

ME8135 — Assignment 1 Solution

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1. \mathbf{x} is a random variable of length K :

$$\mathbf{x} = \mathcal{N}(0, 1) \quad (1)$$

(a) We want to determine what type of random variable \mathbf{y} is given that:

$$\mathbf{y} = \mathbf{x}^T \mathbf{x} \quad (2)$$

We know that \mathbf{x} is a random variable which is *standard normally* distributed.

Thus, random variable \mathbf{y} is *chi-squared* (of order K) when $\mathbf{x} = \mathcal{N}(0, 1)$ is length K .

(b) To compute the mean and variance of \mathbf{y} , we can use *Isserlis'* theorem (2.2.2).

Denoting $\mathbf{x} = [x_1, x_2, \dots, x_K]^T$ and $x_i \in \mathbf{x}$, given that $x_i \sim \mathcal{N}(0, 1)$.

Also, $E\{x_i, x_j\} = 0$ and $E\{x_i, x_i\} = 1$ where $\forall i, j \in [1, K]$ and $i \neq j$.

Mean of \mathbf{y} is computed as follows:

$$E\{\mathbf{x}^T \mathbf{x}\} = E\{x_1 x_1\} + \dots + E\{x_K x_K\} = \sum_{i=1}^K E\{x_i, x_i\} = K \quad (3)$$

Variance of \mathbf{y} is computed as follows:

$$\begin{aligned} \text{Var}(\mathbf{y}) &= E\left\{(\mathbf{x}^T \mathbf{x} - K)(\mathbf{x}^T \mathbf{x} - K)^T\right\} \\ &= E\{\mathbf{x}^T \mathbf{x} \mathbf{x}^T \mathbf{x}\} - 2E\{\mathbf{x} \mathbf{x}^T\} K + K^2 \\ &= E\{(x_1 x_1 + \dots + x_K x_K)(x_1 x_1 + \dots + x_K x_K)\} - K^2 \\ &= E\left\{\sum_{i=1}^K x_i x_i x_i x_i\right\} + E\left\{\sum_{\forall i, j \in [1, K]} x_i x_i x_j x_j\right\} - K^2 \\ &= KE\{x_i x_i x_i x_i\} + (K^2 - K)E\{x_i x_i x_j x_j\} - K^2 \\ &= K(3) + (K^2 - K)(1) - K^2 \\ &= 2K \end{aligned} \quad (4)$$

Where from *Isserlis'* theorem and equation (2.40) of the textbook we have:

$$E\{x_i x_i x_i x_i\} = 3E\{x_i x_i\}E\{x_i x_i\} = 3 \quad (5)$$

$$E\{x_i x_i x_j x_j\} = E\{x_i x_i\}E\{x_j x_j\} + 2E\{x_i x_j\}E\{x_i x_j\} = 1 \quad (6)$$

(c) Please refer to the GitHub repo (A1/Q1.C.ipynb directory).

2. \mathbf{x} is a random variable of length N :

$$\mathbf{x} = \mathcal{N}(\mu, \Sigma) \quad (7)$$

(a) To calculate the mean and covariance of \mathbf{y} , where $\mathbf{y} = A\mathbf{x}$ (A is an $N \times N$ matrix), we do as follows:

$$\mu_y = E[\mathbf{y}] = E[A\mathbf{x}] = AE[\mathbf{x}] = A\mu \quad (8)$$

$$\begin{aligned} \Sigma_{yy} &= E[(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^T] \\ &= AE[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^T]A^T \\ &= A\Sigma A^T \end{aligned} \quad (9)$$

(b) Calculating the mean and covariance of \mathbf{y} , where $\mathbf{y} = A_1\mathbf{x} + A_2\mathbf{x}$:

$$\mu_y = E[\mathbf{y}] = E[A_1\mathbf{x} + A_2\mathbf{x}] = A_1E[\mathbf{x}] + A_2E[\mathbf{x}] = (A_1 + A_2)\mu \quad (10)$$

$$\begin{aligned} \Sigma_{yy} &= E[(\mathbf{y} - \mu_y)(\mathbf{y} - \mu_y)^T] \\ &= E[(A_1\mathbf{x} + A_2\mathbf{x} - (A_1 + A_2)\mu)(A_1\mathbf{x} + A_2\mathbf{x} - (A_1 + A_2)\mu)^T] \\ &= E[((A_1 + A_2)(\mathbf{x} - \mu))((A_1 + A_2)(\mathbf{x} - \mu))^T] \\ &= (A_1 + A_2)E[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^T](A_1 + A_2)^T \\ &= (A_1 + A_2)\Sigma(A_1 + A_2)^T \end{aligned} \quad (11)$$

(c) In the case that \mathbf{x} goes through a nonlinear differentiable function to produce $\mathbf{y} = f(\mathbf{x})$, we can compute the the covariance matrix of \mathbf{y} by *linearization* of the nonlinear map, $f(\cdot)$, in equation (12), and then passing our Gaussian through this linearized function in closed form to complete our approximation:

$$f(\mathbf{x}) \approx f(\mu) + \mathbf{J}(\mathbf{x} - \mu) \quad (12)$$

In equation (12), \mathbf{J} is the Jacobian of $f(\cdot)$ with respect to \mathbf{x} .

Passing a Gaussian PDF, $p(\mathbf{x})$, through a stochastic nonlinearity, we compute:

$$p(y) = \int_{-\infty}^{\infty} p(y|\mathbf{x})p(\mathbf{x})d\mathbf{x} \quad (13)$$

Where we have

$$p(y|x) = \mathcal{N}(f(x), \mathbf{R}) \quad (14)$$

$$p(x) = \mathcal{N}(\mu, \Sigma) \quad (15)$$

In equation (14), our nonlinear map, $f(\cdot)$, is corrupted by zero-mean Gaussian noise with covariance, \mathbf{R} . Computing equation (14), as outlined in section 2.2.8 of the textbook, yields the Gaussian for y :

$$y = \mathcal{N}(f(\mu), \mathbf{R} + \mathbf{J}\Sigma\mathbf{J}^T) \quad (16)$$

Thus, covariance matrix of y is given by $\mathbf{J}\Sigma\mathbf{J}^T$.

(d) To compute the covariance matrix of y analytically, given:

$$x = \begin{bmatrix} \rho \\ \theta \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_{\rho\rho}^2 & \sigma_{\rho\theta}^2 \\ \sigma_{\rho\theta}^2 & \sigma_{\theta\theta}^2 \end{bmatrix}, \text{ and } y = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix} \quad (17)$$

Step 1: Compute the the Jacobian matrix \mathbf{J} :

$$\mathbf{J} = \frac{\partial f}{\partial x} = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} \quad (18)$$

Step 2: Apply part (c) to compute Σ_y :

$$\Sigma_y = \mathbf{J}\Sigma\mathbf{J}^T = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{\rho\rho}^2 & \sigma_{\rho\theta}^2 \\ \sigma_{\rho\theta}^2 & \sigma_{\theta\theta}^2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{bmatrix} \quad (19)$$

(e) Please refer to the GitHub repo (A1/Q2_E.ipynb directory).