Student number : ____________________

2020 **YEAR 12** YEARLY EXAMINATION

CARLINGFORD HIGH SCHOOL

Year 12 Advanced Mathematics

General Instructions

- Working time - 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt all questions
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. What is the value of $f'(x)$ if $f(x) = 3x^4(4-x)^3$?

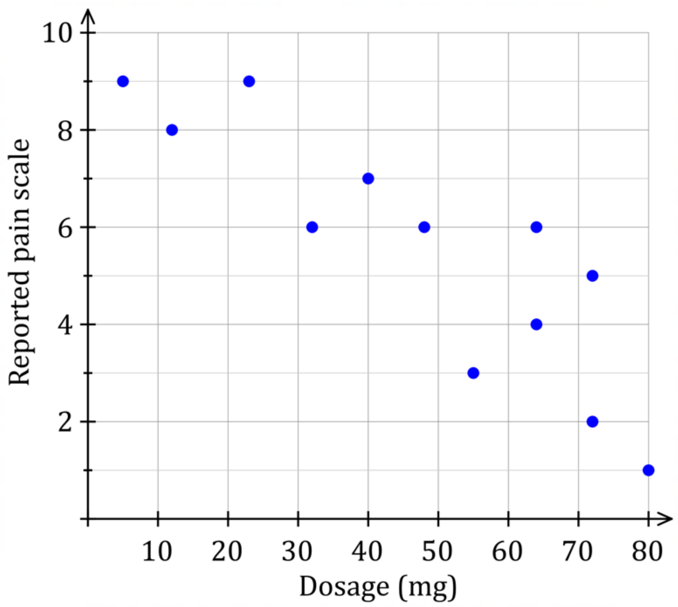
(A) $3x^3(4-x)^3(7x-16)$

(B) $3x^3(4-x)^3(16-7x)$

(C) $3x^3(4-x)^2(7x-16)$

(D) $3x^3(4-x)^2(16-7x)$

2. A scatterplot of pain (as reported by patients) compared to the dosage (in mg) of a drug is shown below.



How could you describe the correlation between the pain and the dosage?

(A) A moderate negative correlation

(B) A moderate positive correlation

(C) A weak positive correlation.

(D) No correlation.

3. What values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

(A) $x < -\dfrac{1}{6}$

(B) $x > -\dfrac{1}{6}$

(C) $x < -6$

(D) $x > 6$

4. What is the period and amplitude for the curve $y = \sin\pi x$?

(A) Amplitude = 1; Period = 2

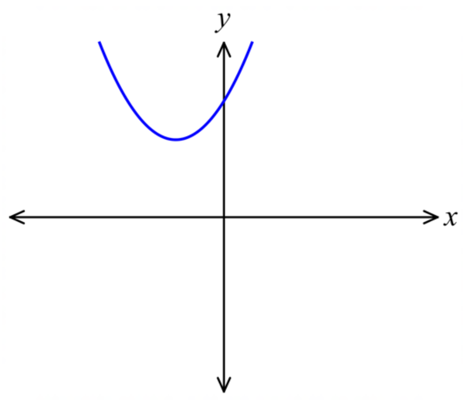
(B) Amplitude = π; Period = 2

(C) Amplitude = 1; Period = 2π

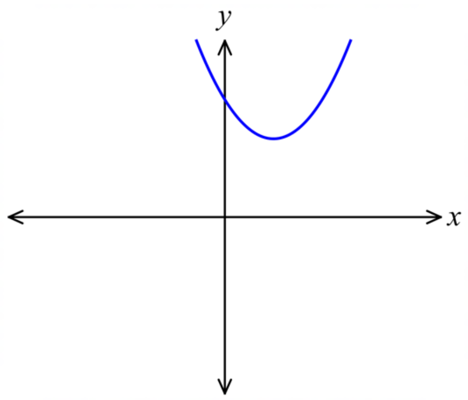
(D) Amplitude = π; Period = 2π

5. Which diagram best shows the graph of the parabola $y = 2 - (x + 1)^2$

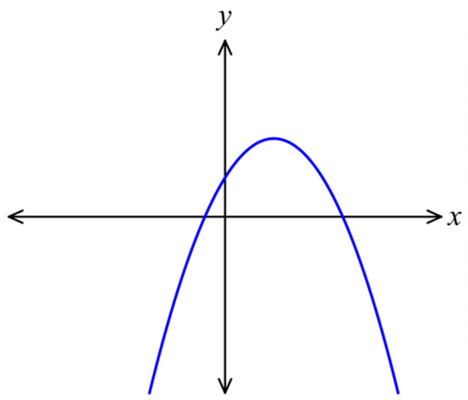
(A)



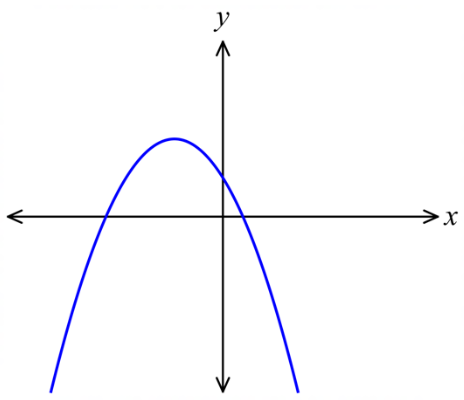
(B)



(C)



(D)



6. The weekly pay for workers at the Prosper Factory is normally distributed, with a mean of \$750 and a standard deviation of \$35.

What percentage of workers earn below \$680 a week?

(A) 0.15%

(B) 2.5%

(C) 5%

(D) 47.5%

7. The table below shows the future value of a $1 annuity.

Future value of $1				
End of year	4%	6%	8%	10%
1	1.00	1.00	1.00	1.00
2	2.04	2.06	2.08	2.10
3	3.12	3.18	3.25	3.31
4	4.25	4.37	4.51	4.64

What amount would need to be invested every month into an account earning 16% p.a. interest compounded quarterly, to be worth $28 475 after a year?

(A) $6137
(B) $6314
(C) $6700
(D) $13 958

8. The continuous random variable X has a normal distribution with a mean 11 and standard deviation 2. If the random variable Z has the standard normal distribution, then the probability that X is greater than 16 is equal to:

(A) $P(Z > 2)$
(B) $P(Z < -2.5)$
(C) $P(Z < 2.5)$
(D) $P(Z > 5)$

9. What are the solutions to the equation $2\sin x + \sqrt{3} = 0$, where $\{x: 0 \le x \le 2\pi\}$?

(A) $\frac{\pi}{3}, \frac{2\pi}{3}$
(B) $\frac{2\pi}{3}, \frac{5\pi}{3}$
(C) $\frac{4\pi}{3}, \frac{5\pi}{3}$
(D) $\frac{7\pi}{3}, \frac{11\pi}{3}$

10. Consider the region bounded by the x-axis, the y-axis, the line with equation $y = 3$ and the curve with equation $y = \ln(x - 1)$. The exact value of the area of this region is:

(A) $3e^3 - e^{-3} + 2$
(B) $e^{-3} - 1$
(C) $3e^2$
(D) $e^3 + 2$

Section II

90 marks
Attempt all questions
Allow about 2 hours and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the back of the examination paper.

Question 11 (2 marks) **Marks**

Find the anti-derivative of $\dfrac{1}{1-2x}$ with respect to x. **2**

Question 12 (3 marks)

Let $f(x) = \dfrac{(x+3)(2x+1)}{\sqrt{x}}, x > 0$

(a) Show that $f(x)$ can be written in the form **2**

$Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}} + Cx^{-\frac{1}{2}}$

Find the values of A, B and C.

(b) Find $f'(x)$ **1**

Question 13 (3 marks) **Marks**

The random variable X has this probability distribution.

X	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

(a) Find $P(1 < X \leq 3)$ **1**

(b) Find the variance of X. **2**

Question 14 (2 marks)

Find $\int 6x^2 + 2 + x^{-\frac{1}{2}}\,dx$, giving each term in its simplest form. **2**

Question 15 (2 marks)

Find the common ratio of a geometric series with a first term of 3 and a limiting sum of 1.8. **2**

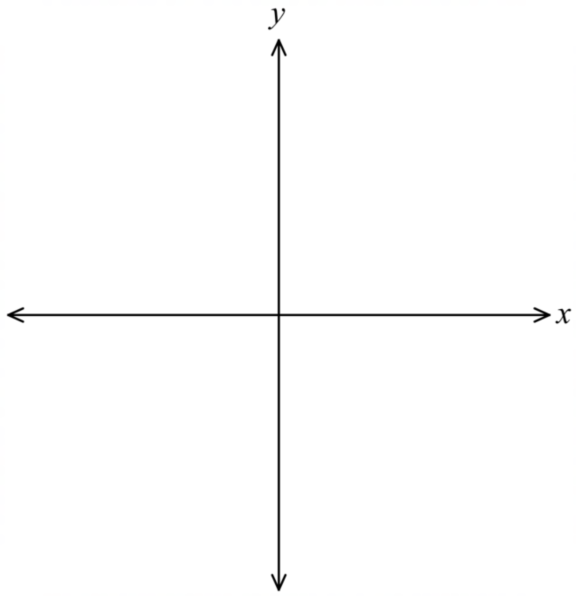
Question 16 (8 marks) **Marks**

Let $f(x) = (x-2)(x^2+1)$

(a) Find where the graph of $y = f(x)$ cuts the x-axis and y axis. **2**

(b) Find the coordinates of the stationary points on the curve with the equation $y = f(x)$ and determine their nature. **3**

(c) Sketch the graphs of $y = f(x)$ and $y = -f(x)$ on the same diagram. **3**



Question 17 (2 marks)

Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$. **2**

Question 18 (5 marks) **Marks**

Given that $f(x) = (x^2 - 6x)(x - 3) + 2x$.

(a) Express $f(x)$ in the form $x(ax^2 + bx + c)$, where a, b and c are constants. **2**

(b) Hence factorise $f(x)$ completely. **1**

(c) Sketch the graph of $y = f(x)$, showing the coordinates of each point at which the graph meets the axes. **2**

Question 19 (3 marks)

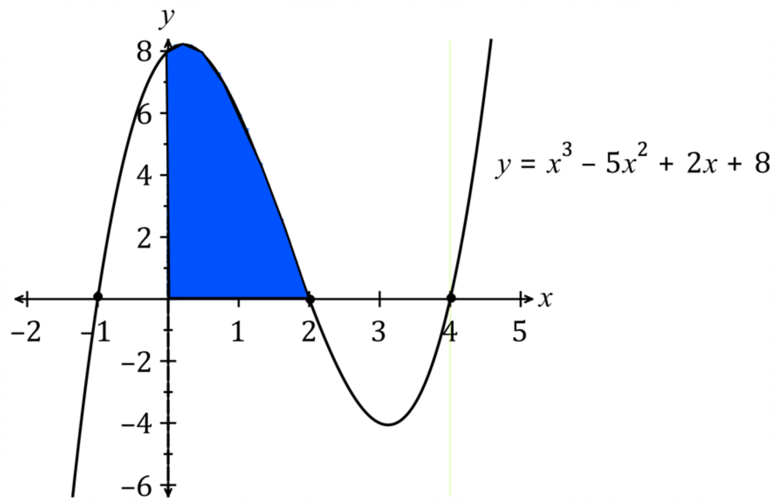
Differentiate with respect to x

(a) $\ln(x^2 + 2)$ **1**

(b) $\dfrac{\sin x}{x^2}$ **2**

Question 20 (2 marks) **Marks**

2



Find the shaded area enclosed by the curve $y = x^3 - 5x^2 + 2x + 8$ and the coordinate axes.

Question 21 (4 marks)

The probability density function for the continuous random variable X is given by:

$$f(x) = \begin{cases} |1-x| & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $P(X \le 1.5)$ **2**

(b) Find $P(1 \le X \le 2)$ **2**

Question 22 (6 marks) **Marks**

Sonny repays a loan over a period of n months. His monthly repayments form an arithmetic sequence. He repays \$119 in the first month, \$117 in the second month, \$115 in the third month, and so on. Sonny makes his final repayment in the nth month.

(a) Find the amount Sonny repays in the 25th month. **2**

(b) Over the n months, he repays a total of \$3200. **2**
Form an equation in n, and show that your equation may be written as
$n^2 - 120n + 3200 = 0$.

(c) State, with a reason, which of the solutions to the equation in part (b) is not a sensible solution to the repayment problem. **2**

Question 23 (2 marks)

If $\tan\theta = \frac{2}{3}$, and θ is acute, find the exact value of $\sin\theta$? **2**

Question 24 (6 marks) **Marks**

A curve with the equation $y = f(x)$, has $\dfrac{dy}{dx} = x^3 + 2x - 7$.

(a) Find $\dfrac{d^2y}{dx^2}$ **1**

(b) Show that $\dfrac{d^2y}{dx^2} \geq 2$ for all values of *x*. **1**

(c) The point $P(2, 4)$ lies on the curve. Find y in terms of x. **2**

(d) Find an equation for the normal to the curve at P, in the form $ax + by + c = 0$, where a, b and c are integers. **2**

Question 25 (4 marks) **Marks**

The students in a school were surveyed on the number of hours of sleep per week. The results were normally distributed. The survey indicated that 95% of students had between 42 and 54 hours of sleep per week.

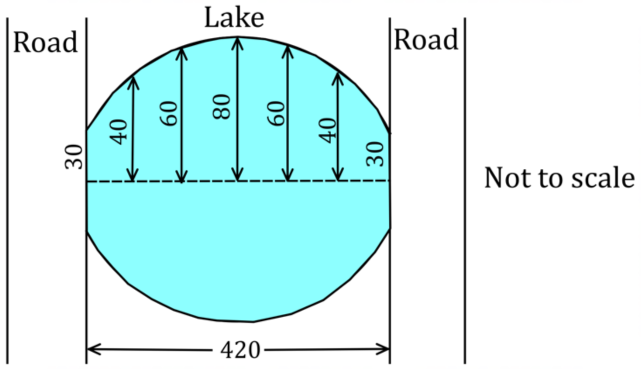
(a) Determine the mean number of hours of sleep per week. **1**

(b) What was the standard deviation? **1**

(c) What percentage of students would have indicated they had between 51 and 57 hours of sleep per week? **2**

Question 26 (3 marks)

A symmetrical lake has two roads, 420 metres apart, forming two of its sides.



Equally spaced measurements of the lake, in metres, are shown on the above diagram. Use the trapezoidal rule to estimate the area of the lake. **3**

Question 27 (5 marks) **Marks**

John started work at 18 and at the beginning of each month he invested \$400 into a superannuation fund. Interest was paid at 6% p.a. compounded monthly on the investment. John retired at 63 after having contributed to the fund for 45 years.

(a) How much did John contribute to the fund over the 45 years? **1**

(b) How much did John's investment amount to after 45 years? **2**

(c) John plans to reinvest some of the money into an account which offers 8% p.a. compound interest compounded annually. He plans to have \$300 000 at the end of the 10 year investment period. How much does John need to reinvest to achieve this amount. Answer correct to the nearest dollar. **2**

Question 28 (3 marks)

The average life cycle of an insect is one month. A viable nest of this insect has between 100 000 to 500 000 insects. The population P of a nest of this insect grows exponentially so that: **3**

$$\frac{dP}{dt} = 1200e^{0\cdot 3t}$$

A nest of these insects had a population of 5000 after one month.
Determine how long it will take the nest to reach the viable stage (i.e. when the population has reached 100 000). Answer correct to the nearest month.

Question 29 (6 marks) **Marks**

A particle moves in a straight line. At time t seconds, its distance x metres from a fixed point 0 on the line is given by $x = 1 - \cos 2t$.

(a) Sketch the graph of x as a function of t for $0 \leq t \leq \pi$. **2**

(b) Using your graph, or otherwise, to find the times when the particle is at rest and the position of the particle at these times. **2**

(c) Find the velocity of the particle when $t = \frac{\pi}{4}$. **1**

(d) When is the particle's velocity greater than 1 m/s? **1**

Question 30 (3 marks)

Show that the curve $y = 2x^2 - \ln\left(\frac{x}{2}\right) - 4$ has a minimum value at $\left(\frac{1}{2}, \ln 4 - 3\frac{1}{2}\right)$ **3**

Question 31 (2 marks) **Marks**

The probability density function for the continuous random variable X is: **2**

$$f(x) = \begin{cases} x^3 - x + 4 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the E(X).

Question 32 (3 marks)

The table below shows the present value interest factors for some monthly interest rates and loan periods in months.

Present value of $1				
Period	0.0060	0.0065	0.0070	0.0075
46	40.09350	39.64965	39.21263	38.78231
47	40.84841	40.38714	39.93310	39.48617
48	41.59882	41.11986	40.64856	40.18478
49	42.34475	41.84785	41.35905	40.87820

(a) Find the present value, if $3200 is contributed per month for 46 months at 0.75% per month. Answer to the nearest cent. **1**

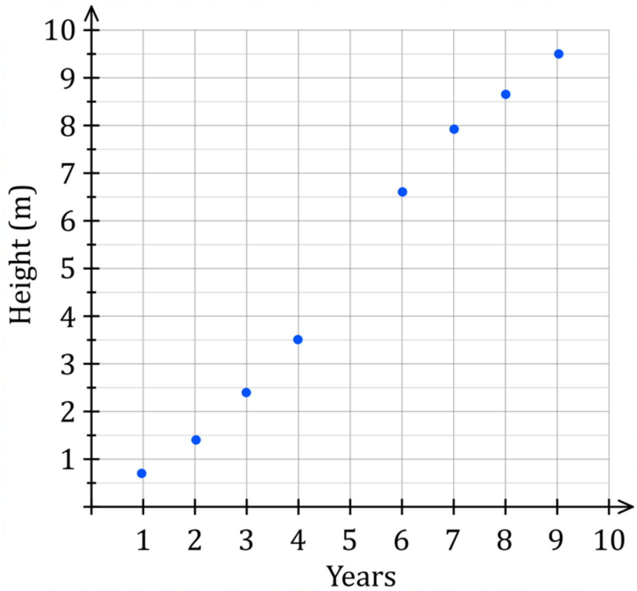
(b) Annabelle borrows $27 000 for a car. She arranges to repay the loan with monthly repayments over 4 years. She is charged 7.8% per annum interest. Find Annabelle's monthly repayment. Answer to the nearest cent. **2**

Question 33 (5 marks) **Marks**

Hayden is an agricultural scientist studying the growth of a particular tree over several years. The data he recorded is shown in the table below.

Years since planting, t	1	2	3	4	5	6	7	8	9
Height of tree, H metres	0.7	1.4	2.4	3.5		6.6	7.9	8.7	9.5

A scatterplot of the data is shown below.



(a) What is Pearson's correlation coefficient? Answer correct to 4 decimal places. **1**

(b) Find the equation of the least-squares line of best fit in terms of years (t) and height (h). Answer correct to 2 decimal places. **1**

(c) Hayden did not record the tree's height after five years. Predict the height after five years, correct to one decimal place. **1**

(d) Use algebra to estimate how many years it will take for the tree to reach a height of 20 metres. Answer correct to 1 decimal place. **1**

(e) Comment on the reliability of your answers in (c) and (d). **1**

Question 34 (6 marks) **Marks**

The height $h(t)$ metres of the tide above the mean sea level on 1st April is given by the following rule:

$$h(t) = 4\sin\left(\frac{\pi}{8}t\right)$$

where t is the number of hours after midnight

(a) Draw a graph of $y = h(t)$ for $0 \leq t \leq 24$. **2**

(b) When was high tide? **1**

(c) What was the height of the high tide? **1**

(d) What was the height of the tide at 10 a.m.? **2**
Answer correct to one decimal place

End of paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\dfrac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

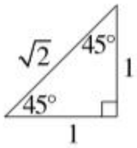
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

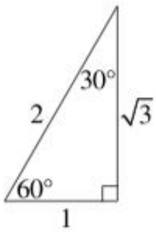


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

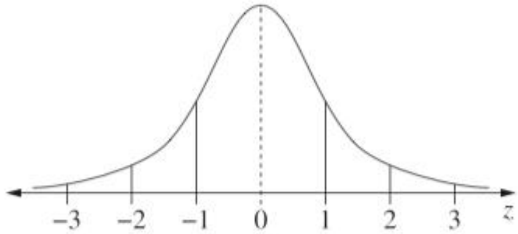
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$