ME8135 — Assignment 7 Solution

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Introduction:

A *Lie group* is an old mathematical abstract object related to the theory of continuous transformation groups. In robotics, parts of the theory have proven to be extremely useful in modern estimation algorithms, especially in the fields of SLAM, visual odometry. In short, a Lie group is a group that is a *differential manifold*, with the property that the group operations are smooth. Here smoothness implies that we can use differential calculus on the manifold.

Typically, we are interested in the well-known manifolds of rotation SO(3) and rigid motion SE(3), also known as *matrix Lie groups*. The *special orthogonal group* (SO), representing rotations, is simply the set of valid rotation matrices. While the *special Euclidean group* (SE), representing poses (i.e., translation and rotation), is simply the set of valid transformation matrices. In this assignment, we will look more deeply into the nature and properties of these lie groups.

1. Problem Statement:

For the following function:

$$f : SO(3) \rightarrow \mathbb{R}^3; f(\mathbf{R}, \mathbf{p}) = \mathbf{R}\mathbf{p} \tag{1}$$

we wish to calculate the left and right derivatives of f with respect to $\mathbf{R}$, by applying the definitions of the left and right derivatives:

$$\frac{{}^{R}Df(\mathbf{R}, \mathbf{p})}{D\mathbf{R}} \tag{2}$$

$$\frac{{}^{L}Df(\mathbf{R}, \mathbf{p})}{D\mathbf{R}} \tag{3}$$

Q1 Solution Formulation:

For the right derivative we have (using (41a) from *Micro Lie Theory* paper and identity $[\mathbf{a}]_\times \mathbf{b} = -[\mathbf{b}]_\times \mathbf{a}$):

$$\begin{aligned}
\frac{{}^{R}Df(\mathbf{R}, \mathbf{p})}{D\mathbf{R}} &= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{(\mathbf{R} \oplus \boldsymbol{\theta}) \cdot \mathbf{p} \ominus \mathbf{R} \cdot \mathbf{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{(\mathbf{R} \cdot \mathrm{Exp}(\boldsymbol{\theta})) \cdot \mathbf{p} - \mathbf{R} \cdot \mathbf{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{\mathbf{R} \cdot (\mathbf{I} + [\boldsymbol{\theta}]_\times) \cdot \mathbf{p} - \mathbf{R} \cdot \mathbf{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{\mathbf{R} \cdot [\boldsymbol{\theta}]_\times \cdot \mathbf{p}}{\boldsymbol{\theta}} \\
&= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{-\mathbf{R} \cdot [\mathbf{p}]_\times \cdot \boldsymbol{\theta}}{\boldsymbol{\theta}} \\
&= -\mathbf{R} \cdot [\mathbf{p}]_\times \in \mathbb{R}^{3\times 3}
\end{aligned} \tag{4}$$

Note that in (4) we used the right-$\oplus$ operator, as defined in (25) of *Micro Lie Theory* paper.

For the left derivative we have (using (44) from *Micro Lie Theory* paper):

$$\begin{aligned} \frac{{}^{L}Df(\mathbf{R},\mathbf{p})}{D\mathbf{R}} &= \lim_{\boldsymbol{\theta}\to 0} \frac{(\boldsymbol{\theta}\oplus\mathbf{R})\cdot\mathbf{p}\ominus\mathbf{R}\cdot\mathbf{p}}{\boldsymbol{\theta}} \\ &= \lim_{\boldsymbol{\theta}\to 0} \frac{(\mathrm{Exp}(\boldsymbol{\theta})\cdot\mathbf{R})\cdot\mathbf{p}-\mathbf{R}\cdot\mathbf{p}}{\boldsymbol{\theta}} \\ &= \lim_{\boldsymbol{\theta}\to 0} \frac{(\mathbf{I}+[\boldsymbol{\theta}]_\times)\cdot\mathbf{R}\cdot\mathbf{p}-\mathbf{R}\cdot\mathbf{p}}{\boldsymbol{\theta}} \\ &= \lim_{\boldsymbol{\theta}\to 0} \frac{[\boldsymbol{\theta}]_\times\cdot\mathbf{R}\cdot\mathbf{p}}{\boldsymbol{\theta}} \\ &= \mathbf{R}\cdot[\mathbf{p}]_\times \in \mathbb{R}^{3\times 3} \end{aligned} \tag{5}$$

Note that in (5) we used the left-$\oplus$ operator, as defined in (27) of *Micro Lie Theory* paper.

2. Problem Statement:

Given a Lie group $\mathcal{M}$ with the composition operation $\circ$, and elements $\mathcal{X},\mathcal{Y}\in\mathcal{M}$, we wish to calculate the derivative of $\mathcal{X}\circ\mathcal{Y}$ with respect to $\mathcal{Y}$:

$$\frac{{}^{\mathcal{Y}}D\mathcal{X}\circ\mathcal{Y}}{D\mathcal{Y}} \tag{6}$$

Q2 Solution Formulation:

For the right Jacobian we have (using (64) from *Micro Lie Theory* paper):

$$\begin{aligned} \mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} &\triangleq \frac{{}^{\mathcal{Y}}D\mathcal{X}\circ\mathcal{Y}}{D\mathcal{Y}} \in \mathbb{R}^{m\times m} \\ \mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} &= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathrm{Log}((\mathcal{X}\mathcal{Y})^{-1}(\mathcal{X}\mathcal{Y}\mathrm{Exp}(\boldsymbol{\theta})))}{\boldsymbol{\theta}} \\ &= \lim_{\boldsymbol{\theta}\to 0} \frac{(\boldsymbol{\theta}^\wedge)^\vee}{\boldsymbol{\theta}} \\ &= \lim_{\boldsymbol{\theta}\to 0} \frac{\boldsymbol{\theta}}{\boldsymbol{\theta}} \\ &= \mathbf{I} \end{aligned} \tag{7}$$

For the left Jacobian we have:

$$\begin{aligned} \mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} &= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathcal{X}(\boldsymbol{\theta}\oplus\mathcal{Y})\ominus\mathcal{X}\mathcal{Y}}{\boldsymbol{\theta}} \\ &= \lim_{\boldsymbol{\theta}\to 0} \frac{\mathrm{Log}((\mathcal{X}\mathcal{Y}\mathrm{Exp}(\boldsymbol{\theta}))(\mathcal{X}\mathcal{Y})^{-1})}{\boldsymbol{\theta}} \\ &= \lim_{\boldsymbol{\theta}\to 0} \frac{(\mathcal{X}(\boldsymbol{\theta})^\wedge\mathcal{X}^{-1})^\vee}{\boldsymbol{\theta}} \\ &= \mathbf{Ad}_{\mathcal{X}} \end{aligned} \tag{8}$$

3. Problem Statement:

Given the Lie group of $M = SE(3)$ with with the composition operation $\circ$, and elements $\mathcal{X},\mathcal{Y}\in M$, we wish to show that:

$$\mathbf{Ad}_{\mathcal{X}}\mathbf{Ad}_{\mathcal{Y}} = \mathbf{Ad}_{\mathcal{X}\circ\mathcal{Y}} \tag{9}$$

$$\mathbf{Ad}_{\mathcal{X}^{-1}} = \mathbf{Ad}_{\mathcal{X}}^{-1} \tag{10}$$

Q3 Solution Formulation:

In Lie groups, it is often necessary to transform a tangent vector from the tangent space around one element to the tangent space of another. The adjoint performs this transformation. One very nice property of Lie groups in general is that this transformation is linear. For an element $\mathcal{X}$ of a Lie group M, the adjoint is written as $\mathbf{Ad}_{\mathcal{X}}$.

For (9) we can show that:

$$\begin{aligned}\mathbf{Ad}_{\mathcal{X}}\mathbf{Ad}_{\mathcal{Y}} &= \begin{bmatrix} R_{\mathcal{X}} & [t]_{\times}R_{\mathcal{X}} \\ 0 & R_{\mathcal{X}} \end{bmatrix}\begin{bmatrix} R_{\mathcal{Y}} & [t]_{\times}R_{\mathcal{Y}} \\ 0 & R_{\mathcal{Y}} \end{bmatrix} \\ &= \begin{bmatrix} R_{\mathcal{X}}R_{\mathcal{Y}} & R_{\mathcal{X}}[t]_{\times}R_{\mathcal{Y}} + [t]_{\times}R_{\mathcal{X}}R_{\mathcal{Y}} \\ 0 & R_{\mathcal{X}}R_{\mathcal{Y}} \end{bmatrix}\end{aligned} \tag{11}$$

$$\begin{aligned}\mathbf{Ad}_{\mathcal{X}\circ\mathcal{Y}} &= \begin{bmatrix} R_{\mathcal{X}\circ\mathcal{Y}} & [t]_{\times}R^2_{\mathcal{X}\circ\mathcal{Y}} \\ 0 & R_{\mathcal{X}\circ\mathcal{Y}} \end{bmatrix} \\ &= \begin{bmatrix} R_{\mathcal{X}}R_{\mathcal{Y}} & R_{\mathcal{X}}[t]_{\times}R_{\mathcal{Y}} + [t]_{\times}R_{\mathcal{X}}R_{\mathcal{Y}} \\ 0 & R_{\mathcal{X}}R_{\mathcal{Y}} \end{bmatrix}\end{aligned} \tag{12}$$

We deduce that (11) is equivalent to (12), hence proving (9).

Lastly, for (10) we can show that (using defintion (62) from *Micro Lie Theory* paper):

$$\begin{aligned}\mathbf{Ad}_{\mathcal{X}^{-1}} &= -\begin{bmatrix} R_{\mathcal{X}} & [t]_{\times}R_{\mathcal{X}} \\ 0 & R_{\mathcal{X}} \end{bmatrix} \\ &= -\mathbf{Ad}_{\mathcal{X}}\end{aligned} \tag{13}$$

We also know that:

$$\mathbf{Ad}_{\mathcal{X}}^{-1} = -\begin{bmatrix} R_{\mathcal{X}} & [t]_{\times}R_{\mathcal{X}} \\ 0 & R_{\mathcal{X}} \end{bmatrix} \tag{14}$$

We deduce that (13) is equivalent to (14), hence proving (10).