Combinatorics Exercises (and Solutions):

For the following set of problems determine what part of Combinatorics we need to use and apply the appropriate formula. Have in mind that there might be more than one correct approach to some (or all) of these questions.

Problem 1:

Imagine you are working at an office and have 5 tasks labelled as “Critical” in Jira to complete by the end of the day. In how many ways can you complete said tasks before the day ends?

** “Jira” is a Project Management software which allows you to create tasks and label them depending on their importance. “Critical” is the highest level of importance and no task with a lower-level can be started once such a task is initiated.

Solution:

We need to arrange all 5 tasks; hence we are looking for the number of Permutation between 5 elements. Thus, we have $5! = 120$ ways of completing our assignments.

Problem 2:

Imagine your company is trying to gain customers by running an online ad campaign. The idea is to focus on a certain demographic which frequently uses social media. Your campaign will run ads on Facebook, Messenger, Instagram, Twitter and Reddit. Your graphical designers have created 8 different versions of the banner you can use. Based on this information:

a) Calculate how many different options you have for the entire campaign, assuming you want to use a different one for each platform.
b) Calculate how many different options you have for the entire campaign, assuming you can use the same banner for some or all the platforms.
c) Calculate how many ways we can pick which of the 8 banners to use, assuming we use different ones for each platform.
d) Calculate how many ways we can pick which of the 8 banners to use, assuming we can use each one multiple times.

Solution:

Now, we have 8 banners at our disposal, and we need to put them on 5 platforms.

a) Using different banners for each platform means we can think of each social media platform as a different position. Hence, we are going to be dealing with Variations. Since we are using different ones for each site/app, we cannot repeat values, so our formula is $V = \frac{n!}{(n-p)!} = \frac{8!}{3!} = 4 \times 5 \times 6 \times 7 \times 8 = 6{,}720.$
b) Same as a), but we ***can*** repeat values, so we have Variations **with** repetition, so $\bar{V} = n^p = 8^5 = 32{,}768.$

c) We need to select 5 out of the 8 banners to use. We use different ones for each platform, so repetition is not allowed. Hence, we use the formula for Combinations without repetition. Thus, $C = \frac{n!}{p!(n-p)!} = \frac{8!}{5!3!} = \frac{6\times7\times8}{6} = 7 \times 8 = 56.$

d) Now, we need to select 5 out of the 8 banners to use. However, we can choose some multiple times. Therefore, we need to use Combinations with repetition, so $\bar{C} = \frac{(n+p-1)!}{p!(n-1)!} = \frac{12!}{5!7!} = \frac{8\times9\times10\times11\times12}{1\times2\times3\times4\times5} = \frac{(8\times9\times11)\times10\times12}{(2\times5)\times(3\times4)} = 8 \times 9 \times 11 = 792$.

   In this case, it is vital to not only know which banners we are using, but also to know how many times we are using each one, so we can assign them accordingly. If instead, we did not care how many times we use each banner we have selected, then we would have to find the sum of $C_5^8 + C_4^8 + C_3^8 + C_2^8 + C_1^8$. That is because we are essentially estimating the number of ways, we can select the banners, assuming we are using 5 different ones, 4 different ones, 3 different ones and so on.

Problem 3:

You are renovating your entire apartment and want to repaint the walls of each room. The flat consists of two bedrooms, a kitchen, a living room, a bathroom, a study and a hall, or 7 rooms in total. You have at your disposal several colors of paint: white, yellow, orange, red, purple, blue, green, grey and pink.

How many different ways can you paint the house, assuming…

a) …you paint all the rooms in different colours?
b) …you paint the bathroom, study and hall in white?
c) …you paint the two bedrooms in identical color?
d) …you can only use grey and yellow?

Solution:

Now, we have 9 colors at our disposal, and 7 rooms we need to paint.

a) Using different colors for each platform means we can think of each room as a different position. Hence, we are going to be dealing with Variations. Since we are using different colors for each room, we cannot repeat values, so our formula is: $V = \frac{n!}{(n-p)!} = \frac{9!}{2!} = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \ = 181,440.$

b) If you paint the bathroom, study and hall in white, you only need to think about the other 4 rooms. Now, this problem can be interpreted several different ways, so let us examine each outcome:
   a. We paint the other 4 rooms in 4 different colors. That means we have Variations without repetition, so $V = \frac{n!}{(n-p)!} = \frac{9!}{5!} = 6 \times 7 \times 8 \times 9 = 3{,}024.$
   b. We paint the other 4 rooms in 4 different colors. That means we have Variations without repetition. Additionally, we have already used white, so we are down to only 8 colors. Thus, $V = \frac{n!}{(n-p)!} = \frac{8!}{4!} = 5 \times 6 \times 7 \times 8 = 1{,}680.$

c. We have no restriction on the colors we plan to use in the remaining 4 rooms. Therefore, we have Variations with repetition, ***and*** we can use "white". Thus, $\bar{V} = n^p = 9^4 = 6{,}561$.

We phrased the question with the idea of going for interpretation "b", but we see merit in the other approaches as well.

c) If we paint the two bedrooms in the same color, we can think of them as a single big room. Thus, the number of rooms becomes 6 instead of 7.
   a. There is no restriction on whether we can repeat any of the colors we use, so we have variations with repetition once more $\bar{V} = n^p = 9^6 = 531{,}441$.
   b. Alternatively, it is not *clearly* stated we **can** repeat values, so let us examine the alternative. If we cannot repeat values, we have variations without repetition, so the formula we use becomes: $V = \frac{n!}{(n-p)!} = \frac{9!}{3!} = 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 60{,}480$.

d) Using only grey and yellow means we have 2 colors to choose from, so $n = 2$. Additionally, to paint all the rooms we **must** repeat one or both colors. Therefore, these are Variations with repetition, so the formula is the following: $\bar{V} = n^p = 2^7 = 128$.

Problem 4:

This year, you are helping organize your college's career fest. There are 11 companies which are participating, and you have just enough room fit all of them. How many ways can you arrange the various firms, assuming…:

a) … no firm has any preference where they want to be positioned?
b) … JP Morgan representatives made a deal, where they have to be located exactly in the middle?
c) … JP Morgan, Citi Bank and Morgan-Stanley must be positioned in the middle 3 spots?
d) … Deutsche Bank representatives cancel, so you can give the additional space to one of the other companies?

Solution:

We have 11 firms and 11 spots where we can place each one.

a) If no look has any preference, then we need to arrange the entire set of 11 firms across the room. Thus, we need to use Permutations, so: $P_n = 11! = 39{,}916{,}800$.
b) If JP Morgan has to be located in the middle, then we only need to arrange the remaining 10 firms around the room. Thus, we can once again use permutation, but this time $n = 10$. Thus, $P_n = 10! = 3{,}628{,}800$.
c) One approach to this problem is by looking at two distinct groups of firms – JP Morgan, Citi Bank and MS as one group, and the other 8 firms as the second group. Then if we find the number of ways, we can set up each group around the room, we just have two events with distinct sample spaces.

   Let's start with arranging the 3 banks in the middle. Since we need to split the 3 middle spots among the 3 banks all we need to do is compute the number of Permutations among 3 elements. Therefore, $P_n = 3! = 6$.

Now, since none of the remaining 8 firms cares too much where they are positioned, we once again need to arrange them around the room. Since we have 8 firms and 8 positions, we once again rely of permutations, so $P_n = 8! = 40{,}320$.

For any of the 40,320 ways we set the 8 firms around the room, we have 6 different ways to arrange the 3 banks in the middle. Therefore, in total, we have $40{,}320 \times 6 = 241{,}920$ ways of setting up the career fest.

d) We have 10 firms, which need to fill out 11 spots. Then, if we start filling up the room in some specific order, then there are going to be 10 options for who gets the first position. Since any firm can be given the additional space provided by DB's withdrawal, then there are once again 10 options for the second spot. Then, there would be 9 different options for the third and so on. This results in having $10 \times 10 \times 9 \times 8 \ldots \times 1 = 10 \times 10! = 36{,}288{,}000$ many options to arrange the firms.

Problem 5:

Your best friend is organizing a birthday party for her twins – Amy and Steve - and she put you in charge of ordering the cakes. The bakery offers several types of cakes – a Cheesecake, Sacher Cake, a Chiffon Cake, a Coconut Cake and a Carrot Cake. How many different ways can you order the cakes, assuming that…

a) … both twins enjoy all the 5 types of cake?
b) … Steve dislikes Coconuts?
c) … Amy loves chocolate (Sacher)?
d) … each cake comes with a generic "Happy Birthday!" wish?
e) … each cake comes with a personalized "Happy Birthday Steve!" or "Happy Birthday Amy!" sign?

Solution:

a) Now, if both twins enjoy all 5 cakes, then need to find the number of different combinations of picking 2 cakes out of these 5. Since we are not explicitly told whether we could get the same cake for both, we should consider both scenarios.
   a. Assuming we wish to get the different cakes, then we use the formula for Combinations without repetition: $C = \frac{n!}{p!(n-p)!} = \frac{5!}{2!(3)!} = 10$.
   b. Assuming, we can get them identical cakes, that means we have 5 more options – Cheese and Cheese, Sacher and Sacher and so on. Therefore, we have 15 different ways of getting these cakes. Additionally, we can use the formula for variations with repetition to get: $\bar{C} = \frac{(n+p-1)!}{p!(n-1)!} = \frac{6!}{2!4!} = \frac{6\times5}{1\times2} = 15$.

b) Since Steve dislikes coconuts, our options are limited to 4 cakes. Then, we need to choose two of the 4 remaining cakes, so $C = \frac{n!}{p!(n-p)!} = \frac{4!}{2!2!} = 6$. If we can get two identical cakes, then we have $\bar{C} = \frac{(n+p-1)!}{p!(n-1)!} = \frac{5!}{2!3!} = 10$ options.

   (Alternatively, we can get one Coconut cake and 1 other cake. That way Steve will still have something else to eat. In that scenario, if we can have two identical cakes, then the only option which we want to avoid is the double Coconut one. Thus, we take the 15 we got in part b of a), and subtract 1, so we get 14 options.

Now, if we want to have 2 different cakes, we need to remove the double Cheesecake, double Sacher, double Chiffon and double Carrot cake options. Therefore, there would be 10 different orders we could make.)

c) If Amy **loves** chocolate, one of the two cakes **must** be Sacher. Then, we only need to think about what the other one is. Since we have 5 different cakes, then we have 5 options for choosing the cakes.
d) Now, if the cakes come with generic "Happy Birthday" wishes, it does not matter who gets each cake. Then, since we are not given any additional indication of preference, we can assume this is equivalent to part a). Thus, there are 15 different orders we can make.
e) Now, since it is vital to select the appropriate wish on each of the two cakes, this means that we are dealing with variations. Once again, we have two approaches depending on whether we wish to get them different cakes.
    a. If we decide to do so, then we have $V = \frac{n!}{(n-p)!} = \frac{5!}{3!} = 4 \times 5 = 20$ different orders.
    b. Now, if we are allowed to get them identical cakes, then we have variations with repetition. Thus, $\bar{V} = 5^2 = 25$.

Problem 6:

You want to go to the gym between lectures every day, but you only have an hour to workout. Knowing this, you decide to do a circuit workout. Your start with 5 minutes of cardio as a warm-up, then you hit two different leg exercises, followed by a chest exercise, as well as one for shoulders. After that, you continue with a bicep exercise and a triceps one, before moving to the back one. You finish the circuit with 2 abdominal exercises like a plank and some crunches. After completing the circuit several times, you end with another 10 minutes of cardio before you stretch and leave.

Now, assuming the gym has ellipticals, treadmills and stationary bikes, you have 3 options for the cardio. Additionally, you have 5 different leg exercises you can do. You have 4 choices of what to do for each of the next 3 muscle groups (chest, shoulders and bicep). For triceps you have heavy preferences towards two specific exercises, so you always do one of the two. The same can be said about the back. When it comes to the abdominal exercises, you have 4 options once again.

Taking everything into consideration, if you want to do a different workout each day, how long will it take you to run out of options?

Solution:

To begin with, notice that this *entire* exercise is just an example of Combinations of events with separate sample spaces. Our best approach would be to go through the workout regime one exercise at a time and determine the size of the sample space at each instance.

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Start with the warm-up cardio. We have 3 options, so we fill out the first position.

$$\underline{3}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

Next, we go over the leg exercises. We want to do 2 different exercises and we care which one we do first. Thus, we have $V = \frac{n!}{(n-p)!} = \frac{5!}{3!} = 4 \times 5 = 20$ options for the legs.

$$\underline{3}\ \underline{20}\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

Then we have 4 alternatives for each of the next 3 groups – chest, shoulders and biceps.

$$\underline{3}\ \underline{20}\ \underline{4}\ \underline{4}\ \underline{4}\ \_\_\ \_\_\ \_\_\ \_\_$$

For triceps and back we have two options each, so we add those as well.

$$\underline{3}\ \underline{20}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{2}\ \underline{2}\ \_\_\ \_\_$$

When it comes to the abdominal exercises, we take the same approach we did with the leg exercises. Thus, we have $V = \frac{n!}{(n-p)!} = \frac{4!}{2!} = 3 \times 4 = 12$ options for the abs.

$$\underline{3}\ \underline{20}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{2}\ \underline{2}\ \underline{12}\ \_\_$$

Our warm-down consists of additional cardio, for which we have 3 options.

$$\underline{3}\ \underline{20}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{2}\ \underline{2}\ \underline{12}\ \underline{3}$$

Now, to solve this, we need to multiply the sizes of the different sample spaces we just defined. Thus, we find the product $\underline{3} \times \underline{20} \times \underline{4} \times \underline{4} \times \underline{4} \times \underline{2} \times \underline{2} \times \underline{12} \times \underline{3}$. This results in 552, 960 different variations of the same circuit workout. Therefore, realistically, you will never run out of options.