$y = f(x)$ $\quad \frac{dy}{dx} \quad \frac{d^2y}{dx^2}$ $\quad \frac{\partial z}{\partial x} \quad \frac{\partial^2 z}{\partial y^2}$ $\quad z = f(x,y)$

Partial Differential Equations

Boundary value problems governed by linear second order partial differential equations:

Over a two dimensional Cartesian domain, let u be the dependent variable. Then a general second order partial differential equation may be written as,

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F(x, y, u, u_x, u_y) + G = 0} \quad \text{----------} \quad (1)$$

(1)

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F(x, y, u, u_x, u_y) + G = 0 \qquad u = f(x,y)$$

where A, B, C are functions of (x, y) and F may be non-linear function, then equation (1) is called a quasi-linear partial differential equation. If F is also a linear function then equation (1) is called a linear partial differential equation. If $G = 0$ then equation (1) is homogeneous otherwise non-homogeneous.

Classification of PDE:

Equation (1) at a point or in a domain is said to be,

Elliptic if $\boxed{B^2 - 4AC < 0}$; Parabolic if $\boxed{B^2 - 4AC = 0}$; Hyperbolic if $\boxed{B^2 - 4AC > 0}$.

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Examples:

$A=1,\ B=0,\ C=1,\ B^2-4AC=-4<0$

1. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$ is called Poisson equation is an elliptic equation

2. $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$, is called wave equation is hyperbolic equation

(x,t)

$A=c^2,\ B=0,\ C=-1$

$B^2-4AC=-4c^2\times -1$

$=4c^2>0$

3. $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$, is called heat equation is parabolic equation

$A=c^2,\ B=0,\ C=0,\ B^2-4AC=0$

4. $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$, is parabolic equation

5. $(1+x^2)\frac{\partial^2 u}{\partial x^2}+(5+2x^2)\frac{\partial^2 u}{\partial x\partial t}+(4+x^2)\frac{\partial^2 u}{\partial t^2}=0$, is Hyperbolic equation.

In ① if $f(x,y)=0$, Laplace equation

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Finite difference method

Differential equation

↓

Difference equation — approximating each derivative by a finite difference approximation

↓

Solve system of eqns

↙

Taylor series

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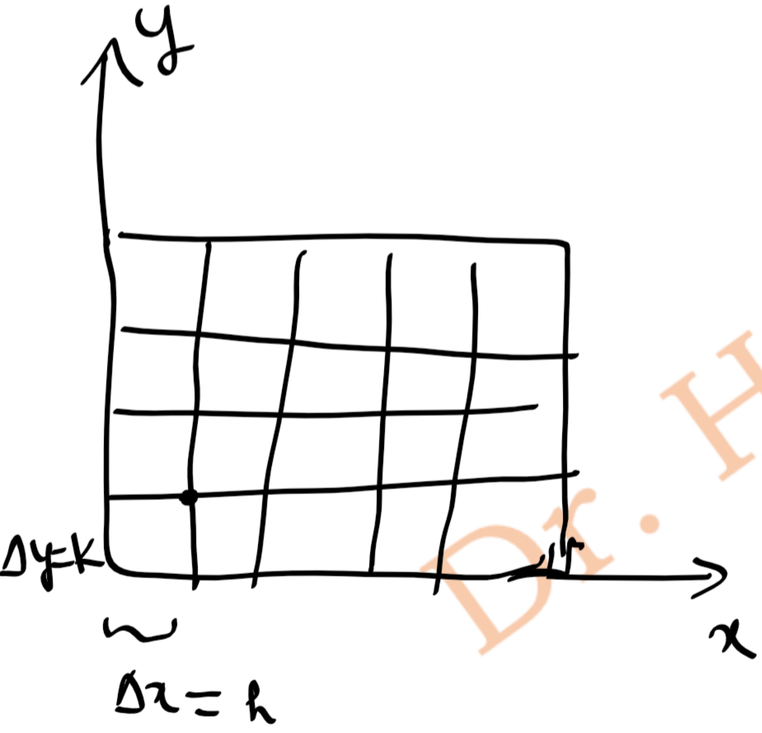
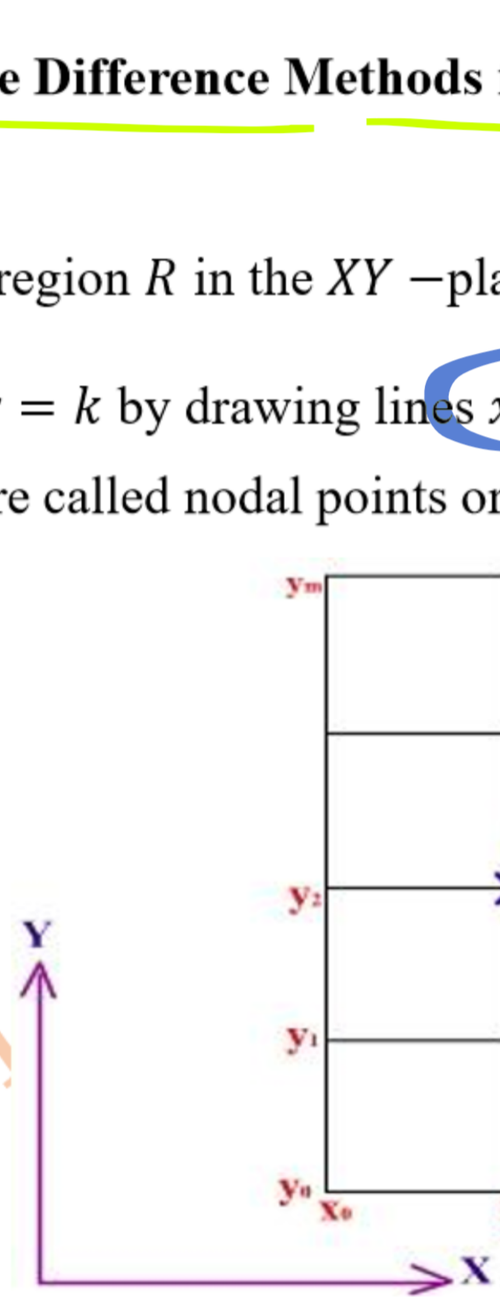
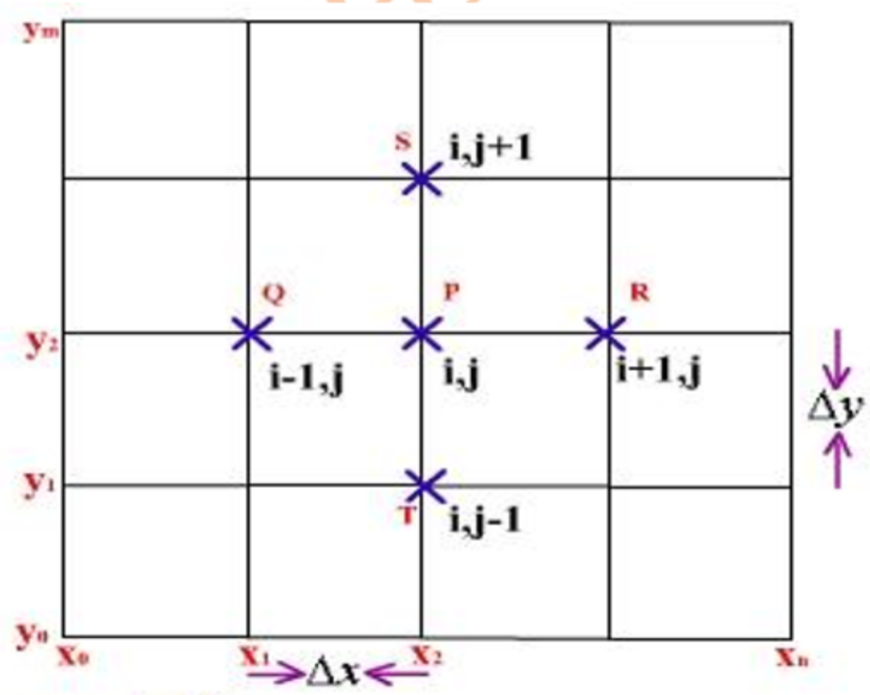
Finite Difference Methods for Partial Differential Equation:

Consider a rectangular region R in the XY −plane. Divide the region into rectangular network of sides $\Delta x = h$ and $\Delta y = k$ by drawing lines $x = ih$ and $y = jk$, where $i, j = 0,1,2,3, \ldots$ Points of intersection are called nodal points or mesh points or grid points.







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$u(x,y) = u(ih, jk)$
$= u_{i,j}$

$u(x+h, y) = u_{i+1,j}$

$u(x, y+k) = u(ih, jk+k) = u_{i,j+1}$

Writing $u(x,y) = u(ih, jk)$ as simply $u_{i,j}$, the finite difference approximations for the first order partial derivatives can be derived using Taylor series expansion for 2 variables as follows.

We have, $u(x+h, y) = u_{i+1,j} = u(x,y) + hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \cdots$ ----------

(2)

$\Delta x = h$

$\Delta y = k$

Thus, the forward difference formula can be written as,

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{h}\left(u_{i+1,j} - u_{i,j}\right) + O(h)} \quad \text{----------(3)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{h}\left(u_{i+1,j} - u_{i,j}\right) + O(h)$$

$h = 0.01$

very small

$h^2 = 0.0001$, $h^3 = 0.000001$

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Similarly, $u(x-h,y) = u_{i-1,j} = u(x,y) - hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \cdots$ ----------(4)

Thus, the backward difference formula can be written as,

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{h}\left(u_{i,j} - u_{i-1,j}\right) + O(h)} \qquad \text{----------(5)}$$

The central difference formula can be obtained by Eq.(2) –Eq.(4) as,

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h}\left(u_{i+1,j} - u_{i-1,j}\right) + O(h^2) \qquad \text{----------(6)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h}\left(u_{i+1,j} - u_{i-1,j}\right)$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{2k}\left(u_{i,j+1} - u_{i,j-1}\right)$$

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$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

The finite difference approximations for the second order partial derivatives can be obtained by adding Eq.(2) and (4).

$$\boxed{\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + O(h^2)} \qquad \text{----------(7)}$$

Similarly,

Forward: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{k}\left(u_{i,j+1} - u_{i,j}\right) + O(k)$ ----------(8)

Backward: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{k}\left(u_{i,j} - u_{i,j-1}\right) + O(k)$ ----------(9)

Central: $\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{2k}\left(u_{i,j+1} - u_{i,j-1}\right) + O(k^2)$ ----------(10)

And $\boxed{\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{1}{k^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) + O(k^2)}$ ----------(11)

Replacing the derivatives in any partial differential equation by their corresponding difference approximation, we obtain the finite-difference analogues of the given equation.

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right]$$

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Elliptic partial differential equation:

Most relevant examples of elliptic PDE are Laplace equation and Poisson equation.

The Poisson equation in Cartesian coordinate system is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad \text{or} \quad \boxed{\nabla^2 u = f(x, y), a \leq x \leq b, c \leq y \leq d}$$

Subject to boundary condition: $u(x, y) = g(x, y)$ (Dirichlet boundary condition).

The Laplace equation is a special case of Poisson equation with $f(x, y) = 0$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad \nabla^2 u = 0$$

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Solution for Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{----------(12)}$$

Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h (assuming that an exact division of R is possible.

Standard five-point formula

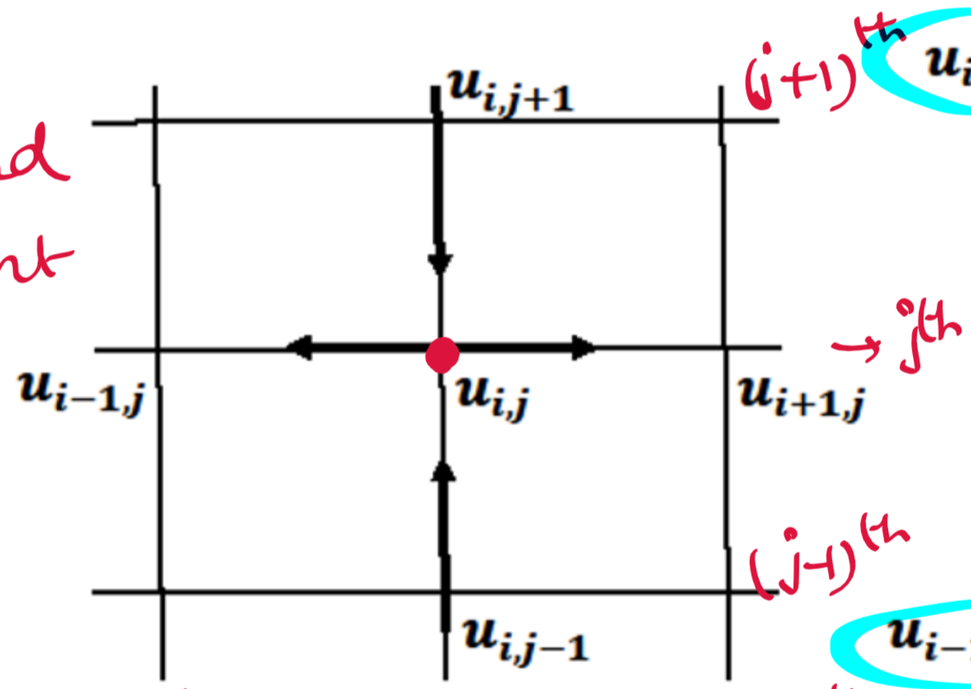


Figure-2.

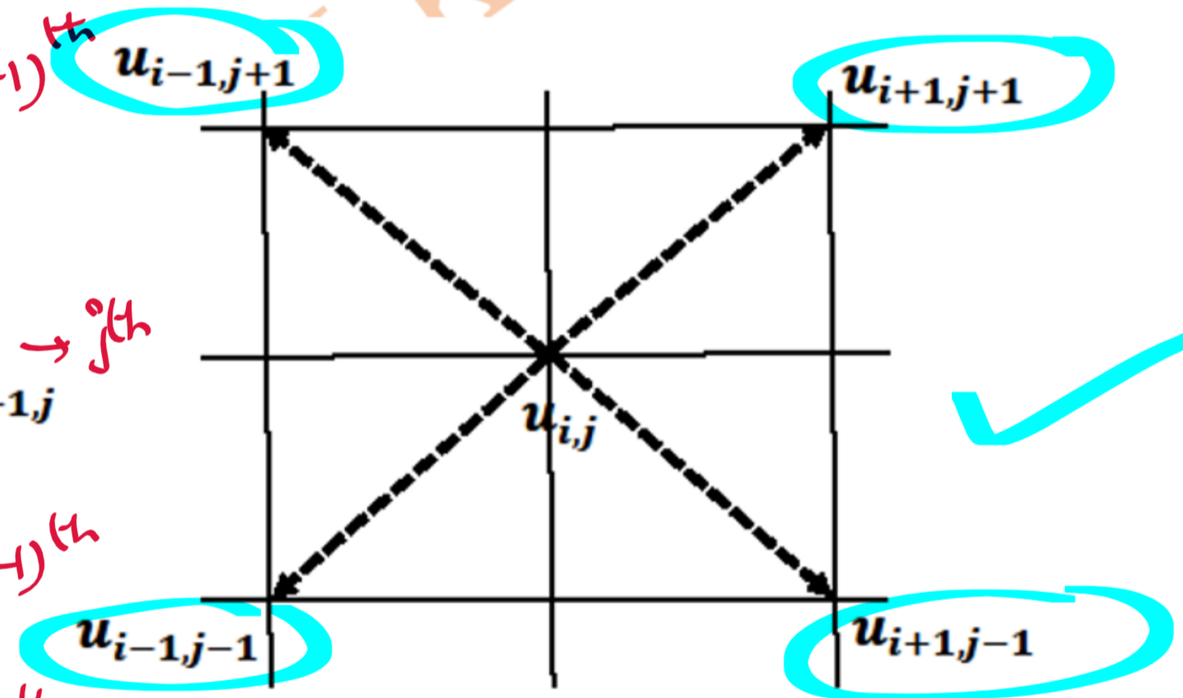
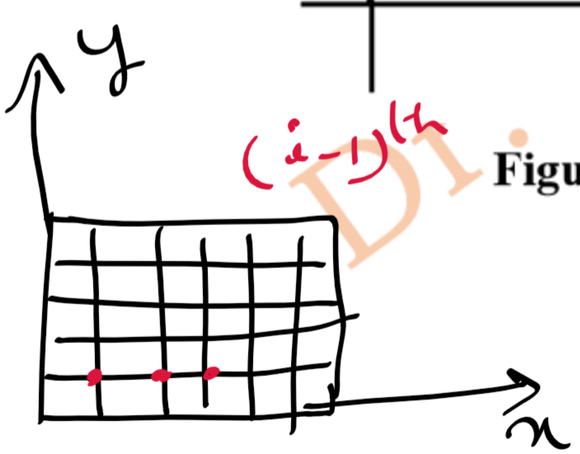
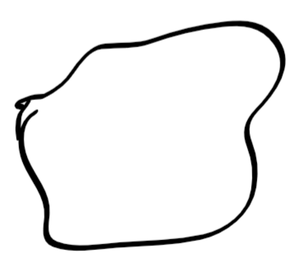
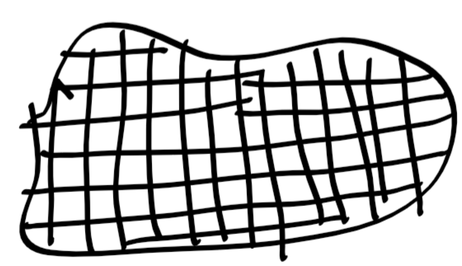


Figure-3.







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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$h = k$

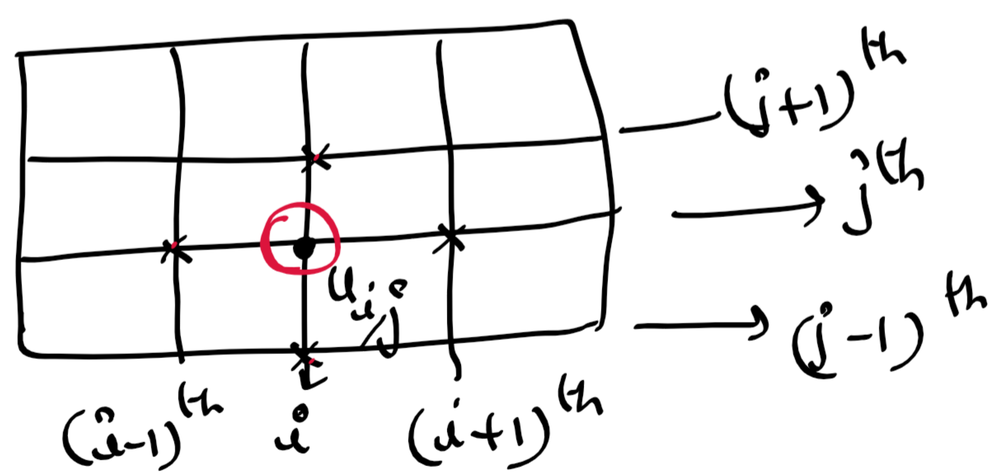
Replacing the derivatives in (12) by their difference approximations, we have

$$\frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \frac{1}{h^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = 0$$

Or $$\boxed{u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)} \qquad \text{----------(13)}$$

This shows that the value of u at any interior mesh point is the average of its values at four neighboring points to the left, right, above and below. Equation (13) is called ***standard 5-point formula*** as shown in figure-2.

$u_{i,j} = \frac{1}{4}(N + E + W + S)$



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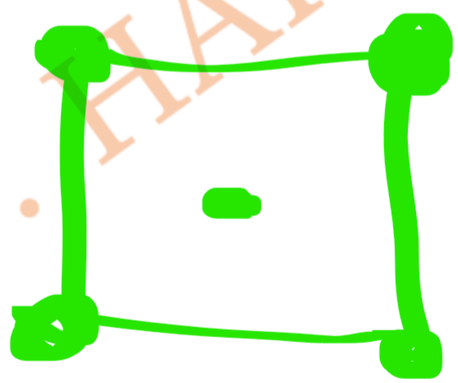
Sometimes a formula similar to this is used which is given by,

$$u_{i,j} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \qquad \text{----------(14)}$$

Which shows that the value of u at any interior mesh point is the average of its values at four neighboring diagonal mesh points. Equation (14) is also called the ***diagonal 5-point formula*** as shown in figure-3. Although this is less accurate than the standard 5-point formula, it is used in getting a good approximation for the starting values at the mesh points.



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By applying 5-point formula at each interior mesh point, we arrive at linear equations in the nodal values $u_{i,j}$. These equations can be solved by Jacobi's iterative method or Gauss-Seidal iterative method. The iterative methods are repeated till the difference between two consecutive iterates become negligible.

1. **Jacobi's method:** Denoting the nth iterative value of $u_{i,j}$ by $u_{i,j}^{n}$, the iterative formula to solve is,

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left(u_{i+1,j}^{(n)} + u_{i-1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}\right) \qquad \text{----------(15)}$$

It gives improved values of $u_{i,j}$ at the interior mesh points.

1. **Gauss-Seidal method or Leibmann's method:** In this method, the iteration formula is,

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left(u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)}\right) \qquad \text{----------(16)}$$

This utilizes the latest iterative values available and evaluates the mesh points symmetrically from left to right along successive rows.

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$h = k$

Note:

- **The accuracy of the calculations depends on the mesh-size i.e. smaller the h, better the accuracy. But if h is too small, it increases the computations.**
- **The error in solving Laplace and Poisson's equations by finite-difference method is of the order $o(h^2)$.**

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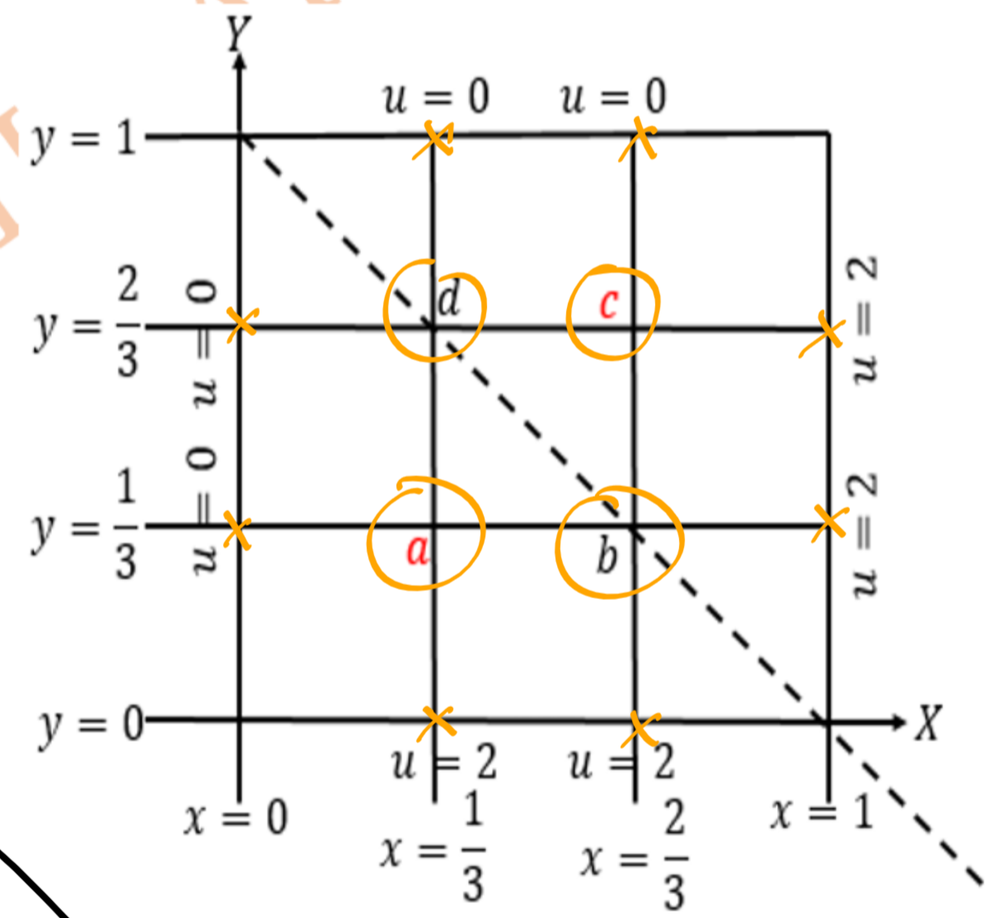
$u(x,1) = 0$

$u(0,y) = 0$

Solved examples:

1. **Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with $h = \frac{1}{3}$, $u(x,1) = u(0,y) = 0, 0 < x, y < 1$ and also $u(x,0) = 9(x - x^2)$, $u(1,y) = 9(y - y^2)$.**

Solution: Solution is symmetric about principal diagonal as shown in figure. Using 5-point formula at each of the unknown mesh points, we have,



$4a = b + d + 2$

$4b = 2a + 4$

$4d = 2a$

$d = \frac{1}{4}(0 + 0 + c + a)$

$4d = 2a$

Solving for unknowns,

$a = c$

$$\boxed{a = c = 1; b = 1.5; d = 0.5}$$

$b = \frac{1}{4}(a + c + 2 + 2)$

$4b = 2a + 4$

$a = \frac{1}{4}(0 + d + b + 2)$

$4a = d + b + 2$

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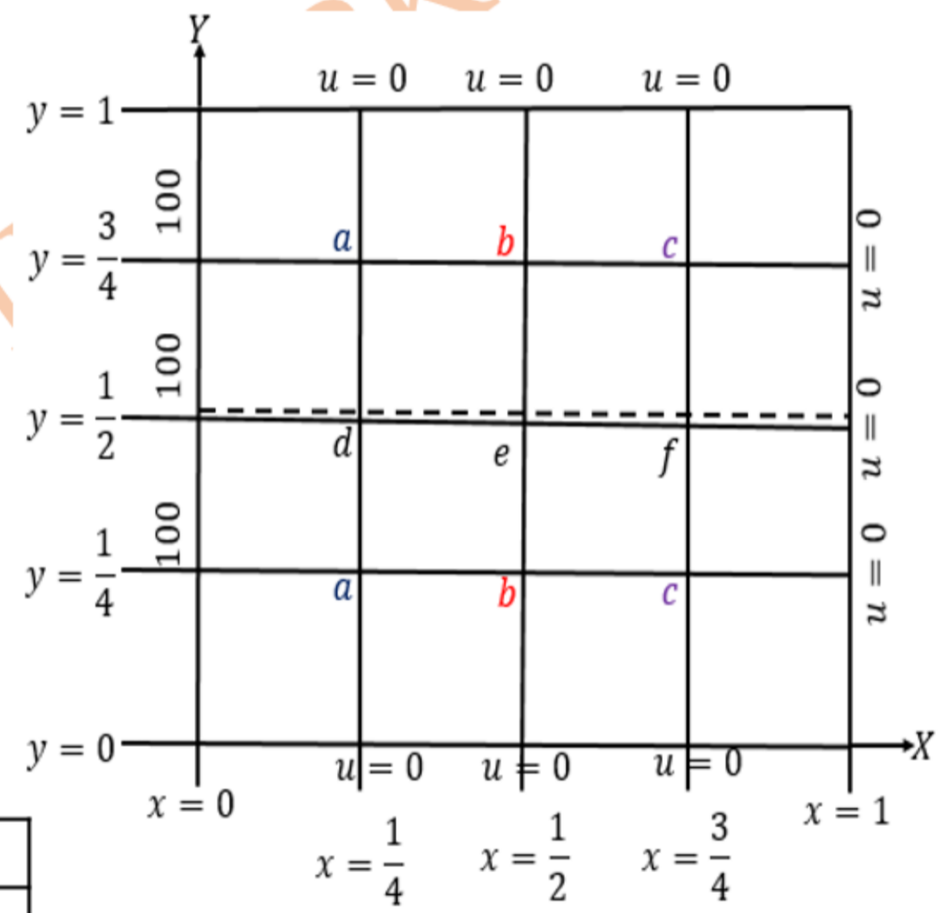
2. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with $h = \frac{1}{4}, u(x,0) = u(x,1) = 0, 0 < x, y < 1$ and also $u(0,y) = 100, u(1,y) = 0$ ~~using Leibmann's method~~.



Solution: With reference to the figure, we have equations in 6 unknowns.

$$
\begin{aligned}
4a &= 100 + b + d \\
4b &= a + e + c \\
4c &= b + f \\
4d &= 100 + 2a + e \\
4e &= 2b + d + f \\
4f &= 2c + e
\end{aligned}
$$

~~Solving using Leibmann's method or Gauss seidal~~ method(3 iterations),

a	b	c	d	e	f
~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~
~~25~~	~~6.25~~	~~1.5625~~	~~37.5~~	~~12.5~~	~~0.78125~~
35.9375	12.5	3.3203	46.09375	17.96875	6.1523

and so on.

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Exercise problems:

1. Solve Laplace equation $\nabla^2 u = 0, 0 < x, y < 2$, with $h = \frac{1}{2}$, $u(x,0) = 0, u(x,2) = 6$, $u(0,y) = 1, u(2,y) = 1$.
2. Solve $\nabla^2 u = 0, |x| < 1, |y| < 1, h = \frac{1}{2}$. Also, $u(\pm 1, y) = y^2, u(x, \pm 1) = x^2$.
3. Solve $\nabla^2 u = 0, 0 < x, y < 4$, with $h = 1, u(x,0) = x^2 + 2x$, $u(0,y) = -2y - y^2, u(4,y) = 24 - y^2 - 2y, u(x,4) = x^2 + 2x - 24$.

(Hint: Negative symmetry to be considered. Number of variables reduces to six.)

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{or} \quad \nabla^2 u = f(x,y)$$

Poisson Equation

The Poisson equation is an elliptic partial differential equation that frequently emerges when modeling electromagnetic systems. However, like many other partial differential equations, exact solutions are difficult to obtain for complex geometries. This motivates the use of numerical methods in order to provide accurate results for real-world systems. One very simple algorithm is the Finite-Difference Method (FDM), which works by replacing the continuous derivative operators with approximate finite differences. Although the Finite-Difference Method is one of the oldest methods ever devised, comprehensive information is difficult to find compiled in a single reference.

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$x = ih, \quad y = jh$

$h = k$

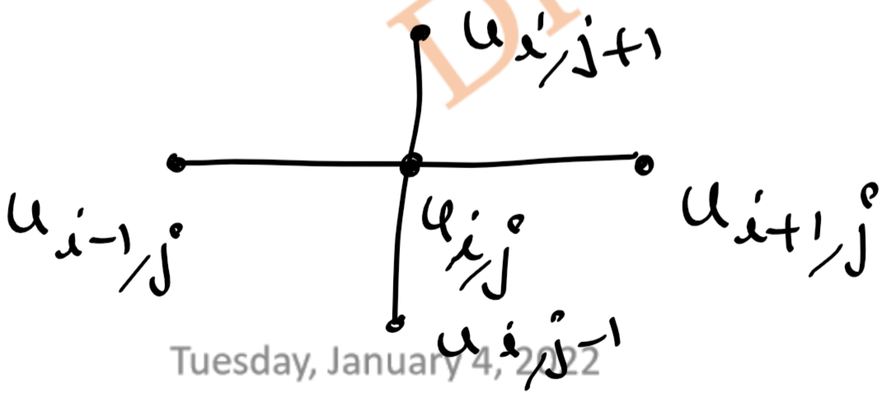
- Poisson differential equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{---------- (1)}$$

- Its method of solution is similar to that of Laplace equation. Here the standard 5-point formula takes the form,

$$\boxed{u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)} \qquad \text{----- (2)}$$

By applying formula (2) at each interior mesh point, we arrive at linear equations in the nodal values $u_{i,j}$. These equations can be solved by Gauss-Seidal iterative method. The iterative methods are repeated till the difference between two consecutive iterates become negligible.



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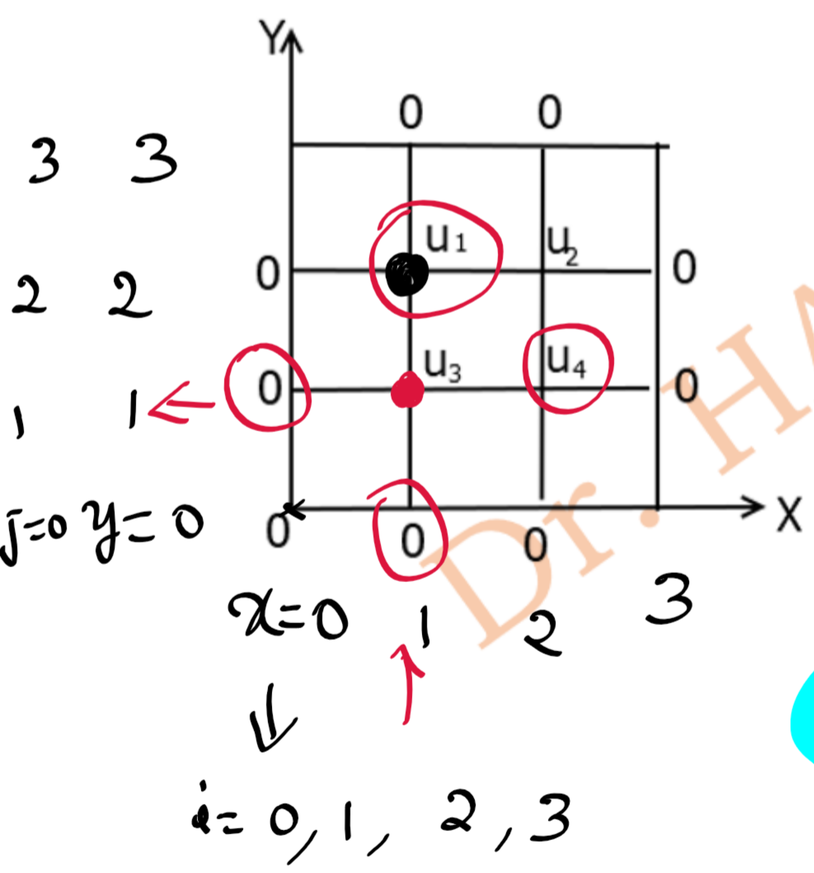
Examples:

$0 \le x, y \le 3$

1) **With $h = 1$, solve** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10)$

$u = 0$ on the boundary

x	0	1	2	3
y	0	1	2	3



By five point formula,

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad \text{-----}(1)$$

$$u_2 = \frac{1}{4}(u_1 + u_4 + 180) \quad \text{-----}(2)$$

$$u_3 = \frac{1}{4}(u_1 + u_4 + 120) \quad \text{-----}(3)$$

$$u_4 = \frac{1}{4}(u_2 + u_3 + 150) \quad \text{-----}(4)$$

$f(x,y) = -10(x^2+y^2+10)$

$f(ih, jh) = -10((ih)^2 + (jh)^2 + 10)$

$h = 1$

$= -10(i^2 + j^2 + 10)$

$u_1 \rightarrow i = 1, j = 2$

$f(ih, jh) = -10(1+4+10)$

$= -150$

$0 + u_3 + u_2 + 0 - 4u_1 = h^2 f(ih, jh)$

$= -150$

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From equation (1) and (4), $u_1 = u_4$.

Hence equations becomes

$$4u_1 - u_2 - u_3 = 150, 4u_2 - 2u_1 = 180,$$
$$4u_3 - 2u_1 = 120.$$

By solving these equations we get,

$$u_1 = u_4 = 75, u_2 = 82.5, u_3 = 67.5.$$

$f(ih, jh) = -10(i^2 + j^2 + 10)$

$0 + 0 + u_4 + u_1 - 4u_3 = h^2 f(ih, jh)$

$= -120$

$\Rightarrow u_3 = \frac{1}{4}(u_1 + u_4 + 120)$

For u_3, $i = 1$, $j = 1$

$f(ih, jh) = -10(1 + 1 + 10)$

$= -120$

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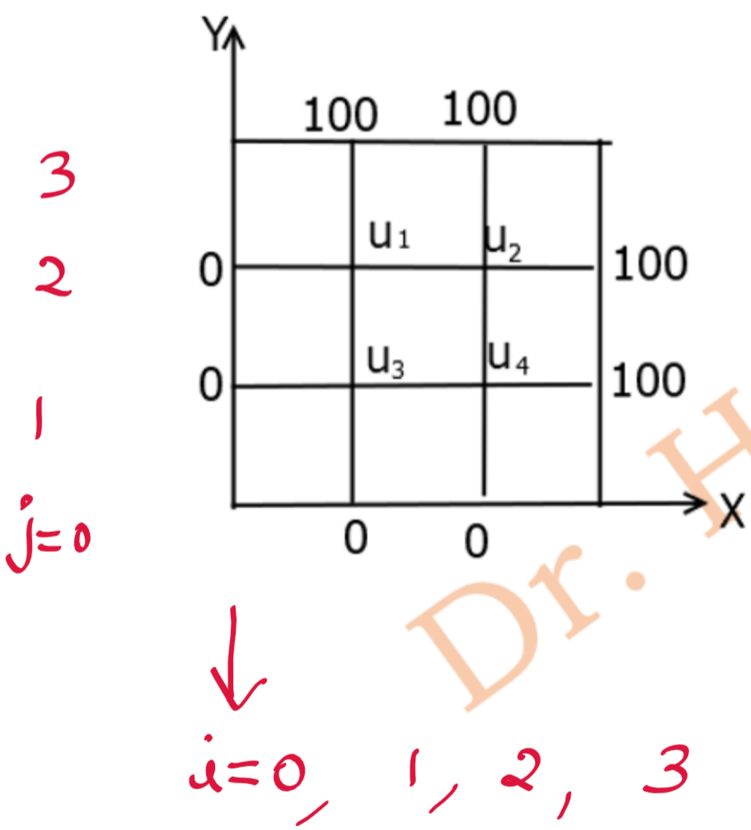
2) **Solve** $\nabla^2 u = -81xy, 0 < x, y < 1, h = \frac{1}{3}$. **Also,** $u(0, y) = u(x, 0) = 0, u(1, y) = u(x, 1) = 100.$

Solution

x	0	1/3	2/3	1
y	0	1/3	2/3	1



By standard five point formula,

$$u_1 = \frac{1}{4}(u_2 + u_3 + 102) \quad ----(1)$$

$$u_2 = \frac{1}{4}(u_1 + u_4 + 204) \quad ----(2)$$

$$u_3 = \frac{1}{4}(u_1 + u_4 + 1) \quad ----(3)$$

$$u_4 = \frac{1}{4}(u_2 + u_3 + 102) \quad ----(4)$$

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From equation (1) and (4), $u_1 = u_4$.

Hence equations are:

$4u_1 - u_2 - u_3 = 102, 2u_1 - 4u_2 = -204, 2u_1 - 4u_3 = -1$

Solving theses equations, we get $u_1 = u_4 = 51, u_2 = 76, u_3 = 26$.

- **<u>Exercise problems:</u>**

1) Solve $\nabla^2 u = -1, |x| < 2, |y| < 2, h = 1$.

Also, $u(-2, y) = u(x, 2) = u(2, y) = u(x, -2) = 0$.

2) Solve $\nabla^2 u = -1, |x| < 2, |y| < 2, h = 1$.

Also, $u(-2, y) = u(x, 2) = u(2, y) = u(x, -2) = 0$.

3) Solve $\nabla^2 u = -1, |x| < 1, |y| < 1, h = \frac{1}{2}$. Also, $u(\pm 1, y) = u(x, \pm 1) = 0$.

Dr. HARINAKSHI KARKERA

Wave Equation

The wave equation is a second-order linear hyperbolic PDE that describes the propagation of a variety of waves, such as sound or water waves. It arises in different fields such as acoustics, electromagnetics, or fluid dynamics.

The wave equation one dimension is of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Dr. HARINAKSHI KARKERA

Numerical Solution of Wave Equation

Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad a < x < b,\ t > 0 \text{ --------- } (1)$$

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \text{ ---------- } (2)$$

$$u_t(a,t) = \varphi(x), \quad u(b, t) = \Psi(x) \text{ -------- } (3)$$

To solve by finite difference method, we divide the interval [a, b] into n equal parts each of breadth h, i.e. $h = \frac{b-a}{n}$

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Let $x_i = x_0 + ih$ [x_0=a], i = 1, 2, …., n -------- (4)

If k is the time internal size, let

$$t_j = jk \text{ ---------(5)}$$

Denote u(x_i, t_j)=u_{ij}

We have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} \text{ ----------(6)}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{k^2} \text{ ----------(7)}$$

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Using (6) and (7) in (1), we get,

$$\frac{u_{ij+1}-2u_{ij}+u_{ij-1}}{k^2} = c^2\,\frac{u_{i+1j}-2u_{ij}+u_{i-1j}}{h^2} \text{----------(8)}$$

If k = h/c, (8) simplifies to

$$u_{ij+1}=u_{i+1j} + u_{i-1j} - u_{ij-1} \text{----------(9)}$$

This is the finite difference scheme for the wave equation.

If we know u at two consecutive time steps t_j and t_{j+1}, (9) gives u at time t = t_{j+1}

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From the initial conditions (2)

$$u_{i0} = fi\ ,\ \ u_{i1} = \frac{1}{2}(f_{i-1} + f_{i+1}) + kg_i$$

From the boundary conditions (3)

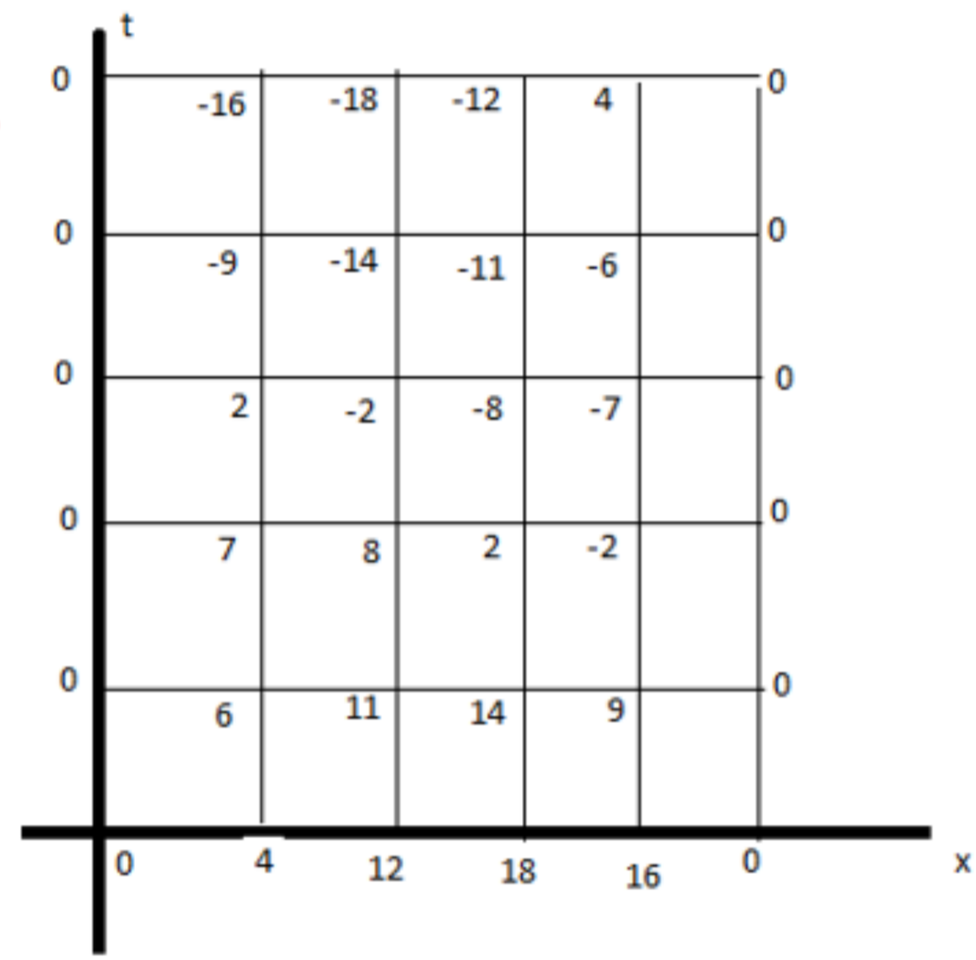
$$u_{0j} = \varphi_j, u_{nj} = \psi_j$$

Dr. HARINAKSHI KARKERA

Evaluate the pivotal values of the equation $u_{tt}=16u_{xx}$, taking h =1 upto t = 1.25. The boundary and initial conditions are

$$u(0, t)=u(5, t) = 0,\ u(x, 0)=x^2(5-x),\ u_t(x,0)=0.$$

Here $c^2= 16$, h=1. We choose k = h/c=1/4

Finite difference scheme is

$$u_{ij+1}=u_{i+1j} + u_{i-1j} - u_{ij-1}$$