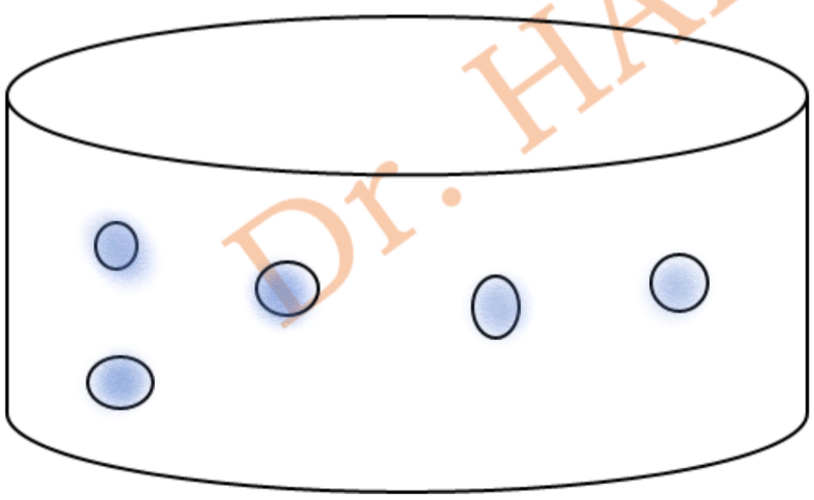
Problems

Two marbles are selected randomly from a box that contains 3 blue marbles, 2 red marbles & 3 green marbles. If X is the no. of blue marbles & Y is the no. of red marbles selected, find the joint probability distribution of (X, Y).



3 B
2 R
3 G
8

2 marbles

X: no. of blue marbles

Y: " red "

$p(x_i, y_j)$

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$X : 0, 1, 2$

$Y : 0, 1, 2$

$p(x_i, y_j)$

Possible values of (X, Y)

$\{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$

$p(0,0) = P[X=0, Y=0]$

$= \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$

$p(2,0) = P[X=2, Y=0]$

$= \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$

$p(1,1) = P[X=1, Y=1]$

$= \frac{({}^3C_1) \times ({}^2C_1)}{{}^8C_2} = \frac{3}{14}$

$|||^{ly}$ $p(0,1) = \frac{{}^2C_1 \times {}^3C_1}{{}^8C_2}$

$p(1,0) = \frac{{}^3C_1 \times {}^3C_1}{{}^8C_2}$

$p(0,2) = \frac{{}^2C_2}{{}^8C_2}$

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$y \backslash x$	0	1	2	row Total
0	$p(0,0)=\frac{3}{28}$	$p(1,0)=\frac{9}{28}$	$p(2,0)=\frac{3}{28}$	$\frac{15}{28}=q(0)$
1	$p(0,1)=\frac{6}{28}$	$p(1,1)=\frac{6}{28}$	$p(2,1)=0$	$\frac{12}{28}=q(1)$
2	$p(0,2)=\frac{1}{28}$	$p(1,2)=0$	$p(2,2)=0$	$\frac{1}{28}=q(2)$
Column total	$\frac{10}{28}=p(0)$	$\frac{15}{28}=p(1)$	$\frac{3}{28}=p(2)$	1

$$\boxed{\sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, y_j) = 1}$$

$$\sum_{j=1}^{n} q(y_j) = 1$$

$$\sum_{i=1}^{n} p(x_i) = 1$$

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* Suppose that the joint pdf of (X, Y) is given by

$$f(x,y) = \begin{cases} e^{-y} & \text{for } x>0,\ y>x \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal pdf of X & Y. Examine for independence.

Marginal pdf of X: $g(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$

$$= \int_{y=-\infty}^{x} 0 \cdot dy + \int_{y=x}^{\infty} e^{-y}\,dy$$

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$$\therefore g(x) = \int_{y=x}^{\infty} e^{-y}\,dy = \left.\frac{e^{-y}}{-1}\right|_{y=x}^{\infty} = -\left[0 - e^{-x}\right]$$

$$= e^{-x}$$

$$\boxed{\therefore g(x) = e^{-x} \quad ; \quad x > 0}$$

Marginal pdf of y :

$$h(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx = \int_{x=-\infty}^{0} 0 \cdot dx + \int_{x=0}^{y} e^{-y}\,dx$$

$$= e^{-y}\,x\Big|_{x=0}^{y} = e^{-y}[y - 0] = y e^{-y}$$

$$\boxed{\therefore h(y) = y e^{-y} \quad ; \quad y > 0}$$

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$$g(x)\cdot h(y) = e^{-x}\cdot y e^{-y}$$

$$= y e^{-(x+y)} \neq f(x,y)$$

$$\therefore g(x)\cdot h(y) \neq f(x,y)$$

$\therefore$ <u>X & Y are</u> <u>not independent</u>

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* Suppose the two-dimensional random variable (X, Y) has joint pdf

$$f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal pdf of X & Y. Examine for independence.

Marginal pdf of X, $g(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$

$$= \int_{y=x}^{1} 8xy\,dy = 8x \left.\frac{y^2}{2}\right|_{y=x}^{1} = 4x(1-x^2)$$

$g(x) = 4x(1-x^2)$, $x > 0$

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Marginal pdf of y: $h(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$

$$= \int_{x=0}^{y} 8xy\,dx = 8y\frac{x^2}{2}\Big|_{x=0}^{y}$$

$$= 4y^3$$

$$\boxed{\therefore h(y) = 4y^3, \quad y > 0}$$

$$g(x) \cdot h(y) = 4x(1-x^2) \cdot 4y^3$$

$$\neq f(x,y)$$

$\therefore$ X & Y are not independent.

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If X & Y are independent then

$$\begin{cases} f(x,y) = g(x)\,h(y) \\ \& \; p(x_i, y_j) = p(x_i)\cdot q(y_j) \end{cases}$$

so If $f(x,y) \neq g(x)\cdot h(y) \Rightarrow X$ & Y are not independent.

Converse

Is this true????

Now:

If $\underbrace{g(x)\cdot h(y) = f(x,y)}$ $\Rightarrow$ X & Y are independent ?

HW

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Expected value of 2 dimensional r.v's

Let (X, Y) be two dimensional r.v.

(i) $g(X, Y)$ — function of (X, Y)

$$\therefore E(g(X,Y)) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(x_i, y_j)\, p(x_i, y_j)$$

— discrete case

$$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy$$

— continuous case

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Note: (i) $g(X,Y) = XY$

$$\therefore E(XY) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (x_i y_j) \, p(x_i, y_j)$$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, f(x,y) \, dx \, dy$$

(ii) $g(X,Y) = X + Y$

$$E(X+Y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (x_i + y_j) \, p(x_i, y_j)$$

$$E(X+Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) \, f(x,y) \, dx \, dy$$

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If (X,Y) — $p(x_i, y_j)$

then $E(X) = \sum_{i=1}^{\infty} x_i \, p(x_i)$

$$p(x_i) = \sum_{j=1}^{\infty} p(x_i, y_j)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i \, p(x_i, y_j)$$

discrete case

$$E(X) = \int_{-\infty}^{\infty} x \, g(x) \, dx$$

$$g(x) = \int_{y=-\infty}^{\infty} f(x,y) \, dy$$

$$E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, f(x,y) \, dx \, dy$$

$|||^{ly}$ $E(Y)$

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$$E(Y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y_j \, p(x_i, y_j)$$ } discrete case

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \, f(x,y) \, dx \, dy$$ } continuous case

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Covariance of (X,Y)

$$\sigma_{xy} = \mathrm{cov}(X,Y) = E\left[(X - E(X))\,(Y - E(Y))\right]$$

The correlation coefficient of (X,Y) is

$$\rho = \rho_{xy} = \frac{\mathrm{cov}(X,Y)}{\sqrt{V(X)\,.\,V(Y)}}$$

$$-1 \le \rho \le 1$$

$$V(X) = E(X^2) - (E(X))^2$$

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If X & Y are independent then $\rho = 0$

$$\sigma_{xy} = \mathrm{cov}(X,Y) = 0 \Rightarrow \boxed{E(XY) = E(X)\,E(Y)}$$

$$\mathrm{cov}(X,Y) = E\left[(X - E(X))(Y - E(Y))\right]$$

$$= E\left[XY - X\,E(Y) - Y\,E(X) + E(X)\,E(Y)\right]$$

$$= E(XY) - E(Y)\,E(X) - E(X)\,E(Y) + E(X)\cdot E(Y)$$

$$\boxed{\mathrm{cov}(X,Y) = E(XY) - E(X)\,E(Y)}$$

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