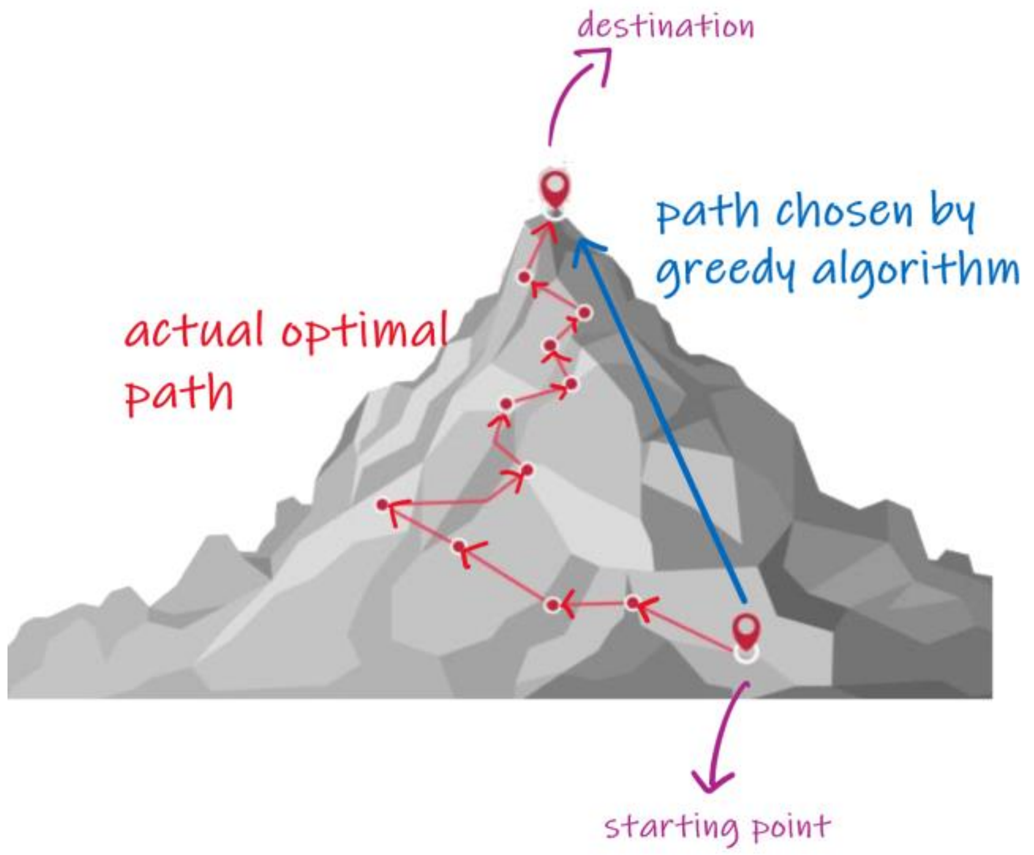
DSE 2256 DESIGN & ANALYSIS OF ALGORITHMS

Lecture 36, 37, 38

Greedy Technique

Prim's Algorithm
Kruskal's Algorithm
Dijkstra's Algorithm



Instructors:

Dr. Savitha G,
Assistant Professor, DSCA, MIT, Manipal

Dr. Abhilash K. Pai,
Assistant Professor, DSCA, MIT, Manipal

Greedy Technique

- Constructs a solution to an optimization problem piece by piece through a sequence of choices that are:
  - **Feasible**, i.e. satisfying the problem constraints
  - **Locally optimal** (It has to be the best local choice among all choices in that step)
  - **Greedy** (in terms of some measure), **and irrevocable** (cannot be changed in subsequent steps)

For some problems, it yields a globally optimal solution for every instance. For most, does not give globally optimal solution, but can be useful for fast approximations.

Applications of the Greedy Strategy

- Optimal solutions:
  - Change making for “normal” coin denominations
  - Minimum spanning tree (MST)
  - Single-source shortest paths
  - Huffman codes

- Approximations/heuristics:
  - Traveling salesman problem (TSP)
  - Knapsack problem
  - Other combinatorial optimization problems

Minimum Spanning Tree (MST)

- **Spanning tree** of a connected graph G = (V, E) is a **connected acyclic subgraph of G** that includes all of G's vertices with the minimum possible no. of edges.

Some useful properties of spanning trees:

- The spanning tree of a graph G has |V| - 1 edges (where, |V| =the total no. of vertices in G).
- For any given graph, multiple spanning trees are possible.
- For a cycle graph (a graph with a single cycle) with *n* vertices, a total of *n* spanning trees are possible.
- For a complete graph (K_n) with n vertices a total of $n^{(n-2)}$ spanning trees are possible.

- **Minimum spanning tree** of a weighted, connected graph G: is a spanning tree of G of the **minimum total weight.**

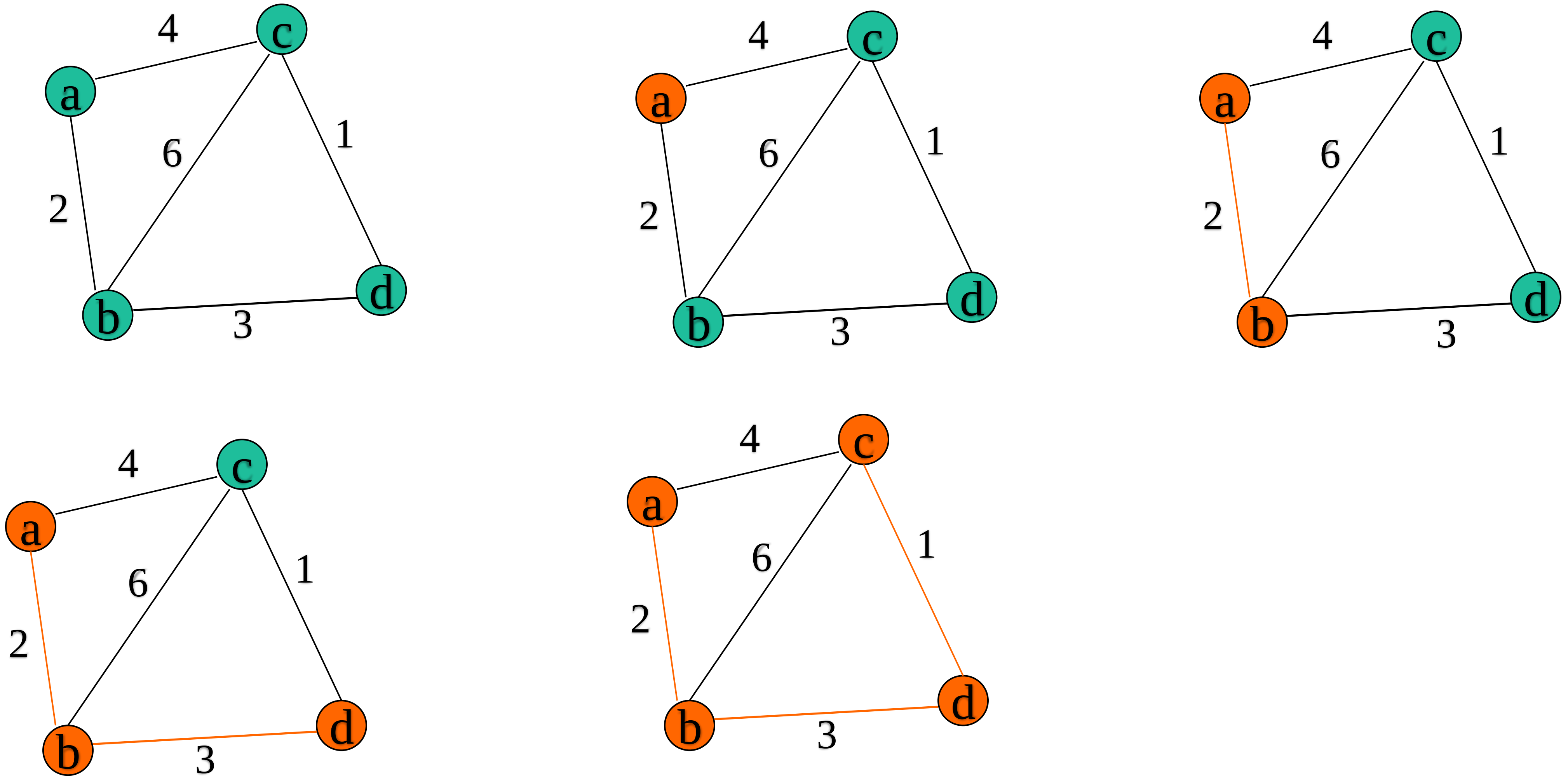
Prim's MST algorithm

- Start with tree T_1 consisting of one (any) vertex and "grow" tree one vertex at a time to produce MST through a series of expanding subtrees $T_1, T_2, \ldots, T_n$

- On each iteration, construct T_{i+1} from T_i by adding vertex not in T_i that is closest to those already in T_i (this is a "greedy" step!)

- Stop when all vertices are included

Prim's MST algorithm: Example



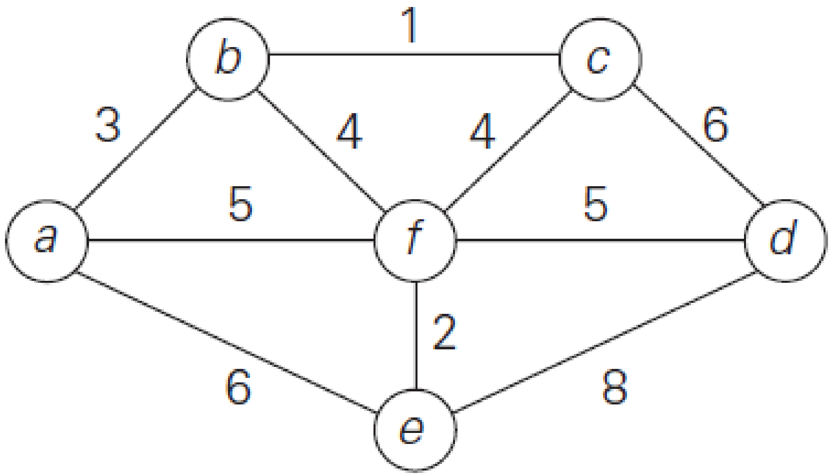
Prim's MST algorithm

ALGORITHM $Prim(G)$

//Prim's algorithm for constructing a minimum spanning tree
//Input: A weighted connected graph $G = \langle V, E \rangle$
//Output: E_T, the set of edges composing a minimum spanning tree of G
$V_T \leftarrow \{v_0\}$ //the set of tree vertices can be initialized with any vertex
$E_T \leftarrow \varnothing$
for $i \leftarrow 1$ **to** $|V| - 1$ **do**
    find a minimum-weight edge $e^* = (v^*, u^*)$ among all the edges (v, u)
    such that v is in V_T and u is in $V - V_T$
    $V_T \leftarrow V_T \cup \{u^*\}$
    $E_T \leftarrow E_T \cup \{e^*\}$
return E_T

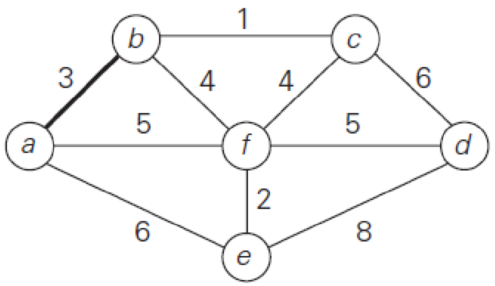
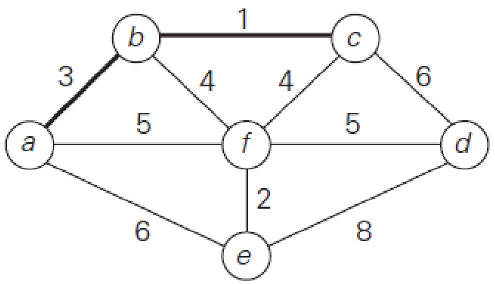
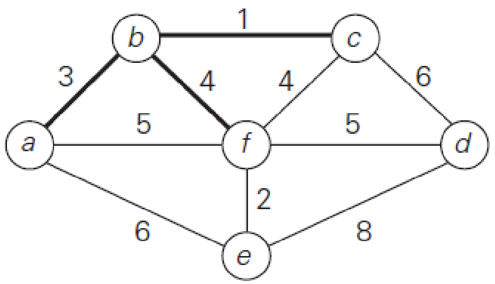
Prim’s MST algorithm: Example

Apply Prim’s algorithm to the following graph to find the minimum spanning tree



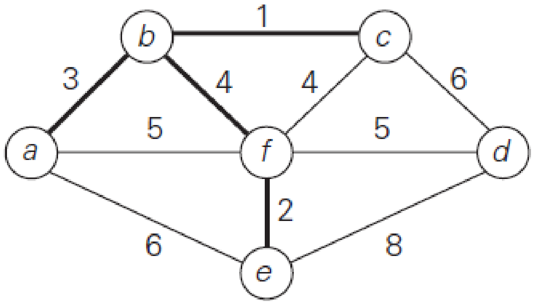
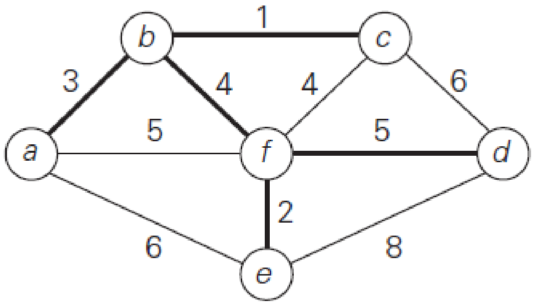
Prim's MST algorithm: Example

Start with any arbitrary vertex.
Here, let us start with **a**

Tree vertices	Remaining vertices	Illustration
a(–, –)	**b(a, 3)** c(–, ∞) d(–, ∞) e(a, 6) f(a, 5)	
b(a, 3)	**c(b, 1)** d(–, ∞) e(a, 6) f(b, 4)	
c(b, 1)	d(c, 6) e(a, 6) **f(b, 4)**	

- At any step check whether there is a connection between all the intermediate tree vertices and the remaining vertices of the input graph.
- If so, choose the connection with the minimum cost.
- Repeat until there are no remaining graph vertices to be considered.
- The resultant tree obtained will be a spanning tree with the minimum total weight (MST).

Prim's MST algorithm: Example

f(b, 4)	d(f, 5) **e(f, 2)**	
e(f, 2)	**d(f, 5)**	
d(f, 5)		

Prim's MST algorithm : Efficiency

- Depends on data structures chosen for graph.
- Generally implemented using priority queue (which contains the priorities of the remaining graph vertices based on their distances to the nearest tree vertices).

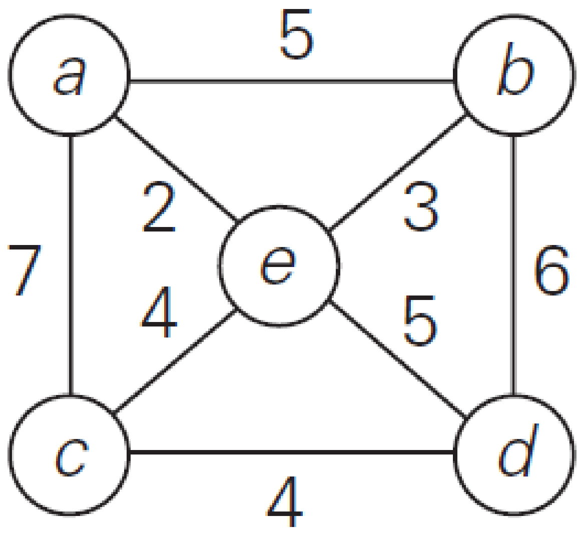
I. If the graph is implemented using weighted adjacency matrix and the priority queue is implemented using arrays, then Prim's algorithm's running time will be: $\mathbf{O(|V|^2)}$

II. If the graph is implemented using adjacency lists and the priority queue is implemented as a min-heap, then the algorithm performs $|V|-1$ deletions of the smallest element and makes $|E|$ verifications. In this case, the running time of the Prim's algorithm is in :

$(|V|-1+|E|)\ O(\log|V|) = \mathbf{O(|E| \log |V|)}$ (because in a connected graph, $|V|-1 <= |E|$)

Exercise I

Apply Prim's algorithm to the following graph to find the minimum spanning tree.



Kruskal's Algorithm

- Another greedy algorithm for computing the Minimum Spanning Tree.

- Developed by Joseph Kruskal in 1956.

Kruskal's Algorithm (Working):

- Sort the edges in nondecreasing order of lengths

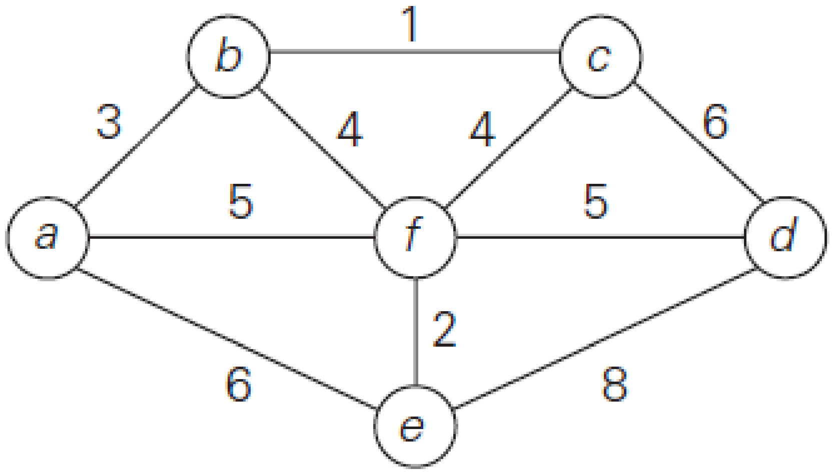
- "Grow" tree one edge at a time to produce MST through a series of expanding forests* $F_1, F_2, \ldots, F_{n-1}$

- On each iteration, add the next edge on the sorted list unless this would create a cycle. (If it would, skip the edge.)

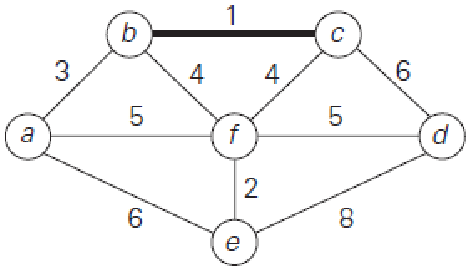
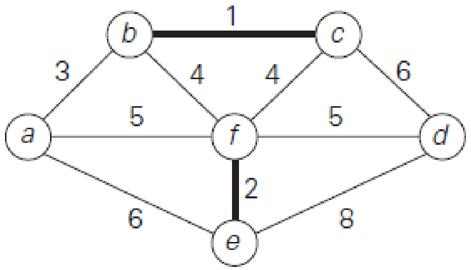
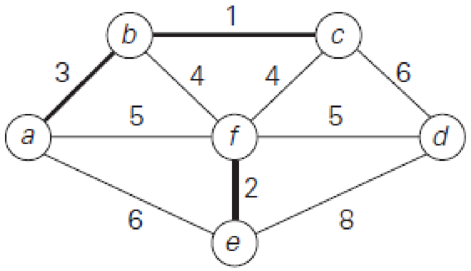
*A forest is a graph where each connected component is a tree.

Kruskal's Algorithm: Example

Apply Kruskal's algorithm to the following graph to find the minimum spanning tree



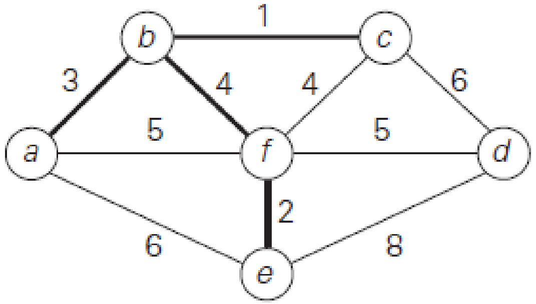
Kruskal's Algorithm: Example

Tree edges	Sorted list of edges	Illustration
	bc ef ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
bc 1	bc **ef** ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
ef 2	bc ef **ab** bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	

Kruskal's Algorithm: Example

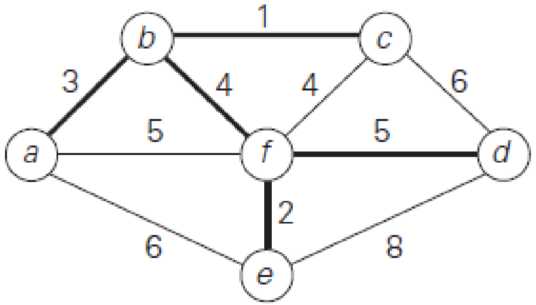
ab
3

bc	ef	ab	**bf**	cf	af	df	ae	cd	de
1	2	3	4	4	5	5	6	6	8



bf
4

bc	ef	ab	bf	cf	af	**df**	ae	cd	de
1	2	3	4	4	5	5	6	6	8



df
5

Kruskal's Algorithm and its efficiency

ALGORITHM *Kruskal(G)*

//Kruskal's algorithm for constructing a minimum spanning tree
//Input: A weighted connected graph $G = \langle V, E \rangle$
//Output: E_T, the set of edges composing a minimum spanning tree of G
sort E in nondecreasing order of the edge weights $w(e_{i_1}) \leq \cdots \leq w(e_{i_{|E|}})$
$E_T \leftarrow \varnothing$; $\quad ecounter \leftarrow 0$ //initialize the set of tree edges and its size
$k \leftarrow 0$ //initialize the number of processed edges
while $ecounter < |V| - 1$ **do**
  $k \leftarrow k + 1$
  if $E_T \cup \{e_{i_k}\}$ is acyclic
    $E_T \leftarrow E_T \cup \{e_{i_k}\}$; $\quad ecounter \leftarrow ecounter + 1$
return E_T

Time Complexity:

$T_{\text{kruskal (worst case)}} = T_{\text{sort\_edges}} + T_{\text{check\_acyclic}}$

Where, $T_{\text{check\_acyclic}} = T_{\text{create\_subsets}} + T_{\text{union\_find}}$

$T_{\text{check\_acyclic}} = O(|V|) + O(|E| \log |E|) = \mathbf{O(|E| \log |E|)}$

and

$T_{\text{sort\_edges}} = O(|E| \log|E|)$

Therefore, $T_{\text{kruskal (worst case)}} = O(|E| \log |E|)$

Dijkstra's Algorithm

- It is applied to **Single source shortest problem**: for a given vertex called the source in weighted connected graph, find shortest paths to its other vertices.

- Introduced by Edsger W. Dijkstra in 1956.

- Applicable to both directed and undirected graphs with non-negative weights.

Applications:

- Transportation planning.
- Packet routing in communication networks.
- Shortest paths social networks, robotics, speech recognition, document formatting, robotics, compilers, airline crew scheduling.
- Path finding in video games, puzzles etc.

Dijkstra's Algorithm

It is similar to Prim's MST algorithm, with a different way of computing numerical labels:

Among vertices not already in the tree, it finds vertex *u* with the smallest sum

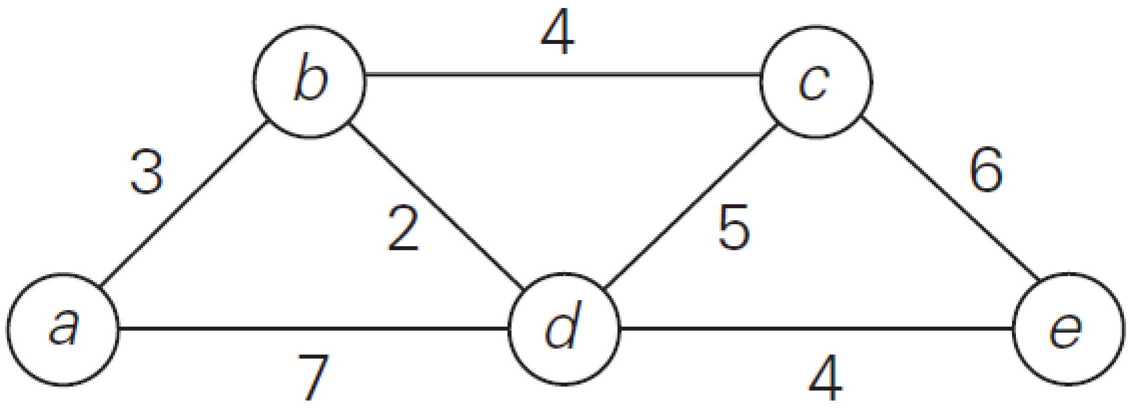
$$d_v + w(v,u)$$

Where,

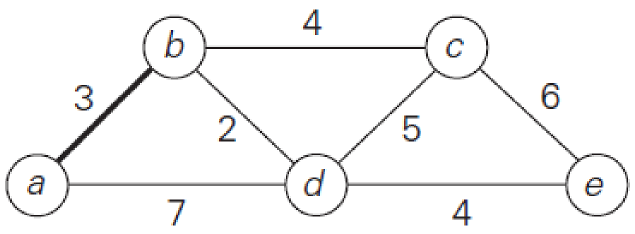
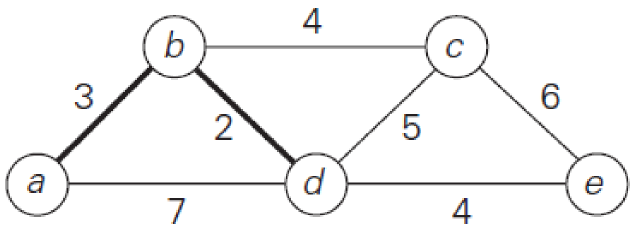
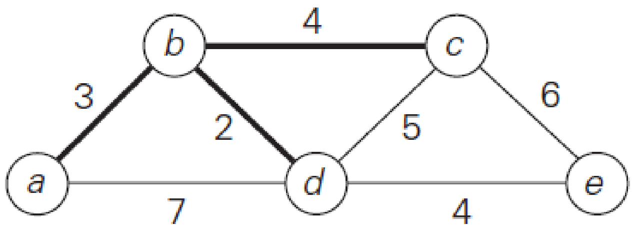
- v is a vertex for which shortest path has been already found on preceding iterations (such vertices form a tree rooted at *s*)
- d_v is the length of the shortest path from source *s* to *v*
- $w(v,u)$ is the length (weight) of edge from *v* to *u*

Dijkstra's Algorithm : Example

Apply Dijkstra's algorithm to the following graph to find the shortest paths from source vertex “a” to all other vertices.

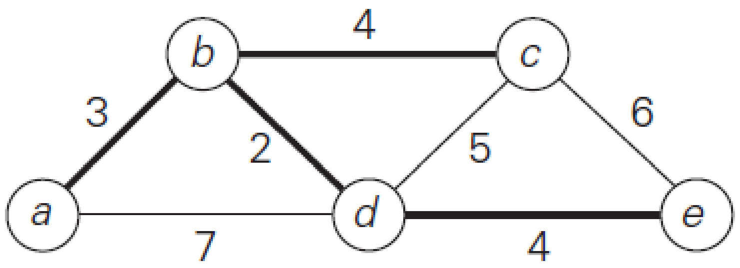


Dijkstra's Algorithm : Example

Tree vertices	Remaining vertices	Illustration
a(−, 0)	**b**(**a**, **3**) c(−, ∞) d(a, 7) e(−, ∞)	
b(a, 3)	c(b, 3 + 4) **d**(**b**, **3 + 2**) e(−, ∞)	
d(b, 5)	**c**(**b**, **7**) e(d, 5 + 4)	

Dijkstra's Algorithm : Example

c(b, 7) **e(d, 9)**



e(d, 9)

---

from a to b : $a - b$ of length 3

from a to d : $a - b - d$ of length 5

from a to c : $a - b - c$ of length 7

from a to e : $a - b - d - e$ of length 9

Dijkstra's Algorithm

ALGORITHM *Dijkstra*(G, s)

```
//Dijkstra's algorithm for single-source shortest paths
//Input: A weighted connected graph G = ⟨V, E⟩ with nonnegative weights
//          and its vertex s
//Output: The length d_v of a shortest path from s to v
//            and its penultimate vertex p_v for every vertex v in V
Initialize(Q)   //initialize priority queue to empty
for every vertex v in V
    d_v ← ∞;   p_v ← null
    Insert(Q, v, d_v)   //initialize vertex priority in the priority queue
d_s ← 0;   Decrease(Q, s, d_s)   //update priority of s with d_s
V_T ← ∅
for i ← 0 to |V| − 1 do
    u* ← DeleteMin(Q)   //delete the minimum priority element
    V_T ← V_T ∪ {u*}
    for every vertex u in V − V_T that is adjacent to u* do
        if d_u* + w(u*, u) < d_u
            d_u ← d_u* + w(u*, u);   p_u ← u*
            Decrease(Q, u, d_u)
```

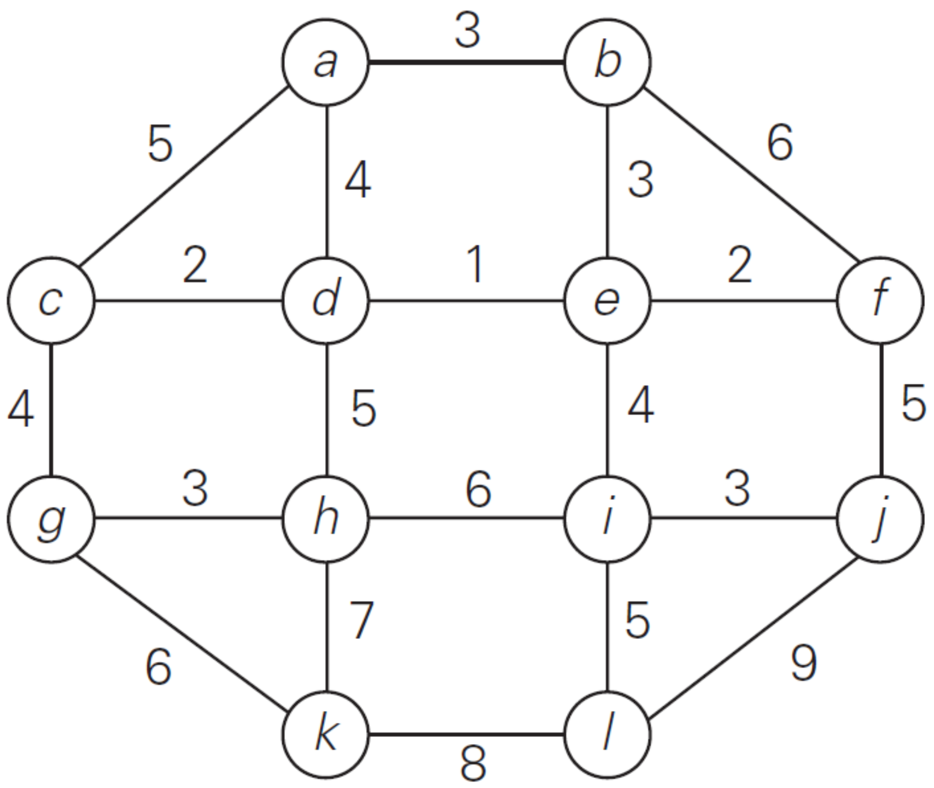
Time Complexity is similar to Prim's Algorithm

If implemented using :

- Weighted adjacency matrix + array-based priority queue, the complexity is $O(|V|^2)$
- Adjacency lists + Min-heap based priority queue, the complexity is $O(|E| \log |E|)$

Exercise II

Apply Prim's, Kruskal's and Dijkstra's algorithms to the following graph (assume starting/source vertex as "a" for Prim's and Dijkstra's).



Thank you!

Any queries?