If $U = AX + B$ & $W = CY + D$ where A, B, C, D are constants. (X, Y – random variables)

then

$$\boxed{\rho_{UW} = \left(\frac{AC}{|AC|}\right)\rho_{XY} = \pm \rho_{XY}}$$

ie $(A \neq 0,\ C \neq 0)$ ✓

X & Y — r.v's

U & W

$U = AX + B$

$W = CY + D$

ρ_{XY} ✓

$$\rho_{UW} = \frac{Cov(U, W)}{\sqrt{Var(U)}\ \sqrt{Var\, W}}$$

$$\rho_{UW} = \left(\frac{AC}{|AC|}\right)\rho_{XY}$$

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$$V(ax+by) = a^2 V(X) + b^2 V(Y) + 2ab\, Cov(X,Y)$$

HW

$$V(ax+by) = E[ax+by]^2 - [E(ax+by)]^2 = _____$$

If X & Y are independent, $Cov(X,Y)=0$

$$\therefore V(ax+by) = a^2 V(X) + b^2 V(Y)$$

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Converse is not true

If X & Y are uncorrelated then X & Y need not be independent.

Ex: uncorrelated but dependent random variables

$X : -1, 0, 1$

$p(x_i) = \frac{1}{3}$ } each

$y = x^2$

$Y : 1, 0, 1$ $\quad p(y_i) = \frac{1}{3}$ } each

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = \underbrace{E(XY) - E(X)}E(Y)$$

$$E(X) = \sum_i x_i p(x_i) = (-1) \times \frac{1}{3} + (0) \times \frac{1}{3} + (1) \times \frac{1}{3} = 0$$

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$$E(Y) = \sum y_i p(y_i) = 1 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{2}{3}$$

$$E(XY) = E(X \cdot X^2)$$

$$= E(X^3) = \sum x_i^3 p(x_i) = (-1)^3 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1^3 \times \frac{1}{3}$$

$$= 0$$

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

$$= 0$$

$$\rho = 0$$

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If $f(x) = \begin{cases} x\, e^{-x^2/2} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}$. Determine $E(X)$ & σ^2

$p(x_i)$ — Discrete

$f(x)$ — continuous] — $f(x)$

HW

$$E(X) = \int_0^\infty x f(x)\, dx = \int_0^\infty x \cdot x\, e^{-x^2/2}\, dx$$

$$E(X^2) = \int_0^\infty x^2 f(x)\, dx$$

$$= \int_0^\infty x^2 x\, e^{-x^2/2}\, dx$$

$$\sigma^2 = V(X) = E(X^2) - [E(X)]^2$$

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Two independent random variables X_1 and X_2 has means 5 & 10 and variances 4 & 9 respectively. Find the covariance between

$$U = 3X_1 + 4X_2 \quad \& \quad V = 3X_1 - X_2$$

X_1 & X_2 $\qquad U = 3X_1 + 4X_2$, $V = 3X_1 - X_2$

$$\mathrm{Cov}(U, V) = E(UV) - E(U)E(V)$$

$$= E\left[(3X_1 + 4X_2)(3X_1 - X_2)\right] - E\left[3X_1 + 4X_2\right]E\left[3X_1 - X_2\right]$$

$$= 9E(X_1)^2 + 9E(X_1X_2) - 4E(X_2)^2 - \left[3E(X_1) + 4E(X_2)\right]\left[3E(X_1) - E(X_2)\right]$$

$$= 9E(X_1)^2 + 9E(X_1X_2) - 4E(X_2)^2 - 9\left[E(X_1)\right]^2 + 3E(X_1)E(X_2) - 12E(X_1)E(X_2) + 4\left[E(X_2)\right]^2$$

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$$\mathrm{Cov}(U,V) = 9V(X_1) - 4V(X_2) + 9\left[E(X_1X_2) - E(X_1)E(X_2)\right]$$

$$= 9 \times 4 - 4 \times 9$$

$$\boxed{\mathrm{Cov}(U,V) = \underline{\underline{0}}}$$

X_1, X_2 — independent r.v

U, V } functions of X_1 & X_2 — ?

HW

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If X, Y and Z are uncorrelated random variables with standard deviations 5, 12 and 9 respectively and if $U = X+Y$ and $V = Y+Z$, evaluate the correlation coefficient between U & V.

$U = X + Y$ $\qquad$ $V = Y + Z$

$\rho_{uv} = ?$ $\qquad$ X, Y, Z — uncorrelated

$\Rightarrow \quad Cov(X,Y) = 0, \quad Cov(X,Z) = 0, \quad Cov(Y,Z) = 0$

$\rho_{xy} = \rho_{xz} = \rho_{yz} = 0$

$$\rho_{uv} = \frac{E(UV) - E(U)E(V)}{\sqrt{Var(U)}\sqrt{Var(V)}}$$

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$$\rho_{uv} = \frac{E[(X+Y)(Y+Z)] - E[X+Y]\,E[Y+Z]}{\sqrt{V(X+Y)\cdot V(Y+Z)}}$$

$V(aX+bY) = a^2V(X) + b^2V(Y) + 2ab\,\mathrm{cov}(X,Y)$

$$= \frac{E[XY+XZ+Y^2+YZ] - [E(X)+E(Y)]\,[E(Y)+E(Z)]}{\sqrt{[V(X)+V(Y)+2\mathrm{cov}(X,Y)]\cdot[V(Y)+V(Z)+2\mathrm{cov}(Y,Z)]}}$$

$$= \frac{E(XY)+E(XZ)+E(Y^2)+E(YZ)-E(X)\cdot E(Y)-E(X)\cdot E(Z) - [E(Y)]^2 - E(Y)\,E(Z)}{\sqrt{[V(X)+V(Y)]\cdot[V(Y)+V(Z)]}}$$

$$= \frac{\overset{=0}{\mathrm{cov}(X,Y)} + \overset{=0}{\mathrm{cov}(X,Z)} + \overset{=0}{\mathrm{cov}(Y,Z)} + \overset{=12^2}{V(Y)}}{\sqrt{(5^2+12^2)(12^2+9^2)}}$$

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$$\rho_{uv} = \frac{12^2}{\sqrt{(5^2+12^2)(12^2+9^2)}} = \frac{144}{195} = \underline{\underline{0.738}}$$

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Change of variables

Discrete case

X — d.r.v , $p(x)$ — discrete probability distn

$U = \phi(X)$ — probability distribution of u?

$X = \psi(U)$

$$\boxed{g(u) = p(\psi(u))}$$

(X, Y) — two dimensional d.r.v

$p(x, y)$ — joint probability distn fn

$U = \phi_1(X, Y)$ & $V = \phi_2(X, Y)$

$X = \psi_1(U, V)$ & $Y = \psi_2(U, V)$

joint probability distn fn of U & V

$$\boxed{g(u, v) = p[\psi_1(u, v), \psi_2(u, v)]}$$

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Continuous case

(Q) X – c.r.v, pdf $f(x)$

$U = \phi(X)$, pdf of U ?

1-D

$x = \chi(u)$

$$\boxed{g(u) = f(\chi(u))\,|\chi'(u)|}$$

$$\chi'(u) = \frac{dx}{du}$$

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(b) (X, Y) – two dimensional c.r.v

joint pdf $f(x, y)$

$U = \phi_1(X, Y)$ $\quad$ $V = \phi_2(X, Y)$

joint pdf U, V ?

$X = \psi_1(U, V)$, $Y = \psi_2(U, V)$

joint pdf of U, V $\quad$ $g(u, v) = f(\psi_1(u, v), \psi_2(u, v))\,|J|$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

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* The probability function of a random variable X is

$$f(x) = \begin{cases} 2^{-x} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Find the probability function for a random variable $U = X^4 + 1$

X is a d.r.v

$U = X^4 + 1$, $X = \sqrt[4]{U-1}$, the real positive root is taken

$U = 2, 17, 82, \ldots$

$\therefore$ probability function of U,

$$g(U) = \begin{cases} 2^{-\sqrt[4]{U-1}} & U = 2, 17, 82, \ldots \\ 0 & \text{otherwise} \end{cases}$$

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* The pdf of a r.v X is given by

$$f(x) = \begin{cases} \frac{x^2}{81} & -3 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of $U = \frac{1}{3}(12 - X)$

$$u = \frac{1}{3}(12 - x)$$

$$\Rightarrow x = 12 - 3u$$

$-3 < x < 6$

$x = -3, \quad u = 5$

$x = 6, \quad u = 2$

$$\underline{\underline{2 < u < 5}}$$

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