CS 331: Theory of Computing

Problem Set 7

Iowa State University
Computer Science Department
March 21, 2013

Due by the midnight of March 29, 2013

Problem 1 (10 points)

Let $\mathscr{K} = \{\langle M \rangle \mid \langle M \rangle \notin \mathscr{L}(M)\}$. Prove that $\mathscr{K}$ is **not** Turing recognizable.

Hint: Russell's Paradox.

Problem 2 (20 points)

Prove the following statements.

1. $\mathscr{L}_1 \leq_m \mathscr{L}_2$ and $\mathscr{L}_2 \leq_m \mathscr{L}_3$ imply $\mathscr{L}_1 \leq_m \mathscr{L}_3$. (10 points)
2. $\mathscr{L}_1 \leq_T \mathscr{L}_2$ implies that $\overline{\mathscr{L}_1} \leq_T \overline{\mathscr{L}_2}$. (10 points)

Note: $\leq_T$ is Turing reduction (introduced in Chapter 5.1) and $\leq_m$ is many-one reduction (introduced in Chapter 5.3).

Problem 3 (10 points)

Prove that $A_{TM} \not\leq_m \overline{A_{TM}}$.

Hint: Proof By Contradiction and Problem 2.

Problem 4 (20 points)

Which of the following PCP problems has a solution? Justify your answer.

$$(1)\left\{ \left[\frac{ab}{a}\right], \left[\frac{bb}{ab}\right], \left[\frac{aa}{ba}\right], \left[\frac{cc}{bc}\right], \left[\frac{aa}{ca}\right], \left[\frac{d}{cd}\right] \right\}$$

$$(2)\left\{ \left[\frac{ab}{a}\right], \left[\frac{bb}{ab}\right], \left[\frac{aa}{ba}\right], \left[\frac{c}{bc}\right], \left[\frac{aa}{ca}\right], \left[\frac{d}{cd}\right] \right\}$$

Problem 5 (20 points)

Does the following PCP problem P have a solution?

$$\left\{ \begin{array}{|c|}\hline a \\ \hline ab \\ \hline\end{array} \begin{array}{|c|}\hline b \\ \hline ccc \\ \hline\end{array} \begin{array}{|c|}\hline c \\ \hline b \\ \hline\end{array} \begin{array}{|c|}\hline c \\ \hline d \\ \hline\end{array} \begin{array}{|c|}\hline dddd \\ \hline \\ \hline\end{array} \begin{array}{|c|}\hline ddde \\ \hline e \\ \hline\end{array} \right\}$$

Hint: Prove that P has a solution if and only if $\exists n\, (3^n \mod 4) = 3$.

Problem 6 (20 points)

Let $\Sigma = \{0, 1, -\}$ (where – denotes whitespace) be the tape alphabet for all TMs in this problem. Define the busy beaver function $BB : \mathbb{N} \to \mathbb{N}$ as follows. For each value of k, consider all k-state TMs that halt when started with a blank tape. Let $BB(k)$ be the maximum number of steps that can be performed by a k-state Turing machine. Show that BB is not a computable function.

Hint: Halting Problem.