Moment Generating Function (mgf)

The moment generating function of the random variable X is given by $E(e^{tX})$ and is denoted by $M_X(t)$

$$\therefore \quad M_X(t) = E(e^{tX}) = \begin{cases} \sum_{x_i} e^{tx_i} p(x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

$$\boxed{M_X(t) = E(e^{tX})}$$

$E(X)$ $V(X)$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \cdots + \frac{(tx)^n}{n!} + \cdots$$

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$$M_X(t) = 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$$

differenting w.r.t t

$$\frac{d\,M_X(t)}{dt} = M'(t) = E(X) + \frac{2t}{2!}E(X^2) + \frac{3t^2}{3!}E(X^3) + \cdots$$

setting $t=0$ ↓

$$\boxed{M'(0) = E(X)}$$

IIIrly diff. again & setting $t=0$

$$\boxed{M''(0) = E(X^2)}$$

In general

$$\boxed{M^{(n)}(0) = E(X^n)}$$

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$\therefore\ E(X) = M'(0)$

$E(x) = \int x\, f(x)\, dx$

$V(X) = E(X^2) - (E(X))^2$

$$\boxed{V(X) = M''(0) - (M'(0))^2}$$

$V(X) = E(X^2) - (E(X))^2$

$\downarrow$

$E(X^2) = \int x^2 f(x)\, dx$

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If X and Y are independent random variables having mgf $M_X(t)$ and $M_Y(t)$ respectively, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$M_X(t) = E(e^{tX})$$

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right)$$

$$= E\left(e^{tX} e^{tY}\right)$$

$$= E(e^{tX})\, E(e^{tY})$$

$$= M_X(t)\, M_Y(t)$$

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Characteristic function

$$C_X(t) = E\left(e^{itX}\right)$$

$i = \sqrt{-1}$, imaginary unit

Details HW

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Uniqueness

X & Y r.v's

$M_X(t)$, $M_Y(t)$] $M_X(t) = M_Y(t)$ iff X & Y have the same distribution

* A random variable X has density function given by $f(x) = \begin{cases} 2e^{-2x} & ; x \geq 0 \\ 0 & x < 0 \end{cases}$

Find the (a) mgf

(b) the first four moments about the origin.

mgf $M_X(t) = E(e^{tX})$

$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

If X – discrete

$\sum_{x_i} e^{tx_i} p(x_i)$

$= \int_0^{\infty} e^{tx} 2e^{-2x}\,dx = 2\int_0^{\infty} e^{-(2-t)x}\,dx$

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$$M_X(t) = 2 \left. \frac{e^{-(2-t)x}}{-(2-t)} \right|_{x=0}^{\infty} = \frac{-2}{(2-t)}\left[0 - 1\right]$$

$$= \frac{2}{2-t} \quad \text{assuming } \underline{t < 2}$$

$$\boxed{M_X(t) = \frac{2}{2-t}}$$

(1) **First method**

$|t| < 2$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$$M_X(t) = \frac{2}{2(1-t/2)} = \left(1 - t/2\right)^{-1}$$

$$M_X(t) = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots$$

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$$M_X(t) = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots$$

$$= 1 + t\left(\frac{1}{2}\right) + \frac{t^2}{2!}\left(\frac{2!}{2^2}\right) + \frac{t^3}{3!}\left(\frac{3!}{2^3}\right) + \frac{t^4}{4!}\left(\frac{4!}{2^4}\right) + \cdots$$

Compare with $M_X(t) = 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$

The first four moments

$E(X) = \frac{1}{2}$, $E(X^2) = \frac{1}{2}$, $E(X^3) = \frac{3}{4}$, $E(X^4) = \frac{3}{2}$

---

(2) **Another method**

$$M_X(t) = 2(2-t)^{-1}$$

$$M'(t) = 2(-1)(2-t)^{-2}(-1)$$

$$\boxed{M'(0) = E(X) = 1/2}$$

$$M''(t) = 2(-2)(2-t)^{-3}(-1)$$

$$\boxed{M''(0) = 1/2 = E(X^2)}$$

$$\boxed{M'''(0) = 3/4 = E(X^3)}$$

$$\boxed{M^{iv}(0) = 3/2 = E(X^4)}$$

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Normal Distribution

A continuous random variable X is said to have a normal distribution if its pdf is of the form

$$N(\mu,\sigma^2;x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \quad ; -\infty < x < \infty$$

$$N(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \quad ; -\infty < x < \infty$$

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* $N(\mu,\sigma^2;x) \geq 0$ ✓
* $\int_{-\infty}^{\infty} N(\mu,\sigma^2;x)\,dx = 1$ ✓

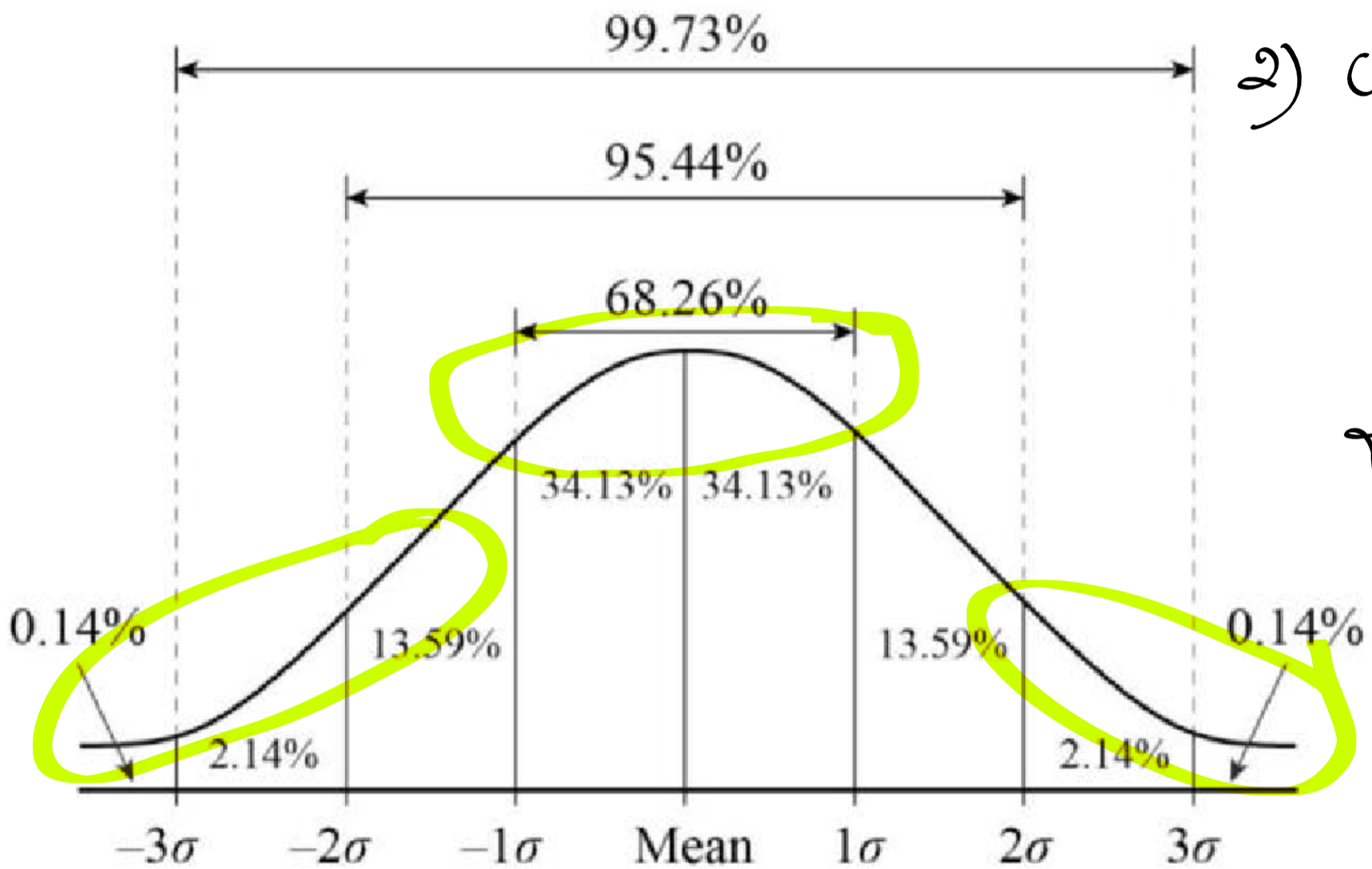
HW

Normal curve

1) Concave downwards

$\mu - \sigma < x < \mu + \sigma$

2) Concave upward otherwise



μ

The total area $= 1$

Properties of normal curve

The total area $= 1$

cdf : $F(X) = P[X \leq x]$

$$= \int_{-\infty}^{x} N(\mu, \sigma^2, x)\, dx$$

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Mean and Variance

Deriving the mean & variance using mgf

1) obtain the mgf

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} N(\mu, \sigma^2, x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sigma} = z$

$x = \mu + \sigma z$

$dx = \sigma dz$

$$\therefore \quad M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \cdot e^{-\frac{1}{2}z^2}\, \sigma\, dz$$

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$$M_X(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[z^2 - 2z(\sigma t) + (\sigma t)^2 - (\sigma t)^2\right]} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(z - \sigma t)^2 - \sigma^2 t^2\right]} dz$$

$$M_X(t) = e^{t\mu} e^{\sigma^2 t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - \sigma t)^2} dz$$

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rewriting $M_X(t)$ in terms of v

Put $z - \sigma t = v$

$dz = dv$

$M_X(t) =$

continue

$$M_X(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

Mean $M'(t) =$

setting $t=0$

$$M'(0) = \mu = E(X)$$

$M''(t)$

$M''(0) = E(X^2)$

$=$

$V(X) = E(X^2) - (E(X))^2$

$= (\sigma^2 + \mu^2) - \mu^2$

$$V(X) = \sigma^2$$

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Standard normal distribution

The normal distribution for which the mean $\mu = 0$ and variance $\sigma^2 = 1$ is called the standard normal distribution.

For a standard normal distribution of a random variable z, the pdf is

$$\phi(z) = N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

where z is standard normal variate

$$\frac{x-\mu}{\sigma} = z$$ — standard normal variable

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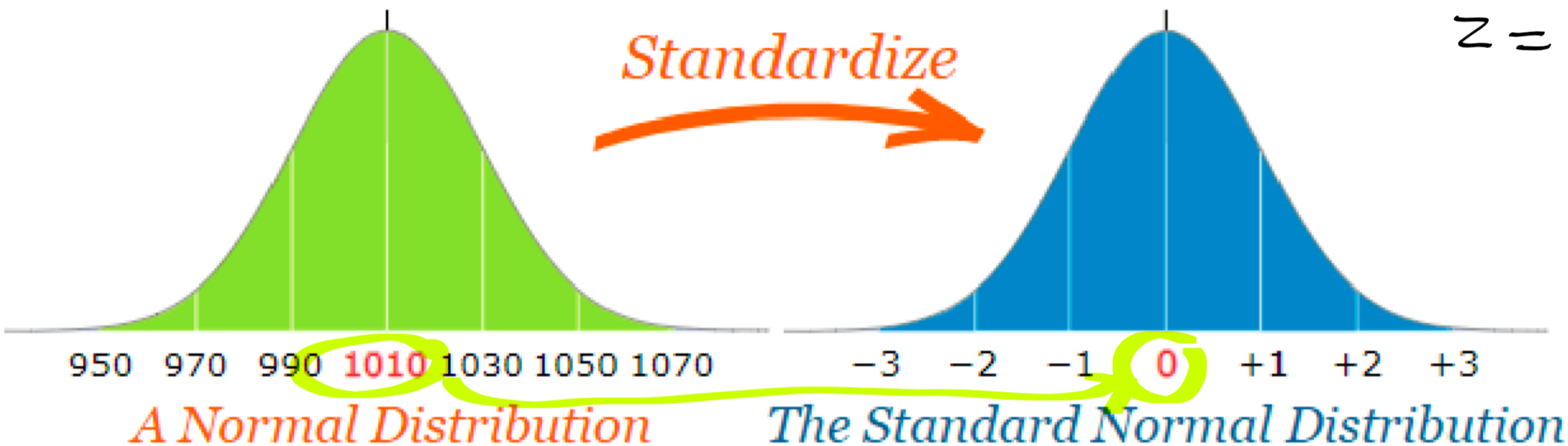
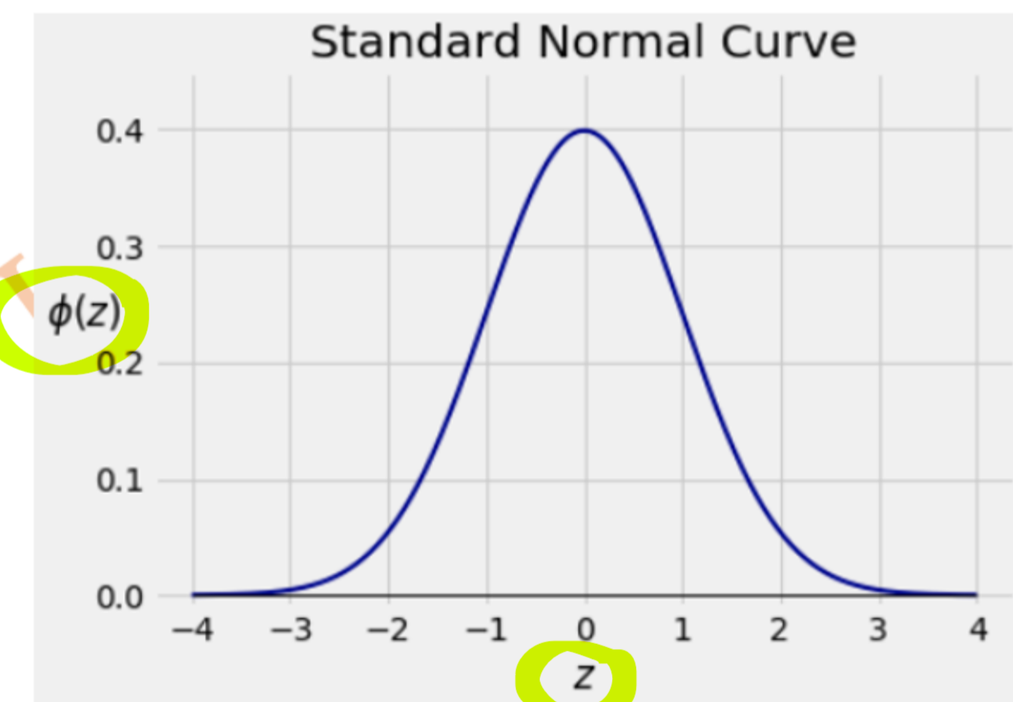
Standard normal curve

$$z = \frac{x-\mu}{\sigma}$$





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Evaluation of P(a≤x≤b)

x — Normal random variable

z — Standard normal variable

$$P[a \le x \le b] = \boxed{\int_a^b N(\mu, \sigma^2, x)\, dx}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma}$

$x = \mu + \sigma z$

$dx = \sigma\, dz$

When $x = a$, $z = z_1$

$x = b$, $z = z_2$

$$\therefore P(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{1}{2}z^2} dz$$

$$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \boxed{\int_{z_1}^{z_2} N(0, 1, z)\, dz}$$

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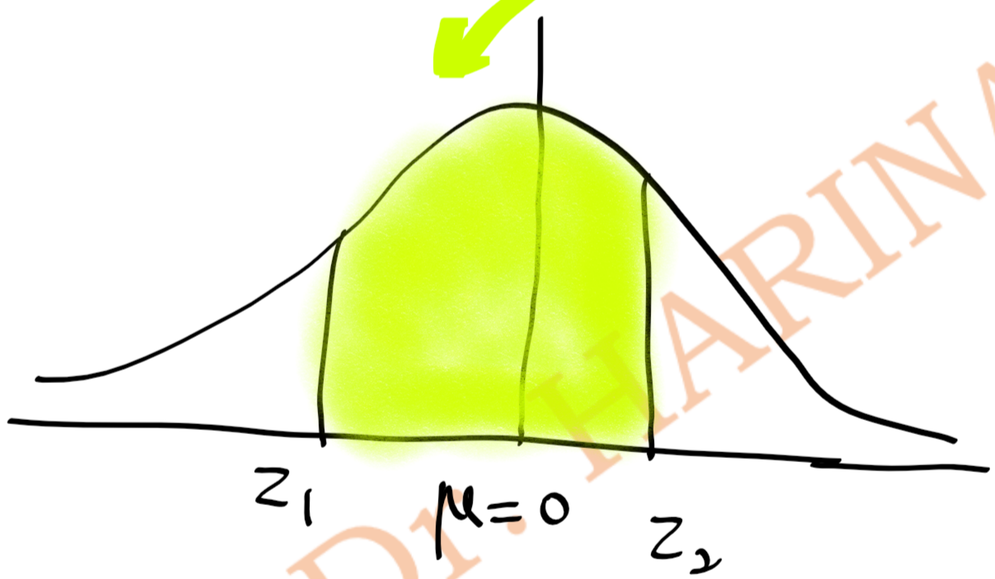
$$\therefore P(z_1 \le z \le z_2) = \int_{z_1}^{z_2} N(0,1,z)\,dz = \int_{z_1}^{z_2} \phi(z)\,dz$$

= Area under the standard normal curve for z between z_1 & z_2



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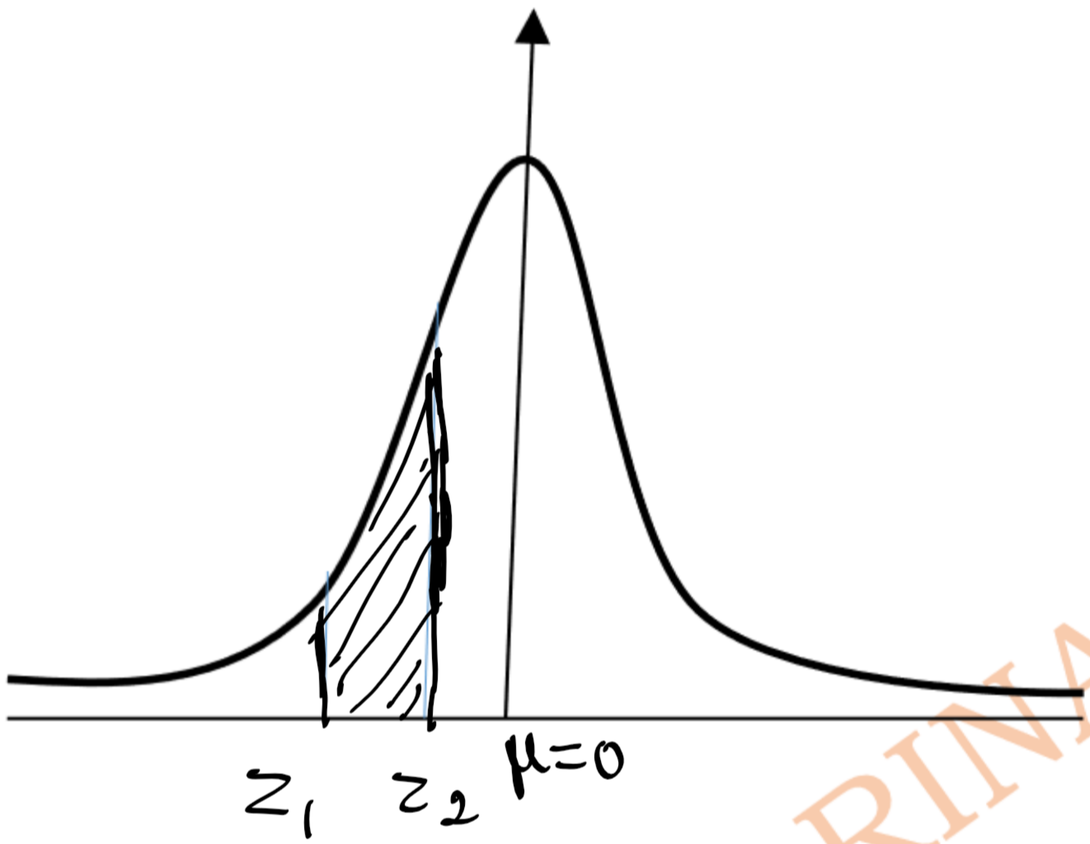
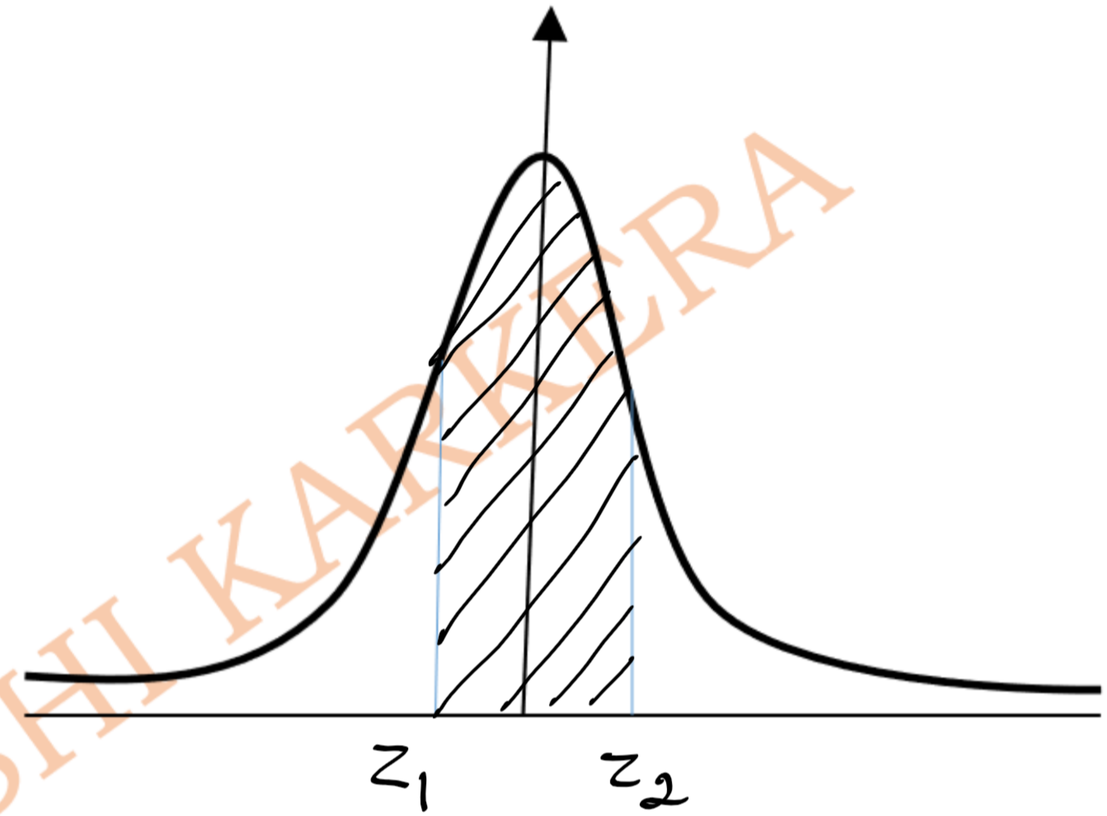
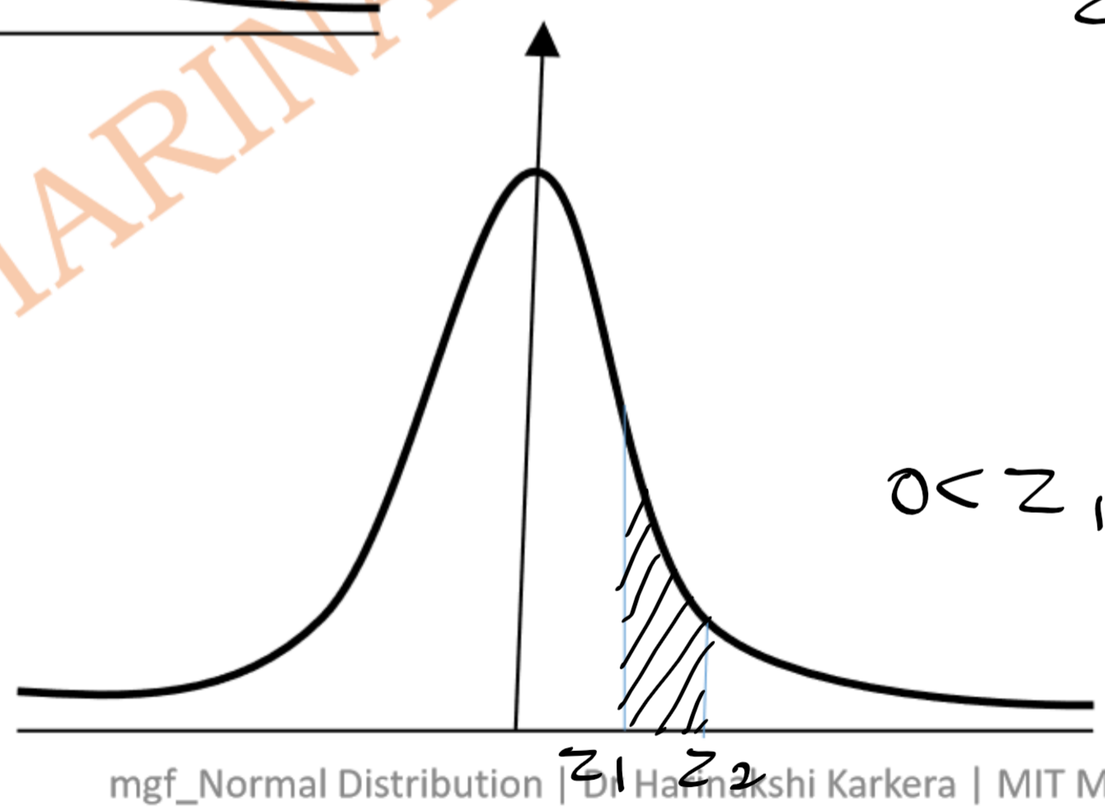
Particular cases



i) $z_1 < z_2 < 0$



$z_1 < 0, \quad z_2 > 0$



$0 < z_1 < z_2$

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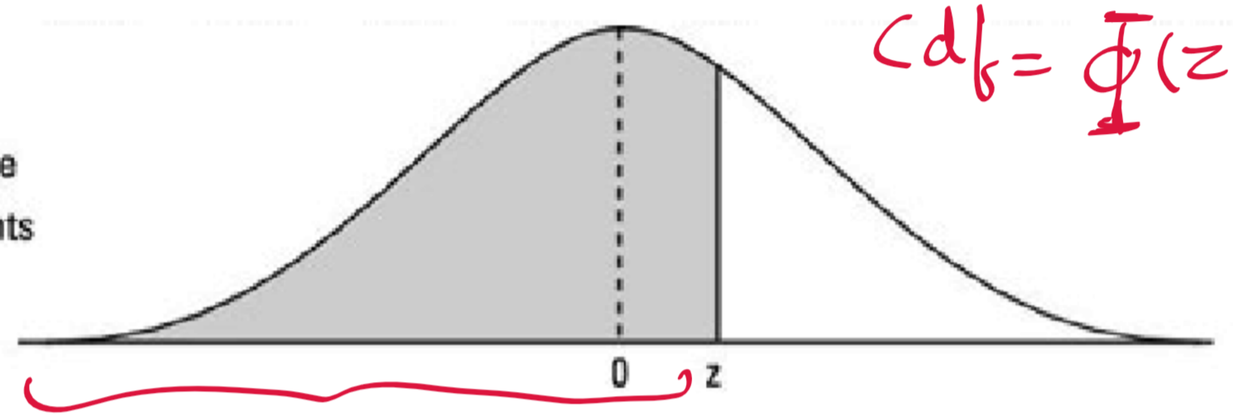
Standard normal table

Number in the table represents $P(Z \le z)$



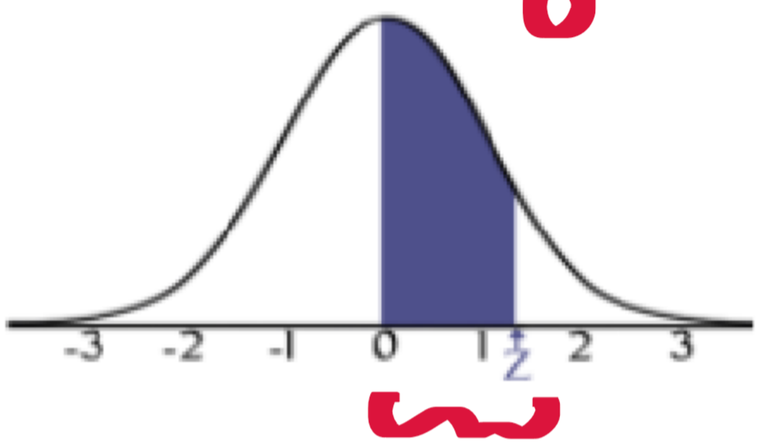
$cdf = \Phi(z) = P[Z \le z]$

$= \int_{-\infty}^{z} N(0,1)\, dz$

$\int_{-\infty}^{z} N(0,1)\, dz$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

$$\int_0^z N(0,1)\,dz$$



STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and z standard deviations above the mean. For example, for $z = 1.25$ the area under the curve between the mean (0) and z is 0.3944.

We are not using this table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830

* Suppose that X has distribution $N(2, 0.16)$.

Evaluate

1) $P[X \geq 2.3]$

2) $P[1.8 \leq X \leq 2.1]$

$\mu = 2$, $\sigma^2 = 0.16$, $\sigma = 0.4$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-2}{0.4}$$

$$\boxed{\therefore\ x = 0.4Z + 2}$$

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$$P(X \geq 2.3) = P(0.4Z + 2 \geq 2.3)$$

$$= P(0.4Z \geq 0.3)$$

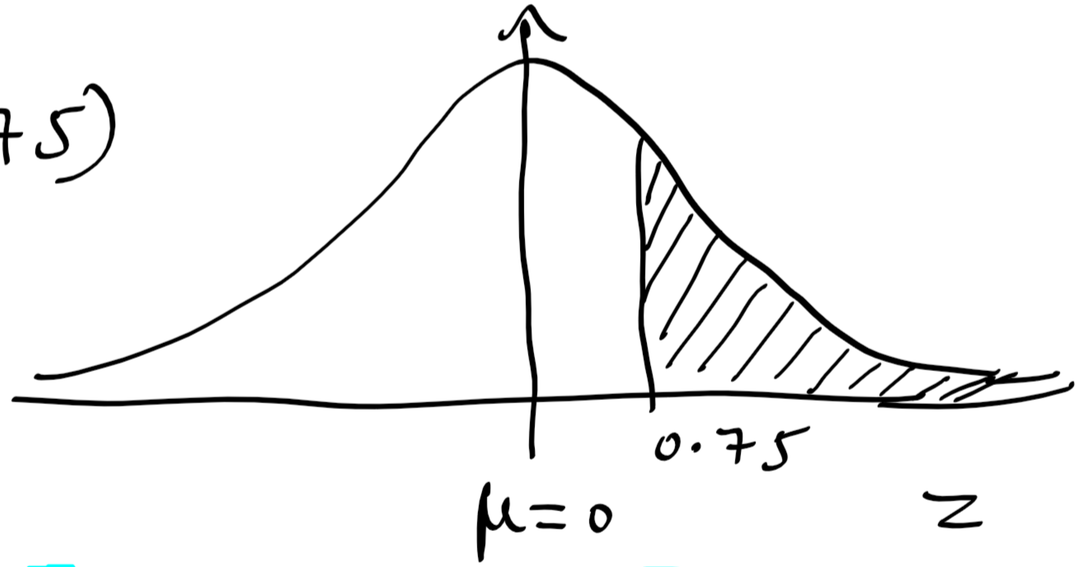
$$= P\left(Z \geq \frac{0.3}{0.4}\right)$$

$$= P(Z \geq 0.75)$$

$$= 1 - P(Z < 0.75)$$

$$= 1 - 0.7734$$

$$\boxed{P(X \geq 2.3) = \underline{\underline{0.2266}}}$$



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(i) $P[1.8 \le X \le 2.1] = P[1.8 \le 0.4Z + 2 \le 2.1]$

$= P(-0.2 \le 0.4Z \le 0.1)$

$= P(-0.5 \le Z \le 0.25)$

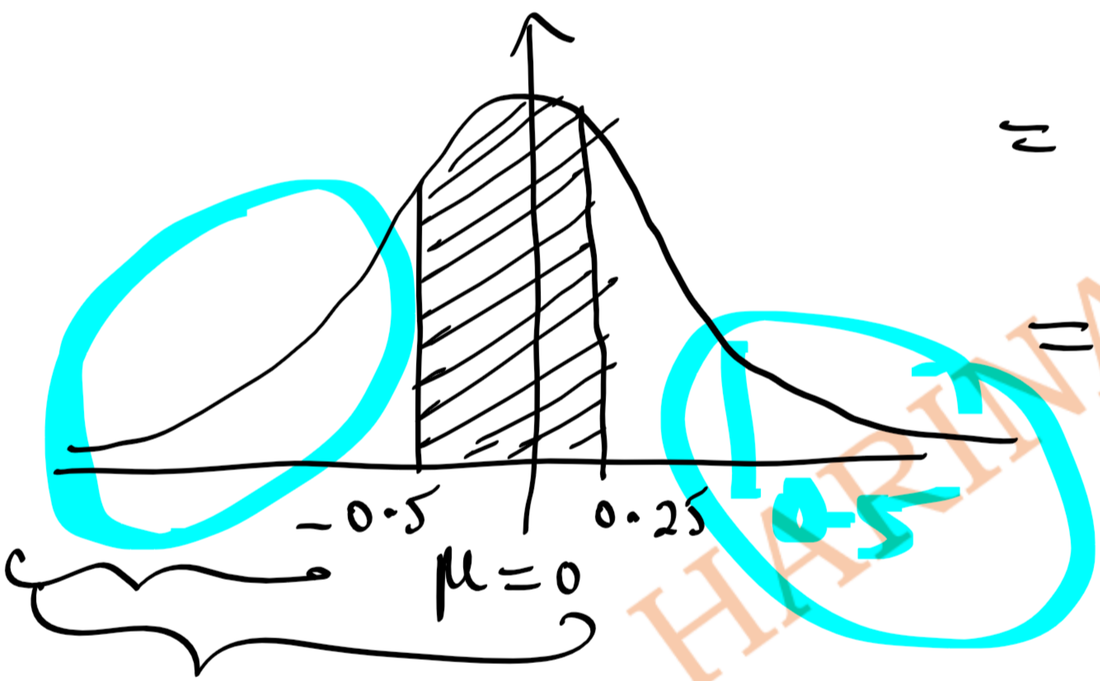
$= P(Z < 0.25) - P(Z < -0.5)$

$= P(Z < 0.25) - P(Z > 0.5)$

$= P(Z < 0.25) - [1 - P(Z < 0.5)]$

$= 0.5987 - 1 + 0.6915$

$$\boxed{= 0.2902}$$



$$\int_{-\infty}^{z} N(0,1)\,dz$$

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