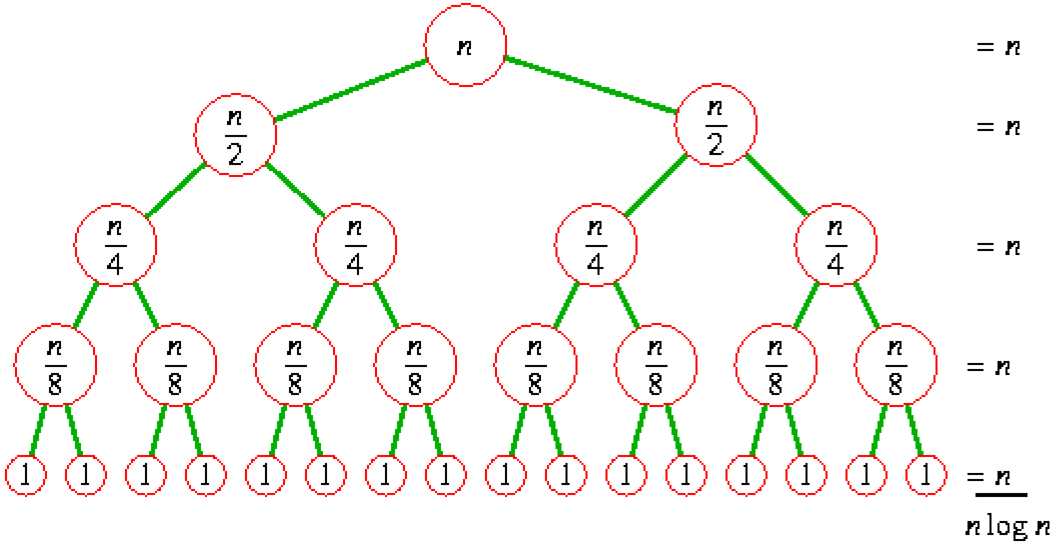
DSE 2256 DESIGN & ANALYSIS OF ALGORITHMS

Lecture 21

Divide-and-Conquer:

Multiplication of large integers
Strassen's Matrix Multiplication



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Recap of L20

- Binary Search

- Binary Tree
  - Computing Height of a Binary Tree
  - Binary Tree Traversals

Multiplication of Large Integers

- Consider the problem of multiplying two (large) n-digit integers represented by arrays of their digits such as:

A = 12345678901357986429   B = 87654321284820912836

The grade-school algorithm:

$$
\begin{array}{r}
a_1\ a_2 \dots\ a_n \\
b_1\ b_2 \dots\ b_n \\
\hline
(d_{10})\, d_{11} d_{12} \dots d_{1n} \quad\quad\\
(d_{20})\, d_{21} d_{22} \dots d_{2n} \quad\quad\quad\quad\quad\quad\\
\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \quad\quad\quad\quad\quad\\
(d_{n0})\, d_{n1} d_{n2} \dots d_{nn} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\\
\hline
\end{array}
$$

Efficiency: $\Theta(n^2)$ single-digit multiplications

Multiplication of Large Integers

$$23 * 14$$

$$23 = 2 \cdot 10^1 + 3 \cdot 10^0 \qquad 14 = 1 \cdot 10^1 + 4 \cdot 10^0$$

- Multiplying 23 and 14 :

$$23 * 14 = (2 \cdot 10^1 + 3 \cdot 10^0) * (1 \cdot 10^1 + 4 \cdot 10^0)$$

$$= (2 * 1)10^2 + \boxed{(2 * 4 + 3 * 1)}10^1 + (3 * 4)10^0 \quad \text{---------} \quad (1)$$

4 multiplications!

- Re-considering Middle part of Equation (1):

Already computed

$$2 * 4 + 3 * 1 = (2 + 3) * (1 + 4) - \boxed{2 * 1} - \boxed{3 * 4}$$

1 multiplication instead of 2 in middle part of Eq. (1)

Multiplication of Large Integers

$$\underset{a_1\,a_0}{\overset{a}{\boxed{23}}} * \underset{b_1\,b_0}{\overset{b}{\boxed{14}}} = \underbrace{\boxed{(2*1)}}_{c_2}10^2 + \underbrace{\boxed{((2+3)*(1+4) - 2*1 - 3*4))}}_{c_1}10^1 + \underbrace{\boxed{(3*4)}}_{c_0}10^0$$

- Based on the above equation:

$$c = a * b = c_2 10^2 + c_1 10^1 + c_0,$$

where $a = a_1 a_0$ and $b = b_1 b_0.$

$c_2 = a_1 * b_1$ is the product of their first digits,

$c_0 = a_0 * b_0$ is the product of their second digits,

$c_1 = (a_1 + a_0) * (b_1 + b_0) - (c_2 + c_0)$ is the product of the sum of the a's digits and the sum of the b's digits minus the sum of c_2 and c_0.

Multiplication of Large Integers

- To multiply two n-digit numbers, apply the divide-and-conquer strategy by dividing both numbers in the middle.

$$
\begin{aligned}
c = a * b &= (a_1 10^{n/2} + a_0) * (b_1 10^{n/2} + b_0) \\
&= (a_1 * b_1)10^n + (a_1 * b_0 + a_0 * b_1)10^{n/2} + (a_0 * b_0) \\
&= c_2 10^n + c_1 10^{n/2} + c_0,
\end{aligned}
$$

where

$c_2 = a_1 * b_1$ is the product of their first halves,

$c_0 = a_0 * b_0$ is the product of their second halves,

$c_1 = (a_1 + a_0) * (b_1 + b_0) - (c_2 + c_0)$ is the product of the sum of the a's halves and the sum of the b's halves minus the sum of c_2 and c_0.

Multiplication of Large Integers

Example: **2101 * 1130**

$C_2 = 21*11$, $C_0 = 01*30$

$C_1 = (21+01)*(11+30) - (C_2+C_0) = 22*41 - 21*11 - 01*30$

For 21*11

$C_2 = 2*1 = 2$

$C_0 = 1*1 = 1$

$C_1 = (2+1)*(1+1) - (2+1)$

$= 3*23 = 3$

$21*11 = 2*10^2 + 3*10^1 + 1$

$= \mathbf{231}$

For 01*30

$C2 = 0*3 = 0$

$C_0 = 1*0 = 0$

$C_1 = (0+1)*(3+0) - (0+0)$

$= 1*3 - 0 = 3$

$01*30 = 0*10^2 + 3*10^1$

$= \mathbf{30}$

For 22*41

$C2 = 2*4 = 8$

$C_0 = 2*1 = 2$

$C_1 = (2+2)*(4+1) - (8+2)$

$= 4*5 - 10 = 10$

$22*41 = 8*10^2 + 10*10^1 + 2$

$= \mathbf{902}$

Hence, $2101*1130 = 231*10^4 + (902-231-30)*10^2 + 30$

$= 2{,}374{,}130$

Multiplication of Large Integers : Analysis

- The recurrence relation for this approach would be:

$$M(n) = 3M(n/2), \quad \text{for } n>1,$$
$$M(1) = 1$$

- Solving it by backward substitution for $n = 2^k$:

$$M(2^k) = 3M(2^{k-1})$$
$$= 3M[3M(2^{k-2})] = 3^2 M(k-2) = \ldots\ldots = 3^i M(2^{k-i}) = \ldots\ldots = 3^k M(2^{k-k}) = \mathbf{3^k}$$

Since, $k = \log_2 n$:

$$M(n) = 3^{\log_2 n} = n^{\log_2 3} \approx \mathbf{n^{1.585}}$$

Matrix Multiplication

- Brute force approach for matrix multiplication

$$\begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} * \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00} * b_{00} + a_{01} * b_{10} & a_{00} * b_{01} + a_{01} * b_{11} \\ a_{10} * b_{00} + a_{11} * b_{10} & a_{10} * b_{01} + a_{11} * b_{11} \end{bmatrix}$$

- Time complexity = $O(n^3)$

Strassen's Matrix Multiplication

- Introduced by Volker Strassen in 1969.

$$\begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} * \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$

$$= \begin{bmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{bmatrix}$$

$m_1 = (a_{00} + a_{11}) * (b_{00} + b_{11})$

$m_2 = (a_{10} + a_{11}) * b_{00}$

$m_3 = a_{00} * (b_{01} - b_{11})$

$m_4 = a_{11} * (b_{10} - b_{00})$

$m_5 = (a_{00} + a_{01}) * b_{11}$

$m_6 = (a_{10} - a_{00}) * (b_{00} + b_{01})$

$m_7 = (a_{01} - a_{11}) * (b_{10} + b_{11})$

Exercise

Apply Strassen's Algorithm to multiply the given matrices:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & 5 & 0 \end{bmatrix}$$

Thank you!

Any queries?