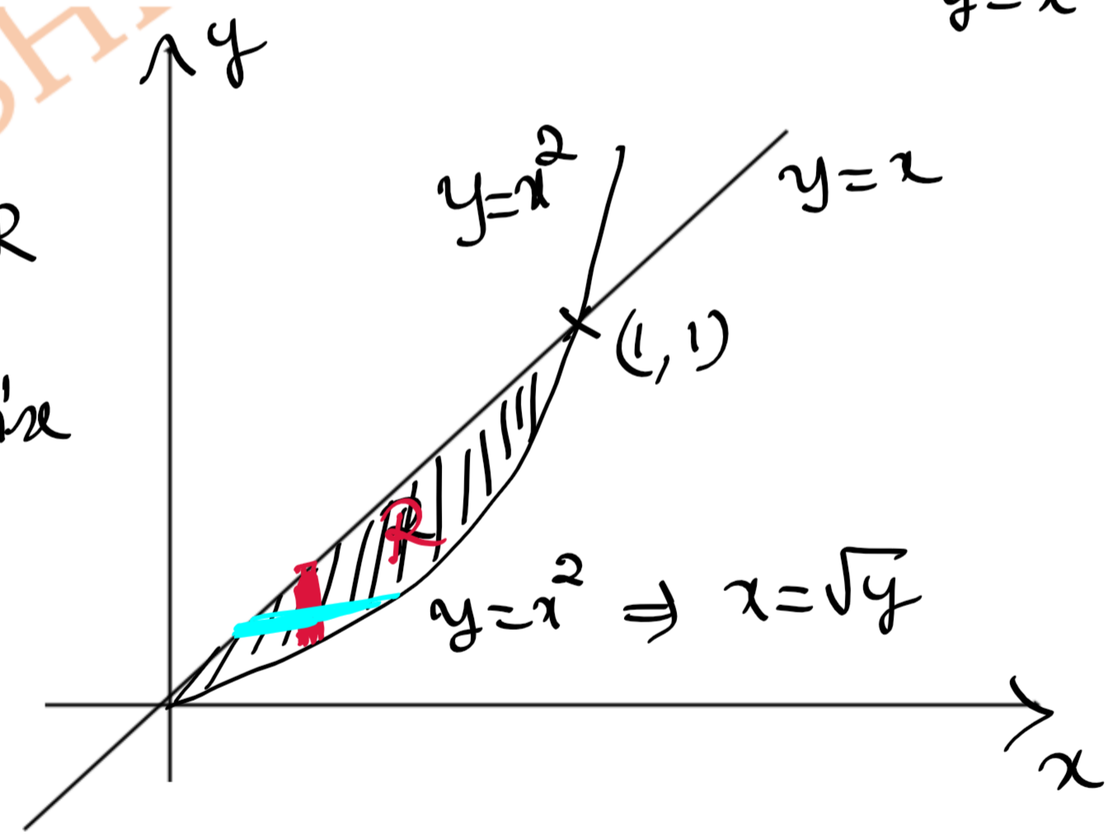
Problems of uniform distribution

* Suppose that the two-dimensional r.v (X,Y) is uniformly distributed over the region R bounded between $y=x$ & $y=x^2$. Find the marginal pdf of X & Y.

$y=x^2$
$y=x$



$$f(x,y)=\begin{cases}\dfrac{1}{\text{area of } R} & ; (x,y)\in R\\ 0 & \text{otherwise}\end{cases}$$

$$\text{Area of } R = \int_{x=0}^{1}\left[\int_{y=x^2}^{x} dy\right] dx$$

Dr. HARINAKSHI

$$= \int_{x=0}^{1} (x - x^2)\, dx = \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{2} - \frac{1}{3} \boxed{= \frac{1}{6}}$$

$$\therefore f(x,y) = \begin{cases} 6 & ; \quad (x,y) \in R \\ 0 & \quad \text{otherwise} \end{cases}$$

The marginal pdf of X is

$$g(x) = \int_{y=-\infty}^{\infty} f(x,y)\, dy = \int_{y=x^2}^{x} 6\, dy \boxed{= 6(x - x^2) \; ; \quad 0 \le x \le 1}$$

Dr. HARINAKKARKERA

$\therefore$ The marginal pdf of Y is

$$h(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx = \int_{x=y}^{\sqrt{y}} 6\,dx = 6(\sqrt{y}-y) \quad ; 0 \le y \le 1$$

Dr. HARINAKSHI KARKERA

Exponential distribution

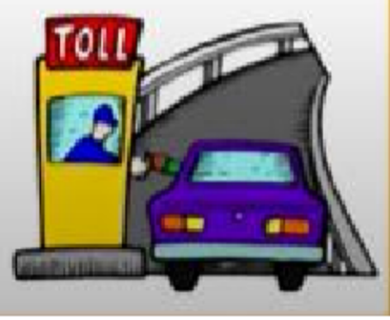
A continuous random variable X is said to have an exponential distribution with parameter $\alpha > 0$ if its pdf is given by

$$E(\alpha, x) = f(x) = \begin{cases} \alpha e^{-\alpha x} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Exponential Probability Distribution

- The exponential probability distribution is useful in describing the time it takes to complete a task.
- The exponential random variables can be used to describe:

Time between vehicle arrivals at a toll booth



Time required to complete a questionnaire

Distance between major defects in a highway

Note:

1) $E(\alpha, x) \geq 0$

2) $$\int_{-\infty}^{\infty} E(\alpha, x)\, dx = \int_{0}^{\infty} \alpha e^{-\alpha x}\, dx$$

$$= \cancel{\alpha} \left. \frac{e^{-\alpha x}}{-\cancel{\alpha}} \right|_{x=0}^{\infty}$$

$$= -[0 - 1] \boxed{= 1}$$

3) cdf $F(x) = P[X \leq x]$

$$\boxed{= 1 - e^{-\alpha x} \; ; x > 0}$$

Dr. HARINAKSHI KARKERA

Mean & Variance

$$E(X) = \int_{-\infty}^{\infty} x\, E(\alpha, x)\, dx$$

} do the calculation

$$\therefore E(X) = \frac{1}{\alpha}$$

Variance,

$$V(X) = \frac{1}{\alpha^2}$$

Dr HARINAKSHI KARKERA

* The length of a telephone conversation has an exponential distribution with a mean of 3 minutes. Find the probability that a call (i) ends in less than 3 minutes

(ii) takes between 3 & 5 minutes.

X: the length of the telephone conversation

Exponential distribution.

$$E(\alpha, x) = \alpha e^{-\alpha x} \quad ; x > 0$$

$$= \frac{1}{3} e^{-(1/3)x} \quad ; x > 0$$

$$\text{mean} = \frac{1}{\alpha} = 3$$

$$\Rightarrow \boxed{\alpha = \frac{1}{3}}$$

Dr. HARINAKSHI KARKERA

(i) $P[x < 3] = \int_{x=0}^{3} \frac{1}{3} \cdot e^{-\frac{1}{3}x} \, dx$

$$= \frac{1}{3} \cdot \left. \frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}} \right|_{x=0}^{3}$$

$$= -\left[e^{-1} - 1\right] = 1 - \frac{1}{e} \boxed{= 0.6321}$$

(ii) $P[3 < x < 5] = \int_{x=3}^{5} \frac{1}{3} e^{-\frac{1}{3}x} \, dx = \frac{1}{3} \left. \frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}} \right|_{x=3}^{5}$

$$= -\left[e^{-5/3} - e^{-1}\right] \boxed{= 0.179}$$

Dr. HARINAKSHI KARKERA

* In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for (i) less than 10 minutes (ii) 10 minutes or more.

HW

Dr. HARINAKSHI KARKERA

The Gamma function, Γ

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} \, dx \quad , \quad p > 0$$

* $\Gamma p = (p-1)\,\Gamma(p-1)$

* $\Gamma p = (p-1)!$ — p is a positive integer

* $\Gamma 1 = 1$

* $\Gamma \frac{1}{2} = \sqrt{\pi}$

Dr. HARINAKSHI KARKERA

The Gamma distribution

Let X be a continuous random variable assuming only nonnegative values. We say that X has a Gamma probability distribution if the pdf is given by

$$f(x) = \begin{cases} \dfrac{\alpha}{\Gamma(r)} (\alpha x)^{r-1} e^{-\alpha x} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$r > 0$, $\alpha > 0$ } two parameters of Gamma distribution

HARINAKSHI KARKERA

Note: 1) If $r=1$, $f(x) = \alpha e^{-\alpha x}$ — exponential distribution.

[a special case of the Gamma distribution]

2) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \frac{\alpha}{\Gamma(r)} (\alpha x)^{r-1} e^{-\alpha x}\,dx$$

$$\boxed{= 1}$$

Dr. HARINAKSHI KARKERA

Mean and Variance

$$E(X) = \frac{r}{\alpha}$$

$$V(X) = \frac{r}{\alpha^2}$$

For exponential distn

$r = 1$ $\quad E(X) = \frac{1}{\alpha}$

$V(X) = \frac{1}{\alpha^2}$

mgf of Gamma distribution

$$M_X(t) = \left(\frac{\alpha}{\alpha - t}\right)^r, \quad \alpha > t$$

HARINAKSHI KARKERA

The chi-square distribution

Taking $r = \frac{n}{2}$ (n is a +ve integer)

$\alpha = \frac{1}{2}$ in the Gamma distribution

special case of

A continuous r.v 'z' said to have a chi-square distribution (χ^2_n) with n degrees of freedom (n is a parameter) if its pdf is

$$f(z) = \frac{1}{2^{n/2}\,\Gamma_{n/2}}\, z^{n/2-1}\, e^{-z/2} \quad ; \quad z>0$$

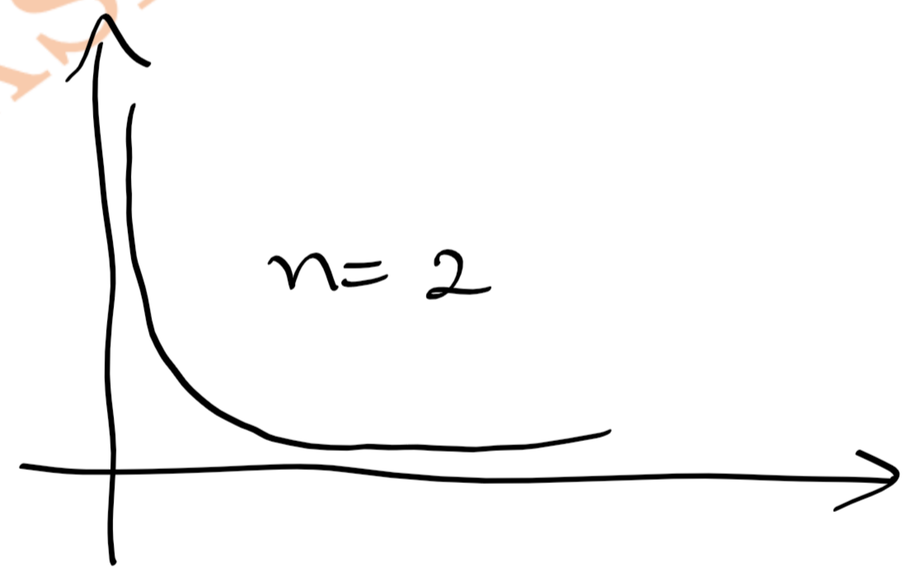
Dr. HARINAKSHI KARKERA

χ_n^2, chi-square distribution

$$f(z) = \frac{1}{2^{n/2}\,\Gamma n/2}\, z^{n/2-1}\, e^{-z/2} \quad ; z>0$$

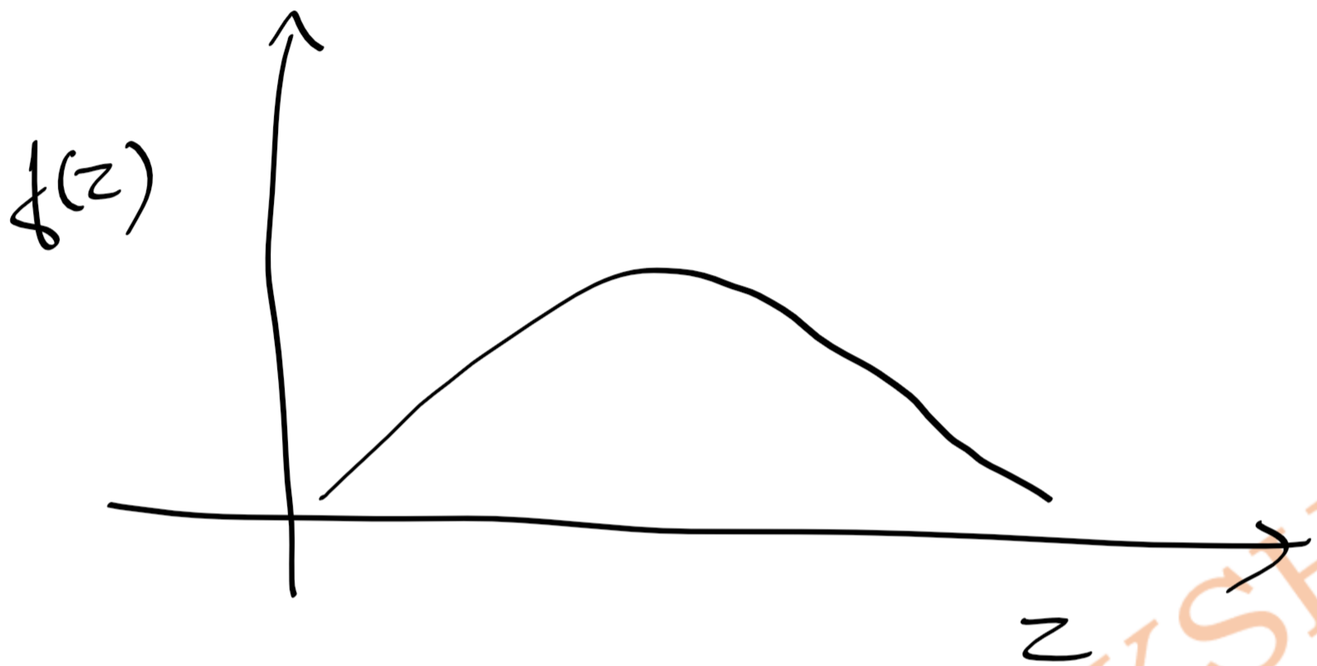
For $n=1$

For $n=2$



Dr. HARINAKSHI KARKERA

For $n > 2$



Dr. HARINAKSHI KARKERA