AMA3020 Short Investigations

Andrew Brown, Myrta Grüning

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Guidance will be provided by Drs Brown and Grüning. Topics chosen for the pair investigations will generally not be available for the solo investigations. In a large class, some projects will probably be chosen by more than one pair or more than one student. However, we would like the class to do as many *different* projects as possible. Although your preferences will be taken into account, there is no guarantee that you will always be assigned a project of your first choice.

Contents

1 No one should miss a penalty!

(Pre-requisite: None)

By tradition, the penalty spot in football is 12 yards (36 feet) from the centre of the goal which measures 24 feet wide by 8 feet high.

Is this the best position to maximise the chance that a goal is scored?

2 Loose chippings

(Pre-requisite: None)

Many local authorities recommend that car drivers limit the speed of the car to 20 mph when travelling over roads which have been newly surfaced with bituminous binder and stone chippings. The advisory limit is intended to avoid the situation where loose stones lifted by the rear wheels of a vehicle in front fly into the path of your vehicle and can cause damage to paintwork or even break the windscreen.

Investigate whether this is a reasonable speed limit, given that the vehicles are separated by a distance recommended by the Highway Code (i.e., a two-second gap).

You may assume that relative to the vehicle the stones are projected with the speed of the vehicle, that air resistance is negligible and that drivers are only worried about stones hitting the front of the bonnet (typically 0.75 metres above the ground). Any other assumptions which you make should be stated clearly.

3 Dipsticks

(Pre-requisite: None)

A domestic tank which can hold 1000 litres of heating oil is in the shape of a right circular cylinder, with its axis horizontal. Calibrate a dipstick to show marks at intervals of 50 litres.

4 Paying off a mortgage

(Pre-requisite: None)

The problem of repaying a mortgage can be cast of the form of a differential equation

$$\frac{dB}{dt} - \alpha B = -r, \tag{1}$$

where B is the outstanding balance on the loan, α is the interest rate (i.e., a fraction of the remaining loan charged per unit time), and r is the rate of mortgage payments, i.e., the money that must be paid to the bank or building society per unit time). Solving this equation for constant α and r allows one to find the solution $B(t)$ that satisfies the conditions $B(0) = B_0$ (initial size of the mortgage) and $B(T) = 0$, where T is the end time of the mortgage. From this, you should be able to find r, i.e., determine the monthly payments (if the month is used as a unit of time).
In practice, the interest rate will usually vary during the lifetime of a mortgage, which means that you need to consider Eq. (1) for α being a function of time, i.e., $\alpha(t)$. As a result, r will no longer be a constant, but using the first part of your investigation, you should be able to find the repayment rate that allows one to pay off the mortgage by the time T. You should thus be able to find the solution of Eq. (1) for an arbitrary $\alpha(t)$. This solution may contain some integrals that can be evaluated numerically (e.g., using Python or Mathematica), allowing you to explore the behaviour of $B(t)$ and r as functions of time for different cases (e.g., growing, descreasing or oscillating $\alpha(t)$).

5 Cycling into the wind

(Pre-requisite: Classical Mechanics)

It often seems that you are more likely to cycle into the wind than to have the wind in your back. While cynics may say that this is a purely psychological effect of the effort you make to overcome the wind, good modelling suggest that the effect is due to air resistance.

- Assume that the air is a collection of particles which move in the same direction and the same (wind) speed. How does the air resistance depend on the wind speed, the cyclist's speed and the angle between the direction of the wind and that of the cyclist?
- If a cyclist can reach a speed of 20 mph in the absence of any wind, and 25 mph with wind fully in the back, what maximum speed will the cyclist reach in full headwind? What speed can be reached for other wind directions?
- What effect does resistance due to the road surface have?

6 Bungee jumping

(Pre-requisite: Classical Mechanics)

In bungee jumping, a person jumps from a tall structure, e.g., a bridge, while attached to a long, heavy elastic band. The elastic band ensures that the person does not hit the surface below. The aim of this project is to find the acceleration the person experiences while falling.

To simplify the problem, you can replace the elastic band with a (non-elastic) rope. Assume also that the rope initially hangs down from the structure and the person.

- How does the speed of the person depend on the distance travelled? Determine the acceleration of the person by taking the derivative of the velocity with respect to time. How does it compare to g? [Hint: use energy conservation.]
- What happens for other initial positions of the rope?
- Extend, for example, by numerically determining the full motion, or by replacing the rope by an elastic band.

7 The error function

(Pre-requisite: Familiarity with integration)

The function

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

occurs in discussions of many probabilistic phenomena, e.g., those involving the normal distribution or diffusion processes. Most elementary statistics texts will provide a set of values of $\Phi(x)$ tabulated at closely spaced values of x.

Investigate ways of evaluating this function to whatever accuracy you care to specify, without recourse to such tables.

8 Flat green bowling

(Pre-requisite: Classical Mechanics)

This is a game so called because it involves rolling balls on a flat horizontal surface, either made of grass or (when played indoors) consisting of long mats. The aim of the game is for one's bowl(s) to reach as near as possible to a specified position (the position of the 'jack').

It is made more interesting by the bowls having their mass distributed non-spherically – they have one axis of rotational symmetry with the centre of mass on this axis but not at the geometrical centre of the bowl. (In practice, the bowls are not quite geometrically spherical either, but this can be thought of as a 'second-order' effect.) This lack of symmetry results in the path of the bowl being curved rather the straight, as it would be for a spherically symmetric bowl.

Find the path of the bowl, for a given initial speed and eccentricity of the centre of mass.

9 Making the best of it

(Pre-requisite: None)

- A wire of unit length is cut into two pieces, which are formed into the shape of a circle and a square respectively. Show that the sum of the two areas so formed is a minimum when

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\text{perimeter of circle}}{\text{perimeter of square}}$$

- What about other shapes? More pieces? Etc.

10 Two coupled spins

(Pre-requisite: Quantum Theory)

An NMR experiment on a molecule containing two nuclei A and B, is described by the spin Hamiltonian

$$\begin{aligned} H &= (\omega_0 + \delta/2)\hat{I}_{Az} + (\omega_0 - \delta/2)\hat{I}_{Bz} + J\,\hat{\mathbf{I}}_A \cdot \hat{\mathbf{I}}_B \\ &= (\omega_0 + \delta/2)\hat{I}_{Az} + (\omega_0 - \delta/2)\hat{I}_{Bz} + J\left[\hat{I}_{Az}\hat{I}_{Bz} + \frac{1}{2}(\hat{I}_{A+}\hat{I}_{B-} + \hat{I}_{A-}\hat{I}_{B+})\right], \end{aligned} \tag{1}$$

where $\hat{\mathbf{I}}_A$ and $\hat{\mathbf{I}}_B$ are the nuclear spin operators with components $\hat{I}_{Az}$ and $\hat{I}_{A\pm} = \hat{I}_{Ax} \pm i\hat{I}_{Ay}$ (and the same for B), and ω_0, δ and J are constants. The first two terms describe the interaction of the spins with a magnetic field in the z direction, and the third term decribes the interaction between the spins.

For spin-$\frac{1}{2}$ nuclei the spin operators for each of the nuclei are Pauli matrices, and for $J = 0$ the system can be found in one of the four states ("up-up", "up-down", "down-up" and "down-down"). Using these as basis states, the eigenvalues of the Hamiltonian (1) can be found by diagonalising a 4×4 matrix.

Hence, set up this matrix and find the spectrum of eigenvalues for $\omega_0 = 1000$, $\delta = 10$, and $J = 0,\ 5,\ 10,\ 20,\ 100$.

See *Nuclear Magnetic Resonance Spectroscopy* by R. K. Harris for background reading.

11 Newton-Raphson in N dimensions

(Pre-requisite: None, but willingness to code a bit)

The Newton-Raphson method is an iterative procedure for finding the zeros of a function $f(x)$ according to the algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Prove this result using the leading terms in a Taylor series expansion of $f(x)$.
- Derive a similar iterative procedure for finding solutions (x_1, x_2) of

$$\begin{aligned} f_1(x_1, x_2) &= 0, \\ f_2(x_1, x_2) &= 0. \end{aligned}$$

- Generalise to N equations in N unknowns.
- Illustrate with examples.

12 Impacts and executive toys
(Pre-requisite: Classical Mechanics)

There are various 'executive toys' involving the collisions of spheres – Newton's cradle is a typical example. Another can be developed as follows.

- Two spheres A and B of unequal mass can move freely on a thin horizontal wire threaded through their centres. There is a buffer at one end of the wire, and both A and B are projected with speed V towards the buffer, with A closer to the buffer than B. Let the coefficient of restitution for all impacts be e. After A and B collide, sphere A is reduced to rest. Determine the ratio of the masses of the two spheres for this effect to occur, and the speed which sphere B attains after its collision with A.
- Suppose there are n spheres, each initially projected with the same speed V towards the buffer, and with a sufficient gap between them that collisions take place in order. If all but the final sphere is reduced to rest by this sequence of impacts, determine its mass as a proportion of the sum of the masses of all the spheres, and its final speed. (You might find it helpful to work with e=1 first and then generalise.)
- Suppose that the wire is now mounted vertically and the buffer is at its bottom end. Discuss this case when the 'executive toy' consists of three spheres.

13 Bloch equations
(Pre-requisite: None)

The Bloch equations for a magnetic resonance experiment

$$\frac{d\mathbf{M}}{dt} = -\gamma\mathbf{B}\times\mathbf{M} - (\mathbf{M}-\mathbf{M}_0)/T$$

describe the time evolution of the magnetisation vector $\mathbf{M}(t)$ in the presence of the magnetic field $\mathbf{B}$. Here the constant γ is the gyromagnetic ratio of the nucleus, T is the relaxation time, and $\mathbf{M}_0$ is the equilibrium magnetisation, assumed to be in the z direction, $\mathbf{M}_0 = (0, 0, M_0)$.

- Determine the behaviour of the x, y and z components of the magnetisation as a function of time

  (a) for $\mathbf{B} = (0, 0, B_0)$ and $\mathbf{M}(0) = (M_0, 0, 0)$ (the free precession experiment);
  (b) for $\mathbf{B} = (0, 0, B_0)$ and $\mathbf{M}(0) = (0, 0, -M_0)$ (the inversion recovery experiment).

- In Nuclear Magnetic Resonance Spectroscopy, a weak radio frequency magnetic field is applied in addition to the strong constant field, so that $\mathbf{B} = (B_1\cos\omega t, -B_1\sin\omega t, B_0)$.

  Transform the equations of motion to rotating axes $\mathbf{i}' = \mathbf{i}\cos\omega t - \mathbf{j}\sin\omega t$, $\mathbf{j}' = \mathbf{j}\cos\omega t + \mathbf{i}\sin\omega t$ and find expressions for the components of magnetisation in these axes in the *steady state* (i.e., when the magnetisation is constant in these rotating axes). Plot the ratio of three components of magnetisation in the rotating axes to M_0 as a function of $\omega/\gamma B_0$ assuming $T\gamma B_1 = 10^{-6}$ and $T\gamma B_0 = 100$.

See *Nuclear Magnetic Resonance Spectroscopy* by R. K. Harris for background reading.

14 Electrons in a laser field
(Pre-requisite: Electromagnetism would help but is not necessary. Familiarity with coding)

When an atom is irradiated by a strong laser pulse, its electric field can detach one of the electrons. What happens next is determined by the motion of this electron in the laser field.

Consider an electron that emerges *with zero velocity* from an atom at the origin ($x = 0$) at $t = 0$ and is acted upon by a periodic electric force in the x direction,

$$F(t) = eE_0\cos(\omega t + \phi),$$

where e is the elementary charge, E_0 is the amplitude of the electric field, ω is its frequency, and ϕ is the initial phase.

- By solving Newton's equation, determine the subsequent motion of the electron.
- Driven by the field, the electron can be directed back towards its parent atom. Investigate the motion of the electron for different phases ϕ and determine whether it returns to the origin. If it does, find the kinetic energy of the electron upon its return, as a function of the phase ϕ (e.g., for ϕ between $-\pi/2$ and $\pi/2$). What is the maximum value of this energy? [Note that this problem may only have a numerical solution and python can be used to find it.]
- In a real experiment, He atoms are irradiated with laser light with wavelength $\lambda = 800$ nm [Th. Weber *et al.*, Phys. Rev. Lett. **84**, 443 (2000)]. Determine the intensity of the laser light that will allow the returning electron to knock out the second electron from the same atom.

15 Models for electrolyte solutions

(Pre-requisite: None)

In the Debye-Hückel model for electrolyte solutions, the electrostatic potential due to an ion A with charge q_A is reduced from the Coulomb value $\phi(r) = q_A/4\pi\varepsilon\varepsilon_0 r$ due to screening by other ions. In this expression r is the distance from the origin, and ε is the dielectric constant of the liquid. To obtain the *screened potential* one combines the Poisson equation for the electrostatic potential,

$$\nabla^2\phi(r) = -\frac{\rho(r)}{\varepsilon\varepsilon_0}, \tag{1}$$

where $\rho(r)$ is the charge density, with the Boltzmann expression for the probability per unit volume of finding an ion B with charge q_B at distance r from ion A

$$p_B = c_B \exp[-q_B\phi(r)/kT]. \tag{2}$$

Here c_B is the average number of ions of type B per unit volume (i.e., their concentration), k is the Boltzmann constant and T is temperature. Combining (1) and (2) one obtains

$$\nabla^2\phi(r) = -\sum_i c_i q_i \exp[-q_i\phi(r)/kT]/(\varepsilon\varepsilon_0), \tag{3}$$

where the sum is over all types of ions in the solution.

(a) Write down this equation for the case where there are two types of ions present in solution with charges $+e$ and $-e$ and concentration c (e.g., a solution of NaCl).

(b) Simplify it by expanding the exponential in powers of $e\phi/kT$ and keeping the lowest order term. Solve this linearised equation for $\phi(r)$.

(c) Discuss the changes that result if the ions have diameter a so that equation (2) only holds for $r > a$ and $p_B = 0$ for $r < a$.

Reading: P. W. Atkins and J. de Paula, *Physical Chemistry*.

16 Damped driven oscillations

(Pre-requisite: Classical Mechanics would help but is not necessary. Familiarity with coding)

The damped, driven oscillator obeys the equation of motion

$$\ddot{x} + b\dot{x} + \sin x = a\cos\omega t,$$

where a, b and ω are constants. For small displacements, i.e., $|x| \ll 1$, one has $\sin x \simeq x$, and this equation can be solved analytically. For large x, however, this nonlinear differential cannot be solved analytically, but it can be solved numerically. The solutions of this equation have some very surprising properties for appropriate values of a, b and ω.

Give an account of the different types of motion possible for this system. [This problem can be investigated numerically using python.]

17 Chaos and nonlinearity

(Pre-requisite: None)

One of the most interesting examples of chaos is given by the logistic equation

$$x_n = r x_{n-1}(1 - x_{n-1}),$$

where, choosing a value $0 < x_0 < 1$ and a fixed number $0 \leq r \leq 4$, we can calculate a sequence of values $\{x_1, x_2, ..., x_N\}$ by repeated use of this equation. As r increases, the regular behaviour changes to chaos.

By investigating the behaviour as a function of r, find out when bifurcations start, and when chaos starts. Can you expand on your findings?

References

J. Gleick, *Chaos: making a new science*.

S. H. Strogatz, *Nonlinear dynamics and chaos*.

18 Harmonic oscillators

(Pre-requisite: None really, but Classical Mechanics would make things easier)

A point P moves so that its position vector $\mathbf{r}$ relative to the origin O satisfies the equation $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$, where ω is a constant. Show that the motion of P is compounded from two harmonic oscillations executed with the same frequency at right angles to each other and that the locus of P is an ellipse with the centre at O.

Show further that if P has Cartesian coordinates (x, y) in this plane and O is the origin of coordinates, the locus of P has the equation

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos(\epsilon_2 - \epsilon_1) = \sin^2(\epsilon_2 - \epsilon_1),$$

where a_1 and a_2 are the amplitudes of the harmonic oscillations and ϵ_1 and ϵ_2 are their respective phases.

Discuss the shape and orientation of this locus for different values of $\epsilon_2 - \epsilon_1$.

If the two harmonic oscillations are now taken to have different frequencies, obtain the equation of the path of P.

19 Random walks on Wall Street

(Pre-requisite: None really, but you need to be willing to code and you should be familiar with differential equations)

The problem in predicting the prices of financial products is an apparently random nature of their variation [1]. One mathematical model of price fluctuations is the *random walk* [1,2], where one assumes that the price jumps (up or down) are given by a normal distribution. This mathematical problem is very closely related to the problem of Brownian motion solved in 1905 by Einstein.

Let $S(t)$ be the price of shares at time t. Let us assume that the change in S during time interval dt can be described by the equation

$$dS = aSdt + bSdW, \tag{1}$$

where a and b are constants. The second term in Eq. (1) is a random jump and dW is a normally distributed random variable with zero mean and variance dt.

This project is about solving equation (1) step-by-step using a computer, for one or two values of a and b. This requires writing a short program that generates the random numbers (jumps) so that you can calculate dS. You can then compare your computed result for $S(t)$ with the exact solution of the equation.

[1] J. C. Hull J C,*Options, futures and other derivatives*.

[2] P. E. Kloeden, E. Platen, and H. Shurz, *Numerical solution of stochastic differential equations through computer experiments*.

20 Weighing on a ship

(Pre-requisite: Classical Mechanics)

The simplest way of determining the mass of an object is by measuring the gravitional force mg on this object. This is fine on land as the Earth is (approximately) an inertial frame of reference and $g \approx$ const. However, a ship rocked by waves is not an inertial frame, and the apparent "gravity" is not constant. This needs to be taken into account when, for example, weighing the catch. In this project you will investigate how this can be done.

G. Kessling, D. Birnbacher and C. Berg, *Meas. Sci. Technol.* **4**, 1035–1042 (1993).

21 Ellipse and its normals

(Pre-requisite: None)

An ellipse can be defined by its equation in Cartesian coordinates

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b ($b \leq a$) are its semimajor and semiminor axes, respectively. Alternatively, it can be defined in the parametric form by $x = a\cos t$, $y = b\sin t$, $0 \leq t \leq 2\pi$.

How many *normals* to the ellipse can you draw from a given point in the xy plane? [The normal is a line perpendicular to the tangent at the point where it meets the curve.]

22 Bouncing ball

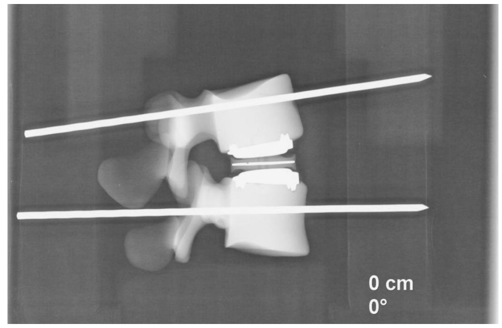
(Pre-requisite: Classical mechanics)

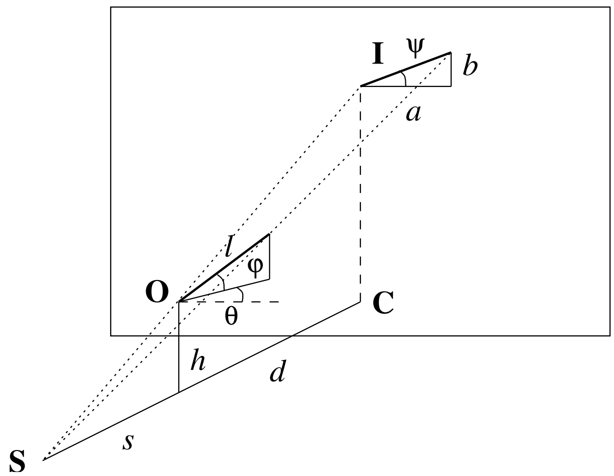
A small ball is released from rest from a point above a large sphere of radius R. If this point lies on the vertical line through the centre of the sphere, the ball will keep bouncing along this line. However, if the point is some distance d away from this line ($d < R$), the ball will bounce off at an angle after colliding with the sphere. In this case it will only make a finite number of bounces on the spherical surface, before jumping off it.

Estimate the number of times the ball will bounce off the sphere. [You may assume that the collisions are perfectly elastic. It might be easier to find an estimate for $d \ll R$. However, an exact (numerical?) solution may also be possible in the general case.]

23 What's the angle between your vertebrae?
(Pre-requisite: None)

Using X-rays, one can measure angles between bones in a body (e.g., between two vertebrae, as shown on the X-ray of a model with pins [1]). However, when measuring an angle on an image, the result may not be the same as the true angle. Thus, in the diagram below the angle φ that an object ("stick" of length l) makes with the horizontal, is not necessarily equal to the angle ψ between its image I and the horizontal line on the screen.





The difference between φ and ψ may depend on the distance s between the X-ray source S and the plane parallel to the screen which contains one end of the object O, and the distance d between O and the plane of the screen. It may also depend on the height h of the end-point O above the line SC perpendicular to the screen, and on the angle θ that the vertical plane through O makes with the direction parallel to the screen.

Using the above diagram or otherwise, show that

$$\tan\psi = \frac{1}{\cos\theta}\tan\varphi - \frac{h}{s}\tan\theta. \tag{2}$$

Investigate how the difference beteeen ψ and φ depends on the angle θ and the vertical displacement h.

You can further investigate if one can find φ from the results of two measurements (yielding ψ_1 and ψ_2), corresponding to two position θ_1 and θ_2, for which only the difference $\Delta\theta = \theta_2 - \theta_1$ is known.

[1] John McManus, *The Influence of X-Ray Technique on the Angle Measurement of an Artificial Disc*, Thesis (Queen's University Belfast, 2006).

24 Two ships at anchor
(Pre-requisite: None)

Two ships, A and B, are at anchor some distance from each other and from shore. A boat launched from A takes a number of crew ashore and then sails to B. What is the quickest path the boat can take?

Start by considering the simplest case of a straight shoreline. Then consider other shapes. Can you find equations that would solve the problem in the general case?

If you want to look at a more difficult problem, add a sea current parallel to the shore!

25 Ladybird lost

(Pre-requisite: Classical mechanics would help, but really none)

In a round room of radius R, a large number of coins N of diameter d are randomly dispersed upon the floor. A ladybird starts from the centre of the room, crawling at speed v.

1. How long (on average) does it take before it meets a coin?
2. Suppose that every time the ladybird meets a coin, it changes direction at random. How long (on average) before it makes it to the wall?
3. Suppose every time it 'hits' a coin, the coin magically disappears. Work out (approximately, and on average) the law of decrease of the number of remaining coins as a function of time. (Assume that if, in the process, the ladybird hits the wall, it is simply 'reflected' back towards the interior of the room, at a random angle.)
4. Calculate the answers for parts 1–3 for a room of typical size, 1p coins and $v = 1$ cm/s, and N of your choice.

Keywords: to work on this problem, you will need to think (or read) about mean free paths, random walks/diffusion/Bro motion.

26 Finding the truth in TEQs

(Pre-requisite: None but willingness to code a bit)

One of the questions that students answer when filling in Teaching Evaluation Questionnaires (TEQ) is about the percentage of lectures they attended. After tallying, these results (in simplified form) are presented in a table like this

% of lectures attended	25%	50%	75%	100%
No. of answers	7	19	42	29

where the total number of answers is 97.

Since the TEQs are filled in in a lecture, the frequencies in the table above are *biased*. Using the data from the table, estimate the true frequencies that would be observed if the TEQ were filled by all 150 students in the class. You can assume that attendances by individual students are uncorrelated random variables. Using your result, check whether the attendance of the lecture where TEQ was taken, was typical for the given class size.

Generalise your approach to an arbitrary number of scores, and to a continuous distribution. In each case, determine the mean and standard deviation od the number of students in class based on the true frequencies that you obtain from the frequencies observed in the lecture.

27 Narrow escape

(Pre-requisite: None)

A circular field is surrounded by a fence. A man standing in the centre of the field can run with speed v. A dog can run along the fence with speed $u = 4v$. Will the man be able to escape, i.e., reach a point in the fence before the dog gets there? (The dog and the man can see each other at all times, and the dog is doing its best to catch the man.) What strategy/path should the man follow?

What is the maximum ratio u/v for the which the man can escape? If the ratio is greater than this, where can the man be at the start, relative to the dog, so that he can still escape?

28 Lenz's potentials

(Pre-requisite: Classical Mechanics)

For a particle moving in a central field $U(r)$, the path can be found using plane polar coordinates r and ϕ in terms of an indefinite integral [1]. However, there are only a handful of potentials for which this integral and the equation of the path can be found analytically. You are familiar with one example, namely the gravitational potential $U(r) = -\alpha/r$, in which the paths are conic sections (ellipse, parabola or hyperbola). Using this as a starting point, investigate the paths of the particle with zero energy ($E = 0$) in *Lenz's potential*

$$U(r) = -\frac{2uR^2}{r^2\left[\left(\frac{r}{R}\right)^\mu + \left(\frac{R}{r}\right)^\mu\right]^2},$$

where u, R and μ are constants, for different values of μ. Such potentials find some unexpected applications in atoms and clusters [V. N. Ostrovsky, Phys. Rev. A **56**, 626 (1997)].

[1] L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford, 2001).

29 Minds swapping

(Pre-requisite: None)

A long time ago in a galaxy far, far away, four friends, Amy, Bob, Charlie and David found a strange device capable of swapping the minds of the two people using it. At first the group find it amusing, in fact Amy swapped with Bob, Charlie swapped with David and finally Amy (in Bob's body) swapped with Charlie.

Unfortunately, they soon realised that the device would not swap back a pair that already used the device. Can the 4 friends swap back their minds to their original bodies?

Two other friends Eliza and Francis go and help their four friends. Is there a sequence of swaps that brings everybody to their original bodies avoiding a pair using the device twice?

Can the problem be solved when N bodies are shuffled?
How many swaps are needed?
Discuss possible extensions.

30 "Impossible" exam question

(Pre-requisite: None)

In January 2015, some final year economics students at Sheffield University claimed that their exam contained an "impossible question" [1]. However, the question, which is reproduced below, should give no trouble to a mathematics student! Moreover, you should also be able to find ways of extending it.

Consider a country with many cities and assume that there are $N > 0$ people in each city. Output per person is $\sigma N^{0.5}$ and there is a coordination cost per person of γN^2. Assume that $\sigma > 0$ and $\gamma > 0$.

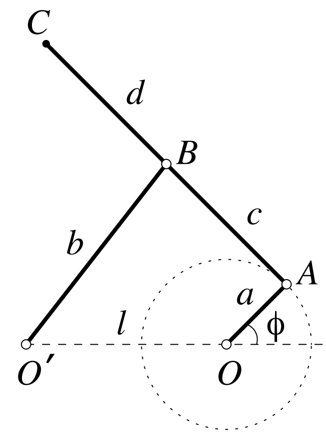
(a) What sort of things does the coordination cost term γN^2 represent? Why does it make sense that the exponent on N is greater than 1? [10 marks]

(b) Draw a graph of per-capita consumption as a function of N and derive the optimal city size N. How does it depend on the parameters σ and γ? Provide intuition for your answers. [10 marks]

(c) Describe which combinations of σ and γ generate a peasant economy, meaning an economy with no cities (or 1-person cities). Why might the values of the parameters σ and γ have changed over time? What do these changes imply in terms of the optimal city size? [10 marks]

[1] http://www.bbc.com/news/education-31057005.

31 Design a robot
(Pre-requisite: None)



Consider a planar mechanism which consists of three bars connected by joints (see diagram). Bars OA and $O'B$ (called cranks) are free to rotate about fixed points O and O', respectively. Find how the coordinates of point C depend on the angle ϕ that rod OA makes with the line $O'O$. Explore how the shape of the path described by point C depends on the lengths of the cranks a and b, and lengths c and d, taking them as fractions of the length l between points O and O'. In particular, if this mechanism is to model the motion of a foot, find the optimal lengths that would make point C move in a curve, part of which is close to a straight line (foot in contact with the ground), the other part being an arc-like path of the foot though the air.