CSSA
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

Centre Number

Student Number

2015
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Thursday July 30 2015

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2 – 7

10 marks
- Attempt Questions 1 – 10
- Allow 15 minutes for this section

Section II Pages 8-16

90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given that $z = 1 + i$, what is the value of z^8?

(A) -16

(B) -8

(C) 8

(D) 16

2 What are the equations of the directrices of the ellipse $\frac{x^2}{4} + y^2 = 1$?

(A) $x = \pm\frac{4}{\sqrt{3}}$

(B) $x = \pm\sqrt{3}$

(C) $x = \pm\frac{\sqrt{5}}{2}$

(D) $x = \pm\frac{2}{\sqrt{5}}$

3 The equation $x^3 - 8x^2 + px + q = 0$ where p and q are real numbers has roots α, β and γ.
What is the value of α if $\alpha = \beta + \gamma$?

(A) -2

(B) 2

(C) -4

(D) 4

4 When the substitution $x = 5\sin\theta$ is used, the integral equivalent to $\int \frac{x^2}{(25-x^2)^{\frac{3}{2}}}dx$ is

(A) $\int \frac{\sin^2\theta}{\cos^3\theta}d\theta$

(B) $\int 5\tan^2\theta\ d\theta$

(C) $\int \sqrt{5}\tan^2\theta\ d\theta$

(D) $\int \tan^2\theta\ d\theta$

5 A wheel of radius 1.5 m revolves at 1000 revolutions per minute. What is the tangential velocity of a point on the rim of the wheel?
Answer to the nearest whole number.

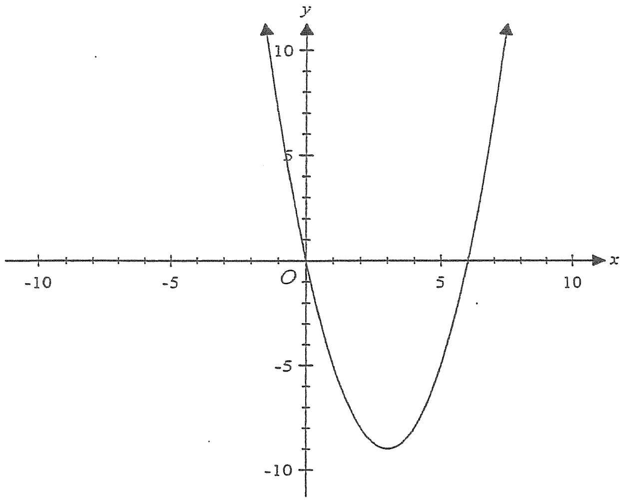
(A) 4 m/s

(B) 50 m/s

(C) 157 m/s

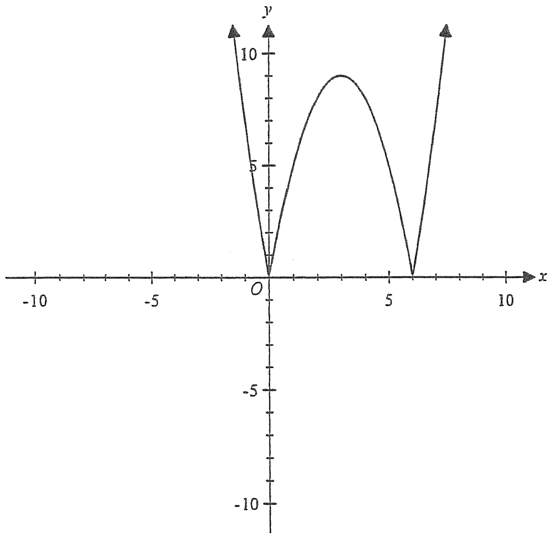
(D) 1570 m/s

6 Consider the graph of $y = f(x)$ drawn below.

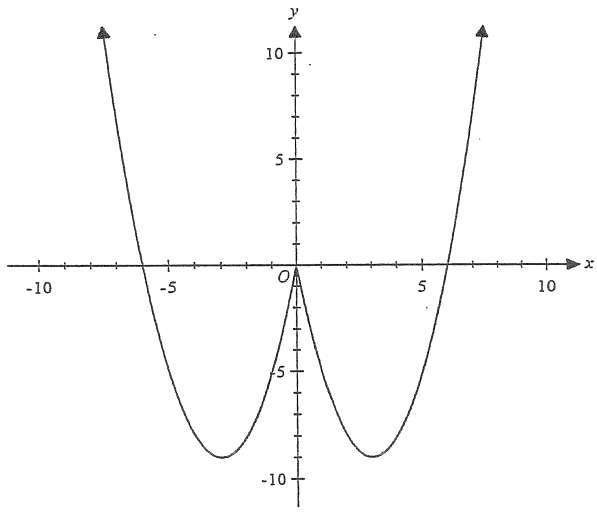


Which one of the following diagrams shows the graph of $y = f(|x|)$?

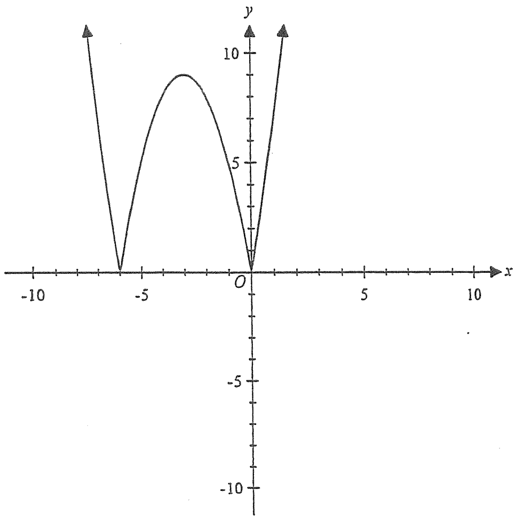
(A)



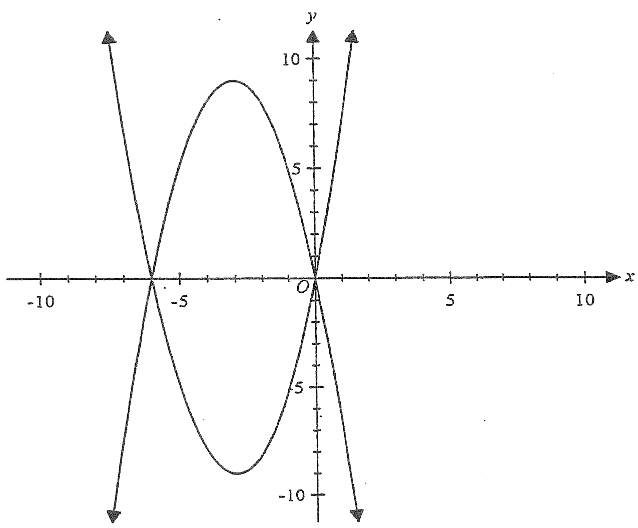
(B)



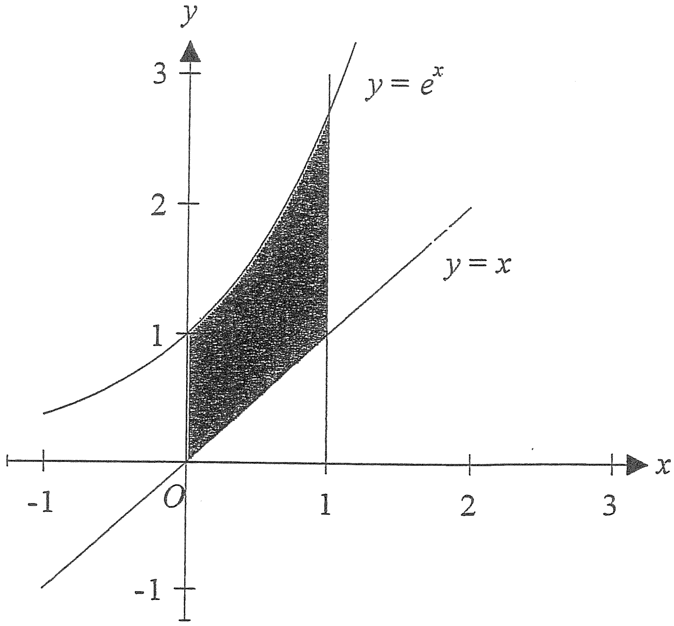
(C)



(D)



7 The region bounded by the curve $y = e^x$, the line $y = x$, the y-axis and the line $x = 1$ is rotated about the y-axis to form a solid.



Using the method of cylindrical shells, which integral represents the volume of this solid?

(A) $2\pi\int_0^1 x(e^x - x)\,dx$

(B) $2\pi\int_0^1 (x - e^x)\,dx$

(C) $2\pi\int_0^1 (e^x - x)\,dx$

(D) $2\pi\int_0^1 x(x - e^x)\,dx$

8 Consider the graph of $x^3 + y^3 = 1$. Which one of the following is FALSE?

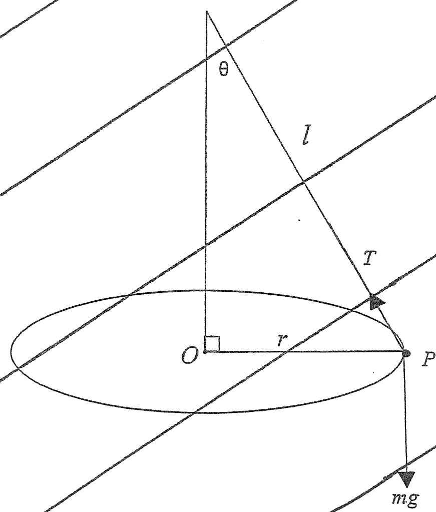
(A) $\frac{dy}{dx} < 0$ for all real values of x and y, except for $x = 0$ and $y = 0$.

(B) $y = x$ is an oblique asymptote.

(C) There is a vertical tangent at (1, 0) and a horizontal tangent at (0, 1).

(D) Both domain and range are the set of all real numbers.

9 A particle P of mass m kg is connected at one end of a light inextensible string of length l. The other end of the string is fixed and the particle moves with constant angular velocity ω in a horizontal circle of radius r and centre O. The system forms a conical pendulum.



The forces acting on the particle are the gravitational force mg and the tension T in the string. The string makes an angle of θ with the vertical.

Which one of the following is TRUE?

(A) $\sin\theta = \frac{g}{l\omega^2}$

(B) $\cos\theta = \frac{g}{l\omega^2}$

(C) $\cos\theta = \frac{l\omega^2}{g}$

(D) $\sin\theta = \frac{l\omega^2}{g}$

10 In how many ways can 10 students be placed in rooms A, B and C subject to the condition that a maximum of 4 can be placed in any one room?

(A) 22050

(B) 25200

(C) 44100

(D) 50400

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $w = 3 - 4i$ and $z = 2 + 2i$.

(i) Find $w\bar{z}$ in the form $x + iy$. **1**

(ii) Find $\text{Im}\left(\dfrac{1}{2-w}\right)$. **2**

(iii) The point representing the complex number z is rotated 270^0 in an anti-clockwise direction about the origin in an Argand diagram. **1**

What is the complex number represented by the new position of the point?

(b) The equation $3x^3 - x^2 + 2x + 1 = 0$ has roots α, β and γ.

(i) Find the cubic equation with integer coefficients whose roots are α^2, β^2 and γ^2. **2**

(ii) Hence, or otherwise, find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

(c) (i) Find the values of a and b such that **2**

$$\frac{6}{(x+2)(x-1)} = \frac{a}{x+2} + \frac{b}{x-1}.$$

(ii) Hence find $\displaystyle\int \frac{dx}{(x+2)(x-1)}$. **2**

(d) Using the substitution $t = \tan\left(\dfrac{x}{2}\right)$, or otherwise, find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$. **4**

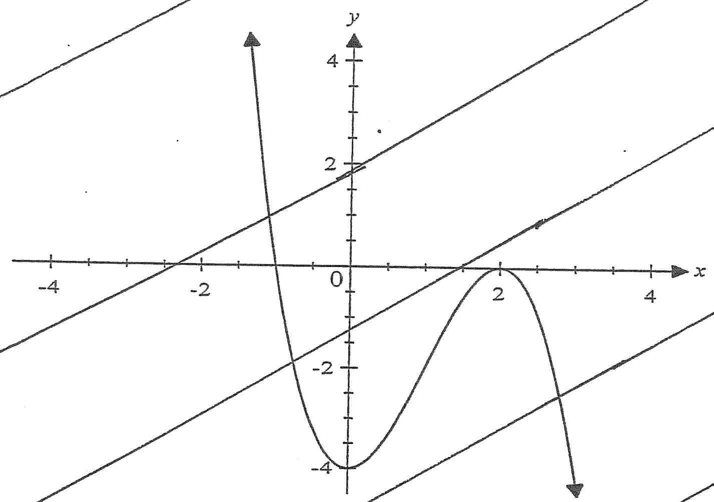
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find the values of real numbers a and b such that $(a+ib)^2 = -8+6i$. **2**

(ii) Hence solve the quadratic equation $2z^2 - (3+i)z + 2 = 0$. **2**

(b) The diagram shows the graph of a function $f(x)$.



Draw a separate half page graph for each of the following functions, showing clearly all intercepts and any asymptotes.

(i) $y = [f(x)]^2$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = e^{f(x)}$ **2**

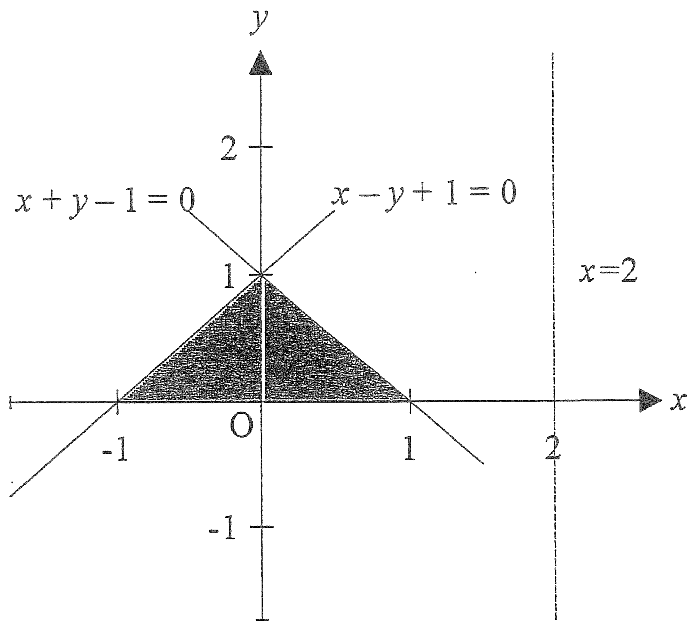
(c) Let $P(x) = x^3 + 4x^2 + px + q$, where p and q are real. **3**

Given that $2+i$ is a complex zero of $P(x)$, express $P(x)$ as a product of real linear and quadratic factors.

Question 12 continues on the next page

Question 12 (continued)

(d) The region bounded by the lines $x - y + 1 = 0$, $x + y - 1 = 0$ and the x-axis is rotated about the line $x = 2$. **3**

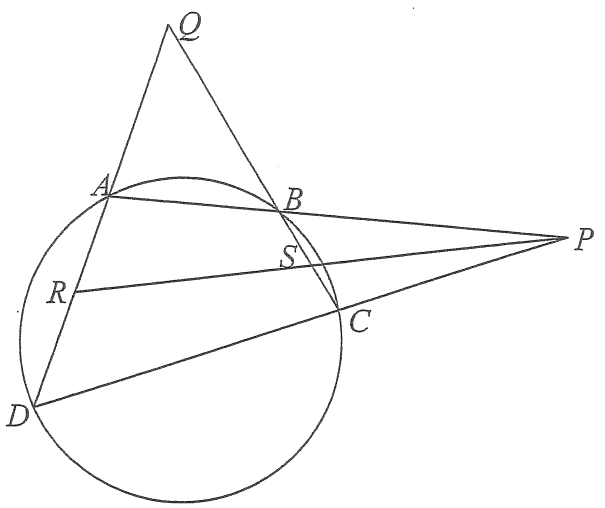


Using slices perpendicular to the axis of rotation, find the volume of the solid that is formed.

End of Question12

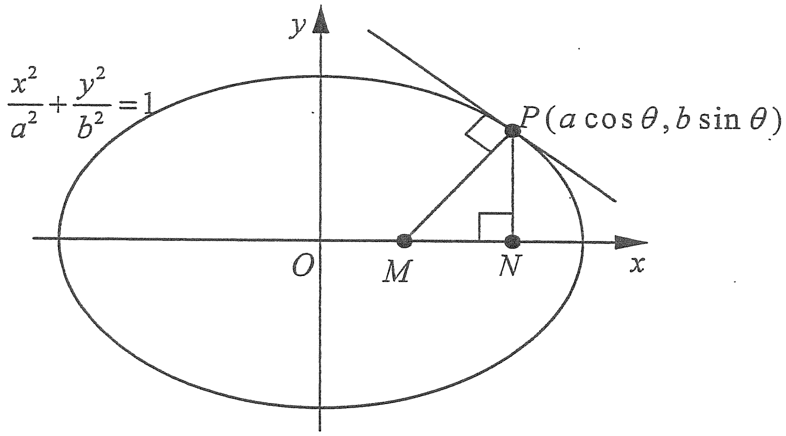
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows a cyclic quadrilateral $ABCD$. **2**
Chords AB and DC produced meet at P and chords DA and CB produced meet at Q.
PR is the internal bisector of $\angle APD$ meeting AD at R and BC at S.



Prove that ΔQRS is isosceles.

(b) $P\,(a\cos\theta, b\sin\theta)$ is a point on the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
The normal at P meets the x-axis at M and PN is perpendicular to the x-axis.



(i) Show that the equation of the normal at P is given by **2**

$$ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta.$$

(ii) Hence show that $MN = \left|\frac{b^2\cos\theta}{a}\right|$. **2**

Question 13 continues on the next page

Question 13 (continued)

(c) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ for all integers $n \geq 0$.

(i) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for integers $n \geq 2$. **3**

(ii) Hence find $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$. **2**

(d) (i) Prove that $c\int_a^b f(ct)dt = \int_{ca}^{cb} f(t)dt$, using the substitution $u = ct$. **2**

(ii) Use the result in (i) to show that $\int_1^{cb} \frac{1}{t} dt = \int_1^b \frac{1}{t} dt + \int_1^c \frac{1}{t} dt$. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) When a polynomial $P(x)$ is divided by $(x-1)$ the remainder is 3 and when divided by $(x-2)$ the remainder is 5. Find the remainder when the polynomial $P(x)$ is divided by $(x-1)(x-2)$. **2**

(b) The complex roots of $z^3 = 1$ are given by 1, ω and ω^2.

(i) Find the value of $(1+\omega^2)^6$. **1**

(ii) Hence show that $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9$. **2**

(c) There are n cars in a parking lot that has n exit gates.
Each of the cars can leave the parking lot through any one of the exit gates.

(i) In how many ways can all of the cars leave the parking lot? **1**

(ii) If all of the cars leave the parking lot, what is the probability that at least 1 exit gate is not chosen by any of the cars? **2**

(d) A particle of mass m kg is projected vertically upward under gravity with initial speed nV ms^{-1} in a medium in which the resistance to the motion is given by mkv^2, where v is the speed at time t seconds. The constants k and n are positive and V is the terminal velocity of the particle in the medium.

(i) Find the time taken for the particle to reach its maximum height. **3**

(ii) Having reached its maximum height, the particle falls to its starting point. The particle is still under the effect of gravity and a resistance which has magnitude of mkv^2. **1**

Show that the terminal velocity is given by $V = \sqrt{\dfrac{g}{k}}$.

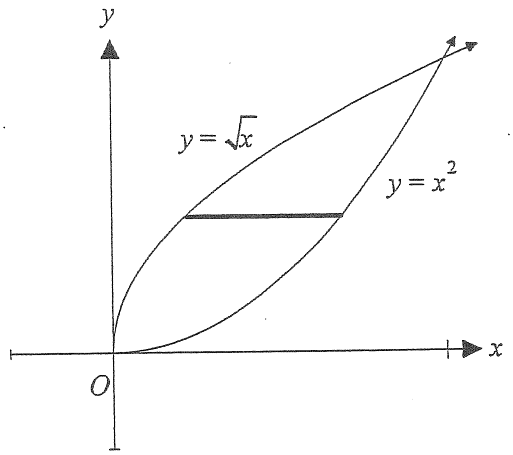
(iii) Find the distance below the maximum height where the particle attains 40% of its terminal velocity. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) By using integration by parts, find $\int_0^1 2\tan^{-1} x \; dx$. **3**

(b) The area between the curves $y = x^2$ and $y = \sqrt{x}$ is the base of a solid S. Cross sections perpendicular to the y-axis are squares. **3**



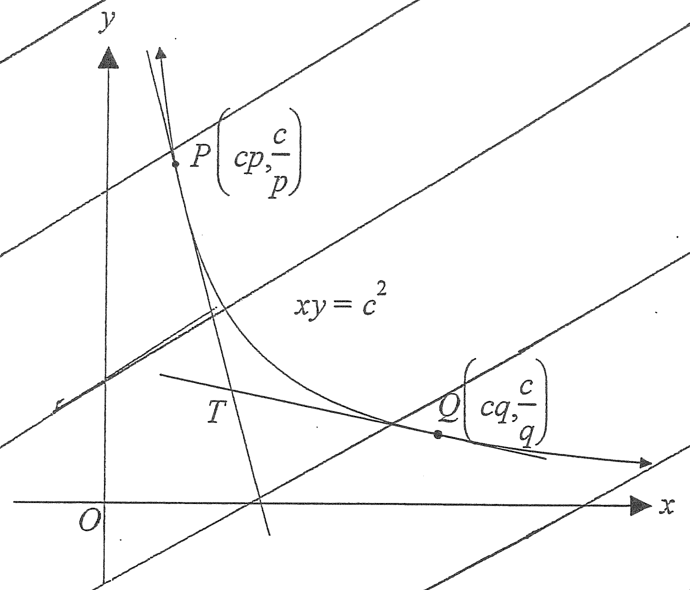
Find the volume of S.

Question 15 continues on the next page

Question 15 (continued)

(c) $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ are two points on the hyperbola $xy = c^2$ where c, p and q are positive.

Tangents at P and Q intersect at T.



(i) Show that the equation of the tangent at P is $x + p^2 y = 2cp$. **1**

(ii) Show that T is the point $(\frac{2cpq}{p+q}, \frac{2c}{p+q})$. **2**

(iii) If $pq = C$, where C is a non-zero constant, find the equation of the locus of T and describe it geometrically. **2**

(d) Let a, b and c be positive real numbers.

(i) Show that $\frac{a}{b} + \frac{b}{a} \geq 2$. **2**

(ii) Hence, or otherwise, show that **2**

$(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geq 9.$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Sketch and shade on an Argand diagram the region R in which **2**

$$|z-4i| \le 3 \text{ and } 0 \le \arg(z+1) \le \frac{\pi}{4} \text{ hold simultaneously.}$$

(ii) Find the value of $\arg(z+1)$ at the point in the region R where $\arg(z+1)$ is a minimum. Answer to the nearest degree. **2**

(b) Using the result $\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$, find the general solution to the equation $\cos 2x + \cos 3x + \cos 4x = 0$. **3**

(c) A sequence is defined so that $U_1 = 5$, $U_2 = 13$ and $U_{n+2} = 5U_{n+1} - 6U_n$ for $n \ge 1$. **3**

Using mathematical induction, prove that $U_n = 2^n + 3^n$ for integers $n \ge 1$.

(d) (i) Let n be a positive integer and z a complex root of the equation **2**

$$(z-1)^n + (z+1)^n = 0 \ldots\ldots\ldots\ldots (1).$$

Show that $|z-1| = |z+1|$ and hence show that all the roots of (1) can be written in the form $z = i\alpha_k$, where α_k is real and $k = 1, 2, \ldots, n$.

(ii) Using the relations between roots and coefficients, show that $\sum_{k=1}^{n}(\alpha_k)^2 = n(n-1)$. **3**

End of Examination

Multiple Choice Q.5 Replacement

5 What is the acute angle between the asymptotes of the hyperbola $\frac{x^2}{3} - y^2 = 1$? 1

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

Multiple Choice Q.9 Replacement

9 A, B , C are three consecutive terms in an arithmetic progression. **1**

Which of the following is a simplification of $\dfrac{\sin(A+C)}{\sin B}$?

(A) $2\cos B$

(B) $\sin 2B$

(C) $\cot B$

(D) 1

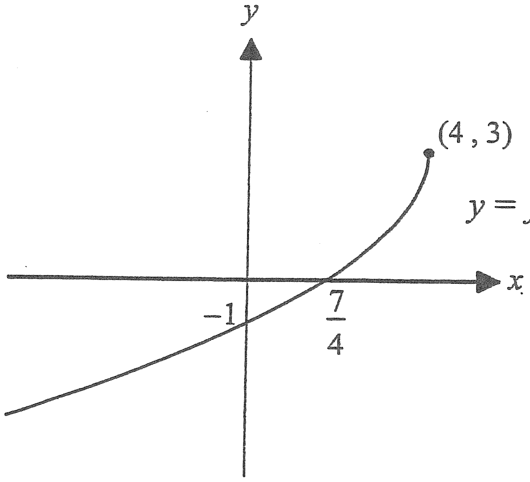
Question 12 Replacement Questions

(a) The equation $x^4+bx^3+cx^2+dx+k=0$ has roots α, $\frac{1}{\alpha}$, β and $\frac{1}{\beta}$.

(i) Show that $k=1$. **1**

(ii) Show that $b=d$. **2**

(b)



In the diagram the curve $y=f(x)$ has equation $f(x)=3-2\sqrt{4-x}$.
On separate diagrams sketch the graphs of the following curves showing the coordinates of any endpoints, any intercepts on the axes and the equations of any asymptotes.

(i) $y=\log_e f(x)$. **1**

(ii) $y=f^{-1}(x)$. **1**

(iii) $y=\dfrac{1}{f(x)}$. **2**

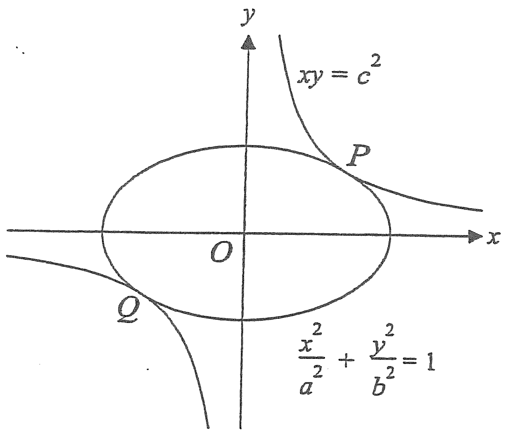
(iv) $y=f(x^2)$. **2**

Now, Don't forget to do Q.12 part (c) on the main paper after this page

Question 14 Replacement Questions

(c) Find the equation of the tangent to the curve $x^2 - xy + y^3 = 1$ at the point $P(1,1)$ on the curve. **3**

(d) The hyperbola $xy = c^2$, $c > 0$ touches the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b > 0$ at points P and Q, where $P\left(cp, \dfrac{c}{p}\right)$ lies in the first quadrant.



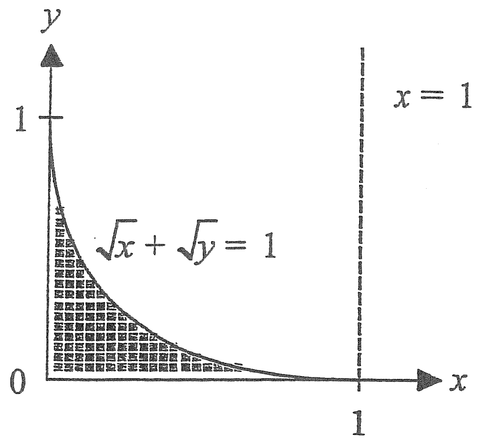
(i) Explain why the equation $(bc)^2 t^4 - (ab)^2 t^2 + (ca)^2 = 0$ has roots $p, p, -p, -p$ where $p > 0$. **2**

(ii) Deduce that $p = \dfrac{a}{c\sqrt{2}}$ and $ab = 2c^2$. **2**

(iii) Show that if S and S' are the foci of the hyperbola $xy = c^2$, then the quadrilateral with vertices P, S, Q and S' has area $2c(a-b)$. **3**

Question 15 Replacement Question

(c)



The region bounded by the curve $\sqrt{x}+\sqrt{y}=1$ and the x axis between $x=0$ and $x=1$ is rotated through 2π radians about the line $x=1$.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V=2\pi\int_0^1(1-x)\left(1-\sqrt{x}\right)^2\,dx$. **2**

(ii) Hence find the value of V in simplest exact form. **3**

Now, Don't forget to do Q.15 part (d) on the main paper after this page

This question replaces Q16 parts (c) and (d)

Question 16

(c)

(i) If n is a positive integer and $c>0$, by considering the stationary values of $f(x)=\frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n+1}$ for $x>0$, show that $\frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n+1} \geq \left(\frac{c}{n}\right)^{n}$ for $x>0$ and state when equality holds. **3**

(ii) Deduce that for $n+1$ positive real numbers x_i, $i=1,2,\ldots,n+1$ **1**

$$\left(\frac{1}{n+1}\sum_{i=1}^{n+1} x_i\right)^{n+1} \geq x_{n+1}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{n}.$$

(iii) For any set of n positive real numbers x_i, $i=1,2,\ldots,n$ where $n\geq 2$, prove by mathematical induction that their arithmetic mean $\frac{1}{n}\left(x_1+x_2+\ldots+x_n\right)$ is greater than or equal to their geometric mean $\left(x_1x_2\ldots x_n\right)^{\frac{1}{n}}$. State when equality holds, justifying your statement. **3**

(iv) Deduce $\left(n!\right)^{\frac{1}{n}} < \frac{1}{2}\left(n+1\right)$. **1**

YEAR 12 EXTENSION 2 MATHEMATICS 2015

TRIAL EXAM SOLUTIONS

Multiple Choice

Q.1. $z = 1+i \quad , z^8 = (\sqrt{2})^8 \operatorname{cis} \frac{8\pi}{4} = 16 \operatorname{cis} 2\pi = 16(\cos 2\pi + i \sin 2\pi) = 16.$

$= \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ (D)

Q.2. $\frac{x^2}{4} + y^2 = 1 \qquad a = 2 \quad b = 1 \qquad b^2 = a^2(1-e^2)$

$1 = 4(1-e^2)$

$e^2 = \frac{3}{4}, \ e = \frac{\sqrt{3}}{2}$

Directrix: $x = \pm \frac{a}{e} = \pm \frac{2}{\sqrt{3}} \times 2 = \pm \frac{4}{\sqrt{3}}$ (A)

Q.3. $x^3 - 8x^2 + px + q = 0. \qquad \alpha, \beta, \gamma$ are the roots.

$\alpha = \beta + \gamma.$

Sum of roots: $\alpha + \beta + \gamma = 8 \qquad \therefore \alpha + \alpha = 8, \ \alpha = 4$ (D)

Q4. $\displaystyle\int \frac{x^2}{(25-x^2)^{3/2}}\, dx = \int \frac{25\sin^2\theta}{(25-25\sin^2\theta)^{3/2}} \cdot \frac{dx}{d\theta} \cdot d\theta$

Let $x = 5\sin\theta$

$\frac{dx}{d\theta} = 5\cos\theta$

$\displaystyle = \int \frac{25\sin^2\theta}{(25\cos^2\theta)^{3/2}} \times 5\cos\theta \ d\theta$

$\displaystyle = \int \frac{125\sin^2\theta\cos\theta}{125\cos^3\theta}\, d\theta$

$\displaystyle = \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$ (D)

$\displaystyle = \int \tan^2\theta \ d\theta.$

Q.5. $\frac{x^2}{3} - y^2 = 1.$

$a=\sqrt{3}$, $b=1$ $\qquad b^2 = a^2(e^2-1)$

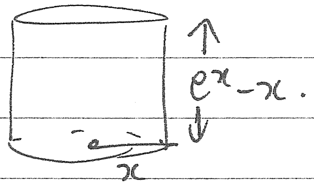
$1 = 3(e^2-1)$

$\frac{1}{3} = e^2 - 1$, $e^2 = \frac{4}{3}$, $e = \frac{2}{\sqrt{3}}$

Eq^ns of asymptotes: $x = \pm \frac{a}{e} = \pm$? (C)

Q6. $y = f|x| =$ (B). (B)

Q7. $V = 2\pi \int_0^1 x(e^x - x)\, dx$ (A)



Q8. $3x^2 + 3y^2 \frac{dy}{dx} = 0.$

$\frac{dy}{dx} = -\frac{x^2}{y^2}$ It's a decreasing f^n ∴ (B)

since $y=x$ is an increasing f^n.

Q. 9. $\frac{\sin(A+C)}{\sin B} = \frac{\sin 2B}{\sin B} = \frac{2\sin B \cos B}{\sin B} = 2\cos B$ (A)

Since arithmetic, $B = \frac{A+C}{2}$

Q10. | 4 | 4 | 2 | + | 4 | 3 | 3 | } arranged in 3! ways.

∴ $3 \times {}^{10}C_4 \times {}^6C_4 \times {}^2C_2$ (A)

$+ 3 \times {}^{10}C_4 \times {}^6C_3 \times {}^3C_3 = 22050$

Q.11. (a) $w = 3-4i$ $\qquad z = 2+2i$

(i) $w.\bar{z} = (3-4i)(2-2i)$

$= 6 - 14i - 8$ (1)

$= -2 - 14i$.

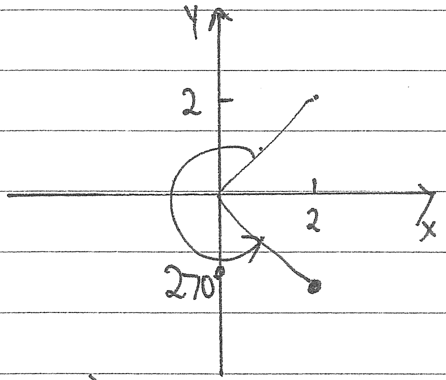
(ii) $\text{Im}\left(\frac{1}{2-w}\right)$ $\qquad 2-w = -1+4i$

$$\frac{1}{2-w} = \frac{1}{-1+4i} \times \frac{-1-4i}{-1-4i} = \frac{-1-4i}{1-16i^2} = \frac{-1-4i}{17}$$

(2)

$\therefore \text{Im}\left(\frac{1}{2-w}\right) = -\frac{4}{17}$.

(iii)



New number is $2-2i$

Algebraically, the rotation corresponds to $i^3 z = -i(2+2i)$

$= -2i + 2$

$= 2-2i$.

(1)

Q.11(b) $\qquad 3x^3 - x^2 + 2x + 1 = 0 \qquad$ — 1

(i) Find eqn with roots $\alpha^2, \beta^2, \gamma^2$.

Let $x = \alpha^2, \beta^2, \gamma^2$

$\therefore$ Replace x in 1 with $\sqrt{x}$.

1 becomes $\qquad 3(\sqrt{x})^3 - (\sqrt{x})^2 + 2\sqrt{x} + 1 = 0$

ie. $\quad 3x\sqrt{x} - x + 2\sqrt{x} + 1 = 0.$

But we want integer coeffs: $\sqrt{x}(3x+2) = \cancel{1-x}\,(x-1)$

Square both sides $\qquad x(3x+2)^2 = \cancel{(1-x)^2}\ (x-1)^2$ $\qquad$ 2

$$x(9x^2+12x+4) = 1 - 2x + x^2$$

$$9x^3 + 12x^2 + 4x = 1 - 2x + x^2$$

$$9x^3 + 11x^2 + 6x - 1 = 0.$$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = \Sigma \text{ roots} = \dfrac{-11}{9}.$ $\qquad$ 1

Q11 (c) (i) $\dfrac{6}{(x+2)(x-1)} = \dfrac{a}{x+2} + \dfrac{b}{x-1}$

$\times (x+2)(x-1)$

$$6 = a(x-1) + b(x+2)$$

If $x=1$, $6 = 3b$ $\therefore b = 2$ (2)

If $x=-2$, $6 = -3a$ $\therefore a = -2$

(ii) $\displaystyle\int \frac{1}{(x+2)(x-1)} = \frac{1}{6}\int\left(-\frac{2}{x+2} + \frac{2}{x-1}\right) dx$

$$= \frac{1}{6}\Big[-2\ln(x+2) + 2\ln(x-1)\Big] + C$$

(2)

$$= \frac{1}{6}\ln\left(\frac{x-1}{x+2}\right)^2 + C$$

(d) Use $t = \tan\left(\frac{x}{2}\right)$ to find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$

$x=0,\ t=0$

$x=\frac{\pi}{2},\ t=1$

$t = \tan\frac{x}{2}$

$\tan^{-1} t = \frac{x}{2}$.

$2\tan^{-1} t = x$

$\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

$$= \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \frac{dx}{dt}\, dt$$

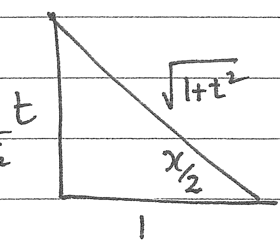
(4)

$$= \int_0^1 \frac{\cancel{1+t^2}}{1+t^2+2t} \cdot \frac{2}{\cancel{1+t^2}}\, dt$$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$= 2\dfrac{t}{\sqrt{1+t^2}} \cdot \dfrac{1}{\sqrt{1+t^2}}$

$= \dfrac{2t}{1+t^2}$



$$= \int_0^1 \frac{2}{1+2t+t^2}\, dt$$

$$= \int_0^1 \frac{2}{(1+t)^2}\, dt = -\frac{2}{1+t}\Big]_0^1 = 1 \ \#$$

12(a) $x^4+bx^3+cx^2+dx+k=0$ has roots $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$.

(i) Show that $k=1$

(ii) Show that $b=d$.

(i) $k=$ product of roots $\therefore$ $k = \alpha \times \frac{1}{\alpha} \times \beta \times \frac{1}{\beta} = 1$ (1)

(ii) To show $b=d$,

$b=$ sum of roots

$d=$ sum of triple products $({}^4C_3 = 4)$.

$$b = \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta}$$

$$= \frac{\alpha^2+1}{\alpha} + \frac{\beta^2+1}{\beta}$$

$$d = \alpha \cdot \frac{1}{\alpha} \cdot \beta + \alpha \frac{1}{\alpha} \cdot \frac{1}{\beta} + \alpha \cdot \beta \cdot \frac{1}{\beta} + \frac{1}{\alpha} \cdot \beta \cdot \frac{1}{\beta}$$

$$= \beta + \frac{1}{\beta} + \alpha + \frac{1}{\alpha}$$

$$= \frac{\alpha^2+1}{\alpha} + \frac{\beta^2+1}{\beta}$$ (2)

$\therefore$ $b=d$ as required.

Q12 (cont)

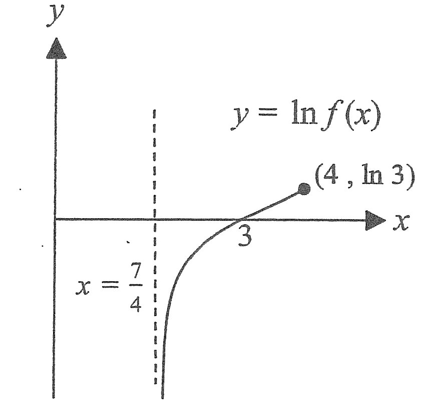
(b) **Outcomes assessed: E6**

Marking Guidelines

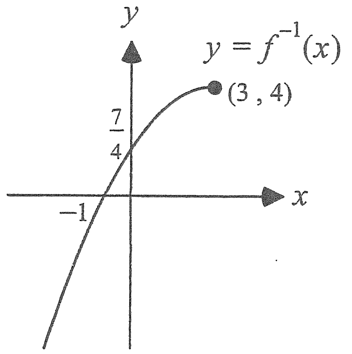
Criteria	Marks
i • correct shape with endpoint, x intercept and asymptote shown	1
ii • correct shape with x and y intercepts and endpoint shown	1
iii • left branch correct shape with asymptote and y intercept shown	1
• second branch correct shape with endpoint shown	1
iv • correct shape and position with y intercept shown	1
• x intercepts and endpoints shown	1

Answer

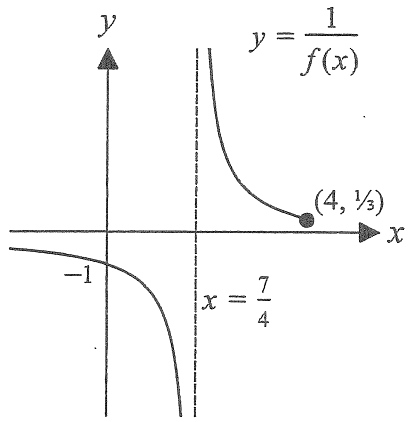
i.



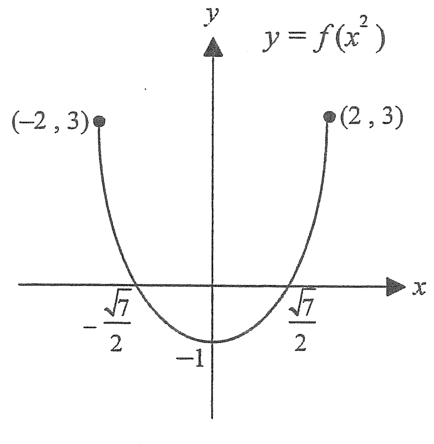
ii



iii.



iv.



12(c) $P(x) = x^3+4x^2+px+q$, p, q real.

$x = 2+i$ is a root. Since all coeffs are real $x = 2-i$ is also a root.

$\therefore P(x) = (x-2-i)(x-2+i)(x-m)$ where m is a real no.

$= (x-(2+i))(x-(2-i))(x-m)$

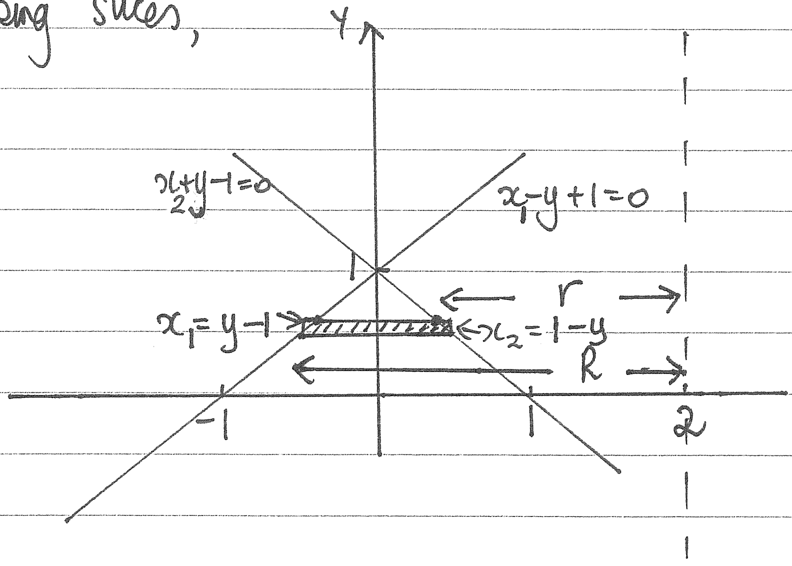
$= (x^2-4x+5)(x+m)$ 3

Now, sum of roots $= -4 = 2+i+2-i+m$

$\therefore -4 = 4+m$, $m=-8$.

$\therefore P(x) = (x^2-4x+5)(x+8)$

12(d) Using slices,



- each slice will be an annulus with inner radius r and outer radius R.

$\therefore r = 2-x_2$ and $R = 2-x_1$

$= 2-(1-y)$ $\quad$ $= 2-(y-1)$

$= 1+y$ $\quad$ $= 3-y$

Thus, the area of the annulus is

$$A = \pi R^2 - \pi r^2$$

$$= \pi(3-y)^2 - \pi(1+y)^2$$

$$= \pi(9-6y+y^2) - \pi(1+2y+y^2)$$

$$= \pi(8-8y)$$

1

Let the thickness of a typical slice be δy

$\therefore$ Volume of a slice $= \pi(8-8y)\,\delta y$ 1.

and the Volume of the solid is given by: 3

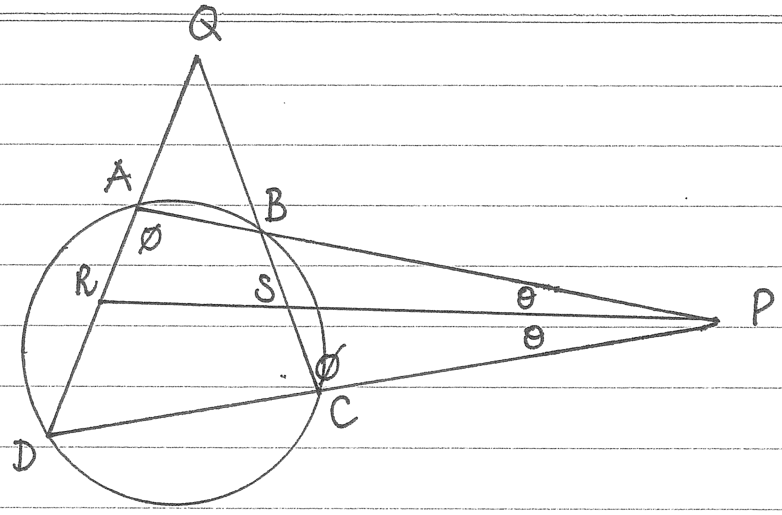
$$V = \int_0^1 \pi(8-8y)\,dy$$

$$= 8\pi \int_0^1 (1-y)\,dy$$

$$= 8\pi \left[y - \frac{y^2}{2} \right]_0^1$$

$$= 8\pi \times \tfrac{1}{2} = 4\pi \text{ cubic units.}$$ 1

Q.13 (a)



To Prove: ΔQRS is isosceles.

Proof: Let $\angle APR = \angle DPR = \theta$ (given PR bisects $\angle APD$)

Let $\angle BCP = \phi$

$\angle BCP = \angle PAR$ (exterior $\angle$ of cyclic quad is equal to opp. interior angle).

$= \phi$

$\therefore \angle QRP = 180 - (\theta + \phi)$.

In ΔPSC, $\angle PSC = 180 - (\theta + \phi)$

But $\angle PSC = \angle QSR$ (vertically opposite angles)

$\therefore \angle QRP = \angle QSR$ (both equal to $180 - (\theta + \phi)$)

$\therefore \Delta QRS$ is isosceles (equal base angles).

QED.

Q.13(b)

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{-2x}{a^2} \times \dfrac{b^2}{2y} = \dfrac{-xb^2}{ya^2}$.

At $P(a\cos\theta, b\sin\theta)$, $\dfrac{dy}{dx} = \dfrac{-a\cos\theta . b^2}{b\sin\theta . a^2} = \dfrac{-b\cos\theta}{a\sin\theta}$

Hence gradient of normal is $\dfrac{a\sin\theta}{b\cos\theta}$.

$\therefore$ eq^n of normal at P is: $y - b\sin\theta = \dfrac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

i.e. $by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

i.e. $ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$.

(ii) To show that $MN = \left|\dfrac{b^2\cos\theta}{a}\right|$,

N is the point $(a\cos\theta, 0)$ and M is a pt. on the normal with $y=0$.

i.e. $ax\sin\theta = (a^2-b^2)\sin\theta\cos\theta \Rightarrow x = \dfrac{(a^2-b^2)}{a}\cos\theta$.

$\therefore MN = \left|\dfrac{(a^2-b^2)\cos\theta}{a} - a\cos\theta\right|$

$= \left|\dfrac{a^2\cos\theta - b^2\cos\theta - a^2\cos\theta}{a}\right| = \left|\dfrac{b^2\cos\theta}{a}\right|$ as required.

Q.13(c)

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx \quad , \quad n \geqslant 0 .$$

(i) To show that $I_n = \frac{1}{n-1} - I_{n-2}$ for $n \geqslant 2$,

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x . \tan^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x . (\sec^2 x - 1) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x . \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x . \sec^2 x \, dx - I_{n-2} .$$

Now let $u = \tan x$ when $x = \frac{\pi}{4}$, $u = 1$

$\frac{du}{dx} = \sec^2 x$ when $x = 0$, $u = 0$.

3

$$\therefore I_n = \int_0^1 u^{n-2} \, du - I_{n-2}$$

$$= \left[\frac{u^{n-1}}{n-1} \right]_0^1$$

$$= \left[\frac{1}{n-1} - 0 \right] - I_{n-2} = \frac{1}{n-1} - I_{n-2}$$

as required.

Q13(c) (ii) Hence find $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$.

Now, $I_5 = \int_0^{\frac{\pi}{4}} \tan^5 x \, dx = \frac{1}{n-1} - I_{n-2}$ $(n=5)$

$= \frac{1}{4} - I_3$

Also $I_3 = \frac{1}{2} - I_1$

and $I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$

$= \left[-\log_e(\cos x)\right]_0^{\frac{\pi}{4}}$

$= -\log_e \frac{1}{\sqrt{2}} + \log_e 1$ 2

$= -\log_e(2^{-\frac{1}{2}}) = \frac{1}{2}\log_e 2.$

$\therefore I_3 = \frac{1}{2} - \frac{1}{2}\log_e 2.$

and $I_5 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\log_e 2\right)$

$= \frac{1}{2}\log_e 2 - \frac{1}{4}$

Q.13(d) (i) To Prove:

$$c\int_a^b f(ct)\,dt = \int_{ca}^{cb} f(t)\,dt, \quad \text{using} \quad ct = u.$$

Proof: If $u = ct$, $\frac{du}{dt} = c$ and when $t = b$, $u = cb$
& when $t = a$, $u = ac$

$$\therefore \quad c\int_a^b f(ct)\,dt = c\int_{ac}^{cb} f(u)\,\frac{dt}{du}.\,du$$

$$= c\int_{ac}^{cb} f(u).\frac{1}{c}\,du$$

$$= \int_{ac}^{cb} f(u)\,du \quad = \int_{ac}^{cb} f(t)\,dt$$

2.

as required.

(ii) To show,
$$\int_1^{cb} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt + \int_1^c \frac{1}{t}\,dt.$$

Now,
$$c\int_a^b f(ct)\,dt = \int_{ac}^{bc} f(t)\,dt \quad \text{from (i)}.$$

$$\therefore \quad \int_c^{bc} \frac{1}{t}\,dt = c\int_1^b \frac{1}{ct}\,dt \quad \text{using (i)}$$

$$= \int_1^b \frac{1}{t}\,dt \quad \text{—— } 1$$

But Also,
$$\int_c^{bc} \frac{1}{t}\,dt = \int_c^1 \frac{1}{t}\,dt + \int_1^{bc} \frac{1}{t}\,dt$$

Q13(d)(ii)

ie. $\int_{c}^{bc} \frac{1}{t}\,dt = -\int_{1}^{c} \frac{1}{t}\,dt + \int_{1}^{bc} \frac{1}{t}\,dt$ — ②

Thus, equating RHS of ① & ② we obtain

$$\int_{1}^{b} \frac{1}{t}\,dt = -\int_{1}^{c} \frac{1}{t}\,dt + \int_{1}^{bc} \frac{1}{t}\,dt$$

ie. $\int_{1}^{bc} \frac{1}{t}\,dt = \int_{1}^{b} \frac{1}{t}\,dt + \int_{1}^{c} \frac{1}{t}\,dt$ as required.

Q.14. (a) $P(1) = 3$
$P(2) = 5$

Let $P(x) = (x-1)(x-2)Q(x) + ax+b$. where $ax+b$ is the remainder.

Given $P(1) = 3$ we obtain $P(1) = a+b = 3$ — ①

Given $P(2)=5$ we obtain $P(2) = 2a+b = 5$ — ②

② − ① yields $a = 2$ $\therefore b=1$.

Hence the remainder is $2x+1$.

2

Q14 (b)

$z^3 = 1$ has roots $1, \omega, \omega^2$.

(i) $(1+\omega^2)^6$ Now, Σ roots $= 0$ $\therefore$ $1+\omega+\omega^2 = 0$

$1+\omega^2 = -\omega$.

$\therefore (1+\omega^2)^6 = (-\omega)^6 = \omega^6 = (\omega^3)^2 = 1$ since $\omega^3 = 1$. 1

(ii) To hence show that:

$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9$.

Now $\omega^4 = \omega^3.\omega = \omega$ and $\omega^5 = \omega^3.\omega^2 = 1.\omega^2$

$$\begin{aligned}
\therefore (1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) &= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2) \\
&= \left[(1-\omega)(1-\omega^2)\right]^2 \\
&= \left[1-\omega^2-\omega+\omega^3\right]^2 \\
&= \left[1+\omega^3-(\omega^2+\omega)\right]^2 \\
&= \left[1+1-(-1)\right]^2 = 3^2 = 9 \text{ as required.}
\end{aligned}$$

2

(c) $x^2 - xy + y^3 = 1$

Differentiating w.r.t. x: $2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$$\frac{dy}{dx}\left(3y^2 - x\right) = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{3y^2-x}$$

At $P(1,1)$, $\frac{dy}{dx} = \frac{1-2}{3-1} = -\frac{1}{2}$

3

$\therefore$ eq^n of tangent at P is : $y-1 = -\frac{1}{2}(x-1)$

$2y-2 = -x+1$ i.e. $x+2y-3=0$.

$y = -\frac{x}{2}+\frac{3}{2}$.

(d) $xy = c^2$, $c>0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $a>b>0$

The hyperbola is tangential to the ellipse, $P(cp, \frac{c}{p})$ is one pt. of contact.

$\therefore$ Substituting $(ct, \frac{c}{t})$ eq^n of ellipse yields:

$$\frac{c^2t^2}{a^2} + \frac{c^2}{t^2b^2} = 1$$

$\times a^2t^2b^2$

$$c^2b^2t^4 + a^2c^2 = a^2t^2b^2$$

i.e. $(bc)^2t^4 - (ab)^2t^2 + (ac)^2 = 0$ —①

Since p is a pt. where the hyperbola touches the ellipse, p is a double solution of the above equation.

Since the eq^n is a quadratic in t^2, $-p$ must also be a double root.

Q.14(d) (ii) To deduce that $p = \frac{a}{c\sqrt{2}}$ and $ab = 2c^2$

Using $(bc)^2 t^4 - (ab)^2 t^2 + (ac)^2 = 0$,

Σ of product of pairs of roots $= \frac{-(ab)^2}{(bc)^2} = p.p + p.-p + p.-p + p.-p + p.-p + -p.-p$.

The roots are $p, p, -p, -p$.

of which there are ${}^4C_2 = 6$.

$= 2p^2 - 4p^2 = -2p^2$.

$\therefore \quad 2p^2 = \frac{(ab)^2}{(bc)^2} = \frac{a^2}{c^2}$

$\Rightarrow p^2 = \frac{a^2}{2c^2} \quad \therefore p = \frac{a}{\sqrt{2}\, c}$.

since $p > 0$.

Product of Roots $= \frac{(ac)^2}{(bc)^2} = p^4$

$\Rightarrow \frac{a^2}{b^2} = p^4$

2

$\frac{a^2}{b^2} = \frac{a^4}{4c^4}$ from above

$\therefore \quad 4c^4 = a^2 b^2$

$ab = 2c^2$ since $a, b, c > 0$.

Q.15.

(a) $$\int_0^1 \underset{dv}{\overset{\uparrow}{2}}\ \underset{u}{\overset{\uparrow}{\tan^{-1}x}}\, dx$$

Let $u = \tan^{-1}x$ $\qquad \frac{du}{dx} = \frac{1}{1+x^2}$ $\qquad$ 1

$\frac{dv}{dx} = 2$ $\qquad v = 2x$

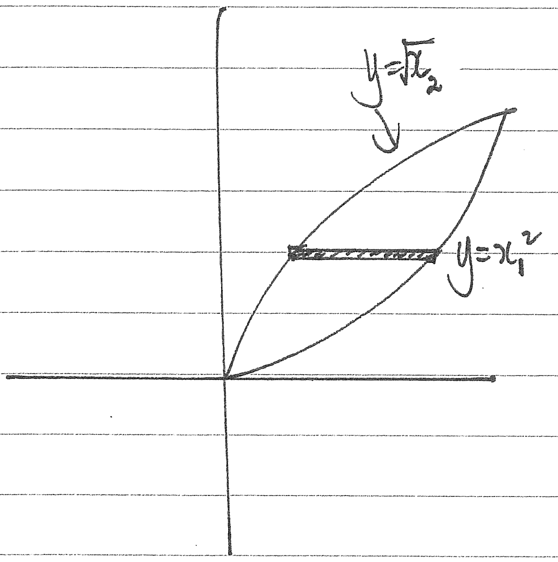
$\therefore\ I = u.v - \int v \frac{du}{dx}\, dx$

$= \left[2x \tan^{-1}x\right]_0^1 - \int_0^1 2x \frac{1}{1+x^2}\, dx$ $\qquad$ 1

$= 2 \times \frac{\pi}{4} - \left[\log_e(1+x^2)\right]_0^1$ $\qquad$ 3

$= \frac{\pi}{2} - \log_e 2$ $\qquad$ 1

(b) $y = x^2$ $\quad$ & $y = \sqrt{x}$



Side length of square cross-sections is $x_1 - x_2$

$y = x_1^2 \quad \therefore x_1 = \sqrt{y}$

$y = \sqrt{x_2} \quad \therefore x_2 = y^2$

$\therefore$ Side length is $(\sqrt{y} - y^2)$

Area of Square is $(\sqrt{y} - y^2)^2$ $\qquad$ 1

and thickness of a slice is δy.

Pts of $\cap$ of curves : $x^2 = \sqrt{x}$ $\quad x^4 = x$, $x(x^3-1)=0$,

$$x=0, \quad x=1.$$

$\therefore$ Volume $= \displaystyle\int_0^1 (\sqrt{y}-y^2)^2 \, dy$ 1

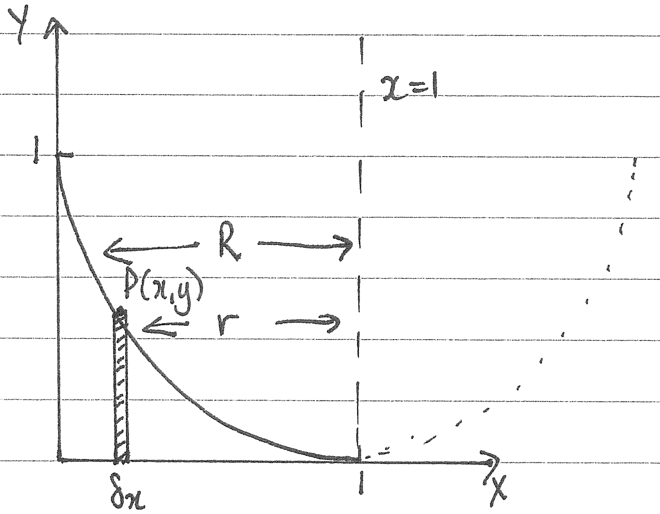
$$= \int_0^1 \left(y - 2y^{5/2} + y^4\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{4}{7}y^{7/2} + \frac{y^5}{5}\right]_0^1$$ 3

$$= \frac{1}{2} - \frac{4}{7} + \frac{1}{5}$$

$= \dfrac{35-40+14}{70} = \dfrac{9}{70}$ cubic units. 1

(c)



Open up a cylinder to give a rectangular prism of depth δx

Area of cross-section $= 2\pi r h$. Where the radius is $(1-x)$

and the height is the y coord at P.

δx corresponds to $R - r$.

Now $R = 1 - x$ and $r = 1 - x - \delta x$

Also, $h = y$ where $\sqrt{x} + \sqrt{y} = 1 \Rightarrow \sqrt{y} = 1 - \sqrt{x}$

$y = (1-\sqrt{x})^2$

$\therefore \delta V = \pi(R^2 - r^2)h$

$= \pi(R+r)(R-r)h$

$= \pi\left(2(1-x) - \delta x\right)(\delta x)(1-\sqrt{x})^2$ 2

$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{x=1} 2\pi(1-x)(1-\sqrt{x})^2 \delta x$ (ignoring 2nd order terms)

$= \int_0^1 2\pi(1-x)(1-\sqrt{x})^2 \, dx$

$= 2\pi \int_0^1 (1-x)(1 - 2\sqrt{x} + x) \, dx$

3

$= 2\pi \int_0^1 (1 - 2\sqrt{x} + \cancel{x} - \cancel{x} + 2x\sqrt{x} - x^2) \, dx$

$= 2\pi \left[x - \frac{4}{3}x^{3/2} + \frac{4}{5}x^{5/2} - \frac{x^3}{3} \right]_0^1$

$= 2\pi \left[1 - \frac{4}{3} + \frac{4}{5} - \frac{1}{3} \right]$

$= 2\pi \left[-\frac{2}{3} + \frac{4}{5} \right]$

$= \frac{4\pi}{15}$ cubic units.

Q 15 (d)

$a, b, c > 0.$

(i) To show $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$

$$LHS = \frac{a^2+b^2}{ab} \quad \cancel{= \frac{(a-b)^2 + 2ab}{ab}}.$$

Now $(a-b)^2 \geq 0$

i.e. $a^2 - 2ab + b^2 \geq 0$

i.e. $a^2 + b^2 \geq 2ab$

$\therefore \dfrac{a^2}{ab} + \dfrac{b^2}{ab} \geq 2$

2

$\therefore \dfrac{a}{b} + \dfrac{b}{a} \geq 2.$

(ii) Hence or otherwise show that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9.$$

2.

Now, $LHS = 1 + \dfrac{a}{b} + \dfrac{a}{c} + \dfrac{b}{a} + 1 + \dfrac{b}{c} + \dfrac{c}{a} + \dfrac{c}{b} + 1$

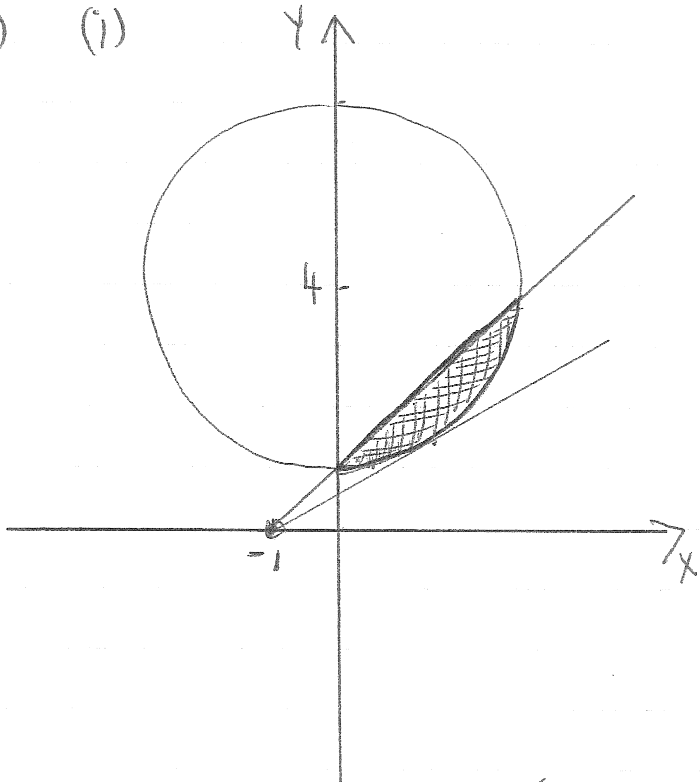
$$= 3 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{a}{c} + \frac{c}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right)$$

Using $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$ from (i) $\therefore \dfrac{a}{c} + \dfrac{c}{a} \geq 2$ and $\dfrac{b}{c} + \dfrac{c}{b} \geq 2$, $\therefore LHS \geq 9$.

as required.

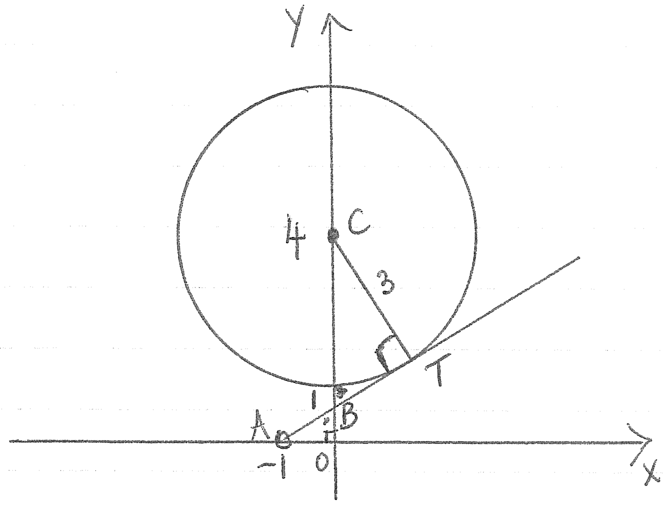
Q.16

(a) (i)



(2)

(ii) This occurs at the point where the ray of $\arg(z+1)$ is a tangent to the circle.



Need to use similar triangles.

$\triangle AOB \,|||\, \triangle BTC$ since they are equiangular.

$$\frac{AB}{BC} = \frac{AO}{CT} \qquad \frac{AB}{BC} = \frac{1}{3} \quad -(1)$$

Also, $OB + BC = 4$.

i.e. $OB = 4 - BC \quad -(2)$

Using Pythagoras' th^m on ΔAOB,

$$AB^2 = AO^2 + OB^2$$
$$= 1 + (4 - BC)^2.$$

But from ①, $BC = 3AB$.

$$\therefore \quad AB^2 = 1 + (4 - 3AB)^2$$

i.e. $AB^2 = 1 + 16 - 24AB + 9AB^2$

$$8AB^2 - 24AB + 17 = 0.$$

$$AB = \frac{24 \pm \sqrt{24^2 - 4 \times 8 \times 17}}{16}$$
$$= \frac{24 \pm \sqrt{32}}{16}$$
$$= \frac{24 \pm 4\sqrt{2}}{16} = \frac{6 \pm \sqrt{2}}{4}$$

$\therefore$ To find the min^m value of $\angle BAO$, letting $\angle BAO = \theta$

$$\cos\theta = \frac{1}{(6+\sqrt{2})/4} \quad \text{or} \quad \cos\theta = \frac{1}{(6-\sqrt{2})/4}$$
$$= \frac{4}{6+\sqrt{2}} \quad \text{or} \quad = \frac{4}{6-\sqrt{2}}$$ ②

$$\theta = 57° \quad \text{or} \quad = 29°$$

to nearest degree.

Using

(b) $\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$

solve $\cos 2x + \cos 3x + \cos 4x = 0.$

Solution: $\underbrace{\cos 2x + \cos 4x} + \cos 3x = 0.$

$2\cos 3x \cos x + \cos 3x = 0.$

(3).

$\Rightarrow \cos 3x\,(2\cos x + 1) = 0$

$\Rightarrow \cos 3x = 0$ or $\cos x = -\frac{1}{2}.$

$\Rightarrow 3x = 2n\pi \pm \cos^{-1} 0$ or $x = 2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right).$

(n integer)

$\Rightarrow x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$ or $x = 2n\pi \pm \frac{2\pi}{3}$

Q.16 (c)

$$f(x) = \frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n+1} \qquad x>0.$$

$f'(x) = 0$ for stationary points.

$$f'(x) = -\frac{1}{x^2}\left(\frac{x+c}{n+1}\right)^{n+1} + \frac{n+1}{(n+1)^{n+1}}(x+c)^n \cdot \frac{1}{x}$$

$$= \frac{1}{x} \cdot \frac{1}{(n+1)^n}\left[-\frac{1}{x}\frac{(x+c)^{n+1}}{n+1} + (x+c)^n\right]$$

$$= \frac{(x+c)^n}{x(n+1)^n}\left[1 - \frac{1(x+c)}{x(n+1)}\right] = \frac{(x+c)^n}{x(n+1)^n}\left[\frac{x(n+1)-(x+c)}{x(n+1)}\right]$$

$$= \frac{(x+c)^n}{x(n+1)^n}\left[\frac{xn-c}{x(n+1)}\right].$$

$= 0$ when $x = -c$ or when $x+c = x(n+1)$

ie. $x(n+1-1) = c$

$xn = c$

$x = \frac{c}{n}.$

Since $c>0$, $x>0$

only viable solution is $x = \frac{c}{n}$.

If $0<x<\frac{c}{n}$, $f'(x)<0$ and if $x>\frac{c}{n}$ $f'(x)>0$.

$\therefore$ $f(x)$ has a minimum at $x=\frac{c}{n}$.

Since $f(x)$ has a min^m at $x = \frac{c}{n}$,

$f(x) \geq f\left(\frac{c}{n}\right)$ for $x>0$.

The value of $f\left(\frac{c}{n}\right) = \left(\frac{n}{c}\right)\left(\frac{\frac{c}{n}+c}{n+1}\right)^{n+1}$

$$= \left(\frac{n}{c}\right)\left(\frac{c+cn}{n(n+1)}\right)^{n+1}$$

③.

$$= \left(\frac{n}{c}\right)\left(\frac{c\cancel{(n+1)}}{n\cancel{(n+1)}}\right)^{n+1}$$

$$= \left(\frac{n}{c}\right)\left(\frac{c}{n}\right)^{n+1} = \left(\frac{c}{n}\right)^{n} \quad \text{for } x>0.$$

(ii) For $n+1$ positive reals x_i, $i = 1, 2, 3, \ldots, n+1$.

$$\left[\frac{1}{n+1}\sum_{i=1}^{n+1} x_i\right]^{n+1} \geq x_{n+1}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]^{n}$$

Now, we know $f(x) \geq f\left(\frac{c}{n}\right) = \left(\frac{c}{n}\right)^{n}$

ie. $x = \frac{c}{n} = \frac{1}{n} \times c.$

Now, $\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n} x_i + x_{n+1}.$

$\therefore$ If $x = \frac{c}{n} = \frac{1}{n}\times c$ and $c = \sum_{i=1}^{n} x_i$

then $\sum_{i=1}^{n+1} x_i = c + x_{n+1}.$

and $x = \frac{1}{n}\times c = \frac{1}{n}\sum_{i=1}^{n} x_i$

(iii) $S(n): \frac{1}{n}(x_1 + x_2 + \dots + x_n) \geqslant (x_1 x_2 \dots x_n)^{1/n}$ $\quad n = 2, 3, 4, \dots$

$S(2): \left(\frac{1}{2}(x_1 + x_2)\right) \geqslant (x_1 x_2)^{1/2}$

Q15 (cont)

b. Outcomes assessed: HE2, E2

Marking Guidelines

Criteria	Marks
i • derives $f(x)$ with respect to x	1
• finds the value of $x > 0$ for which f takes its minimum value	1
• finds this minimum value of f and states when it is attained	1
ii • selects appropriate values of x and c to deduce result	1
iii • defines an appropriate sequence of statements and verifies the first is true	1
• shows that if the k^{th} statement is true, then the $(k+1)^{\text{th}}$ is conditionally true	1
• completes the induction process and states with explanation when equality holds	1
iv • selects an appropriate set of positive real numbers to deduce the result	1

Answer

i. $$f(x)=\frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n+1}$$

$$f'(x)=\frac{-1}{x^2}\left(\frac{x+c}{n+1}\right)^{n+1}+\frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n}$$

$$=\frac{1}{x^2}\left(\frac{x+c}{n+1}\right)^{n}\left(x-\frac{x+c}{n+1}\right)$$

$$=\frac{1}{x^2}\left(\frac{x+c}{n+1}\right)^{n}\left(\frac{nx-c}{n+1}\right)$$

For $x>0$, only stationary value is for $nx=c$.

$$0<x<\frac{c}{n}\Rightarrow f'(x)<0 \quad \text{and} \quad x>\frac{c}{n}\Rightarrow f'(x)>0$$

$\therefore f(x)$ has minimum stationary value at $x=\dfrac{c}{n}$

$$\therefore f(x)\ge f\left(\frac{c}{n}\right)=\frac{n}{c}\left(\frac{c}{n}\right)^{n+1}=\left(\frac{c}{n}\right)^{n} \text{ for } x>0$$

with equality if and only if $x=\dfrac{c}{n}$.

ii. If $c=\sum_{i=1}^{n}x_i$ and $x=x_{n+1}$, then $x_{n+1}+c=\sum_{i=1}^{n+1}x_1$ and $\dfrac{1}{x_{n+1}}\left(\dfrac{1}{n+1}\sum_{i=1}^{n+1}x_i\right)^{n+1}\ge\left(\dfrac{1}{n}\sum_{i=1}^{n}x_i\right)^{n}$.

Hence $\left(\dfrac{1}{n+1}\sum_{i=1}^{n+1}x_i\right)^{n+1}\ge x_{n+1}\left(\dfrac{1}{n}\sum_{i=1}^{n}x_i\right)^{n}$ (with equality if and only if $x_{n+1}=\dfrac{1}{n}\sum_{i=1}^{n}x_1$).

iii. Let $S(n)$, $n=2,3,\ldots$ be the sequence of statements defined by

$S(n)$: $\frac{1}{n}(x_1+x_2+\ldots+x_n)\ge(x_1x_2\ldots x_n)^{\frac{1}{n}}$ *for any set of* n *positive real numbers* x_i, $i=1,2,\ldots,n$.

Consider $S(2)$: $\left\{\frac{1}{2}(x_1+x_2)\right\}^2\ge x_2(x_1)^1$ from (ii), hence $\frac{1}{2}(x_1+x_2)\ge(x_1x_2)^{\frac{1}{2}}$ and $S(2)$ is true.

If $S(k)$ *is true* : $\frac{1}{k}(x_1+x_2+\ldots+x_k)\ge(x_1x_2\ldots x_k)^{\frac{1}{k}}$ *

Consider $S(k+1)$: $\left\{\frac{1}{k+1}(x_1+x_2+\ldots+x_{k+1})\right\}^{k+1}\ge x_{k+1}\left\{\frac{1}{k}(x_1+x_2+\ldots+x_k)\right\}^k$ from (ii)

$\ge x_{k+1}(x_1x_2\ldots x_k)$ *if* $S(k)$ *is true using* *

giving $\frac{1}{k+1}(x_1+x_2+\ldots+x_{k+1})\ge(x_1x_2\ldots x_{k+1})^{\frac{1}{k+1}}$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(2)$ is true, then $S(3)$ is true and so on. Hence by mathematical induction, $S(n)$ is true for positive integers n, $n\ge 2$.
The arithmetic mean is equal to the geometric mean if and only if $x_1=x_2=\ldots=x_n$, since result in (ii) requires that $x_2=x_1$, $x_3=\frac{1}{2}(x_1+x_2)$, $x_4=\frac{1}{3}(x_1+x_2+x_3)$, ... in induction process for equality to hold.

iv. Considering the numbers $1, 2, 3, \ldots, n$: $\frac{1}{n}(1+2+3+\ldots+n)>(n!)^{\frac{1}{n}}$ $\quad\therefore (n!)^{\frac{1}{n}}<\frac{1}{2}(n+1)$.