CS331: Homework #1

Due on January 31, 2012 at 11:59pm

Professor Zhang 9:00am

Josh Davis

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Problem 1

Show that for any integer $k \geq 2$, $\sqrt[k]{2}$ is an irrational number.

Proof. To prove by contradiction, suppose that $\sqrt[k]{2}$ is rational. Then

$$\sqrt[k]{2} = \frac{p}{q}$$

where p and q are integers and co-prime. If we are to raise both sides to k then we get

$$2 = \left(\frac{p}{q}\right)^k$$

Which we can write as

$$2 = \frac{p^k}{q^k}$$

We multiply each side by q^k and get

$$2q^k = p^k$$

Thus p is even because any number times 2 is even. Let $p = 2j$ for $j \in \mathbb{Z}$. Then

$$\begin{aligned} 2q^k &= (2j)^k \\ &= 2^k j^k \end{aligned}$$

dividing both sides by 2 yields

$$\begin{aligned} q^k &= 2^{k-1} j^k \\ q^k &= 2(2^{k-2} j^k) \end{aligned}$$

since $k \geq 1$, q is even because any number multiplied by 2 is even. This is a contradiction because earlier p and q were co-prime meaning there were no numbers that could be divided into both of them. Thus p and q can't both be even. $\square$

Problem 2

Show that for every $n \geq 0$ a depth n perfect binary tree has $2^{n+1} - 1$ nodes.

Proof. We will do a proof by induction to prove that for every $n \geq 0$ a perfect binary tree of has $2^{n+1}-1$ nodes.

Base For the base case, we have a perfect binary tree of height $= 0$. Then

$$2^{n+1} - 1 = 2^{0+1} - 1 = 1 \text{ node}$$

which is true because a tree of height 0 is a single root node.

Induction Step We will prove that for every $n \geq 0$ a perfect binary tree of height $n+1$ has $2^{(n+1)+1} - 1$ nodes.
Consider a tree, T with height h. To create a perfect binary tree of height $h+1$, we can take two of T and connect it to a single root node. Thus

$$\begin{aligned} \text{nodes} &= T + T + 1 \\ &= 2T + 1 \end{aligned}$$

By the induction hypothesis

$$\begin{aligned} \text{nodes} &= 2(2^{n+1} - 1) + 1 \\ &= 2 \times 2^{n+1} - 2 + 1 \\ &= 2 \times 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \\ &= 2^{(n+1)+1} - 1 \end{aligned}$$

Thus we have concluded our proof by showing that a perfect binary tree of height $n+1$ has $2^{(n+1)+1} - 1$ nodes. $\square$

Problem 3

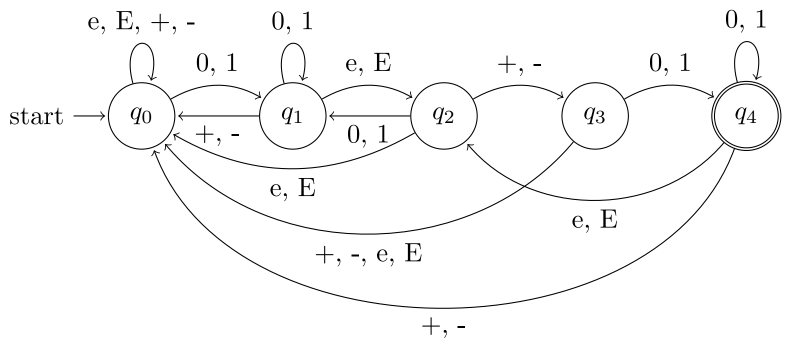
Problem 4

Let $\Sigma = \{0, 1\}$. What language is defined by the following regular expression? Define it in one or two sentences.

1. $\Sigma^*0\Sigma^*1\Sigma^*$
   The language is the set of all words that have at least one 0 and one 1 in them.

2. 00^*1^*
   The language is the set of all words that begin with a zero. The word also ends with any number of zeroes (including none) followed by any number of ones (including none).

Problem 5



Problem 6