CS 331: Theory of Computing

Problem Set 3

Iowa State University
Computer Science Department
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Due by the midnight of February 14, 2013

Problem 1 (20 points)

If $\mathscr{L}$ is any language, let $\mathscr{L}_{\frac{1}{2}-}$ be the set of all first halves of strings in $\mathscr{L}$ so that

$$L_{\frac{1}{2}-} = \{\, w \;\mid\; \textit{for some } w', |w| = |w'| \;\&\; ww' \in L \,\}.$$

Prove that, if $\mathscr{L}$ is regular, then so is $\mathscr{L}_{\frac{1}{2}-}$.

Hint: Meet In The Middle. Let $\mathcal{A}$, $\mathcal{B}$ be automata such that $\mathscr{L}(\mathcal{A}) = \mathscr{L}$ and $\mathscr{L}(\mathcal{B}) = \mathscr{L}^{\mathcal{R}}$ ($\mathscr{L}^{\mathcal{R}}$ is defined as in Problem 2 in HW2). Use $\mathcal{A}$ and $\mathcal{B}$ to construct an automaton $\mathcal{C}$ that recognizes $\mathscr{L}_{\frac{1}{2}-}$.

Problem 2 (20 points)

A **universal** finite automaton (UFA) $\mathcal{A}$ is a 5-tuple $\langle Q, \Sigma, Q_0, \delta, F\rangle$ (syntactically just like an NFA) that accepts a word w if **every** run of $\mathcal{A}$ over w ends in F. Note that an NFA accepts a word if there **exists** a run of $\mathcal{A}$ over w that ends in F.

Prove that UFAs recognize the class of regular languages, that is, a language $\mathscr{L}$ is recognized by a UFA if and only if $\mathscr{L}$ is regular.

Hint: Given a UFA $\mathcal{A}$, can you construct an NFA that recognizes $\overline{\mathscr{L}(\mathcal{A})}$, the complement of $\mathscr{L}(\mathcal{A})$?

Problem 3 (20 points)

Let $\Sigma = \{0, 1\}$. A palindrome is a word that reads the same forward and backward. For example, "ABLE WAS I ERE I SAW ELBA" is palindromic. Prove that the following language

$$\{\, w \in \Sigma^* \mid w \text{ is not a palindrome} \,\}$$

is not regular.

Problem 4 (20 points)

Let $k > 1$ and $\mathscr{L}_k = \{\epsilon, a, aa, \ldots, a^{k-2}\}$. Prove the following statements:

1. $\mathscr{L}_k$ can be recognized by a DFA with k states. (5 points)
2. $\mathscr{L}_k$ cannot be recognized by any DFA with $k - 1$ states. (15 points)

Hint: Use the pigeonhole principle as in the proof of the Pumping Lemma to prove Part 2.

Problem 5 (20 points)

Let $\mathscr{L}_n = \{w \in \{0,1\}^* \mid$ *the n-th letter of w from the end is* $1\ \}$ for $n \geq 1$. Prove that

- an NFA with $n+1$ recognizes $\mathscr{L}_n$ (5 points).
- any DFA that recognizes $\mathscr{L}_n$ needs at least 2^n states (15 points).

Hint: Let S_n be the set of all words of length n. Show that any DFA $\mathcal{A}$ that recognizes $\mathscr{L}_n$ should have a distinctive state q_w for each $w \in S_n$ such that whenever $\mathcal{A}$ reaches q_w, w is the suffix of the word that has been read by $\mathcal{A}$.