门函数卷积的分析*

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摘　要：对门函数卷积进行了分析，当两个不同宽度门函数卷积时，其结果为梯形函数，梯形函数的高为宽度窄的门函数脉宽，其底分别为两个门函数脉宽差的绝对值和脉宽和；两个相同宽度的门函数卷积时，其结果为三角函数，三角函数的高为门函数脉宽，其底为两个门函数脉宽和.

关键词：卷积；门函数；三角函数；梯形函数
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卷积积分在信号与系统理论中占有十分重要的地位. 在信号与系统课程教学中，卷积是学生学习的难点之一. 当学生利用卷积定理求三角形函数的频谱[1] 时，教材中利用两个完全相同的门函数卷积可得到三角函数这一结论进行求解，学生对这一结论的来源感到迷惑，作者分别查阅了其他的相关教材[2]，也没有对此结论进行分析. 本文着重分析两个门函数的卷积，这对于卷积定义及卷积定理的理解有着非常重要的意义.

1　门函数的定义

如图 1 所示，幅度为 1，宽度为 τ 的函数为门函数(又称矩形脉冲)，用符号 $g_\tau(t)$ 表示.

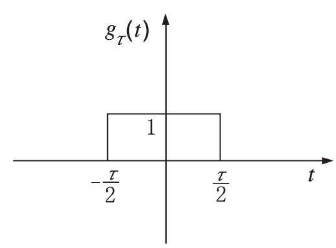

图1　门函数 $g_\tau(t)$

2　两个门函数卷积

设两个门函数分别为 $g_{\tau 1}(t)$ 和 $g_{\tau 2}(t)$，$\tau 1$、$\tau 2$ 分别为门函数的宽度，其图形为图 2 和图 3. 下面分别用卷积的性质和作图法给出这两个门函数卷积的推导过程.

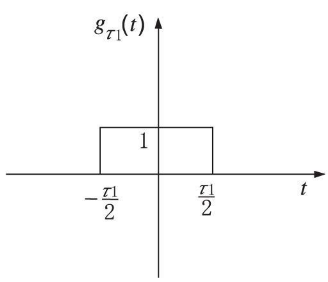

图2　门函数 $g_{\tau 1}(t)$

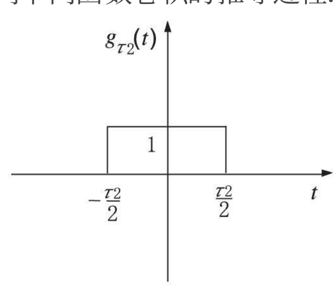

图3　门函数 $g_{\tau 2}(t)$

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2.1　利用卷积性质

卷积积分性质：如果 $f_1(t) * f_2(t) = f(t)$，则 $f_1(t - t1) * f_2(t - t2) = f(t - t1 - t2)$，由于两个门函数都可以用阶跃函数来表示，

$$g_{\tau1}(t) = \varepsilon\left(t + \frac{\tau1}{2}\right) - \varepsilon\left(t - \frac{\tau1}{2}\right),\quad g_{\tau2}(t) = \varepsilon\left(t + \frac{\tau2}{2}\right) - \varepsilon\left(t - \frac{\tau2}{2}\right).$$

又由于

$$\varepsilon(t) * \varepsilon(t) = t\varepsilon(t),$$

$$g_{\tau1}(t) * g_{\tau2}(t) = \left[\varepsilon\left(t + \frac{\tau1}{2}\right) - \varepsilon\left(t - \frac{\tau1}{2}\right)\right] * \left[\varepsilon\left(t + \frac{\tau2}{2}\right) - \varepsilon\left(t - \frac{\tau2}{2}\right)\right] =$$

$$\varepsilon\left(t + \frac{\tau1}{2}\right) * \varepsilon\left(t + \frac{\tau2}{2}\right) - \varepsilon\left(t + \frac{\tau1}{2}\right) * \varepsilon\left(t - \frac{\tau2}{2}\right) - \varepsilon\left(t - \frac{\tau1}{2}\right) * \varepsilon\left(t + \frac{\tau2}{2}\right) + \varepsilon\left(t - \frac{\tau1}{2}\right) * \varepsilon\left(t - \frac{\tau2}{2}\right) =$$

$$\left(t + \frac{\tau1}{2} + \frac{\tau2}{2}\right)\varepsilon\left(t + \frac{\tau1}{2} + \frac{\tau2}{2}\right) - \left(t + \frac{\tau1}{2} - \frac{\tau2}{2}\right)\varepsilon\left(t + \frac{\tau1}{2} - \frac{\tau2}{2}\right) -$$
$$\left(t - \frac{\tau1}{2} + \frac{\tau2}{2}\right)\varepsilon\left(t - \frac{\tau1}{2} + \frac{\tau2}{2}\right) + \left(t - \frac{\tau1}{2} - \frac{\tau2}{2}\right)\varepsilon\left(t - \frac{\tau1}{2} - \frac{\tau2}{2}\right).$$

(1) 当 $\tau1 > \tau2$ 时，

$$g_{\tau1}(t) * g_{\tau2}(t) =$$
$$\left(t + \frac{\tau1}{2} + \frac{\tau2}{2}\right)\left[\varepsilon\left(t + \frac{\tau1}{2} + \frac{\tau2}{2}\right) - \varepsilon\left(t + \frac{\tau1}{2} - \frac{\tau2}{2}\right)\right] - \tau2\left[\varepsilon\left(t + \frac{\tau1}{2} - \frac{\tau2}{2}\right) - \varepsilon\left(t - \frac{\tau1}{2} + \frac{\tau2}{2}\right)\right] +$$
$$\left(-t + \frac{\tau1}{2} + \frac{\tau2}{2}\right)\left[\varepsilon\left(t - \frac{\tau1}{2} + \frac{\tau2}{2}\right) - \varepsilon\left(t - \frac{\tau1}{2} - \frac{\tau2}{2}\right)\right].$$

(2) 当 $\tau2 > \tau1$ 时，

$$g_{\tau1}(t) * g_{\tau2}(t) =$$
$$\left(t + \frac{\tau1}{2} + \frac{\tau2}{2}\right)\left[\varepsilon\left(t + \frac{\tau1}{2} + \frac{\tau2}{2}\right) - \varepsilon\left(t + \frac{\tau2}{2} - \frac{\tau1}{2}\right)\right] - \tau2\left[\varepsilon\left(t + \frac{\tau2}{2} - \frac{\tau1}{2}\right) - \varepsilon\left(t - \frac{\tau2}{2} + \frac{\tau1}{2}\right)\right] +$$
$$\left(-t + \frac{\tau1}{2} + \frac{\tau2}{2}\right)\left[\varepsilon\left(t - \frac{\tau2}{2} + \frac{\tau1}{2}\right) - \varepsilon\left(t - \frac{\tau1}{2} - \frac{\tau2}{2}\right)\right].$$

(3) 当 $\tau1 = \tau2 = \tau$ 时，

$$g_{\tau1}(t) * g_{\tau2}(t) = (t + \tau)[\varepsilon(t + \tau) - \varepsilon(t)] + (-t + \tau)[\varepsilon(t) - \varepsilon(t - \tau)].$$

2.2　利用作图法

由于 $g_{\tau1}(t) * g_{\tau2}(t) = \int_{-\infty}^{+\infty} g_{\tau1}(\tau) g_{\tau2}(t - \tau)\,\mathrm{d}\tau$，首先将两个门函数的自变量改为 τ，其波形如图 4、图 5，然后将 $g_{\tau2}(\tau)$ 函数反转并右移 t 个单位得到 $g_{\tau2}(t - \tau)$，如图 6，最后求其积分.

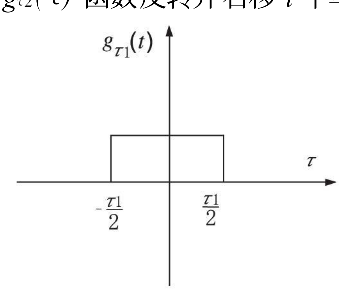

图 4　门函数 $g_{\tau}(t)$

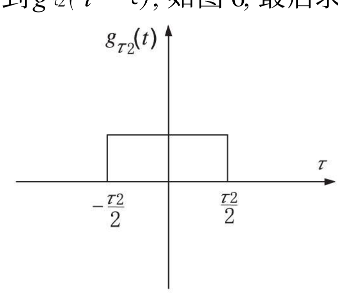

图 5　门函数 $g_{\tau1}(t)$

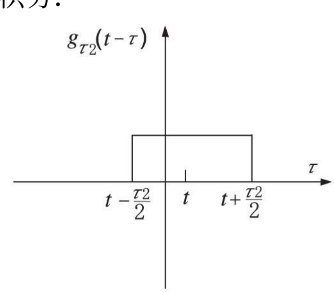

图 6　门函数 $g_{\tau2}(t-\tau)$

(1) $\tau1 > \tau2$ 时：

a. 当 $t < -\frac{\tau1}{2} - \frac{\tau2}{2}$ 时，$g_{\tau1}(t) * g_{\tau2}(t) = 0$.

b. 当$-\frac{\tau1}{2}-\frac{\tau2}{2}<t<-\frac{\tau1}{2}+\frac{\tau2}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{-\frac{\tau1}{2}}^{t+\frac{\tau2}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=t+\frac{\tau2}{2}+\frac{\tau1}{2}$.

c. 当$-\frac{\tau1}{2}+\frac{\tau2}{2}<t<\frac{\tau1}{2}-\frac{\tau2}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{t-\frac{\tau2}{2}}^{t+\frac{\tau2}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=\tau2$.

d. 当$\frac{\tau1}{2}-\frac{\tau2}{2}<t<\frac{\tau1}{2}+\frac{\tau2}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{t-\frac{\tau2}{2}}^{\frac{\tau1}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=\frac{\tau1}{2}+\frac{\tau2}{2}-t$.

e. 当$t>\frac{\tau1}{2}+\frac{\tau2}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=0$.

(2) 当$\tau2>\tau1$时:

a. 当$t<-\frac{\tau1}{2}-\frac{\tau2}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=0$.

b. 当$-\frac{\tau1}{2}-\frac{\tau2}{2}<t<-\frac{\tau2}{2}+\frac{\tau1}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{-\frac{\tau1}{2}}^{t+\frac{\tau2}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=t+\frac{\tau2}{2}+\frac{\tau1}{2}$.

c. 当$-\frac{\tau2}{2}+\frac{\tau1}{2}<t<\frac{\tau2}{2}-\frac{\tau1}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{t-\frac{\tau1}{2}}^{t+\frac{\tau1}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=\tau1$.

d. 当$\frac{\tau1}{2}-\frac{\tau2}{2}<t<\frac{\tau1}{2}+\frac{\tau2}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{t-\frac{\tau1}{2}}^{\frac{\tau2}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=\frac{\tau1}{2}+\frac{\tau2}{2}-t$.

e. 当$t>\frac{\tau1}{2}+\frac{\tau2}{2}$时, $g_{\tau1}(t)*g_{\tau2}(t)=0$.

(3) 当$\tau2=\tau1=\tau$时:

a. 当$t<-\tau$时, $g_{\tau1}(t)*g_{\tau2}(t)=0$.

b. 当$-\tau<t<0$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{-\frac{\tau}{2}}^{t+\frac{\tau}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=t+\tau$.

c. 当$0<t<\tau$时, $g_{\tau1}(t)*g_{\tau2}(t)=\int_{t-\frac{\tau}{2}}^{\frac{\tau}{2}}g_{\tau1}(\tau)g_{\tau2}(\tau)\mathrm{d}\tau=\tau-t$.

d. 当$t>\tau$时, $g_{\tau1}(t)*g_{\tau2}(t)=0$.

由此结果画出如下波形图, 见图7、图8和图9.

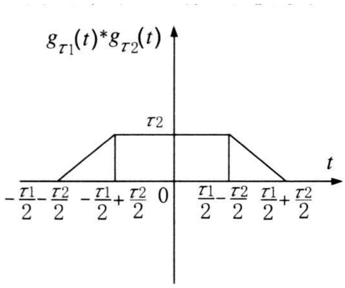

图7 $\tau1>\tau2$

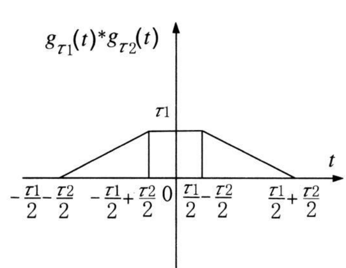

图8 $\tau2>\tau1$

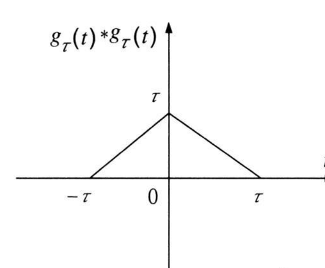

图9 $\tau2=\tau1=\tau$

3　结 论

通过上述分析可以得到: 两个不同宽度的门函数卷积时, 其结果为梯形函数, 梯形函数的高度为宽度窄的门函数脉宽, 其上底为两个门函数宽度之差绝对值, 下底为两个门函数脉宽之和, 其腰分别为$\frac{\tau1+\tau2}{2}+t$, $\frac{\tau1+\tau2}{2}-t$; 两个相同宽度的门函数卷积时, 其结果为三角函数, 三角函数的高度为门函数脉宽, 底为两个门函数脉宽之和.

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An Analysis of the Convolution Integral of Gatefunction

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Abstract: The convolution integral of gatefunction with different width is trapezoidal function, with the height being the width of gatefunction with narrow width, the top being the absolute value of the difference of both gatefunction width, the bottom being the sum of both gatefunction width, the convolution integral of two identical gatefunction being trigonometric function, the height of trigonometric function being the width of gatefunction an the bottom the sum of the width of gatefunction.

Key words: convolution; gatefunction; trigonometric function; Trapezoidal function