CS331: Homework #1

Due on February 1, 2013 at 11:59pm

Professor Zhang 9:00am

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Problem 1

Show that for any integer $k \geq 2$, $\sqrt[k]{2}$ is an irrational number.

Proof. To prove by contradiction, suppose that $\sqrt[k]{2}$ is rational. Then

$$\sqrt[k]{2} = \frac{p}{q}$$

where p and q are integers and co-prime. If we are to raise both sides to k then we get

$$2 = \left(\frac{p}{q}\right)^k$$

Which we can write as

$$2 = \frac{p^k}{q^k}$$

We multiply each side by q^k and get

$$2q^k = p^k$$

Thus p is even because any number times 2 is even. Let $p = 2j$ for $j \in \mathbb{Z}$. Then

$$\begin{aligned} 2q^k &= (2j)^k \\ &= 2^k j^k \end{aligned}$$

dividing both sides by 2 yields

$$\begin{aligned} q^k &= 2^{k-1} j^k \\ q^k &= 2(2^{k-2} j^k) \end{aligned}$$

since $k \geq 1$, q is even because any number multiplied by 2 is even. This is a contradiction because earlier p and q were co-prime meaning there were no numbers that could be divided into both of them. Thus p and q can't both be even. $\square$

Problem 2

Show that for every $n \geq 0$ a depth n perfect binary tree has $2^{n+1} - 1$ nodes.

Proof. We will do a proof by induction to prove that for every $n \geq 0$ a perfect binary tree of has $2^{n+1} - 1$ nodes.

Base For the base case, we have a perfect binary tree of height $= 0$. Then

$$2^{n+1} - 1 = 2^{0+1} - 1 = 1 \text{ node}$$

which is true because a tree of height 0 is a single root node.

Induction Step We will prove that for every $n \geq 0$ a perfect binary tree of height $n + 1$ has $2^{(n+1)+1} - 1$ nodes.

Consider a tree, T with height h. To create a perfect binary tree of height $h + 1$, we can take two of T and connect it to a single root node. Thus

$$\begin{aligned} \text{nodes} &= T + T + 1 \\ &= 2T + 1 \end{aligned}$$

By the induction hypothesis

$$\begin{aligned} \text{nodes} &= 2(2^{n+1} - 1) + 1 \\ &= 2 \times 2^{n+1} - 2 + 1 \\ &= 2 \times 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \\ &= 2^{(n+1)+1} - 1 \end{aligned}$$

Thus we have concluded our proof by showing that a perfect binary tree of height $n + 1$ has $2^{(n+1)+1} - 1$ nodes. $\square$

Problem 3

Proof. We will use proof by induction to show that for any truth assignment M and M' such that $M \leq M'$ and any positive propositional formula φ, if $M \models \varphi$, then $M' \models \varphi$.

Base For the base case, we will consier the propositional formula φ with just one variable, $\varphi = p$, and two truth assignments, M and M' such that $M \leq M'$.

We know that $M(p) \leq M'(p)$, therefore we have the following possibilities:

$M(p)$	$M'(p)$
0	0
0	1
1	1

If $M \models \varphi$, then $M(p)$ must be equal to 1, which according to the truth table, $M'(p)$ is also equal to 1. Therefore $M' \models \varphi$. This concludes the base step.

Induction Step Given the positive propositional formula φ_0 and two truth assignments, M and M' such that $M \leq M'$. We will show that for any positive propositional formula φ, if $M \models \varphi$, then $M' \models \varphi$.

Taking φ_0, we can do one of two things to extend the propositional formula becuase it is positive. We can conjunct or disjunct it with any other propositional variable.

Case 1 Let φ be the conjunction of the propositional variable p to it, $\varphi = (\varphi_0 \wedge p)$.

By assuming the induction hypothesis, we can assume that $M \models \varphi_0$. Then we know that M is true under φ_0 making it equal to 1.

Since $M \leq M'$, we have three values to represent like the table in the base step. We can represent it like so:

φ_0	$M(p)$	$M'(p)$	$\varphi_0 \wedge M(p)$	$\varphi_0 \wedge M'(p)$
1	0	0	0	0
1	0	1	0	0
1	1	1	1	1

We can see that the only time M is true over φ ($M \models \varphi$) is also when M' is true over φ ($M' \models \varphi$). Therefore we have proven that for conjunction, $M' \models \varphi$.

Problem 3 continued on next page. . .

Case 2 Let φ be the disjunction of the propositional variable p to it, $\varphi = (\varphi_0 \vee p)$.

By assuming the induction hypothesis, we can assume that $M \models \varphi_0$. Then we know that M is true under φ_0 making it equal to 1.

Since $M \leq M'$, we have three values to represent like the table in the base step. We can represent it like so:

φ_0	$M(p)$	$M'(p)$	$\varphi_0 \vee M(p)$	$\varphi_0 \vee M'(p)$
1	0	0	1	1
1	0	1	1	1
1	1	1	1	1

We can see that the disjunction of any value with φ_0 will result in the truth assignment being true over φ. Therefore we have proven that for disjunction, $M' \models \varphi$.

We have concluded both cases therefore our proof is finished. $\square$

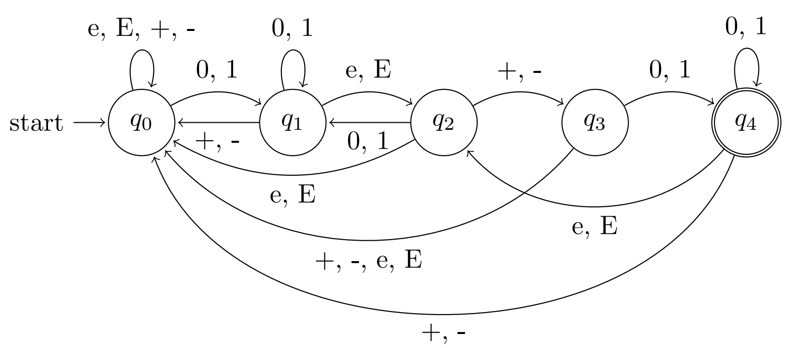
Problem 4

Let $\Sigma = \{0, 1\}$. What language is defined by the following regular expression? Define it in one or two sentences.

1. $\Sigma^*0\Sigma^*1\Sigma^*$
   The language is the set of all words that have at least one 0 and one 1 in them and the 0 comes before the 1.

2. 00^*1^*
   The language is the set of all words that begin with a zero. The word also ends with any number of zeroes (including none) followed by any number of ones (including none).

Problem 5



Problem 6

Let $A = \langle Q, \Sigma, \delta, q_0, F \rangle$ and $B = \langle Q, \Sigma, q_0, F' \rangle$ be two nondeterministic finite automata such that $F' = Q \setminus F$. Prove or disprove: The language of $L(B)$ is the complement of the language $L(A)$.

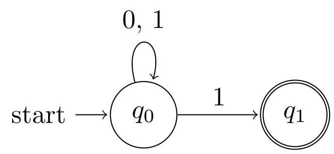


Figure 1: Automata A

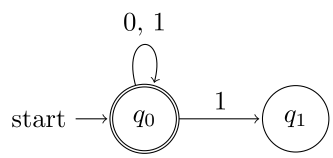


Figure 2: Automata B

As a counterexample, consider the automata A and B pictured in Figure 1 and Figure 2 respectively.

The only difference between the two automata is that the final state set of B, F' is just $F' = Q \setminus F$ where Q is the set of states in A and F is the set of final states in A.

This is a counterexample because the word $\omega = 1$ is accepted by both automata. Thus $\omega \in L(A)$ and $\omega \in L(B)$ therefore $L(A)$ is not the complement of $L(B)$.