Cumulative Summation Control Charts

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Introduction

- To this point, we have primarily been discussing Shewhart-style charts
- The $\bar{x}$, R, s, s^2, p, C, and U charts are all Shewhart-style which means we have:
  1. A sample plotting statistic
  2. An upper and possibly lower control limit based on the in-control sampling distribution of the plotting statistic
  3. An assumption of independence between observations

Introduction

- Shewhart-style charts are useful for (1) efficiently detecting large shifts ($> 2\sigma$) and (2) ease of interpretation and set up.

- However, it also has limitations:
  1. Since we assume independence, the only information Shewhart-style charts take into consideration is contained in the most recently observed point (the whole sequence is not considered).
  2. It is inefficient at detecting smaller shifts ($< 1.5\sigma$).
  3. Use of runs rules/warning limits etc., impede the interpretability of the charts
  4. Not effective when $n = 1$

- This leads to the need for new types of control charts which address these two limitations, one of which is called the "Cumulative Summation (CUSUM)" control chart.

CUSUM Chart

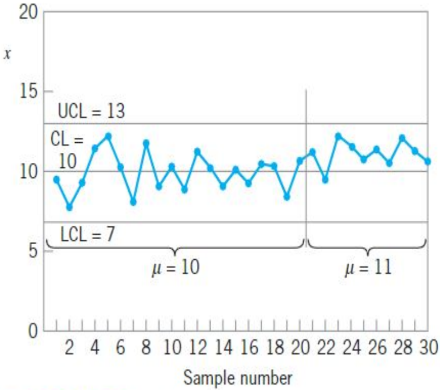

FIGURE 9.1 A Shewhart control chart for the data in Table 9.1.

CUSUM Chart

- As we can see in this prior example, the first 20 observations come from a normal distribution with mean 10 and standard deviation 1. The following 10 come from a normal distribution with mean 11 and standard deviation 1.
- If we only rely upon the control limits, we would not stop the process to search for an assignable cause despite a shift having taken place.
- So how does the CUSUM chart solve this problem?

CUSUM Chart

- The CUSUM chart was proposed by Page (1954). Here's the concept:
- Let's say we want to monitor the mean of a process and this process has a target value, say μ_0. To monitor the process, we still take random samples and obtain an estimate of the sample mean, say $\bar{x}_i$.
- Instead of plotting the individual value of $\bar{x}_i$ itself, we take the deviance between the target value and the sample mean, $\bar{x}_i - \mu_0$.

CUSUM Chart

- Then, the cumulative summation component is that the plotting statistic is a function of the cumulative sum of the deviances.

$$C_i = \sum_{j=1}^{i} (\bar{x}_i - \mu_0)$$

- Here, the idea is that if the process starts to drift upwards or downwards, those cumulative differences will eventually signal to us that a shift has occurred. Notice that we are taking all $\bar{x}_i$'s into consideration instead of just the latest one!

CUSUM Chart

- What you'll likely notice from the previous C_i is that its distribution is non-stationary. If you've taken time series, you'll know that this means that either the mean or the variance change with time.
- If we assume $\bar{x}_i$ is normally distributed with mean μ_0 and variance σ^2/n, then:

$$C_i = \sum_{j=1}^{i}(\bar{x}_i - \mu_0) \dot{\sim} N\left(0, \frac{i\sigma^2}{n}\right)$$

CUSUM Chart

- This may serve as a limitation if we were to construct control limits in a Shewhart-style manner as the limits will change over time and also won't approach an asymptote.
- Consequently, there have been two proposed variants to the CUSUM chart to help solve this problem, but we will only focus on the one that's mostly used in practice: the **Tabular** CUSUM.

CUSUM Chart

- The plotting statistic for the Tabular CUSUM is actually two statistics, an upper and a lower CUSUM:

$$C^+ = \max[0, \bar{x}_i - (\mu_0 + K) + C^+_{i-1}]$$
$$C^- = \max[0, (\mu_0 - K) - \bar{x}_i + C^-_{i-1}]$$

where, $C_0^+ = C_0^- = 0$ and K is referred to as the “slack” or “allowance” value.

CUSUM Chart

- Typically, K is chosen to be the midpoint between our target value, μ_0 and an out-of-control value, μ_1, that we're interested in quickly detecting:

$$K = \frac{|\mu_1 - \mu_0|}{2}$$

- C^+ is for detecting upward shifts ($\mu_1 > mu_0$) and C^- is for detecting downward shifts ($\mu_1 < \mu_0$).

- If either C_i^+ or C_i^- exceed some control limit, say H, this signals to us that the process may be OOC.

CUSUM Chart

- Let's look at an example using the data used in the example where the process mean shifted from 10 to 11.

CUSUM Chart

- If a shift has taken place, it is often useful information to see what the new mean (or other characteristic that you're monitoring) has shifted to.

- To do this, your text recommends using:

$$\hat{\mu} = \mu_0 + K + \frac{C_i^+}{N^+}, \quad C_i^+ > H$$

$$\hat{\mu} = \mu_0 - K - \frac{C_i^-}{N^-}, \quad C_i^- > H$$

- C_i^+ and C_i^- denote the upper and lower CUSUM values associated with the first OOC point, and N^+ and N^- denote the number of non-zero C_i^+ and C_i^- values in the sequence which eventually lead to the OOC point. Let's go ahead and estimate the new mean in our example.

CUSUM Chart Design

- In the last example, we used a CUSUM control chart where $H = 5$ and $K = 0.50$. How do we choose those values?
- Generally, the rule of thumb is to define $H = h\sigma$ and $K = k\sigma$, where $h = 4$ or 5 and $k = 0.50$. So for a given known or estimated value of σ, we can obtain H and K.

CUSUM Chart Design

TABLE 9.3

ARL Performance of the Tabular CUSUM with $k = \frac{1}{2}$ and $h = 4$ or $h = 5$

Shift in Mean (multiple of σ)	$h = 4$	$h = 5$
0	168	465
0.25	74.2	139
0.50	26.6	38.0
0.75	13.3	17.0
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

TABLE 9.4

Values of k and the Corresponding Values of h That Give $ARL_0 = 370$ for the Two-Sided Tabular CUSUM [from Hawkins (1993a)]

k	0.25	0.5	0.75	1.0	1.25	1.5
h	8.01	4.77	3.34	2.52	1.99	1.61

CUSUM Chart Usage

- So far we have seen that CUSUM can be used for monitoring sample means when $n \geq 1$.
- However, CUSUM can be used for other statistics as well, including the sample standard deviation, sample proportions, number of nonconforming items, etc.
- It can also be used for variable sample sizes by standardizing observations.

CUSUM ARL Estimation

- Unlike the Shewhart-style charts, the plotting statistics for the CUSUM chart are not independent of each other. Thus, the typical $ARL_0 = 1/\alpha$ and $ARL_1 = 1/(1-\beta)$ calculations aren't valid.
- Estimating ARL for CUSUM charts can be quite complicated. However, we do have some rough estimates that we can use in Phase I when making decisions about chart construction.
- For a one-sided CUSUM, Siegmund (1985) provided an approximation:

CUSUM ARL Estimation

$$ARL = \frac{\exp(-2\Delta b) + 2\Delta b - 1}{2\Delta^2}$$

- where $\Delta = \delta^* - k$ for the upper CUSUM, $\Delta = -\delta^* - k$ for the lower CUSUM, $b = h + 1.166$ and $\delta^* = (\mu_1 - \mu_0)/\sigma$.
- Then, if we're using a two-sided CUSUM, we can estimate the overall ARL by:

$$\frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}$$

CUSUM ARL Estimation

- Let's use the piston rings Phase I data to estimate the ARL_1 where we want to protect against shifts of 1.5σ in either the positive or negative direction.