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COMPOUND-RETURN MEAN-VARIANCE EFFICIENT PORTFOLIOS NEVER RISK RUIN*†

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The implications of concentrating on the lowest moment(s) of average compound return over N periods in making investment decisions have recently been examined. In particular, maximization of expected average compound return has been shown to imply the existence of a utility of wealth function in each period with the "right" properties for all finite $N \geqslant 2$ as well as in the limit. More importantly, for large N a close (or exact) approximation to the set of mean-variance efficient portfolios (with respect to average compound return) is obtainable via a subset of the isoelastic class of utility of wealth functions. The properties of this class render it both empirically plausible and highly attractive analytically: among them are monotonicity, strict concavity, and decreasing risk aversion; moreover, the optimal mix of risky assets is independent of initial wealth (providing a basis for the formation of mutual funds) and the optimal investment policy is myopic. The purpose of this paper is to extend the class of return distributions for which the preceding results hold and to demonstrate that portfolios which are efficient with respect to average compound return, at least for large N, do not risk ruin either in a short-run or a long-run sense.

1. Introduction

In an earlier paper [4], one of the authors examined the implications of concentrating the lowest moment(s) of average compound return over N periods (abbreviated ACRN) in making investment decisions. In particular, maximization of expected average compound return was shown to imply the existence of a utility of wealth function in each period with the "right" properties for all finite $N \geqslant 2$ as well as in the limit. More importantly, mean-variance efficient portfolios with respect to average compound return were also shown to be closely approximated by portfolios obtained by use of the class of utility of wealth functions $(1/\gamma)x^{\gamma}$, $\gamma \leqslant 1/N$, when N is large. The properties of this class render it both empirically plausible and highly attractive analytically: among them are monotonicity, strict concavity, and decreasing risk aversion. Moreover, the optimal mix of risky assets is independent of initial wealth (providing a basis for the formation of mutual funds) and the optimal investment policy is myopic. The purpose of this paper is to extend the class of return distributions for which the preceding results hold and to demonstrate that portfolios which are efficient with respect to average compound return do not risk ruin either in a short-run or a long-run sense.

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As will become apparent in §3, exactly efficient portfolio sequences are difficult to compute. For this reason, we wish to emphasize at the outset that the problem addressed in this paper (and in [4]) is that of finding approximations to the exactly efficient sequences that ease the computational task. No attempt is made to examine the possible usefulness of ACRN-efficiency to an expected utility maximizer. This "reverse" question, however, is addressed by Samuelson and Merton [7].

2. Preliminaries

The following notation is continued from [4]:

x_j = amount of investment capital at decision point j (the beginning of the jth period) $(x_1 > 0)$,

M_j = the number of investment opportunities available in period j, where $M_j \leqslant M$,

S_j = the subset of investment opportunities which it is possible to sell short in period j,

$r_j - 1$ = rate of interest in period j,

β_{ij} = proceeds per unit of capital invested in opportunity i, where $i = 2, \ldots, M_j$, in the jth period (random variable),

z_{1j} = amount lent in period j (negative z_{1j} indicate borrowing) (decision variable),

z_{ij} = amount invested in opportunity i, $i = 2, \ldots, M_j$, at the beginning of the jth period (decision variable),

$$F_j(y_2, y_3, \ldots, y_{M_j}) \equiv \Pr\{\beta_{2j} \leqslant y_2, \beta_{3j} \leqslant y_3, \ldots, \beta_{M_j j} \leqslant y_{M_j}\},$$

$$z_j \equiv (z_{2j}, \ldots, z_{M_j j}),$$

$$v_{ij} \equiv z_{ij}/x_j, \qquad i = 1, \ldots, M_j,$$

$$v_j \equiv (v_{2j}, \ldots, v_{M_j}),$$

$$\langle v_N \rangle \equiv v_1, v_2, \ldots, v_N,$$

$$E_j \equiv \left\{ v_{ij} : v_{ij} \geqslant 0,\, i \notin S_j,\, \sum_{i=2}^{M_j} |v_{ij}| = 1 \right\}.$$

v_{ij} clearly denotes the proportion of capital x_j invested in opportunity i at the beginning of period j. Thus, $\langle v_N \rangle$ uniquely identifies the investment policy over the first N periods.

We assume that the return distributions F_j are independent and satisfy the boundedness conditions

$$\beta_{ij} \geqslant 0 \quad \text{all} \quad i, j, \tag{1}$$

$$E[\beta_{ij}] \leqslant K \quad \text{all} \quad i, j, \tag{2}$$

$$E[\beta_{ij}] \geqslant r_j + \eta_2, \quad \text{where} \quad r_j \geqslant \eta_1 > 1,\ \eta_2 > 0,$$

$$\times \text{and} \quad \beta_{ij} \leqslant L, \quad \textit{some} \quad i, \quad \text{all} \quad j, \tag{3a}$$

$$\beta_{ij} \leqslant L, \quad i \in S_j, \quad \text{all} \quad j, \tag{3b}$$

and the "no-easy-money condition"

$$\Pr\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j)\theta_i < \delta_1 \right\} > \delta_2 \quad \text{for all} \quad j \quad \text{and all} \quad \theta_i \tag{4}$$

such that $\sum_{i=2}^{M_j} |\theta_i| = 1$ and $\theta_i \geqslant 0$ for all $i \notin S_j$, where $\delta_1 < 0$, $\delta_2 > 0$. (1)–(4) are weaker than the corresponding conditions in [4], where the β_{ij}'s themselves (as opposed to their first moments) were assumed to be uniformly bounded and to assume their values with a (joint) probability at least equal to some number $p > 0$.

We also assume, as before, that the investor must remain solvent in each period, i.e., that he must satisfy the solvency constraints

$$\Pr\{x_{j+1} \geqslant 0\} = 1, \qquad j = 1, 2, \ldots . \tag{5}$$

This constraint is necessary to achieve consistency with the standard assumption of risk-free lending and to cope with the requirements of a multiperiod model of the reinvestment type.

The basic difference equation when all proceeds are reinvested is

$$\begin{aligned} x_{j+1} &= \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j x_j, \qquad && j = 1, 2, \ldots, \\ &= x_j R_j(v_j), && j = 1, 2, \ldots, \end{aligned} \tag{6}$$

where

$$R_j(v_j) \equiv \sum_{i=2}^{M_j} (\beta_{ij} - r_j) v_{ij} + r_j. \tag{7}$$

Note that $R_j(v_j)$ is 1 plus the return on the whole portfolio v_j in period j. Clearly, $v_{1j} = 1 - \sum_{i=2}^{M_j} v_{ij}$. When $x_j > 0$, the solvency constraint (5) is equivalent to the constraint

$$\Pr\{R_j(v_j) \geqslant 0\} = 1, \qquad j = 1, 2, \ldots . \tag{8}$$

Solving (6) recursively, we obtain

$$x_{N+1} = x_1 \prod_{j=1}^{N} R_j(v_j), \qquad N = 1, 2, \ldots . \tag{9}$$

Denoting the average compound return over N periods, ACRN, by $C_N(\langle v_N \rangle) - 1$, where

$$C_N(\langle v_N \rangle) \equiv \left(\prod_{j=1}^{N} R_j(v_j) \right)^{1/N}, \qquad N = 1, 2, \ldots, \tag{10}$$

(9) may now be written

$$x_{N+1} = x_1 C_N(\langle v_N \rangle)^N, \qquad N = 1, 2, \ldots . \tag{11}$$

In the following, it will be convenient to define the variables

$$S_N(\langle v_N \rangle) = \sum_{j=1}^{N} \log R_j(v_j), \qquad N = 1, 2, \ldots, \tag{12}$$

$$G_N(\langle v_N \rangle) = S_N(\langle v_N \rangle)/N, \qquad N = 1, 2, \ldots, \tag{13}$$

which gives

$$C_N(\langle v_N \rangle) = \exp\{G_N(\langle v_N \rangle)\}, \qquad N = 1, 2, \ldots . \tag{14}$$

By (9), we may now write

$$x_{N+1} = x_1 \exp\{S_N(\langle v_N \rangle)\} = x_1 \exp\{N G_N(\langle v_N \rangle)\}, \qquad N = 1, 2, \ldots . \tag{15}$$

3. Exact and Approximate Efficiency

The portfolio sequence $\langle v_N \rangle$ is said to be efficient with respect to ACRN if there exists no other (feasible) sequence $\langle v'_N \rangle$ such that

$$E[C_N(\langle v'_N \rangle)] \geqslant E[C_N(\langle v_N \rangle)], \qquad \mathrm{Var}[C_N(\langle v'_N \rangle)] \leqslant \mathrm{Var}[C_N(\langle v_N \rangle)] \tag{16}$$

where one of the inequalities holds strictly.

For any finite N, the efficient sequences at the two "extremes" are readily identified. By (4) and the independence of the distribution functions F_j, there is only one efficient sequence for which $\mathrm{Var}[C_N(\langle v_N \rangle) - 1] = 0$, namely $v_1, v_2, \dots, v_N = (0, 0, \dots, 0)$; its expected ACRN is, using (3a),

$$E[C_N(0, 0, \dots, 0)] - 1 = (r_1 r_2 \cdots r_N)^{1/N} - 1 \geqslant \eta_1 - 1 > 0. \tag{17}$$

There is also only one sequence which achieves the highest expected ACRN; the jth component of that sequence is that portfolio v_j which maximizes

$$E\Big[R_j(v_j)^{1/N}\Big] \tag{18}$$

subject to (8) and

$$v_{ij} \geqslant 0, \qquad i \notin S_j. \tag{19}$$

This result was derived in [4, IV] and is not dependent on the special assumptions of that paper.[1] Maximization of (18) is equivalent to maximization of $E[u(x_{j+1})]$ at each decision point $j = 1, 2, \dots, N$, where

$$u(x) = x^{1/N}. \tag{20}$$

Thus, the efficient sequence at the "upper extreme" is obtained by myopic application of the single-period utility of wealth function (20). Among the properties of (20) are monotonicity, strict concavity, and decreasing risk aversion,[2] i.e., $-u''(x)/u'(x)$ is decreasing in x, and the optimal investment policy is myopic and satisfies the separation property [4], i.e., the optimal mix of assets is independent of wealth.

Turning now to the remaining efficient sequences, we observe that the set of all exactly efficient portfolio sequences, including those at the two "extremes," is obtained by solving

Pl: $\mathrm{Max}_{\langle v_N \rangle} E[C_N(\langle v_N \rangle)]$ subject to $\mathrm{Var}[C_N(\langle v_N \rangle)] = a$ for all values of a in the interval $[0, A_N]$, where A_N is the variance of ACRN for the sequence obtained by maximizing (18) for $j = 1, \dots, N$.

For each a, we let $\langle v_{Na} \rangle$ denote the solution(s) to Pl; thus, the set of all $\langle v_{Na} \rangle$ represents the set of all exactly efficient sequences. Analogously, we write for their means and variances $E_N^a \equiv E[C_N(\langle v_{Na} \rangle)]$, $V_N^a \equiv \mathrm{Var}[C_N(\langle v_{Na} \rangle)]$.

Since the distribution functions $F_1, \dots, F_N$ are independent, Pl may be written

P2: $\mathrm{Max}_{\langle v_N \rangle} E[R_1(v_1)^{1/N}] \cdots E[R_N(v_N)^{1/N}]$ subject to $E[R_1(v_1)^{2/N}] \cdots E[R_N(v_N)^{2/N} - E^2[R_1(v_1)^{1/N}] \cdots E^2[R_N(v_N)^{1/N}] = a$ for all values of $a \in [0, A_N]$.

By Hölder's inequality, $E[R_j(v_j)^{2/N}] \geqslant E^2[R_j(v_j)^{1/N}]$ all j, where the largest expectation for which equality holds is $v_j = (0, \dots, 0)$ by (4). Thus, since $E[R_j(0, \dots, 0)^{1/N}] = r_j^{1/N}$, we obtain, for every exactly efficient sequence $\langle v_{Na} \rangle$,

$$r_j^{1/N} \leqslant E\Big[R_j(v_{ja})^{1/N}\Big] \leqslant E_{j\,\max}, \qquad j = 1, \dots, N,$$

where $E_{j\,\max}$ is the maximum value $E[R_j(v_j)^{1/N}]$ can assume (see Lemma 1).

[1] The possibility that (19) may be binding under the present assumptions is of no import.

[2] The empirical significance of this property is described in Arrow [1, pp. 35–43].

Furthermore, for each $E_{ja} \equiv E[R_j(v_{ja})^{1/N}]$, v_{ja} has the property (see P2 or its reverse) that it is also the solution to

P3: Min $E[R_j(v_j)^{2/N}]$ subject to $E[R_j(v_j)^{1/N}] = E_{ja}$.

Except for the two extremes, the sequences $\langle v_{Na} \rangle$ appear hard to come by since P1 is not easy to solve. It is therefore natural to search for approximate solutions that are easier to generate. In [4], it was suggested that when the central limit law operates, close approximations to the "nonextreme" efficient sequences could be obtained by the myopic application of the members of the set of single-period utility of wealth functions $u(x_{j+1}) = x_{j+1}^{\gamma}/\gamma$, $\gamma < 1/N$. This approximation is obtained by solving the sequence of problems

$$\operatorname*{Max}_{v_j} E\Big[R_j(v_j)^{\gamma}/\gamma\Big], \qquad j = 1, \ldots, N, \tag{21}$$

subject to (8) and (19), where each γ yields a unique solution (Lemma 1), which we denote $\langle v_{N\gamma} \rangle$. For the corresponding means and variances, we write $E_{N\gamma} \equiv E[C_N(\langle v_{N\gamma} \rangle)]$, $V_{N\gamma} \equiv \mathrm{Var}[C_N(\langle v_{N\gamma} \rangle)]$.

When the sequence $\log R_1(v_1), \ldots, \log R_N(v_N)$ obeys the central limit law, the distribution of $G_N(\langle v_N \rangle)$ tends to normality and the distribution of $C_N(\langle v_N \rangle)$ to lognormality (see (14)). Suppose, first, that $C_N(\langle v_N \rangle)$ is exactly lognormal for all efficient and near-efficient (risky) portfolio sequences. We then obtain, denoting the mean and variance of $S_N(\langle v_N \rangle)$ by μ_N and s_N^2,

$$E\big[C_N(\langle v_N \rangle)\big] = \exp\{\mu_N/N + s_N^2/2N^2\},$$

$$\mathrm{Var}\big[C_N(\langle v_N \rangle)\big] = \exp\{2\mu_N/N + s_N^2/N^2\}\big(\exp\{s_N^2/N^2\} - 1\big).$$

In this case, the exactly efficient sequences $\langle v_{Na} \rangle$ were shown in [4] to be obtained by maximizing

$$\mu_N + b s_N^2/2N \tag{22}$$

subject to (8) and (19) for all values of $b \leqslant 1$; these sequences, in turn, were found to be generated by repeated application of the members of the class of single-period utility of wealth functions

$$u(x) = x^{\gamma}/\gamma, \qquad \gamma \leqslant 1/N. \tag{23}$$

But (23) yields the sequence $\langle v_{N\gamma} \rangle$. Under exact lognormality, then, the sequences $\langle v_{N\gamma} \rangle$ are identical to the exactly efficient sequences $\langle v_{Na} \rangle$ for every $N > 1$. In the absence of exact lognormality, the preceding is true only when there is a single risky asset.

While for many sequences both x_{N+1} and $C_N(\langle v_N \rangle)$ are approximately lognormally distributed for large N, it should be noted that x_{N+1} and $C_N(\langle v_N \rangle)$ would generally not be exactly lognormal unless $R_j(v_j)$ is $j = 1, \ldots, N$. But $R_j(v_j)$ would not be exactly lognormal even if the β_{ij} were, since the lognormal distribution does not reproduce itself under additon (see (7)). Thus, we must consider the validity of the approximations $\langle v_{N\gamma} \rangle$ in the absence of exact lognormality.

By Theorem 4 and Corollary 3, $E_{N\gamma} \geqslant (r_1 \cdots r_N)^{1/N}$ all $\gamma \leqslant 1/N$, $V_{N\gamma} \geqslant 0$, with equality holding in the limit as $\gamma \to -\infty$, and $E_{N\gamma}$ and $V_{N\gamma}$ are continuous in γ for $\gamma \leqslant 1/N$. Thus, $E_{N\gamma}$ and E_N^a span the same interval, since their upper endpoints are identical (see (18)) for all values of N. By Theorem 2, we obtain, for $\epsilon > 0$ and N sufficiently large, $V_{N\gamma} < \epsilon$ all $\gamma < 1$ and hence $V_N^a < \epsilon$ all a since $V_N^a \leqslant V_{N\gamma}$ whenever $E_N^a = E_{N\gamma}$.

There are clearly many approximations $\langle v_{Nx} \rangle$ for which the set of expectations $E[C_N(\langle v_{Nx} \rangle)]$ equals the set of exact expectations and for which $\mathrm{Var}[C_N(\langle v_{Nx} \rangle)] < \epsilon$. But in view of the exact lognormality case, there is no smaller class than (23) which will always do. Neither can we expect to find a simpler approximation since (23) implies the repeated use of the *same* myopic objective function in each period.

An unresolved problem bears on the size, when $E_N^a = E_{N\gamma}$, of the error $V_{N\gamma} - V_N^a$ in relation to the exact variance V_N^a. Based on limited computational evidence, we conjecture that the approximation error $V_{N\gamma} - V_N^a$ is small compared to V_N^a even when N is quite small.

4. The Central Limit Problem

In this section we establish that the sequences $\log R_1(v_1), \log R_2(v_2), \ldots$ generated by each of the functions (23) have uniformly bounded means and variances and that they satisfy both the central limit law and the law of large numbers. To do this, we require some lemmas. Due to space limitations, the proofs of Lemmas 1, 2, and 3 have been placed in a supplement, which may be obtained from The Institute of Management Sciences (for detailed instructions, see footnote†).

LEMMA 1. *Let $v_{1\gamma}, v_{2\gamma}, \ldots$ be a portfolio sequence which maximizes*

$$h_j(v_j) = E\left[R_j(v_j)^\gamma\right]/\gamma, \qquad j = 1, 2, \ldots, \tag{24}$$

subject to (8) *and* (19) *for some fixed γ, $-\infty < \gamma < 1$. Then the maximizing sequence is unique and there exist numbers K_1 and K_2 independent of γ such that*

$$E\left[R_j(v_{j\gamma})\right] \leqslant K_1 \quad all \quad j, \tag{25}$$

$$E\left[R_j(v_{j\gamma})^{\gamma-1}\right] \leqslant K_2 \quad all \quad j. \tag{26}$$

LEMMA 2. *Let X be a random variable such that*
(a) $X \geqslant c > 0$ *and*
(b) $\mathrm{Var}[X^\gamma/\gamma] \geqslant \epsilon_1(\gamma) > 0$ *for* $-\infty < \gamma < 0$.
Then $\mathrm{Var}[\log X] \geqslant \epsilon_2(\epsilon_1(\gamma), \gamma) > 0$.

LEMMA 3. *Let $v_{1\gamma}, v_{2\gamma}, \ldots$ be defined as in Lemma* 1. *Then* $\mathrm{Var}[\log R_j(v_j)] \geqslant K_3(\gamma) > 0$ *all j.*

THEOREM 1. *Let $v_{1\gamma}, v_{2\gamma}, \ldots,$ be a portfolio sequence which maximizes* (24) *subject to* (8) *and* (19) *for some fixed γ, $-\infty < \gamma < 1$. Then the variables* $\log R_1(v_{1\gamma})$, $\log R_2(v_{2\gamma}), \ldots,$ *where* $\mu_{j\gamma} = E[\log(R_j(v_{j\gamma}))]$ *and, as before,* $s_{N\gamma}^2 = \mathrm{Var}[S_N(\langle v_{N\gamma}\rangle)]$, *satisfy*

$$N^{-2} \sum_{j=1}^{N} E\left[|\log R_j(v_{j\gamma}) - \mu_{j\gamma}|^2\right] \to 0, \tag{34}$$

$$s_{N\gamma}^{-3} \sum_{j=1}^{N} E\left[|\log R_j(v_{j\gamma}) - \mu_{j\gamma}|^3\right] \to 0, \tag{35}$$

i.e. they obey the law of large numbers and the central limit theorem (*Loéve* [6, *p.* 275]).

PROOF. For large y, $y \geqslant \log^3 y$ and for small y, $|y^{\gamma-1}| \geqslant |\log y|^3$. Therefore Lemma 1 implies that the second and third moments of $|\log R_j(v_{j\gamma}) - \mu_{j\gamma}|$ are uniformly bounded (call this bound B). By Lemma 3, $s_{N\gamma} \geqslant (NK_3(\gamma))^{1/2}$; upon insertion into the left side of (35) we obtain $NB/(N^{3/2}K_3(\gamma)^{3/2}) \to 0$. The proof of (34) is even simpler and is omitted.

If we strengthen assumption (2) to read

$$E[\beta_{ij}^{2+\delta}] \leqslant K \quad \text{for some} \quad \delta > 0, \quad \text{all} \quad i, j, \tag{36}$$

then we can also establish

THEOREM 2. *Let* $v_{1\gamma}, v_{2\gamma}, \ldots$ *be a sequence of Theorem* 1 *and assume that* (36) *holds. Then* $\mathrm{Var}[C_N(\langle v_{N\gamma}\rangle)] \to 0$.

PROOF. Letting $\mu_N = E[S_N(\langle v_{N\gamma}\rangle)]$ and $y_N = S_N(\langle v_{N\gamma}\rangle) - \mu_N$, we obtain

$$\begin{aligned}\mathrm{Var}[C_N(\langle v_{N\gamma}\rangle)] &= \mathrm{Var}\left[\exp(G_N(\langle v_{N\gamma}\rangle))\right]\\ &\leqslant E\left[(\exp((y_N + \mu_N)/N) - \exp(\mu_N/N))^2\right]\\ &= E\left[\exp(2(\mu_N/N))(\exp(2(y_N/N)) + 1 - 2\exp(y_N/N))\right].\end{aligned}$$

The factor $\exp(2(\mu_N/N))$ is uniformly bounded by $(\bar{\lambda}K)^2$. (y_N/N), and hence $2(y_N/N)$, both converge in probability to the point 0 by Theorem 1. Since e^y is a continuous function, $\exp(y_N/N)$ and $\exp(2y_N/N)$ both converge in probability to the point 1 (Billingsley [2, Corollary 2, p. 31]). The proof will be complete when we establish that $\exp(y_N/N)$ and $\exp(2y_N/N)$ are uniformly integrable and hence $E[\exp(y_N/N)]$ and $E[\exp(2y_N/N)] \to E[1] = 1$ (Billingsley [2, Theorem 5.4, p. 32]). In particular we will establish that $\sup_N E[(\exp(2y_N/N))^{1+\epsilon}] < \infty$ for some positive ϵ, which is a sufficient condition for $\exp(2y_N/N)$ to be uniformly integrable (Billingsley [2, p. 32]). Write

$$E\left[(\exp(2y_N/N))^{1+\delta/2}\right] = \exp(-(2+\delta)\mu_N/N)E\left[\exp((2+\delta)(y_N + \mu_N)/N)\right].$$

The first factor is bounded since μ_N is bounded below ((26) and the proof of Theorem 1). The second factor equals

$$E\left[\left(\prod_{j=1}^{N} R_j(v_{j\gamma})^{1/N}\right)^{2+\delta}\right] \leqslant E\left[\left(\sum_{j=1}^{N} R_j(v_{j\gamma})/N\right)^{2+\delta}\right]$$

using the arithmetic-geometric mean inequality. The latter is less than or equal to $E[\sum_{j=1}^{N} R_j(v_{j\gamma})^{2+\delta}/N]$ by the convexity of $y^{2+\delta}$. But

$$E\left[(R_j(v_{j\gamma}))^{2+\delta}\right] = E\left[\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j)v_{ij\gamma} + r_j\right)^{2+\delta}\right]$$

is uniformly bounded since $E[\beta_{ij}^{2+\delta}] \leqslant K$ and $v_{j\gamma}$ is uniformly bounded. This proves the $\exp(2y_N/N)$ case; the $\exp(y_N/N)$ case is similar.

Thus, while condition (2) (as Theorem 1 shows) is sufficient for the distribution of ACRN to tend to its mean, the stronger condition (36) is necessary for the variance of ACRN to vanish with respect to the distributions generated by the class (23).

5. Efficiency in the Limit

A portfolio sequence $\langle v_N\rangle$ is said to be efficient in the limit with respect to ACRN if there exists no other (feasible) sequence $\langle v_N'\rangle$ such that

$$\begin{aligned}&\limsup\left(E[C_N(\langle v_N'\rangle)] - E[C_N(\langle v_N\rangle)]\right) \geqslant 0,\\ &\limsup\left(\mathrm{Var}[C_N(\langle v_N\rangle)] - \mathrm{Var}[C_N(\langle v_N'\rangle)]\right) \geqslant 0,\end{aligned} \tag{37}$$

with one of the inequalities holding strictly.

THEOREM 3. *Let $v_1^*, v_2^*, \ldots,$ be the portfolio sequence which maximizes*

$$E[\log R_j(v_j)], \qquad j = 1, 2, \ldots, \tag{38}$$

subject to (8) *and* (19) (*the* $\gamma = 0$ *case*), *and strengthen assumption* (2) *to* $E[\beta_{ij}^{1+\delta}] \leqslant K$ *for some* $\delta > 0$. *Then for any feasible sequence* $v'_1, v'_2, \ldots, v'_N$ *we have*

$$\limsup\big(E[C_N(\langle v'_N\rangle)] - E[C_N(\langle v_N^*\rangle)]\big) \leqslant 0.$$

PROOF. We begin by considering the portfolio sequence $v_1^\epsilon, v_2^\epsilon, \ldots, v_N^\epsilon$ defined by $v_j^\epsilon = v'_j(1-\epsilon)$, $j = 1, 2, \ldots, N$, where v'_j is any feasible sequence and $\epsilon > 0$ and small. The point of introducing $\langle v_N^\epsilon\rangle$ is that $\mathrm{Var}[\log R_j(v_j^\epsilon)]$ is uniformly bounded since the fact that (8) holds for $\langle v'_N\rangle$ implies $\Pr\{R_j(v_j^\epsilon) \geqslant \epsilon r_j\} \geqslant 1$, $j = 1, \ldots, N$.

For $j = 1, \ldots, N$,

$$(1/(1-\epsilon))\left(\sum_{i=2}^{M_j}(\beta_{ij}(\omega) - r_j)v'_{ij}(1-\epsilon) + r_j\right) > \sum_{i=2}^{M_j}(\beta_{ij}(\omega) - r_j)v'_{ij} + r_j$$

which is equivalent to $(1/(1-\epsilon))R_j(v_j^\epsilon, \omega) > R_j(v'_j, \omega)$ for every ω. Hence

$$\log(1/(1-\epsilon)) + \log\big(R_j(v_j^\epsilon, \omega)\big) > \log R_j(v'_j, \omega)$$

for every ω. Since e^y is monotone,

$$E\left[\exp\left(\sum_{j=1}^{N}\big(\log(1/(1-\epsilon)) + \log R_j(v_j^\epsilon)\big)/N\right)\right] > E\left[\exp\left(\sum_{j=1}^{N}\log R_j(v'_j)/N\right)\right], \tag{39}$$

$$(1/(1-\epsilon))E[C_N(\langle v_N^\epsilon\rangle)] > E[C_N(\langle v'_N\rangle)].$$

Returning to the two original policies,

$$\begin{aligned}
&\limsup\big(E[C_N(\langle v'_N\rangle)] - E[C_N(\langle v_N^*\rangle)]\big)\\
&\quad< \limsup\big((1+\epsilon/(1-\epsilon))E[C_N(\langle v_N^\epsilon\rangle)] - E[C_N(\langle v_N^*\rangle)]\big)\\
&\quad= \limsup E\big[(1+\epsilon/(1-\epsilon))\exp\{(\mu_N^\epsilon + y_N^\epsilon)/N\} - \exp\{(\mu_N^* + y_N^*)/N\}\big]
\end{aligned} \tag{40}$$

where $\mu_N^\epsilon = E[S_N(\langle v_N^\epsilon\rangle)]$ and $y_N^\epsilon = S_N(\langle v_N^\epsilon\rangle) - \mu_N^\epsilon$, with analogous definitions for μ_N^* and y_N^*. By the definition of $v_1^*, v_2^*, \ldots, \mu_N^* \geqslant \mu_N^\epsilon$, so that (40) is less than or equal to

$$\begin{aligned}
\limsup E\big[&\exp(\mu_N^*/N)(\exp(y_N^\epsilon/N) - \exp(y_N^*/N))\\
&+ (\epsilon/(1-\epsilon))\exp((\mu_N^\epsilon + y_N^\epsilon)/N)\big].
\end{aligned}$$

The lim sup of the expression in front of the plus sign is zero since $\exp(\mu_N^*/N)$ is uniformly bounded by $\bar{\lambda}K$ and $E[\exp(y_N^\epsilon/N) - \exp(y_N^*/N)] \to 0$ by the arguments used in Theorem 2. The second term is bounded by $\epsilon\bar{\lambda}K$ and since ϵ is arbitrary the proof is complete.

Under the assumption (36), we have the following two corollaries.

COROLLARY 1. *The portfolio sequence $v_1^*, v_2^*, \ldots$ of Theorem* 3 *is efficient in the limit when* (36) *holds.*

PROOF. Apply Theorems 3 and 2.

COROLLARY 2. *No portfolio sequence $\langle v'_j\rangle$ such that* $\limsup \mathrm{Var}[C_j(\langle v'_j\rangle)] > 0$ *is efficient in the limit when* (36) *holds.*

PROOF. Let $\langle v_j^*\rangle$ be the portfolio sequence of Theorem 3. We have

$$\liminf\big(E[C_j(\langle v_j^*\rangle)] - E[C_j(\langle v'_j\rangle)]\big) \geqslant 0$$

and a fortiori

$$\limsup\Big(E\Big[\,C_j(\langle v_j^*\rangle)\Big] - E\Big[\,C_j(\langle v_j'\rangle)\Big]\Big) \geqslant 0.$$

Since $\operatorname{Var}[C_j(\langle v_j^*\rangle)] \to 0$ by Theorem 2,

$$\limsup\Big(\operatorname{Var}\Big[\,C_j(\langle v_j'\rangle)\Big] - \operatorname{Var}\Big[\,C_j(\langle v_j^*\rangle)\Big]\Big) > 0.$$

Thus, the results of this section confirm that the crucial quantity in the limit is $\lim E[G_N(\langle v_N\rangle)]$. As a result, any portfolio sequence which is not asymptotically close to the unique sequence $v_1^*, v_2^*, \ldots$ generated by (38) cannot be efficient in the limit.

6. Efficiency and the Risk of Ruin

From (11), we see that, in general, the investor's probability of ultimate ruin cannot be greater than the probability that $C_N(\langle v_N\rangle) < 1$. We shall now show that for efficient sequences, $\Pr\{C_N(\langle v_N\rangle) < 1\}$ can be made arbitrarily small; in fact, that for every $\epsilon > 0$ we obtain, by choosing N sufficiently large,

$$\Pr\{\,C_N(\langle v_{Na}\rangle) < \eta_1\} < \epsilon \quad \text{all} \quad a \tag{41}$$

where $\eta_1 > 1$ by (3a). The following lemma will be needed.

Lemma 4. *For every* γ, $v_j \neq 0$, *and* $\delta > 0$,

$$r_j^{\delta} E\Big[\,R_j(v_j)^{\gamma}\big(R_j(v_j) - r_j\big)\Big] < E\Big[\,R_j(v_j)^{\gamma+\delta}\big(R_j(v_j) - r_j\big)\Big].$$

Proof. We need to show that

$$E\Big[\,R_j(v_j, \omega)^{\gamma}\big(R_j(v_j, \omega) - r_j\big)\big(R_j(v_j, \omega)^{\delta} - r_j^{\delta}\big)\Big] > 0.$$

The expression in brackets is strictly positive for every ω such that $R_j(v_j, \omega) \neq 0$ or r_j, which, by (4), completes the proof.

Theorem 4. $E[C_N(\langle v_{N\gamma}\rangle)] \geqslant (r_1 \cdots r_N)^{1/N}$ *and is continuous in* γ *for* $\gamma \leqslant 1/N$. *Moreover*,

$$\lim_{\gamma\to-\infty} E\big[\,C_N(\langle v_{N\gamma}\rangle)\big] = (r_1 \cdots r_N)^{1/N}. \tag{42}$$

Proof. The continuity of $E[C_N(\langle v_{N\gamma}\rangle)]$ and (42) follows easily from the arguments of Lemma 1. Turning then to the inequality

$$E\big[\,C_N(\langle v_{N\gamma}\rangle)\big] \geqslant (r_1 \cdots r_N)^{1/N}, \qquad \gamma \leqslant 1/N, \tag{43}$$

we recall that $E[C_N(\langle v_{N\gamma}\rangle)] = E[R_1(v_{1\gamma})^{1/N}] \cdots E[R_N(v_{N\gamma})^{1/N}]$. Thus, it is sufficient, to demonstrate (43), to show that

$$E\Big[\,R_j(v_{j\gamma})^{1/N}\Big] > r_j^{1/N}, \qquad \gamma \leqslant 1/N, \tag{44}$$

for all j.

By Lemma 3, $v_{j\gamma} \neq (0, \ldots, 0)$. By Lemma 1, we obtain, upon differentiating $E[R_j(\lambda v_{j\gamma})^{\gamma}]/\gamma$ with respect to λ,

$$E\Big[\,R_j(\lambda v_{j\gamma})^{\gamma-1}\big(R_j(v_{j\gamma}) - r_j\big)\Big] \equiv d\,(\lambda; \gamma) \geqslant 0, \qquad 0 < \lambda \leqslant 1.$$

But by Lemma 4,

$$E\Big[\,R_j(v_{j\gamma})^{1/N-1}\big(R_j(v_{j\gamma}) - r_j\big)\Big] > d\,(1; \gamma) r_j^{1/N-\gamma} \geqslant 0, \qquad \gamma < 1/N.$$

Thus,

$$E\left[R_j(\lambda v_{j\gamma})^{1/N-1}\left(R_j(v_{j\gamma}) - r_j\right)\right] > 0, \qquad 0 < \lambda \leqslant 1,$$

which, since $E[R_j(0, \ldots, 0)^{1/N}] = r_j^{1/N}$, implies (44).

COROLLARY 3. $\mathrm{Var}[C_N(\langle v_{N\gamma}\rangle)]$ *is continuous in* γ *for* $\gamma \leqslant 1/N$ *and*

$$\lim_{\gamma \to -\infty} \mathrm{Var}\left[C_N(\langle v_{N\gamma}\rangle)\right] = 0.$$

What Theorem 4 says, then, is that for each expected ACRN for an exactly efficient sequence, E_N^a, we can find an approximating sequence $\langle v_{N\gamma}\rangle$ with the same expected ACRN, $E_{N\gamma}$, and vice versa. But for $E_N^a = E_{N\gamma}$, $V_N^a \leqslant V_{N\gamma}$, and since $V_{N\gamma}$ can be made arbitrarily small by choosing N sufficiently large (Theorem 2), (41) holds, i.e., portfolios which are mean-variance efficient with respect to ACRN do not risk ruin. (41) is a property not shared by portfolio sequences which are efficient under the traditional (single-period) mean-variance criterion [5]. For such policies we may have $\Pr\{C_N(\langle v_N\rangle) < 1\} \to 1$ and hence, for any $\alpha > 0$, $\Pr\{x_{N+1} < \alpha\} \to 1$.

It may be reassuring to consider the "optimal" flirtation with ruin in the individual periods of the portfolio sequence $\langle v_{N\gamma}\rangle$. In view of (28) and (30) in the proof of Lemma 1 (see Supplement), we obtain $\Pr\{R_j(v_{j\gamma}) = 0\} = 0$, $j = 1, \ldots, N$. In fact, when, as in [4], the possible values assumed by $R_j(v_j)$ have a probability of at least $p > 0$, there is a lower bound $b_\gamma > 0$ such that $\Pr\{R_j(v_{j\gamma}) \geqslant b_\gamma\} = 1$ for all j; by (26) $b_\gamma^{\gamma-1} p \leqslant K_2$, which gives $b_\gamma \geqslant (K_2/p)^{1/(\gamma-1)} > 0$.

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