ME8135 — Assignment 1 Solution

Student: Arash Basirat Tabrizi
Submitted to: Dr. Sajad Saeedi
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1. $\mathbf{x}$ is a random variable of length K:

$$\mathrm{x} = \mathcal{N}(0,1) \tag{1}$$

(a) We want to determine what type of random variable $\mathbf{y}$ is given that:

$$\mathrm{y} = \mathrm{x}^{\mathrm{T}}\mathrm{x} \tag{2}$$

We know that $\mathbf{x}$ is a random variable which is *standard normally* distributed.

Thus, random variable $\mathbf{y}$ is *chi-squared* (of order K) when $\mathrm{x} = \mathcal{N}(0,1)$ is length K.

(b) To compute the mean and variance of $\mathbf{y}$, we can use *Isserlis'* theorem (2.2.2).

Denoting $\mathbf{x} = [x_1, x_2, ..., x_K]^{\mathrm{T}}$ and $x_i \in \mathbf{x}$, given that $x_i \sim \mathcal{N}(0,1)$.

Also, $E\{x_i, x_j\} = 0$ and $E\{x_i, x_i\} = 1$ where $\forall i, j \in [1, K]$ and $i \neq j$.

Mean of $\mathbf{y}$ is computed as follows:

$$E\left\{\mathbf{x}^T\mathbf{x}\right\} = E\{x_1x_1\} + ... + E\{x_Kx_K\} = \sum_{i=1}^{K} E\{x_i, x_i\} = K \tag{3}$$

Variance of $\mathbf{y}$ is computed as follows:

$$\begin{aligned}
Var(\mathbf{y}) &= E\left\{\left(\mathbf{x}^T\mathbf{x} - K\right)\left(\mathbf{x}^T\mathbf{x} - K\right)^T\right\} \\
&= E\left\{\mathbf{x}^T\mathbf{x}\mathbf{x}^T\mathbf{x}\right\} - 2E\left\{\mathbf{x}\mathbf{x}^T\right\}K + K^2 \\
&= E\left\{(x_1x_1 + ... + x_Kx_K)(x_1x_1 + ... + x_Kx_K)\right\} - K^2 \\
&= E\left\{\sum_{i=1}^{K} x_ix_ix_ix_i\right\} + E\left\{\sum_{\forall i,j \in [1,K]} x_ix_ix_jx_j\right\} - K^2 \\
&= KE\{x_ix_ix_ix_i\} + \left(K^2 - K\right)E\{x_ix_ix_jx_j\} - K^2 \\
&= K(3) + \left(K^2 - K\right)(1) - K^2 \\
&= 2K
\end{aligned} \tag{4}$$

Where from *Isserlis'* theorem and equation (2.40) of the textbook we have:

$$E\{x_ix_ix_ix_i\} = 3E\{x_ix_i\}E\{x_ix_i\} = 3 \tag{5}$$

$$E\{x_ix_ix_jx_j\} = E\{x_ix_i\}E\{x_jx_j\} + 2E\{x_ix_j\}E\{x_ix_j\} = 1 \tag{6}$$

(c) Please refer to the GitHub repo (A1/Q1_C.ipynb directory).

2. $\mathbf{x}$ is a random variable of length N:

$$\mathrm{x} = \mathcal{N}(\mu, \Sigma) \tag{7}$$

(a) To calculate the mean and covariance of $\mathbf{y}$, where $\mathbf{y} = A\mathbf{x}$ (A is an $N \times N$ matrix), we do as follows:

$$\mu_y = E\left[\mathbf{y}\right] = E\left[A\mathbf{x}\right] = AE\left[\mathbf{x}\right] = A\mu \tag{8}$$

$$\begin{aligned}\Sigma_{yy} &= E\left[\left(\mathrm{y} - \mu_y\right)\left(\mathrm{y} - \mu_y\right)^T\right] \\ &= AE\left[\left(\mathrm{x} - \mu\right)\left(\mathrm{x} - \mu\right)^T\right]A^T \\ &= A\Sigma A^T\end{aligned} \tag{9}$$

(b) Calculating the mean and covariance of $\mathbf{y}$, where $\mathbf{y} = A_1\mathbf{x} + A_2\mathbf{x}$:

$$\mu_y = E\left[\mathbf{y}\right] = E\left[A_1\mathbf{x} + A_2\mathbf{x}\right] = A_1E\left[\mathbf{x}\right] + A_2E\left[\mathbf{x}\right] = (A_1 + A_2)\mu \tag{10}$$

$$\begin{aligned}\Sigma_{yy} &= E\left[\left(\mathrm{y} - \mu_y\right)\left(\mathrm{y} - \mu_y\right)^T\right] \\ &= E\left[\left(A_1\mathrm{x} + A_2\mathrm{x} - (A_1 + A_2)\mu\right)\left(A_1\mathrm{x} + A_2\mathrm{x} - (A_1 + A_2)\mu\right)^T\right] \\ &= E\left[\left((A_1 + A_2)\left(\mathrm{x} - \mu\right)\right)\left((A_1 + A_2)\left(\mathrm{x} - \mu\right)\right)^T\right] \\ &= (A_1 + A_2)E\left[\left(\mathrm{x} - \mu\right)\left(\mathrm{x} - \mu\right)^T\right](A_1 + A_2)^T \\ &= (A_1 + A_2)\Sigma(A_1 + A_2)^T\end{aligned} \tag{11}$$

(c) In the case that $\mathbf{x}$ goes through a nonlinear differentiable function to produce $\mathbf{y} = \mathrm{f}(\mathbf{x})$, we can compute the the covariance matrix of $\mathbf{y}$ by *linearization* of the nonlinear map, f(.), in equation (12), and then passing our Gaussian through this linearized function in closed form to complete our approximation:

$$\mathrm{f}(\mathrm{x}) \approx \mathrm{f}(\mu) + \mathbf{J}\left(\mathrm{x} - \mu\right) \tag{12}$$

In equation (12), $\mathbf{J}$ is the Jacobian of f(.) with respect to x.

Passing a Gaussian PDF, $p(\mathrm{x})$, through a stochastic nonlinearity, we compute:

$$p(\mathrm{y}) = \int_{-\infty}^{\infty} p(\mathrm{y}|\mathrm{x})p(\mathrm{x})d\mathrm{x} \tag{13}$$

Where we have

$$p(\mathrm{y}|\mathrm{x}) = \mathcal{N}(\mathrm{f}(\mathrm{x}), \mathbf{R}) \tag{14}$$

$$p(\mathrm{x}) = \mathcal{N}(\mu, \Sigma) \tag{15}$$

In equation (14), our nonlinear map, f(.), is corrupted by zero-mean Gaussian noise with covariance, **R**. Computing equation (14), as outlined in section 2.2.8 of the textbook, yields the Gaussian for y:

$$\mathrm{y} = \mathcal{N}(\mathrm{f}(\mu), \mathrm{R} + \mathrm{J}\Sigma\mathrm{J}^T) \tag{16}$$

Thus, covariance matrix of y is given by $\mathrm{J}\Sigma\mathrm{J}^T$.

(d) To compute the covariance matrix of y analytically, given:

$$\mathrm{x} = \begin{bmatrix} \rho \\ \theta \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix}, \text{and } \mathrm{y} = \begin{bmatrix} \rho cos\theta \\ \rho sin\theta \end{bmatrix} \tag{17}$$

Step 1: Compute the the Jacobian matrix J:

$$\mathrm{J} = \frac{\partial \mathrm{f}}{\partial \mathrm{x}} = \begin{bmatrix} cos\theta & -\rho sin\theta \\ sin\theta & \rho cos\theta \end{bmatrix} \tag{18}$$

Step 2: Apply part (c) to compute Σ_y:

$$\Sigma_y = \mathrm{J}\Sigma\mathrm{J}^T = \begin{bmatrix} cos\theta & -\rho sin\theta \\ sin\theta & \rho cos\theta \end{bmatrix} \begin{bmatrix} \sigma^2_{\rho\rho} & \sigma^2_{\rho\theta} \\ \sigma^2_{\rho\theta} & \sigma^2_{\theta\theta} \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -\rho sin\theta & \rho cos\theta \end{bmatrix} \tag{19}$$

(e) Please refer to the GitHub repo (A1/Q2_E.ipynb directory).