YEAR 12 EXTENSION 2 TRIAL 2020 SOLUTIONS

Q.11.(a) $(-i)^2 - ib + 1 - i = 0$ 1

$\cancel{-1} - ib + \cancel{1} - i = 0$

$-i(b+1) = 0$, $b = -1$ 1

(2)

(b) $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ $\quad \underset{\sim}{b} = 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{c} = m\underset{\sim}{i} + n\underset{\sim}{j}$

(i) $\underset{\sim}{a} + \underset{\sim}{c} \parallel \underset{\sim}{b}$

$\therefore (2+m)\underset{\sim}{i} + (2+n)\underset{\sim}{j} + \underset{\sim}{k} = K(2\underset{\sim}{j} + 2\underset{\sim}{k})$

Equating components:

(2)

$2 + m = 0$, $m = -2$

$2 + n = K2$ 1

$1 = 2K$ $\therefore K = \frac{1}{2}$ and $n = -1$

Solutions: $\underline{\underline{m = -2}}$, $\underline{\underline{n = -1}}$ 1

(ii) $|\underset{\sim}{c}| = 1$ $\quad \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$\sqrt{m^2 + n^2} = 1$

$\therefore m^2 + n^2 = 1$ — (1)

$(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) \cdot (m\underset{\sim}{i} + n\underset{\sim}{j}) = 0$

need both equations for 1 mark

$2m + 2n = 0 \Rightarrow m = -n$ — (2)

Subst. (2) into (1) $\quad 2n^2 = 1$

$n^2 = \frac{1}{2}$

(ii) (cont)

$n = \pm \frac{1}{\sqrt{2}}$

If $n = \frac{1}{\sqrt{2}}$, $m = -\frac{1}{\sqrt{2}}$ (2)

If $n = -\frac{1}{\sqrt{2}}$, $m = \frac{1}{\sqrt{2}}$ 1

(c) $u = 1 - \sin 2x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 - \sin 2x}\,(1 - 2\cos^2 x)\,dx$$

$2\cos^2 x = \cos 2x + 1$

$\therefore 1 - 2\cos^2 x = -\cos 2x$ 1

$u = 1 - \sin 2x$ $\quad \frac{du}{dx} = -2\cos 2x$ 1

When $x = \frac{\pi}{2}$, $u = 1$

$x = \frac{\pi}{4}$, $u = 0$ 1

$$I = \int_0^1 \frac{1}{2} u^{1/2}\,du$$

$$= \frac{1}{3}\left[u^{3/2}\right]_0^1 = \frac{1}{3}$$ 1

(4)

Q.11

(d) $z = \sqrt{3} + i$

(i) $|z| = \sqrt{3+1} = 2$ ②

$\text{Arg} z = \frac{\pi}{6}$ 1 $\cos\theta = \frac{\sqrt{3}}{2}$

$\sin\theta = \frac{1}{2}$

$\therefore z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ 1

(ii) $z^n - \bar{z}^n = 0$

$\bar{z} = \sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$$\left.\begin{aligned}\bar{z}^n &= 2^n\left(\cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)\\ z^n &= 2^n\left(\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right)\right)\end{aligned}\right\} 1$$

$\therefore z^n - \bar{z}^n$ ③

$= 2^n\left[\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right) - 2^n\cos\left(\frac{n\pi}{6}\right) + i\sin\left(\frac{n\pi}{6}\right)\right]$

$= 2^n . 2i\sin\left(\frac{n\pi}{6}\right)$ 1

Using $\cos(-\theta) = \cos\theta$ & $\sin(-\theta) = -\sin\theta$

$= 2^{n+1} i\sin\frac{n\pi}{6} . = 0$

When $\sin\frac{n\pi}{6} = 0$

$\sin\frac{n\pi}{6} = \sin k\pi$ 1

If $k = 1$, $n = 6$

Hence smallest possible positive integer is 6.

Q.12.

(a) $\text{Im}\, u = 2$ $\text{Re}\, v = -1$

$u + v = -uv$

Let $u = x + 2i$ $v = -1 + yi$

$u + v = -uv$

$\therefore (x-1) + (2+y)i = -\left(-x + xyi - 2i - 2y\right)$

$= x + 2y + (2 - xy)i$ 1

Equating real & imaginary parts:

$x - 1 = x + 2y \Rightarrow 2y = -1$ $y = -\frac{1}{2}$ 1

$2 + y = 2 - xy \Rightarrow y = -xy$, $x = -1$ ②

$\therefore u = -1 + 2i$ $v = -1 - \frac{1}{2}i$

(b) $|e^{2i\theta} + e^{-2i\theta}| = 1$ $-\pi < \theta \leq \pi$

$e^{2i\theta} = \cos 2\theta + i\sin 2\theta$ 1

$e^{-2i\theta} = \cos(-2\theta) + i\sin(-2\theta)$

$= \cos 2\theta - i\sin 2\theta$. ③

$\therefore |e^{2i\theta} + e^{-2i\theta}| = |2\cos 2\theta| = 2|\cos 2\theta| = 1$

$|\cos 2\theta| = \frac{1}{2}$, $\cos 2\theta = \pm\frac{1}{2}$. 1

$\theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$ 1

Q.12(c)

$$y = \frac{1}{1+x}$$

To Prove: $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^n n!}{(1+x)^{n+1}}$

$n > 0$

n an integer.

PROOF:

Show true for $n=1$.

$$\frac{dy}{dx} = \frac{-1}{(1+x)^2}$$

and $\dfrac{(-1)^1 . 1!}{(1+x)^2} = -\dfrac{1}{(1+x)^2} = \dfrac{dy}{dx}$

1

$\therefore$ True for $n=1$

Assume true for $n=k$

i.e. $\dfrac{d^k y}{dx^k} = \dfrac{(-1)^k . k!}{(1+x)^{k+1}}$

Show it is true for $n=k+1$

i.e. $\dfrac{d^{k+1} y}{dx^{k+1}} = \dfrac{(-1)^{k+1} . (k+1)!}{(1+x)^{k+2}}$

$$\text{LHS} = \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$\text{LHS} = \frac{d}{dx}\left(\frac{(-1)^k . k!}{(1+x)^{k+1}}\right)$$

$$= (-1)^k . k! \left(-(k+1)(1+x)^{-(k+2)}\right) \quad 1$$

$$= \frac{(-1)^{k+1} . (k+1)!}{(1+x)^{k+2}}$$

1   3

$$= \text{RHS}.$$

thus it is true for $n=k+1$ if true for $n=k$

$\therefore$ True for $n \geq 1$ by mathematical Induction.

(d)(i) $\text{Arg}(z-2i) = \frac{\pi}{6}$ $\theta = \frac{\pi}{6}$

Let $z = x+iy$ $\therefore z-2i = x + (y-2)i$

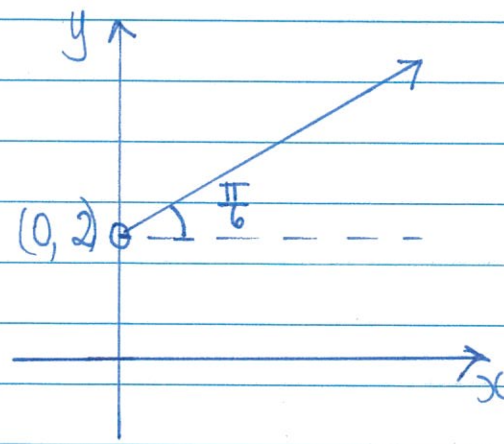
$\cos\frac{\pi}{6} = x$ $\sin\frac{\pi}{6} = y-2$

2

$\therefore \tan\frac{\pi}{6} = \dfrac{y-2}{x} = \dfrac{1}{\sqrt{3}}$

i.e. $\sqrt{3}(y-2) = x$ i.e. $y = \dfrac{x}{\sqrt{3}} + 2$

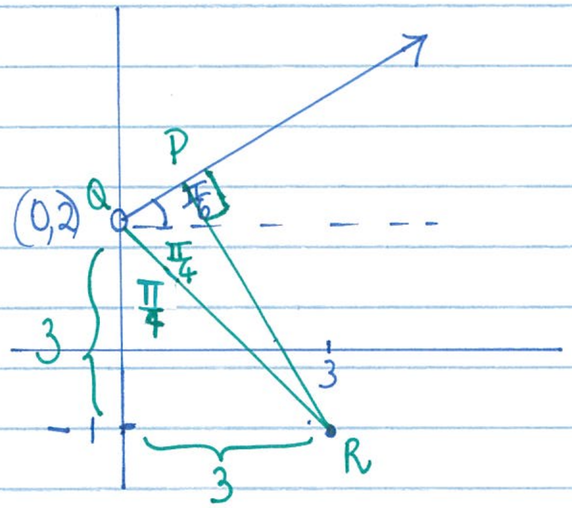
(ii)



1

Q.12(d)

(iii) Find least possible exact value of $|z-3+i|$



This is $\perp$ distance from ray to the point $(3,-1)$.

$\perp$ distance formula is not in the syllabus but we can use another method.

In ΔPQR, $\angle PQR = \frac{\pi}{6}+\frac{\pi}{4} = \frac{5\pi}{12}$ 1

$d(QR) = \sqrt{3^2+3^2} = 3\sqrt{2}$ 1

$\therefore \sin\frac{5\pi}{12} = \frac{PR}{QR}$

ie. $PR = \underline{QR\sin\frac{5\pi}{12}}$

We want exact values here

$\therefore \sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$ 1

$= \sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$\sin\frac{5\pi}{12} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$ 4.

$= \underline{\frac{\sqrt{3}+1}{2\sqrt{2}}}$ 1

$\therefore PR = QR\sin\frac{5\pi}{12} = \frac{3\sqrt{2}(\sqrt{3}+1)}{2\sqrt{2}} = \underline{\frac{3(\sqrt{3}+1)}{2}}$

Q.13.(a) To Prove: $\log_2 5$ is irrational.

Proof: Using Proof by Contradiction,

Assume there exists p, q both integers

Such that $\log_2 5 = \frac{p}{q}$ where $q \geq 1$

and the H.C.F. of p and q is 1.

$\therefore \log_2 5 = \frac{p}{q} \Rightarrow 5 = 2^{\frac{p}{q}}$

Raising both sides to power q, we get

$5^q = 2^p$

Since this requires 5 to be a factor of 2^p which it cannot be, or else 2 to be a factor of 5^p, which it is not, & since 5^p is always odd and 2^p is always even, therefore this cannot ever be true!

Thus by contradiction, $\log_2 5$ must be irrational.

Q 13(b)

SHM: $T = \frac{2\pi}{n}$ amp $= a$.

At $t=0$, $x = \frac{a}{2}$

(i) Using $x = a\sin(nt+\alpha)$

When $t=0$, $x = \frac{a}{2}$ $\therefore \frac{a}{2} = a\sin\alpha$

i.e. $\sin\alpha = \frac{1}{2}$, $\alpha = \frac{\pi}{6}$.

$\therefore x = a\sin\left(nt+\frac{\pi}{6}\right)$ 1

(ii) Particle first reaches an extreme point when $x = a$

i.e. $a = a\sin\left(nt+\frac{\pi}{6}\right)$

$\sin\left(nt+\frac{\pi}{6}\right) = 1$ 1

$nt + \frac{\pi}{6} = \frac{\pi}{2}$, $nt = \frac{\pi}{3}$, $t = \frac{\pi}{3n}$

(iii) $|v| = V$ m/s at $x = \frac{2a}{3}$

($\frac{a}{3}$ m from extreme)

Max^{m} speed occurs at centre of motion.

Using $v^2 = n^2(a^2 - x^2)$

$|v|_{max} = \sqrt{n^2a^2}$

i.e. $|v|_{max} = na$

But we want this in terms of V.

Since $|v| = V$ at $x = \frac{2a}{3}$.

Using $v^2 = n^2(a^2 - x^2)$

$V^2 = n^2\left(a^2 - \frac{4a^2}{9}\right)$

$V^2 = n^2 \frac{5a^2}{9}$ 1

$\Rightarrow 9V^2 = 5n^2a^2$

$\Rightarrow a^2 = \frac{9V^2}{5n^2}$, $a = \frac{3V}{\sqrt{5}n}$

$\therefore |v|_{max} = \frac{\cancel{n}3V}{\sqrt{5}\cancel{n}} = \frac{3V}{\sqrt{5}}$ m/s. 1

QUESTION 13 (continued)

(b) (iii) At $x = \frac{2a}{3}$ $|v| = V$

and at $x = -\frac{2a}{3}$ $|v| = V$

Using $v^2 = n^2(a^2 - x^2)$

$$V^2 = n^2\left(a^2 - \frac{4a^2}{9}\right)$$

ie. $V^2 = n^2 \frac{5a^2}{9}$ 1

$\Rightarrow 9V^2 = 5n^2a^2$ — ①

Now $|v|_{max}$ occurs when

$v = na$.

from ① $a = \sqrt{\frac{9V^2}{5n^2}} = \frac{3V}{\sqrt{5}\,n}$

$\therefore v_{max} = na = \frac{3V}{\sqrt{5}}$ m/s. 1

2

(c)(i)

$$l_1 = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + k \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

QUESTION 13

(a) Prove that $\log_2 5$ is irrational

By Contradiction:

Proof: Assume there exists $p, q \in N$

such that $\log_2 5 = \frac{p}{q}$,

$q \geq 1$ and the HCF of p and q

is 1.

$\therefore \log_2 5 = \frac{p}{q}$

$2^{\log_2 5} = 2^{p/q}$

i.e. $5 = 2^{p/q}$

$5^q = 2^p$

this statement implies that

either 5 is a factor of both

5^q and 2^p

or 2 is a factor of both

5^q and 2^p

or that they are both even or both odd.

Since this is untrue,

$\log_2 5$ cannot equal $\frac{p}{q}$

and therefore must be irrational.

(b) $T = \frac{2\pi}{n}$, $a = a$

At $t=0$, $x = \frac{a}{2}$

(i) Using $x = a\sin(nt+\alpha)$

At $t=0$, $x=\frac{a}{2}$ $\therefore \frac{a}{2} = a\sin\alpha$

$\Rightarrow \sin\alpha = \frac{1}{2}$, $\alpha = \frac{\pi}{6}$

$\therefore x = a\sin\left(nt+\frac{\pi}{6}\right)$ 1

(ii) Extreme pos^n $x=a$ since a starts at $a/2$ and moves away from the origin.

$a = a\sin\left(nt+\frac{\pi}{6}\right)$

$\Rightarrow \sin\left(nt+\frac{\pi}{6}\right) = 1$

$nt+\frac{\pi}{6} = \frac{\pi}{2}$ $\Rightarrow nt = \frac{\pi}{3}$

$t = \frac{\pi}{3n}$ 1

Q.13(C)

(i) $l_1 = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + k\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

$\uparrow$ position vector $\quad \uparrow$ direction vector

1

$l_2 = (-t+1)\underset{\sim}{i} + (2t-2)\underset{\sim}{j} + (3t+6)\underset{\sim}{k}$

(ii) vector equation of l_2 can be written as:

$$\underset{\sim}{r}_2 = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\nwarrow$ direction vector 1

So $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ is a vector parallel to $\underset{\sim}{l}_2$.

(iii) For point of intersection of l_1 and l_2:

Equate the $\underset{\sim}{i}$ $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components of $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$

ie. $k - 1 = -t + 1 \quad -1$

1

$-2k + 2 = 2t - 2 \quad -2$

$2k - 3 = 3t + 6 \quad -3$

2 + 3 yields $\quad -1 = 5t + 4$

ie. $5t = -5, \; t = -1$

1

Subst. $t = -1$ in 1 $\quad k = 3$

Substitute for k in $\underset{\sim}{r}_1$ or t in $\underset{\sim}{r}_2$ to obtain:

$$r_1 = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \quad 1$$

Remember:
Check the other value in the other eqn to get the same point!
This is our point of intersection
You should show your check.

(iv) For acute angle between l_1 and l_2:

we will find the angle between their direction vectors.

For l_1, $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ & for l_2 $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

$\boxed{3}$

1

$$\cos\theta = \frac{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}}{\left|\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right| \left|\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}\right|} = \frac{-1-4+6}{\sqrt{9}\cdot\sqrt{14}}$$

1

$$= \frac{1}{3\sqrt{14}}$$

$$\theta = \cos^{-1}\frac{1}{3\sqrt{14}} = 84.9^\circ \quad 1$$

QUESTION 14

(a)

(i) $z = x+iy$ $\quad e^{iz} = e^{i(x+iy)} = e^{-y+ix}$

$$\therefore \ |e^{iz}| = |e^{-y+ix}| = |e^{-y} \cdot e^{ix}| = e^{-y}$$

since $|e^{ix}| = 1$

$$\arg(e^{iz}) = \arg(e^{-y} \cdot e^{ix})$$

1

$$= \arg(e^{ix}) = x$$

(ii) For $e^{iz} = 3i$

$$|e^{iz}| = |3i| = 3$$

and $\arg(e^{iz}) = \arg(3i) = \frac{\pi}{2} + 2k\pi$, k an integer

1

We need to find all $z = x+iy$ and there are infinitely many solutions.

But $|e^{iz}| = e^{-y} = 3$ and $\arg(e^{iz}) = x = \frac{\pi}{2} + 2k\pi$.

Now, since $e^{-y} = 3$ 1

take ln of both sides yields

$$-y = \ln 3 \quad \therefore y = -\ln 3$$

1

$$\therefore \ z = x+iy = \left(\frac{\pi}{2} + 2k\pi\right) - i\ln 3, \quad k \text{ any integer.}$$

Q.14 (b) $d > 1$

(i) Show that $\frac{1}{d^2} < \frac{1}{d(d-1)}$

Since $d > 1$, $d > d-1$

Also, $(\times d)$ $d^2 > d(d-1)$

$\therefore$ $\frac{1}{d^2} < \frac{1}{d(d-1)}$

(ii) $\frac{1}{d^2-d} = \frac{1}{d-1} - \frac{1}{d}$ n +ve integer.

Show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 2$

Proof: We have $\frac{1}{d(d-1)} = \frac{1}{d-1} - \frac{1}{d}$

Also, from (i) $\frac{1}{d^2} < \frac{1}{d(d-1)} = \frac{1}{d-1} - \frac{1}{d}$

$\therefore$ If $d=1$, $\frac{1}{1^2} < \frac{1}{(1-1)} - \frac{1}{1}$ undefined. $\therefore$ treat as equal to 1.

If $d=2$, $\frac{1}{2^2} < \frac{1}{2-1} - \frac{1}{2} = \frac{1}{1} - \frac{1}{2}$

If $d=3$, $\frac{1}{3^2} < \frac{1}{2} - \frac{1}{3}$

If $d=4$, $\frac{1}{4^2} < \frac{1}{3} - \frac{1}{4}$ etc.

Q.14(b) (cont.)

$\therefore$ If $d = n$, we have

$$\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}$$

Summing these terms we obtain:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

$$+ \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \frac{1}{n-1} - \frac{1}{n}$$

$$< 2 - \frac{1}{n}$$

Since n is a positive integer $\frac{1}{n} < 1$ and therefore

we have $2 - \frac{1}{n} < 2$

$\therefore$ $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$ as required.

14(c) (i) $z^5 = (z+1)^5$

$$= z^5 + 5z^4 + 10z^3 + \dots$$

1

the z^5 terms cancel leaving a quartic eq^n with four roots.

(ii) To solve:

METHOD 1 - 1st SOLUTION

$$z^5 = (z+1)^5$$

We are looking for the four roots.

Beginning with $z^5 = (z+1)^5$ and using $e^{i\theta} = 1$ when $\theta = 2k\pi$,

we have $e^{2k\pi i} = 1$

$$\therefore \quad e^{2k\pi i} z^5 = (z+1)^5$$

Taking the fifth root of both sides

$$e^{\frac{2k\pi i}{5}} z = z+1 \qquad \text{for } k = 1, 2, 3, 4$$

Exclude $k=0$ since $z \neq z+1$

Now, $e^{\frac{2\pi k i}{5}} . z = z+1$

i.e. $z\left(e^{\frac{2\pi k i}{5}} - 1\right) = 1$

$$z = \frac{1}{e^{\frac{2\pi k i}{5}} - 1}$$

1

We need this in the form $a + bi\cot\theta$

14(c) (cont)

$$z = \frac{1}{e^{\frac{2k\pi i}{5}} - 1}$$

Multiplying top and bottom by $e^{-\frac{k\pi i}{5}}$ we get

$$z = \frac{e^{-\frac{k\pi i}{5}}}{e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}}} \quad — ③$$ 1

Now $e^{i\theta} = \cos\theta + i\sin\theta$ —①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ —②

①-② yields $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.

Using this property, $e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}} = 2i\sin\frac{k\pi}{5}$ 1

$\therefore$ ③ becomes $z = \dfrac{e^{-\frac{k\pi i}{5}}}{2i\sin\frac{k\pi}{5}} = \dfrac{\cos\left(-\frac{k\pi}{5}\right) + i\sin\left(-\frac{k\pi}{5}\right)}{2i\sin\frac{k\pi}{5}}$ ④

$$= \frac{\cos\frac{k\pi}{5} - i\sin\frac{k\pi}{5}}{2i\sin\frac{k\pi}{5}}$$

$$= \frac{1}{2i}\cot\frac{k\pi}{5} - \frac{1}{2}$$

$$\frac{1}{2i} \times \frac{i}{i} = -\frac{i}{2}$$

$$= -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}$$ 1

Q14(c)(ii)

OR

METHOD 2 — 2nd SOLUTION

$$z^5 = (z+1)^5$$

$$\frac{z^5}{(z+1)^5} = 1$$

$$\left(\frac{z}{z+1}\right)^5 = 1 = r^5 \operatorname{cis} 5\theta \qquad r=1,\ 5\theta = 0 + 2k\pi,\ k \in R$$

$$\theta = \frac{2k\pi}{5} \qquad k \in R.$$

$$z_n = \frac{z}{z+1} = r \operatorname{cis}\theta$$

$$\therefore \frac{z}{z+1} = \operatorname{cis}\frac{2k\pi}{5}. \qquad k \in R.$$

$$z = (z+1)\operatorname{cis}\frac{2k\pi}{5}$$

$$\Rightarrow z\left(1 - \operatorname{cis}\frac{2k\pi}{5}\right) = \operatorname{cis}\frac{2k\pi}{5}.$$

$$z = \frac{\operatorname{cis}\frac{2k\pi}{5}}{1 - \operatorname{cis}\frac{2k\pi}{5}} = \frac{\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}}{1 - \left(\cos\frac{2k\pi}{5} - i\sin\frac{2k\pi}{5}\right)}$$

$$= \frac{\left(\cos\frac{2\pi k}{5} + i\sin\frac{2k\pi}{5}\right)\left(1 - \cos\frac{2k\pi}{5} + i\sin\frac{2\pi k}{5}\right)}{\left(1 - \cos\frac{2k\pi}{5} - i\sin\frac{2k\pi}{5}\right)\left(1 - \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right)}$$

$$= \frac{\cos\frac{2k\pi}{5} - \cos^2\frac{2k\pi}{5} + \cancel{i\sin\frac{2k\pi}{5}\cos\frac{2k\pi}{5}} + i\sin\frac{2k\pi}{5} - \sin^2\frac{2k\pi}{5} \cancel{- i\sin\frac{2\pi k}{5}\cos\frac{2k\pi}{5}}}{\left(1 - \cos\frac{2k\pi}{5}\right)^2 + \sin^2\frac{2k\pi}{5}}$$

$$= \frac{\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5} - 1}{1 - 2\cos\frac{2k\pi}{5} + \cos^2\frac{2k\pi}{5} + \sin^2\frac{2k\pi}{5}}$$

Q 14(c)(ii) (continued)

$$= \frac{\cos\frac{2k\pi}{5} - 1 + i\sin\frac{2k\pi}{5}}{2 - 2\cos\frac{2k\pi}{5}}$$

$$= \frac{\left(\cos\frac{2k\pi}{5} - 1\right) + i\sin\frac{2k\pi}{5}}{-2\left(\cos\frac{2k\pi}{5} - 1\right)}$$

$$= -\frac{1}{2} - \frac{i\sin\frac{2k\pi}{5}}{2\cos\frac{2k\pi}{5} - 2}$$

$$= -\frac{1}{2} - \frac{\frac{i\sin\frac{2k\pi}{5}}{\sin\frac{2k\pi}{5}}}{\frac{2\cos\frac{2k\pi}{5} - 2}{\sin\frac{2k\pi}{5}}}$$

$$= -\frac{1}{2} - \frac{i\sin\frac{2k\pi}{5} \times \left(\cos\frac{2k\pi}{5} + 1\right)}{2\left(\cos\frac{2k\pi}{5} - 1\right) \times \left(\cos\frac{2k\pi}{5} + 1\right)}$$

$$= -\frac{1}{2} - \frac{i}{2\cot\frac{2k\pi}{5} -}$$

$$= -\frac{1}{2} - \frac{i\sin\frac{2k\pi}{5}\cos\frac{2k\pi}{5} + i\sin\frac{2\pi}{5}}{2\left[\cos^2\frac{2k\pi}{5} - 1\right]}$$

$$\cos 2\theta - 1 = -2\sin^2\theta$$

$$= -\frac{1}{2} - i\,\frac{2\sin\frac{k\pi}{5}\cdot\cos\frac{k\pi}{5}}{2\left[\cos\frac{2k\pi}{5} - 1\right]}.$$

$$= -\frac{1}{2} - i\,\frac{\cancel{2}\sin\cancel{\frac{k\pi}{5}}\cdot\cos\frac{k\pi}{5}}{\cancel{2}\times(-2)\sin^{\cancel{2}}\frac{k\pi}{5}}$$

$$= -\frac{1}{2} + \frac{1}{2}i\cot\frac{k\pi}{5}$$

✓.

Q14(c) (iii) $iz^5 = (iz+1)^5$ and $z^5 = (z+1)^5$

The roots of $z^5 = (z+1)^5$ are $z = -\frac{1}{2} - \frac{1}{2} i \cot \frac{k\pi}{5}$, k an integer.

The roots of $iz^5 = (iz+1)^5$ are $iz = -\frac{1}{2} - \frac{1}{2} i \cot \frac{k\pi}{5}$

$\Rightarrow z = \dfrac{\left(-\frac{1}{2} - \frac{1}{2} i \cot \frac{k\pi}{5}\right)}{i}$

$\div i$ ie. $\frac{1}{i} \times \frac{i}{i} = -i$

So, dividing by i is the same as multiplying by $-i$.

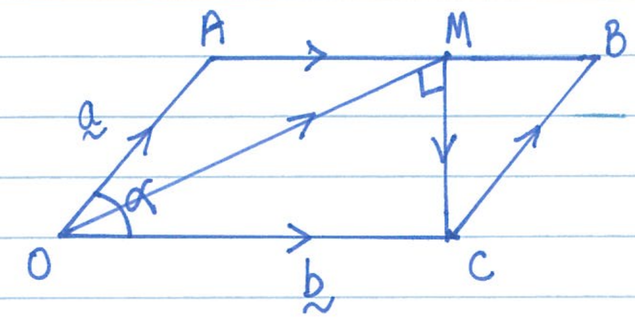
$\therefore$ the roots of $iz^5 = (iz+1)^5$ are $-i\left(-\frac{1}{2} - \frac{1}{2} i \cot \frac{k\pi}{5}\right)$

Now multiplying by $-i$ is equivalent to a clockwise rotation through $\frac{\pi}{2}$ about the origin

(Remember: $\times i$ is an anticlockwise rotation through $\frac{\pi}{2}$ about the origin).

QUESTION 15

(a)



$$|\vec{OC}| = 2|\vec{OA}|$$

$$\vec{AM} = k\vec{AB} \qquad 0 \leq k \leq 1 \qquad \vec{OM} . \vec{MC} = 0$$

(i) Show that $\quad |\underset{\sim}{a}|^2 (1-2k)(2\cos\alpha - (1-2k)) = 0$

Solution: $\quad \vec{OM} = \vec{OA} + \vec{AM} = \underset{\sim}{a} + k\underset{\sim}{b} \qquad 1$

$\vec{MC} = \vec{MB} + \vec{BC} = (1-k)\underset{\sim}{b} - \underset{\sim}{a}$

Since $\vec{OM}.\vec{MC} = 0$, $\quad (\underset{\sim}{a} + k\underset{\sim}{b})[(1-k)\underset{\sim}{b} - \underset{\sim}{a}] = 0$

ie $\underset{\sim}{a}.\underset{\sim}{b}(1-k) - \underset{\sim}{a}.\underset{\sim}{a} + k(1-k)\underset{\sim}{b}.\underset{\sim}{b} - k\underset{\sim}{a}.\underset{\sim}{b} = 0 \qquad 1$

ie $\quad |\underset{\sim}{a}||\underset{\sim}{b}|\cos\alpha(1-k) - |\underset{\sim}{a}|^2 + k(1-k)|\underset{\sim}{b}|^2 - k|\underset{\sim}{a}||\underset{\sim}{b}|\cos\alpha = 0.$

$\Rightarrow |\underset{\sim}{a}||\underset{\sim}{b}|\cos\alpha - 2k|\underset{\sim}{a}||\underset{\sim}{b}|\cos\alpha - |\underset{\sim}{a}|^2 + k(1-k)|\underset{\sim}{b}|^2 = 0. \qquad -(1)$

Also, $|\vec{OC}| = 2|\vec{OA}| \quad \therefore |\underset{\sim}{b}| = 2|\underset{\sim}{a}| \qquad -(2)$

1

Subst (2) into (1) $\quad 2|\underset{\sim}{a}|^2\cos\alpha - 4k|\underset{\sim}{a}|^2\cos\alpha - |\underset{\sim}{a}|^2 + k(1-k)4|\underset{\sim}{a}|^2 = 0$

$\Rightarrow 2|\underset{\sim}{a}|^2\cos\alpha - 4k|\underset{\sim}{a}|^2\cos\alpha - |\underset{\sim}{a}|^2 + 4k|\underset{\sim}{a}|^2 - 4k^2|\underset{\sim}{a}|^2 = 0$

Q.15(a)(i) (cont.)

$$|\underset{\sim}{a}|^2\left[2\cos\alpha - 4k\cos\alpha - 1 + 4k - 4k^2\right] = 0$$ 4

$$|\underset{\sim}{a}|^2\left[2\cos\alpha(1-2k) - (1-2k)^2\right] = 0$$

$$|\underset{\sim}{a}|^2\left[(1-2k)(2\cos\alpha - (1-2k)\right] = 0 \quad \text{as required.}$$ 1

(ii) Find α such that 2 positions for M.

From (i) Equation is true if either

$$(1-2k) = 0 \quad \text{or} \quad 2\cos\alpha = 1-2k, \quad \text{since } |\underset{\sim}{a}| \neq 0.$$

$$\therefore \; k = \tfrac{1}{2} \quad \text{or} \quad 2k = 1 - 2\cos\alpha, \; k = \tfrac{1}{2}(1-2\cos\alpha)$$
$$= \tfrac{1}{2} - \cos\alpha$$ 1

Since $0 \leqslant k \leqslant 1$,

$$0 \leqslant \tfrac{1}{2} - \cos\alpha \leqslant 1$$

i.e. $-\frac{1}{2} < -\cos\alpha < \frac{1}{2}$

$\times -1$ $\quad \frac{1}{2} \geqslant \cos\alpha \geqslant -\frac{1}{2}$ 2

i.e. $-\frac{1}{2} \leqslant \cos\alpha \leqslant \frac{1}{2}$.

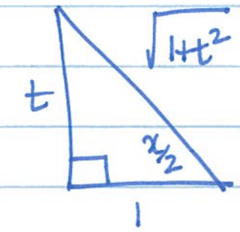
$\therefore$ Since α must be positive, and $\alpha \neq \frac{\pi}{2}$ b/c this makes the // ogram a square with only one solution for M, α is either acute or obtuse, i.e. $\frac{\pi}{3} \leqslant \alpha \leqslant \frac{2\pi}{3}$. 1

Q15 (b) $\quad I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{2}{3+5\cos x}\,dx$

(i) $t = \tan\frac{x}{2}$ $\quad$ show that $I = \displaystyle\int_0^1 \frac{2}{4-t^2}\,dt$.

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(1+t^2)$



$\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$= \frac{1-t^2}{1+t^2}$

$x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$

$x = 0$, $t = \tan 0 = 0$

$\frac{x}{2} = \tan^{-1} t$, $x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$I = \displaystyle\int_0^1 \frac{2}{3+\frac{5(1-t^2)}{1+t^2}}\,\frac{dx}{dt}\,dt$ $\quad$ 1

$= \displaystyle\int_0^1 \frac{2\cancel{(1+t^2)}}{3(1+t^2)+5(1-t^2)} \cdot \frac{2}{\cancel{1+t^2}}\,dt$

$= \displaystyle\int_0^1 \frac{4}{8-2t^2}\,dt$ $\quad$ 2

$= \displaystyle\int_0^1 \frac{2}{4-t^2}\,dt$ $\quad$ 1

(ii) $\frac{2}{4-t^2} = \frac{a}{2-t} + \frac{b}{2+t}$

$\Rightarrow 2 = a(2+t) + b(2-t)$

$t=2$, $2 = 4a$ $\quad a = \frac{1}{2}$ $\quad$ 1

$t=-2$, $2 = 4b$ $\quad b = \frac{1}{2}$

$I = \frac{1}{2}\displaystyle\int_0^1 \left(\frac{1}{2-t} + \frac{1}{2+t}\right) dt$ $\quad$ 2

$= \frac{1}{2}\Big[-\ln(2-t) + \ln(2+t)\Big]_0^1$ $\quad$ 1

$= \frac{1}{2}\left[\ln\frac{2+t}{2-t}\right]_0^1 = \frac{1}{2}\ln 3 = \ln\sqrt{3}$

		Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c)	(i)	We are required to find T such that $\frac{g}{4} + gT\sin\theta - \frac{g}{2}T^2 = 0$. $T = \dfrac{-g\sin\theta \pm \sqrt{g^2\sin^2\theta - 4\left(-\frac{g}{2}\right)\left(\frac{g}{4}\right)}}{-g}$ $= \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$ $\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta}$ $(\sin\theta > 0)$ Substituting for $2\sin^2\theta$ and $\sin\theta$ into $T = \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ gives: $T = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta} \pm \frac{1}{\sqrt{2}}\sqrt{2 - \cos 2\theta}$ $\sqrt{2 - \cos 2\theta} > \sqrt{1 - \cos 2\theta}$. We require $T > 0$ and so $T = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta} + \frac{1}{\sqrt{2}}\sqrt{2 - \cos 2\theta}$. Hence $T = \frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta})$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution .... 3 • Substitutes for $2\sin^2\theta$ and $\sin\theta$ into $T = \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ OR equivalent merit .... 2 • Attempts to solve $\frac{g}{4} + gT\sin\theta - \frac{g}{2}T^2 = 0$ for T 1
	(ii)	$R = (g\cos\theta)T$ where $T = \frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta})$. $\cos 2\theta = 2\cos^2\theta - 1$ $\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}\sqrt{1 + \cos 2\theta}$ $(\cos\theta > 0)$ Substituting for T and $\cos\theta$ into $R = (g\cos\theta)T$ gives: $R = \frac{g}{\sqrt{2}}\sqrt{1 + \cos 2\theta}\left(\frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta})\right)$ So $R = \frac{g}{2}(\sqrt{1 - \cos^2 2\theta} + \sqrt{2 + \cos 2\theta - \cos^2 2\theta})$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) When $\theta = 45°$, $R = \frac{g}{2}(1 + \sqrt{2})$. When $\cos 2\theta = \frac{1}{5}$, $R = \frac{g}{2}\left(\sqrt{\frac{24}{25}} + \sqrt{\frac{54}{25}}\right)$. Let d represent the extra distance attained. $\sqrt{24} = 2\sqrt{6}$ and $\sqrt{54} = 3\sqrt{6}$. $d = \frac{g}{2}\left(\frac{2\sqrt{6}}{5} + \frac{3\sqrt{6}}{5}\right) - \frac{g}{2}(1 + \sqrt{2})$ $= \frac{g}{2}(\sqrt{6} - \sqrt{2} - 1)$ So the extra distance attained is $\frac{g}{2}(\sqrt{6} - \sqrt{2} - 1)$ metres.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution. 1

Q.16 (a)

$$I_n = \int \frac{dx}{(x^m+1)^n}$$

(i)

Let $\frac{dv}{dx} = 1 \qquad v = x$

and $u = \frac{1}{(x^m+1)^n} \qquad \frac{du}{dx} = -n(x^m+1)^{-n-1} \cdot mx^{m-1}$

$$= \frac{-mnx^{m-1}}{(x^m+1)^{n+1}} \qquad 1$$

$$\therefore I_n = \frac{x}{(x^m+1)^n} + mn\int \frac{x^m}{(x^m+1)^{n+1}}\,dx$$

$$\frac{x^m+1}{(x^m+1)^{n+1}} = \frac{1}{(x^m+1)^n} = I_n$$

$$= \frac{x}{(x^m+1)^n} + mn\int \frac{x^m+1}{(x^m+1)^{n+1}}\,dx - mn\int \frac{1}{(x^m+1)^{n+1}}\,dx \qquad 1$$

$$= \frac{x}{(x^m+1)^n} + mn\,I_n - mn\,I_{n+1} \qquad 1$$

$$\Rightarrow mn\,I_{n+1} = \frac{x}{(x^m+1)^n} + I_n(mn-1)$$

$$\div mn \qquad I_{n+1} = \frac{x}{mn(x^m+1)^n} + \frac{mn-1}{mn} I_n \quad \text{as required.} \qquad 1$$

Q.16(a)

(ii)

For $\int \frac{dx}{(x^2+1)^3}$, $m=2$, $n=2$

$$I_3 = \frac{x}{4(x^2+1)^2} + \frac{3}{4} I_2$$ 1

For $m=2$, $n=1$

$$I_2 = \frac{x}{2(x^2+1)} + \frac{1}{2} I_1$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{(x^2+1)}$$ 1

$$\therefore I = \int \frac{dx}{(x^2+1)^3} = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\left[\frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x\right] + c$$

$$= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\tan^{-1}x + c$$ 1

Q.16(b)

(i) To Prove: $\frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2}$

$a_1, a_2 \geq 0$. ①

PROOF: $(\sqrt{a_1}-\sqrt{a_2})^2 \geq 0$

since squares cannot be negative.

$\therefore (\sqrt{a_1}-\sqrt{a_2})^2 = a_1 - 2\sqrt{a_1a_2} + a_2 \geq 0$

i.e. $a_1+a_2 \geq 2\sqrt{a_1a_2}$

i.e. $\frac{a_1+a_2}{2} \geq \sqrt{a_1a_2}$

as required.

(ii) GIVEN $a_1, a_2, a_3 \dots a_n$ +ve reals

If $a_1a_2a_3\dots a_n = 1$, then $a_1+a_2+\dots+a_n \geq n$

(ii) Prove that

$$\frac{a_1+a_2+\dots+a_n}{n} \geq (a_1a_2\dots a_n)^{1/n}$$

Let $x = (a_1a_2\dots a_n)^{1/n}$ —①

$\therefore x^n = a_1a_2\dots a_n$

Now, we are given that if $a_1a_2\dots a_n = 1$

then $a_1+a_2+\dots+a_n \geq n$

$\therefore x^n = a_1a_2\dots a_n = 1$

i.e. $\frac{a_1a_2\dots a_n}{x^n} = \frac{1}{1} = 1$

i.e. $\frac{a_1}{x} \cdot \frac{a_2}{x} \dots \frac{a_n}{x} = 1$

and $\frac{a_1}{x}, \frac{a_2}{x} \dots \frac{a_n}{x} > 0$ (i.e. all >0)

Also, $a_1, a_2, \dots a_n$ all +ve

and $a_1+a_2+\dots+a_n \geq n$

$\therefore \frac{a_1}{x}+\frac{a_2}{x}+\dots+\frac{a_n}{x} \geq n$

i.e. $\frac{a_1+a_2+\dots+a_n}{n} \geq x$

i.e. $\frac{a_1+a_2+\dots+a_n}{n} \geq (a_1a_2\dots a_n)^{1/n}$

from ①

(iii) Hence prove that

$2^n - 1 > n\sqrt{2^{n-1}}$ $n \geq 1$

Now, $2^n - 1 = (2-1)[2^{n-1}+2^{n-2}+\dots+2+1$

using $(a^n-b^n) = (a-b)(a^{n-1}b + a^{n-2}b^2 + \dots)$

$a=2,\ b=1$

Q16(b)(cont.)

$2^n - 1 > n\sqrt{2^{n-1}}$

$\therefore 2^n - 1 = 2^{n-1} + 2^{n-2} + \dots + 2 + 1$

Also $2^{n-1}, 2^{n-2}, \dots, 2, 1$ are all +ve reals.

Using $\dfrac{a_1 + a_2 + \dots + a_n}{n} > (a_1 a_2 \dots a_n)^{1/n}$

$$\frac{2^{n-1} + 2^{n-2} + \dots + 2 + 1}{n} > (2^{n-1} . 2^{n-2} . \dots 2.1)^{1/n}$$ 1

ie. $$\frac{2^n - 1}{n} > (2^{n-1} . 2^{n-2} . \dots 2.1)^{1/n}$$

$$= 2^{(\underbrace{(n-1)+(n-2)+\dots+2+1+0}_{\text{we need to sum this A.P.}})\frac{1}{n}}$$

* Big NOTE!!

The last index in the sum of indices is zero!!

since the last term is

$2^{0} = 1$

For $\therefore 2^{((n-1)+(n-2)+\dots+2+1+0)^{1/n}}$

For $\Sigma\ (n-1)+(n-2)+\dots+2+1+0$ 1

Using Sum of A.P. $= \frac{n}{2}(a+l)$,

there are n terms,

$a = n-1$, $l = 0$

$\therefore \Sigma = \frac{n}{2}(n-1)$

$$\therefore \frac{2^n - 1}{n} = 2^{\left[\frac{n}{2}(n-1)\right]\frac{1}{n}} = 2^{\frac{n-1}{2}}$$ 1

$$= \sqrt{2^{n-1}}$$

ie. $\dfrac{2^n - 1}{n} > \sqrt{2^{n-1}}$

ie. $2^n - 1 > n\sqrt{2^{n-1}}$ for $n \geq 1$

as required.