27/8/24.

UNIT - 1.

SETS AND PROPOSITIONS.

* **Sets** -

1. What is a set?

— ① A set is a collection of distinct objects considered as an object in its own right.

② For ex. set of vowels in the English language can be,

$A = \{a, e, i, o, u\}$

③. Elements: The objects in the sets are called as elements or members. If an object x is in set A we write $x \in A$. (x belongs to A).

* **Empty set**:

— The set with no elements is an empty set denoted by ϕ or $\{\}$.

• **Subset**

— A set 'A' is a subset of set 'B' if all elements of A are also elements of B. we denote this as $A \subseteq B$.

• **Union**

— The union of two sets A and B is the set of elements that are in either A, B or

or both. it's denoted as $A \cup B$.

- **Intersection.**
  The intersection of two sets A and B is the set of elements that are in both A and B its denoted as $A \cap B$.

- **Complement -**
  Complement of set A (relative to universal set U) is the set of all elements in U that are not in A. It's denoted as A' or $\bar{A}$.

- **Difference -**
  The difference between two sets A and B written as $A - B$ is a set containing elements that are in but not in B.

  $\therefore$ ex. -

  If $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$ then $A - B = \{1, 2\}$

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* Given the set $A = \{1, 2, 3, 4, 5\}$ answer the following questions.

i) Is the number 3 an element of set A.

ii) Is number 6 is an element of A.

iii) List all elements of set A.

Soln: i) Yes number 3 is an element of set A. so $3 \in A$.

ii) No number 6 is not an element of set A, so $6 \notin A$.

iii) The elements of set A are 1, 2, 3, 4, 5

$\therefore A = \{1, 2, 3, 4, 5\}$

* Let $A = \{2, 4, 6, 8\}$ and $B = \{4, 5, 6, 7\}$, Find:

1. The union of sets A and B.
2. The intersection of sets A and B.

Soln: (i) Union:

$A \cup B = \{2, 4, 6, 8, 5, 7\}$

(ii) Intersection:

$A \cap B = \{4, 6\}$ — common elements

* Consider the sets $C = \{a, b, c\}$ and $D = \{a, b, c, d\}$ Determine:
1. C is subset of D.
2. Is D subset of C.

→ 1. C is subset of D, because, all elements of C are present in D. so $C \subseteq D$.

2. D is not subset of C, because, all elements of D are not present in C. so $D \not\subseteq C$.

* Combination of sets.

Combination of sets typically refers to operation like union, intersection, difference and complement.

* Complement

Given that universal set $U = \{1, 2, 3, 4, 5, 6, 7\}$ and set $C = \{2, 4, 6\}$ find complement of C.

Solution -

- Complement of C (C' or $\overline{C}$)

$C' = \{1, 3, 5, 7\}$

* Venn Diagram-

A venn Diagram is a visual tool used to represent sets and their relationships, such as union, intersection and difference. Each set is depicted as a circle, with overlaping areas showing the common elements between sets.

- Basic Venn Diagram concept.

1. Single set -
   - A single circle represent one set.
   - The elements inside the circle belong to set while the numbers outside it do not.

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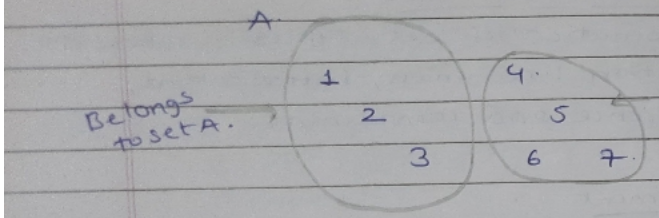
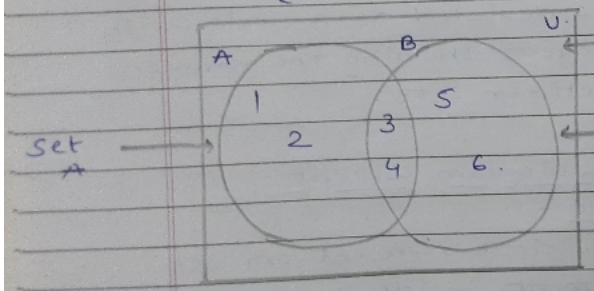
$\therefore A = \{1, 2, 3\}$.



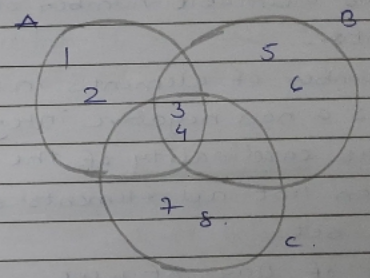
2. Two sets.

- When two sets are involved their circles may overlap depending upon whether they have common elements.
- The overlapping area represents the intersection (common elements) of two sets.
- The total area covered by circle represent union of sets.
- $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$.



3. Three sets:

- For three sets, three circles are drawn potentially all overlapping.



$A = \{1, 2, 3, 4\}$
$B = \{3, 4, 5, 6\}$
$C = \{3, 4, 7, 8\}$

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* Finite and Infinite sets.

1. Finite sets.

- A set is called finite if it contains countable number of elements.
- The number of elements in a finite set is a non negative integer, known as cardinality of the set.
- You can list all elements of a finite set.

ex - Set of Days of week.
$W = \{M, T, W, T, F, S, Sun\}$

2. Infinite sets.

- A set is called infinite it has an uncountable or endless number of elements.
- Infinite elements can go indefinitely and you cannot list all their elements.

ex - $N = \{1, 2, 3, 4, 5, ----\}$
N is set of natural numbers hence is infinite.

- The set of points on a line is infinite because there is no limit to the number of points.
- The set of real numbers between 0 and 1, $R = \{x \mid 0 < x < 1\}$, is infinite because there are many decimal numbers between 0 & 1.

* Countable sets -

1. ~~Fi~~ A set is called countable if its elements can be put into a one-to-one correspondence with the set of natural numbers $N = \{1, 2, 3, \ldots\}$. In other words, if you can count the elements of the set, even if it takes forever, the set is considered countable.

* Types.

1. Finite sets.

- Every finite set is countable because you can list its elements one by one until you reach last element.
- Ex - set $A = \{2, 4, 6\}$

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2. Infinite countable sets.

- An infinite set is countable if its elements can be arranged in a sequence where each element corresponds to a unique natural numbers.
- ex - $N = \{1, 2, 3, \cdots\}$ is an infinite countable set of natural numbers
  Even though it has infinite elements you can count them.

* Uncountable sets.

Some infinite sets are uncountable set. An uncountable is so large that you cannot list its elements in a sequence that matches the natural numbers.

- ex - set of real numbers between 0 and 1. There are so many real numbers in this interval that they cannot be matched one-to-one with natural numbers.

* Multisets -

- A multiset (is also known as a bag) is similar to a set, but it allows for multiple occurence of the same element.
- In a regular set, each element can only occur once, while in multisets elements can occur more than once.
- The number of times an element has occured is called its multiplicity.

1. Key concept
   1. Elements and multiplicity
      In a multiset each element has a multiplicity, which is a positive integer that indicates how many times a number has repeated.
      - ex, $m = \{a, a, b, b, c\}$ the element a has 2 multiplicity, b has 2 and c has 1.

2. Notation -

- Multisets can be written using curly brackets with repetition
- ex, $m = \{a, a, b, b, b, c\}$.

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- Alternatively they can be written using a format that indicates multiplicity, explicity, ~~e~~
- ex - $m = \{(a,2), (b,3), (c,1)\}$ where pairs show elements and its multiplicity.

* Operations on multisets.

1. Union -

The union of two multisets combines elements from both, taking the maximum multiplicity for each element.

- ex - If $m_1 = \{a, a, b\}$ &
  $m_2 = \{a, b, b, c\}$ then,
  $m_1 \cup m_2 = \cancel{\{a, b, c\}}\ \{a, a, b, b, c\}$

2. Intersection. -

- It takes minimum multiplicity for each element.
- ex - $m_1 = \{a, a, b\}$ &
  $m_2 = \{a, b, b, c\}$ then
  $m_1 \cap m_2 = \{a, b\}$

3. Addition -

- Adding two ~~elements~~ multisets combines their elements adding their ~~multiplications~~ ~~multiplicities~~.
- ex - $M_1 = \{a, a, b\}$ &
  $m_2 = \{a, b, b, c\}$ then
  $m_1 + m_2 = \{a, a, a, b, b, b, c\}$

4. Subtraction -

- Subtracting one multiset from another reduces the multiplicity of each element by the corresponding amount in other multiset, but never below zero.
- ex - $M_1 = \{a, a, b, b\}$ &
  $m_2 = \{a, b, b, c\}$ then
  $m_1 - m_2 = \{a, b\}$ ~~c~~

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* Applications of multisets.

- Data Analysis
  Effective when dealing with frequencies such as counting occurence of item in data set.

- Computer science -
  Multiset can represent bags of resources or inventory where duplicates are allowed.

- Combinatorics
  Choosing combinations with repetitions.

* Principle of Inclusion and Exclusion (PIE)

- The PIE is a fundamental counting technique used to calculate the size of union of multiple sets by considering the overlaps (intersections) among the sets. It helps avoid double counting elements that belong to more than one set.

* Two sets.

- Union of Two sets - The union $A \cup B$ represents all elements that are in A, B or both.

The formula is -

$$|A \cup B| = |A| + |B| - |A \cap B|$$

number of elements in A — number of elements in B — number of elements common to both.

* Three sets.

- Union of three sets. - The union $A \cup B \cup C$ represents all elements that are in atleast one of the sets A, B or C

- formula -

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

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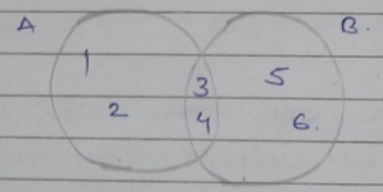
ex. Given two sets.

$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

Find $|A \cup B|$.

Soln:- Venn Diagram.



$|A| = 4$

$|B| = 4$

$|A \cap B| = \{3, 4\}$ so $|A \cap B| = 2$

Using formula,

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B| = 4 + 4 - 2$ — given.

$|A \cup B| = 8 - 2$

$\boxed{|A \cup B| = 6}$

ex Given three sets.

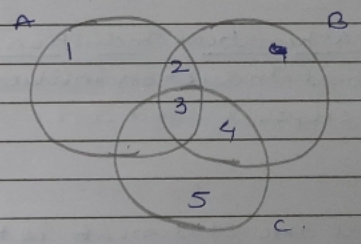
$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

$C = \{3, 4, 5\}$

Find $|A \cup B \cup C|$.

Soln: Venn Diagram.



$\therefore |A| = 3$

$\therefore |B| = 3$

$\therefore |C| = 3$

$\therefore |A \cap B| = \{2, 3\}$ so $= 2$

$\therefore |B \cap C| = \{3, 4\}$ so, 2

$\therefore |A \cap C| = \{3\}$ so, 1.

$\therefore |A \cap B \cap C| = \{3\}$ so, 1

Using formula,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 3 + 3 + 3 + 2 - 2 - 1 + 1$

$\boxed{|A \cup B \cup C| = 5}$.

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* Mathematical Induction -

- Mathematical Induction is a powerful proof technique used to prove statements about integers, typically of the form "a certain property holds for all natural numbers". Its similar to a chain of reaction.

* Steps of Mathematical Induction.

Mathematical Induction involves two main steps.

1. Base case.
   - Prove that the statement is true for the initial value, often $n=1$
   - This step establish the foundation of proof.

2. Inductive ~~case~~ step.
   - Assume the statement is true for some arbitary value $n=k$ (this is called inductive hypothesis).
   - Then prove that if the statement is true for $n=k$, it must also be true for $n=k+1$.
   - This step implies shows that the truth of the statement for one value n implies its truth for next value.
   - If both steps are successfully completed, the statement is true for all 'n' starting from the base case.

ex- Prove that the sum of first n natural numbers is given by formula $\frac{n(n+1)}{2}$.

- Step 1: Base case.

  start with $n=1$

  $$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

  Base case holds truth.

- Step 2: Inductive step.

  Assume statement is true for some $n=k$

  That is assume,

  $$1+2+3+\dots+k+(k+1) = \frac{(k+1)(k+1)}{2}$$

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Start with the left hand side using the inductive hypothesis.

$$1+2+3+\dots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

factor $k+1$ out of right-hand side.

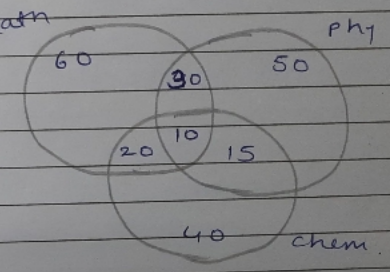
$$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

This matches the right hand side of what we wanted to prove.

ex- In a group of 100 students.

- 60 students are taking maths.
- 50 physics
- 40 chemistry.
- 30 both maths and phy.
- 20 both maths and chem.
- 15 both phy and chem.
- 10 all math, phy, chem.
- How many are taking atleast one of three subjects, maths, phy, chem.

Soln:



$|M| = 60$

$|P| = 50$

$|C| = 40$

$|M \cap P| = 30$

$|M \cap C| = 20$

$|P \cap C| = 15.$

$|M \cap P \cap C| = 10.$

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$$|M \cup P \cup C| = |M| + |P| + |C| - |M \cap P| - |P \cap C| - |M \cap C| + |M \cap P \cap C|$$

$$= 60 + 50 + 40 - 30 - 15 - 20 + 10$$

$$\boxed{|M \cup P \cup C| = 95.}$$

* PROPOSITIONS.

- What is a propositions?
  - A proposition is a declarative statement that is true or false but not both.
  - ex - "The sky is blue" - it is either true or false.
    - $2+2=4$
    - open the door - this is not proposition as it is a command & can't be true or false.

- Types of propositions. -

① Atomic proposition - A proposition that cannot be broken down into simpler proposition. (denoted by small $p, q, r, \ldots$)
primary / primitive
ex - It is raining today.

② Compound proposition - A proposition formed by combining one or more atomic propositions using logical connectives.
- ex - It is raining and it is cold.

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* Propositional variable -
- letters (lower case) used to denote propositional statements like p, q, r ... etc
- It has no truth value and hence is not a statement.

* Logical connectives -
- It is possible to form new and rather complicated statements from the primary statements by using certain connecting words, with whose usage; we are already familiar in English language.
- These statements are called compound statements.
- not, and, or, but, while, etc.

* Negation -
- The negation of statement is formed either by introducing the word "not" at a proper place or by prefixing the statement with the phrase "It is not the case that".

- If p denotes a statement then negation of p is denoted by "$\sim p$" or ($\bar{p}$)
- If $\sim p$ is true then $\sim p$ is false
- If p is false then $\bar{p}$ is true.
- If p: "I am going for a walk".
  $\sim p$: "I am not going for a walk".
  or "It is not the case that I am going for a walk".

* Conjunction - ("And") -
- If p and q are the statements, the compound statement "p and q" is called as conjunction of p and q; and is denoted by $p \wedge q$.
- ex - p: The sun is shining.
  q: The birds are singing.
  Then $p \wedge q$ is the statements "The sun is shining and the birds are singing".

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* Disjunction ("or") -

- If p and q are statement then the compound statement "p or q" is called the disjunction of p and q and is denoted by $p \vee q$.

- ex - p : There is an error in the program
  q : The data is wrong.
  $p \vee q$: There is an error in the program or the data is wrong.

- In the above example, the connectivity "or" is used in the inclusive sense, i.e. atleast one possibility exists or even both the possibilities exist.

- Alternatively "or" can be used exclusively i.e. either one or the other activity can happen, but not both.
  'v' - inclusive sense
  '$\bar{v}$' - exclusive sense.

- ex - I will read a book or go to sleep.
  $p \bar{v} q$.

* conditional (IF ... THEN).

- If p and q are statements, then compound statement "If p then q" denoted by $p \to q$ is called a conditional statement or implication.
- p is called antecedent or hypothesis, while q is called consequent.
- The converse of $p \to q$ is $q \to p$ and the contrapositive of $p \to q$ is the conditional $\sim q \to \sim p$.

$p \to q$

- ex - p : Hari works hard.
  q : Hari will pass the exam.
  $p \to q$: If Hari works hard, then he will pass the exam.

ex2 - p : It rains
  q : I carry an umbrella.
  $q \to p$ (converse) : If I carry an umbrella, then it rains.

  $\sim q \to \sim p$ (contrapositive) :
  If I do not carry an umbrella, then it does not rain.

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* **Biconditional (If and only if)**

- If p and q are statements the compound statement "p if and only if q", denoted by $p \leftrightarrow q$, is called a bi-conditional statement.
- often If and only if is shortened as "iff".
- $p \leftrightarrow q$ is also read as.
  "If p then q and conversly".

ex - An integer is even if and only if it is divisible by 2.
- Two lines are parallel if and only if they have same slope.

* **Propositional or statement form.**

p: "The sky is cloudy".
q: "It is raining"
r: "The ground is wet".
s: "The plants are green.

→ $\therefore (p \wedge \sim q)$: (The sky is cloudy) (and) (it is not raining)
p ∧ ~q

$\therefore (r \rightarrow s)$: (If) (the ground is wet) (then) (plants are growing)
r → s

$r \rightarrow s$

∴ Either the sky is cloudy and it is not raining or if the ground is wet then plants are growing.

→ $(p \wedge \sim q) \vee (r \rightarrow s)$

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ex- p: Mohan is rich
q: mohan is happy.

① Mohan is rich but unhappy.
→ $(p \wedge \sim q)$.

② Mohan is poor but happy.
→ $(\sim p \wedge q)$

③ Mohan is neither rich nor happy.
→ $(\sim p \wedge \sim q)$

④ Mohan is poor or he is both rich and unhappy
→ $\sim p \vee (p \wedge \sim q)$.

* Truth Table
- A table giving all possible truth values of a statement form, corresponding to truth values assigned to its variables, is called a truth table.

* $(\sim p \vee q) \rightarrow q$.

p	q	~p	~p∨q	(~p∨q)→q
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

* $\sim(p \wedge q) \vee (p \times q)$

p	q	p∧q	~(p∧q)	p×q	~(p∧q)∨p×q
T	T	T	F	T	T
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	T	T	T

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$((\sim p \wedge q) \vee (q \wedge r)) \rightarrow r.$

p	q	r	$\sim p$	$\sim p \wedge q$	$q \wedge r$	$(\sim p \wedge q) \vee (q \wedge r)$	$((\sim p \wedge q) \vee (q \wedge r)) \rightarrow r.$
T	T	T	F	F	T	T	T
T	T	F	F	F	F	F	F
T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F
F	F	T	T	F	F	F	T
F	F	F	T	F	F	F	F

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* Logical Equivalence.

— The concept of equivalence of statement forms is similar to the concept of equality of algebraic expressions in mathematics.

1. Ex -

1. p and $p \wedge p$ are logically equivalent

Idempotence.

→.

p	$p \wedge p$
T	T
F	F

2.

(i) $p \wedge q$ and $q \wedge p$ are logically equivalent

(ii) $p \wedge q$.

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(ii) $p \to q$ and $\sim q \to \sim p$ are logically equivalent (contrapositive).

p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

$p \to q$
T
F
T
T.

$\sim q \to \sim p$
T
F
T
T

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$\vee$ - or
$\wedge$ - and.

* Quantifiers.

$p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent (Distributivity).

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T
T	F	T
F	T	T
F	F	F
F	F	F
F	F	F
F	F	F
F	F	F

* Predicates

Consider,

(i) "x is tall and handsome"

(ii) "$x+3=5$"

(iii) "$x+4 \geq 10$"

- These statements are not propositions since they do not have any truth value. However if values are assign to them they become propositions.
- An assertion that contain one or more variable is a predicate, its truth value is predicated after assigning truth values to its variables.

* Normal Forms

- Reducing the statement form, to so called Normal Forms.

* Laws

1. $p \wedge q \equiv q \wedge p$; $p \vee q \equiv q \vee p$ — Commutative l.
2. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$; $p \vee (q \vee r) \equiv (p \vee q) \vee r$ — Associative l.
3. $p \wedge p \equiv p$; $q \vee q \equiv q$ — Idempotent law.
4. $\sim(p \vee q) \equiv \sim p \wedge \sim q$; $\sim(p \wedge q) \equiv \sim p \vee \sim q$ — Demorgan's l.
5. $\sim(\sim p) \equiv p$ — Involution law
6. $p \rightarrow q \equiv \sim p \vee q$ — Conditional law.
7. $p \wedge (p \wedge q) \equiv p$; $p \vee (p \wedge q) \equiv p$ — Absorption law.
8. $p \wedge (q \vee r) \equiv p(\wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv p(\vee q) \wedge (p \vee r)$ } — Distributive law.
9. $p \wedge \sim p \equiv F$; $p \vee \sim p \equiv T$ — Complement law.
10. $p \wedge T \equiv p$; $p \vee T \equiv T$, $p \wedge F \equiv F$ $p \vee F \equiv F$ — Identity law.

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