Two-dimensional random variables

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Two-dimensional random variable

Defn: Let 'E' be an experiment & 'S' a sample space associated with E. Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcome $s \in S$. We call $\boxed{(X, Y)}$ a two-dimensional random variable (or random vector)

2 selected

12 [3 R, 4 W, 5 G]

X : no. of red marbles

Y : no. of white marbles

Z : no. of green marbles

(X, Y)

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n-dimensional random variable

If $X_1 = X_1(s)$, $X_2 = X_2(s)$, ..., $X_n = X_n(s)$ are 'n' functions each assigning a real number to every outcome $s \in S$, we call $(X_1, X_2, \ldots, X_n)$ an n-dimensional random variable (or an n-dimensional random vector).

$(X_1, X_2, \ldots, X_n)$

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Discrete & continuous two-dimensional random variables

* (X,Y) is a $\boxed{\text{two-dimensional discrete random variable}}$ if the possible values of (X,Y) is finite or countably infinite. That is, the possible values of (X,Y) may be represented as (x_i, y_j), $i = 1, 2, \ldots, n, \ldots$; $j = 1, 2, \ldots, m, \ldots$

12 $\left[\begin{array}{l} 3\,R \\ 4\,W \\ 5\,G \end{array}\right]$ — 2 selected

X : no. of red marbles

Y : no. of green marbles

$(X,Y) : \{(0,0), (0,1), (1,0), (1,1), (0,2), (2,0)\}$

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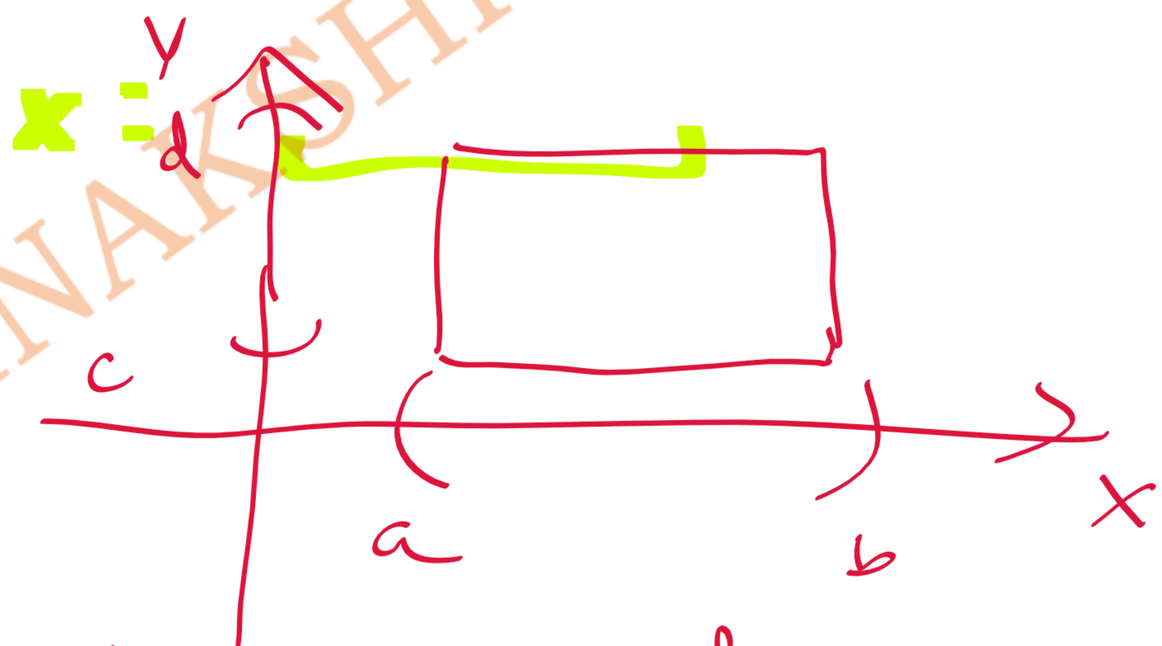
Discrete & continuous two-dimensional random variables

* (X,Y) is a $\boxed{\text{two-dimensional continuous random variable}}$ if (X,Y) can assume all values in some noncountable set of the Euclidean plane.

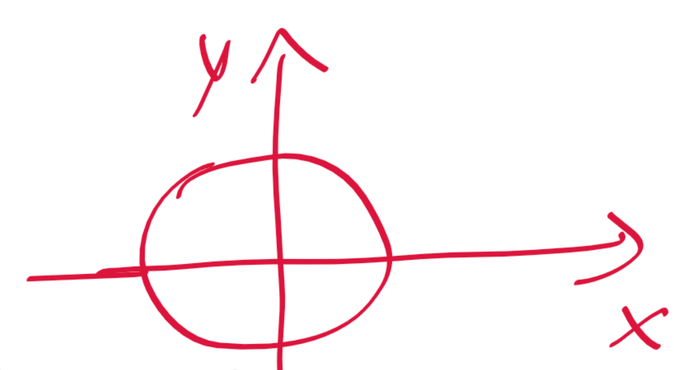
(X,Y)

x :



(i) $(X,Y) = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

$(X,Y) = \{(x,y) \mid x^2+y^2 \le 1\}$



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Probability distribution of (X,Y)

Let (X,Y) be a two dimensional discrete random variable. With each possible outcome (x_i, y_j) we associate a number $p(x_i, y_j)$ representing $P(X=x_i, Y=y_j)$ and satisfying the following conditions:

(a) $p(x_i, y_j) \geq 0$ for all (x, y)

$p(x_i, y_j) = P[X=x_i, Y=y_j]$

(b) $$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p(x_i, y_j) = 1$$

The function p defined for all (x_i, y_j) in the range space of (X,Y) is called the **probability function of (X,Y)**.

The set of triples $(x_i, y_j, p(x_i, y_j))$, $i, j = 1, 2, \ldots$ is called the **probability distribution of (X,Y)**.

$y_j \backslash x_i$	0	1	2
0	$p(0,0)$	$p(1,0)$	$p(2,0)$
1			
2			

Joint probability density function (joint pdf) of (X,Y)

Let (X,Y) be a continuous random variable assuming all values in some region R of the Euclidean plane.

(X,Y)

The $\boxed{\text{joint probability density function}}$ 'f' is a function satisfying the following conditions:

$f(x,y)$

(a) $f(x,y) \geq 0$ for all $(x,y) \in R$

(b) $\iint_R f(x,y)\, dx\, dy = 1.$

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Cumulative distribution function (cdf) F of (X,Y)

Let (X,Y) be a two-dimensional random variable. The cumulative distribution function (cdf) 'F' of two-dimensional r.v (X,Y) is defined by

$$F(x,y) = P(X \le x, Y \le y)$$

$$F(x,y) = P(X \le x, Y \le y)$$

(X,Y) — joint pdf $f(x,y)$

$$\boxed{\frac{\partial^2 F(x,y)}{\partial x \, \partial y} = f(x,y)}$$

whenever $F(x,y)$ is differentiable.

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Marginal probability distributions

Let (X, Y) be a two-dimensional r.v.

Discrete case:

(a) The marginal probability distribution of X, $\boxed{p(x_i) = P(X = x_i)}$

$$= P(X = x_i, Y = y_1 \text{ or } X = x_i, Y = y_2 \text{ or } \ldots)$$

$$p(x_i) = \sum_{j=1}^{\infty} p(x_i, y_j)$$

The function p defined for $x_1, x_2, \ldots$ represents the $\boxed{\text{marginal pdf of } X}$

$$p(x_i) = P[X = x_i] = P[X = x_i, Y = y_1 \text{ or } X = x_i, Y = y_2 \text{ or } \ldots]$$

$$= \sum_{j=1}^{\infty} p(x_i, y_j)$$

Marginal pdf of X and Y

(b) The marginal probability distribution of Y,

$$\boxed{q(y_j) = P(Y = y_j)}$$

$$= P[X = x_1, Y = y_j \text{ or } X = x_2, Y = y_j \text{ or } \ldots]$$

$$q(y_j) = \sum_{i=1}^{\infty} p(x_i, y_j)$$

The function q defined for $y_1, y_2, \ldots$ represents the $\boxed{\text{marginal pdf of } Y.}$

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Marginal pdf of X and Y

Continuous case :

Let $f(x,y)$ be the joint pdf of the continuous two-dimensional random variable (X,Y)

(a) The marginal probability density function of X is defined as

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \rightarrow f_1(x)$$

(b) The marginal probability density function of Y is defined as

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

$$\rightarrow f_2(y)$$

(X, Y) — 2d. c. r. v $\qquad f(x,y)$ — joint pdf

$$P[c \le X \le d] = P[c \le X \le d, \ -\infty < Y < \infty]$$

$$= \int_{x=c}^{d} \left[\int_{y=-\infty}^{\infty} f(x,y)\, dy \right] dx$$

$$\boxed{P[c \le X \le d] = \int_{x=c}^{d} g(x)\, dx}$$

III^{ly}

$$\boxed{P[a \le Y \le b] = \int_{y=a}^{b} \left[\int_{x=-\infty}^{\infty} f(x,y)\, dx \right] dy = \int_{y=a}^{b} h(y)\, dy}$$

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Conditional probability

(X, Y) — 2.d.r.v

(i) **Discrete Case:**

The probability of $X = x_i$ given that $Y = y_j$ is

$$p(x_i | y_j) = P[X = x_i \mid Y = y_j]$$

$$= \frac{p(x_i, y_j)}{q(y_j)} \quad \text{if } q(y_j) > 0$$

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The probability of $Y = y_j$ given that $X = x_i$ is

$$q(y_j \mid x_i) = P[Y = y_j \mid X = x_i]$$

$$= \frac{p(x_i, y_j)}{p(x_i)} \quad \text{if } p(x_i) > 0$$

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NOTE

(i) $p(x_i | y_j) \geq 0$

(ii) $$\sum_{i=1}^{\infty} p(x_i | y_j) = \sum_{i=1}^{\infty} \frac{p(x_i, y_j)}{q(y_j)}$$

$$= \frac{1}{q(y_j)} \sum_{i=1}^{\infty} p(x_i, y_j)$$

$$= \frac{1}{q(y_j)} \cdot q(y_j)$$

$$\therefore \sum_{i=1}^{\infty} p(x_i | y_j) = 1$$

(i) $q(y_j | x_i) \geq 0$

(ii) $$\sum_{j=1}^{\infty} q(y_j | x_i) = 1$$

$$\left[\because \sum_{j=1}^{\infty} q(y_j | x_i) = \sum_{j=1}^{\infty} \frac{p(x_i, y_j)}{p(x_i)} \right.$$

$$= \frac{1}{p(x_i)} \sum_{j=1}^{\infty} p(x_i, y_j)$$

$$\left. = \frac{1}{p(x_i)} p(x_i) = 1 \right]$$

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(ii) **Continuous case**:

(X, Y) — 2.d.c.r.v

$f(x, y)$ — joint pdf of (X, Y)

$g(x)$ — marginal pdf of X

$h(y)$ — marginal pdf of Y

The conditional pdf of X for given $Y = y$

$$\boxed{g(x|y) = \frac{f(x,y)}{h(y)}, \quad h(y) > 0}$$

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The conditional pdf of Y for given $X = x$

$$\boxed{h(y|x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0}$$

NOTE

(i) $g(x|y) \geq 0$

(ii) $\int_{x=-\infty}^{\infty} g(x|y)\,dx = 1$

$$\left[\because = \int_{-\infty}^{\infty} \frac{f(x,y)}{h(y)}\,dx = \frac{1}{h(y)} \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dx}_{= \frac{h(y)}{h(y)} = 1}\right]$$

(i) $h(y|x) \geq 0$

(ii) $\int_{y=-\infty}^{\infty} h(y|x)\,dy = 1$

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Independent Random Variable

(i) (X, Y) — 2.d. discrete r.v.

$\boxed{X \text{ \& } Y \text{ are independent}}$

$p(x_i | y_j) = p(x_i)$ for all i & j

& $q(y_j | x_i) = q(y_j)$ for all i & j

$\boxed{\therefore p(x_i, y_j) = p(x_i) \cdot q(y_j)}$ — Discrete case

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Independent Random Variable

(ii) (X, Y) — 2.d. continuous r.v.

$\boxed{X \text{ \& } Y \text{ are independent}}$

$$g(x|y) = g(x) \quad \text{for all } (x,y)$$

& $$h(y|x) = h(y) \quad \text{for all } (x,y)$$

$$\boxed{\therefore f(x,y) = g(x) \cdot h(y)}$$ — Continuous case

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