

# CS331: Homework #1

Due on January 31, 2012 at 11:59pm

*Professor Zhang 9:00am*

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## Problem 1

Show that for any integer  $k \geq 2$ ,  $\sqrt[k]{2}$  is an irrational number.

*Proof.* To prove by contradiction, suppose that  $\sqrt[k]{2}$  is rational. Then

$$\sqrt[k]{2} = \frac{p}{q}$$

where  $p$  and  $q$  are integers and co-prime. If we are to raise both sides to  $k$  then we get

$$2 = \left(\frac{p}{q}\right)^k$$

Which we can write as

$$2 = \frac{p^k}{q^k}$$

We multiply each side by  $q^k$  and get

$$2q^k = p^k$$

Thus  $p$  is even because any number times 2 is even. Let  $p = 2j$  for  $j \in \mathbb{Z}$ . Then

$$\begin{aligned} 2q^k &= (2j)^k \\ &= 2^k j^k \end{aligned}$$

dividing both sides by 2 yields

$$\begin{aligned} q^k &= 2^{k-1} j^k \\ q^k &= 2(2^{k-2} j^k) \end{aligned}$$

since  $k \geq 1$ ,  $q$  is even because any number multiplied by 2 is even. This is a contradiction because earlier  $p$  and  $q$  were co-prime meaning there were no numbers that could be divided into both of them. Thus  $p$  and  $q$  can't both be even.  $\square$

## Problem 2

Show that for every  $n \geq 0$  a depth  $n$  perfect binary tree has  $2^{n+1} - 1$  nodes.

*Proof.* We will do a proof by induction to prove that for every  $n \geq 0$  a perfect binary tree of has  $2^{n+1} - 1$  nodes.

**Base** For the base case, we have a perfect binary tree of height = 0. Then

$$2^{n+1} - 1 = 2^{0+1} - 1 = 1 \text{ node}$$

which is true because a tree of height 0 is a single root node.

**Induction Step** We will prove that for every  $n \geq 0$  a perfect binary tree of height  $n + 1$  has  $2^{(n+1)+1} - 1$  nodes.

Consider a tree,  $T$  with height  $h$ . To create a perfect binary tree of height  $h + 1$ , we can take two of  $T$  and connect it to a single root node. Thus

$$\begin{aligned} \text{nodes} &= T + T + 1 \\ &= 2T + 1 \end{aligned}$$

By the induction hypothesis

$$\begin{aligned} \text{nodes} &= 2(2^{n+1} - 1) + 1 \\ &= 2 \times 2^{n+1} - 2 + 1 \\ &= 2 \times 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \\ &= 2^{(n+1)+1} - 1 \end{aligned}$$

Thus we have concluded our proof by showing that a perfect binary tree of height  $n + 1$  has  $2^{(n+1)+1} - 1$  nodes.  $\square$

## Problem 3

## Problem 4

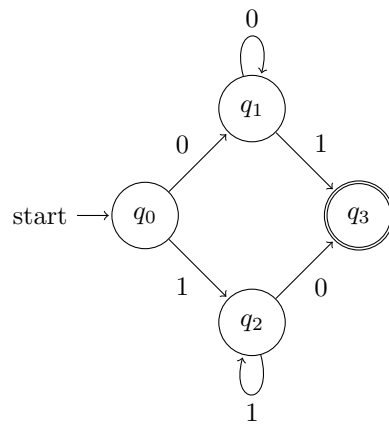
Let  $\Sigma = \{0, 1\}$ . What language is defined by the following regular expression? Define it in one or two sentences.

1.  $\Sigma^*0\Sigma^*1\Sigma^*$

The language is the set of all words that have at least one 0 and one 1 in them.

2.  $00^*1^*$

The language is the set of all words that begin with a zero. The word also ends with any number of zeroes (including none) followed by any number of ones (including none).

**Problem 5**

## Problem 6