Analysis 1 Practicals

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Contents

Chapter 1

Complex Numbers

1.1 Algebraic form

1.1.1 Definition(Algebraic Form)

The algebraic form is defined as follows:

$$z = x + iy \tag{1.1}$$

1.1.2 Properties

Algebraic form makes addition, subtraction, rationalization easier. The following properties are worth remembering:

$$|z| \geq 0, |z| = 0 \Leftrightarrow z = 0 \tag{1.2}$$

$$\bar{z} = (x - iy) \Longrightarrow \tag{1.3}$$

$$z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y^2 = \left|z^2\right| = \left|\bar{z}^2\right| \tag{1.4}$$

$$\mathrm{Re}(x + iy) = x \Longrightarrow \mathrm{Re}(z) = \mathrm{Re}(\bar{z}) \tag{1.5}$$

$$\mathrm{Im}(x + iy) = y \Longrightarrow \mathrm{Im}(z) = -\mathrm{Im}(\bar{z}) \tag{1.6}$$

$$|z| = \sqrt{(x^2 + y^2)} \Longrightarrow |z| = |\bar{z}| \tag{1.7}$$

$$\text{but it is not valid for comparison operators:} \tag{1.8}$$

$$|z + w| \leq |z| + |w| \tag{1.9}$$

Conjugate properties carry over operations:

$$\overline{(z + w)} = \bar{z} + \overline{w} \tag{1.10}$$

$$\overline{(z \cdot w)} = \bar{z} \cdot \overline{w} \tag{1.11}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\overline{w}} \tag{1.12}$$

1.2 Polar coordinates & Trigonometric form

1.2.1 Polar coordinates

Definition (Polar coordinates) We define the polar coordinates as follows:

$$z = (x + iy) = \begin{cases} x = \rho \cos(\vartheta) \\ y = \rho \sin(\vartheta) \end{cases}$$

1.2.2 Trigonometric form

Definition (Trigonometric form) Each $z \in \mathbb{C}$ can be expressed as follows using polar forms:

$$z = \rho(\cos(\vartheta) + \imath \sin(\vartheta)) \text{ with} \tag{1.13}$$

$$p = \sqrt{x^2 + y^2} \tag{1.14}$$

$$\tag{1.15}$$

$$\vartheta = \begin{cases} \arctan(\frac{y}{x}), & \text{if } x > 0 \text{ (First Quadrant)} \\ \pi + \arctan(\frac{y}{x}), & \text{if } x < 0, y \geq 0 \text{ (Fourth Quadrant)} \\ -\pi + \arctan(\frac{y}{x}), & \text{if } x < 0, y < 0 \text{ (First Quadrant)} \\ \frac{\pi}{2}, & \text{if } x = 0, y > 0 \\ -\frac{\pi}{2}, & \text{if } x = 0, y < 0 \end{cases}$$

The trick to remembering the aforementioned definitions is to refer to the unit circle and the uniquely defined tangents in the first and fourth quadrant of the unit circle.

Properties of the Trigonometric form

Multiplication and division The trigonometric form is useful with multiplication and division:

$$z_1 \cdot z_2 = \rho_1 \cdot \rho_2 \left[\cos(\vartheta_1 + \vartheta_2) + i \sin(\vartheta_1 + \vartheta_2)\right] \tag{1.16}$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \left[\cos(\vartheta_1 - \vartheta_2) + i \sin(\vartheta_1 - \vartheta_2)\right] \tag{1.17}$$

De Moivre's Formula

$$z^n = \rho^n \left[\cos(n \cdot \vartheta) + i \sin(n \cdot \vartheta)\right] \tag{1.18}$$

1.3 Exponential form

1.3.1 Definition (Exponential Form)

The exponential form is defined as follows:

$$z = |z| \cdot e^{i \cdot \theta} = \rho e^{i\theta} \tag{1.19}$$

1.3.2 Multiplication

The exponential form is both useful for multiplication and division (?), and also for root calculation. Multiplication is as follows:

$$z_1 = \rho_1 \cdot e^{i\theta_1} \tag{1.20}$$

$$z_2 = \rho_2 \cdot e^{i\theta_2} \tag{1.21}$$

$$z_1 \cdot z_2 = \rho_1 \cdot e^{i\theta_1} \cdot \rho_2 \cdot e^{i\theta_2} = \rho_1 \rho_2 e^{i(\theta_1+\theta_2)} \tag{1.22}$$

$$z^n = \rho^n e^{in\theta} \tag{1.23}$$

1.3.3 Root calculation

Let $z = \rho e^{i\vartheta}, w = ke^{i\alpha}$, then we can find the n roots as follows:

$$z^n = w \Longrightarrow \rho^n e^{in\vartheta} = ke^{i\alpha} \Longrightarrow \tag{1.24}$$

$$\Longrightarrow \begin{cases} \rho^n = k \\ \cos(n\vartheta) = \cos(\alpha) \\ \sin(n\vartheta) = \sin(\alpha) \end{cases} \tag{1.25}$$

$$\Longrightarrow \begin{cases} \rho = \sqrt[n]{k} \\ \vartheta = \left(\frac{\alpha}{n} + \frac{2c\pi}{n}\right), \quad \text{con } c = 0, 1, \cdots, n-1 \end{cases} \tag{1.26}$$

Chapter 2

Sequences

2.1 General Limit Theory

2.1.1 Infinities

Before hopping into indeterminate forms, here are all the cases:

$$\lim_{n \to +\infty} \frac{P_k(n)}{Q_k(n)} \text{ with } k \in \mathbb{N} = \frac{a_k}{b_k} \tag{2.1}$$

if infinities are of the same order and eliminable. Otherwise the following applies:

Sumtraction

If we are evaluating $\lim_{n \to +\infty} a_n \pm b_n$:

- if $a_n \overset{+\infty}{\to} +\infty$ and $b_n \overset{+\infty}{\to} c \in \mathbb{R}$, then $a_n \pm b_n = +\infty$
- if $a_n \overset{+\infty}{\to} -\infty$ and $b_n \overset{+\infty}{\to} c \in \mathbb{R}$, then $a_n \pm b_n = -\infty$
- if $a_n \overset{+\infty}{\to} +\infty$ and $b_n \overset{+\infty}{\to} +\infty$, then $a_n + b_n = +\infty$
- if $a_n \overset{+\infty}{\to} -\infty$ and $b_n \overset{+\infty}{\to} -\infty$, then $a_n + b_n = -\infty$

Multiplication

If we are evaluating $\lim_{n \to +\infty} a_n \cdot b_n$:

- if $a_n \overset{+\infty}{\to} +\infty$ and $b_n \overset{+\infty}{\to} c \in \mathbb{R} \setminus \{0\}$ then $a_n \cdot b_n = \pm\infty$
- if $a_n \overset{+\infty}{\to} \pm\infty$ and $b_n \overset{+\infty}{\to} \pm\infty$ then $a_n \cdot b_n = \pm\infty$
  (Sign rule applies)

Division

If we are evaluating $\lim_{n\to+\infty} \frac{a_n}{b_n}$

- if $a_n \overset{+\infty}{\to} a \in \mathbb{R}$ and $b_n \overset{+\infty}{\to} \pm\infty$ then $\frac{a_n}{b_n} = 0$
- if $a_n \overset{+\infty}{\to} \pm\infty$ and $b_n \overset{+\infty}{\to} b \in \mathbb{R} \setminus \{0\}$ then $\frac{a_n}{b_n} = \pm\infty$
- if $a_n \overset{+\infty}{\to} a \in \mathbb{R} \setminus \{0\}$ and $b_n \overset{+\infty}{\to} 0$ then $\frac{a_n}{b_n} = \pm\infty$ ($b_n > 0$ and $b_n < 0$)

Keeping in mind that the rule of signs still applies.

Limited sequences and infinity

If $(a_n)_{n\in\mathbb{N}} \to 0$ and $(b_n)_{n\in\mathbb{N}}$ is a limited sequence, then $(a_n \cdot b_n)_{n\in\mathbb{N}}$ also converges to 0 due to the squeeze theorem.

Indeterminate Forms

Nothing can be said about the following forms:

- if $a_n \overset{+\infty}{\to} +\infty$, $b_n \overset{+\infty}{\to} -\infty$ then $a_n + b_n = ?$
- if $a_n \overset{+\infty}{\to} \pm\infty$, $b_n \overset{+\infty}{\to} 0$ then $a_n \cdot b_n = ?$
- if $a_n \overset{+\infty}{\to} \pm\infty$, $b_n \overset{+\infty}{\to} \pm\infty$ then $\frac{a_n}{b_n} = ?$
  which is similar to the previous case.
- if $a_n \overset{+\infty}{\to} 0$, $b_n \overset{+\infty}{\to} 0$ then $\frac{a_n}{b_n} = ?$
- 0^0
- ∞^0
- 1^∞

Extracted Sequences

A sequence admits a limit if and only if all its extracted sequence admit the same limit.

2.1.2 Notable Limits

Geometric Sequence

Every sequence of type a^n with $a \in \mathbb{R}$ is called geometric sequence.
The limit is as follows:

$$\lim_{n\to+\infty} a^n = \begin{cases} +\infty & \text{if } a > 1 \\ 0 & \text{if } |a| < 1 \\ 1 & \text{if } a = 1 \\ \nexists & \text{if } a \leq -1 \end{cases} \tag{2.2}$$

Roots

For each $a > 0$ we have that $\lim_{n\to+\infty} \sqrt[n]{a} = 1$
We also have that $\lim_{n\to+\infty} \sqrt[n]{n} = 1$

Exponential

The sequence $\left((1+\frac{1}{n})^n\right)_{n\in\mathbb{N}}$ admits limit:

$$\exists \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = \sup_{n\in\mathbb{N}} \left(1+\frac{1}{n}\right)^n = e \in \mathbb{R} \tag{2.3}$$

therefore, it follows that if:

$$a_n \to +\infty \Rightarrow \lim_{n\to+\infty} \left(1+\frac{1}{a_n}\right)^{a_n} = e \tag{2.4}$$

$$a_n \to 0 \Rightarrow \lim_{n\to+\infty} (1+a_n)^{\frac{1}{a_n}} = e \tag{2.5}$$

Logarithm

For each $n > 3$ we have that $\lim_{n\to+\infty} \frac{\log n}{n} = 0$

Trigonometry

Let a_n be an infinitesimal sequence. Then:

- $\lim_{n\to+\infty} \sin(a_n) = 0$ due to the fact that $|\sin(x)| \leq |x| \, \forall x \in \mathbb{R}$
  and by our proposition $a_n \to 0$
- $\lim_{n\to+\infty} \frac{\sin(a_n)}{a_n} = 1$ due to the first comparison theorem.

Chapter 3

Function Limits

3.1 Basic Limits

3.1.1 Examples

1. **(Arctan)** We have that $\lim_{x\to+\infty} \arctan x = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$
2. **(Exponential)** We have that $\lim_{x\to+\infty} a^x = +\infty$ and $\lim_{x\to-\infty} a^x = 0$ from $a^{-\alpha} = \frac{1}{a^{\alpha}},\ \alpha \in \mathbb{R}$
3. **(Power)** We have that $\lim_{x\to+\infty} x^n = +\infty$ and $\lim_{x\to-\infty} x^n = \begin{cases} +\infty & \text{if n is even} \\ -\infty & \text{if n is not even} \end{cases}$

3.1.2 Theorem (Characterisation of limits with sequences)

Let $\mathbb{X} \in \mathbb{R}$, let $x_0 \in \mathbb{R}$ be an accumulation point of $\mathbb{X}$ and $f : \mathbb{X} \to \mathbb{R}$ a function. Then

$$\exists \lim_{x\to x_0} f(x) = l \in \bar{\mathbb{R}} \Leftrightarrow \forall (x_n)_{n\in\mathbb{N}} : \lim_{n\to+\infty} x_n = x_0 \Rightarrow \lim_{n\to+\infty} f(x_n) = l \tag{3.1}$$

Which, alongside other consequences, implies that a function admits a limit if an only if all of its extracted sequences do:

$$\nexists \lim_{x\to+\infty} \sin(x) \text{ due to limits to infinity of the sequences } x_n = \frac{\pi}{2} + 2k\pi, y_n = 2k\pi \tag{3.2}$$

Examples

We demonstrated that for each infinitesimal sequence a_n we have that $\lim_{n\to+\infty} \sin(a_n) = 0$, $\lim_{n\to+\infty} \cos(a_n) = 1$, $\lim_{n\to+\infty} \frac{\sin(a_n)}{a_n} = 1$

As a result of the previous theorem, we have

$$\lim_{x \to 0} \sin(x) = 0 \tag{3.3}$$

$$\lim_{x \to 0} \cos(x) = 1 \tag{3.4}$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \tag{3.5}$$

3.1.3 Continuous Functions

Weierstraß Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function in $[a, b]$. Then f is limited and has minimum and maximum in $[a, b]$, specifically:

$$\exists x_1, x_2 \in [a, b] \text{ such that } \forall x \in [a, b]: \ f(x_1) \leq f(x) \leq f(x2) \tag{3.6}$$

Theorem of the existence of zeroes

Let $f : [a, b] \to \mathbb{R}$ be a function continuous in $[a, b]$ such that $f(a) \cdot f(b) < 0$. Then there exists at least one $x_0 \in (a, b)$ such that $f(x_0) = 0$.
If f is also monotonous in $[a, b]$ then x_0 is unique.

Note that as a corollary, the same applies to intervals of continuous functions. This theorem is useful for establishing zeroes in non easily resolvable equations.

Discontinuity

There exist 3 types of discontinuity:

1. $\exists \lim_{x \to x_0} f(x) = l \in \mathbb{R}$ with $l \neq f(x_0)$
   Example:
   $f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ with an eliminable discontinuity at 0.
2. **Discontinuity of first species:**
   $\exists \lim_{x \to x_0^+} f(x) = l_2 \in \mathbb{R} \neq l_3 \in \mathbb{R} = \exists \lim_{x \to x_0^-} f(x)$
   with jump of value the subtraction of the two limits.
3. **All the remaining cases:**
   Limits to infinity, only one-sided limits to infinity, one side not existing and the other existing, and so on, are called discontinuity of second species.

3.1.4 Notable Limits

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x\to 0} \frac{\tan x}{x} = 1$
3. $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
4. $\lim_{x\to 0} \frac{\arctan x}{x} = 1$
5. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
6. $\lim_{x\to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$
7. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
8. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
9. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
10. $\lim_{x\to +\infty} \frac{\log x}{x} = 0$

Chapter 4

Derivation

4.1 Definitions of Derivative

$$\exists\, f'(x_0) \Leftrightarrow \exists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \Leftrightarrow \exists \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \tag{4.1}$$

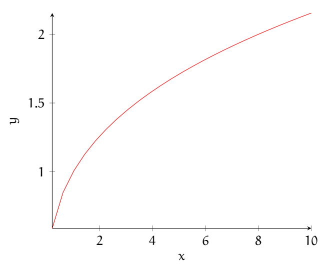
4.2 Geometric meaning of derivation

There are 5 situations that can develop when evaluating the definition of derivative in a function:

1. **Tangent line to the point:** $\exists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$
(Derivable)

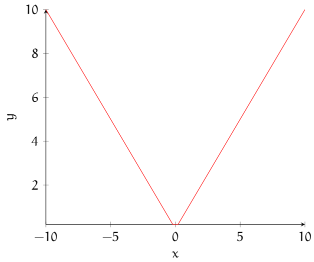
2. **Non derivable:** if f is not derivable in x_0, then either
$\exists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \pm\infty$ or $\nexists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$

3. **Vertical flex tangent:**
In case f is continuous in x_0 and $\exists \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \pm\infty$
Then the tangent assumes a position parallel to the y-axis.
Example: $\sqrt[3]{x}$ in 0

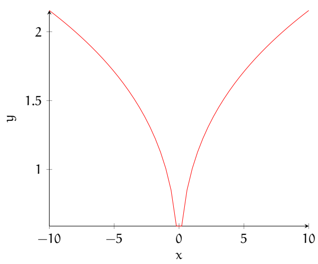


4. **Angular point:**
In case f is continuous in x_0 and
$\exists \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = m \in \mathbb{R}$, $\exists \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = m' \in \mathbb{R}$

with $m \neq m'$. Example: $|x|$ in 0:



5. **Pinnacle point:**
   In case f is continuous in x_0 and
   $\exists \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = +\infty$, $\exists \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = -\infty$
   Then both the left and right derivative assume a position parallel to the y-axis but they do not coincide due to the direction of the tangent coming from left or right. Example: $\sqrt[3]{|x|}$ in 0:



4.2.1 Identifying non-derivable points

Having described the different derivation points, there are a few steps to go over during function studies to identify non-derivable points:

1. Identify possible points on non derivability, avoiding derivable points (sum, subtraction, product, quotient, composition of derivable functions.

2. Identify our function's domain.

3. Calculate the derivative of our function, and determine its domain.

4. Calculate $\text{Dom}(f) \cap \text{Dom}(f')$ (intersection of the two domains) because it does not make sense to talk about derivation points in the points in which even f is not defined.

5. In the points in which f' is continuous we have no issues, as the necessary condition of derivability is fulfilled.

6. If we have a function defined in tracts:
   $f(x) = \begin{cases} f_1(x) & \text{for } x \in [a,b] \\ f_2(x) & \text{for } x \in (b,c] \end{cases}$

then the joint point $x = b$ is a potential non-derivability point.
Example: Calculating the limit definition of derivation with absolute value functions.

7. The points we identified in steps 1-6 are possible points on non-derivability. Calculate the left and right limit of the incremental product to see which types of points they are.

Example 1: Absolute function

Let's analyse the function $f(x) = \left|9 - x^2\right|$ which is continuous in all its domain, that is, $\mathrm{Dom}(f) = \mathbb{R}$.
However, the absolute value function $f(x) = |x|$ is a non-derivable function in its only point in which the argument is null, due to the left limit and right limit in the limit definition of derivative existing but having different values. Therefore, we will analyse the points where the argument cancels:
$9 - x^2 = 0 \to x = \pm 3$
Furthermore, there are no more points to take into consideration:
$f'(x) = \frac{|9-x^2|}{9-x^2} \cdot (-2x)$
which is defined for each $x \in \mathrm{Dom}(f)$, $x \neq \pm 3$, or rather:
$\mathrm{Dom}(f) \cap \mathrm{Dom}(f') = \mathbb{R} \setminus \{\pm 3\}$

Having identified our possible points in which our function is not derivable, let's analyse them:
$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h}$, $\lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h}$
$\lim_{h\to 0^+} \frac{f(-3+h)-f(-3)}{h}$, $\lim_{h\to 0^-} \frac{f(-3+h)-f(-3)}{h}$
and we will find that ± 3 are angle points for f because these limits (taken in couples) assume finite but different values.
For example:
$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{|9-(3+h)^2|-0}{h} = \lim_{h\to 0} \frac{|-6h-h^2|}{h}$, let's distinguish the left and right cases:
To the left of $h = 0$ we have $-6h - h^2 > 0$ and therefore $\left|-6h - h^2\right| = +\left(-6h - h^2\right)$, and therefore
$\lim_{h\to 0^-} \frac{-6h-h^2}{h} = \lim_{h\to 0^-} (-6 - h) = -6$
On the other hand, at the right side of $h = 0$ we have $-6h - h^2 < 0$ and therefore $\left|-6h - h^2\right| = -\left(-6h - h^2\right)$, lastly
$\lim_{h\to 0^+} \frac{|-6h-h^2|}{h} = \lim_{h\to 0^+} \frac{6h+h^2}{h} = +6.$

4.3 Calculating Derivatives

4.3.1 Basic derivation rules

1. $(f \pm g)' = f' \pm g'$
2. $(f \cdot g)' = f' \cdot g \;+\; g' \cdot f$
3. $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
4. $\forall c \in \mathbb{R},\; (c \cdot f)' = c \cdot (f)'$
5. $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
6. $(g \circ f)' = g'(f) \cdot f'$
7. $\left(f^{-1}\right)' = \frac{1}{f'}$

4.3.2 Basic functions derivates

1. **Constants:** $\forall k \in \mathbb{R},\; k' = 0$

   Natural Powers:

2. $\forall n \in \mathbb{N},\; (x^n)' = n \cdot x^{n-1}$
3. $f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x^1 + b_0}$ is derivable in $\mathbb{R} \setminus \left\{x \in \mathbb{R} : \; b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x^1 + b_0 = 0\right\}$
   Example: $\left(\frac{1}{x^n}\right)' = -\frac{n}{x^{n+1}},\; \forall x \in \mathbb{R} \setminus \{0\}$

   Trigonometric Functions:

4. $(\sin x)' = \cos x$
5. $(\cos x)' = -\sin x$
6. $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$
7. $\forall x \in (-1, 1),\; (\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$
8. $\forall x \in (-1, 1),\; (\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$
9. $\forall x \in \mathbb{R},\; (\arctan x)' = \frac{1}{1 + x^2}$

4.4 Applications of derivatives in function study

4.4.1 Convexity and concavity

Let $f : [a, b] \to \mathbb{R}$ be a function derivable in $[a, b]$ and 2 times derivable in (a, b).
Then the following properties are valid:

1. The following conditions are equivalent:
   (a) f is convex in $[a, b]$
   (b) f' is increasing in $[a, b]$
   (c) $f'' \geq 0$ for each $x \in (a, b)$
2. The following conditions are equivalent:
   (a) f is concave in $[a, b]$
   (b) f' is decreasing in $[a, b]$
   (c) $f'' \leq 0$ for each $x \in (a, b)$

4.4.2 Maximum & Minimum points

First derivatives allow us to find both relative and absolute maximum & minimum points of a function, and to establish in which intervals of a function's domain the function increases or decreases.
We know that absolute maximum & minimum points are also relative maximum & minimum points.

Fermat's Theorem

Let $f(x)$ be a function with domain $\mathrm{Dom}(f) \subseteq \mathbb{R}$.
If $x_0 \in \mathrm{Dom}(f)$ is an extreme relative point for f, and the function is derivable in that point, then

$$f'(x_0) = 0 \tag{4.2}$$

Fermat's theorem is a necessary but not sufficient condition for a point to be relative or absolute maximum or minimum.
It states that stationary points are those where the derivative is null.
Some of these points are absolute or relative extreme points, i.e. absolute or relative minimums or maximums.

(Theorem) Test of the first derivative

$$\begin{aligned} x_0 \text{ is a maximum point } &\Leftrightarrow f(x) \text{ is increasing at the left of } x_0 && (4.3)\\ &\text{and decreasing at the right of } x_0 && (4.4) \end{aligned}$$

$$\begin{aligned} x_0 \text{ is a minimum point } &\Leftrightarrow f(x) \text{ is decreasing at the left of } x_0 && (4.5)\\ &\text{and increasing at the right of } x_0 && (4.6) \end{aligned}$$

Flex points

4.5 Taylor's Formula

4.5.1 Landau notation

Definition("Little o") Given two functions, f and g defined in a range around x_0, we say that

$$f(x) = o(g(x)), \text{ for } x \to x_0 \tag{4.7}$$

which is read as "$f(x)$ is little o of $g(x)$ for $x \to x_0$" if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \tag{4.8}$$

If g is a non-null function in $I \setminus \{x_0\}$ and $\lim\limits_{x \to x_0} g(x) = 0$, meaning that g is an infinitesimal for $x \to x_0$, $f(x) = o(g(x))$ for $x \to x_0$ implies that
f tends to 0 for $x \to x_0$ but faster than g: "$0 \leq f(x) \leq g(x)$ as $x \to x_0$".
The Landau notation expresses a family of functions such that the limit tends to 0.
The following properties follow:

1. $o(g) + o(g) = o(g)$
2. $c \cdot o(g) = o(g)$
3. $g_1 \cdot o(g_2) = o(g_1 \cdot g_2)$
4. $o(g_1) \cdot o(g_2) = o(g_1 \cdot g_2)$
5. $|o(g)|^\alpha = o\left(|g|^\alpha\right)$
6. $o(g + o(g)) = o(g)$
7. $o(o(g)) = o(g)$

Chapter 5

Integration

5.1 Indefinite Integration of irrational functions

In these integrals we try to turn irrational integrals through substitution into rational ones. For example:

$$\int R\left(x, \sqrt{ax+b}\right) dx \ \Rightarrow \ \text{Let } z^2 = ax+b \ \rightarrow dx = \frac{2}{a} z dz \ \Rightarrow \ \int R\left(\frac{z^2-b}{a}, z\right) \frac{2}{a} z dz \ .$$

If we have irrational second grade polynomials: $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$,we complete the square which puts us into one of 3 cases.

Completing the square:

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a} \tag{5.1}$$

1. $\int R\left(x, \sqrt{a^2-x^2}\right) dx$:
   We use the substitution $x = a\sin z \ \rightarrow \ dx = a\cos z dz$.
   Therefore $\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin z)^2} = |a\cos z|$
   We then have left to integrate
   $\int R\left(a\sin z, |a\cos z|\right) a\cos z dz$
   which can be further rationalised through the substitution $t = \tan\frac{z}{2}$.

2. $\int R\left(x, \sqrt{a^2+x^2}\right) dx$:
   We use the hyperbolic substitution $x = a\sinh z \ \rightarrow \ dx = a\cosh z dz$.
   Therefore $\sqrt{a^2+x^2} = \sqrt{a^2+(a\sinh z)^2} = |a|\cosh z$
   We then have left to integrate
   $\int R\left(a\sinh z, |a|\cosh z\right) a\cosh z dz$
   Which is generally solved by parts or through the substitution $t = e^z$. To return to the x variable, we use the inverse hyperbolic function:
   $z = \operatorname{sett}\sinh\frac{x}{a} = \log\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2+1}\right)$

3. $\int R\left(x, \sqrt{x^2 - a^2}\right) dx$:
In this case we can also use the hyperbolic substitution $x = a\cosh z \ \rightarrow \ dx = a\sinh z dz$.
Therefore $\sqrt{x^2 - a^2} = \sqrt{(a\cosh z)^2 - a^2} = |a\sinh z|$.
We then have left to integrate
$\int R\left(a\cosh z, |a\sinh z|\right) a\sinh z dz$
Which can be calculated like the previous one. To return to the x variable we also use an inverse hyperbolic function:
$z = sett\cosh\frac{x}{a} = \log\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right)$