotational symmetry with the centre of mass on this axis but not at the geometrical centre of the bowl. (In practice, the bowls are not quite geometrically spherical either, but this can be thought of as a 'second-order' effect.) This lack of symmetry results in the path of the bowl being curved rather the straight, as it would be for a spherically symmetric bowl.

Find the path of the bowl, for a given initial speed and eccentricity of the centre of mass.

11 Happy birthday!

(Pre-requisite: None)

(a) It is well-known that 23 is the smallest number of people in a group for which the probability that at least two of them have the same birthday is greater than 0.5. Prove this result.

(b) How many people must you query so that the probability of finding at least one with *your* birthday is more than 0.5?

(c) What is the relationship between the situations in (a) and (b)?

(d) If the group in (a) consisted of equal numbers of men and women, what is the minimum group size if the pair sharing the common birthday is a man and a woman?

(e) Extend.

12 A model for giving up smoking

(Pre-requisite: Familiarity with differential equations)

Let $S(t)$ denote the number of smokers at time t, $P(t)$ the number of potential smokers, i.e., those who have not taken it up yet, and $Q(t)$ the number of smokers who have quit permanently, and let $N = S(t)+P(t)+Q(t)$ be the total population size which is assumed to be constant. A possible model of the variation of $S(t)$, $P(t)$, and $Q(t)$ with time is given by

$$\begin{aligned}
\frac{dP(t)}{dt} &= \mu N - \beta P(t)\frac{S(t)}{N} - \mu P(t),\\
\frac{dS(t)}{dt} &= \beta P(t)\frac{S(t)}{N} - (\mu+\gamma)S(t),\\
\frac{dQ(t)}{dt} &= \gamma S(t) - \mu Q(t),
\end{aligned}$$

where β determines the rate at which potential smokers take up smoking due to contact with smokers, $1/\gamma$ is the average time as a smoker, and $1/\mu$ is the average time in the population.

- Explain each term in the above differential equations. What assumptions are being made in this model?
- Re-write the equations in terms of the population fraction variables

  $$x(t) = \frac{P(t)}{N}, \quad y(t) = \frac{S(t)}{N}, \quad z(t) = \frac{Q(t)}{N},$$

  and show by changing the time variable to $\tau = \mu t$, that the behaviour of the system depends only on β/μ and γ/μ.
- Study the equilibrium positions of this system of equations, and the behaviour of x, y, z close to equilibrium positions.
- Investigate simple modifications to the model.

13 Making the best of it

(Pre-requisite: None)

- A wire of unit length is cut into two pieces, which are formed into the shape of a circle and a square respectively. Show that the sum of the two areas so formed is a minimum when

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\text{perimeter of circle}}{\text{perimeter of square}}$$

- What about other shapes? More pieces? Etc.

14 Newton-Raphson in N dimensions

(Pre-requisite: None, but willingness to code a bit)

The Newton-Raphson method is an iterative procedure for finding the zeros of a function $f(x)$ according to the algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Prove this result using the leading terms in a Taylor series expansion of $f(x)$.
- Derive a similar iterative procedure for finding solutions (x_1, x_2) of

$$\begin{aligned} f_1(x_1, x_2) &= 0, \\ f_2(x_1, x_2) &= 0. \end{aligned}$$

- Generalise to N equations in N unknowns.
- Illustrate with examples.

15 Impacts and executive toys

(Pre-requisite: Classical Mechanics)

There are various ‘executive toys’ involving the collisions of spheres – Newton’s cradle is a typical example. Another can be developed as follows.

- Two spheres A and B of unequal mass can move freely on a thin horizontal wire threaded through their centres. There is a buffer at one end of the wire, and both A and B are projected with speed V towards the buffer, with A closer to the buffer than B. Let the coefficient of restitution for all impacts be e. After A and B collide, sphere A is reduced to rest. Determine the ratio of the masses of the two spheres for this effect to occur, and the speed which sphere B attains after its collision with A.
- Suppose there are n spheres, each initially projected with the same speed V towards the buffer, and with a sufficient gap between them that collisions take place in order. If all but the final sphere is reduced to rest by this sequence of impacts, determine its mass as a proportion of the sum of the masses of all the spheres, and its final speed. (You might find it helpful to work with e=1 first and then generalise.)
- Suppose that the wire is now mounted vertically and the buffer is at its bottom end. Discuss this case when the ‘executive toy’ consists of three spheres.

16 Bloch equations

(Pre-requisite: None)

The Bloch equations for a magnetic resonance experiment

$$\frac{d\mathbf{M}}{dt} = -\gamma \mathbf{B} \times \mathbf{M} - (\mathbf{M} - \mathbf{M}_0)/T$$

describe the time evolution of the magnetisation vector $\mathbf{M}(t)$ in the presence of the magnetic field $\mathbf{B}$. Here the constant γ is the gyromagnetic ratio of the nucleus, T is the relaxation time, and $\mathbf{M}_0$ is the equilibrium magnetisation, assumed to be in the z direction, $\mathbf{M}_0 = (0, 0, M_0)$.

- Determine the behaviour of the x, y and z components of the magnetisation as a function of time
  (a) for $\mathbf{B} = (0, 0, B_0)$ and $\mathbf{M}(0) = (M_0, 0, 0)$ (the free precession experiment);
  (b) for $\mathbf{B} = (0, 0, B_0)$ and $\mathbf{M}(0) = (0, 0, -M_0)$ (the inversion recovery experiment).
- In Nuclear Magnetic Resonance Spectroscopy, a weak radio frequency magnetic field is applied in addition to the strong constant field, so that $\mathbf{B} = (B_1 \cos\omega t, -B_1 \sin\omega t, B_0)$.

  Transform the equations of motion to rotating axes $\mathbf{i}' = \mathbf{i}\cos\omega t - \mathbf{j}\sin\omega t$, $\mathbf{j}' = \mathbf{j}\cos\omega t + \mathbf{i}\sin\omega t$ and find expressions for the components of magnetisation in these axes in the *steady state* (i.e., when the magnetisation is constant in these rotating axes). Plot the ratio of three components of magnetisation in the rotating axes to M_0 as a function of $\omega/\gamma B_0$ assuming $T\gamma B_1 = 10^{-6}$ and $T\gamma B_0 = 100$.

See *Nuclear Magnetic Resonance Spectroscopy* by R. K. Harris for background reading.

17 Models for electrolyte solutions
(Pre-requisite: None)

In the Debye-Hückel model for electrolyte solutions, the electrostatic potential due to an ion A with charge q_A is reduced from the Coulomb value $\phi(r) = q_A/4\pi\varepsilon\varepsilon_0 r$ due to screening by other ions. In this expression r is the distance from the origin, and ε is the dielectric constant of the liquid. To obtain the *screened potential* one combines the Poisson equation for the electrostatic potential,

$$\nabla^2\phi(r) = -\frac{\rho(r)}{\varepsilon\varepsilon_0}, \tag{1}$$

where $\rho(r)$ is the charge density, with the Boltzmann expression for the probability per unit volume of finding an ion B with charge q_B at distance r from ion A

$$p_B = c_B \exp[-q_B\phi(r)/kT]. \tag{2}$$

Here c_B is the average number of ions of type B per unit volume (i.e., their concentration), k is the Boltzmann constant and T is temperature. Combining (1) and (2) one obtains

$$\nabla^2\phi(r) = -\sum_i c_i q_i \exp[-q_i\phi(r)/kT]/(\varepsilon\varepsilon_0), \tag{3}$$

where the sum is over all types of ions in the solution.

(a) Write down this equation for the case where there are two types of ions present in solution with charges $+e$ and $-e$ and concentration c (e.g., a solution of NaCl).

(b) Simplify it by expanding the exponential in powers of $e\phi/kT$ and keeping the lowest order term. Solve this linearised equation for $\phi(r)$.

(c) Discuss the changes that result if the ions have diameter a so that equation (2) only holds for $r > a$ and $p_B = 0$ for $r < a$.

Reading: P. W. Atkins and J. de Paula, *Physical Chemistry*.

18 Control of protein synthesis
(Pre-requisite: None)

Messenger ribonucleic acid (mRNA) plays a vital role in protein synthesis. It carries the information which determines the structure of a protein to the ribosomes, the spherical particles on which protein synthesis occurs (see, e.g., J. D. Watson, *Molecular Biology of the Gene*, Ch. 12 and 14).

The simplest model for the control of protein synthesis is given by the differential equations

$$\begin{aligned}\frac{dY}{dt} &= \frac{c}{a+bM} - kY,\\ \frac{dZ}{dt} &= eY - fZ,\\ \frac{dM}{dt} &= gZ - hM,\end{aligned}$$

where Y is the concentration of the mRNA in the nucleus at time t and Z is the concentration of the protein. The protein catalyses the production of a repressor molecule, with concentration M, which prevents the manufacture of mRNA, and a, b, c, e, f, g, h and k are positive constants (see J. Maynard Smith, *Mathematical Ideas in Biology*, p. 107 *et seq.*).

Analyse in detail the predictions this model makes about the behaviour of Y, Z and M with time. Consider any possible improvements that can be made to the model.

19 Damped driven oscillations
(Pre-requisite: Classical Mechanics would help but is not necessary. Familiarity with coding)

The damped, driven oscillator obeys the equation of motion

$$\ddot{x} + b\dot{x} + \sin x = a\cos\omega t,$$

where a, b and ω are constants. For small displacements, i.e., $|x| \ll 1$, one has $\sin x \simeq x$, and this equation can be solved analytically. For large x, however, this nonlinear differential cannot be solved analytically, but it can be solved numerically. The solutions of this equation have some very surprising properties for appropriate values of a, b and ω.

Give an account of the different types of motion possible for this system. [This problem can be investigated numerically using python.]

20 Harmonic oscillators
(Pre-requisite: None really, but Classical Mechanics would make things easier)

A point P moves so that its position vector $\mathbf{r}$ relative to the origin O satisfies the equation $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$, where ω is a constant. Show that the motion of P is compounded from two harmonic oscillations executed with the same frequency at right angles to each other and that the locus of P is an ellipse with the centre at O.

Show further that if P has Cartesian coordinates (x, y) in this plane and O is the origin of coordinates, the locus of P has the equation

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1a_2}\cos(\epsilon_2 - \epsilon_1) = \sin^2(\epsilon_2 - \epsilon_1),$$

where a_1 and a_2 are the amplitudes of the harmonic oscillations and ϵ_1 and ϵ_2 are their respective phases.

Discuss the shape and orientation of this locus for different values of $\epsilon_2 - \epsilon_1$.

If the two harmonic oscillations are now taken to have different frequencies, obtain the equation of the path of P.

21 Weighing on a ship
(Pre-requisite: Classical Mechanics)

The simplest way of determining the mass of an object is by measuring the gravitional force mg on this object. This is fine on land as the Earth is (approximately) an inertial frame of reference and $g \approx$ const. However, a ship rocked by waves is not an inertial frame, and the apparent "gravity" is not constant. This needs to be taken into account when, for example, weighing the catch. In this project you will investigate how this can be done.

G. Kessling, D. Birnbacher and C. Berg, *Meas. Sci. Technol.* **4**, 1035–1042 (1993).

22 Time and tides
(Pre-requisite: None)

The times of low and high tides are important for holidaymakers as well as fishermen at sea. Tidal information can also be used to spot tsunamis well in advance of them reaching shore. This project investigates the mathematics behind so-called tide tables.

The height of the tide $h(t)$ at any point on the Earth depends on many factors including its geographical location, coastline, ocean currents and storms. The primary cause of the tides is the force of gravity due to the Moon and the Sun, that combine to give the daily, monthly and seasonal tides. The three main cycles are: the daily rotation of the Earth ω_1, the monthly rotation of the moon around the Earth ω_2, and the annual motion of the Earth around the sun ω_3. There are also two small lower-frequency corrections for the precession of the Earth-Moon orbit (the perigee and orbital plane) but these can be neglected.

The simplest model of the tide height is given by

$$h(t) = h_0 + a\cos\Omega t + b\sin\Omega t,$$

with h_0 the average sea level. The frequency Ω has a well-known value [1]

$$\Omega = 2(\omega_1 - \omega_2 + \omega_3) = 28.984 \text{ deg/hr},$$

while the remaining parameters h_0, a and b depend on the location of the port [2]. This investigation requires you to find them. This can be done by using data for $h(t)$ from tide tables for a few days, and solving the linear equations to determine h_0, a and b. Once these are found, you can *predict* the times for high and low tides over the whole week or fortnight. Do this for Bristol and Donaghadee and then compare your forecast with the forecasts or measurements made at these harbours. Can you suggest any improvements to your model, and how these might be implemented?

You may also use data from the NOAA Tsunami center [3].

[1] `www.math.sunysb.edu/~tony/tides/harmonic.html`

[2] `http://news.bbc.co.uk/weather/coast_and_sea/tide_tables/`

[3] `www.ndbc.noaa.gov/dart.shtml`

23 Ellipse and its normals
(Pre-requisite: None)

An ellipse can be defined by its equation in Cartesian coordinates

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b ($b \le a$) are its semimajor and semiminor axes, respectively. Alternatively, it can be defined in the parametric form by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.

How many *normals* to the ellipse can you draw from a given point in the xy plane? [The normal is a line perpendicular to the tangent at the point where it meets the curve.]

24 Bouncing ball

(Pre-requisite: Classical mechanics)

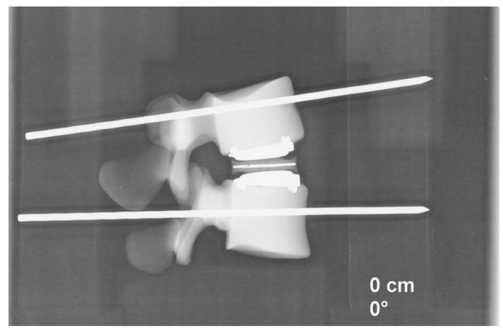
A small ball is released from rest from a point above a large sphere of radius R. If this point lies on the vertical line through the centre of the sphere, the ball will keep bouncing along this line. However, if the point is some distance d away from this line ($d < R$), the ball will bounce off at an angle after colliding with the sphere. In this case it will only make a finite number of bounces on the spherical surface, before jumping off it.

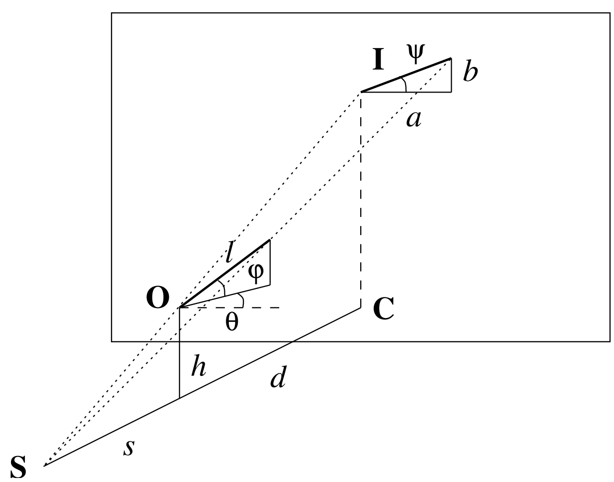
Estimate the number of times the ball will bounce off the sphere. [You may assume that the collisions are perfectly elastic. It might be easier to find an estimate for $d \ll R$. However, an exact (numerical?) solution may also be possible in the general case.]

25 What's the angle between your vertebrae?

(Pre-requisite: None)

Using X-rays, one can measure angles between bones in a body (e.g., between two vertebrae, as shown on the X-ray of a model with pins [1]). However, when measuring an angle on an image, the result may not be the same as the true angle. Thus, in the diagram below the angle φ that an object ("stick" of length l) makes with the horizontal, is not necessarily equal to the angle ψ between its image I and the horizontal line on the screen.





The difference between φ and ψ may depend on the distance s between the X-ray source S and the plane parallel to the screen which contains one end of the object O, and the distance d between O and the plane of the screen. It may also depend on the height h of the end-point O above the line SC perpendicular to the screen, and on the angle θ that the vertical plane through O makes with the direction parallel to the screen.

Using the above diagram or otherwise, show that

$$\tan\psi = \frac{1}{\cos\theta}\tan\varphi - \frac{h}{s}\tan\theta. \tag{4}$$

Investigate how the difference beteeen ψ and φ depends on the angle θ and the vertical displacement h.

You can further investigate if one can find φ from the results of two measurements (yielding ψ_1 and ψ_2), corresponding to two position θ_1 and θ_2, for which only the difference $\Delta\theta = \theta_2 - \theta_1$ is known.

[1] John McManus, *The Influence of X-Ray Technique on the Angle Measurement of an Artificial Disc*, Thesis (Queen's University Belfast, 2006).

26 Two ships at anchor

(Pre-requisite: None)

Two ships, A and B, are at anchor some distance from each other and from shore. A boat launched from A takes a number of crew ashore and then sails to B. What is the quickest path the boat can take?

Start by considering the simplest case of a straight shoreline. Then consider other shapes. Can you find equations that would solve the problem in the general case?

If you want to look at a more difficult problem, add a sea current parallel to the shore!

27 Double pendulum

(Pre-requisite: Classical Mechanics)

A double pendulum is a pendulum with a second pendulum attached to it. We assume that each pendulum consists of a mass m at the end of a massless rod of length ℓ. Their positions are described by the angles ϕ_1 and ϕ_2 that each pendulum makes with the vertical.

The Lagrangian of the system reads (verify!)

$$L = \frac{1}{2} m\ell^2 \left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2 \cos(\phi_1 - \phi_2) \right] + mg\ell(2\cos\phi_1 + \cos\phi_2)$$

1. List the conserved quantities.
2. Describe qualitatively the motion of the pendulums with simple initial conditions, e.g., one at rest and the other one with some initial velocity.
3. Write Lagrange's equations of motion for the variables ϕ_i. Investigate the motion for small ϕ_i.
4. Use the Runge-Kutta [1] method to solve the equations of motion for arbitrary initial conditions, using python or a programming language of your choice.
5. Test your program with the initial conditions you discussed in 2 and comment.

[1] See for example: R. L. Burden and J. D. Faires, *Numerical Analysis*

28 Ladybird lost

(Pre-requisite: Classical mechanics would help, but really none)

In a round room of radius R, a large number of coins N of diameter d are randomly dispersed upon the floor. A ladybird starts from the centre of the room, crawling at speed v.

1. How long (on average) does it take before it meets a coin?
2. Suppose that every time the ladybird meets a coin, it changes direction at random. How long (on average) before it makes it to the wall?
3. Suppose every time it 'hits' a coin, the coin magically disappears. Work out (approximately, and on average) the law of decrease of the number of remaining coins as a function of time. (Assume that if, in the process, the ladybird hits the wall, it is simply 'reflected' back towards the interior of the room, at a random angle.)

4. Calculate the answers for parts 1–3 for a room of typical size, 1p coins and $v = 1$ cm/s, and N of your choice.

Keywords: to work on this problem, you will need to think (or read) about mean free paths, random walks/diffusion/Bro motion.

29 Finding the truth in TEQs

(Pre-requisite: None but willingness to code a bit)

One of the questions that students answer when filling in Teaching Evaluation Questionnaires (TEQ) is about the percentage of lectures they attended. After tallying, these results (in simplified form) are presented in a table like this

% of lectures attended	25%	50%	75%	100%
No. of answers	7	19	42	29

where the total number of answers is 97.

Since the TEQs are filled in in a lecture, the frequencies in the table above are *biased*. Using the data from the table, estimate the true frequencies that would be observed if the TEQ were filled by all 150 students in the class. You can assume that attendances by individual students are uncorrelated random variables. Using your result, check whether the attendance of the lecture where TEQ was taken, was typical for the given class size.

Generalise your approach to an arbitrary number of scores, and to a continuous distribution. In each case, determine the mean and standard deviation od the number of students in class based on the true frequencies that you obtain from the frequencies observed in the lecture.

30 Assigning investigations

(Pre-requisite: None)

In order to be assigned an investigation project from a list, students in a module are asked for their 5 choices without particular preferences. You are faced with the task of assigning each student one of the projects of their choice in a way that no two students are assigned the same topic (*satisfiability problem*). Assume that the number of projects K is greater than or equal to the number of students N. Devise an algorithm based on backtracking or other forms of search that finds a feasible solution to this problem. Discuss examples.

Assume now that the 5 students' choices are ordered from their best preference to the least one. In this case use the Hungarian algorithm, or any other of your choice, to find the assignment that maximises the overall students' preferences. Discuss the adequacy of 5 preferences for assigning projects and the relationship between the number of students and number of projects available. (Note that some projects can be much more popular than others).

31 Lenz's potentials

(Pre-requisite: Classical Mechanics)

For a particle moving in a central field $U(r)$, the path can be found using plane polar coordinates r and ϕ in terms of an indefinite integral [1]. However, there are only a handful of potentials for which this integral and the equation of the path can be found analytically. You are familiar with one example, namely the gravitational potential $U(r) = -\alpha/r$, in which the paths are conic sections (ellipse, parabola or hyperbola). Using this as a starting point, investigate the paths of the particle with zero energy ($E = 0$) in *Lenz's potential*

$$U(r) = -\frac{2uR^2}{r^2\left[\left(\frac{r}{R}\right)^\mu + \left(\frac{R}{r}\right)^\mu\right]^2},$$

where u, R and μ are constants, for different values of μ. Such potentials find some unexpected applications in atoms and clusters [V. N. Ostrovsky, Phys. Rev. A **56**, 626 (1997)].

[1] L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford, 2001).

32 "Impossible" exam question

(Pre-requisite: None)

In January 2015, some final year economics students at Sheffield University claimed that their exam contained an "impossible question" [1]. However, the question, which is reproduced below, should give no trouble to a mathematics student! Moreover, you should also be able to find ways of extending it.

Consider a country with many cities and assume that there are $N > 0$ people in each city. Output per person is $\sigma N^{0.5}$ and there is a coordination cost per person of γN^2. Assume that $\sigma > 0$ and $\gamma > 0$.

(a) What sort of things does the coordination cost term γN^2 represent? Why does it make sense that the exponent on N is greater than 1? [10 marks]

(b) Draw a graph of per-capita consumption as a function of N and derive the optimal city size N. How does it depend on the parameters σ and γ? Provide intuition for your answers. [10 marks]

(c) Describe which combinations of σ and γ generate a peasant economy, meaning an economy with no cities (or 1-person cities). Why might the values of the parameters σ and γ have changed over time? What do these changes imply in terms of the optimal city size? [10 marks]

[1] http://www.bbc.com/news/education-31057005.

33 Design a robot

(Pre-requisite: None)

Consider a planar mechanism which consists of three bars connected by joints (see diagram). Bars OA and $O'B$ (called cranks) are free to rotate about fixed points O and O', respectively. Find how the coordinates of point C depend on the angle ϕ that rod OA makes with the line $O'O$. Explore how the shape of the path described by point C depends on the lengths of the cranks a and b, and lengths c and d, taking them as fractions of the length l between points O and O'. In particular, if this mechanism is to model the motion of a foot, find the optimal lengths that would make point C move in a curve, part of which is close to a straight line (foot in contact with the ground), the other part being an arc-like path of the foot though the air.