# CARLINGFORD HIGH SCHOOL

## DEPARTMENT OF MATHEMATICS

### Year 12 Mathematics Ext 1

### Term 2 Examination 2013

**Time allowed :** ***55 mins***

**Name :** ________________________________ **Class : 12M** _____

**Teacher : Ms Strilakos / Mr Gong / Mr Cheng**

## Instructions

- Start each question on a **new page** and write on one side of the paper only
- Board approved calculators may be used
- Show all necessary working by using blue/ black pen except graphs/diagrams
- Marks may be deducted for untidy setting out

| Outcomes | MC | Q4 | Q5 | Q6 | Q7 | Q8 | Q9 | Q10 | Q11 | Q12 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| HE4 | /2 | /11 | /6 | /3 | | | | | | /7 | /29 |
| H5 | /1 | | | | /2 | /3 | /1 | /4 | | | /11 |
| HE6 | | | | | | | | | /11 | | /11 |
| Total | /3 | /11 | /6 | /3 | /2 | /3 | /1 | /4 | /11 | /7 | /51 |

**Mutiple Choice ( 1 mark each) 3 marks**

**Question 1**

$\tan^{-1}\frac{3}{5}$ is equal to:

(A) $\sin^{-1}\frac{3}{4}$ (B) $\cos^{-1}\frac{3}{4}$ (C) $\sin^{-1}\frac{3}{\sqrt{34}}$ (D) $\cos^{-1}\frac{3}{\sqrt{34}}$

**Question 2**

Which of the following represents the inverse function of $f(x)=\frac{2}{5x+10}+1$?

(A) $f^{-1}(x)=\frac{2}{5x-5}-2$ (B) $f^{-1}(x)=5-\frac{2}{5x-5}$

(C) $f^{-1}(x)=2-\frac{1}{5x-5}$ (D) $f^{-1}(x)=\frac{2}{5x-5}+2$

**Question 3**

By considering the sketches of $y=\sin 2\theta$ and $y=\cos\theta$ or otherwise, determine how many solutions the equation $\sin 2\theta=\cos\theta$ has in the domain $0 \leq \theta \leq 2\pi$.

(A) 2 (B) 3 (C) 4 (D) 5

**Question 4**

a) Consider the function $f(x) = 3\sin^{-1}2x$

i) State the **domain and range** of f(x) **2**

ii) **Sketch** the **graph** of f(x) **2**

b) Find the **exact value** of

i) $\cos^{-1}(\cos\frac{-5\pi}{3})$ **1**

ii) $\cos(\cos^{-1}(\frac{4}{7}) + \tan^{-1}(\frac{-4}{3}))$ **2**

c) Find the **derivative** of

i) $f(x) = \sin^{-1}(3x+2)$ **2**

ii) $y = \log_e(\tan^{-1}(3x^2))$ **2**

**Question 5**

Consider the function $f(x) = x^2 - 6x$.

i) **Sketch** the graph of f(x) **1**

ii) State the **domain** and **range** of f(x). **2**

iii) The **domain** of this function is **restricted** so that an **inverse function** $f^{-1}(x)$ exists whose **range** is entirely **positive.** Write down the **largest domain** for this inverse function. **1**

iv) **Sketch** this inverse function. **2**

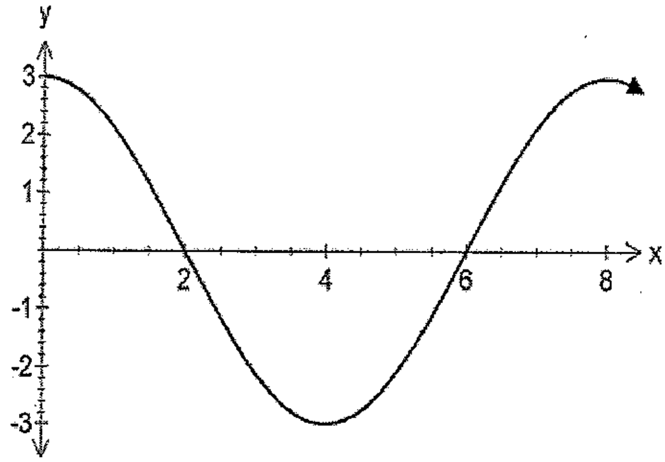
**Question 6**

The **area** between the curve $y = \frac{1}{\sqrt{1+x^2}}$ and the x-axis, from x = 0 to x = 3 is **rotated** about the **x-axis.** Find the **exact volume** of revolution of the solid formed. **3**

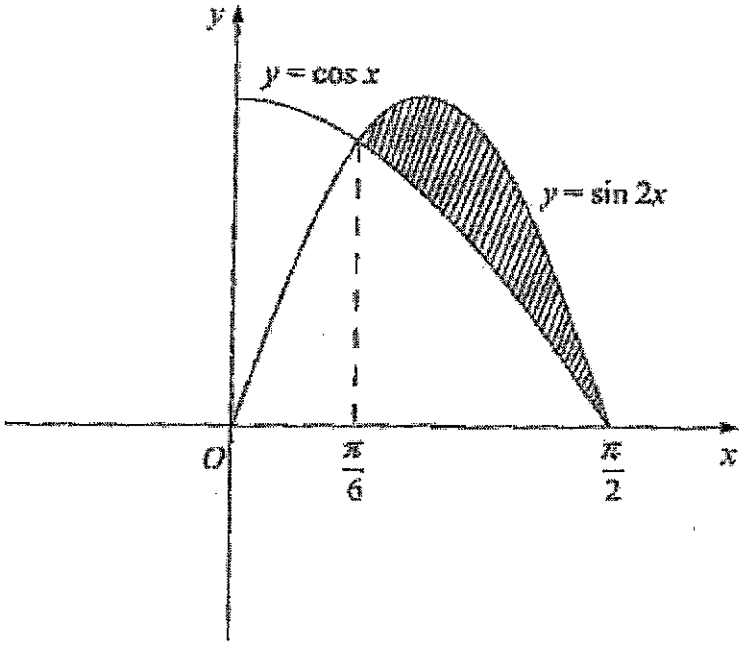
**Question 7**

The graph of $y = A\cos(nx)$ is drawn on the right.

Find the values of $A$ and $n$. 2




**Question 8** 3




The diagram shows the graphs of the functions $y = \cos x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{2}$. The two graphs intersect at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Calculate the area of the shaded region.

**Question 9** **1**

Evaluate $\lim_{x \to 0} \frac{\sin 3x}{5x}$.

**Question 10**

i) Write down the expansion of **cos2x** in terms of $\sin^2 x$. **1**

ii) Hence find $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \left( \sin^2(2x) - \frac{1}{2} \right) dx$ **3**

**Question 11**

(a) Find $\int cotx\, dx$ **2**

(b) Write down the general solutions of $cos\theta = \sqrt{3} sin\theta$ **3**

(c) By using the substitution $x = 2sin\theta$ find $\int \frac{x^2 dx}{\sqrt{4-x^2}}$ **3**

(d) By using the substitution $u = 1 + x$ find $15\int_{-1}^{0} x\sqrt{1+x}dx$ **3**

## Question 12

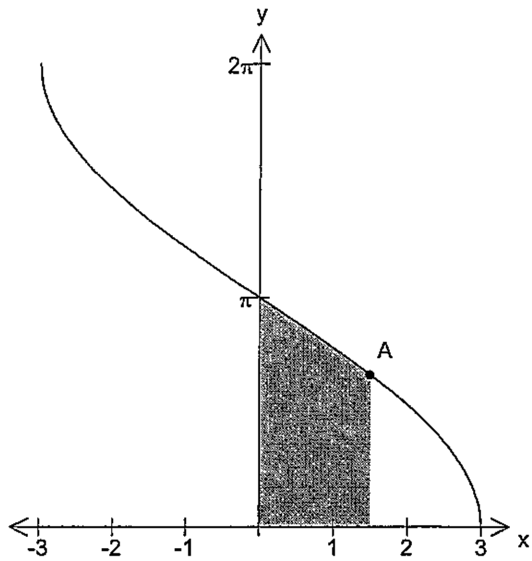
The curve on the right has equation

$f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$.

The point A has coordinates $\left(\frac{3}{2}, \frac{2\pi}{3}\right)$.

Hilary wanted to find the shaded area. However she realised that she could not integrate $f(x)$.

Her friend Barack suggested she use the inverse function.




(i) Find the inverse function $f^{-1}(x)$ **2**

(ii) Sketch the inverse function $f^{-1}(x)$ and shade the corresponding area. You must be sure your sketch clearly shows the domain and range of $f^{-1}(x)$ **2**

(iii) Hence evaluate the shaded area. **3**

# END OF EXAM

# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

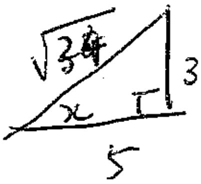
$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Q1

Let $x = \tan^{-1}\frac{3}{5}$

$\tan x = \frac{3}{5}$




$\sin x = \frac{3}{\sqrt{34}}$ $\quad \cos x = \frac{5}{\sqrt{34}}$

$x = \sin^{-1}\left(\frac{3}{\sqrt{34}}\right) \Rightarrow$ (C)

Q2 $y = \frac{2}{5x+10} + 1$

Interchange $x$ and $y$

$x = \frac{2}{5y+10} + 1$

$x - 1 = \frac{2}{5y+10}$

$(x-1)(5y+10) = 2$

$5y(x-1) + 10(x-1) = 2$

$5y = \frac{10 - 10x + 2}{5(x-1)}$

$5y = \frac{12 - 10x}{5(x-1)}$

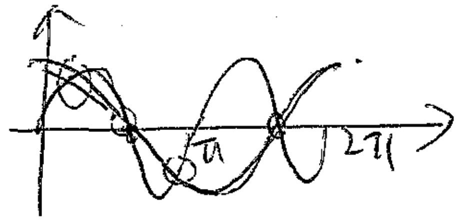
$y = \frac{12 - 10x}{5x - 5}$

$= -\frac{(10x+10)}{5x-5} + \frac{2}{5x-5}$

$= -2 + \frac{2}{5x-5}$

$\Rightarrow$ (A)

Q3




(4) (C)

Q4

~~R: $-\frac{\pi}{2} \leq 3\sin^{-1}2x \leq \frac{\pi}{2}$~~

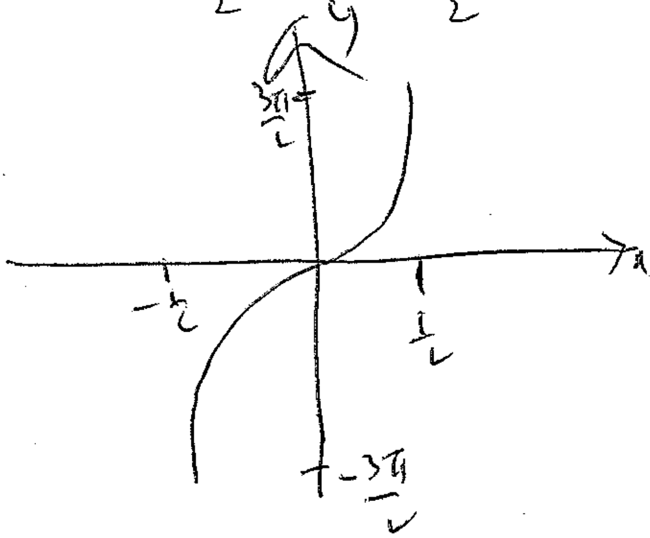
(a) $-1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq 2$ Domain

$-\frac{\pi}{2} \leq \sin^{-1}2x \leq \frac{\pi}{2}$

$-\frac{3\pi}{2} \leq 3\sin^{-1}2x \leq \frac{3\pi}{2}$ Range

$-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$



b (i) Let $x = \cos\left(-\frac{5\pi}{3}\right)$

$= \cos\frac{5\pi}{3}$

$= \cos\left(2\pi - \frac{\pi}{3}\right)$

$= \cos\frac{\pi}{3}$

$= \frac{1}{2}$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$ #

(ii) Let $x = \cos^{-1}\left(\frac{4}{7}\right)$ $y = \tan^{-1}\left(\frac{4}{3}\right)$

$\cos x = \frac{4}{7}$ $\tan y = \frac{4}{3}$.

$\sin x = \frac{\sqrt{33}}{7}$

$\cos y = \frac{3}{5}$

$\sin y = \frac{4}{5}$

(iii) $\cos(x+y)$

$= \cos x \cos y - \sin x \sin y$

$= \frac{4}{7} \times \frac{3}{5} - \frac{\sqrt{33}}{7} \times \frac{4}{5}$

$= \frac{12}{35} - \frac{4\sqrt{33}}{35}$

$= \frac{12 - 4\sqrt{33}}{35}$

Q5

(i) $x^2 - 6x = 0$

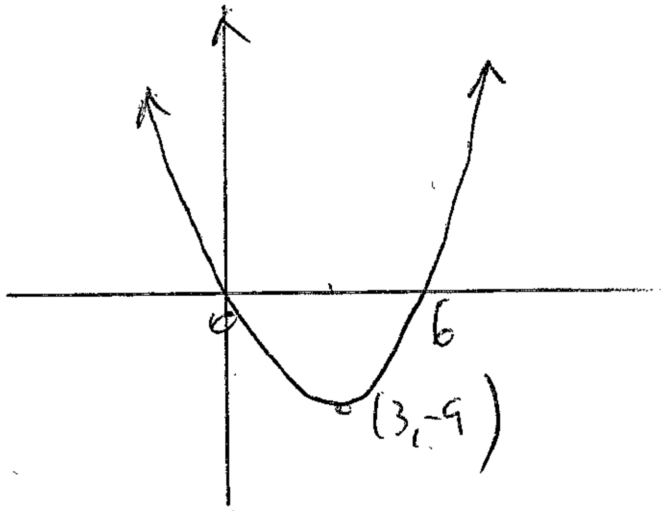
$x(x-6) = 0$

$x = 0$ or $x = 6$

$f'(x) = 2x - 6$

$2x - 6 = 0$

$x = 3$ $(-9)$

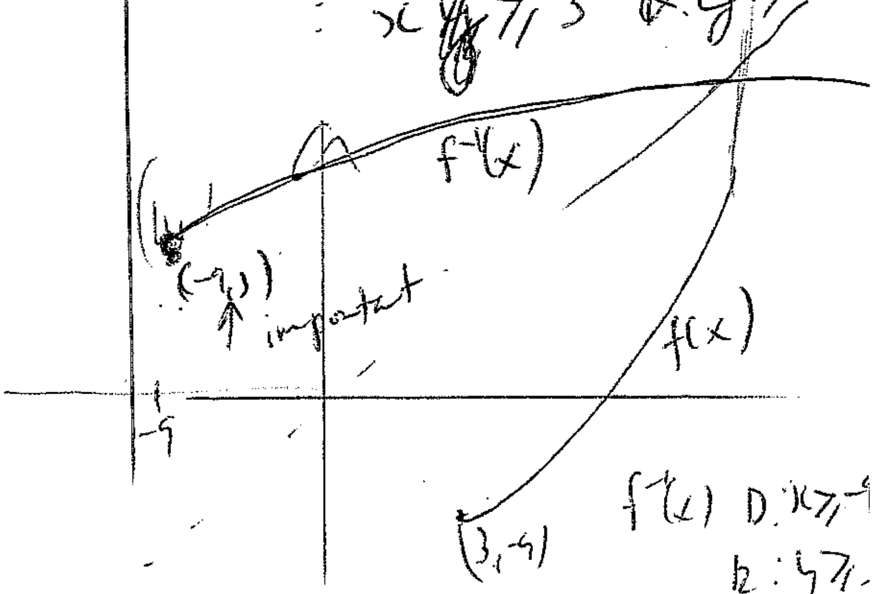



(ii) Domain: All real number of $x$

Range: $y \geq -9$.

(iii) Largest possible domain

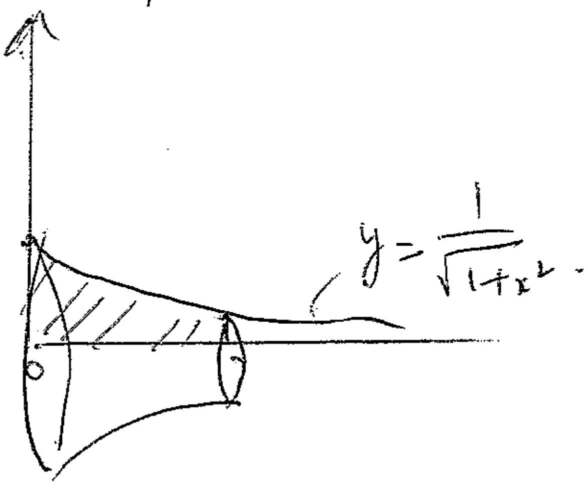
: $x \geq 3$




$f^{-1}(x)$ D: $x \geq -9$

R: $y \geq$

## Q6




$$V = \pi \int y^2 dx$$

$$= \pi \int_0^3 \frac{1}{1+x^2} dx$$

$$= \pi \left[\tan^{-1}\right]_0^3$$

$$= \pi \left[\tan^{-1} 3 - \tan^{-1} 0\right]$$

$$= \underline{\underline{\pi \tan^{-1} 3}} \text{ units}^3$$

## Q7

$\underline{\underline{A = 3}}$ $\quad P = \frac{2\pi}{n}$

$$= \frac{2\pi}{8}$$

$$= \underline{\underline{\frac{\pi}{4}}}$$

## Q8

$$A = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)\, dx$$

$$= \left[-\frac{\cos 2x}{2} - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[-\frac{\cos \pi}{2} - \sin\frac{\pi}{2}\right] - \left[-\frac{\cos\frac{\pi}{3}}{2} - \sin\right.$$

$$= [1 - 1] - \left[-\frac{1}{4} - \frac{1}{2}\right]$$

$$= 0 - \left[-\frac{3}{4}\right]$$

$$= \frac{3}{4} \text{ unit}^2.$$

## Q9.

$$\lim_{x \to 0} \frac{\sin 3x}{5x}.$$

$$= \frac{3}{5} \lim_{x \to 0} \frac{\sin 3x}{3x} = 1$$

$$= \frac{3}{5}.$$

Q10

(i) $\cos 2x = \cos^2 x - \sin^2 x$

$= (1-\sin^2 x) - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$\therefore$

$\cos 2(2x) = 1 - 2\sin^2 2x$

$\cos 4x = 1 - 2\sin^2 2x.$

(ii) $$-\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (1 - 2\sin^2 2x)\,dx$$

$$= -\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \cos 4x \, dx$$

$$= -\left[\frac{\sin 4x}{4}\right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$= -\left[\frac{\sin\frac{3\pi}{2}}{4} - \frac{\sin\frac{\pi}{2}}{4}\right]$$

$$= -\left[-\frac{1}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{2}$$

Q11

(a) $$\int \cot x \, dx$$

$$= \int \frac{\cos x}{\sin x}\,dx$$

Let $u = \sin x$

$du = \cos x\,dx$ ✓

$$= \int \frac{du}{u}$$

$= \ln u + C$

$= \ln(\sin x) + C$ ✓

(b) $\cos\theta = \sqrt{3}\sin\theta$

$1 = \sqrt{3}\,\frac{\sin\theta}{\cos\theta}$

$\frac{1}{\sqrt{3}} = \tan\theta$ ✓

BA $\theta = \frac{\pi}{6}$. ✓

General soln

$\therefore \theta = n\pi + \frac{\pi}{6}$ where $n$ is any integer ✓

Q11

(c) $\int \frac{x^2\,dx}{\sqrt{4-x^2}}$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta}$

$= 2\sqrt{\cos^2\theta}$

$= 2\cos\theta$

$$\int \frac{4\sin^2\theta \cdot \cancel{2\cos\theta}\,d\theta}{\cancel{2\cos\theta}}$$

$$= \int 4\sin^2\theta\,d\theta$$

$2\sin^2\theta = 1-\cos 2\theta$

$4\sin^2\theta = 2(1-\cos 2\theta)$

$$= 2\int (1-\cos 2\theta)\,d\theta$$

$$= 2\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

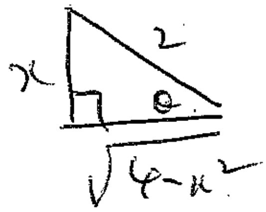
But $2\sin\theta = x$

$\sin\theta = \frac{x}{2}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = \frac{x}{2}$




$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

$$I = 2\theta - \sin 2\theta + c$$

$$= 2\cdot\sin^{-1}\left(\frac{x}{2}\right) - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + c$$

(d) $u = 1+x \Rightarrow x = u-1$

$du = dx$

limit of integration

$x=-1 \quad u=0$

$x=0 \quad u=1$

$$I = 15\int_0^1 (u-1)\sqrt{u}\,du$$

$$15\int_0^1 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= 15\left[\frac{2u^{5/2}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right]_0^1$$

$$= 15\left[\left(\frac{2}{5}-\frac{2}{3}\right) - 0\right]$$

$$= 15\left[\frac{6-10}{15}\right] = \frac{-4}{15}\times 15$$

$$= -4 \; \#$$

Q12

i) let $y = 2\cos^{-1}\left(\frac{x}{3}\right)$

interchange $x$ and $y$

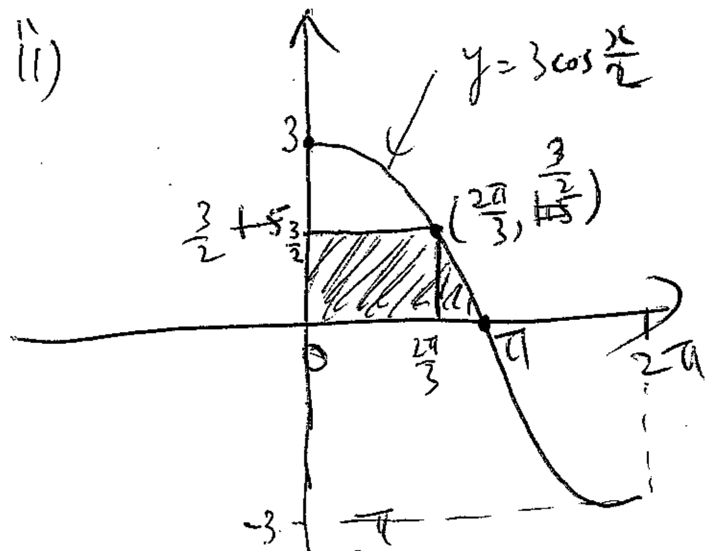
$$x = 2\cos^{-1}\left(\frac{y}{3}\right) \checkmark$$

$$\frac{x}{2} = \cos^{-1}\left(\frac{y}{3}\right)$$

$$\cos\frac{x}{2} = \frac{y}{3}$$

$$\therefore\ y = 3\cos\frac{x}{2} \checkmark$$

ii)




- 1mk for domain, range & all critical point
- 1 mk for correct shape & shaded area.

$$A = \frac{3}{2} \times \frac{2\pi}{3} + \int_{\frac{2\pi}{3}}^{\pi} 3\cos\frac{x}{2}\,dx \checkmark$$

$$= \pi + \left[3 \cdot \sin\left(\frac{x}{2}\right)(2)\right]_{\frac{2\pi}{3}}^{\pi} \checkmark$$

$$= \pi + 6\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right]$$

$$= \pi + 6\left[1 - \frac{\sqrt{3}}{2}\right]$$

$$\therefore A = (\pi + 6 - 3\sqrt{3})\ \text{units}^2 \checkmark$$