If X & Y are independent then

$$\begin{cases} f(x,y) = g(x)\,h(y) \\ \& \; p(x_i, y_j) = p(x_i)\cdot q(y_j) \end{cases}$$

so If $f(x,y) \neq g(x)\cdot h(y) \Rightarrow X$ & Y are not independent.

Converse

Is this true????

Now:

If $g(x)\cdot h(y) = f(x,y)$ then X & Y need not independent

$\boxed{\text{Converse is not true}}$

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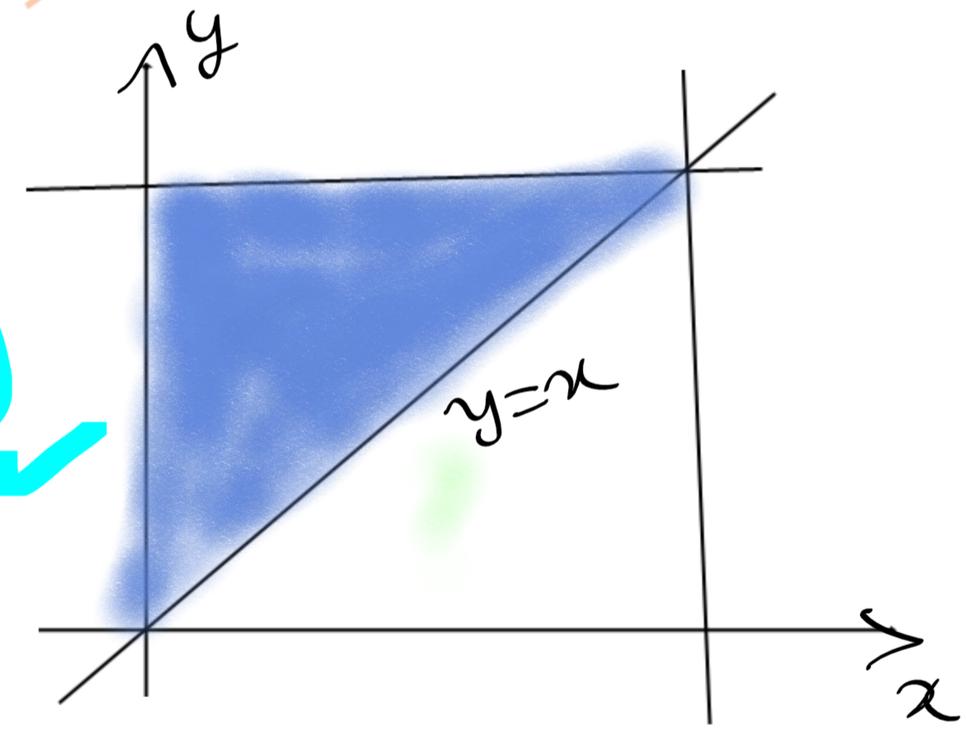
Ex: (X, Y)

$$f(x,y) = \begin{cases} 8xy, & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$1 \ge y \ge x$

$0 \le x \le y$



$f(x,y) = (8x)(y) = x(8y)$

product of $g(x)$ & $h(y)$

$\boxed{f(x,y) = g(x) \cdot h(y)}$

But X & Y are not independent

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Chebyshev's inequality

Let X be a r.v with $E(X)=\mu$ & let c be any real number. Then if $E(X-c)^2$ is finite & ϵ is any +ve number.

We have
$$P[|X-c| \geq \epsilon] \leq \frac{1}{\epsilon^2} E(X-c)^2$$

(i) $$P[|X-c| < \epsilon] \geq 1 - \frac{1}{\epsilon^2} E(X-c)^2$$

Case (i)

If $c=\mu$ & $\epsilon = k\sigma$, $\sigma^2 = Var\, X > 0$

$$P[|X-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

$$P[|X-\mu| < k\sigma] \geq 1 - \frac{1}{k^2}$$

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Case (i)

If $c = \mu$ & $\epsilon = k\sigma$, $\sigma^2 = Var\, X > 0$

$$P[\,|X-\mu| \geq k\sigma\,] \leq \frac{1}{k^2}$$

Case (ii):

If $c = \mu$

$$P[\,|X-\mu| \geq \epsilon\,] \leq \frac{Var\, X}{\epsilon^2}$$

$$\boxed{E(X-\mu)^2 = \sigma^2}$$

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If $f(x) = \begin{cases} 2e^{-2x} & ; x > 0 \\ 0 & \text{otherwise} \end{cases}$

verify Chebyshev's inequality for obtaining $P(|X-\mu| \geq 1)$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{\infty} x\, 2e^{-2x}\,dx = 2\int_{0}^{\infty} x e^{-2x}\,dx$$

$$= 2\left[x \cdot \frac{e^{-2x}}{-2} \Big|_0^{\infty} - (1)\ \frac{e^{-2x}}{4} \Big|_0^{\infty} \right]$$

$$= 2\left[(0-0) - \frac{1}{4}(0-1) \right]$$

$$E(X) = \frac{1}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{\infty} x^2\, 2e^{-2x}\,dx$$

$$V(X) =$$

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$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2\, 2e^{-2x}\,dx = 2\int_0^{\infty} \underbrace{x^2}_{I}\, \underbrace{e^{-2x}}_{II}\,dx$$

$$= 2\left[(x)^2\left(\frac{e^{-2x}}{-2}\right)\Big|_0^{\infty} - (2x)\left(\frac{e^{-2x}}{4}\right)\Big|_0^{\infty} + (2)\left(\frac{e^{-2x}}{-8}\right)\Big|_0^{\infty}\right]$$

$$= 2\left[-\frac{1}{4}(0-1)\right]$$

$$\boxed{E(X^2) = \frac{1}{2}}$$

Using generalized integration by part

$$\int \underbrace{u}_{I}\,\underbrace{v}_{II}\,dx = u\int v - du\iint v + d^2u\iiint v$$

$$= u v_1 - u' v_2 + u'' v_3 - \cdots\cdots$$

$v_1 = \int v\,dx$

$v_2 = \int v_1\,dx$

$v_3 = \int v_2\,dx$

$$V(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Direct method

$$P[|X-\mu| \geq 1] = P\left[\left|X - \frac{1}{2}\right| \geq 1\right]$$

$$= 1 - P\left[\left|X - \tfrac{1}{2}\right| < 1\right]$$

$$= 1 - \int_0^{3/2} f(x)\,dx$$

$$\left|X - \frac{1}{2}\right| \leq 1$$

$$-1 \leq X - \tfrac{1}{2} \leq 1$$

$$-\tfrac{1}{2} \leq X \leq \tfrac{3}{2}$$

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Direct method

$$P[|X-\mu| \geq 1] = P[|X-\tfrac{1}{2}| \geq 1]$$

$$= 1 - P[|X-1/2| < 1]$$

$$= 1 - \int_0^{3/2} f(x)\,dx = 1 - \int_0^{3/2} 2e^{-2x}\,dx$$

$$= 1 - 2\left.\frac{e^{-2x}}{-2}\right|_0^{3/2}$$

$$= 1 + (e^{-3} - 1) = e^{-3}$$

$$\therefore P[|X-\mu| \geq 1] = e^{-3}$$

$$= \underline{\underline{0.0497}}$$

By Chebyshev's inequality

$$P[|X-\mu| \geq 1] \leq \frac{V(X)}{1^2} = \underline{\underline{0.25}}$$

$$\boxed{P[|X-\mu| \geq \epsilon] \leq \frac{Var\,X}{\epsilon^2}}$$

$$\epsilon = 1$$

$$\Rightarrow \boxed{0.0497 \leq 0.25}$$

actual value < predicted value by Chebyshev's inequality

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If $\mu = 10$, $\sigma^2 = 4$. Find bound for $P[5 < X < 15]$

$E(X) = \mu = 10$, $\quad V(X) = \sigma^2 = 4$

By Chebyshev's inequality, $P[|X-\mu| \geq \epsilon] \leq \dfrac{Var\,X}{\epsilon^2}$

$$P[|X-10| \geq \epsilon] \leq \frac{4}{\epsilon^2}$$

$$1 - P[|X-10| < \epsilon] \leq \frac{4}{\epsilon^2}$$

$$\Rightarrow 1 - P[-5+10 < X < 5+10] \leq \frac{4}{25}$$

$$\Rightarrow 1 - P[5 < X < 15] \leq \frac{4}{25}$$

$$\Rightarrow 1 - \frac{4}{25} \leq P[5 < X < 15]$$

$$\therefore P[5 < X < 15] \geq \frac{21}{25}$$

$P[5 < X < 15]$

$|X-10| < \epsilon$

$-\epsilon < X - 10 < \epsilon$

$-\epsilon + 10 < X < \epsilon + 10$

$\Rightarrow \epsilon = 5$

$5 < X < 15$

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If two dice are thrown. Let X denote sum of the numbers showing up. Verify Chebyshev's inequality for $P(|X-7| \geq 3)$

Direct method: $E(X) = 7$, $E(X^2) = ?$ $V(X) = ?$

X : sum of the numbers showing up

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

$\Sigma = 1$

$$\begin{array}{c} \\ \\ \\ 6 \end{array}\left\{\begin{array}{l} (1,1), (1,2) \ldots \} 6 \\ (2,1) \ldots \} 6 \\ \vdots \\ (6,1) \ldots \} 6 \end{array}\right.$$

(1,3)	(1,4)	(1,5)
(3,1)	(4,1)	(5,1)
(2,2)	(2,3)	(3,3)
	(3,2)	(4,2)
		(2,4)

HW

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Suppose that the joint pdf of the two-dimensional random variable (X,Y) is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & \underbrace{0<x<1}, \ \underbrace{0<y<2} \\ 0 & \text{elsewhere} \end{cases}$$

Compute (a) $P(X > 1/2)$ (b) $P(Y < X)$ (c) $P(X+Y \geq 1)$
(d) $P(Y < 1/2 \mid X < 1/2)$

(a) $$P(X > 1/2) = \int_{1/2}^{1} g(x)\,dx = \int_{x=1/2}^{1} \left[\int_{y=-\infty}^{\infty} f(x,y)\,dy \right] dx$$

$$= \int_{x=1/2}^{1} \left[\int_{y=0}^{2} \left(x^2 + \frac{xy}{3} \right) dy \right] dx$$

$$P(X > 1/2) = ?$$

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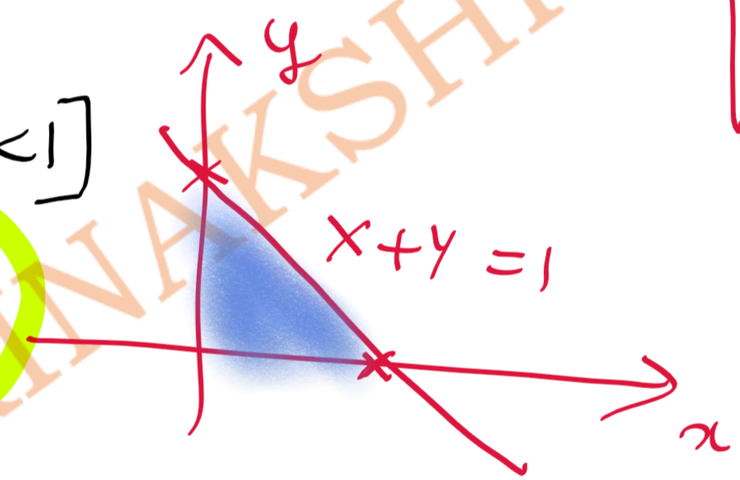
(b) $P[Y<X] = \int_{y=-\infty}^{\infty} h(y)\,dy = \int_{y=0}^{x}\left[\int_{x=-\infty}^{\infty} f(x,y)\,dx\right]dy$

$$P[Y<X] = \int_{y=0}^{x}\left[\int_{x=0}^{1} f(x,y)\,dx\right]dy = \int_{x=0}^{1}\left[\int_{y=0}^{x}\left(x^2+\frac{xy}{3}\right)dy\right]dx$$

$= ?$

(c) $P[X+Y\geq 1] = 1-P[X+Y<1]$



$= 1-P[Y<1-X]$

$$= 1-\int_{y=0}^{1-x}\left\{\int_{x=0}^{1} f(x,y)\,dx\right\}dy = 1-\int_{x=0}^{1}\left[\int_{y=0}^{1-x}\left(x^2+\frac{xy}{3}\right)dy\right]dx$$

$\therefore P[X+Y\geq 1] = ?$

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(d) $$P[Y<1/2 \mid X<1/2] = \frac{P[Y<1/2,\ X<1/2]}{P[X<1/2]}$$

$$= \frac{\int_{y=0}^{1/2}\left[\int_{x=0}^{1/2} f(x,y)\,dx\right]dy}{\int_{x=0}^{1/2} g(x)\,dx}$$

$$= \frac{\int_{y=0}^{1/2}\left[\int_{x=0}^{1/2} f(x,y)\,dx\right]dy}{\int_{x=0}^{1/2}\left[\int_{y=0}^{2} f(x,y)\,dy\right]dx}$$

HW

?

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