Problem 1

a) Given,

$$a_j^l = f(z_j^l)$$

$$z_j^l = \sum_k a_k^{l-1} w_{kj}^l + b_j^l$$

$$C = \frac{1}{2} \left\langle \sum_{j=1}^{n_L} (a_j^L - y_j)^2 \right\rangle$$

$$\Delta_j^l = \frac{\partial}{\partial z_j^l} C$$

Now,

$$\Delta_j^L = \sum_{k=1}^{n_L} (a_k^L - y_k) \frac{\partial}{\partial z_j^L} a_k^L$$

$$= \sum_{k=1}^{n_L} (a_k^L - y_k) \left(\frac{\partial}{\partial z_j^L} a_k^L \right) \left(\frac{\partial z_k^L}{\partial z_j^L} \right)$$

$$= \sum_{k=1}^{n_L} (a_k^L - y_k) \, \delta_{kj}$$

$$= \frac{\partial C}{\partial a_j^L} f'(z_j^L)$$

$$\Delta_j^l = \sum_{k=1}^{n_L} \left(a_k^L - y_k \right) \frac{\partial a_k^L}{\partial z_j^L}$$

$$= \Delta_k^L \left(\sum_{i=1}^{n_{L-1}} \frac{\partial a_i^{L-1}}{\partial z_i^l} \frac{\partial z_i^{L-1}}{\partial z_i^l} W_{ik}^L \right)$$

$$= \Delta_k^L \left[\sum_{i=1}^{n_{L-1}} \sum_{h=1}^{n_{L-2}} f'(z_i^{L-1}) W_{ik}^L \frac{\partial a_h^{L-2}}{\partial z_h^{L-2}} \frac{\partial z_h^{L-2}}{\partial z_i^l} W_{hi}^L \right.$$

$$= \Delta_k^L \left[\sum_{i_1=1}^{n_{L-1}} \cdots \sum_{i_{L-1}=1}^{n_l} \left(f'(z_{i_1}^{L-1}) W_{i_1 k}^L \right) \cdots \left(f'(z_{i_{L-l}}^l) W^l_{i_{L-l}\, i_{L-l+1}} \frac{\partial z^l_{i_{L-l}}}{\partial z_j^l} \right. \right.$$

$$\Rightarrow \Delta_j^{L-1} = \sum_{k=1}^{n_L} \Delta_k^L \sum_{i=1}^{n_{L-1}} f'(z_i^{L-1}) W_{ik}^L \delta_{ij}$$

$$= \sum_{k=1}^{n_L} \Delta_k^{l+1} \left[W^{l+1\,T} \right]_{kj} f'(z_j^l)$$

$$\Rightarrow \Delta_j^{L-2} = \sum_{k=1}^{n_L} \Delta_k^L \sum_{i=1}^{n_{L-1}} f'(z_i^{L-1}) W_{ik}^L \sum_{h=1}^{n_{L-2}} f'(z_h^{L-2}) W_{hi}^{L-2} \delta_{hj}$$

$$= \sum_{i=1}^{n_{L-1}} \left[\sum_{k=1}^{n_L} \Delta_k^L \left[W^{L\,T} \right]_{ki} f'(z_i^{L-1}) \right] \left[W^{L-1\,T} \right]_{ij} f'(z_j^{L-2})$$

$$= \sum_{i=1}^{n_{l+1}} \Delta_i^{l+1} \left[W^{l+1\,T} \right]_{ij} f'(z_j^l)$$

$$\Rightarrow \Delta_j^l = \sum_k \Delta_k^{l+1} \left[W^{l+1\,T} \right]_{kj} f'(z_j^l)$$

b)

$$\frac{\partial C}{\partial w_{ij}^{l}} = \frac{\partial C}{\partial z_j^{l}} \frac{\partial z_j^{l}}{\partial w_{ij}^{l}}$$

$$= a_i^{l-1} \Delta_j^{l}$$

$$\frac{\partial C}{\partial b_j^{l}} = \frac{\partial C}{\partial z_j^{l}} \frac{\partial z_j^{l}}{\partial b_j^{l}}$$

$$= \Delta_j^{l}$$

Problem 2

For $f(z_j^l) = z_j^l$

$$\Rightarrow f'(z_j^l) = 1$$

Then, plugging this into the previous problem's expression, we get

$$\Delta_j^L = \frac{\partial C}{\partial a_j^L} = (a_j^L - y_j)$$

$$\Delta_j^l = \sum_k \Delta_k^{l+1} \left[W^{l+1\,T} \right]_{kj}$$

$$\frac{\partial C}{\partial W_{ij}^l} = a_i^{l-1} \Delta_j^l$$

$$\frac{\partial C}{\partial b_j^l} = \Delta_j^l$$