

Semantic Web

— to Artificial Intelligence

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Reasoning Techniques for Expressive DLs

Tableau-based algorithms (1990–1995)

- ▶ first systems employing such algorithms : Kris and Crack
- ▶ Highly optimized systems : FaCT, Race, DLP

Alternative approaches (2000–2005)

- ▶ resolution-based approaches (use an optimized translation of DLs into first-order predicate logic and then apply appropriate first-order resolution provers)
- ▶ automata-based approaches (often more convenient for showing ExpTime complexity upper-bounds than tableau-based approaches)

Tableau method

Tableau method: most popular approach to reasoning in expressive DLs

- ▶ implemented in state-of-the-art DL reasoners

Tableau algorithms are used to **decide satisfiability**.

- ▶ can also be used for other reasoning tasks (e.g. instance checking) that can be reduced to satisfiability

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Idea: to determine whether a given (concept or KB) Ψ is satisfiable, **try to construct a (representation of a) model of Ψ**

- ▶ if we succeed, then we have shown that Ψ is satisfiable
- ▶ if we fail despite having **considered all possibilities**, then we have proven that Ψ is unsatisfiable

\mathcal{ALC} -concepts

Recall that \mathcal{ALC} -concepts are built using the following constructors:

$T \quad \perp \quad \neg \quad \sqcup \quad \sqcap \quad \forall R.C \quad \exists R.C$

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$\top \perp \neg \sqcup \sqcap \forall R.C \exists R.C$

We say that an \mathcal{ALC} -concept C is in **negation normal form (NNF)** if the symbol \neg only appears directly in front of atomic concepts.

- ▶ in NNF: $A \sqcap \neg B, \exists R.\neg A, \neg A \sqcup \neg B$
- ▶ not in NNF: $\neg(A \sqcap B), \exists R.\neg(\forall S.B), A \sqcup \neg\forall R.B, \neg\top$

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Fact. Every \mathcal{ALC} -concept C can be transformed into an equivalent concept in NNF in linear time by applying the following rewriting rules:

$$\begin{array}{lll} \neg\top \rightsquigarrow \perp & \neg(C \sqcap D) \rightsquigarrow \neg C \sqcup \neg D & \neg(\forall R.C) \rightsquigarrow \exists R.\neg C \\ \neg\perp \rightsquigarrow \top & \neg(C \sqcup D) \rightsquigarrow \neg C \sqcap \neg D & \neg(\exists R.C) \rightsquigarrow \forall R.\neg C \end{array}$$

Note: say C and D are equivalent if the empty TBox entails $C \equiv D$.

Satisfiability of \mathcal{ALC} -concepts via tableau

We start by giving a tableau algorithm for deciding satisfiability of \mathcal{ALC} -concepts in NNF w.r.t. the empty TBox.

Procedure for testing satisfiability of C_0 :

- ▶ We work with a set S of ABoxes
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(note: rules are detailed on next slide)

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- ▶ Stop applying rules when either:
 - ▶ every $\mathcal{A} \in S$ contains a **clash**, i.e. an assertion $\perp(b)$ or a pair of assertions $\{B(b), \neg B(b)\}$
 - ▶ some $\mathcal{A} \in S$ is **clash-free** and **complete**: no rule can be applied to \mathcal{A}

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- ▶ Return “yes” if some $\mathcal{A} \in S$ is clash-free, else “no”.

Tableau rules for \mathcal{ALC}

\sqcap -rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$ and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$
then replace \mathcal{A} with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$

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\exists -rule: if $\{\exists R.C(a)\} \in \mathcal{A}$ and there is no b with $\{R(a, b), C(b)\} \subseteq \mathcal{A}$
then **pick a new individual name d** and
replace \mathcal{A} with $\mathcal{A} \cup \{R(a, d), C(d)\}$

Tableau example: only \sqcap and \sqcup

Let's use the tableau procedure to test satisfiability of

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\mathcal{A}_4 is complete, so we can stop.

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Apply \sqcap -rule to \mathcal{A}_2 :

get $S = \{ \mathcal{A}'_1, \mathcal{A}'_2 \}$ where $\mathcal{A}'_2 = \mathcal{A}_2 \cup \{ (\neg B \sqcup D)(a_0), \neg A(a_0) \}$

Apply \sqcup -rule to \mathcal{A}'_2 :

get $S = \{ \mathcal{A}'_1, \mathcal{A}_3, \mathcal{A}_4 \}$ where $\mathcal{A}_3 = \mathcal{A}'_2 \cup \{ \neg B(a_0) \}$, $\mathcal{A}_4 = \mathcal{A}'_2 \cup \{ D(a_0) \}$

\mathcal{A}_4 contains no clash $\Rightarrow C_0$ is satisfiable

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0)$$

Previous example in a picture

$$\begin{array}{c} (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0) \\ ((\neg B \sqcup D) \sqcap \neg A) (a_0) \qquad \qquad \text{\color{blue} \sqcap-rule} \\ (A \sqcup B) (a_0) \end{array}$$

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0)$$
$$((\neg B \sqcup D) \sqcap \neg A) (a_0)$$
$$(A \sqcup B) (a_0)$$
$$A (a_0)$$
$$B (a_0)$$

⊟-rule



Previous example in a picture

| | |
|---------------|---|
| | $(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0)$ |
| | $((\neg B \sqcup D) \sqcap \neg A) (a_0)$ |
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$$A (a_0)$$

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$$\neg A (a_0)$$

$$B (a_0)$$

$$(\neg B \sqcup D) (a_0)$$

⊓-rule

$$\neg A (a_0)$$

Previous example in a picture

| | |
|---|---------------------------|
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| $A (a_0)$ | $B (a_0)$ |
| $(\neg B \sqcup D) (a_0)$ | $(\neg B \sqcup D) (a_0)$ |
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| | $\neg B (a_0)$ |
| | $D (a_0)$ □-rule |

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$$D (a_0)$$

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$$\neg A (a_0)$$

X

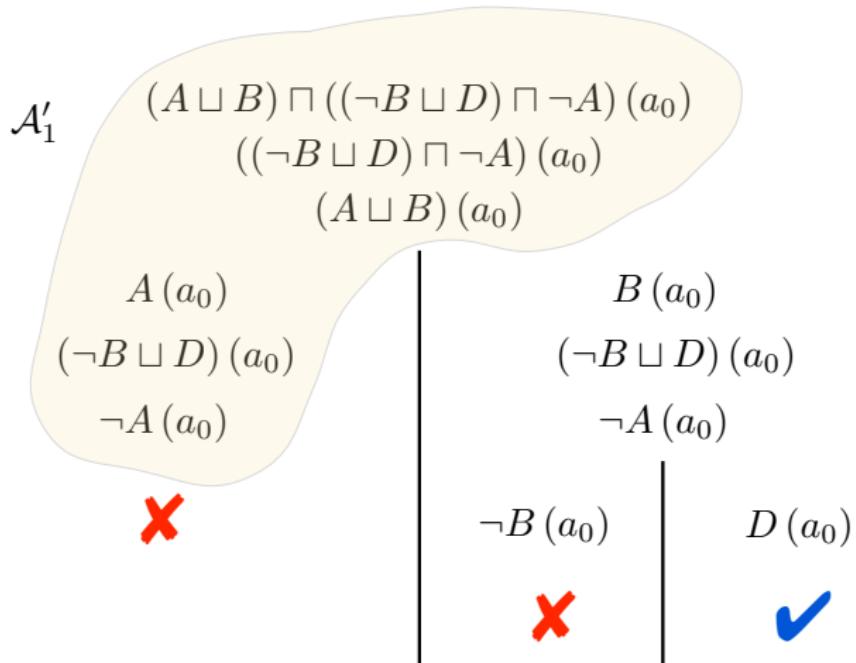
$$B (a_0)$$
$$(\neg B \sqcup D) (a_0)$$
$$\neg A (a_0)$$
$$\neg B (a_0)$$

X

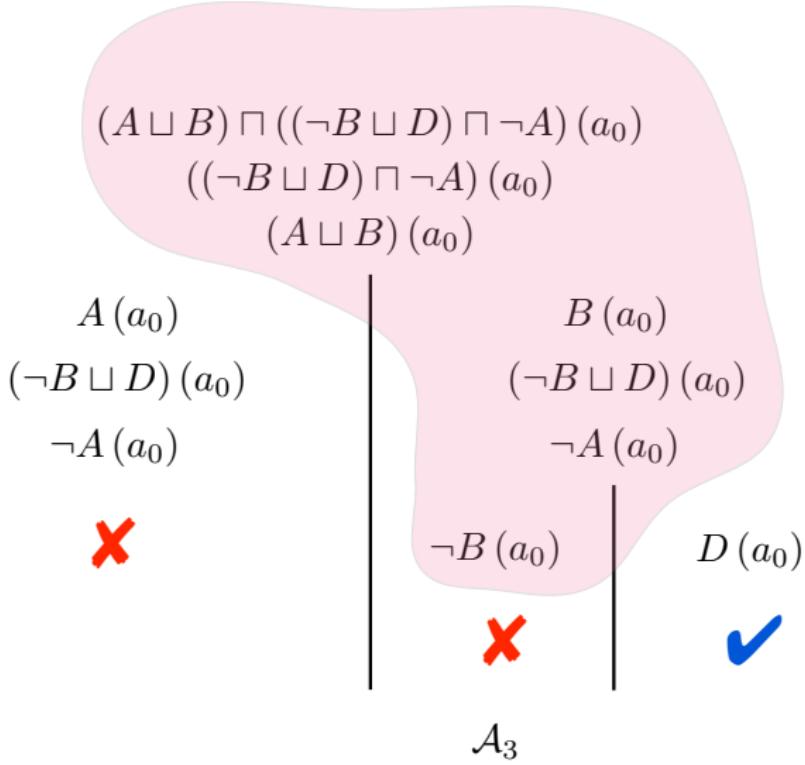
$$D (a_0)$$

✓

Previous example in a picture



Previous example in a picture



Previous example in a picture

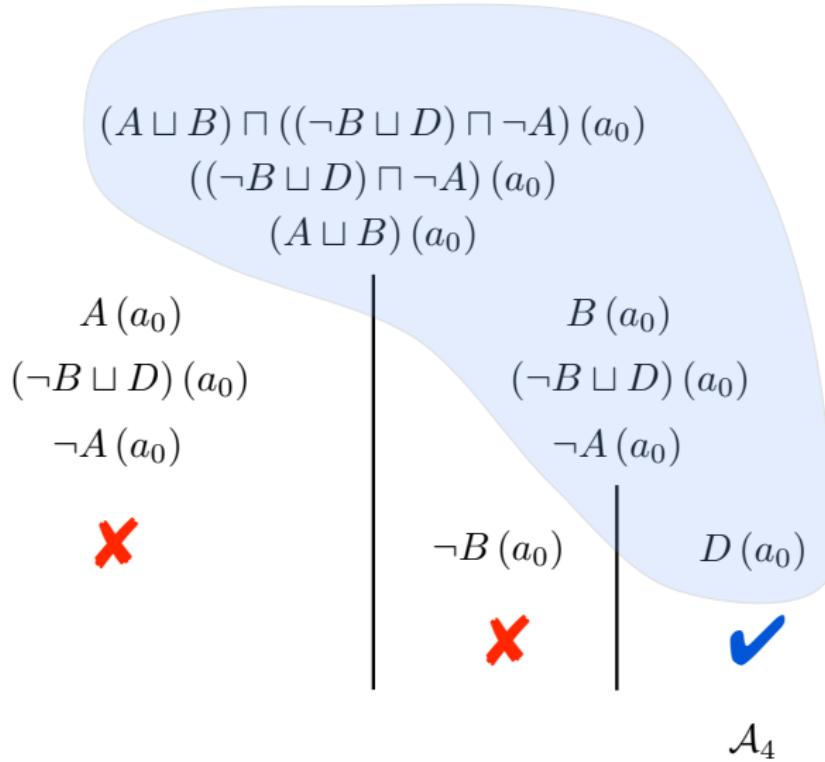


Tableau example: only \sqcap and \sqcup

In our example, we had the complete and clash-free ABox \mathcal{A}_4 :

$$\begin{array}{ll} ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) & (A \sqcup B)(a_0) \\ ((\neg B \sqcup D) \sqcap \neg A)(a_0) & B(a_0) \quad (\neg B \sqcup D)(a_0) \quad \neg A(a_0) \quad D(a_0) \end{array}$$

Can build from \mathcal{A}_4 the interpretation \mathcal{I} with:

- ▶ $\Delta^{\mathcal{I}} = \{a_0\}$ use individuals from \mathcal{A}_4
- ▶ $A^{\mathcal{I}} = \emptyset$ since \mathcal{A}_4 does not contain $A(a_0)$
- ▶ $B^{\mathcal{I}} = D^{\mathcal{I}} = \{a_0\}$ since \mathcal{A}_4 contains $B(a_0)$ and $D(a_0)$

We can verify that $(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)^{\mathcal{I}} = \{a_0\}$.

- ▶ \mathcal{I} witnesses the satisfiability of $C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$

Tableau example: \forall and \exists

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R. A \sqcap \forall R. \neg A$$

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Let's use the tableau procedure to test satisfiability of

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Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ (\exists R.A \sqcap \forall R.\neg A)(a_0) \}.$

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Apply \sqcap -rule to \mathcal{A}_0 :

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\mathcal{A}'''_0 contains clash $\{ A(a_1), \neg A(a_1) \}!$

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get $S = \{ \mathcal{A}'''_0 \}$ where $\mathcal{A}'''_0 = \mathcal{A}''_0 \cup \{ \neg A(a_1) \}$.

The only set in S contains a clash $\Rightarrow C_0$ is unsatisfiable

Previous example in a picture

$$(\exists R.A \sqcap \forall R.\neg A)(a_0)$$

Previous example in a picture

$$\begin{array}{c} (\exists R.A \sqcap \forall R.\neg A) (a_0) \\ (\exists R.A) (a_0) \qquad \qquad \text{□-rule} \\ (\forall R.\neg A) (a_0) \end{array}$$

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$R(a_0, a_1)$ **\exists -rule**

$A(a_1)$

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$A(a_1)$ **\forall -rule**

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Tableau example: \forall and \exists

Suppose that we consider a slightly different concept

$$C_0 = \exists R.A \sqcap \forall R.\neg B$$

Now the tableau algorithm yields the following complete, clash-free ABox:

$$(\exists R.A \sqcap \forall R.\neg B)(a_0) \quad (\exists R.A)(a_0) \quad (\forall R.\neg B)(a_0) \quad R(a_0, a_1) \quad A(a_1) \quad \neg B(a_1)$$

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Corresponding interpretation \mathcal{I} :

- ▶ $\Delta^{\mathcal{I}} = \{a_0, a_1\}$
- ▶ $A^{\mathcal{I}} = \{a_1\}$
- ▶ $B^{\mathcal{I}} = \emptyset$
- ▶ $R^{\mathcal{I}} = \{(a_0, a_1)\}$

Can check that \mathcal{I} is such that $C_0^{\mathcal{I}} = \{a_0\}$.

Properties of the tableau algorithm

Let's call our tableau algorithm CSat (for concept satisfiability).

To show that CSat is a decision procedure, we must show:

Termination: The algorithm CSat always terminates.

Soundness: CSat outputs “yes” on input $C_0 \Rightarrow C_0$ is satisfiable.

Completeness: C_0 satisfiable \Rightarrow CSat will output “yes”.

Preliminary definitions

The **size of a concept C** , denoted $|C|$, is defined as the total number of occurrences of atomic concepts, atomic roles, and constructors in C .

For example : $|A \sqcap B| = 3$, $|\neg(A \sqcap B)| = 4$, $|\neg\neg A| = 3$, $|\exists R.A| = 3$

Preliminary definitions

Subconcepts of a concept:

$$\text{sub}(A) = \{A\}$$

$$\text{sub}(\neg C) = \{\neg C\} \cup \text{sub}(C)$$

$$\text{sub}(\exists R.C) = \{\exists R.C\} \cup \text{sub}(C)$$

$$\text{sub}(\forall R.C) = \{\forall R.C\} \cup \text{sub}(C)$$

$$\text{sub}(C_1 \sqcup C_2) = \{C_1 \sqcup C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2)$$

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Role depth of a concept:

$$\text{depth}(A) = 0$$

$$\text{depth}(\neg C) = \text{depth}(C)$$

$$\text{depth}(\exists R.C) = \text{depth}(\forall R.C) = \text{depth}(C) + 1$$

$$\text{depth}(C_1 \sqcup C_2) = \text{depth}(C_1 \sqcap C_2) = \max(\text{depth}(C_1), \text{depth}(C_2))$$

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1. if $D(b) \in \mathcal{A}$, then $D \in \text{sub}(C_0)$
 - ▶ \mathcal{A} contains at most $|C_0|$ concept assertions per individual

So, there can only be a finite number of rule applications per individual

Next to show that the number of newly introduced individuals in a chain of rule

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Thus: **bound on the size of ABoxes** generated by the procedure.

The tableau procedure **only adds assertions** to ABoxes

⇒ **eventually all ABoxes will contain a clash or will be complete**

Soundness and Completeness :

CSat returns “yes” on input C_0 . $\iff C_0$ is satisfiable.

CSat returns “yes” on input C_0 . $\implies C_0$ is satisfiable. (Soundness)
(I.e., C_0 is unsatisfiable. \implies CSat returns “no” on input C_0 . That is, the positive answers of the algorithm are correct : if the algorithm says “consistent”, then the input concept is indeed satisfiable.)

CSat returns “yes” on input C_0 . $\iff C_0$ is satisfiable. (Completeness)
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Soundness of CSat (1)

Suppose that CSat returns “yes” on input C_0 .

Then S must contain a complete and clash-free ABox \mathcal{A} .

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Claim: \mathcal{I} is such that $C_0^{\mathcal{I}} \neq \emptyset$

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Claim: \mathcal{I} is such that $C_0^{\mathcal{I}} \neq \emptyset$

To show the claim, we prove by induction on the size of concepts that:

$$D(b) \in \mathcal{A} \quad \Rightarrow \quad b \in D^{\mathcal{I}}$$

Soundness of CSat (2)

Base case: $D = A$ or $D = \neg A$

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If $D = A$, then $b \in A^{\mathcal{I}}$.

If $D = \neg A$, then $A(b) \notin \mathcal{A}$, so $b \in \neg A^{\mathcal{I}}$.

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Again, many cases to consider:

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- ▶ $D = \exists R.E$: since \mathcal{A} is complete, there is some c such that $R(b, c) \in \mathcal{A}$ and $E(c) \in \mathcal{A}$. Then $(b, c) \in R^{\mathcal{I}}$ and by the IH, we get $c \in E^{\mathcal{I}}$, so $b \in (\exists R.E)^{\mathcal{I}}$

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- ▶ $D = E \sqcup F$: left as practice
- ▶ $D = \forall R.E$: left as practice

Completeness of CSat

Suppose that the concept C_0 is satisfiable.

Then the ABox $\{C_0(a_0)\}$ must be satisfiable too.

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- ▶ If an ABox \mathcal{A} is satisfiable and \mathcal{A}' is the result of applying a rule to \mathcal{A} , then \mathcal{A}' is also satisfiable.
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We start with a satisfiable ABox and the rules are satisfiability-preserving, so eventually we will reach a complete, satisfiable (thus: clash-free) ABox.

Complexity of CSat

Bad news: our algorithm may require exponential time and space...

To see why, consider what happens if we run CSat on the concept

$$\prod_{0 \leq i < n} \underbrace{\forall R. \dots \forall R.}_{i \text{ times}} (\exists R. B \sqcap \exists R. \neg B)$$

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Good news: can **modify the procedure so it runs in polynomial space**

- ▶ instead of a set of ABoxes, **keep only one ABox in memory at a time**
 - ▶ when apply the \sqcup -rule, first examine A_1 , then afterwards examine A_2
 - ▶ remember that second disjunct stills needs to be checked
- ▶ explore the children of an individual one at a time
 - ▶ possible because no interaction between the different “branches”
 - ▶ store which $\exists R. C$ concepts have been tested, which are left to do
- ▶ this allows us to keep at most $|C_0|$ individuals in memory at a time

Complexity of \mathcal{ALC} concept satisfiability

Hierarchy of complexity classes

$\text{PTIME} \subseteq \text{NP} \subseteq \dots \subseteq \text{PSPACE} \subseteq \text{EXPTIME} \subseteq \text{NEXPTIME} \subseteq \dots \subseteq \text{EXPSPACE} \dots$

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Theorem: \mathcal{ALC} concept satisfiability (no TBox) is PSPACE-complete.

- ▶ Membership in PSPACE shown using modified tableau procedure
- ▶ Hardness for PSPACE shown by giving a reduction from some known PSPACE-hard problem (e.g. QBF validity)

Homework

Exercise 1: Concept satisfiability via tableau

Use the tableau algorithm to decide which of the following concepts is satisfiable:

1. $\exists R.(A \sqcap B) \sqcap \forall R.(\neg A \sqcup D)$
2. $(\exists R.\exists S.A) \sqcap (\forall R.\forall S.\neg A)$
3. $\exists R.B \sqcap \forall R.(\forall R.A) \sqcap \forall R.\neg A$

If a concept is found to be satisfiable, use the result to construct an interpretation in which the concept is non-empty.

Exercise 2: Negation normal form

Consider the following recursive algorithm for putting a concept in NNF.

Algorithm NNF

Input: \mathcal{ALC} -concept C

If $C = A$ or $C = \neg A$ (with A atomic concept), then output C

If $C = D_1 \sqcap D_2$, then output $\text{NNF}(D_1) \sqcap \text{NNF}(D_2)$

If $C = D_1 \sqcup D_2$, then output $\text{NNF}(D_1) \sqcup \text{NNF}(D_2)$

If $C = \exists R.D$, then output $\exists R.\text{NNF}(D)$

If $C = \forall R.D$, then output $\forall R.\text{NNF}(D)$

If $C = \neg(D_1 \sqcap D_2)$, then output $\text{NNF}(\neg D_1) \sqcup \text{NNF}(\neg D_2)$

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If $C = \neg \forall R.D$, then output $\exists R.\text{NNF}(\neg D)$

If $C = \neg(\neg D)$, then output $\text{NNF}(\neg D)$

(we use $\text{NNF}(E)$ to denote output of NNF on input E)

Exercise 2: Negation normal form

Show that the algorithm NNF has the following properties:

1. NNF terminates on every input.
2. $\text{NNF}(C)$ is in negation normal form.
3. $\text{NNF}(C)$ is equivalent to C .

Hint: you may want to use the size of C , which is denoted $|C|$ and defined as the total number of occurrences of atomic concepts, atomic roles, and constructors in C .

Example: $|A \sqcap B| = 3$, $|\neg(A \sqcap B)| = 4$, $|\neg\neg A| = 3$, $|\exists R.A| = 3$

Exercise 3: Adding acyclic terminologies (1)

In this exercise, we see how to test concept satisfiability w.r.t. a special kind of TBox called an acyclic terminology.

A **terminology** is a set of **concept definitions**

$$A \equiv C$$

where A is an atomic concept and C a (possibly complex) concept, and *no two concept definitions have the same atomic concept on the left.*

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- ▶ replace any atomic concept A such that $A \equiv D \in \mathcal{T}$ by D

Example: the unfolding of $A \sqcap \exists R.B$ w.r.t. the terminology

$$\{A \equiv \forall S. \forall R. B \quad B \equiv E \sqcap F \quad F \equiv \neg G\}$$

is the concept $\forall S. \forall R. (E \sqcap \neg G) \sqcap \exists R. (E \sqcap \neg G)$.

Note: for some terminologies, the process may never stop!

- ▶ take $C = A$ and $\mathcal{T} = \{A \equiv \exists R.A\}$

A terminology is called **acyclic** if the **unfolding procedure always halts**.

Solutions to the exercises 2 and 3

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Solution to exercise 2: Termination

The proof is by induction on the size of the input concept.

Base case: input concept C with $|C| \leq 2$

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Induction hypothesis (IH): NNF terminates on all inputs C with $|C| \leq k$

Induction step: show NNF terminates on input C with $|C| = k + 1$

Let C be a concept with $|C| = k + 1$. There are several cases to consider:

- ▶ $C = D_1 \sqcap D_2$: call NNF on D_1 and D_2 , both calls terminate since $|D_1| < |C|$ and $|D_2| < |C|$, hence $|D_1| \leq k$ and $|D_2| \leq k$
- ▶ $C = \neg \exists R.D$: call NNF on $\neg D$, and the call terminates since $|\neg D| \leq k$
- ▶ $C = \dots$

Solution to exercise 2: NNF

The proof is by induction on the size of the input concept C .

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We must have $C = A$ or $C = \neg A$. In both cases, NNF will output C , which is in NNF.

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Induction hypothesis (IH): $\text{NNF}(C)$ is in NNF, whenever $|C| \leq k$

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Induction step: show $\text{NNF}(C)$ is in NNF when $|C| = k + 1$

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- ▶ $C = D_1 \sqcap D_2$: by the IH, both $\text{NNF}(D_1)$ and $\text{NNF}(D_2)$ are in NNF, hence the output $\text{NNF}(D_1) \sqcap \text{NNF}(D_2)$ must be too.
- ▶ $C = \neg \exists R.D$: by the IH, $\text{NNF}(\neg D)$ is in NNF. Thus, $\forall R.\text{NNF}(\neg D)$ must also be in NNF.
- ▶ $C = \dots$

Solution to exercise 2: Equivalence

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Let C be a concept with $|C| = k + 1$. There are several cases to consider:

- ▶ $C = \neg \exists R.D$: by the IH, $\text{NNF}(\neg D)$ is equivalent to $\neg D$. Let \mathcal{I} be an interpretation. We have:

$$\begin{aligned} C^{\mathcal{I}} &= (\neg \exists R.D)^{\mathcal{I}} \\ &= \{d \in \Delta^{\mathcal{I}} \mid \text{there is no } e \text{ with } (d, e) \in R^{\mathcal{I}} \text{ and } e \in D^{\mathcal{I}}\} \\ &= \{d \in \Delta^{\mathcal{I}} \mid \text{for every } e \text{ with } (d, e) \in R^{\mathcal{I}} \text{ we have } e \in (\neg D)^{\mathcal{I}}\} \\ &= (\forall R.\neg D)^{\mathcal{I}} \\ &= (\forall R.\text{NNF}(\neg D))^{\mathcal{I}} \\ &= (\text{NNF}(C))^{\mathcal{I}} \end{aligned}$$

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Exercise 3: Adding acyclic terminologies (1)

In this exercise, we see how to test concept satisfiability w.r.t. a special kind of TBox called an acyclic terminology.

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Example: the unfolding of $A \sqcap \exists R.B$ w.r.t. the terminology

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Note: for some terminologies, the process may never stop!

- ▶ take $C = A$ and $\mathcal{T} = \{A \equiv \exists R.A\}$

A terminology is called **acyclic** if the **unfolding procedure always halts**.

Exercise 3: Adding acyclic terminologies (3)

We can test satisfiability of C w.r.t. acyclic terminology \mathcal{T} as follows:

Step 1 Compute the unfolding U of C w.r.t. \mathcal{T} .

Step 2 Put U into negation normal form.

Step 3 Run CSat on U , and output the same answer as CSat.

First task: Use the above procedure to test satisfiability of the concepts:

$$1. B \sqcap \forall R. \neg G$$

$$2. A \sqcap \forall R. \exists R. G$$

with respect to the following acyclic terminology:

$$\{A \equiv \exists R. B \sqcap \exists S. \neg B \quad B \equiv \exists R. D \sqcap \exists R. E \quad G \equiv D \sqcap E\}$$

Second task: explain why the above procedure is a decision procedure for \mathcal{ALC} concept satisfiability w.r.t. acyclic terminologies.

(note: I am not asking for a formal proof, just a clear explanation)

Solution to exercise 3: first task (1)

For the first concept, we first unfold w.r.t. \mathcal{T} to get:

$$(\exists R.D \sqcap \exists R.E) \sqcap \forall R.\neg(D \sqcap E)$$

and then put the latter concept into NNF:

$$(\exists R.D \sqcap \exists R.E) \sqcap \forall R.(\neg D \sqcup \neg E)$$

The tableau algorithm returns “yes” on the preceding concept, so the **first concept is satisfiable w.r.t. \mathcal{T} .**

Solution to exercise 3: first task (2)

For the second concept, we first unfold w.r.t. \mathcal{T} to get:

$$\exists R.(\exists R.D \sqcap \exists R.E) \sqcap \exists S.(\neg(\exists R.D \sqcap \exists R.E)) \sqcap \forall R.\exists R.(D \sqcap E)$$

and then put the latter concept into NNF:

$$\exists R.(\exists R.D \sqcap \exists R.E) \sqcap \exists S.(\forall R.\neg D \sqcup \forall R.\neg E) \sqcap \forall R.\exists R.(D \sqcap E)$$

The tableau algorithm returns “yes” on the preceding concept, so the **second concept is also satisfiable w.r.t. \mathcal{T} .**

Solution to exercise 3: second task

Suppose we run the procedure on concept C and acyclic terminology \mathcal{T} .

Termination: Step 1 terminates because \mathcal{T} is acyclic. Steps 2 and 3 terminate because we know that NNF and CSat both terminate.

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Suppose we run the procedure on concept C and acyclic terminology \mathcal{T} .

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Soundness and completeness of CSat (previous lecture)

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$$\begin{aligned} \text{CSat returns "yes" on } C_2 &\Leftrightarrow C_2 \text{ is satisfiable} \\ &\Leftrightarrow C_1 \text{ is satisfiable} \end{aligned}$$

NNF preserves equivalence (previous lecture)

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$$\begin{aligned} \text{CSat returns "yes" on } C_2 &\Leftrightarrow C_2 \text{ is satisfiable} \\ &\Leftrightarrow C_1 \text{ is satisfiable} \\ &\Leftrightarrow C \text{ is satisfiable w.r.t. } \mathcal{T} \end{aligned}$$

C_1 satisfiable \Rightarrow take \mathcal{I} with $C_1^{\mathcal{I}} \neq \emptyset$, then reinterpret the defined concepts according to \mathcal{T} in order to get model of \mathcal{T} with C non-empty

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C satisfiable w.r.t. $\mathcal{T} \Rightarrow$ take model \mathcal{I} of \mathcal{T} with $C^{\mathcal{I}} \neq \emptyset$, then because of the definitions in \mathcal{T} , must have $C_1^{\mathcal{I}} \neq \emptyset$

Back to tableau algorithms

Extension to KB satisfiability

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Idea: if we have $C \sqsubseteq D$, then **every element must satisfy either $\neg C$ or D**

Concretely, we might try adding the following rule:

TBox rule: if a is in \mathcal{A} , $C \sqsubseteq D \in \mathcal{T}$, & $(\text{NNF}(\neg C) \sqcup \text{NNF}(D))(a) \notin \mathcal{A}$

then replace \mathcal{A} with $\mathcal{A} \cup \{(\text{NNF}(\neg C) \sqcup \text{NNF}(D))(a)\}$

Examples: KB satisfiability

Let's try the modified procedure on the following KBs

1. $(\mathcal{T}, \{A(a)\})$
2. $(\mathcal{T}, \{R(c, a), B(a)\})$

where \mathcal{T} is the following TBox:

$$\{ \quad A \sqsubseteq \exists R.B \quad B \sqsubseteq D \quad \exists R.D \sqsubseteq \neg A \quad \}$$

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Now try it on the KB $(\{F \sqsubseteq \exists S.F\}, \{F(a)\})$.

Seems we have a problem... How can we ensure termination?

Blocking

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Formally, given individuals a, b in \mathcal{A} , we say that b **blocks** a if:

- ▶ $\{C \mid C(a) \in \mathcal{A}\} \subseteq \{C \mid C(b) \in \mathcal{A}\}$
- ▶ b was present in \mathcal{A} before a was introduced

Say that individual a is **blocked** (in \mathcal{A}) if some b blocks a .

Blocking

Basic idea: if two individuals “look the same”, then it is unnecessary to explore both of them

Formally, give individuals a, b in \mathcal{A} , we say that b blocks a if:

- ▶ $\{C \mid C(a) \in \mathcal{A}\} \subseteq \{C \mid C(b) \in \mathcal{A}\}$
- ▶ b was present in \mathcal{A} before a was introduced

Say that individual a is blocked (in \mathcal{A}) if some b blocks a .

Modify rules so that they **only apply to unblocked individuals**.

Tableau rules for KBS

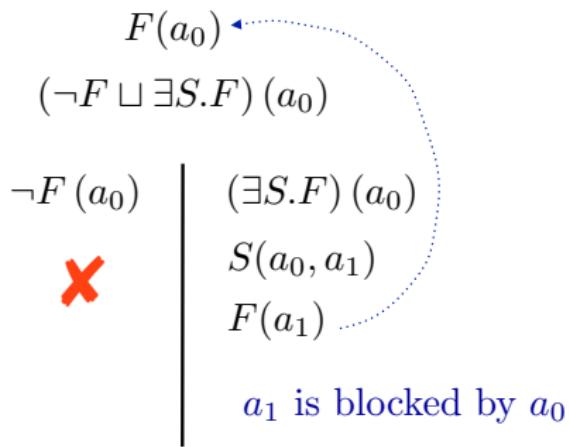
- \sqcap -rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$, **a is not blocked**, and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$,
then replace \mathcal{A} with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$
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- \forall -rule: if $\{\forall R.C(a), R(a, b)\} \in \mathcal{A}$, **a is not blocked**, and $C(b) \notin \mathcal{A}$,
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- \exists -rule: if $\{\exists R.C(a)\} \in \mathcal{A}$, **a is not blocked**, and no $\{R(a, b), C(b)\} \subseteq \mathcal{A}$,
then **pick a new individual name d** and replace \mathcal{A} with
 $\mathcal{A} \cup \{R(a, d), C(d)\}$
- \sqsubseteq -rule: if a appears in \mathcal{A} and **a is not blocked**, $C \sqsubseteq D \in \mathcal{T}$, and
 $(\text{NNF}(\neg C) \sqcup \text{NNF}(D))(a) \notin \mathcal{A}$,
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Example: blocking

Let's try blocking on the problematic KB ($\{F \sqsubseteq \exists S.F, \{F(a)\}\}$).

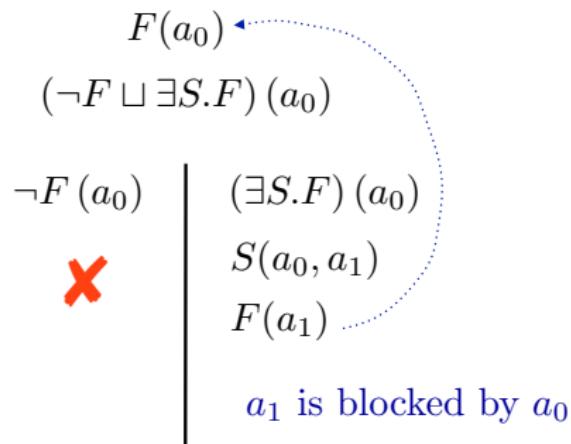
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We obtain a complete and clash-free ABox \Rightarrow the KB is satisfiable !

Another blocking example

Consider the following TBox

$$\mathcal{T} = \{A \sqsubseteq \exists R.A, A \sqsubseteq B, \exists R.B \sqsubseteq D\}$$

and suppose we want to test whether $\mathcal{T} \models A \sqsubseteq D$.

We can do this by running the algorithm on the KB $(\mathcal{T}, \{(A \sqcap \neg D)(a_0)\})$.

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Result: the KB is unsatisfiable $\Rightarrow \mathcal{T} \models A \sqsubseteq D$

Observation: an individual can be blocked, then later become unblocked

Properties of the tableau algorithm

Let's call our new tableau algorithm KBSat (for KB satisfiability).

Termination: The algorithm KBSat always terminates.

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So: KBSat is a decision procedure for KB satisfiability.

Complexity of KB satisfiability

The KBSat algorithm generates ABoxes of at most exponential size

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- ▶ *single branch may be exponentially long*

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Theorem: \mathcal{ALC} KB satisfiability is EXPTIME-complete.

This result means **no polynomial-time algorithm can ever be found**.

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Absorption (1)

When $\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$, we get n disjunctions per individual:

$$(\text{NNF}(\neg C_1) \sqcup \text{NNF}(D_1))(a), \dots, (\text{NNF}(\neg C_n) \sqcup \text{NNF}(D_n))(a)$$

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So for inclusions with atomic left-hand side, can replace \sqsubseteq -rule by:

\sqsubseteq^{at} -rule: if $A(a) \in \mathcal{A}$, a is not blocked, $A \sqsubseteq D \in \mathcal{T}$ (with A atomic),
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Good news: we've lowered the number of disjunctions!

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Absorption technique:

1. preprocess the TBox by replacing inclusions with equivalent inclusions with atomic concept on left, whenever possible
2. when running tableau algorithm
 - ▶ use new \sqsubseteq^{at} -rule for inclusions $A \sqsubseteq D$ with A atomic
 - ▶ use regular \sqsubseteq -rule for the other TBox inclusions

Example: Absorption

Let's use absorption on the KB $(\mathcal{T}, \{A(a)\})$ from earlier with:

$$\{ \quad A \sqsubseteq \exists R.B \quad B \sqsubseteq D \quad \exists R.D \sqsubseteq \neg A \quad \}$$

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Result: completely avoid disjunction, algorithm terminates much faster!

Optimizations for classification

Classification: find all pairs of atomic concepts A, B with $\mathcal{T} \models A \sqsubseteq B$

Naïve approach: test satisfiability of $A \sqcap \neg B$ w.r.t. \mathcal{T} for all pairs A, B

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Some ideas:

- ▶ some subsumptions are obvious
 - ▶ $A \sqsubseteq A$ and subsumptions that are explicitly stated in \mathcal{T}
- ▶ can use simple reasoning to obtain new (non-)subsumptions
 - ▶ if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \models B \sqsubseteq D$, then $\mathcal{T} \models A \sqsubseteq D$
 - ▶ if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \not\models A \sqsubseteq D$, then $\mathcal{T} \not\models B \sqsubseteq D$

Homework

Exercise 4: KB satisfiability via tableau

Consider the following TBox (from a previous homework):

$$\mathcal{T} = \{A \sqsubseteq \forall R.B, B \sqsubseteq \neg F, E \sqsubseteq G, A \sqsubseteq D \sqcup E, D \sqsubseteq \exists R.F, \exists R.\neg B \sqsubseteq G\}$$

Use the KBSat algorithm to decide whether:

1. $\mathcal{T} \models A \sqsubseteq E$
2. $\mathcal{T} \models E \sqsubseteq G$
3. $\mathcal{T} \models E \sqsubseteq F$
4. $\mathcal{T} \models A \sqsubseteq G$
5. $\mathcal{T} \models D \sqsubseteq G$
6. $\mathcal{T} \models G \sqsubseteq F$

Please explain your steps. You are encouraged to use the optimizations introduced in the course.