One – Dimensional Heat Conduction Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le l,\ t > 0$$

parabolic

$B^2 - 4AC = 0$

$B = 0$

$A = c^2$

$C = 0$

Initial conditions : $u(x, 0) = f(x),\ 0 \le x \le l$

Boundary Conditions : $u(0, t) = g(t),\ \ u(l,t) = h(t),\ \ t > 0$

- c^2 is the ***diffusivity of the substance***.
- $u(x, t)$ is a ***temperature function*** which is defined for values of x from 0 to l (length of the bar) and for values of time t from 0 to ∞.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution of one-dimensional Heat equation

1. Explicit method:

$$\frac{\partial u}{\partial t} = \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right]$$

Consider a rectangular mesh in the xt-plane with spacing h along x direction and k along the t direction.

The mesh point $u(x,t)$ is denoted as $\mathbf{u(x,t) = u(ih, jk) = u_{i,j}}$.

Using finite difference approximations, we have

$$\frac{\partial u}{\partial t} = \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Substituting in heat equation, we get

$$\frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] = \frac{c^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Or

$$\boxed{u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j}}$$

Schmidt explicit formula or two-level formula

where $\lambda = \frac{kc^2}{h^2}$; $0 < \lambda \le \frac{1}{2}$.

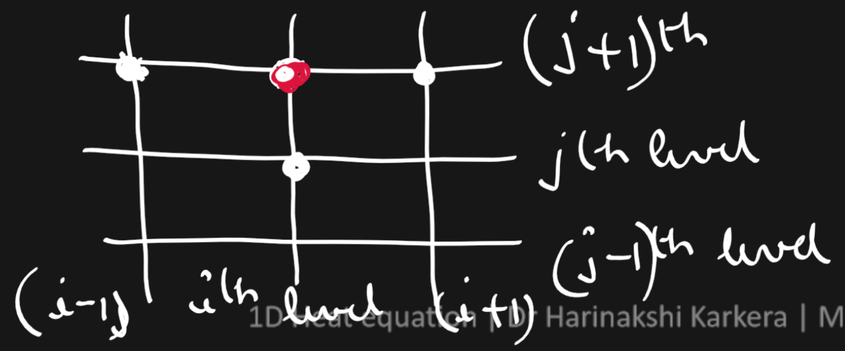
λ is the ***mesh ratio parameter***, h mesh length along x-axis and k mesh length along t-axis.

Note: (i) For $\lambda = \frac{1}{2}$,

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

Bendre – Schmidt recurrence relation

(ii) For $\lambda = \frac{1}{6}$, the error in Schmidt's formula is least.



$$u_{i,j+1} = \lambda\, u_{i-1,j} + (1-2\lambda)\, u_{i,j} + \lambda\, u_{i+1,j}$$

Solution of one-dimensional Heat equation

2. Implicit method:

Consider the Schmidt explicit finite difference scheme of Heat equation

$$\frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] = \frac{c^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

or

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] \text{----------(1)}$$

If the backward difference approximation to $\frac{\partial u}{\partial t}$ is used, then the heat equation reduces to

$$\frac{1}{k}\left[u_{i,j} - u_{i,j-1}\right] = \frac{c^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Replacing j by j + 1, we get

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2}\left[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right] \text{--------(2)}$$

Adding equations (1) & (2), we obtain

$$\left(u_{i,j+1} - u_{i,j}\right) = \frac{kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right]$$

Or

$$\boxed{-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}}$$

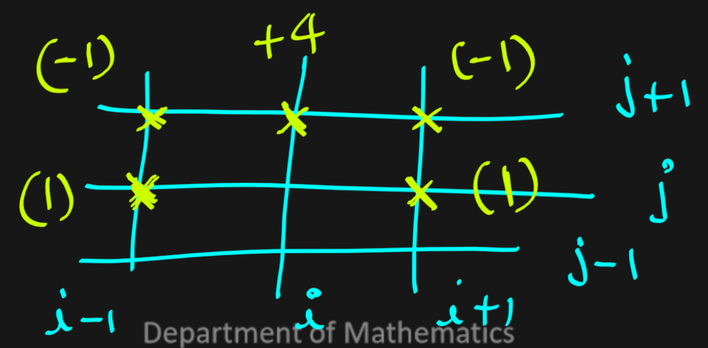
Crank – Nicolson's implicit finite difference scheme

Note:

(i) In ***Schmidt's method,*** $0 < \lambda \leq \frac{1}{2}$, but there is no restriction on λ in ***Crank – Nicolson's method***

(ii) For $\lambda = 1$

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$



Problems:

- Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 2$, $t > 0$; $u(x,0) = 0$, $u(0,t) = sin\frac{8\pi}{3}t$, $u(2,t) = 0$. Choose $h = 0.5$ and compute u for three-time steps.

  (i) $\lambda = \frac{1}{2}$ (ii) $\lambda = \frac{1}{3}$

(i) $\lambda = \frac{1}{2}$, $c^2 = 1$, $h = 0.5$

$\therefore k = \frac{\lambda h^2}{c^2} = \frac{1}{8}$ and $x_i = ih = i \times 0.5$, $i = 0,1,2,3,4$

$t_j = jk = j \times \frac{1}{8}$

$u(0,t) = \sin\frac{8\pi}{3}t$

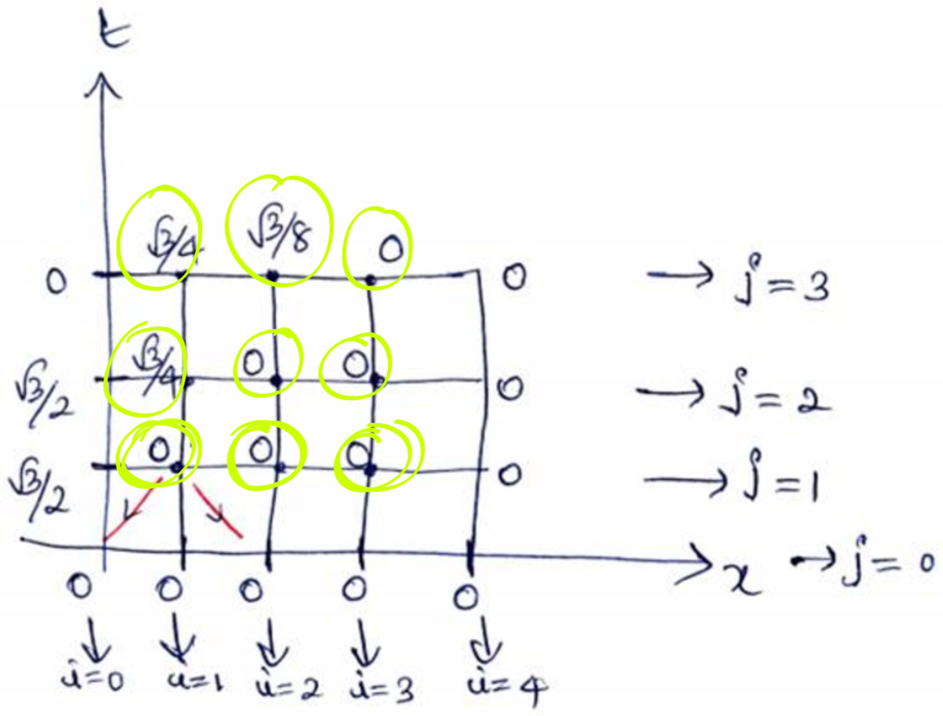
$u_{0,j} = \sin\frac{8\pi}{3}(jk)$

For $j = 0$, $u_{0,0} = 0$

For $j = 1$, $u_{0,1} = \frac{\sqrt{3}}{2}$

For $j = 2$, $u_{0,2} = \frac{\sqrt{3}}{2}$

For $j = 3$, $u_{0,3} = 0$



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  (i) $\lambda = \frac{1}{2}$ (ii) $\lambda = \frac{1}{3}$

(ii) $\lambda = \frac{1}{3}$, $k = \frac{\lambda h^2}{c^2} = \frac{1}{12}$

$$\therefore u_{i,j+1} = \frac{1}{3}\left[u_{i-1,j} + u_{i,j} + u_{i+1,j}\right]$$

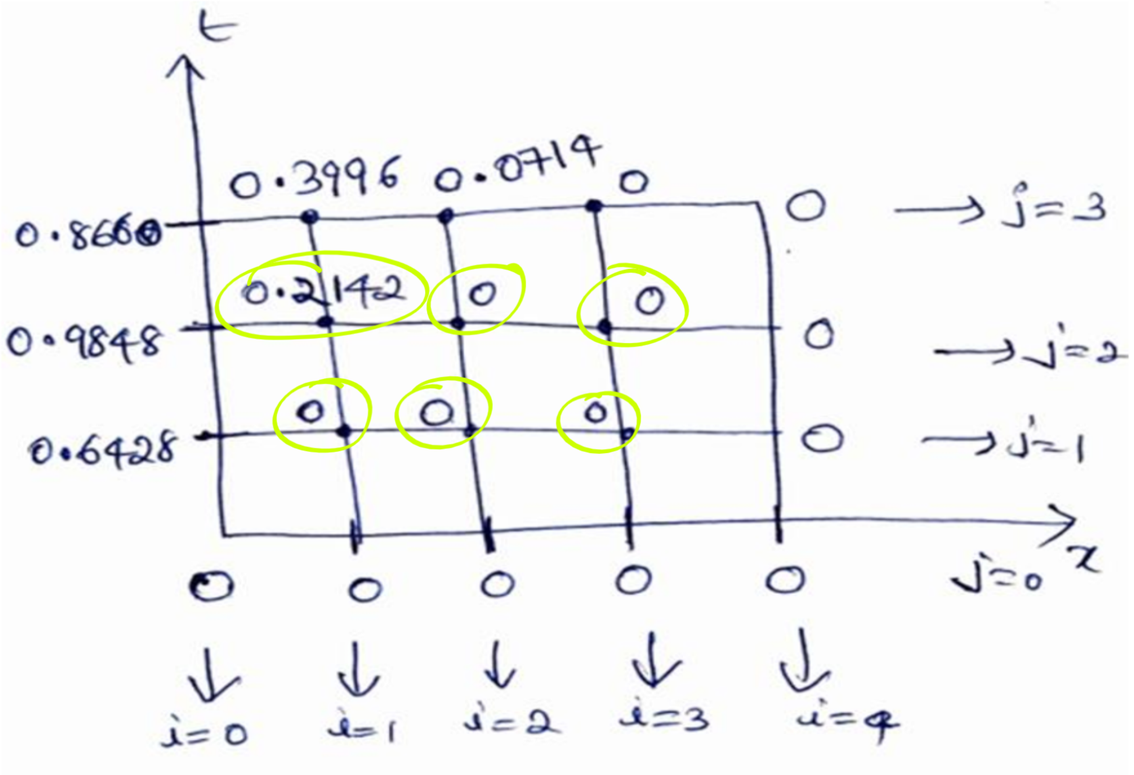
$$u_{0,j} = \sin\frac{8\pi}{3}(jk)$$

For $j = 0$, $u_{0,0} = 0$

For $j = 1$, $u_{0,1} = 0.6428$

For $j = 2$, $u_{0,2} = 0.9848$

For $j = 3$, $u_{0,3} = 0.8660$



Problems:

- **Solve by Crank-Nicolson's scheme:**

$$\frac{\partial u}{\partial t} = \frac{1}{16}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1\,,\ t > 0\ ;\ u(x,0) = 0,\ u(0,t) = 0,\ u(1,t) = 100t.$$ **Compute u for one time step with** $h = \frac{1}{4}$.

$c^2 = \frac{1}{16}$, $h = \frac{1}{4}$. Taking $\lambda = 1$

$\therefore k = \frac{\lambda h^2}{c^2} = 1$ and $x_i = ih = i \times 0.25$, $i = 0,1,2,3,4$.

$$t_j = jk = j \times 1$$

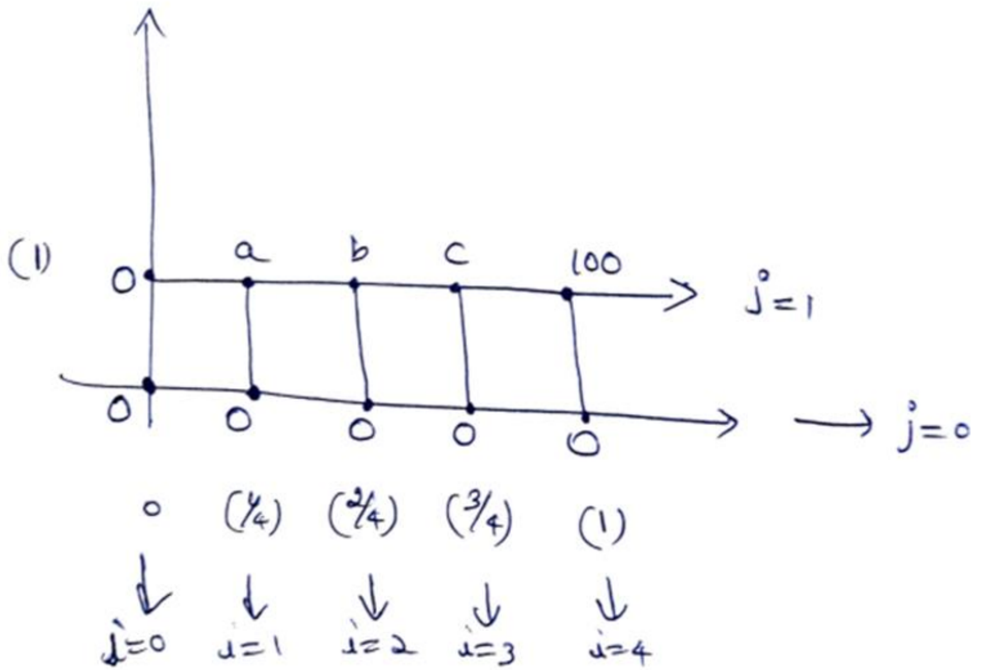
$$u(1,t) = 100t$$

Using Crank-Nicolson's formula

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

Let $u\left(\frac{1}{4},1\right) = a$, $u\left(\frac{1}{2},1\right) = b$, $u\left(\frac{3}{4},1\right) = c$

Then the system of equations are

$$0 + 4a - b = 0$$
$$-a + 4b - c = 0$$
$$-b + 4c - 100 = 0$$

$\Rightarrow$ $a = 1.79$, $b = 7.14$, $c = 26.79$