Carlingford High School
Mathematics Extension 1
Higher School Certificate
Trial Examination 2020

NESA Number: ________________________

General Instructions	• Working time - 120 minutes • Write using black pen • NESA approved calculators may be used • A reference sheet is provided at the back of this paper • In questions 11-14, show relevant mathematical reasoning and/or calculations
Total marks: 70	**Section I – 10 marks** • Attempt Questions 1-10 • Allow about 15 minutes for this section **Section II – 60 marks** • Attempt questions 11-14 • Allow about 1 hour and 45 minutes for this section

	MC	Q11	Q12	Q13	Q14	Total
Further work with functions	/1	/2	/4			**/7**
Polynomials	/1					**/1**
Inverse Trigonometric Functions			/6			**/6**
Further Trigonometric Identities		/3		/5		**/8**
Rates of change	/1			/6		**/7**
Combinatorics	/1					**/1**
Proof	/1	/3				**/4**
Vectors	/2	/2	/5		/6	**/15**
Trigonometric Equations	/1					**/1**
Further Calculus Skills	/2	/7				**9**
Applications of Calculus				/4	/7	**/11**
Total	/10	/17	/15	/15	/13	/70

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j}$. What is the angle between the two vectors.

A. $\frac{\pi}{2}$

B. $\frac{\pi}{4}$

C. π

D. 2π

2. Write the expression $y = \sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$.

A. $2\sin\left(x + \frac{\pi}{4}\right)$

B. $\sin\left(x + \frac{\pi}{3}\right)$

C. $2\sin\left(x + \frac{\pi}{3}\right)$

D. $2\sin\left(x - \frac{\pi}{3}\right)$

3. Given that $f(x) = e^x - 1$ and $y = f^{-1}(x)$, find an expression for $\frac{dy}{dx}$.

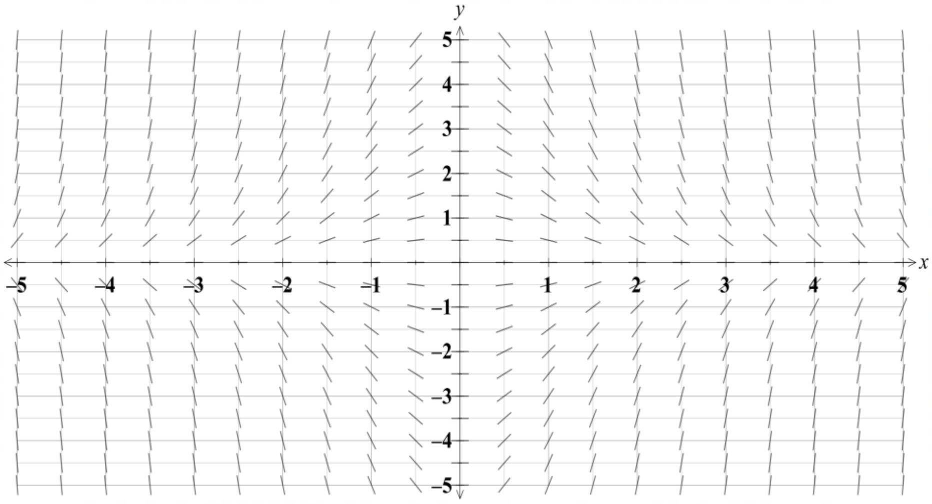
A. $\frac{1}{e^x - 1}$

B. $\frac{1}{x + 1}$

C. $\ln x$

D. $\ln(x + 1)$

4. Which of the following differential equations could be represented by the slope field diagram below?



A. $y' = -xy$

B. $y' = xy$

C. $y' = -x^2y$

D. $y' = x^2y$

5. Which of the following is an expression for $\int \frac{x}{\sqrt{9-x^2}}dx$?

Use the substitution $u = 9 - x^2$

A. $-\sqrt{9-x^2} + C$

B. $-2\sqrt{9-x^2} + C$

C. $\sqrt{9-x^2} + C$

D. $2\sqrt{9-x^2} + C$

6. The polynomial $x^4 + ax^3 - 3x^2 + bx - 2$ has roots -1 and 2, one of which is a triple root.

Find the values of a and b.

A. $a = -1 \;\; b = 2$

B. $a = 2 \;\; b = -5$

C. $a = 1 \;\; b = 3$

D. $a = 1 \;\; b = -5$

7. A body of still water has suffered an oil spill and a circular oil slick is floating on the surface of the water.

The area of the oil slick is increasing by $0.1\ \mathrm{m}^2/\text{minute}$.

At what rate is the radius increasing when the area is $0.3\ \mathrm{m}^2$?

A. 0.0098 m/min

B. 0.03 m/min

C. 0.0515 m/min

D. 0.0531 m/min

8. Find the vector projection of $\underset{\sim}{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ onto $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

A. $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

B. $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

C. $\begin{pmatrix} -2\sqrt{5} \\ 4\sqrt{5} \end{pmatrix}$

D. $\begin{pmatrix} 2\sqrt{5} \\ -4\sqrt{5} \end{pmatrix}$

9. A four-digit security code is to be created for a building alarm, using any selection of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The code must be entered in the correct order to disarm the alarm when entering the building.

Digits ***may*** be repeated.

It has been decided that the code will contain exactly two different digits, for example 4224 or 7177.

If an intruder, who knew about this restriction, tried to guess the alarm code, what is the probability they would get it correct?

A. $\dfrac{1}{10\,000}$

B. $\dfrac{1}{5\,040}$

C. $\dfrac{1}{2\,100}$

D. $\dfrac{1}{630}$

10. Emma made an error proving that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. The proof is shown below.

Step 1: To prove $2^n + (-1)^{n+1}$ is divisible by 3 (n is an integer)

To prove true for $n = 1$

$$2^1 + (-1)^{1+1} = 2 + 1$$

$= 3 \times 1$ Line 1

Result is true for $n = 1$

Step 2: Assume true for $n = k$

$2^k + (-1)^{k+1} = 3m$ (m is an integer) Line 2

Step 3: To prove true for $n = k + 1$

$2^{k+1} + (-1)^{k+1+1} = 2\left(2^k\right) + (-1)^{k+2}$ Line 3

$= 2[3m + (-1)^{k+1}] + (-1)^{k+2}$ Line 4

$= 2 \times 3m + 2 \times (-1)^{k+2} + (-1)^{k+2}$

$= 3[2m + (-1)^{k+2}]$

Which is a multiple of 3 since m and k are integers.

Step 4: True by induction

In which line did Emma make an error?

A. Line 1

B. Line 2

C. Line 3

D. Line 4

Section II

60 marks
Attempt questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 marks) Use a SEPARATE writing booklet. **Marks**

a) Find $\dfrac{d}{dx}(e^x \tan^{-1} x)$ **2**

b) **i)** Write an expression for $\sin 5x \sin x$ in terms of $\cos 4x$ and $\cos 6x$. **1**

ii) Hence, find $\displaystyle\int_0^{\frac{\pi}{4}} \sin 5x \ \sin x \ dx$ **2**

c) Evaluate $\displaystyle\int_{-8}^{0} \frac{x}{\sqrt{1-x}} dx$ using the substitution $u = 1 - x$. **3**

d) A curve C has parametric equations $x = cos^2 t$ and $y = 4sin^2 t$ for $t \in R$.

i) What is the Cartesian equation of C? **1**

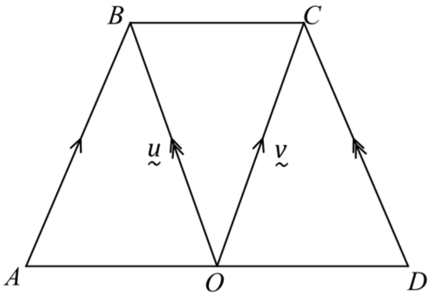
ii) What is the domain of the equation C? **1**

e) Find $\displaystyle\int 2\cos^2 x dx$ **2**

f) Prove by mathematical induction that $4^n + 14$ is divisible by 6 for all positive integers n $(n \geq 1)$. **3**

Question 11 continues on the next page

g) AB is a parallel to OC, DC is parallel OB, $\overrightarrow{OB} = \underset{\sim}{u}$, $\overrightarrow{OC} = \underset{\sim}{v}$ and $AB = OB = OC = DC$.



i) Express $\overrightarrow{AD}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **1**

ii) Express $\overrightarrow{BD}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **1**

End of question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

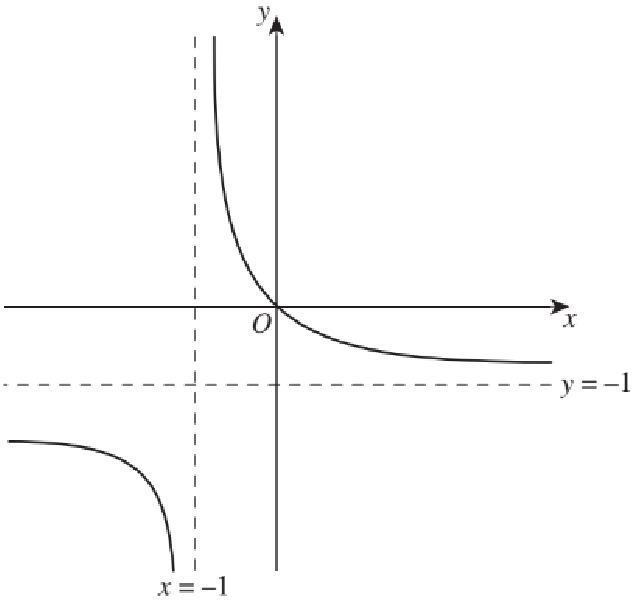
a) The work done, W, by a constant force, $\underset{\sim}{F}$, in moving a particle through a displacement, s, is defined by the formula $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$.
A force described by the vector $\underset{\sim}{F} = \binom{4}{-2}$ moves a particle along the line l from $P(-1,2)$ to $Q(2,-2)$

i) Find $\underset{\sim}{s} = \overrightarrow{PQ}$ and hence find the value of W. **2**

ii) Hence, verify that W is also given by $W = (\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})|\underset{\sim}{s}|$. **2**

iii) Find the component of $\underset{\sim}{F}$ in the direction of l. **1**

b) The diagram below is a sketch of the graph of the function
$f(x) = -\frac{x}{x+1}$.



i) Copy the graph of $f(x)$ into your answer booklet. On the same graph, sketch $y = (f(x))^2$, showing all asymptotes and intercepts. Clearly label each graph. **2**

ii) Solve the equation $(f(x))^2 = f(x)$ **2**

Question 12 is continued on the next page

c) Consider the function $y = \cos^{-1}(\sin x)$.

i)

Show that $\dfrac{dy}{dx} = \mp 1$ **1**

ii) What does your answer to part (a) tell you about stationary points for this function? **1**

iii) Find the range and explain why the domain is: *all real x*. **2**

iv) Sketch the function over the domain $0 \le x \le 2\pi$. **2**

End of question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3. The surface area of the cube is increasing at a constant rate of 8 cm^{2}s^{-1}.

i) Show that $\frac{dx}{dt} = \frac{k}{x}$ where k is a constant. **1**

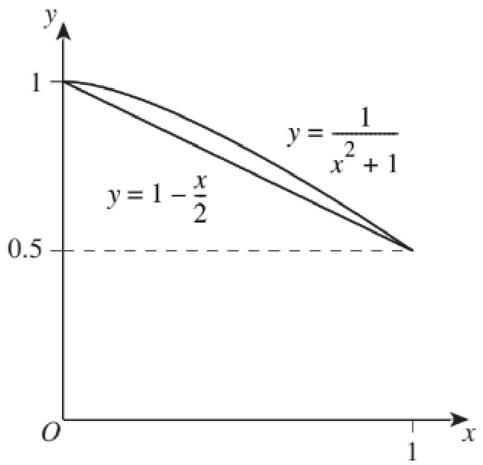
ii) Show that $\frac{dV}{dt} = 2V^{\frac{1}{3}}$ **2**

iii) Given that $V = 8$ when $t = 0$, solve the differential equation in part (ii), and find the value of t when $V = 16\sqrt{2}$. **3**

b) i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\text{cosec}x + \cot x = \cot\frac{x}{2}$. **2**

ii) Hence evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\text{cosec}x + \cot x)dx$. Answer is simplest exact form. **3**

c) The diagram shows the graph of $y = \frac{1}{x^2+1}$ and the graph of $y = 1 - \frac{x}{2}$ for $0 \le x \le 1$. **4**



Find the exact volume of the solid of revolution formed when the region bounded by $y = \frac{1}{x^2+1}$ and $y = 1 - \frac{x}{2}$ is rotated 360^0 about the y-axis.

End of Question 13

Question 14 (13 marks) Use a SEPARATE writing booklet.

a) The area $A\ cm^2$ is occupied by a bacterial colony. The colony has its growth modelled by the logistic equation $\frac{dA}{dt} = \frac{1}{25}A(50 - A)$ where $t \geq 0$ and t is measured in days. At time $t = 0$, the area occupied by the bacteria colony is $\frac{1}{2}cm^2$.

i) Show that $\frac{1}{A(50-A)} = \frac{1}{50}(\frac{1}{A} + \frac{1}{50-A})$ **1**

ii) Using the result from (a) (i), solve the logistic equation and hence show that $A = \frac{50}{1+99e^{-2t}}$. **3**

iii) According to this model, what is the limiting area of the bacteria colony? **1**

iv) Find the exact time when the rate of change in the area occupied by the bacterial colony is at its maximum. **2**

b) A golf ball is hit at a velocity of 110 ms^{-1} at an angle θ to the horizontal.

The position vector *s(t)*, from the starting point, of the ball after *t* seconds is given by

$$s = 110t\cos\theta\ \underset{\sim}{\mathbf{i}} + (110t\sin\theta - 4{\cdot}9\,t^2)\underset{\sim}{\mathbf{j}}$$

i) Using gravity of $9{\cdot}8 \text{ ms}^{-2}$ show that the maximum horizontal range of the ball is $\frac{12100\sin 2\theta}{9{\cdot}8}$ metres. **2**

ii) To ensure that the ball lands on the green, it must travel between 400 and 450 metres. **2**

What values of θ, correct to the nearest minute, would allow this to happen?

iii) The golfer aims accurately and hits the ball directly towards the green. **2**

After 3.4 seconds of flight, at a point 8 metres above the ground, the ball hits a low flying TV drone.

If it had not hit the drone or any other obstacles, would the ball have landed on the green?

End of paper

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}\left(a + b\right)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$\left(x - h\right)^2 + \left(y - k\right)^2 = r^2$$

Financial Mathematics

$$A = P\left(1 + r\right)^n$$

Sequences and series

$$T_n = a + \left(n - 1\right)d$$

$$S_n = \frac{n}{2}\left[2a + \left(n - 1\right)d\right] = \frac{n}{2}\left(a + l\right)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} = \frac{a\left(r^n - 1\right)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

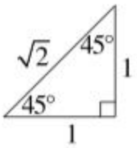
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

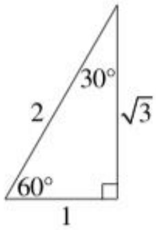


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2} r^2 \theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

If $t = \tan \dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A \cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A \sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A \cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A \sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

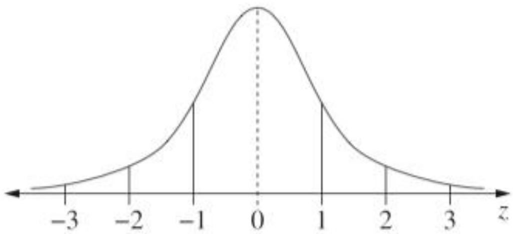
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E(X^2) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

Trial HSC Examination 2020
Mathematics Extension 1 Course

NESA Number ________________________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○