MFDS

Stochastic Process

$\{x(t), t \in T\}$

→ collection of random variables

$t \in T$, $x(t)$ → a random variable

t → time parameter (not necessary for it to be just time, could be other parameter)

$x(t)$ → state of process at time t.

Examples

→ A queueing problem

Bus facility AB5 ⟶ recreation club

→ bus has a capacity of K persons

→ K, < K persons, the bus will depart as per time t

→ > K, first K will depart

At each time period t during a day the number of students left behind → stochastic process

→ customer behaviours

A, B, C → soaps

often change brand, affects the sales

→ new soap to be manufacture, manufacture studies other soaps

→ behaviour in one month (change after a month)

through survey

A → A 60%	B → A 50%	C → A 40%
A → B 30%	B → B 30%	C → B 40%
A → C 10%	B → C 20%	C → C 20%

→ different questions → which soap is brought most
→ how many customer retained

→ can be looked at as stochastic process

→ Barber shop

→ 1 seat only

→ limited 6 waiting capacity

$x(t)$: number of customers in the barber shop at time t

→ what could be the waiting time?
→ how to reduce waiting time?
→ how many customer lost?

→ can be looked at as stochastic process

→ Insurance policy

→ specific capital
→ introduce schemes

→ what should be the capital fixed for the company to be profitable...

In such systems 3 essential aspects

1) behavioral
2) statistical
3) operational

Definations

$\{x(t), t \in T\}$

$T \rightarrow$ parameter space, set of possible values
$t \rightarrow$ indexing parameter.

$T \rightarrow$ usually time period, could be length, thickness etc

$T \rightarrow$ discrete $\rightarrow$ finite
$\rightarrow$ continuous $\rightarrow$ countably infinite.

Notation: denoted by n $\{x_n, n = 0, 1, 2 \ldots\}$ for discrete.

continuous remains same as mentioned above.

state space: possible values of $x(t)$ — states

* The set of all possible values random variable $x(t)$ can assume.

* The process $\{x(t)$ is in state x at epoch t.

$\downarrow$

time period 't'

stochastic process $\rightarrow$ collection of random variables $x(t) \rightarrow t$

$\rightarrow$ collection of the realizations of the process ~~(x)~~ $x(t)$

$\rightarrow$ stochastic sequence

$\rightarrow$ If parameter space <u>discrete</u> $\rightarrow \{x_n, n = 0, 1, 2 - -\}$

$\rightarrow$ If parameter space cont. $\rightarrow \{x(t), t \in T\}$, $\{x_t, t \in T\}$ } finite or infinite intervals

$\{x(t), t \in T\}$

$x(t) \rightarrow$ one dimensional $\rightarrow$ our syllabus.

could be two-dimensional

could be multi dimensional

$$x(t) = (x_1(t), x_2(t))$$

$x_1(t) \rightarrow$ max temp in given t

$x_2(t) \rightarrow$ min temp in given t

complex values stochastic process

$$x(t) = x_1(t) + i\, x_2(t)$$

Classification of stochastic process

→ based on nature of parameter space and state space.

(1) discrete time, discrete state stochastic process

(1) x_n = no. of customers in the barber shop after n^{th} customer depart from the shop

$T = \{1, 2 ----\}$ discrete time

↪ parameter is no. of customer

$S = \{0, 1, 2, ----6\}$ discrete space

(2) Brand switching

x_n = customer preferance observed on a monthly basis

(3) throwing a die.

x_n = outcome of the nth throw

$\{x_n, n \geq 1\} \rightarrow$ DT DS ST

x_n = number of sixes in the first throw

(2) continuous time discrete state

(1) $x(t)$: no. of students waiting for a bus at any time of day.

$$T = \{ t \mid t \geq 0 \}$$

$$S = \{ 0, 1, 2, ---- \}$$

$$\{ x(t), t \in T \} \rightarrow CT \quad DS \quad SP$$

(2) $x(t)$ = no. of customers eating food in canteen at any time t

$$T \; \{ t \mid t \geq 0 \}$$

$$S = \{ 0, 1, 2 ---- \}$$

(3) Discrete time continuous state stochastic process

(1) x_n = the content of a dam observed at n^{th} time unit

monthly ← hourly → weekly

$T = \{1, 2, ---\}$

$S = \{x \mid x \geq 0\}$

$\{x_n, n = 1, 2, ---\}$ DT CT SP

(2) x_n = stock market price of a stock at n^{th} time unit in a day

5m ← 1h → 4h

$T = \{1, 2, 3 ---\}$

$S = \{x \mid x \geq 0\}$

$\{x_n, n = 1, 2, 3 ---\}$

(4) Continuous time continuous state stochastic process.

(1) $x(t)$: content of dam observed at any time t

$$T = \{t \mid t \geq 0\}$$

$$S = \{x \mid x \geq 0\}$$

$$\{x(t),\ t \in T\}$$

(2) $x(t)$: temp of a particular city at any time t

By imposing suitable restrictions on n-dimensional distribution functions

(1) Bernoulli process

$\{x_i, i = 1, 2, ---\}$

$x_i \sim \underbrace{i.i.d}_{\text{independent and identically distributed.}}$ random variables

Each $x_i \rightarrow$ bernoulli's distribution with parameter p

$x_i \sim \text{Bernoulli}(1, p)$

$S_n = \sum_{i=1}^{n} x_i$

$E(S_n) = np$
$V(S_n) = np(1-p)$ } mean and variance of binomial distribution

$P(x_i = k) = \begin{cases} 1-p, & k = 0 \\ p, & k = 1 \end{cases}$

$\therefore S_n \sim \text{Binomial}(n, p)$

$\{S_n, n = 1, 2, ---\}$ Binomial process

(1)

x_i = arrival occurs in the ith trial or not

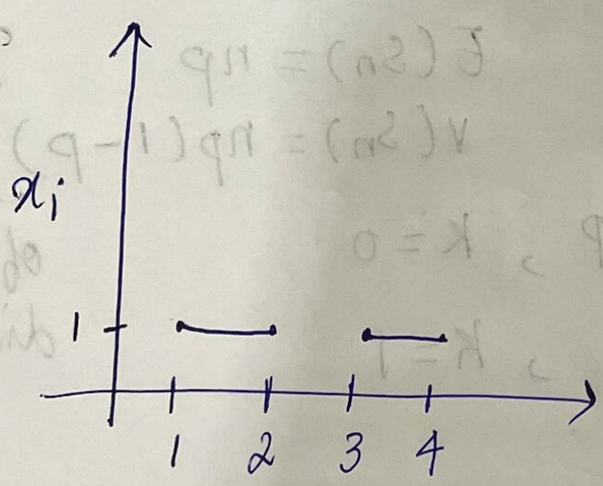
$$\begin{cases} x_i = 0 \rightarrow i^{th} \text{ trial, no arrival takes place} \\ x_i = 1 \rightarrow i^{th} \text{ trial, arrival takes place} \end{cases}$$

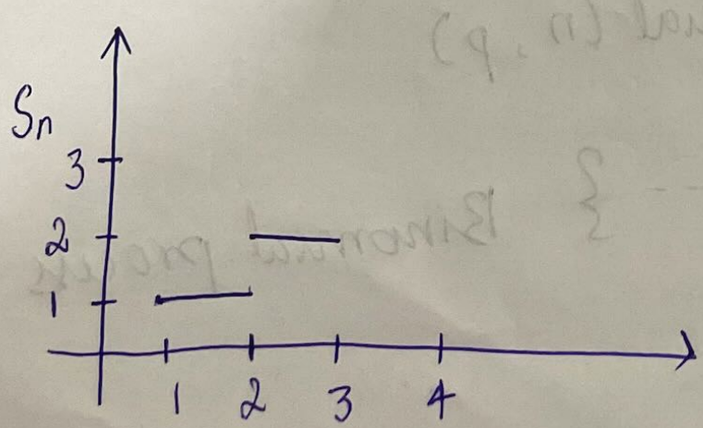
$$S_n = \sum_{i=1}^{n} x_i$$

S_n = no. of arrivals in n trials

$\{S_n, n = 1, 2, \ldots\} \rightarrow$ Binomial process

DT, DS SP





2) Poisson process:

$X(t)$ = number of arrivals occur during the interval $[0, t]$

$\{x(t), t \geq 0\}$ — discrete state continuous time

$[0, t] \rightarrow$ divide in n equal part

$$P = \lambda \frac{t}{n}$$

$$P(x(t) = k) = {}^{n}C_{k} \left(\lambda \frac{t}{n}\right)^{k} \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

$$\lim_{n \to \infty} P(x(t) = k) = \frac{e^{-\lambda t} (\lambda t)^{k}}{k!}$$

$x(t) \sim$ poisson distribution with parameter λt

$\{x(t), t \geq 0\} \rightarrow$ poisson process

(3) Gaussian process

→ later on in multivariate analysis.

(4) Markov process

→ Tossing the coin infinitely many times

$P(\text{head}) = p$

$P(\text{Tail}) = 1 - p \qquad 0 < p < 1$

X_n : outcome of nth trial → 0 {tail, → 1 {head

$P(x_n = 0) = 1 - p \quad \{x_1, x_2, x_3 \ldots\}$

↓ tail

$P(x_n = 1) = p \quad \{x_i;\ i = 0, 1, 2 \ldots\}$

↓ head.

$S_n = X_1 + X_2 \ldots X_n$

S_n : no. of heads appear in the first n trials

$\{S_n, n=1,2,\ldots\}$ $\quad S_{n+1} = S_n + X_{n+1}$

$P(S_{n+1} = k+1 \mid S_n = k) = p$

$\rightarrow$ head in $n+1^{th}$ trial

$P(S_{n+1} = k \mid S_n = k) = 1-p \quad (n \geq 1)$

$\rightarrow$ tail in $n+1^{th}$ trial

$$\boxed{\begin{aligned} &P(S_{n+1} = k \mid S_1 = i_1, S_2 = i_2 \ldots S_n = k) = \\ &P(S_{n+1} = k \mid S_n = k) = 1-p \end{aligned}}$$

$\hookrightarrow$ memory less property / Markov property

$\{S_n, n=1,2\ldots\} \rightarrow$ Markov process

$\{x(t), t \in T\} \quad t_1 < t_2 < t_3 \ldots t_n < t$

$P[a \leq x(t) \leq b \mid x(t_1) = x_1 \ldots x(t_n) = x_n]$

$= P[a \leq x(t) \leq b \mid x(t_n) = x_n]$

by only depending on the previous trial.

① When state space → discrete

Markov process → Markov chain

$\{S_n = 1, 2, 3, 4 \ldots\}$ → Markov chain

parameter space

discrete → discrete time Markov chain

continuous → continuous time Markov chain

<u>Discrete Time Markov Chain</u>

$$P\{X_{n+1} = j \mid X_0 = i_0, X_1 = i_1 \ldots X_n = i\}$$

$$= P\{X_{n+1} = j \mid X_n = i\}$$

$n \geq 0$

$\{X_n, n = 0, 1, 2 \ldots\}$ → discrete time markov chain

We are studying the distribution to predict

X_n distribution $n \geq 1$

$n \to \infty$ limiting distribution

X_0

what is the definition

$X_0 \to X_1 \to X_2$

transition

* Probablity mass function $P_j^{(n)}$ or $P_j(n)$ of a random variable X_n

$$P_j^{(n)} = \text{Prob}\{X_n = j\} \quad j \leftarrow S$$

* Conditional probablity mass func

$$P_{jk}(m,n) = \text{prob}\{X_n = k \mid X_m = j\} \quad 0 \leq m \leq n$$

n^{th} step $\quad m^{th}$ step $\quad j, k \in S$

transition probablity function.

Time homogenious

$$P_{jk}^{(n)} = \text{Prob}\{X_{m+n} = k \mid X_m = j\}$$

at m_{th} step $\rightarrow i^{th}$ state

at $m+n^{th}$ step $\rightarrow k^{th}$ state

$\rightarrow$ n-step transition probablity function

$$P_{jk}^{(1)} = \text{prob}\{X_{m+1} = k \mid X_m = j\}$$

m^{th} step - state j

$m+1^{th}$ step $\rightarrow$ state k

$(m \geq 1)$

$\rightarrow$ one step transition probablity function

$$P_{jk}^{(2)} = \text{prob}\{X_{m+2} = k \mid X_m = j\}$$

$\rightarrow$ two step transition probablity function

One step transition probablity function

$$[P_{jk}] - \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{S \times S} \Bigg\} \text{ Stochastic matrix}$$

(1) $P_{jk} \geq 0, \ \forall j, k \leftarrow S$

(2) $\sum_{k \in S} P_{jk} = 1 \quad j \leftarrow S \Big\}$ row sum $= 1$

Q. Factory has 2 machines and 1 repair crew. P(any one machine breaking down on a particular day)

Assume repair crew is working on a machine the P(complete repairs in a day) = β. For simplicity ignore P(repair completion or a breakdown except at the end of the day)

Let X_n → number of machines in operation, at the end of the nth day.

can we model behaviour of X_n as a Markov chain.

A. X_n → no. of machines in operation at the end of nth day

$X_n = 0, 1, 2$

$\boxed{S = 0, 1, 2}$ → discrete state

$\{X_n, n = 1, 2, 3 \ldots\}$ → discrete time

→ depends on how many machines work on the previous day

Discrete time Markov chain

one step transition prob. matrix

$$[P_{jk}]_{s\times s} = [P_{jk}]_{3\times 3} \quad (s = 0, 1, 2)$$

$$P = \begin{array}{c|ccc} & & & X_{n+1} \\ & & 0 & 1 & 2 \\ \hline X_n & 0 & P_{00}^{(1)} & P_{01}^{(1)} & P_{02}^{(1)} \\ & 1 & P_{10}^{(1)} & P_{11}^{(1)} & P_{12}^{(1)} \\ & 2 & P_{20}^{(1)} & P_{21}^{(1)} & P_{22}^{(1)} \end{array}$$

n^{th} step → $n+1^{th}$ step

$$P_{00}^{(1)} = \text{prob}\{x_{n+1} = 0 \mid x_n = 0\} = 1-\beta$$

state 0 → both machines are under repair

P (both machines breaks down)

$$P_{01}^{(1)} = \text{prob}\{x_{n+1} = 1 \mid x_n = 0\} = \beta$$

$$P_{02}^{(1)} = P\{x_{n+1} = 2 \mid x_n = 0\}$$

previous day 0 machines working next day both working, its impossible

$$P_{02}^{(1)} = 0$$

$$P_{10}^{(1)} = P\{x_{n+1} = 0 \mid x_n = 1\} = \alpha(1-$$

one machine breaking down

another machine breaks down

not ... to ... expo...

$$P_{20}^{(1)} = P\{x_{n+1} = 0 \mid x_n = 2\} = \alpha^2$$

Both machines failed

$$P_{11} = P\{x_{n+1} = 1 \mid x_n = 1\} = (1-\alpha)(1-\beta) + \alpha\beta$$

one couldnt get repaired

machines switches places

$$P_{12}^{(1)} = P\{x_{n+1} = 2 \mid x_n = 1\} = \beta(1-\alpha)$$

$$P_{21}^{(1)} = P\{x_{n+1} = 1, x_n = 2\} = \alpha(1-\alpha) + (1-\alpha)\alpha$$

$\alpha \quad (1-\alpha)$

$\left.\begin{array}{l} \cancel{A} \; B \\ B \; \cancel{A} \end{array}\right\}$ 2 cases

$(1-\alpha) \quad \alpha$

one breaks down, other works or vice versa

$$= 2\alpha(1-\alpha)$$

$$P_{22}^{(1)} = P\{x_{n+1} = 2, x_n = 2\} = (1-\alpha)^2$$

$$P = \begin{array}{c|ccc} & & x_{n+1} & \\ & 0 & 1 & 2 \\ \hline x_n \; 0 & 1-\beta & \beta & 0 \\ 1 & \alpha(1-\beta & (1-\alpha)(1-\beta)+\alpha\beta & \beta(1-\alpha) \\ 2 & \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{array}$$

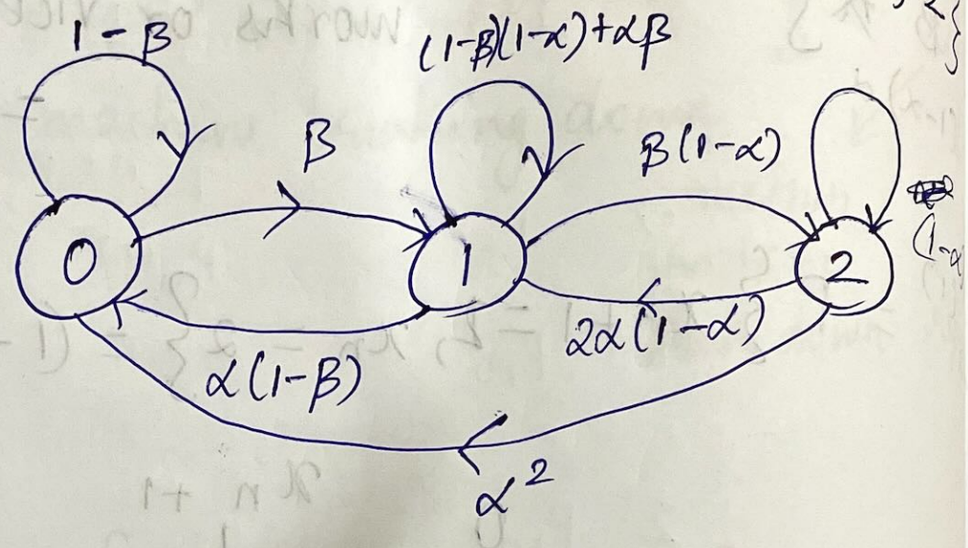
$P_{10} = \alpha(1-\beta)$ → another didnt repair

one broke down

State transition diagram

$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 1-\beta & \beta & 0 \\
1 & \alpha(1-\beta) & (1-\alpha)(1-\beta)+\alpha\beta & \beta(1-\alpha) \\
2 & \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2
\end{array}
$$

3 vertices because $S = \{0, 1, 2\}$



2 step transition probablity function

$$P_{ij}^{(2)} = \text{prob}\{x_{n+2} = j \mid X_n = i\}$$

n step transition probablity function

$$P_{ij}^{(n)} = \text{prob}\{x_{m+n} = j \mid X_m = i\}$$

Probablity mass function

$$P_j(n) = \text{Prob}\{x_n = j\} \quad j \in S, \quad n = 1, 2, 3 \ldots$$

system will be in a state j at n^{th} step

$$= \sum_{i \in S} P(X_0 = i)\, P(X_n = j \mid X_0 = i)$$

summation of possible ways in which the system can reach the stat j from the state i

$P(X_0 = i)$ ↑ initial probablity vector

$P(X_n = j \mid X_0 = i)$ ↑ n step transition probablity where $m = 0$

① $P(X_0 = i)$

0th step $\rightarrow$ state i

$\downarrow$

$p(0) = [P(X_0 = 0) \quad P(X_0 = 1) \quad P(X_0 = 2) \quad \ldots]$

depending on the state space.

initial probablity vector

② $P_{ij}^{(n)} = \text{Prob}\{X_n = j \mid X_0 = i\}$ n step transition

start with 2-step transition probablity

$$P_{ij}^{(2)} = \text{Prob}\{x_{n+2} = j \mid x_n = i\}$$

$$= \sum_{k \in S} \text{Prob}\{x_{n+2} = j, x_{n+1} = k \mid x_n = i\}$$

$$= \sum_{k \in S} \frac{\text{Prob}\{x_{n+2} = j, x_{n+1} = k \mid x_n = i\}}{\text{Prob}\{x_n = i\}}$$

$$= \sum_{k \in S} \frac{P\{X_{n+2}=j \mid X_{n+1}=k,\ X_n=i\}\, P\{X_{n+1}=k,\ X_n=i\}}{P\{X_n=i\}}$$

By using Markov property

$$= \sum_{k \in S} \frac{P(X_{n+2}=j \mid X_{n+1}=k) \cdot P(X_{n+1}=k \mid X_n=i)\, \cancel{P(X_n=i)}}{\cancel{(P_{X_n=i})}}$$

$$= \sum_k P_{ik} P_{kj}$$

$$P_{ij}^{(2)} = \sum_k P_{ik} P_{kj}$$

Generalising

$i \to k \to l \to j$

$$P_{ij}^{(3)} = \sum_k$$

$$P_{ij}^{(n+m)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n}$$

$$P_{ij}^{(m+1)} = \sum_k P_{ik} P_{kj}^{(m)}$$

$$= \sum_k P_{ik}^{(m)} P_{kj}$$

$P = [P_{ij}] \rightarrow$ one step transition

$$P^{(2)} = P^2$$

$$P^{(3)} = P^2 \cdot P = P^3$$

$$P^{(n)} = P^n, \quad n \geq 1$$

$$\boxed{P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)}}$$

$\downarrow$

chapman - Kolmangolor equation.

Distribution of X_n

$$P_j(n) = \text{Prob}\{x_n = j\}$$

$$= \sum_i P(X_0 = i)\, P(X_n = j \mid X_0 = i)$$

$$= \sum_i P_i(0)\, P_{ij}^{(n)}$$

Matrix form

$$P(n) = [P(X_n=0) \;\; P(X_n=1) \;\; P(X_n=2) \; - \; - \; - \; -]$$

$$P(n) = P(0)\, p^{(n)} = P(0)\, p^n$$

sub question (Machine question)

→ If the system starts out with both the machine operating, what is the probablity that both will be in operation, two days later.

X_n → no. of machines in operation at the nth day

We assume that in the zeroth step

At $t=0$ (0th step) → both machines are operating

$X_0 = 2$ probablity $= 1$

$P(X_0 = 2) = 1$ → Given.

$$P(0) = [P(X_0 = 0) \quad P(X_0 = 1) \quad P(X_0 = 2)]$$

→ initial probability vector

$$P(0) = [\,0 \quad 0 \quad 1\,]$$

$$P(0) = [0 \;\; 0 \;\; 1]$$

Find probablity that both machines are working after 2 days

$$P(X_2 = 2) = ?$$

$$P(X_2 = 2) = \sum_{i \in S} P(X_0 = i)\, P(X_2 = 2 \mid X_0 = i)$$

$$P(X_0 = 0) = 0 \qquad P(X_0 = 1) = 0$$

$$= P(X_0 = 2)\, P(X_2 = 2 \mid X_0 = 2)$$

$$1$$

$$\Rightarrow \boxed{P(X_2 = 2 \mid X_0 = 2)}$$

$P(X_2=2) = P(X_2=2 \mid X_0=2)$

two step transition probablity

$= p_{22}^{(2)} \rightarrow$ in two step

was in state 2 goes to state 2

$p^{(2)} = p^2$

$p_{22}^{(2)} = x_n$

$$\begin{array}{c|ccc} & & X_{n+1} & \\ & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & & \\ 2 & & & \end{array}$$

$P(X_2=2) = [P^2]_{(3,3)}$

Take $\alpha = 0.4$ $\beta = 0.7$

$$P = \begin{pmatrix} 1-\beta & \beta & 0 \\ \alpha(1-\beta) & (1-\alpha)(1-\beta)+\alpha\beta & \beta(1-\alpha) \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{pmatrix} = \begin{matrix} 0.3 & 0.7 & 0 \\ 0.4 & & \end{matrix}$$

find p^2 substituting α, β.

Q. Suppose that the chance of rain tomorrow depends on previous weather conditions only, whether or not it is raining today & not on past weather conditions. Suppose, if it rains today, then it will rain tomorrow with probability 0.7 and if it does not rain today, it will rain tomorrow with probability 0.4

Calculate the probablity that it will rain 4 days from today given that it is raining today

→ Markov process

→ state space → ~~continious~~ discrete

→ parameter space → discrete

Discrete time Markov chain

$S = \{0, 1\}$ $\quad$ 0 → rains, 1 → doesnt rain

$$P = \begin{array}{c|cc} & X_{n+1}: 0 & 1 \\ \hline X_n: 0 & \cancel{0.6}\ 0.7 & \cancel{0.4}\ 0.3 \\ 1 & \cancel{0.3}\ 0.4 & \cancel{0.7}\ 0.6 \end{array} \qquad P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}$$

$P_{00}^{(4)} = ?$

$P^{(4)} = P^4 = (P^2)^2$

$P_{00}^{(4)} = [P^4]_{(1,1)}$

Q) Let $\{X_n, n=0,1,2,---\}$ be a markov chain with state space $\{0,1,2\}$, the initial probablity vector $P(0) = [1/4, 1/2\ 1/4]$ and one step transition probablity matrix P is given by

$$P = \begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{bmatrix}$$

1) Find $P(X_0=0, X_1=1, X_2=1)$

2) Find $P(X_2=2)$

A1) $P(X_0=0, X_1=1\ X_2=1) = P(X_2=1 \mid X_1=1)\, P(X_1=1 \mid X_0=0)\, P(X_0=2)$

from matrix → from matrix

$P(X_0=0) = 1/4$ → initial probablity matrix

$P(X_1=1 \mid X_0=0) = 3/4$

$P(X_2=1 \mid X_1=1) = 1/3$

$$\frac{1}{3} \times \frac{3}{4} \times \frac{1}{4} = \frac{1}{16}$$

$P(X_0=0, X_1=1, X_2=1) = 1/16.$

A2) $P(X_2=1) = \sum_{i \in S} P(X_0=i)\underbrace{P(X_2=1 \mid X_0=i)}_{\text{2-step transition prob. matrix}}$

$$P^2 = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$P(X_2=1) = \sum_i P(X_0=i) \quad P(X_2=1 \mid X_0=i)$$

initial probablity vector

$P(X_0=0), P(X_0=1), P(X_0=2)$

from matrix

$P(X_2=1 \mid X_0=0) = P^{(2)}_{01}$

$P(X_2=1 \mid X_0=1) = P^{(2)}_{11}$

$P(X_2=1 \mid X_0=2) = P^{(2)}_{12}$

3) Find $P(X_7=0 \mid X_5=0)$

only calculate 2 step transition probablity matrix

→ time homogenous → DTMC

Simple Random Walk

X_i → integer valued random variables
→ i.i.d

$X_i \quad i = 1, 2, 3, \ldots$

$$P(X_i = k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = -1 \end{cases}$$

cond. ✱

$$S_n = \sum_{i=1}^{n} X_i$$

$S_n = 3$ → $\{X_0 = 1, X_1 = 1, X_2 = -1, X_3 = 1, X_4 = 1\}$

Generalised random walk → 'k' otherwise 1

$$P(X_i = k) = \begin{cases} 1/2 & k = 1 \\ 1/2 & k = -1 \end{cases}$$

& systematic random walk

$\boxed{p = 1/2}$

Ex:

	A	B
Head	+~~2~~ 1	-~~2~~ 1
Tail	-~~2~~ 1	+~~2~~ 1

X_i : amount that person A earning at the ith game $(1, -1)$

$$S_n = \sum_{i=1}^{n} X_i$$

S_n : amount earned by person A at the end of nth game

This type of problem is a simple random walk.

Classification of States

Accessible: State i is accessible from state j

if $P_{ji}^{(n)} > 0$

$P_{ji}^{(n)} = 0$

Communicate:

$$i \longleftrightarrow j$$

we can access i from j or vice versa

Satisfies

(a) Reflexive $i \longleftrightarrow j$

(b) symmetric $i \rightleftarrows j$

(c) Transitive $i \longleftrightarrow j \mid j \longleftrightarrow k$, $\therefore i \longleftrightarrow k$

Class

$S \rightarrow$ state space $(\text{'}\Omega\text{'}, \text{'}E\text{'})$

Subset of state space such that every state in that class communicates with every other state in that class.

$\{0, \quad , \quad \}$

all states satisfy the same property inside a class.

Periodicity

* return state, $P_{ii}^{(n)} > 0$ for some $n \geq 1$

* The period d_i of a return state i

$d_i = g.c.d$ of $\{m : P_{ii}^{(m)} > 0\}$

* If $d_i = 1 \rightarrow$ aperiodic state.

Class

$S \rightarrow$ state space $(\text{'}\Omega\text{'}, \text{'}E\text{'})$

Subset of state space such that every state in that class communicates with every other state in that class.

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Periodicity

* return state, $P_{ii}^{(n)} > 0$ for some $n \geq 1$

* The period d_i of a return state i

$d_i = g.c.d$ of $\{m : P_{ii}^{(m)} > 0\}$

* If $d_i = 1 \rightarrow$ aperiodic state.

* If $d_i > 1 \rightarrow$ periodic

Closed ~~state~~ set of states

→ Set of states, such that no state outside this set can be reached from any state in this state.

→ $\{ \quad \} \rightarrow c$

↳ only one state, its called absorbing state.

↳ no other state is accessible from it

or

$P_{ii}^{(1)} = 1$

Irreducible Markov chain

$S = \{ \quad \}$

cannot partition the state space into more than 1 closed set.

if it can be partitioned → reducible MC

First Visit

Probablity mass function

$f_{jk}^{(n)}$ = Prob { state k for the first time at the nth step / system starts at state j initially

$$P_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} P_{kk}^{(n-r)}, \quad n \geq 1$$

$P_{kk} = 1$, $f_{jk}^{(0)} = 0$, $f_{jk}^{(1)} = P_{jk}^{(1)} = P_{jk}$

First Passage time distribution

F_{jk} = Prob(the system start with state j will ever reach state k)

$$F_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}$$

$F_{jk} < 1$ $\quad$ $1 - F_{jk}$ Possibility that we would never visit the state k if we start from state j

$F_{jk} = 1 \rightarrow$ possibility the we will always reach the state k if we start from the state j

Mean recurrence time (Mean first passage time)

$$\mu_{jk} = \sum_{n=1}^{\infty} n f_{jk}^{(n)}$$

* $j = k$

$f_{jj}^{(n)}$ = dist^n of the recurrence time of the state j

$$\mu_j = \mu_{jj} = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$$ → mean recurrance for the state j

$F_{jj} = 1$ → the return to the state j is certain.

Classification of States

(1) Recurrent state (or persistent)

state j if $F_{jj} = 1 \rightarrow$ always returns

$$F_{jj} = \sum_{n=1}^{\infty} f_{jj}^{(n)}$$

(a) Null recurrent:

$\mu_{jj} = \infty$, the first passage time is ∞

(b) Positive recurrent: If $\mu_{jj} < \infty$

Note: positive recurrent + aperiodic
= Ergodic state.

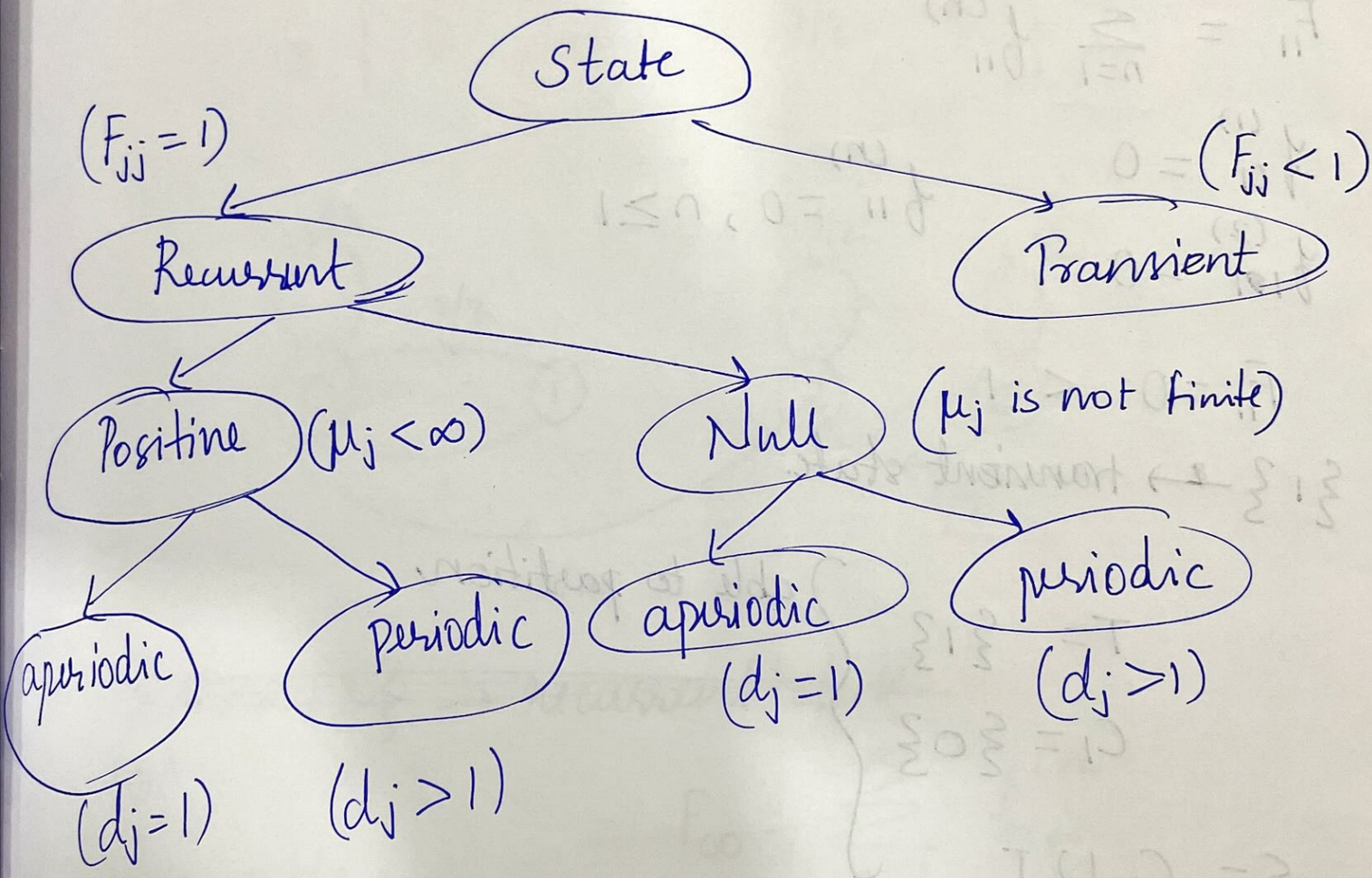
In MC → all states are ergodic state → ergodic MC

(2) Transient state

State j if $F_{jj} < 1$

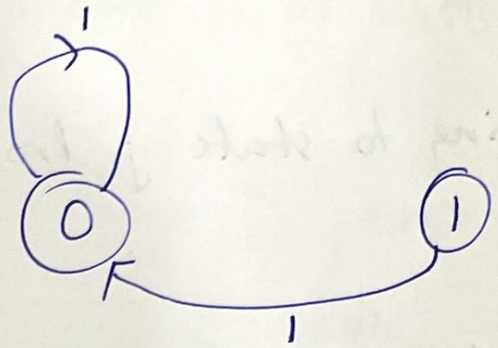
Probability of returning to state j from state j is <u>not certain</u>

Identify with state transition diagram



Ex: $P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}$ $\quad S = \{0, 1\}$

DTMC



$P_{00}^{(1)} = 1 \rightarrow \{0\} \rightarrow$ absorbing state

$$F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)}$$

$f_{11}^{(1)} = 0 \qquad f_{11}^{(n)} = 0, \; n \geq 1$

$f_{11}^{(2)} = 0$

$F_{11} = 0 < 1$

$\{1\} \rightarrow$ transient state.

$$\left.\begin{array}{l} T = \{1\} \\ C_1 = \{0\} \end{array}\right\} \text{able to partition}$$

$S = C_1 \cup T$

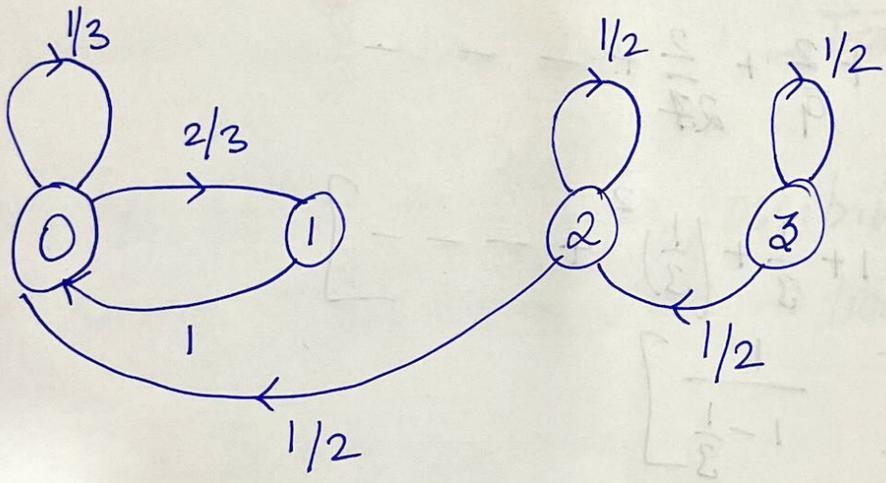
$\hookrightarrow$ reducible MC

DTMC

Ex:

$$S = \{0, 1, 2, 3\}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$



~~$\{0, 2, 3\} \rightarrow$ recurrent state~~

'0'

$f_{00}^{(1)} = 1/3$

$f_{00}^{(2)} = \frac{2}{3} \times 1 = \frac{2}{3}$

$f_{00}^{(3)} = 0$

first visit $f_{00}^{(4)} = 0$

$$\boxed{f_{00}^{(n)} = 0, \; n \geq 3}$$

$$F_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)}$$

$$= 2/3 + 1/3$$

$$\boxed{F_{00} = 1}$$

$\{0\} \rightarrow$ recurrent state.

$\{1\}$ is also a recurrent state as it is closed communicating with $\{0\}$

$$f_{11}^{(1)} = 0$$

$$f_{11}^{(2)} = 2/3$$

$$f_{11}^{(3)} = 1 \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$$

$$f_{11}^{(4)} = 1 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$$

$$F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)}$$

$$= \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots$$

$$= \frac{2}{3}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right]$$

$$= \frac{2}{3}\left[\frac{1}{1-\frac{1}{3}}\right]$$

$$= \frac{2}{3} \times \frac{3}{2} = 1$$

$\boxed{F_{11} = 1}$ → recurrent state

class → $\{0, 1\}$

'2'

$f_{22}^{(1)} = 1/2$

$f_{(22)}^{(2)} = 0$

$f_{(22)}^{(n)} = 0 \quad n \geq 2$

$F_{22} = 1/2 < 1$

$\{2\} \rightarrow$ transient state

'3'

$f_{33}^{(1)} = 1/2$

$f_{33}^{n} = 0, \quad n > 1$

$F_{22} = 0$

$\{3\} \rightarrow$ transient state.

From '0' we always can come back through every possible way, same for '1' $\rightarrow$ recurrent state

But from '2' and '3' there is a possibility that we dont return to the same state $\rightarrow$ transient state.

$S = \{0, 1\} \cup \{2, 3\}$

$\rightarrow$ reducible MC

Periodicity

'0'

$d_i = g\,c\,d\,\{m, P_{ii}^{(m)} > 0$

$d_0 = g.c.d.\ \{1, 2, 3, 4 - - -\}$

$\boxed{d_0 = 1}$

$\{0\} \rightarrow$ aperiodic + positive recurrent = ergodic state.

$\{1\} \rightarrow$ ergodic state $\rightarrow$ same closed communicating class.

Ex: DTMC $\longrightarrow$ countably infinite state s.

Let $\{Y_n, n = 1, 2, 3 \ldots\}$ be a sequence of indep. random variables with prob $\{Y_n = 1\} = p$

$= 1 - \text{Prob}\{Y_n = -1\}$

$n = 1, 2, 3 \ldots$

$0 < p < 1$

Let X_n be defined by

$$X_0 = 0, \quad X_{n+1} = X_n + Y_{n+1} \quad n = 0, 1, 2 \ldots$$

Question: check $\{X_n, n = 1, 2 \ldots\}$ is a DTMC and classify the states.

A. This is a simple random walk

$\{X_n, n = 1, 2, \ldots\}$

$X_0 = 0$

$X_{n+1} = X_n + Y_{n+1}$ Range of $Y_n = \{1, -1\}$

state space S of $\{X_n, n = 1, 2, \ldots\}$

$n = 0 \quad X_1 = X_0 + Y_1$

$= 0 + 1$ or $= 0 - 1$

$= 1$ $\quad = -1$

$S = \{0, \pm 1, \pm 2, \pm 3 \ldots\}$

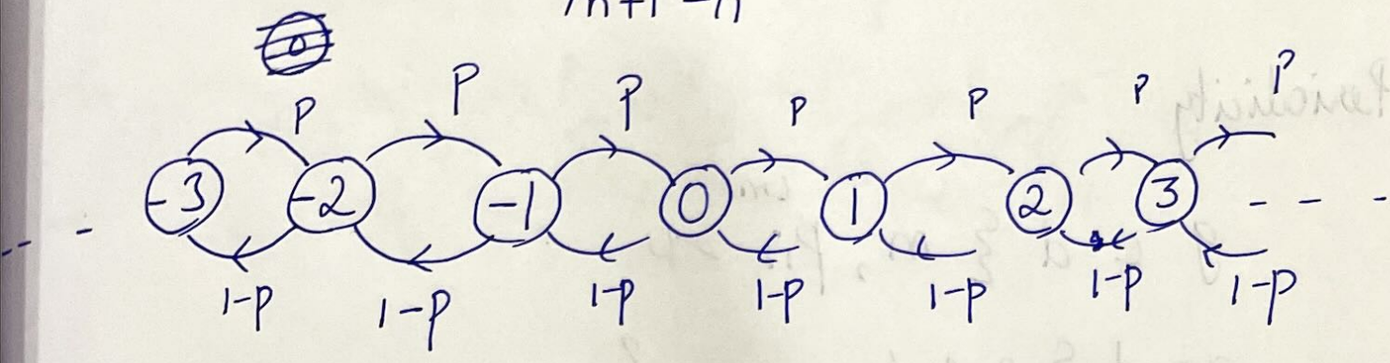
$\hookrightarrow$ countably infi

Given $P(Y_n = 1) = p$

$P(Y_n = -1) = 1-p \quad 0 \le p \le 1$

$n = 0$

$Y_{n+1} = X_1 \quad X_{n+1} = X_1$



X_{n+1} depends only on X_n & not on the previous X_{n-1}, X_{n-2} - - -

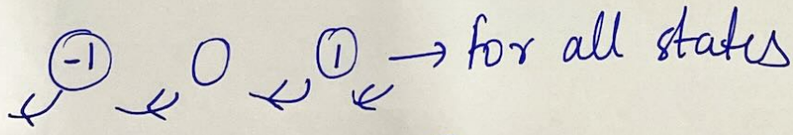
$\{X_n, n = 1, 2, - - - \}$ ~~st~~ satisfies Markov property

DTMC

classification of states

~~all of the states are recurrent~~

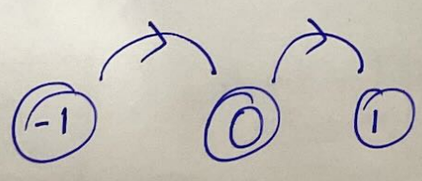
when $p = 0$,



→ for all states

$\{0, \pm 1, \pm 2, \pm 3 - - - \}$ → transient state.

when $p = 1$



$\{0, \pm 1, \pm 2, \pm 3 - - \}$ → transient state

when $0 < p < 1$

There is always a probability of coming

$F_{ii} = 1 \rightarrow$ recurrent state

Periodicity

$$g c d \{ m, p_{ii}^{(m)} > 0$$

$$= g c d \{ 2, 4, 6, - - - \}$$

$$= 2$$

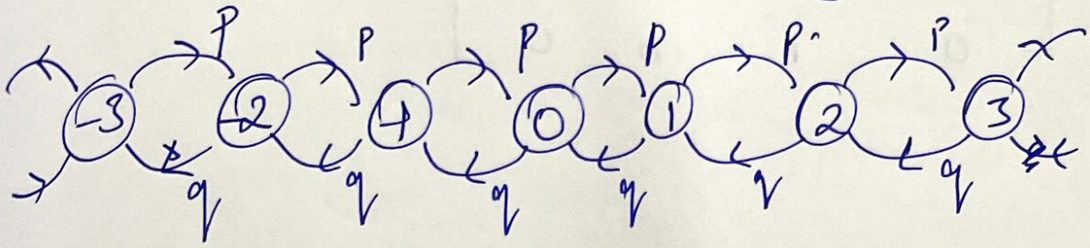
$$P = 1/3 \rightarrow \mu_{jj} = \sum n f_{jj}^{(n)}$$

* DTMC $\rightarrow$ irreducible MC

Q. systematic random walk

$P(X_1 = 1) = 0.5 \qquad P(X_1 = -1) = 0.5$

$S = \{0, \pm 1, \pm 2 \pm 3 \; ---\}$



$$P_{ij}^{(n)} = \begin{cases} {}^{n}C_{\frac{n+j-i}{2}} \; p^{\frac{n+j-1}{2}} \; q^{\frac{n-j+i}{2}}, & \text{if } n+j-i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$P(X_6 = 5 \mid X_0 = 1) = ?$

$P_{15}^{(6)} = {}^{6}C_5 (1/2)^2 (1/2)^1$

$$\boxed{P_{15}^{(6)} = \frac{6}{2^6}}$$

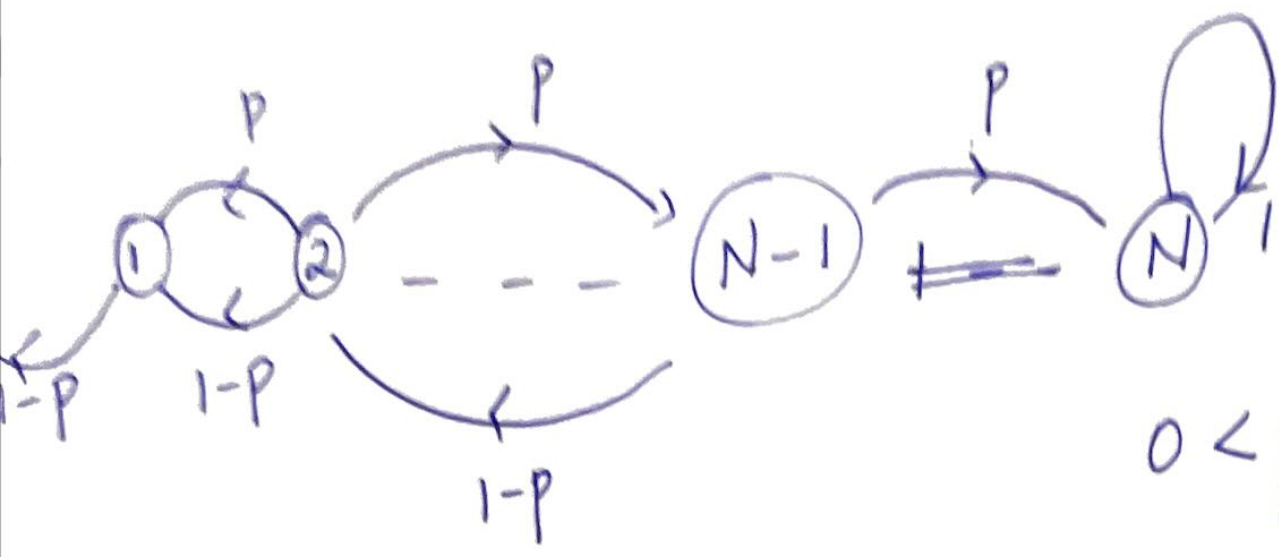
* $P(X_6 = 4 \mid X_0 = 1)$

$$\boxed{P_{14}^{(6)} = 0}$$

$n+j-1 = 10 = 9 \text{ (odd)}$

Ex: MC $S = \{0, 1, 2, 3\}$

$$P = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

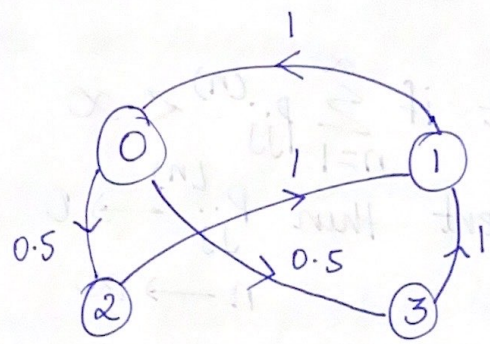


$0 < p < 1$

Q.

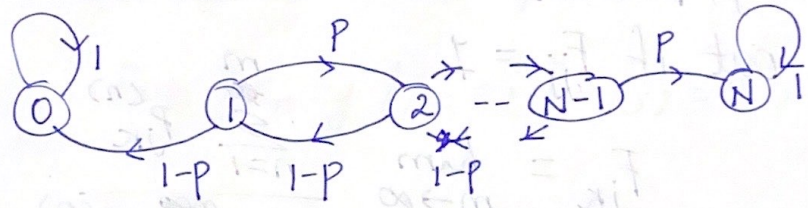
MC, $S = \{0, 1, 2, 3\}$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$



→ All states are recurrent.

Q.



$\{0, N\}$ → absorbing states

$\{1, 2, --- N-1\}$ → Transient states

DTMC is a reducible MC

Limiting distribution

Results:

(1) State j is recurrent if and only if

$$\sum_{n=1}^{\infty} P_{jj}^{(n)} = \infty$$

(2) State j is transient if $\sum_{n=1}^{\infty} P_{jj}^{(n)} < \infty$

Also if j is transient then $P_{jj}^{(n)} \longrightarrow 0$ as $n \longrightarrow \infty$

By Doeblin's formula we can find out if $F_{jj} = 1$

$$F_{jk} = \lim_{m \to \infty} \frac{\sum_{n=1}^{m} P_{jk}^{(n)}}{1 + \sum_{n=1}^{m} P_{kk}^{(n)}}$$

Particular case

$$F_{jj} = 1 - \lim_{m \to \infty} \frac{1}{1 + \sum_{n=1}^{m} P_{jj}^{(n)}}$$

(3) For an irreducible MC, all states are of the same type

(4) For a finite MC, atleast one state must be positive recurrent.

(5) For an irreducible finite MC, all states are +ve recurrent.

Limiting distribution

Limiting state probablities

→ what is the prob that starting from state i and reaches the state j as $n \to \infty$

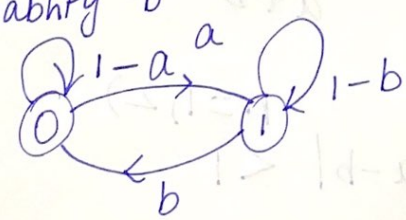
$$\lim_{n \to \infty} \text{Prob}\{X_n = j \mid X_0 = i\} \quad i, j = 0, 1, 2 \ldots$$

Weather forcast example

$0 \longrightarrow$ rains
$1 \longrightarrow$ does not rain

If it rains then it will rain tomorrow with the probablity $1-a$

If it does not rain today, it will rain tomorrow with probablity 'b'



$$\boxed{0 < a, b < 1}$$

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}$$

→ It is an irreducible finite MC

→ All states are +ve recurrent.

we can also write limiting state probablity as

$$\lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} P^{n}$$

$$P^2 = (\quad)$$

$$\vdots$$

$$P^n = (\quad)$$

$$P^n = \begin{pmatrix} \frac{b+a(1-a-b)^n}{a+b} & \frac{a-a(1-a-b)^n}{a+b} \\ \frac{b-b(1-a-b)^n}{a+b} & \frac{a+b(1-a-b)^n}{a+b} \end{pmatrix}$$

$n = 1, 2)$

provided $|1-a-b| < 1$

$$\therefore \lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} P^{n} = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

limiting probablity matrix

$$\pi = (\pi_0 \quad \pi_1)$$

$$\pi_0 = b/_{a+b} \qquad \pi_1 = a/_{a+b}$$

→ For a long run $(n\to\infty)$, the prob of $0 = b/_{a+b}$ and prob $1 = a/_{a+b}$

→ Its independent of the initial state.

Ergodicity

It is a necessary and sufficient condition for the existence of $\{V_j\}$ satisfying

$$V_j = \sum V_j P_{ij} \; ; \; \sum V_j = 1$$ in case of an irreducible and aperiodic MC.

Then we say that the system is ergodic.

Stationary Distribution

The vector π is called a stationary dist. of the discrete time MC if

$$\pi = (\pi_0 \ \pi_1 \ \pi_2 \ ----) \text{ satisfies}$$

(1) $\pi_j \geq 0 \quad \forall j$

(2) $\sum \pi_j = 1$

(3) $\pi = \pi p$ where $p \rightarrow$ one step T. P. M

Note: Irreducible aperiodic positive recurrent M.C then the limiting dist. is same as the stationary dist.

limiting dist = stationary dist^n

for the weather example.

Some important results:

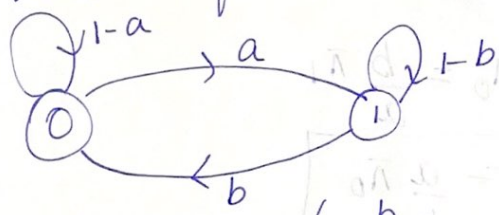
(1) For an irreducible aperiodic +ve recurrent m.c the stationary dist exist and it is unique

Then π is uniquely determined by

$$\pi = \pi P$$

i.e $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ along with $\sum \pi_j = 1$.

Weather forcast problem



limiting dist = $\left(\frac{b}{a+b}, \frac{a}{a+b}\right)$

$\pi = (\pi_0, \pi_1)$

Stationary dist

1) $\sum \pi_i = 1$

2) $\pi = \pi p$.

$$\pi_0 \quad \pi_1 = (\pi_0 \quad \pi_1)\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

sys of eq.

$\pi_0 = \pi_0(1-a) + \pi_1 b$ —— (1)

$\pi_1 = \pi_0 a + \pi_1(1-b)$ —— (2)

Eq ①

$$\cancel{\pi_0} = \cancel{\pi_0} - \pi_0 a + \pi_1 b$$

$$\boxed{\pi_1 = \frac{a}{b}\pi_0} \quad — ③$$

Eq ②

$$\cancel{\pi_1} = \pi_0 a + \cancel{\pi_1} - \pi_1 b$$

$$\cancel{\pi_1 = a}$$

$$\pi_0 a = \pi_1 b$$

$$\cancel{\pi_0 = \frac{b}{a}} \times \quad \pi_0 = \frac{b}{a}\pi_1$$

$$\boxed{\pi_1 = \frac{a}{b}\pi_0}$$

& $\sum \pi_i = 1$ (use to avoid trivial solution.)

$$\pi_0 + \pi_1 = 1$$

$$\pi_0 + \frac{a}{b}\pi_0 = 1$$

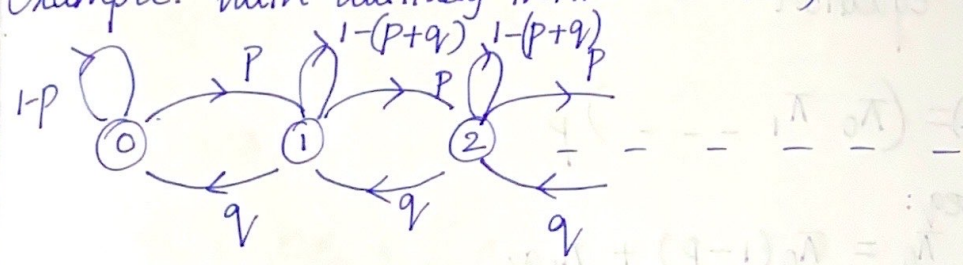
$$\pi_0 = \frac{1}{\left(1 + \frac{a}{b}\right)}$$

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b}\right) \quad \boxed{\pi_0 = \frac{b}{a+b}}$$

same as limiting distribution

$$\boxed{\pi_1 = \frac{a}{a+b}}$$

Example: with countably infinite states)



$0 < p < 1$

$0 < q < 1$

$S = \{0, 1, 2, ----\}$

→ irreducible

$d_i = g.c.d(m, P_{ii}^{(m)} > 0)$

$d_i = g.c.d(1, 2, 3 ----)$

$\boxed{d_i = 1}$

→ Aperiodic

→ Assume that all the states are +ve recurrent

→ ~~By the result~~ So we have irreducible aperiodic positive recurrent MC. so by the result stationary distribution exists and is unique

$\pi = \pi P \quad , \sum \pi_i = 1$

where $\pi = (\pi_0 \ \pi_1 \ \pi_2 \ ----)$

$P \rightarrow$ one step transition prob matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & - & - & - & - & - \end{array} \\ \begin{bmatrix} 1-p & p & 0 & 0 & - & - & - & - & - \\ q & 1-p+q & p & 0 & 0 & - & - & - & - \\ 0 & q & 1-p+q & p & - & - & - & - & - \\ & \ddots & & \ddots & & & & & \end{bmatrix} \end{array}$$

First Equation

$$(\pi_{0\ldots}) = (\pi_0\ \pi_1\ \ldots)\,P$$

First eq:

$$\pi_0 = \pi_0(1-p) + \pi_1 q$$

$$\cancel{\pi_0} = \cancel{\pi_0} - \pi_0 p + \pi_1 q$$

$$\boxed{\pi_1 = p/q\ \pi_0} \quad \text{—— (1)}$$

second eq:

$$\pi_1 = \pi_0(p) + \pi_1(1-p-q) + \pi_2 q$$

$$\cancel{\pi_1} = \pi_0 p + \cancel{\pi_1} - p\pi_1 - q\pi_1 + \pi_2 q$$

$$\pi_1(p+q) = \pi_0 p + \pi_2 q$$

$$\pi_2 = \frac{p^2}{q^2}\pi_0 \qquad \text{using (1)}$$

III^{ly} using 3rd eq

$$\pi_3 = \frac{p^3}{q^3}\pi_0$$

$$\pi_n = \frac{p^n}{q^n}\pi_0$$

Find π_0

$\sum \pi_i = 1$

$\pi_0 + \pi_1 - - - - = 1$

$\pi_0 + \frac{p}{q}\pi_0 + \frac{p^2}{q^2}\pi_0 - - - = 1$

$\pi_0 \left(1 + p/q + p^2/q^2 - - -\right) = 1$

$$\pi_0 = \frac{1}{\left(1 + p/q + p^2/q^2 + - - -\right)}$$

If $p/q < 1$, then $1 + p/q + p^2/q^2 + - - -$ converges

↳ why.

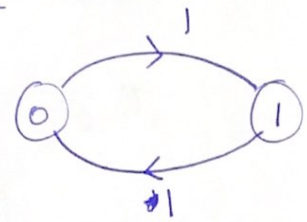
$\pi_n = \frac{p^n}{q^n}\pi_0$, provided $p/q < 1$

$\pi_n = (p/q)^n \pi_0$

$$S = 1\left(\frac{(p/q)^n}{\left(p/q - 1\right)}\right)$$

DOUBT

Example



$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

row sum = 1 (always)

here column sum = 1

$\hookrightarrow$ doubly stochastic matrix.

DTMC $\rightarrow$ irreducible

$d_0 = 2 \quad d_1 = 2$

By the result

Both states are +ve recurrent.

Result:

Consider a discrete time MC with

$P = [P_{ij}]$

$$\underbrace{\sum_{i=1}^{\infty} P_{ij}}_{\text{column sum.} = 1} = 1 \quad j = 1, 2, \ldots n$$

If a finite irreducible DTMC has a doubly stochastic matrix then all the stationary probablities are equal.

$$\pi = (\pi_0 \ \pi_1 \ - - - \ \pi_n)$$

$$\boxed{\pi_j = 1/n} \quad j = 1, 2, 3 \ - - - n$$

for doubly stochastic matrix

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

$\therefore$ Stationary distribution

$$\pi = (\pi_0 \ \pi_1)$$

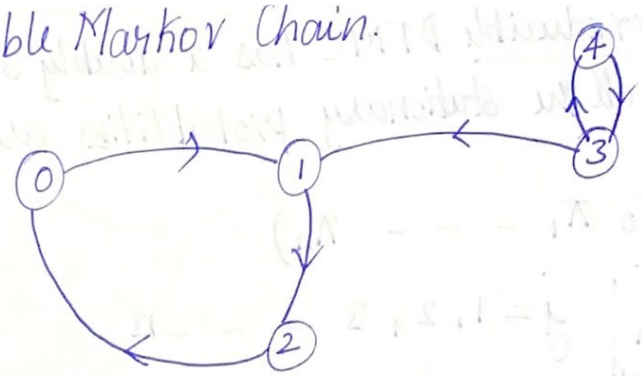
$$\pi = (1/2 \ \ 1/2)$$

limiting state probablity

$$\lim_{n \to \infty} P^n \text{ does not exist.}$$

Reducible Markov Chain.

→



$S = \{0, 1, 2, 3, 4\}$

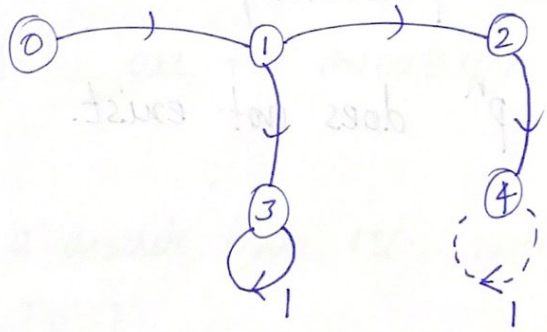
$C_1 = \{0, 1, 2\}$ → recurrent

T ~~C_2~~ $= \{3, 4\}$ → transient

$S = C_1 \cup T$

Reducible MC with one closed communicating class

→



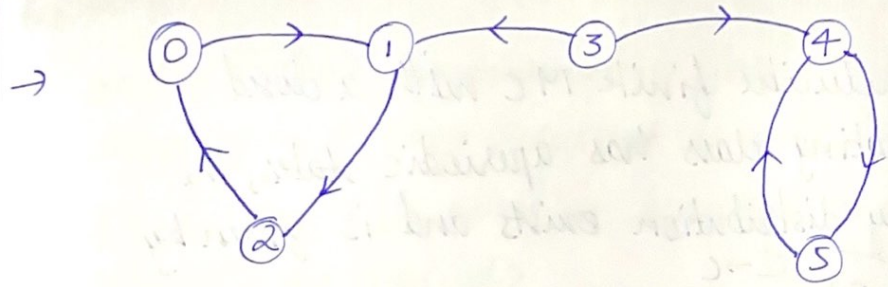
They may not give the self loop

we have 2 absorbing states.

$C_1 = \{3\}$ $C_2 = \{4\}$

$T = \{0, 1, 2\}$

Type 2: Reducible MC with one or more absorbing states.



$C_1 = \{0, 1, 2\}$

$C_2 = \{4, 5\}$

$T = \{3\}$

Type 3: Reducible MC more than one closed communicating class of states

---

Stationary distribution for first type (Type 1)

one closed communicating class

$S = C \cup T$

Further assumptions: 1) S with finite states

2) The states of C are aperiodic

Canonical form $P = \begin{array}{c|cc} & C & T \\ \hline C & P_1 & 0 \\ T & R_1 & Q \end{array}$ ← zero matrix

Re-arrange the 1-step tran. prob. matrix to the canonical form.

Result 1:

For a reducible finite MC with a closed communicating class has aperiodic states, the stationary distribution exists and is given by

$$V = (V_1, 0) \quad C \to C$$

$\uparrow$

Ergodic theorem for the reducible MC

~~Result 2:~~

The n-step transition probability matrix for reducible finite MC

$$P^n = \begin{array}{c c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{pmatrix} P_1^n & 0 \\ R_n & Q^n \end{pmatrix} \end{array}$$

$P_1^n \to$ stochastic sub-matrix

$Q^n \to$ submatrix for $T \to T$

$R_n \to$ function of n

$0 \to$ zero matrix

As $n \to \infty$, $P_1^n \to eV_1$

$Q^n \to 0$

$\therefore$ Stationary dist^n $V = (V_1, 0)$

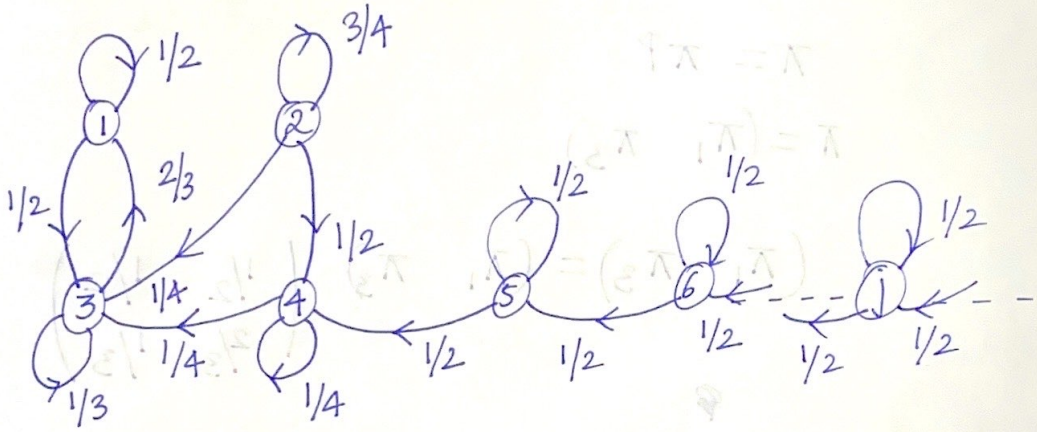
Result 2:

with a closed communicating class c (one more assumption: positive recurrent is added)

Assume that c is positive recurrent and the states of c are aperiodic

Then $\lim_{n\to\infty} P_{ij}^{(n)} = \begin{cases} \gamma_j, & j \to \text{positive recurrent} \\ 0, & j \to \text{transient} \end{cases}$

is independent of initial state i



$C = \{1, 3\}$

$T = \{2, 4, 5, 6 - - - -\}$

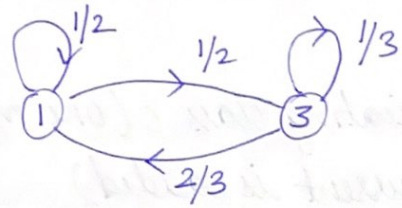
$S = C \cup T$

Type - 1

→ aperiodic

→ positive recurrent

To find stationary distribution, only focus on closed communicating class



$$P_1 = \begin{array}{c|cc} & 1 & 3 \\ \hline 1 & 1/2 & 1/2 \\ 3 & 2/3 & 1/3 \end{array}$$

$\cancel{\pi P_1 = P_1}$

$\sum \pi_i = 1$

$\pi = \pi P$

$\pi = (\pi_1 \quad \pi_3)$

$$(\pi_1 \quad \pi_3) = (\pi_1 \quad \pi_3) \begin{pmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{pmatrix}$$

$$(\pi_1 \quad \pi_3) = \left(\frac{\pi_1}{2} + \frac{2}{3}\pi_3 \quad \frac{\pi_1}{2} + \frac{\pi_3}{3}\right)$$

$$\pi_1 = \frac{\pi_1}{2} + \frac{2}{3}\pi_3 \rightarrow \boxed{\frac{\pi_1}{2} = \frac{2}{3}\pi_3}$$

$$\pi_3 = \frac{\pi_1}{2} + \frac{\pi_3}{3} \qquad \frac{2\pi_3}{3} = \frac{\pi_1}{2}$$

$$\cancel{\pi_1 = \frac{\pi_1}{2} + \frac{\pi_1}{2} + \frac{2}{3}\pi} \qquad \pi_1 = \frac{4}{3}\pi_3$$

$$\boxed{\pi_1 + \pi_3 = 1}$$

$$\pi_1 = 1 - \pi_3$$

$$1 - \pi_3 = \frac{4}{3}\pi_3$$

$$1 = \frac{7}{3}\pi_3$$

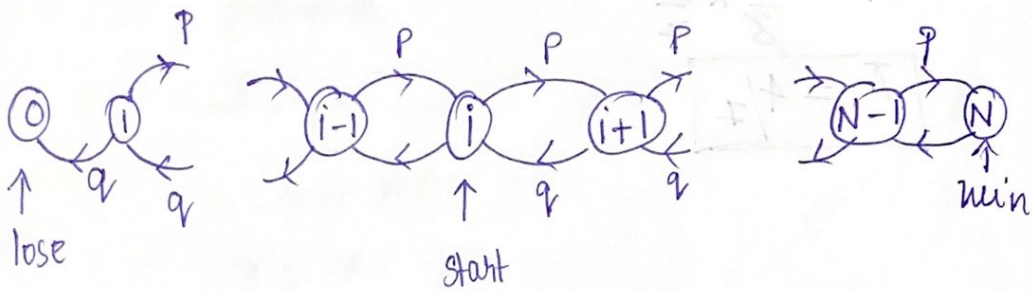
$$\boxed{\pi_3 = 3/7}$$

$$\pi_1 = \frac{4}{\cancel{3}} \times \frac{\cancel{3}}{7}$$

$$\boxed{\pi_1 = 4/7}$$

Gambler's Ruin problem.

Consider a gambler, who at each play of the game has probablity 'p' of winning 1 unit & probablity '$q = 1-p$' of losing 1 unit. Assuming successive plays of the game are independent, what is the probablity that, starting with 'i' units the gambler's fortune will reach N ~~before 0~~ reaching 0.



$\{0\}, \{N\} \rightarrow$ absorbing state

$\{1, 2, \ldots N-1\} \rightarrow$ transient state.

Let X_n denote the player's fortune at time n

then $\{X_n, n = 0, 1, 2 \ldots\} \rightarrow$ DTMC with transition probabilities $P_{00} = P_{NN} = 1$

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 1, 2, \ldots N-1$$

Let P_i denote the probability that, starting with i, the gambler's fortune will eventually reach N

$P_0 = 0 \qquad P_N = 1$

cant start with 0

for $1 \le i \le N-1$ $\boxed{P_i = p P_{i+1} + q P_{i-1}}$

$p+q=1$

$$P_i q_1 + q P_i = p P_{i+1} + q P_{i-1}$$

$$P_{i+1} - P_i = q/p \, (P_i - P_{i-1}) \quad , \; i = 1, 0, \ldots N-1$$

using $P_0 = 0$ for $i = 1, 2, \ldots, N-1$

$i=1$ & $P_2 - P_1 = q/p \, (P_1 - P_0)$

$$P_2 - P_1 = q/p \, (P_1)$$

$$P_2 = (q/p + 1) P_1$$

$i=2$ & $P_3 - P_2 = q/p \, (P_2 - P_1)$

$$P_3 - P_2 = (q/p)^2 P_1$$

$\vdots$

$i = i-1$ & $P_i - P_{i-1} = q/p \, (P_{i-1} - P_{i-2} = (q/p)^{i-1} P_1$

$\vdots$

$i = N-1$ & $P_N - P_{N-1} = q/p^{N-1} P_1$

Adding the first $i-1$ of these equations &

$$P_i - P_1 = P_1 [q/p + (q/p)^2 + \ldots + (q/p)^i]$$

$$P_1 = \begin{cases} \dfrac{1-(q/p)^2}{1-q/p}, & \text{if } q/p \neq 1 \\ i P_1 & \text{if } q/p = 1 \end{cases}$$

using $P_N = 1$ ✓

$$P_1 = \begin{cases} \dfrac{1-q/p}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ 1/N & \text{if } p = 1/2 \end{cases}$$

P_i

What would be the expected wealth at the end of n games

$$0 \cdot (1 - P_i) + N \cdot P_i = N P_i$$

$$= \begin{cases} N \cdot i/n = i & \\ N \cdot \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p = 1/2 \\ & \text{if } p = 1/2 \end{cases}$$

How long does the gambler expect to stay in the game?

Let μ_i denote the mean no. of games that must be played until gambler either goes broke or wins complete fortune N in the game given that it starts with N i.e expected no. of plays

For $i = 0, N$ $\mu_i = 0$

$1 \leq i \leq N-1$ $\mu_i = 1 + p\mu_{i+1} + q\mu_{i-1}$

In general

$$\mu_i = 1 + \sum P_{ij} \mu_j$$

$$\mu_i = \begin{cases} i(N-i), & p = q = 1/2 \\ \dfrac{i-N}{q-p} \dfrac{1-(q/p)^i}{1-(q/p)^N} & p \neq q \end{cases}$$

if $r = q/p$

$$\mu_i = \begin{cases} i(N-i) & p = q = 1/2 \\ \left(\dfrac{r+1}{r-1}\right)\left(i - N\dfrac{1-r^i}{1-r^N}\right), & p \neq q \end{cases}$$

$\mu_{ij} = E[\text{No. of transitions into state } j \mid \text{starts in } i]$

$= f_{ij}\, \mu_{jj}$

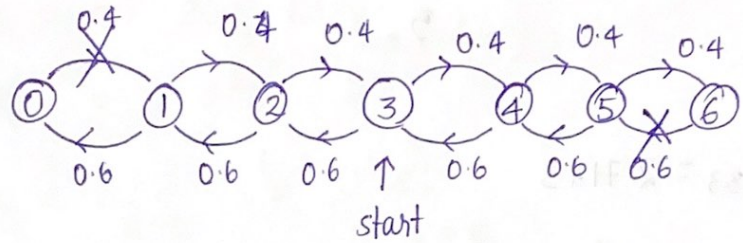
where f_{ij} - Probablity of ever making a transition into state j given that the chain starts ~~with~~ in state i

μ_{jj} - expected no. of time periods spent in state j

Q. Consider the gambler's ruin problem with $p = 0.4$ and $n = 6$. Starting in state 3, determine

a) The expected amount of time spent in state 3

b) The expected no. of visits to state 2



$$
\begin{array}{c|ccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 \\
2 & 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 \\
3 & 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 \\
4 & 0 & 0 & 0 & 0.6 & 0 & 0.4 & 0 \\
5 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0.4 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

This matrix can be rewritten for 1st state to 5th state leaving the 0th and 6th state as they are absorbing states, the question does not really focus on them.

$$[\mu_{ij}] = [I - Q]^{-1} = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1.5865 & 0.9774 & 0.5714 & 0.3008 & 0.1203 \\ 2 & 1.4657 & 2.4436 & 1.4686 & 0.7519 & 0.3008 \\ 3 & 1.2857 & 2.1429 & 2.7143 & 1.4286 & 0.5714 \\ 4 & 1.0150 & & & & \\ 5 & & & & & \end{array}$$

a) $\mu_{33} = 2.7143$

b) $\mu_{32} = 2.1421$

c) Probablity of ever visiting state 4

$$f_{3,4} = \frac{\mu_{3,4}}{\mu_{44}} = \frac{1.4286}{2.4436} = 0.5846.$$

9 - 10
10:30 - 12:30 } class

1 - 4:30 → Data Structures (Trees)
5:30 - 7:00 → Gym
8:30 - 11:30 → DAA

Mon - DBMS
Tue - DAA
Wed - DCN
Thurs - ML
Fri - Maths

Some more examples for finding stationary dist.

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \beta & 1-\beta \end{array}$$

$\pi \cdot P = \pi \qquad \pi = (\pi_0, \pi_1)$

$\sum \pi_1 = 1$

$\pi_0 = \beta/_{\alpha+\beta} \qquad \pi_1 = \alpha/_{\alpha+\beta}$

$$\pi = \left[\beta/_{\alpha+\beta} , \alpha/_{\alpha+\beta} \right]$$

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

$$\pi_0 , \pi_1 = \pi_0 \ \pi_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\pi_0 = \pi_0 \qquad \pi_0 = 1 - \pi_1$

$\pi_1 = \pi_1$

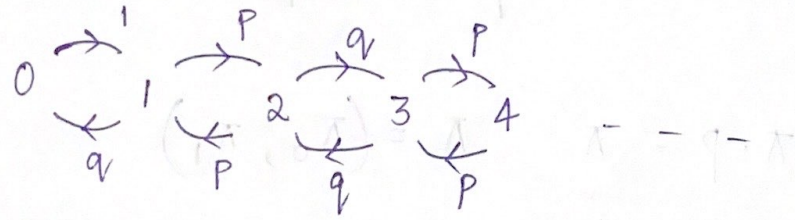
acc to π_0, π_1 can be decided or vice versa

$\pi = (a, 1-a) \qquad a \in [0, 1]$

Consider MC with following transition diagram

stationary dist.

$p+q=1 \quad 0<p<1$



$\pi P = \pi \qquad \sum \pi_i = 1$

$\pi_0 = q\pi_1$ — ① $\qquad \pi_1 = \pi_0 + p\pi_2$

$= q\pi_1 + p\pi_2$

$\pi_1(1-q) = p\pi_2$

$\pi_1 = \pi_2$

|||ly

$\pi_3 = \pi_4 = \pi_5 \ldots$

$\sum_{i=0}^{\infty} \pi_i = 1$

if $\pi_i > 0$ not possible.

$\pi_i = 0$

not possible.

stationary dist does not exist.

* If $\{X_n \; n \geq 0\}$ is a M.C with finite state space, then there exist atleast one stationary dist

*

→ For an irreducible MC, if state j is the recurrent, then as $n \to \infty$

(i) $P_{jj}^{(n)} \to t/\mu_{jj}$ state j is periodic with period t

(ii) $P_{jj}^{(n)} \to 1/\mu_{jj}$ state j is aperiodic

$P_{jj}^{(n)} \to 0$ when j is transient

* In case of state j is null recurrent

$\mu_{jj} = \infty$

then $P_{jj}^{(n)} \to 0$ as $n \to \infty$

→ Continuous time Markov Chain

A discrete state continuous time stochastic process $\{X(t), t \geq 0\}$ is called CTMC if $0 \leq t_0 < t_1 < t_2 \ldots < t_n < t$, it is conditional distribution satifies

$$P(X(t) \leq x \mid X(t_0) = x_0, X(t_1) = x_1 \ldots x(t_n) = x_n]$$

$$= P[X(t) \leq x \mid X(t_n) = x_n]$$

★ Notation

- $P_{ij}(t) = \text{Prob}[X(t) = j \mid X(0) = i]$
  — Stationary transition probability

- $P_{ij}(t) = P[X(t+T) = j \mid X(T) = i]$ for any $T \geq 0$ does not depend on T
  — Time homogeneous

- $\pi_j(t) = \text{Prob}[X(t) = j]$
  — Unconditional Probability

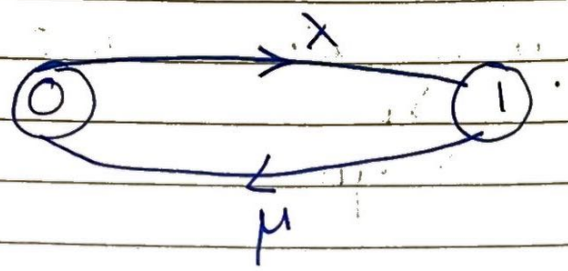
  What is the probability that the system was in state j at time 't'.

Intial state probability vector

$$\pi(0) = [\pi_0(0), \pi_1(0), \pi_2(0) \ldots]$$

where $\pi_j(0) = \text{Prob}[X(0) = j]$

→ Eg 1



Note :: Mean of exponential distance is $\frac{1}{\text{parameter}}$

· average time spent in state 0 before moving into state 1 $= \frac{1}{\lambda}$

The average time spent in state 1 before moving into state 0 $= \frac{1}{\mu}$

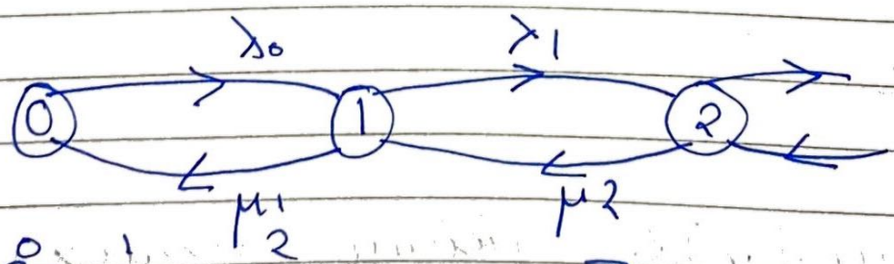
$$Q = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & -\lambda & \lambda \\ 1 & \mu & -\mu \end{array} \quad \left.\begin{array}{l} \Sigma = 0 \\ \Sigma = 0 \end{array}\right\} \text{Infinite decimal Generator matrix}$$

$Q = [q_{ij}]$

1) $q_{ij} \geq 0 \quad , j \neq i$

2) $q_{ii} = -\sum_{i \neq j} q_{ij}$

→ Eg 2



$$Q \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & \end{array} \\ \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 \quad 0 \cdots \\ & & & \end{bmatrix} \end{array}$$

· Probability that the system moving into the state 2 before moving into state 0

$$\frac{\lambda_1}{\lambda_1 + \mu_1}$$

· Probability that the system moving into state 0 before moving into state 2 = $\frac{\mu_1}{\lambda_1 + \mu_1}$

→ Infinitesimal Generator Matrix (or Rate Matrix)

$Q = [q_{ij}]$

such that 1) $q_{ij} \geq 0, \quad i \neq j$

2) $q_{ii} = -\sum_{j \neq i} q_{ij}$

Define

$q_{ij} = \frac{d}{dt} P_{ij}(t)$

$q_{ii} = \frac{d}{dt} P_{ii}(t) \Big|_{t=0}$

→ Stationary distributions of CTMC

A vector π is called the stationary dist of the CTMC if $\pi (\pi_0, \pi_1 \ldots)$ ~~sto~~ satisfies

i) $\pi_j \geq 0$

ii) $\sum_j \pi_j = 1$

iii $\pi Q = 0$

· Result

For an irreducible positive recurrent CTMC the stationary distribution π exist and it's unique.

The vector $\pi = (\pi_0, \pi_1, \ldots)$ uniquely determined by

$$\pi Q = 0 \; ; \quad \sum \pi_j = 1$$

→ Queuing Model
. Railway tickets