CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Mathematics Extension 1

Half Yearly Examination 2015

Time allowed: 2 hours

Name: ________________________________ **Class: 12MA1**______

Teacher: 12MA11 (Ms Strilakos) 12MA12 (Mr Cheng) 12MA13 (Ms Wilson)

Instructions:

- All questions may be attempted.
- Show ALL necessary working out.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- **Answer questions 1 to 6 in the same booklet**, then questions 7 to 10 in separate booklets.

	MC	Q6	Q7	Q8	Q9	Q10	TOTAL
H3			/9				/9
H5	/5	/14			/7		/26
H6				/10	/12		/22
H8				/4		/10	/14
TOTAL	/5	/14	/9	/14	/19	/10	/71

Section 1 (Multiple Choice)

Question 1

For the graph $y = \frac{x^3+1}{x^3-1}$, which of the following is true?

(A) There are no x-intercepts

(B) There are three vertical asymptotes

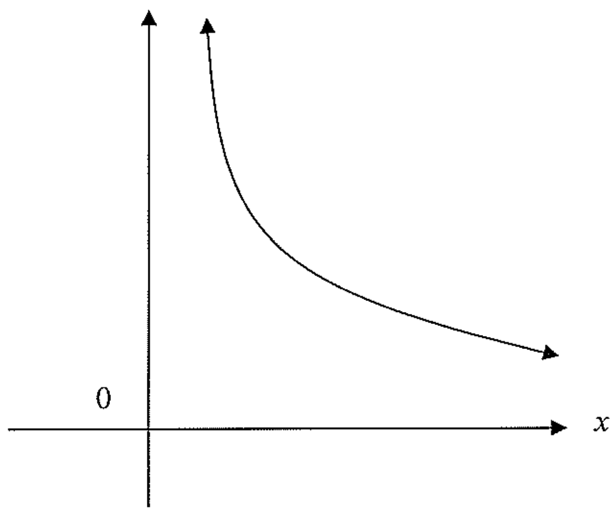
(C) It is an odd function

(D) $y = 1$ is a horizontal asymptote

Question 2

The graph of $y = \sqrt{\frac{2}{x}}$ is shown below.

When this graph is rotated about the y-axis, a solid of revolution is formed. The exact volume of the solid bounded by $= 1$, $y = 3$ and the y-axis is given by



(A) $\pi \int_1^3 \frac{2}{x} dx$ (B) $\pi \int_1^3 \frac{2}{y^2} dy$ (C) $\pi \int_1^3 \frac{2}{y^4} dy$ (D) $\pi \int_1^3 \frac{4}{y^4} dy$

Question 3

Use the trapezoidal rule with 5 function values (4 subintervals) to find an approximation to $\int_0^2 \sqrt{9-x^2}dx$

(A) 5.501

(B) 7.672

(C) 5.21

(D) 9.801

Question 4

If $\int_{-a}^{a} f(x)dx = 0$, which of the following statements is true?

(A) $f(x)$ is an even function

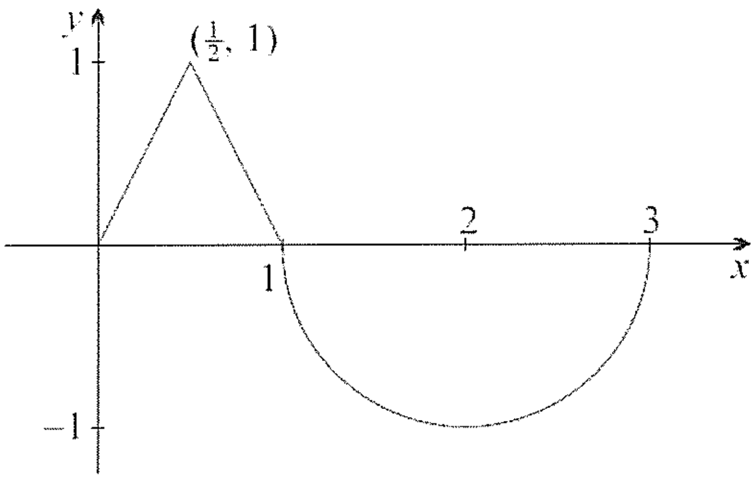
(B) $f(x)$ is an odd function

(C) $f(x)$ is neither an even or an odd function

(D) $f(x)$ is a special function

Question 5

The diagram below shows the graph of $y = f(x)$ in the domain $0 \le x \le 3$



Given that the arc is a semi-circle, the value of $\int_0^3 f(x)\, dx$ is:

(A) $\frac{1}{2} + \frac{\pi}{2}$ (B) $1 + 2\pi$ (C) $\frac{1}{2} - 2\pi$ (D) $\frac{1}{2}(1-\pi)$

Section 2

Question 6 (Answer questions 1 to 6 in the same booklet) **Marks**

(a) Differentiate $y = \ell n\left(\dfrac{2x + 1}{2x - 1}\right)$ with respect to x **2**

(b) (i) $\int e^{4x + 1} dx$ **2**

(ii) $\int_1^e \dfrac{x + 1}{x} dx$ (Leave your answer in exact form) **3**

(iii) $\int \dfrac{1}{(2x - 1)^3} dx$ **2**

(iv) $\int \dfrac{x^2}{x^3 - 1} dx$ **2**

(c) Given that $y = e^{3x^2}$ find,

(i) $\dfrac{dy}{dx}$ **1**

(ii) Hence, find $\int_0^1 x e^{3x^2} dx$ **2**

Question 7 (Start a new booklet)

(a) Given $\log_x a = 3 \cdot 6$ and $\log_x b = 2$, find $2\log_x a + \log_x b^3$ **2**

(b) Solve the following equations for x.

(i) $\log_{27} x = -\dfrac{1}{3}$ **1**

(ii) $\log_x 25 - \log_x 5 = \dfrac{\log_x 25}{\log_x 5}$ **3**

(c) Write $e^{2\ell n x}$ in simplest form. **1**

(d) Show that $5\log_{32} x = \log_2 x$ **2**

Question 8 (Start a new booklet)

(a) The area under the curve $y = e^{x} + e^{-x}$ and bounded by $x = -2$ and $x = 2$ is revolved around the x-axis. Find the exact volume of the solid of revolution. **4**

(b) (i) Write down the domain of $y = \dfrac{\ell nx}{x}$ **1**

(ii) Find where the graph of this function cuts the x-axis. **1**

(iii) It is known that

$$\frac{dy}{dx} = \frac{1 - \ell nx}{x^2} \quad \text{(DO NOT PROVE THIS)}$$

Hence, show that $\dfrac{d^2y}{dx^2} = \dfrac{2\ell nx - 3}{x^3}$ **2**

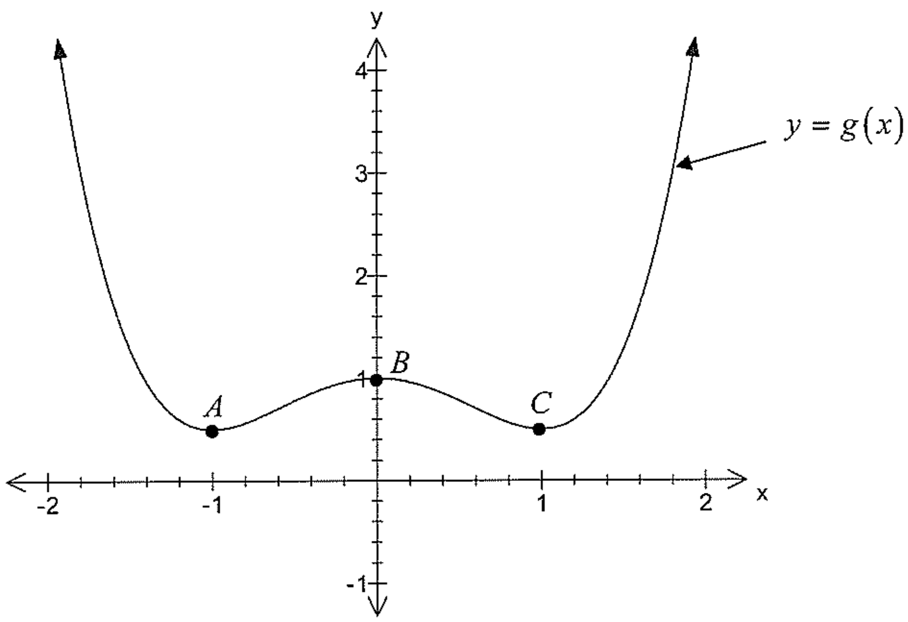
(iv) Find the only stationary point and determine its nature. **2**

(v) Find the exact coordinates of the only point of inflexion. **2**

(vi) Hence sketch the curve. **2**

Question 9 (Start a new booklet)

(a) The graph of $y = g(x)$ is sketched below. The points A, B and C are the stationary points of this curve. Draw a sketch of the gradient function, $y = g'(x)$, of this curve. **2**



(b) A function $f(x)$ is defined by $f(x) = 2x^3 + ax^2 + bx + 3$

(i) If this function $y = f(x)$ has stationary points when $x = 1$ and $x = -2$, show that $a = 3$ and $b = -12$. **2**

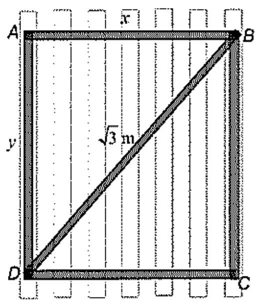
(ii) Hence, find the coordinates of the stationary points of $y = f(x)$ and determine their nature. **3**

(iii) Find the coordinates of the point of inflexion. **2**

(iv) Hence, sketch the graph $y = f(x)$ identifying all key features, including any turning points and points of inflexion. **2**

(v) For what values of x is $f(x) = 2x^3 + 3x^2 - 12x + 3$ concave up? **1**

(c) A swinging gate is to be constructed from timber palings. It will require a support frame using 5 pieces of timber shown in this diagram.



Given $AB \| CD$, $AD \| BC$, $AB = CD = x$ m, $AD = BC = y$ m and $BD = \sqrt{3}$ m,

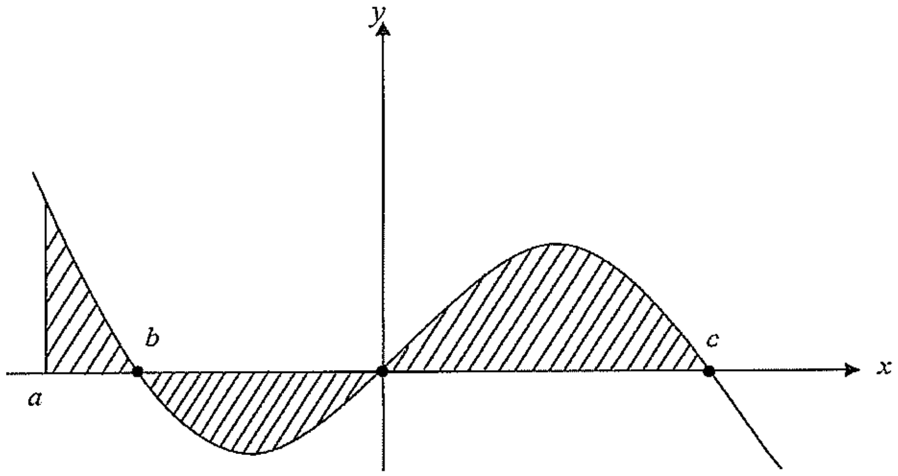
(i) Find an expression for y in terms of x. **2**

(ii) Show that the total length (L) of the timber pieces in the support frame is represented by $L = 2\left(x + \sqrt{3 - x^2} + \frac{\sqrt{3}}{2}\right)$. **2**

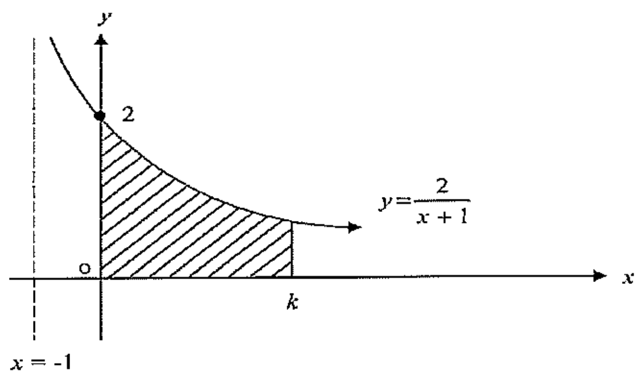
(iii) The gate will have its maximum strength when the length of its support frame is maximized. For what value of x will the gate have maximum strength? **3**

Question 10 (Start a new booklet)

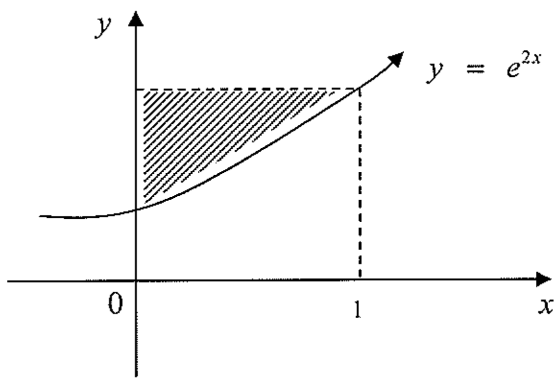
(a) If the equation of this graph is $y = f(x)$, write an expression for the shaded area shown. **2**



(b) Find the value of k such that the area under the curve $y = \dfrac{2}{x+1}$, above the x axis, between $x = 0$ and $x = k$, equals to 6 square units. **4**



(c) Given the graph of $y = e^{2x}$,



(i) Find the area under the curve $y = e^{2x}$, above the x axis and between $x = 0$ and $x = 1$ **2**

(ii) Hence, or otherwise, find the shaded area. **2**

END OF TEST

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Multiple Choice

1A 2D 3A 4B 5D

Mathematics Ext 1 Y12
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Question 6

a) $y = \ln\left(\frac{2x+1}{2x-1}\right) = \ln(2x+1) - \ln(2x-1)$

MC + Q9 (24) Wilson
Q6 + Q7 (23) Strickland
Q8 + Q10 (24) Chong

$\frac{dy}{dx} = \frac{2}{2x+1} - \frac{2}{2x-1}$

$= 2\left(\frac{(2x-1) - 2(x+1)}{(2x+1)(2x-1)}\right)$

$= \frac{-4}{(4x^2-1)}$

b) $\frac{d}{dx}\left(\ln(x^2)\right)$ $\qquad$ $\frac{d}{dx}\left[(\ln x)^2\right]$

$= \frac{d}{dx} 2\ln(x)$ $\qquad$ $= \frac{d}{dx}(\ln x \cdot \ln x)$

$= \frac{2}{x}$ $\qquad$ $= \frac{2\ln x}{x}$

$\therefore \frac{d}{dx}\left(\ln(x^2)\right) \neq \frac{d}{dx}\left[(\ln x)^2\right]$ for $\ln x \neq 1$

b) i) $\int e^{4x+1}\,dx$

$= \frac{1}{4}e^{4x+1} + C$

ii) $\int_1^e \frac{x+1}{x}\,dx = \int_1^e 1 + \frac{1}{x}\,dx$

$= \left[x + \ln(x)\right]_1^e$

$= (e+1) - (1+0)$

$= e$

iii) $\int \frac{1}{(2x-1)^3}\,dx = -\frac{1}{4}(2x-3)^{-2} + C$

iv) $\int \frac{x^2}{x^3-1}\,dx = \frac{1}{3}\ln(x^3-1) + C$

d) $y = e^{3x^2}$

i) $\frac{dy}{dx} = 6x e^{3x^2}$

ii) $\int_0^1 x e^{3x^2} \, dx = \frac{1}{6} \int_0^1 6x e^{3x^2} \, dx$

$= \frac{1}{6} \left[e^{3x^2} \right]_0^1$ from i)

$= \frac{1}{6} \left[e^3 - 1 \right]$

7 a) $2\log_x a + \log_x b^3$

$= 2\log_x a + 3\log_x b$

$= 2 \times 3.6 + 3 \times 2$

$= 13.2$

b) i) $\log_{27} x = -1/3$

$x = 27^{-1/3}$

$= 1/3$

ii $\log_x 25 - \log_x 5 = \frac{\log_x 25}{\log_x 5}$

$\log_x 5 = \cancel{\frac{\log_x 25}{\log_x 5}} \quad \frac{2\log_x 5}{\log_x 5}$

$\log_x 5 = 2$

$5 = x^2$

$x = \sqrt{5}$

c) $e^{2\ln(x)} = e^{\ln(x^2)}$

$= x^2$

d) $5\log_{32} x = \frac{5\log_2 x}{\log_2 32}$ (change of base)

$= \frac{5\log_2 x}{5}$

$= \log_2 x$

8a) $y = e^{x} + e^{-x}$

$y^2 = e^{2x} + e^{-2x} + 2$

$A = \pi \int_a^b y^2 \, dx$

$A = \pi \int_{-2}^{2} e^{2x} + e^{-2x} + 2 \, dx$

$= \pi \left[\frac{1}{2} e^{2x} - \frac{1}{2} e^{-2x} + 2x \right]_{-2}^{2}$

H8

$= \pi \left[\left(\frac{e^4}{2} - \frac{e^{-4}}{2} + 4 \right) - \left(\frac{e^{-4}}{2} - \frac{e^{4}}{2} - 4 \right) \right]$

$= \pi \left(e^4 - e^{-4} + 8 \right)$

b) i) all real $x > 0$

ii) graph cuts x axis at $y = 0$

H6 $\frac{\ln x}{x} = 0 \quad \therefore \ln x = 0$

graph cuts x axis at $x = 1$

iii $\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{1 - \ln x}{x^2}$

$u = 1 - \ln x \quad u' = -\frac{1}{x}$

$v = x^2 \quad v' = 2x$

By the chain rule $\frac{d^2y}{dx^2} = \frac{-\frac{1}{x} \times x^2 - 2x(1 - \ln x)}{x^4}$

$= \frac{-\cancel{x} - 2\cancel{x}(1 - \ln x)}{x^{\cancel{4}3}}$

$= \frac{2\ln x - 3}{x^3}$ as required

iv $\frac{dy}{dx} = 0$ at stationary point

$\therefore \frac{1 - \ln x}{x^2} = 0$

$\ln x = 1$

$x = e$

when $x = e$, $y = \frac{\ln e}{e} = \frac{1}{e}$

stationary point $= (e, \frac{1}{e})$

At $(e, \frac{1}{e})$ $\frac{d^2y}{dx^2} = \frac{2\ln(e) - 3}{e^3} = -\frac{1}{e^3} < 0$

$\therefore$ The stationary point is a local maximum by the second derivative test

v) At a point of inflexion $\frac{d^2y}{dx^2} = 0$

$\therefore \frac{2\ln(x) - 3}{x^3} = 0$

$\therefore 2\ln(x) - 3 = 0$

$\ln(x^2) = 3$

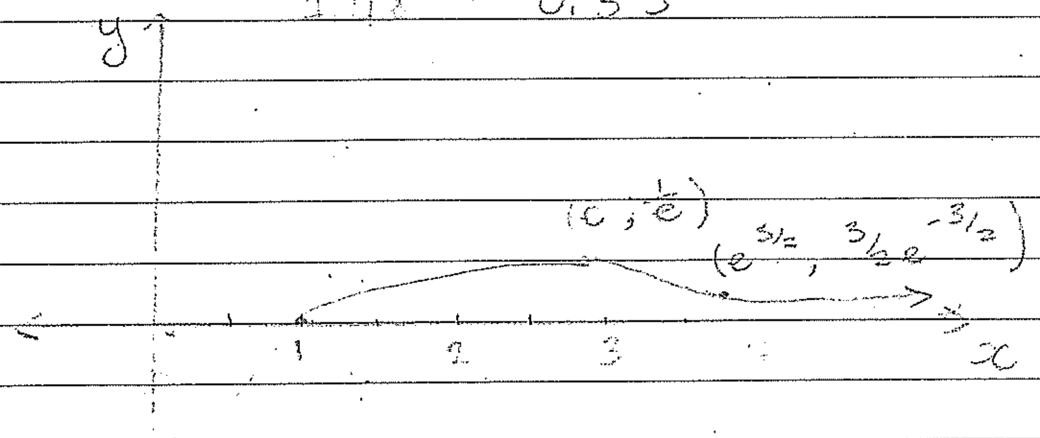
$x^2 = e^3$

$x = e^{3/2}$

When $x = e^{3/2}$, $y = \frac{\ln(e^{3/2})}{e^{3/2}} = \frac{3}{2} e^{-3/2}$

Pt of inflexion $(e^{3/2}, \frac{3}{2}e^{-3/2})$

0.33



c) i $x^2 + y^2 = 3$

$y^2 = 3 - x^2$

$y = \sqrt{3 - x^2}$

ii) $L = 2x + 2y + \sqrt{3}$

$= 2\left(x + \sqrt{3-x^2} + \frac{\sqrt{3}}{2}\right)$

iii) $\frac{dL}{dx} = 2\left(1 - \frac{2x}{2}(3-x^2)^{-1/2}\right)$

$= 2\left(1 - x(3-x^2)^{-1/2}\right)$

Set $\frac{dL}{dx} = 0$ for a maximum

$2\left(1 - x(3-x^2)^{-1/2}\right) = 0$

$\therefore \quad x(3-x^2)^{-1/2} = 1$

$x = \sqrt{3-x^2}$

$x^2 = 3 - x^2$

$2x^2 = 3$

$x = \sqrt{3/2}$

Check this is a maximum: for $x = 1$

$\frac{dL}{dx} = 2\left(1 - \frac{1}{\sqrt{2}}\right) > 0$

for $x = \frac{3}{2}$ $\frac{dL}{dx} = 2\left(1 - 3/2(\sqrt{3/4})^{-1}\right) < 0$

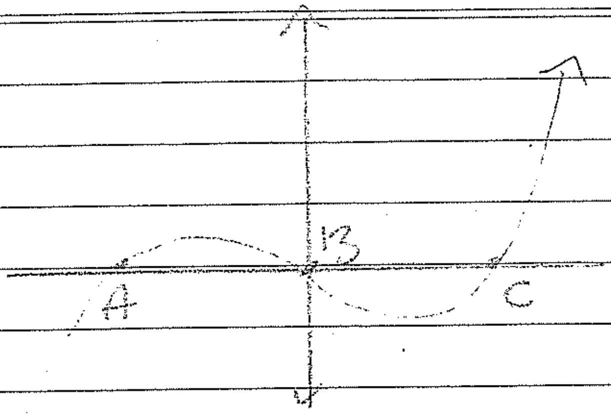
$\therefore$ max

The gate has maximum strenglt for

$x = \sqrt{3/2}$ m

9a and B are H6   9c is H5

9 a)



b) $f(x) = 2x^3 + ax^2 + bx + 3$

i) $f'(x) = 6x^2 + 2ax + b$

$f'(1) = 6 + 2a + b = 0 \quad -①$

$f'(-2) = 24 - 4a + b = 0 \quad -②$

② − ① $\quad 18 - 6a = 0$

$a = 3$

sub $a = 3$ into ① $\quad 6 + 6 + b = 0$

$\therefore b = -12$

ii) from (i) $f(x) = 2x^3 + 3x^2 - 12x + 3$

$f(1) = 2 + 3 - 12 + 3 = -4$

$f(-2) = -16 + 12 + 24 + 3 = 23$

The stationary points are $(1, -4)$ and $(-2, 23)$

$f'(x) = 6x^2 + 6x - 12$

$f''(x) = 12x + 6$

at $x = 1$ $f''(x) = 18 > 0 \therefore (1, -4)$ is a minimum

at $x = -2$ $f''(x) = -18 < 0 \therefore (-2, 23)$ is a ma

Point of inflexion $f''(x) = 0$

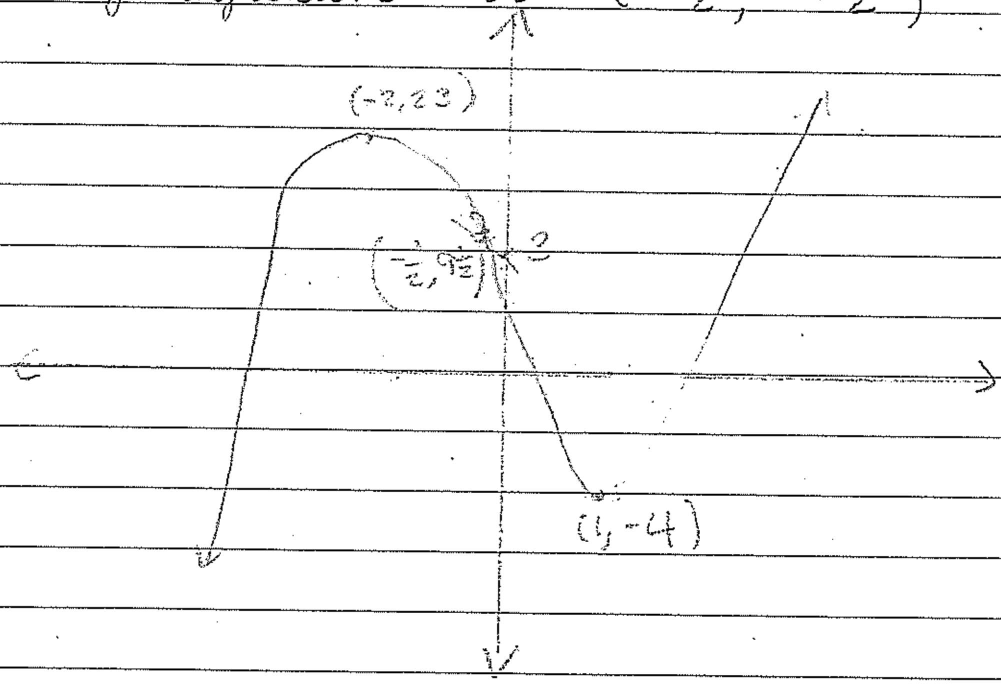
$\therefore \quad 12x + 6 = 0$

$x = -\frac{1}{2}$

$x = -\frac{1}{2} \quad f(-\frac{1}{2}) = -\frac{1}{4} + \frac{3}{4} + 6 + 3$

$= 9\frac{1}{2}$

The point of inflexion is $(-\frac{1}{2}, 9\frac{1}{2})$



Concave up ; $f''(x) > 0$

$\therefore \quad 12x + 6 > 0$

$x > -\frac{1}{2}$

10 a) $A = \int_a^b f(x)\,dx - \int_b^0 f(x)\,dx + \int_0^c f(x)\,dx$

(other sol^{n}s possible)

b) $\int_0^k \frac{2}{x+1}\,dx = 6$

$\cancel{LHS} = \int_0^k \frac{2}{x+1}\,dx = 2\left[\ln(x+1)\right]_0^k$

$= 2\ln(k+1)$

$\therefore 2\ln(k+1) = 6$

$\ln(k+1) = 3$

$k+1 = e^3$

$k = e^3 - 1$

c) $y = e^{2x}$

$A_1 = \int_0^1 e^{2x}\,dx$

$= \frac{1}{2}\left[e^{2x}\right]_0^1$

$= \frac{1}{2}(e^2 - 1)$

Shaded area $= e^2 - \frac{1}{2}(e^2 - 1)$

$= \frac{1}{2}(e^2 + 1)$