Graphs Part One

Content covered Term 4 2017, with completed examples.

NB: some examples may differ slightly from those done in class.

Curve Sketching

The concept of a function and its graph is of great importance in all branches of Mathematics.

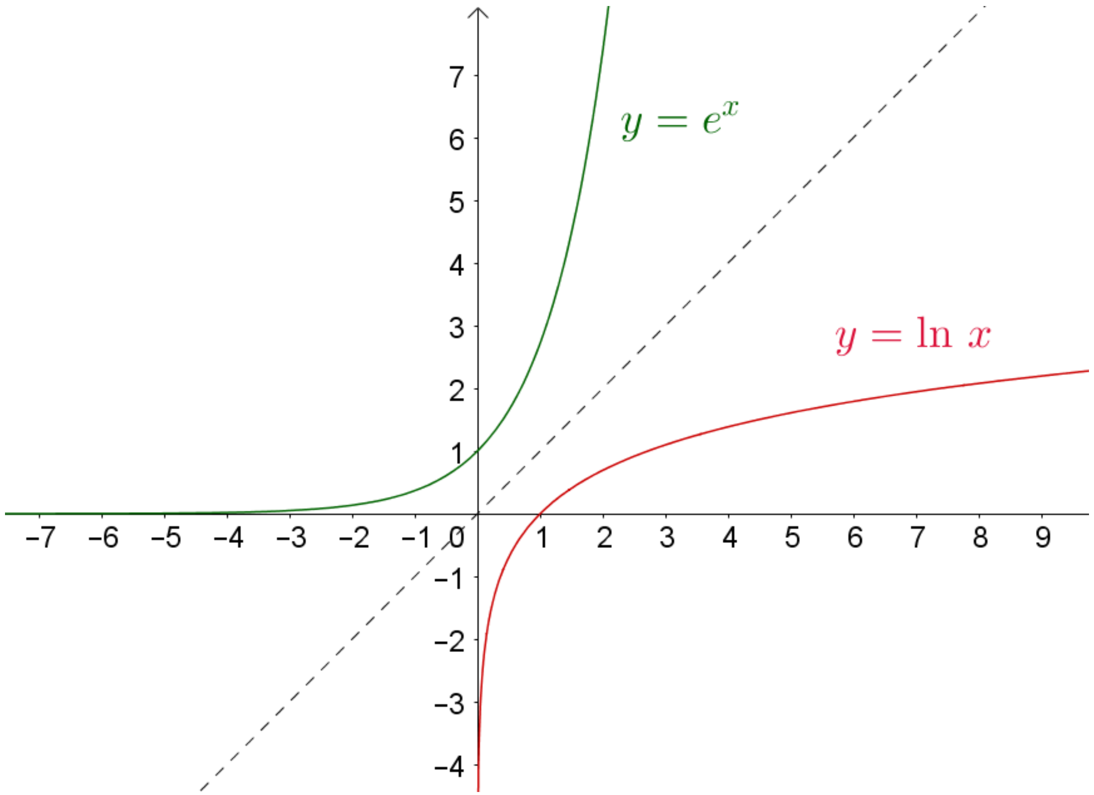
Graphs are a valuable aid to mathematical understanding and solving difficult problems.

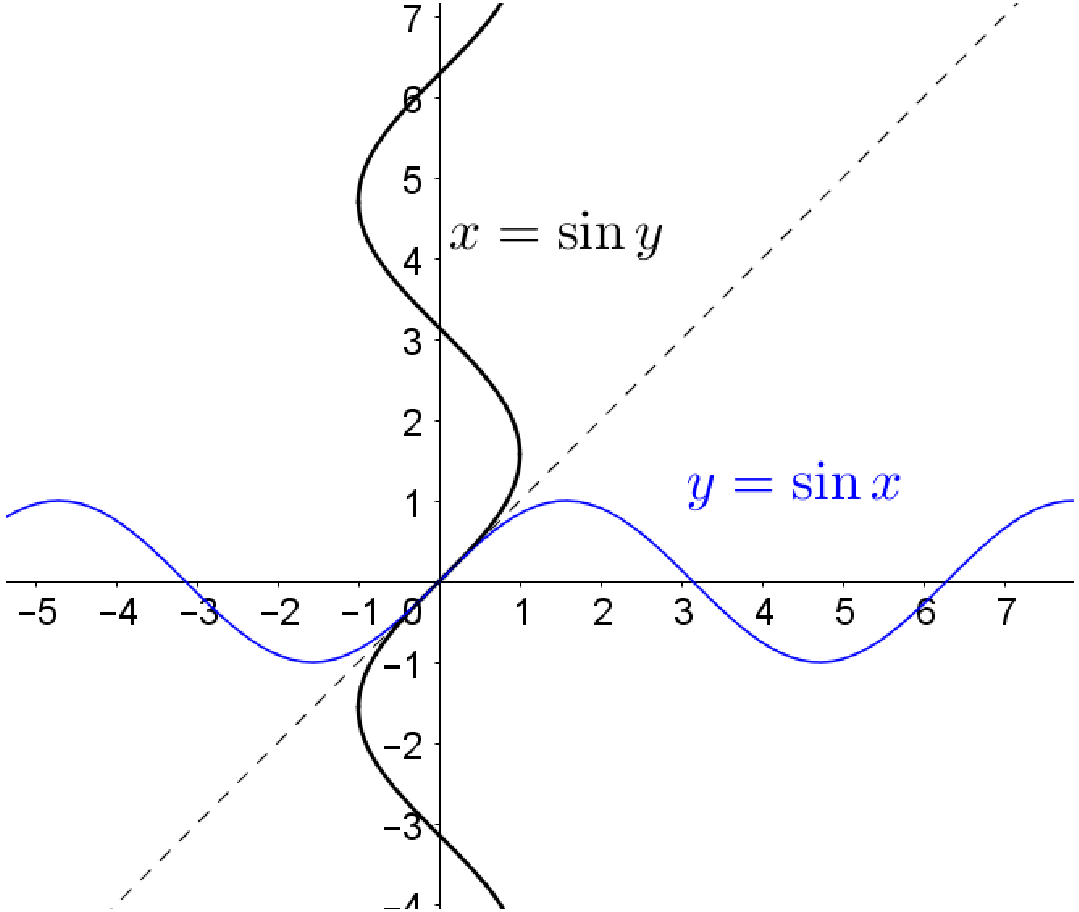
There are many basic graphs you are expected to recognise and be able to sketch. See Fitzpatrick p.37-41 and handout for a list…

Inverse Trigonometric Functions

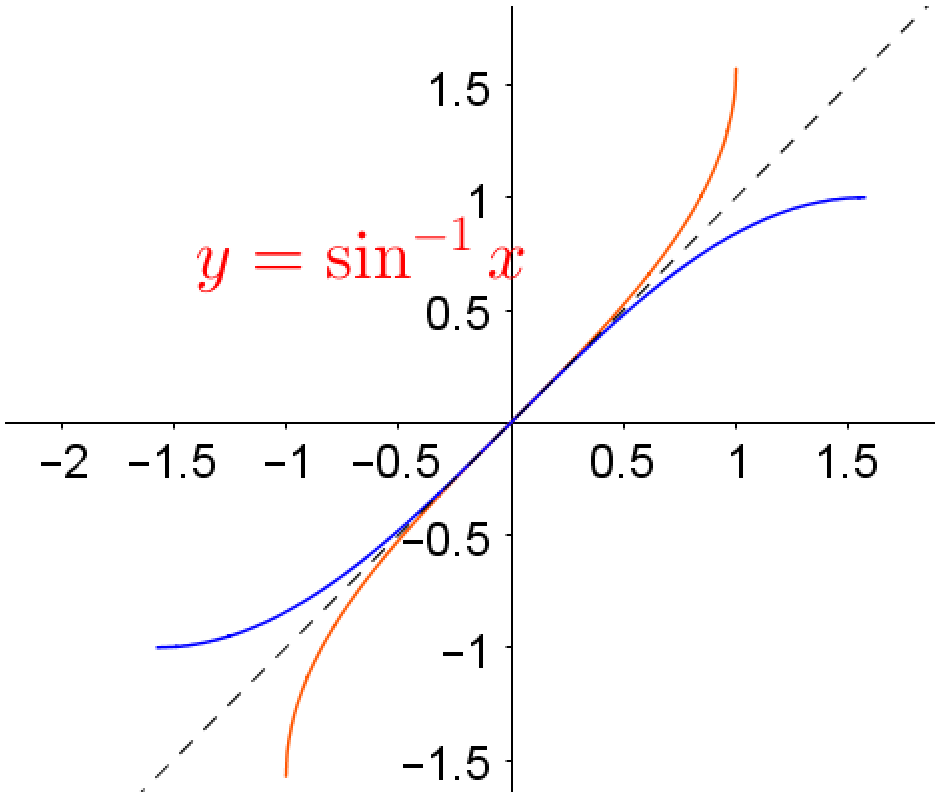
We find the graph of the inverse of a function by reflecting across the line $y = x$, or by interchanging the xs and y's in the equation.

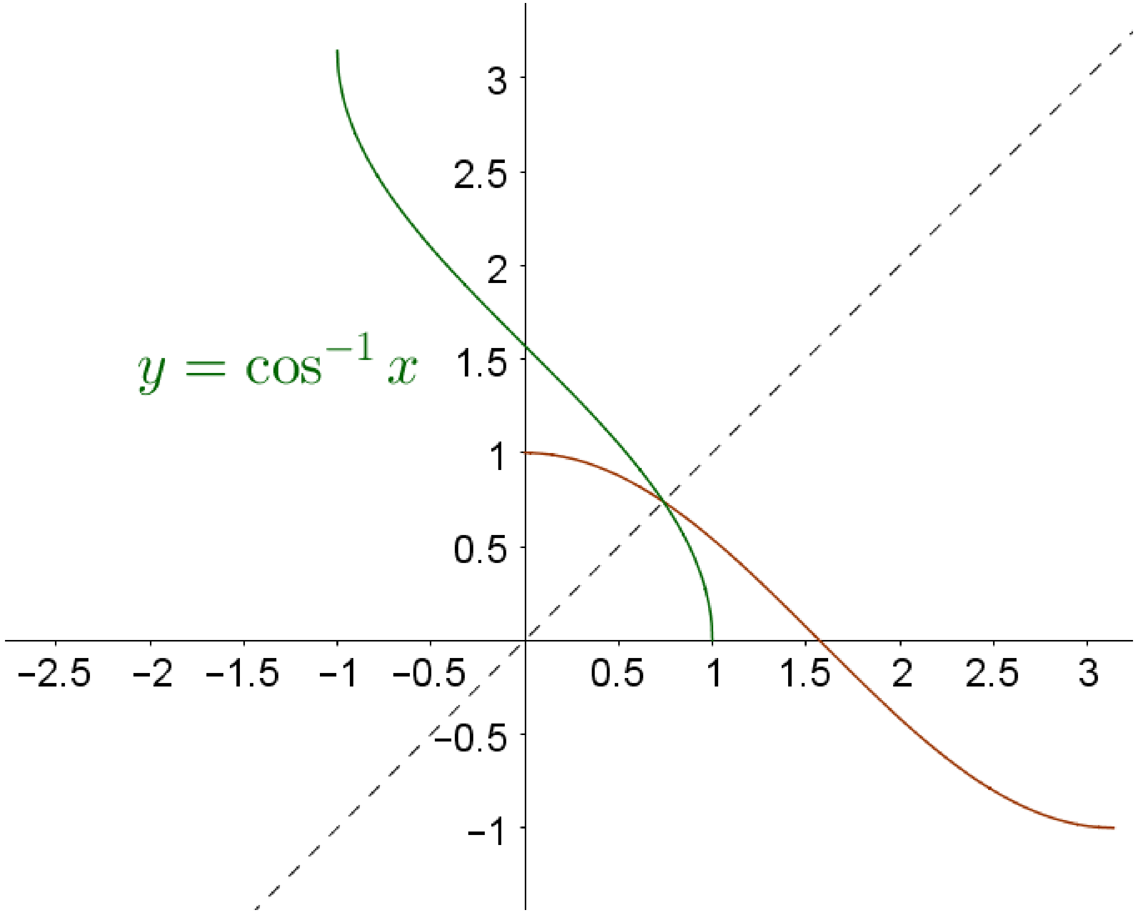
For example, $y = e^x$ and $y = \ln x$ are inverse functions.



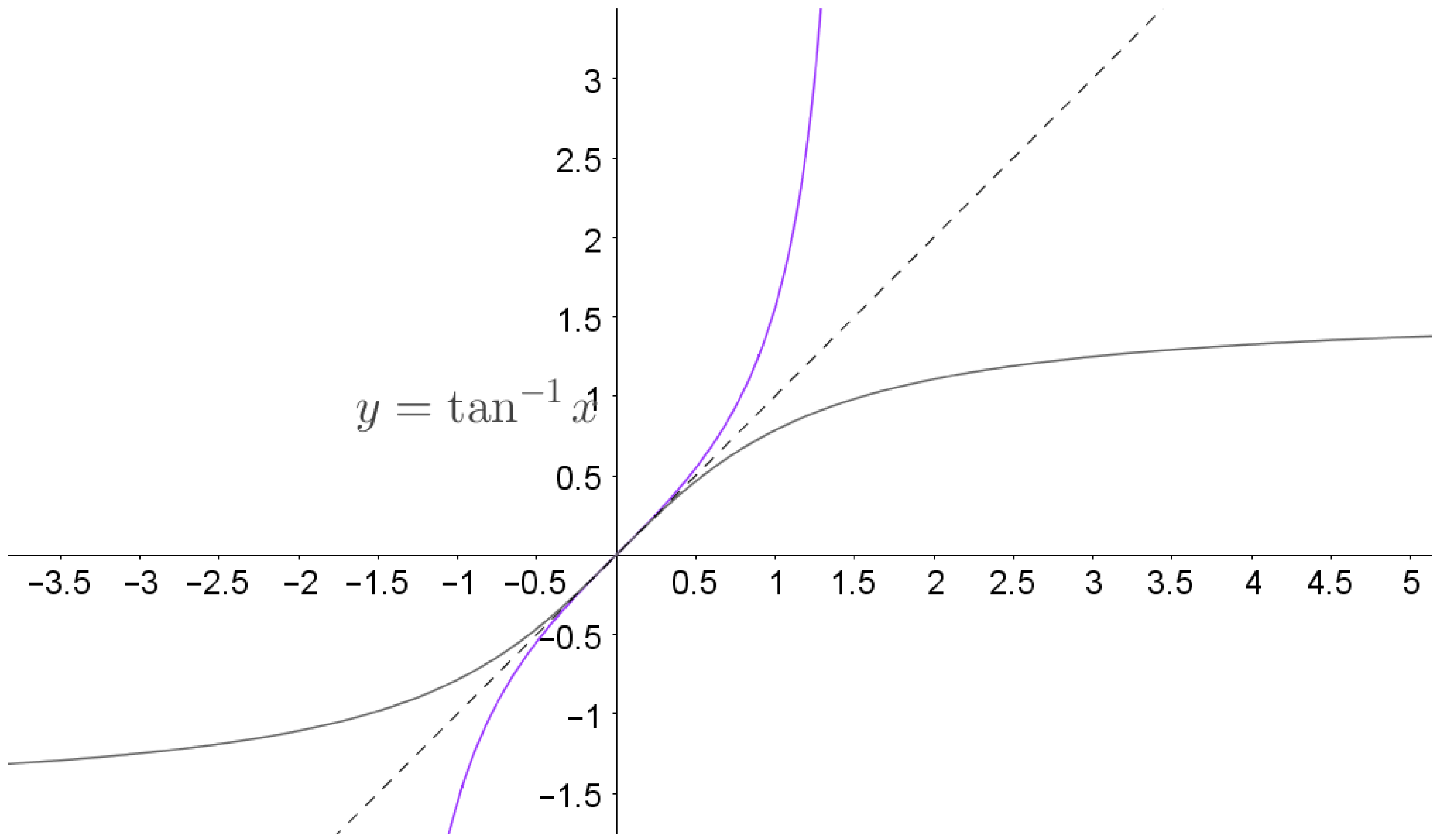


In order to get a function when we reflect the trig functions, we restrict the domain to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ for sin, $0 \leq x \leq \pi$ for cos and $-\frac{\pi}{2} < x < \frac{\pi}{2}$ for tan.





Note that $y = \tan^{-1} x$ has horizontal asymptotes.



General rules applicable to every curve

1 Find the DOMAIN and RANGE if easy to find.

2 Find the x and y INTERCEPTS.

3 Decide if the curve has SYMMETRY:

(a) LINE symmetry for EVEN functions

(b) POINT symmetry for ODD functions

4 Decide on the values of x where y changes sign and decide where y is POSITIVE or NEGATIVE.

5 Find any ASYMPTOTES to the curve;

that is, find y as $x \to \pm\infty$

and find x as $y \to \pm\infty$

Note: Any value of x that makes the denominator zero gives a vertical asymptote.

6 Find any STATIONARY points; that is, where $f'(x) = 0$, and find their nature.

7 Plot a few SPECIAL points to help with the general shape.

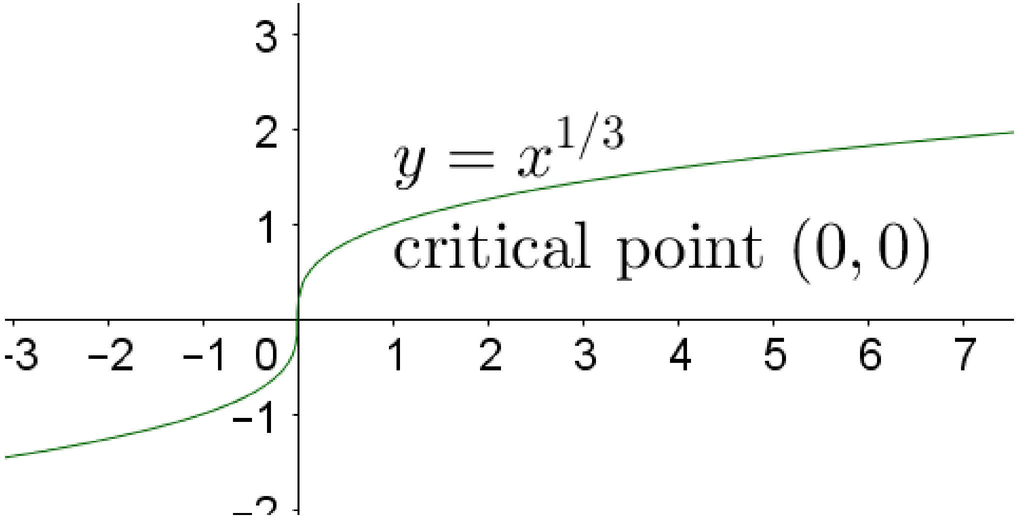
In this topic we will learn techniques which mean we don't need to follow these steps all the time.

Critical Points on Curves

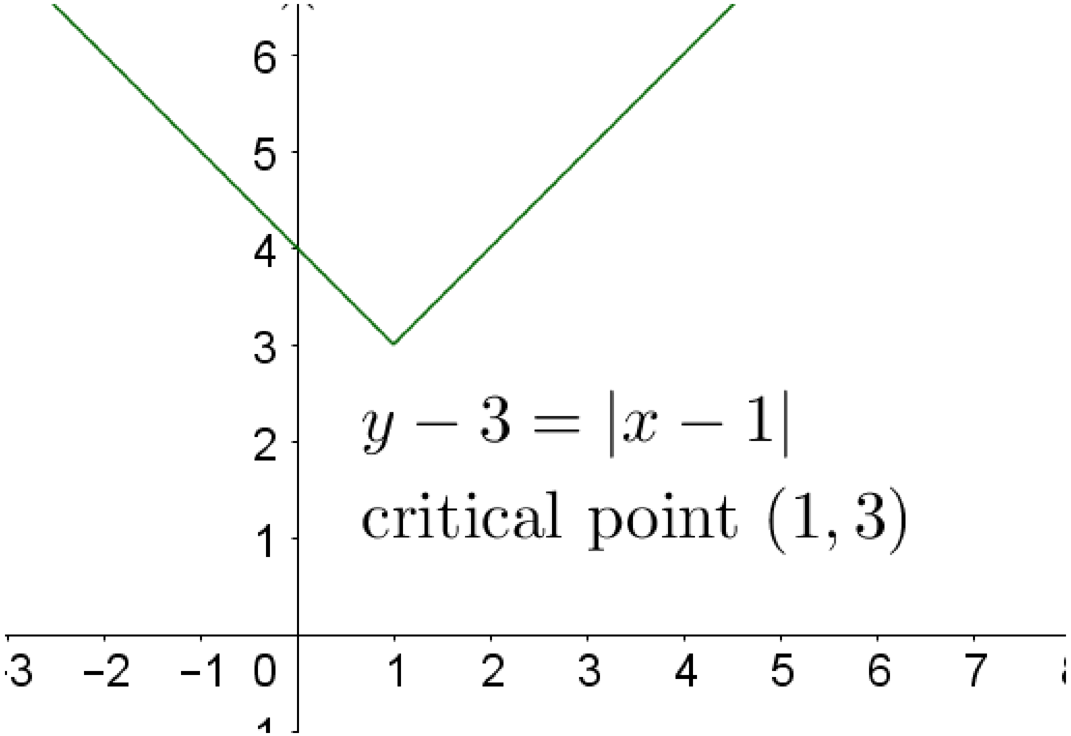
A critical point on the curve $y = f(x)$ is one at which the derivative $f'(x)$ is not defined.

For the derivative to be defined at a point $A(a, f(a))$, the function f must be continuous at $x = a$, and the gradient of the tangent to the curve must have a limiting value which is the same when we approach A from right to left or left to right.

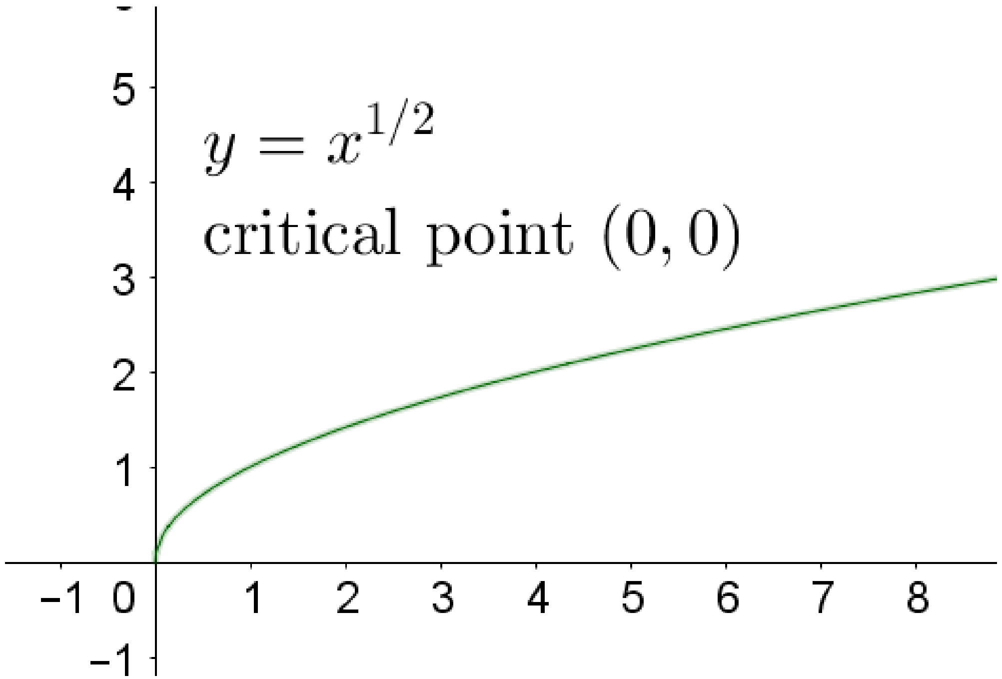
Examples:



$y = x^{1/3}$
critical point $(0, 0)$



$y - 3 = |x - 1|$
critical point $(1, 3)$



$y = x^{1/2}$
critical point $(0, 0)$

Reflecting Graphs in the Coordinate Axes

Reflecting the graph of $y = f(x)$ in the y-axis results in the graph of $y = f(-x)$.

Reflecting the graph of $y = f(x)$ in the x-axis results in the graph of $y = -f(x)$.

Example. Let $f(x) = (x + 2)(x - 4)$. Sketch the graph of

a) $y = f(x)$ b) $y = -f(x)$ c) $y = f(-x)$ d) $y = |f(x)|$

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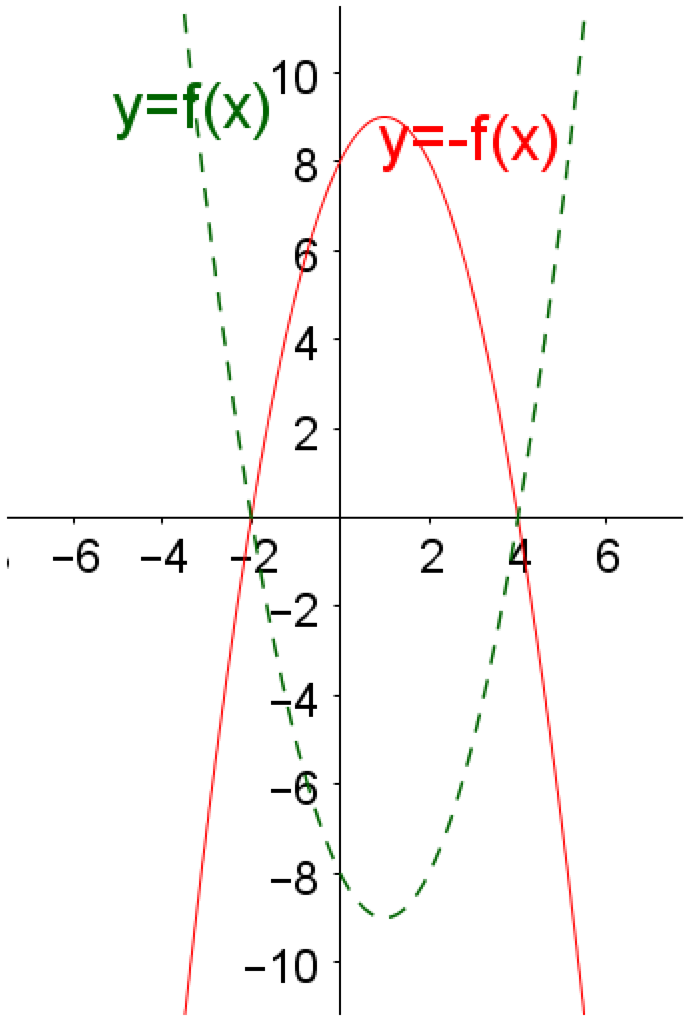
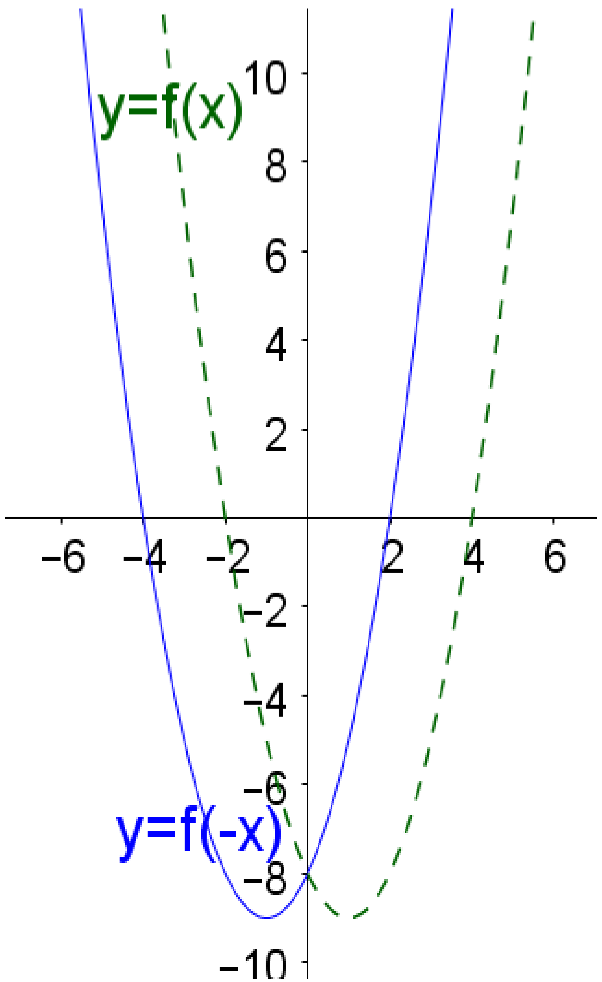
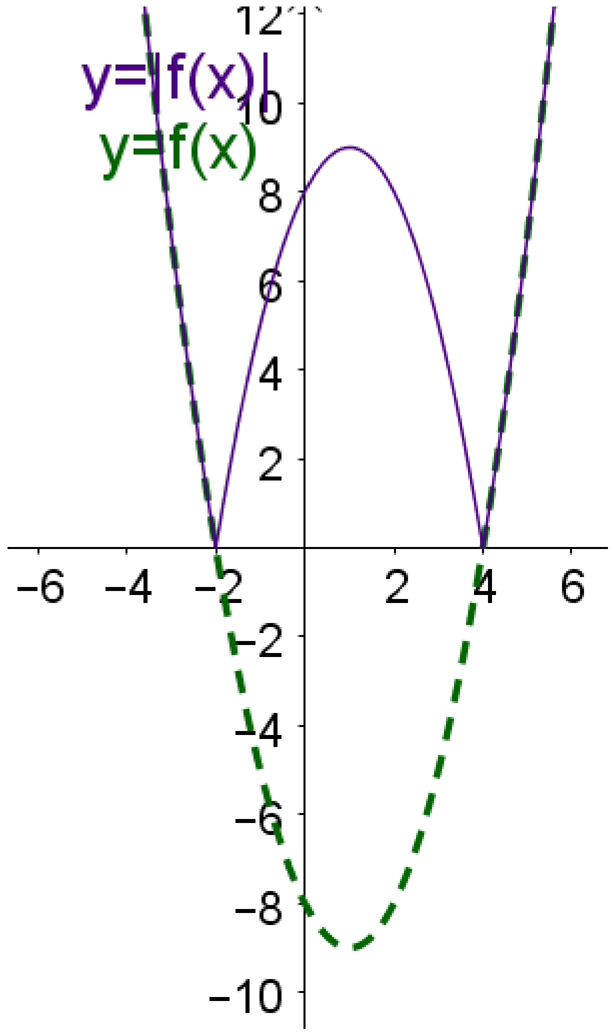






Addition and Subtraction of Ordinates: Graphing $y = f(x) \pm g(x)$

Sketch $f(x)$ and $g(x)$ on the same axes and add or subtract the y-values (the ordinates) for matching values of x.

The domain is the intersection of the domains of $f(x)$ and $g(x)$.

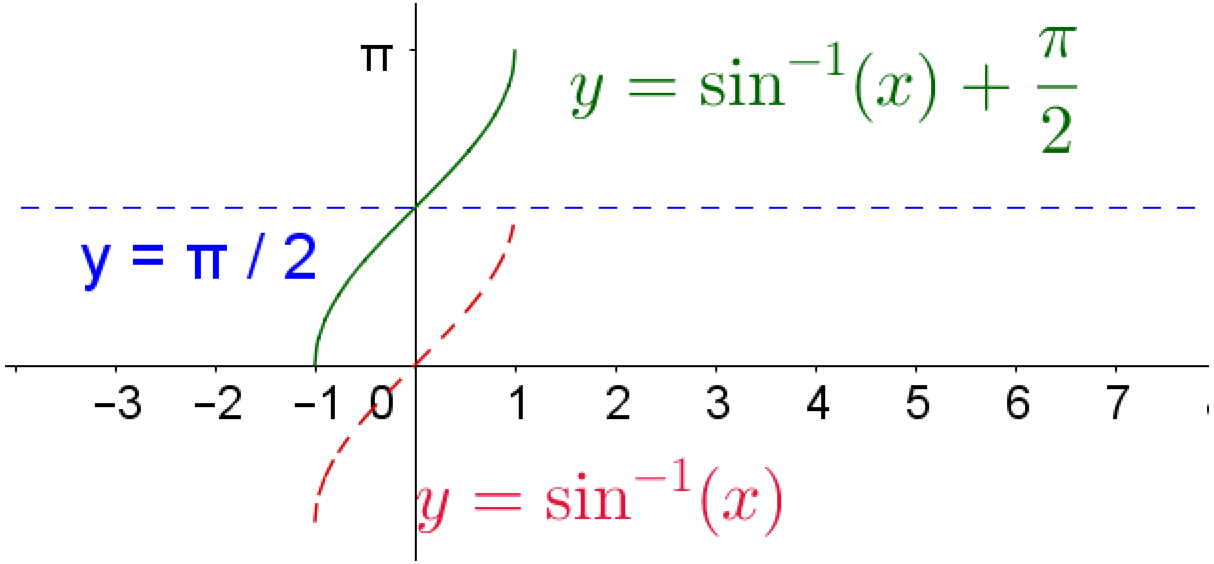
Example. Sketch by addition of ordinates and state the domain and range

a) $y = \sin^{-1} x + \frac{\pi}{2}$

b) $y = x^2 + \frac{1}{x^2}$

c) $y = x^2 - \frac{1}{x^2}$

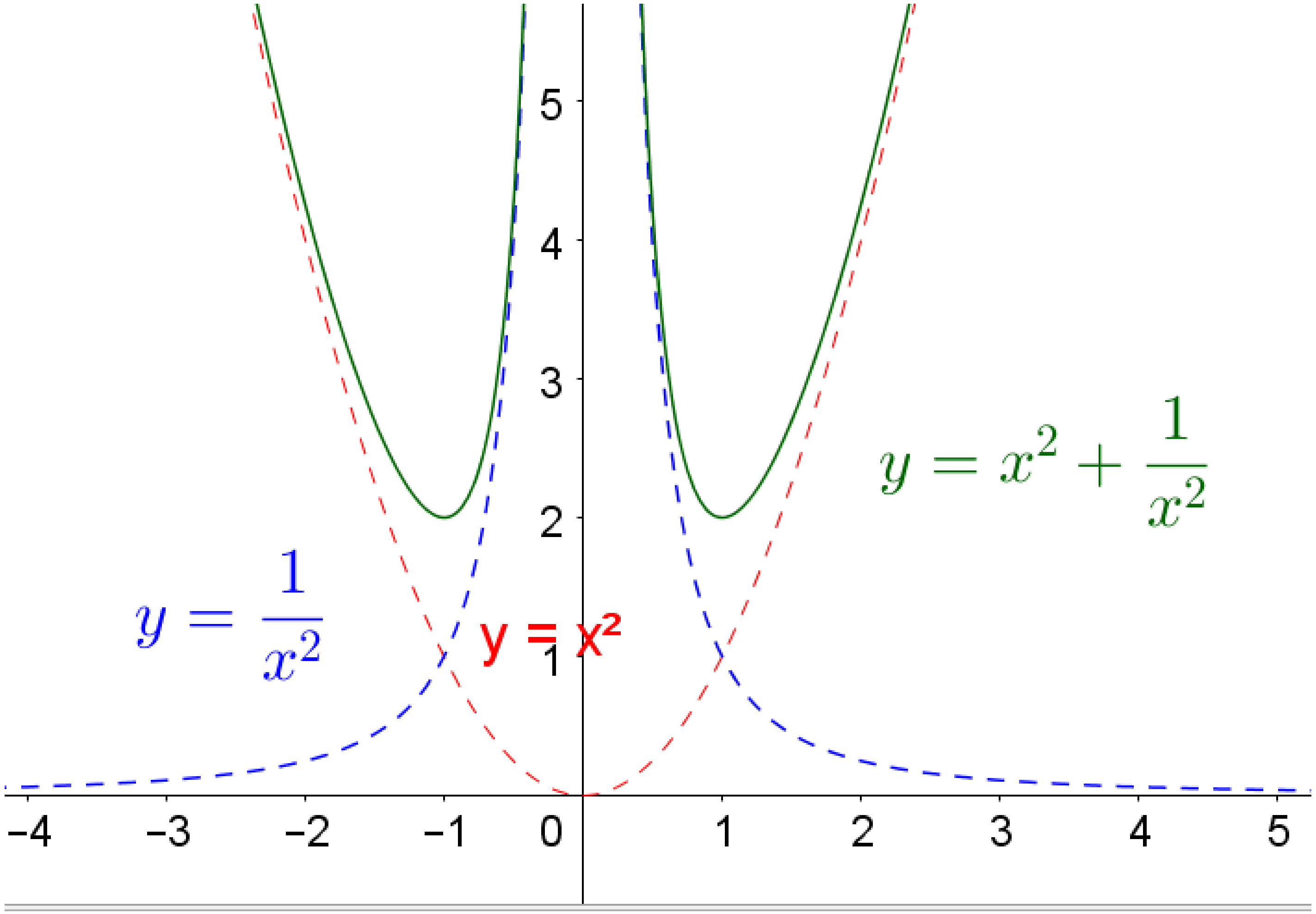
Example. Sketch by addition of ordinates and state the domain and range a) $y = \sin^{-1} x + \frac{\pi}{2}$



Domain: $-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$

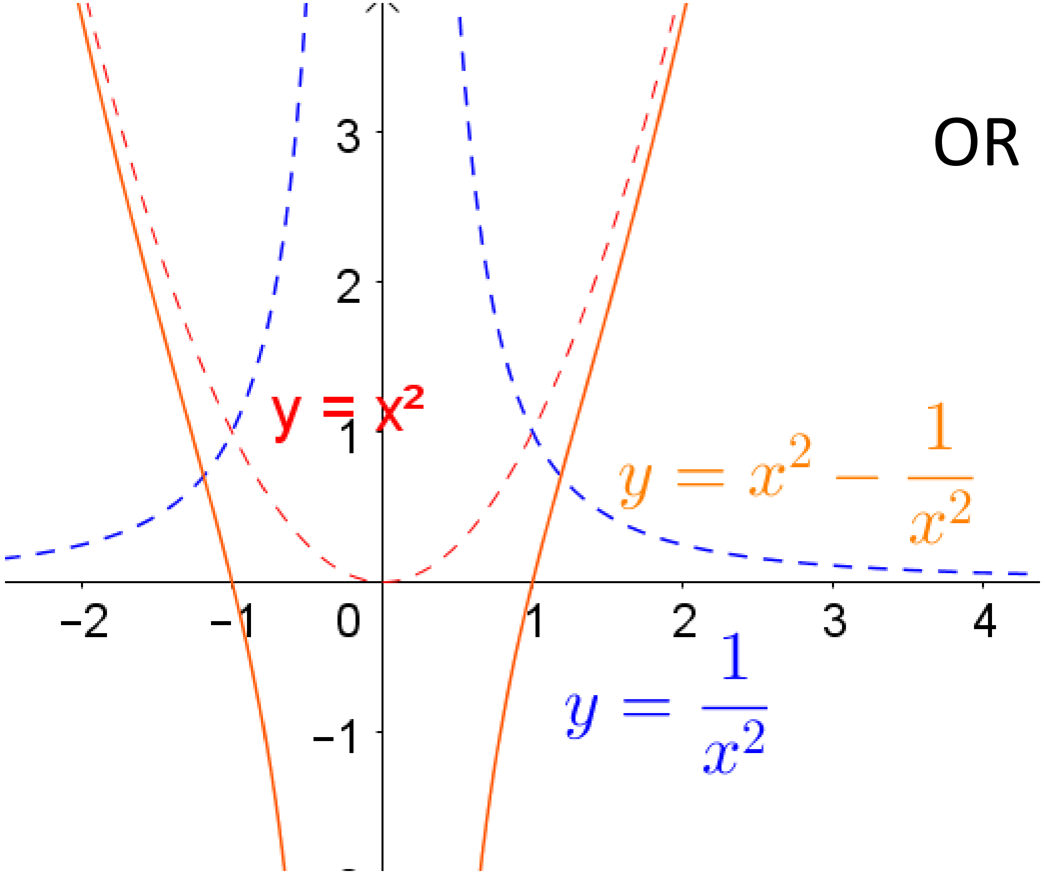
Example. Sketch by addition of ordinates and state the domain and range b) $y = x^2 + \frac{1}{x^2}$ c) $y = x^2 - \frac{1}{x^2}$



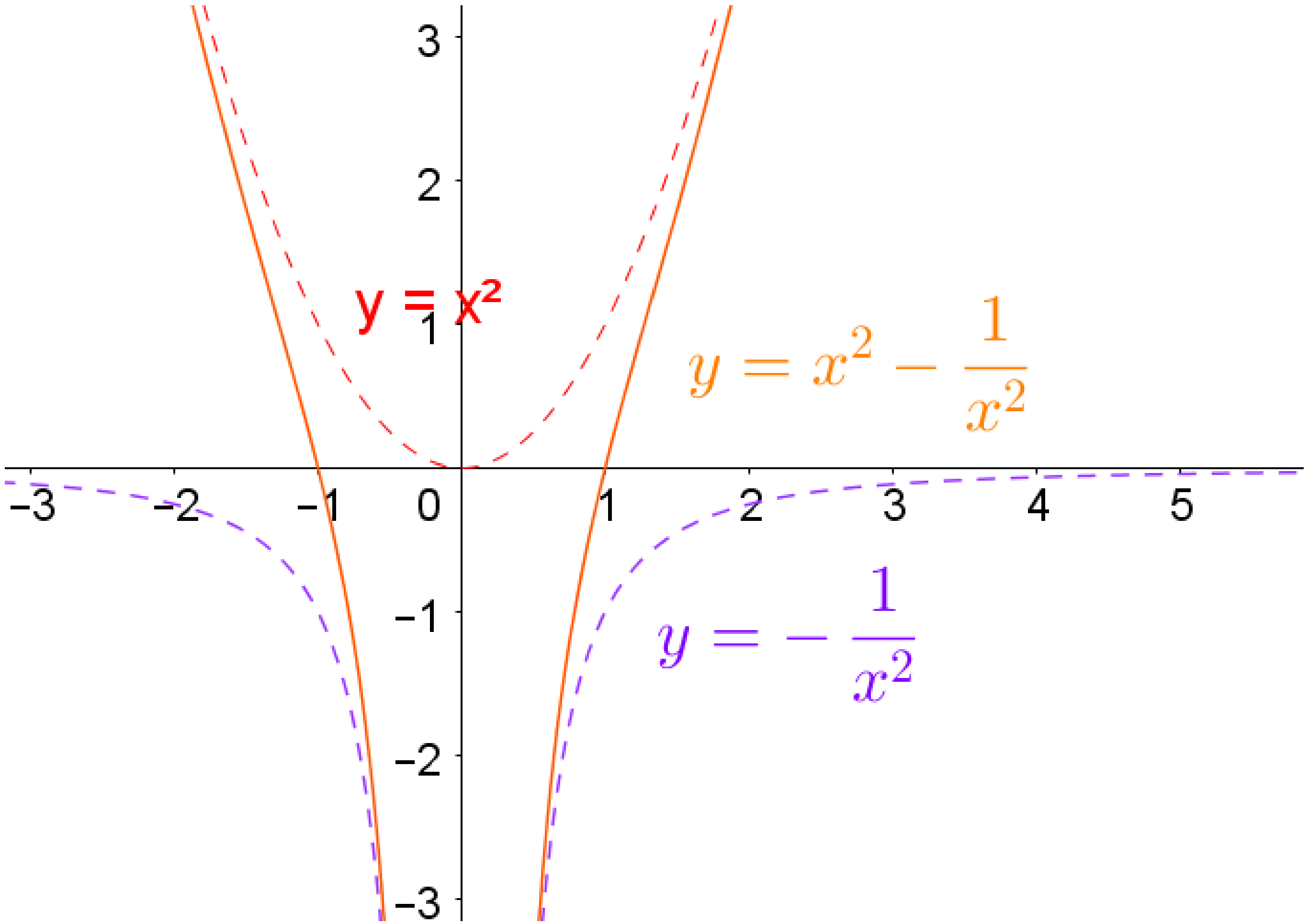
Domain: All real $x, x \neq 0$

Range: $y \geq 2$

Example. Sketch by addition of ordinates and state the domain and range c) $y = x - \frac{1}{x^2}$



OR



Domain: All real $x, x \neq 0$

Range: All real y

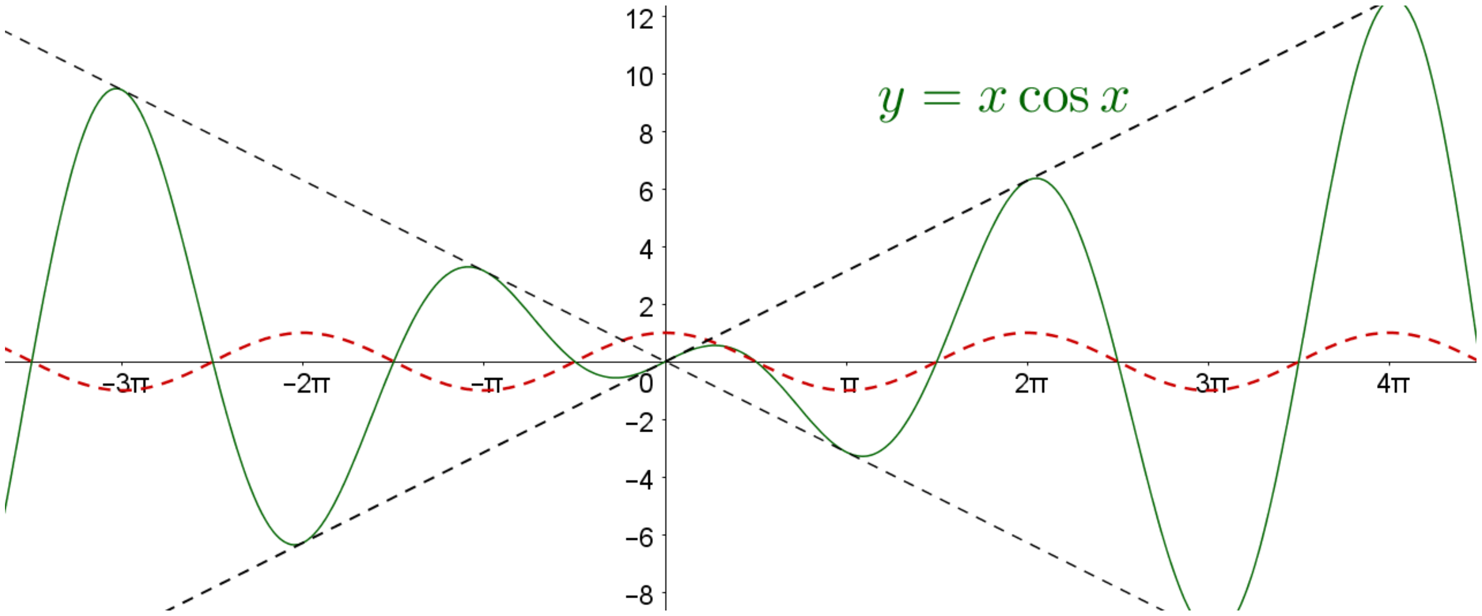
Multiplication of Ordinates

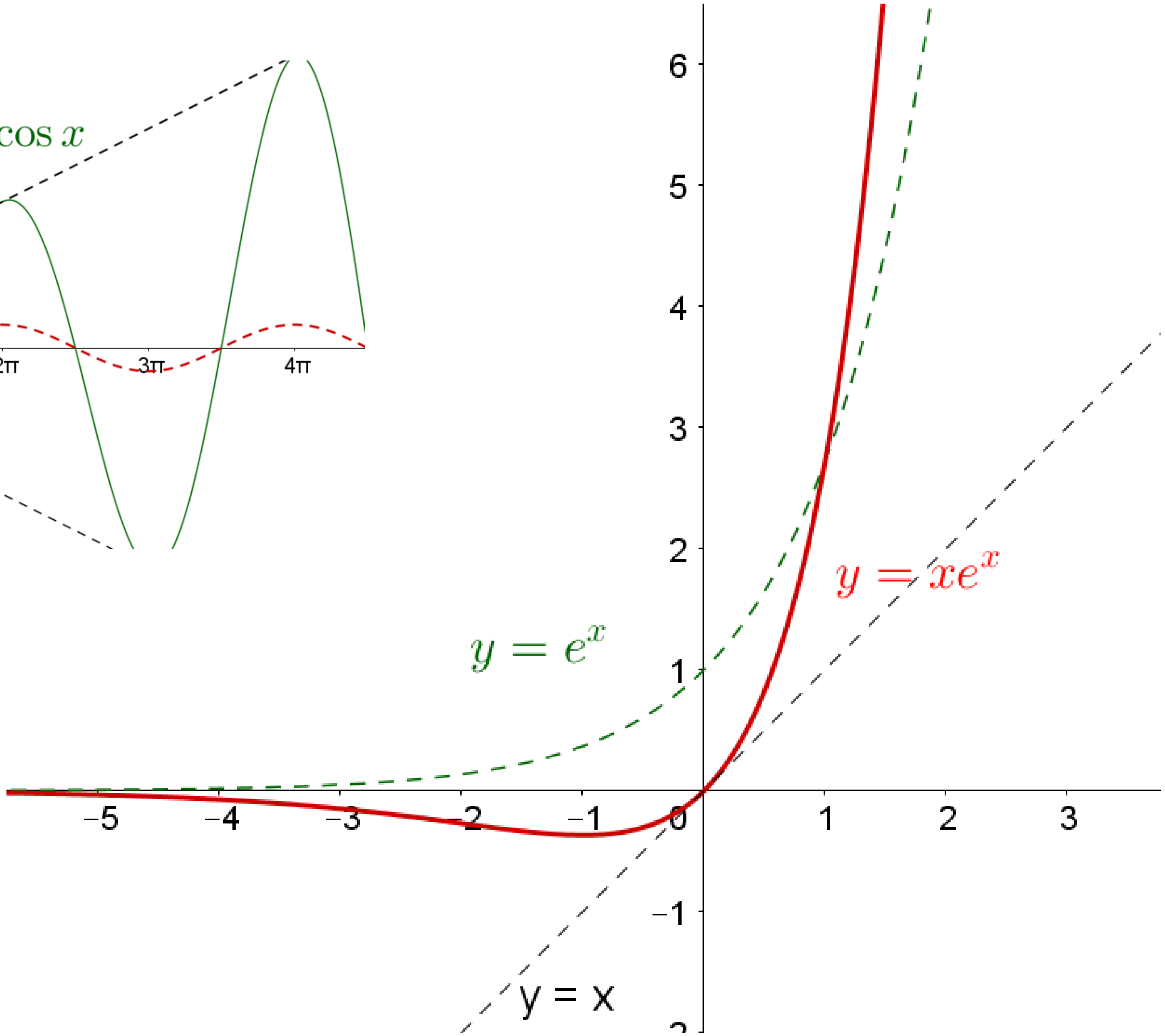
If $f(x) = g(x)h(x)$, then the zeroes of $f(x)$ will be the union of the zeroes of $g(x)$ and the zeroes of $h(x)$.

Example 1. a) $y = xe^x \quad b)\ y = x\cos x$

Note that $y = x$ is an odd function, $y = \cos x$ is even and $y = x\cos x$ is odd.

In general, odd × even = odd, odd × odd =even and even ×even=even.


$y = x\cos x$
12
10
8
6
4
2
0
-2
-4
-6
-8
−3π
−2π
−π
π
2π
3π
4π


$y = e^x$
$y = xe^x$
y = x
6
5
4
3
2
1
0
−1
−5
−4
−3
−2
−1
1
2
3

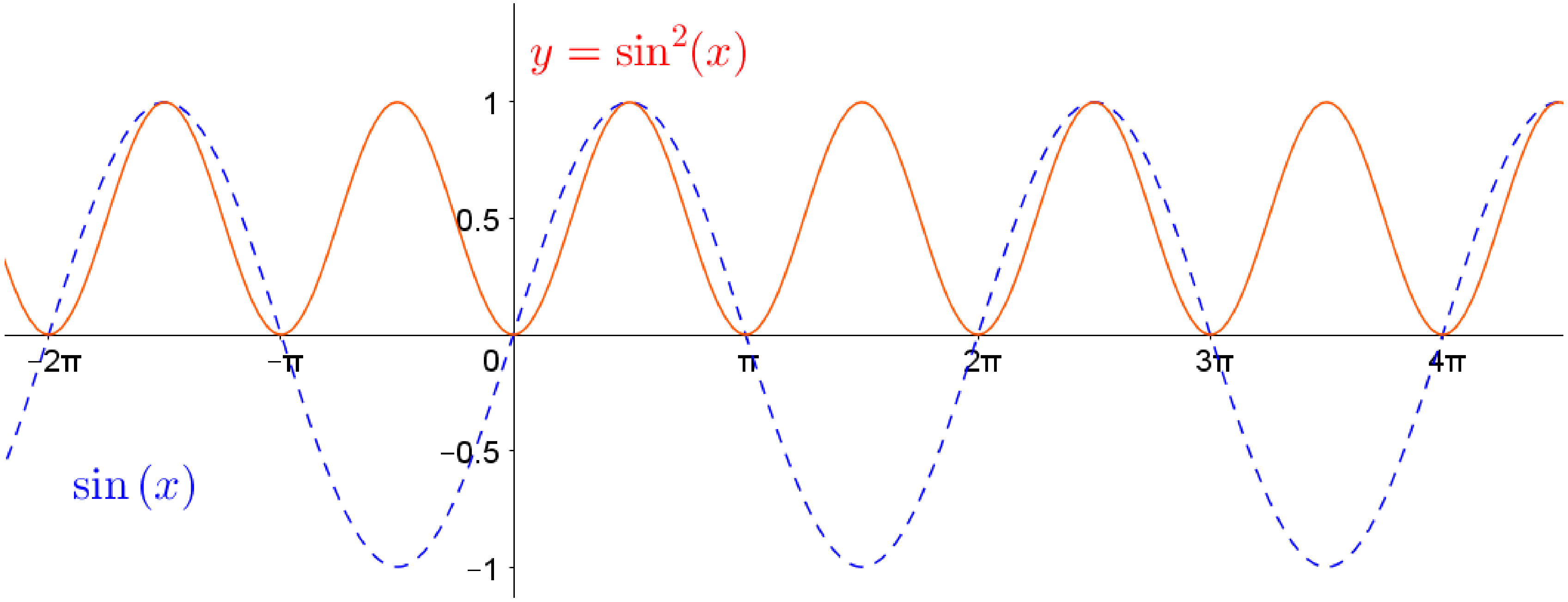
$y = f^n(x), n \ an \ integer > 1$

The x-intercepts of $f(x)$ are intercepts of $f^n(x)$ and stationary points of $f^n(x)$. Proof…

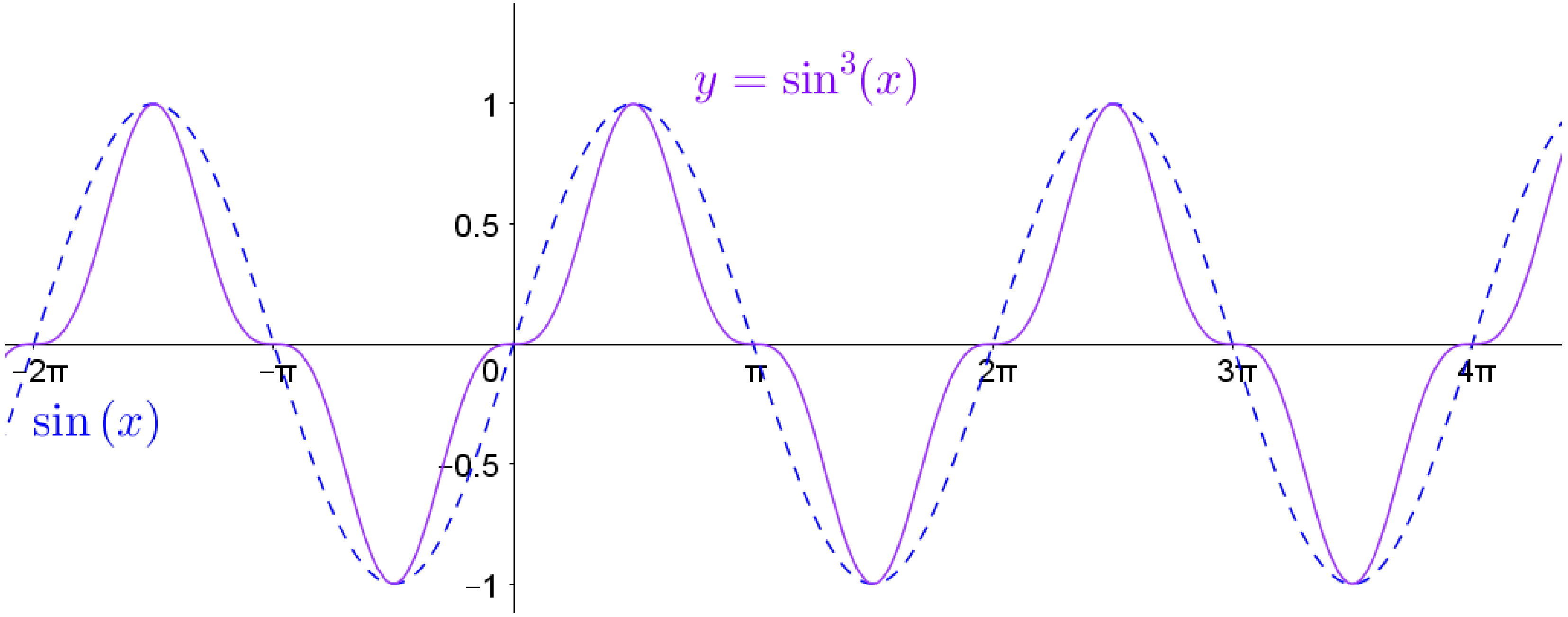
- If n is even, they are also turning points of $f^n(x)$.
- The turning points of $f(x)$ are also turning points of $f^n(x)$.
- If $|f(x)| < 1$ then $|f^n(x)| < |f(x)|$.
- If $|f(x)| > 1$ then $|f^n(x)| > |f(x)|$.

Example 2. Sketch

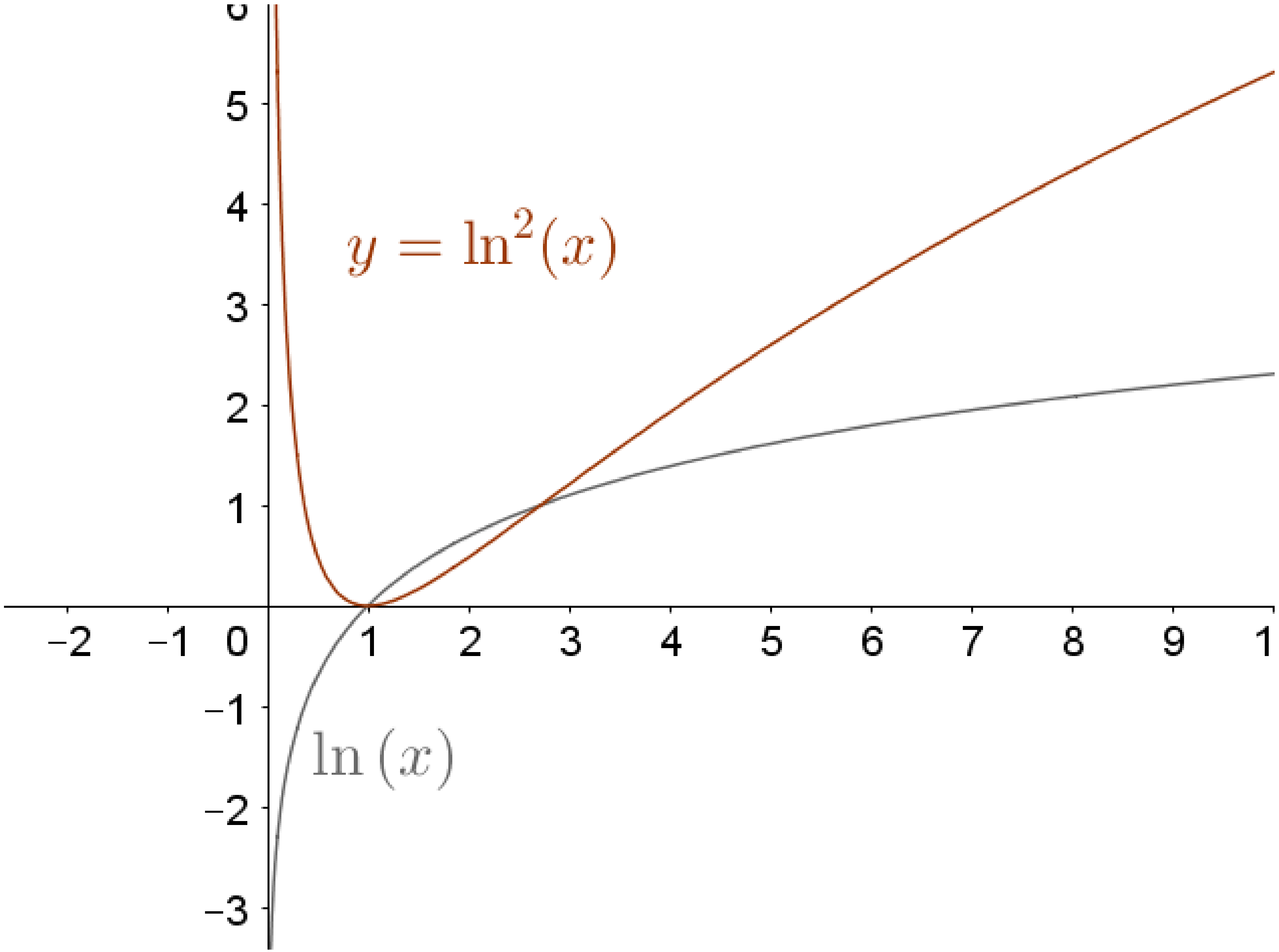
a) $y = \sin^2 x$ b) $y = \sin^3 x$ c) $y = \ln^2 x$



$y = \sin^2(x)$
1
0.5
0
−0.5
−1
−2π
−π
π
2π
3π
4π
$\sin(x)$



$y = \sin^3(x)$
1
0.5
0
−0.5
−1
−2π
−π
π
2π
3π
4π
$\sin(x)$



$y = \ln^2(x)$
$\ln(x)$

Work set so far…

- Ex 2.1 (Fitzpatrick)
- Ex 1.1, 1.3, 1.4, 1.6 (Cambridge)