Poisson Distribution

Let X be a discrete random variable assuming the possible values $0, 1, 2, \ldots\ldots$

If $$\boxed{P(X=x) = p(\lambda, x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; \; x = 0, 1, 2, \ldots\ldots}$$

$(X, p(\lambda, x))$ Poisson distribution with parameter $\lambda > 0$

X — Poisson random variable

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Note:

1) $p(\lambda, x_i) \geq 0$ for $x_i = 0, 1, 2, \ldots$

2) $$\sum_{x_i=0}^{\infty} p(\lambda, x_i) = p(\lambda, 0) + p(\lambda, 1) + p(\lambda, 2) + \cdots$$

$$= e^{-\lambda} + e^{-\lambda}\lambda + \frac{e^{-\lambda}\lambda^2}{2!} + \cdots$$

$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda}$$

$$\boxed{= 1}$$

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Mean and Variance

Mean:

Consider $E(X) = \sum_{x_i=0}^{\infty} x_i \, p(\lambda, x_i)$

$$= 0 \cdot p(\lambda, 0) + 1 \cdot p(\lambda, 1) + 2 p(\lambda, 2) + \cdots$$

$$p(\lambda, x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \cdot \lambda + 2 \cdot e^{-\lambda} \cdot \frac{\lambda^2}{2!} + 3 \cdot e^{-\lambda} \cdot \frac{\lambda^3}{3!} + 4 \cdot e^{-\lambda} \cdot \frac{\lambda^4}{4!} + \cdots$$

$$= e^{-\lambda} \cdot \lambda \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$= \lambda$$

$$\therefore \; E(X) = \lambda$$

$$V(X) = E(X^2) - (E(X))^2$$

Variance:

$$E(X^2) = \sum_{x_i=0}^{\infty} x_i^2 \, P(\lambda, x_i)$$

$$= \sum_{x_i=0}^{\infty} x_i^2 \frac{e^{-\lambda} \cdot \lambda^{x_i}}{x_i!} = \sum_{x_i=0}^{\infty} \cancel{x_i} \, x_i \frac{e^{-\lambda} \lambda^{x_i}}{\cancel{x_i}(x_i-1)!}$$

$$= \sum_{x_i=1}^{\infty} \frac{x_i \, e^{-\lambda} \lambda^{x_i}}{(x_i-1)!}$$

$$= \sum_{x_i=1}^{\infty} \frac{[(x_i-1)+1] \, e^{-\lambda} \cdot \lambda^{(x_i-1)+1}}{(x_i-1)!}$$

$$\lambda^{x_i-1} \cdot \lambda$$

$$= \lambda \left[\underbrace{\sum_{x_i=1}^{\infty} \frac{(x_i-1)\, e^{-\lambda} \lambda^{x_i-1}}{(x_i-1)!}} + \underbrace{\sum_{x_i=1}^{\infty} \frac{e^{-\lambda} \lambda^{x_i-1}}{(x_i-1)!}} \right]$$

$$= \lambda [\lambda + 1]$$

$$= \lambda^2 + \lambda$$

$$e^{-\lambda}[e^{\lambda}] = 1$$

$$\boxed{\therefore V(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}}$$

$$\sum_{x_i=0}^{\infty} x_i \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

$$= \sum_{x_i=1}^{\infty} \frac{(x_i-1)\, e^{-\lambda} \lambda^{(x_i-1)}}{(x_i-1)!}$$

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Theorem: Poisson distribution is a limiting form of binomial distribution.

ie if p is very small & n is very large compared to p, but np remains fixed (constant)

$$b(n, p, x) \simeq p(\lambda, x) \quad \text{with} \quad \lambda = np$$

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Proof: Consider the Binomial distribution

$$P(X=x) = b(n,p,x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} p^x \cdot (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots[n-(x-1)]}{x!} p^x (1-p)^{n-x} \quad \text{--- (1)}$$

Let $\boxed{np=\lambda}$, then $p = \frac{\lambda}{n}$, $1-p = 1-\frac{\lambda}{n}$

$$= \frac{n-\lambda}{n}$$

using this in Eqn (1),

$$P(X=x) = \frac{n(n-1)(n-2)\cdots(n-(x-1))}{x!}\left(\frac{\lambda}{n}\right)^x\left(\frac{n-\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\left[\frac{n(n-1)(n-2)\cdots(n-(x-1))}{n^x} \times \left(\frac{n-\lambda}{n}\right)^{n-x}\right]$$

$$= \frac{\lambda^x}{x!}\left[1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]\left(1-\frac{\lambda}{n}\right)^{n-x}\Bigg]$$

$$= \frac{\lambda^x}{x!}\left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x} \quad \text{--- (2)}$$

letting $n \longrightarrow \infty$ in such a way that $np = \lambda$ remains fixed. This implies as $n \longrightarrow \infty$, $p \longrightarrow 0$ $[\&\ np \longrightarrow \lambda]$.

$$\therefore \lim_{n\to\infty} P[X=x] = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]\left(1-\frac{\lambda}{n}\right)^{n} * \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\cdot \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}$$

$$= \frac{\lambda^x}{x!}\, e^{-\lambda}$$

$$\boxed{\because \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}}$$

$$\boxed{= \frac{e^{-\lambda}\cdot\lambda^x}{x!}}$$, which is a Poisson distribution with parameter λ.

* Suppose that the probability that an item produced by a particular machine is defective equals 0.2. If ten items produced from this machine are selected at random, what is the probability that not more than one defective is found?

$p = P[\text{an item produced is defective}] = 0.2$

$n = 10$

X: no. of defective items out of 10

$$P[X=x] = b(10, 0.2, x) = \binom{10}{x} (0.2)^x (1-0.2)^{10-x}$$

$$= \binom{10}{x} (0.2)^x (0.8)^{10-x}$$

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P[not more than 1 defective is found]

$$= P[X \leq 1]$$

$$= b(10, 0.2, 0) + b(10, 0.2, 1)$$

$$= \binom{10}{0}(0.2)^0 (0.8)^{10} + \binom{10}{1}(0.2)^1 (0.8)^9$$

$$= (0.8)^{10} + 10(0.2)(0.8)^9$$

$$\boxed{= 0.3758}$$

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Use both binomial & Poisson distribution & compare the answers.

$$\lambda = np = 10 \times 0.2 = \underline{\underline{2}}$$

$\therefore P[$ not more than 1 defective is found$]$

$$= P[X \leq 1]$$

$$= \frac{e^{-2} 2^0}{0!} + \frac{e^{-2} 2^1}{1!}$$

$$= e^{-2} + e^{-2} \times 2$$

$$\boxed{= 0.4060}$$

Here Binomial distribution is preferred.

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* An insurance company has discovered that only about 0.1% of the population is involved in a certain type of accident each year. If its 10,000 policy holders were randomly selected from the population, what is the probability that not more than 5 of its clients are involved in such an accident next year?

X: no. of clients will be involved in an accident in next year

$p = P[\text{a client involved in an accident}]$

$= \frac{0.1}{100} = \underline{\underline{0.001}}$

$\lambda = np = 10{,}000 \times 0.001$

$= \underline{\underline{10}}$

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P[not more than 5 clients will involve in an accident next year] $= P[X \leq 5]$

$$= \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^3}{3!} + \frac{e^{-\lambda}\lambda^4}{4!} + \frac{e^{-\lambda}\lambda^5}{5!}$$

$\lambda = 10$

$$= e^{-10}\left[1 + 10 + \frac{10^2}{2} + \frac{10^3}{6} + \frac{10^4}{24} + \frac{10^5}{120}\right]$$

$$\boxed{P[X \leq 5] = 0.06708}$$

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* Six coins are tossed. Find the probability of getting
  i) exactly 3 heads
  ii) almost 3 heads
  iii) atleast 3 heads
  iv) atleast 1 head

X: no. of heads occuring in tossing 6 coins

$p = 1/2$

$n = 6$

$$P[X=x] = b(6, 1/2, x) = {}^6C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{6-x}$$

$$= {}^6C_x \left(\tfrac{1}{2}\right)^6$$

$$= \binom{6}{x} \frac{1}{64} \; ; x = 0, 1, 2, \ldots, 6$$

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i) $P[X=3] = b(6, 1/2, 3)$

$$= \binom{6}{3}\frac{1}{64} \boxed{= 0.3125}$$

ii) $P[X \leq 3] = \frac{1}{64}\left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3}\right]$

$=$

iii) $P[X \geq 3] = 1 - P[X < 3]$

$$= 1 - \frac{1}{64}\left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2}\right]$$

$=$

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iv) $P[X \geq 1] = 1 - P[X < 1]$

$$= 1 - \frac{1}{64}\binom{6}{0}$$

$$\boxed{= 0.984}$$

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* Suppose that X has a Poisson distribution. If $P(X=2) = \frac{2}{3} P(X=1)$. Evaluate $P(X=0)$ & $P(X=3)$

$$P(X=2) = \frac{2}{3} P(X=1)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{2}{3}\frac{e^{-\lambda}\lambda^1}{1!}$$

find λ

HW

$P(X=0) =$

$P(X=3) =$

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* Mean & Variance of binomial distribution are 4 & 4/3 respectively. Find $P[X \geq 1]$

$np = 4$, $npq = \frac{4}{3}$

find n, p, q

HW

$b(n, p, x) = \binom{n}{x} p^x q^{n-x}$

$P[X \geq 1] =$

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