Product of the roots:

$$\cos 0 \cdot \cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5} \cdot \cos\frac{6\pi}{5} \cdot \cos\frac{8\pi}{5} = -\frac{-1}{16}$$

$$\cos^2\frac{2\pi}{5} \cdot \cos^2\frac{4\pi}{5} = \frac{1}{16}$$

$$\cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5} = -\frac{1}{4} \quad \left(\cos\frac{2\pi}{5} > 0,\ \cos\frac{4\pi}{5} < 0\right)$$

Let $a = \cos\frac{2\pi}{5}$ and $b = \cos\frac{4\pi}{5}$:

$$a+b = -\frac{1}{2} \qquad b = -\frac{1}{2} - a$$

$$ab = -\frac{1}{4}$$

$$a\left(-\frac{1}{2} - a\right) = -\frac{1}{4}$$

$$4a^2 + 2a - 1 = 0$$

$$a = \frac{-2 \pm \sqrt{4+16}}{8}$$

$$a = \frac{-1 \pm \sqrt{5}}{4}$$

$$\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4},\ \cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$$

Topic 8
Harder Extension 1 Topics

1 Show that if $x \ge 0,\ y \ge 0$ then

(i) $x^2 + y^2 \ge 2xy$

(ii) $x^3 + y^3 \ge xy(x+y)$

(iii) hence or otherwise,

$$2(x^3 + y^3 + z^3) \ge xy(x+y) + yz(y+z) + zx(z+x)$$

2 (i) Prove that $(a^2 - b^2)(c^2 - d^2) \le (ac - bd)^2$.

(ii) Show that $(a^2 - b^2)(a^4 - b^4) \le (a^3 - b^3)^2$.

3 It is given that

$$a+b+c = 1 \text{ and } a+b+c \ge 3\sqrt[3]{abc},$$

where a, b, and c are positive real numbers.

(i) Prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge 9$.

(ii) Hence, or otherwise, find the smallest possible value of

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right).$$

4 For a sequence of numbers $a_1 = 2,\ a_2 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for all integers $n \ge 3$.

Prove by mathematical induction that

$$a_n = 2^{n-1} + 1.$$

5 (i) Show that $2k+3 > 2\sqrt{(k+1)(k+2)}$ for all $k > 0$.

(ii) Hence prove for $n \ge 1$ that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right).$$

6 Each term T_n, $n = 1, 2, 3, \ldots$, of a sequence of numbers is given by $T_1 = 1$, $T_2 = 3$ and $T_n = T_{n-1} + T_{n-2}$, $n = 3, 4, 5, \ldots$

Show by the method of mathematical induction that $T_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$, $n = 1, 2, 3, \ldots$

7 Prove by mathematical induction that

$$1.1! + 2.2! + 3.3! + \cdots + n.n! = (n+1)! - 1$$

for $n \ge 1$.

8 Use mathematical induction to prove that

$$\sin(n\pi + x) = (-1)^n \sin x$$

for all positive integral values of n.

9 A sequence $\{\alpha_n\}$ is defined by $\alpha_1 = 1$ and $\alpha_{n+1} = \alpha_n^2 + \alpha_n$ for all $n \ge 1$.

(i) Write down the values of $\alpha_2,\ \alpha_3,\ \alpha_4$.

(ii) Prove by mathematical induction that for all n: $\quad \alpha_{n+1} - 1 = \sum_{k=1}^{n} \alpha_k^2$

(iii) Show that

$$(2\alpha_{n+1} + 1)^2 = (2\alpha_n + 1)^2 + (2\alpha_{n+1})^2$$

and deduce that

$$(2\alpha_{n+1} + 1)^2 = (2\alpha_1 + 1)^2 + \sum_{k=2}^{n+1} (2\alpha_n)^2.$$

(iv) Find the value of α_5 and express it as the sum of five positive squares.

10 (i) Show that $\tan\left(A + \frac{\pi}{2}\right) = -\cot A$.

(ii) Use mathematical induction to show that

$$\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n$$

for all integers $n \ge 1$.

11 (i) If $\tan\frac{A}{2} = t$, prove that $\cos A = \frac{1-t^2}{1+t^2}$.

(ii) If also $\tan B = \frac{t}{2}$ and $\cos C = \frac{5\cos A + 3}{3\cos A + 5}$, where A, B and C are acute angles, prove that $\angle C = 2\angle B$.

12 It is known that $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1-x)$ are acute.

(i) Show that $\sin\left(\sin^{-1}x - \cos^{-1}x\right) = 2x^2 - 1$.

(ii) Solve the equation

$$\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x).$$

13 (i) Express $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$, $\tan C$.

(ii) Hence, or otherwise, show that

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

(iii) If A, B and C are the angles of a triangle, show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

14 Solve the equation

$$\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2).$$

15 (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all real θ.

(ii) Hence find in surd form the values of $\cot\frac{\pi}{8}$ and $\cot\frac{\pi}{12}$, and show that

$$\operatorname{cosec}\frac{2\pi}{15} + \operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} = 0.$$

16 A general-knowledge quiz is made up of twenty true or false questions. A contestant is allowed to toss a coin to choose an answer if they do not know the correct answer.

Find the probability that a contestant who knows the correct answers to ten of the questions, but answers the remaining ten by tossing a coin, will obtain a score of at least 85% on the quiz.

17 A group of n people are to be seated around a circular table.

(i) Find the number of possible arrangements if 3 nominated people are to sit together as a group.

(ii) Making use of the result in (i), find the probability, in simplest terms, that 3 particular people sit together if n people are to be randomly seated around a circular table.

18 Two ordinary dice are used in a game of chance played by two people A and B. The two dice are repeatedly thrown until either a total of 7 or a total of 8 is scored on the uppermost faces of the two dice.

Player A wins if a total of 8 occurs first and player B wins if a total of 7 occurs first.

Find the probability that

(i) Player A wins on the first throw

(ii) Player B wins before the sixth throw

(iii) Player A wins eventually.

19 Two teams A and B play in a tennis tournament. Each team consists of four players. The tournament consists of four matches where each member of one team is paired with exactly one member of the other team.

Each match has three possible outcomes. The points allocation is tabled below.

Outcome	*Points allocated*	
A win for the player from team A	1 for A	0 for B
A win for the player from team B	0 for A	1 for B
A draw	$\frac{1}{2}$ for A	$\frac{1}{2}$ for B

(i) Find the number of different ways in which the players of team A and team B can be paired.

(ii) For a particular arrangement of pairs of players, find all the different possible ways in which the outcomes of the four matches can occur.

(iii) Suppose that all players in both teams are evenly paired so that in each of the four matches, the outcomes 'a win for A', 'a win for B' and 'a draw' are equally likely.
For this particular pairing, find the probability that the overall tournament results in a draw for both teams (i.e. each team obtains a total of 2 points).

20 A box contains chocolates which are wrapped in the same coloured foil. The chocolates are either dark or light. Margaret draws a chocolate, attempting to select a dark one. If she fails, the light one is rewrapped and returned to the box and an additional light one (wrapped in foil) is added. She then makes a further attempt.

This continues until she succeeds. The box initially contains one dark and one light chocolate. The identity

$$\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$$

may be used in the following questions.

Find, giving reasons, the probability that she will succeed

(i) on her third attempt

(ii) on her nth attempt

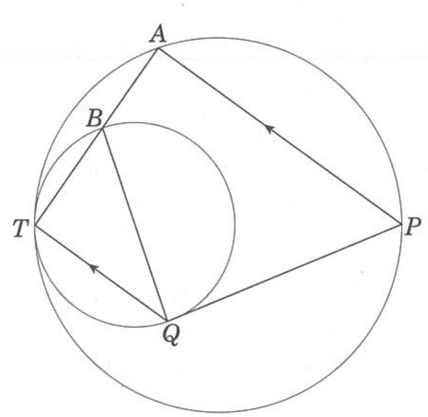
(iii) before her $(n+1)$th attempt.

(iv) Explain how the answer to (iii) behaves as the value of n continues to increase.

21 The quadrilateral $ABCD$ is cyclic and its diagonals intersect at E. A circle passes through A, B and E.

Prove that the tangent at E to this circle is parallel to CD.

22



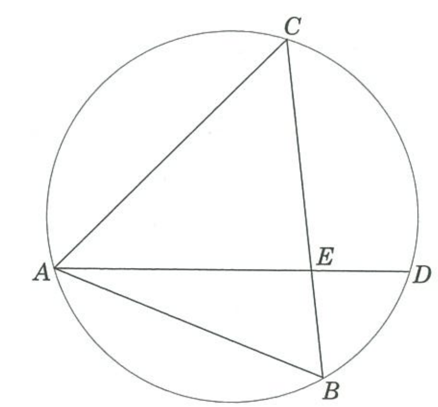
Two circles touch internally at T. From P the tangent PQ is drawn to the smaller circle and the chord PA is drawn parallel to QT. The chord AT meets the smaller circle at B.

(i) Prove that $PABQ$ is a cyclic quadrilateral.

(ii) Copy the diagram into your writing booklet. Draw PQ', the other tangent from P to the smaller circle, and locate points A' and B' on the circles, corresponding to A and B respectively to form a quadrilateral $PA'B'Q'$ which you may assume to be cyclic.

(iii) If BQ and $B'Q'$ meet at R, prove that $PQRQ'$ is a cyclic quadrilateral.

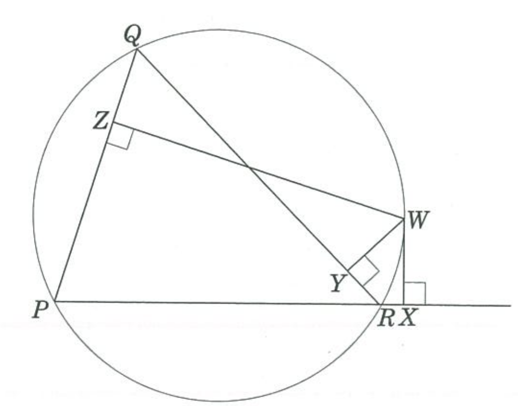
23



Triangle ABC is inscribed in a circle, with sides $AB = AC$. The chord AD intersects the side CB at E.

Prove that $AB^2 - AE^2 = BE \, . \, EC$.

24

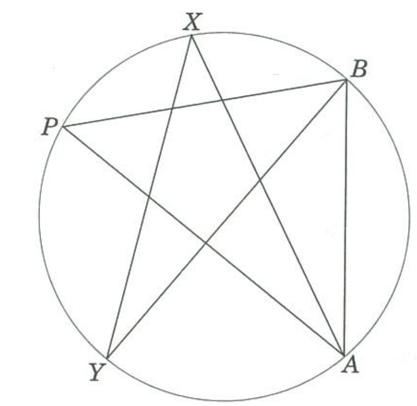


The triangle PQR is inscribed in a circle with W a point on the arc QR. WX is perpendicular to PR produced, and WZ is perpendicular to PQ.

(i) Show that $YRXW$ and $QZYW$ are cyclic quadrilaterals.

(ii) Show that Z, Y and X form a set of collinear points.

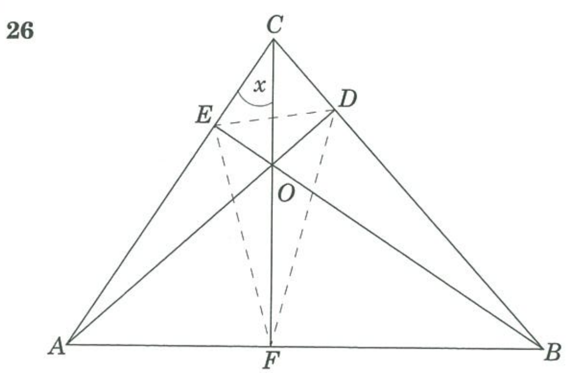
25



AB is a chord of fixed length of a circle. P is any point on the major arc. Angles PAB and PBA are bisected and their bisectors meet the circle at X and Y respectively.

Prove that XY is a fixed length.

26



In the figure above, ABC is an acute-angled triangle with AD, BE and CF perpendicular to the sides CB, CA and AB respectively.

AD, BE and CF are concurrent at O.
The points D, E and F form a triangle.

(i) By letting $\angle ECO = x$ and considering the Δs ABE, ACF, show that $\angle ECO = \angle FBO$.

(ii) Prove that the points C, E, O, D are concyclic.

(iii) Show that $\angle ECO = \angle EDO$.

(iv) Prove that AD bisects $\angle EDF$.

27 An acute-angled triangle has vertices L, M and N. The perpendicular from L meets the side MN at P and the perpendicular from N meets the side LM at Q. The lines QN and LP intersect at X.

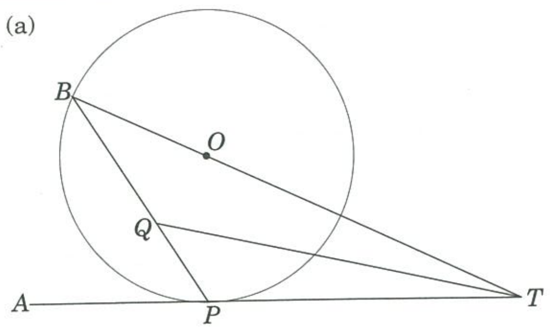
(i) Draw a diagram to show the above information.

(ii) Prove that $\angle PXM = \angle PQM$.

(iii) Prove that $\angle PXM = \angle LNM$.

(iv) Produce MX to intersect LN at R.
Prove that the intervals MR and LN are pendicular.

28 (a)



The line TPA is a tangent from an external point T to the circle, centre O. The line TO is produced to meet the circle at B. The bisector of the angle PTB meets the chord BP at Q.

Let $\angle PTQ = \alpha$ (in radians).

Show that (i) $\angle POT = \frac{\pi}{2} - 2\alpha$

(ii) $\angle PBT = \frac{\pi}{4} - \alpha$

(iii) $\angle PQT = \frac{\pi}{4}$

(b) ABC is a triangle inscribed in a circle. From an external point T a tangent TA is drawn making contact at A. A line parallel to AC cuts side BC at P.

(i) Draw a diagram to show the above information.

(ii) Prove $\angle BAT = \angle BPT$.

(iii) Prove $\angle ATB = \angle APC$.

29 If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ form an arithmetic progression, then the numbers a, b and c are said to be in harmonic progression and b is said to be the harmonic mean of a and c.

(i) Show that the numbers 5, 8 and 20 are in harmonic progression.

(ii) Show that the harmonic mean of a and c is equal to $\frac{2ac}{a+c}$.

(iii) If $a > 0$, $c > 0$, show that the harmonic mean $\frac{2ac}{a+c}$ is less than or equal to the geometric mean $\sqrt{ac}$.

30 (i) The point $P(x_1, y_1)$ lies on the curve with equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Find the equation of the tangent to the curve at the point P.

(ii) The tangent described in (i) meets the coordinate axes at S and T.

Prove that $OS + OT = a$.

31 The function $y = f(x)$ is given by

$$f(x) = 2\cos^{-1}\frac{x}{\sqrt{2}} - \sin^{-1}\left(1-x^2\right).$$

(i) Find the derivative $f'(x)$ of $y = f(x)$.

(ii) Hence, or otherwise, show that $f(x) = \frac{\pi}{2}$.

32 If $f(-x) = -f(x)$ then $f(x)$ is said to be an odd function. Use this definition to show that

$$f(x) = \log_e\left(x + \sqrt{x^2+1}\right)$$

is an odd function.

33 A series S is given by $S = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \cdots$

Find the limiting sum of the series as n increases without bound.

34 (i) Let $f(x) = \frac{1}{1+x^2}$.

(a) Prove that $f(x)$ is a decreasing function for all $x > 0$.

(b) Hence or otherwise prove that $\frac{1}{2} < \frac{1}{1+x^2} < 1$ for $0 < x < 1$.

(ii) Find all terms of the polynomial $Q(x)$ and the constant K if

$$x^4(1-x)^4 = \left(1+x^2\right)Q(x) + K.$$

(iii) Show that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$.

(iv) Hence or otherwise show that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

35 The function $y = f(x)$ is defined by

$$f(x) = \sqrt{2-\sqrt{x}}.$$

(i) State the domain of $y = f(x)$.

(ii) Show that $y = f(x)$ is a decreasing function and determine the range of $y = f(x)$.

(iii) Sketch the graph of $y = f(x)$ for the range and domain determined above.

(iv) Prove that $\displaystyle\int_0^4 \sqrt{2-\sqrt{x}}\,dx = \frac{32\sqrt{2}}{15}$.

36 A particle P is projected from the origin O with a velocity V at an angle α to the horizontal. From a point at a height h vertically above O another particle Q is projected horizontally at the same instant with a velocity U.

Both particles move in the same vertical plane. Air resistance is neglected for both particles. Let g m s^{-2} be the acceleration due to gravity.

(a) Write down the coordinates of both particles at time t, relative to the horizontal and vertical axes at the origin.

(b) Show that if the particles collide, then $V > U$. Find the time, in terms of h, V and U, at which the particles collide.

(c) Show that if the particles collide at the same point on the horizontal plane through 0, then $V^2 - U^2 = \frac{1}{2}gh$.

(d) The highest point of the trajectory of P has coordinates (X, Y), relative to the horizontal and vertical axes at 0.

Show that

(i) the angle of projection of P is given by

$$\tan^{-1}\left(\frac{2Y}{X}\right)$$

(ii) the speed of projection is given by

$$V^2 = \frac{g}{2Y}\left(4Y^2 + X^2\right).$$

37 A projectile sited on a horizontal plane is fired with an initial velocity U m s^{-1} at an acute angle θ and hits a target on the same horizontal plane at a distance of

$$\frac{U^2 \sin 2\theta}{g} \text{ metres}$$

from the point of projection.

If the angle of projection had been α or β, both acute, the projectile would have landed a metres short or b metres beyond the target respectively.

The range R along a horizontal plane may be assumed to be $R = \frac{U^2}{g}\sin 2\theta$.

Prove that

(i) $a + b = \frac{U^2}{g}(\sin 2\beta - \sin 2\alpha)$

(ii) $(a+b)\sin 2\theta = a\sin 2\beta + b\sin 2\alpha$.

38 (i) A particle is projected at an angle of elevation α from a point 0 with speed V. Taking the acceleration due to gravity to be g, and neglecting air resistance, write down the equations of motion of the particle in the horizontal and vertical directions.

(ii) Let $P(x, y)$ be a point on the trajectory of the particle. If coordinate axes are taken through 0, prove that

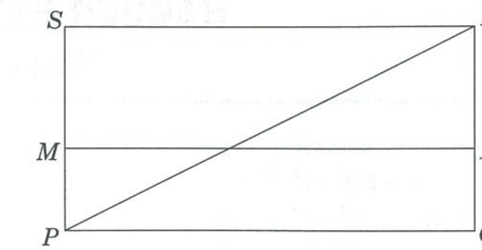
$$y = x\left(1 - \frac{x}{R}\right)\tan\alpha,$$

where R is the horizontal range.

(iii) If the distance between two points on the trajectory which are of the same height h above the horizontal is $2a$, show that

$$R(R - 4h\cot\alpha) = 4a^2.$$

39

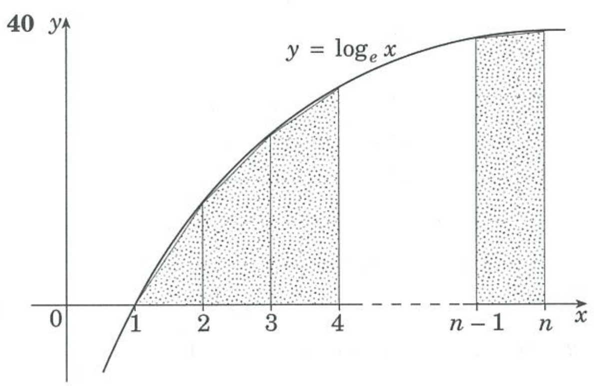


$PQRS$ is a rectangle of constant area A square metres. MN is drawn parallel to the sides PQ and SR. A diagonal is drawn from P to R.

Find, in terms of A, the dimensions of the rectangle so that the sum of the lengths of MN and PR is a minimum.

(You may assume that the second derivative of the expression you find for the sum of the lengths of MN and PR is positive.)

40



The diagram above shows the graph of the curve $y = \log_e x$.

(i) Show that the curve is concave down for all $x > 0$.

(ii) Show that the area under the curve $y = \log_e x$ from $x = 1$ to $x = n$ is given by $n\log_e n - n + 1$.

(iii) The interval $1 \le x \le n$ is divided into $(n-1)$ equal intervals. A series of trapezia are formed under the curve $y = \log_e x$, as shown in the diagram. By finding the area of the trapezia, show that

$$n! < \frac{en^{n+\frac{1}{2}}}{e^n}.$$

HARDER EXTENSION 1 TOPICS
Worked solutions

1 (i)
$$(x-y)^2 \geq 0$$
$$x^2-2xy+y^2 \geq 0$$
$$\therefore \quad x^2+y^2 \geq 2xy$$

(ii)
$$\begin{aligned}(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ &= x^3+y^3+3xy(x+y)\\ \therefore \quad x^3+y^3 &= (x+y)^3-3xy(x+y)\\ &= (x+y)\left[(x+y)^2-3xy\right]\\ &= (x+y)\left(x^2+2xy+y^2-3xy\right)\\ &= (x+y)\left(x^2+y^2-xy\right)\\ &\geq (x+y)(2xy-xy), \quad \text{using } x^2+y^2 \geq 2xy\\ &= (x+y)(xy)\end{aligned}$$
$$\therefore \quad x^3+y^3 \geq xy(x+y)$$

(ii) Alternative method:
$$\begin{aligned}\left(x^2+y^2\right)(x+y) &\geq 2xy(x+y)\\ x^3+xy^2+xy^2+y^3 &\geq 2xy(x+y)\\ x^3+y^3+xy(x+y) &\geq 2xy(x+y)\\ x^3+y^3 &\geq xy(x+y)\end{aligned}$$

(iii)
$$x^3+y^3 \geq xy(x+y)$$
$$y^3+z^3 \geq yz(z+y)$$
$$x^3+z^3 \geq xz(x+z)$$

Adding, $2x^3+2y^3+2z^3 \geq xy(x+y)+yz(z+y)+xz(x+z)$

i.e. $2\left(x^3+y^3+z^3\right) \geq xy(x+y)+yz(z+y)+xz(x+z)$

2 (i)
$$\begin{aligned}&\left(a^2-b^2\right)\left(c^2-d^2\right)-(ac-bd)^2\\ &= a^2c^2-a^2d^2-b^2c^2+b^2d^2-\left(a^2c^2-2abcd+b^2d^2\right)\\ &= a^2c^2-a^2d^2-b^2c^2+b^2d^2-a^2c^2+2abcd-b^2d^2\\ &= -a^2d^2-b^2c^2+2abcd\\ &= -\left(a^2d^2-2abcd+b^2c^2\right)\\ &= -(ad-bc)^2\\ &\leq 0 \quad \text{since } (ad-bc)^2 > 0\end{aligned}$$
$$\therefore \quad \left(a^2-b^2\right)\left(c^2-d^2\right) \leq (ac-bd)^2$$

(ii) In general, $\left(p^2-q^2\right)\left(r^2-s^2\right) \leq (pr-qs)^2$

Let $p=a,\ q=b,\ r=a^2,\ s=b^2$

$$\begin{aligned}\therefore \quad \left(a^2-b^2\right)\left(a^4-b^4\right) &\leq \left(a\cdot a^2-b\cdot b^2\right)^2\\ &= \left(a^3-b^3\right)^2\end{aligned}$$

i.e. $\left(a^2-b^2\right)\left(a^4-b^4\right) \leq \left(a^3-b^3\right)^2$

3 (i) Since $a+b+c=1$ and $a+b+c \geq 3\sqrt[3]{abc}$

$$\therefore \quad 1 \geq 3\sqrt[3]{abc}$$
$$\frac{1}{3} \geq \sqrt[3]{abc}$$

continued . . .

$$\frac{1}{\sqrt[3]{abc}} \geq 3$$

Let $x = \frac{1}{a},\ y = \frac{1}{b},\ z = \frac{1}{c}$.

Since $x+y+z \geq 3\sqrt[3]{xyz}$

$$\begin{aligned}\frac{1}{a}+\frac{1}{b}+\frac{1}{c} &\geq 3\sqrt[3]{\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\left(\frac{1}{c}\right)}\\ &\geq 3\times\frac{1}{\sqrt[3]{abc}}\\ &\geq 3\times 3\\ &\geq 9\end{aligned}$$

(ii)
$$\begin{aligned}\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) &= \left(\frac{1-a}{a}\right)\left(\frac{1-b}{b}\right)\left(\frac{1-c}{c}\right)\\ &= \frac{b+c}{a}\cdot\frac{a+c}{b}\cdot\frac{a+b}{c}\\ &\geq \frac{2\sqrt{bc}}{a}\cdot\frac{2\sqrt{ac}}{b}\cdot\frac{2\sqrt{ab}}{c}, \quad \text{using } x+y \geq 2\sqrt{xy}\\ &= \frac{8\sqrt{a^2b^2c^2}}{abc}\\ &= 8\end{aligned}$$

$$\begin{aligned}a+b+c &= 1\\ \Rightarrow \quad 1-a &= b+c\\ 1-b &= a+c\\ 1-c &= a+b\end{aligned}$$

i.e. $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$

4 Let S_n be the statement $a_n = 2^{n-1}+1$

For $n=1$, S_1: $\quad a_1 = 2^{1-1}+1 = 1+1 = 2$

For $n=2$, S_2: $\quad a_2 = 2^{2-1}+1 = 2+1 = 3$

$\therefore$ S_n is true for $n = 1, 2$

Assume S_n is true for $n = k$

$\therefore$ S_k : $\quad a_k = 2^{k-1}+1$

$$\begin{aligned}S_{k+1}: \quad a_{k+1} &= 3a_{(k+1)-1}-2a_{(k+1)-2}\\ &= 3a_k-2a_{k-1}\\ &= 3\left(2^{k-1}+1\right)-2\left(2^{(k-1)-1}+1\right)\\ &= 3\cdot2^{k-1}+3-2\cdot2^{k-2}-2\\ &= 3\cdot2^{k-1}-2^{k-1}+1\\ &= 2\cdot2^{k-1}+1\\ &= 2^k+1\\ &= 2^{(k+1)-1}+1\end{aligned}$$

Use $a_n = 3a_{n-1}-2a_{n-2}$

$\therefore$ S_n is true for $n = k+1$

Since S_n is true for $n=1$, $n=2$, it is true for $n = 2+1 = 3$, and so on.

$\therefore$ $a_n = 2^{n-1}+1$ for all $n \geq 3$

5 (i) Consider $(2k+3)^2-\left[2\sqrt{(k+1)(k+2)}\right]^2$

$$\begin{aligned}&= 4k^2+12k+9-\left[4\left(k^2+3k+8\right)\right]\\ &= 4k^2+12k+9-4k^2-12k-8\\ &= 1\\ &> 0\end{aligned}$$

$\therefore$ $(2k+3)^2 > \left[2\sqrt{(k+1)(k+2)}\right]^2$

i.e. $2k+3 > 2\sqrt{(k+1)(k+2)}$

(ii) Assume $S_n = 1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$ is true for $n \geq 1$

When $n = 1$, LHS $= 1$

RHS $= 2\left(\sqrt{2}-1\right) < 1$

$\therefore$ true for $n = 1$

Assume S_n is true for $n = k$.

$$\therefore\ S_k = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}} > 2\left(\sqrt{k+1}-1\right)$$

$$\begin{aligned}
S_{k+1} &= 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}}\\
&= \frac{2\left(\sqrt{k+1}\right)\left(\sqrt{k+1}-1\right)+1}{\sqrt{k+1}}\\
&= \frac{2(k+1)-2\sqrt{k+1}+1}{\sqrt{k+1}}\\
&= \frac{2k+3-2\sqrt{k+1}}{\sqrt{k+1}} > \frac{2\sqrt{(k+1)(k+2)}-2\sqrt{k+1}}{\sqrt{k+1}}, \text{ from (i)}\\
&= \frac{\left(\sqrt{k+1}\right)2\left(\sqrt{k+2}-1\right)}{\sqrt{k+1}}\\
&= 2\left(\sqrt{k+2}-1\right)
\end{aligned}$$

i.e. $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{k+1}} > 2\left(\sqrt{(k+1)+1}-1\right)$ for $n = k+1$

Since the result is true for $n = 1$, it is also true for $n = 2$.
Since it is true for $n = 2$, it is true for $n = 2+1$, i.e. $n = 3$, and so on. Hence the result is true for all $n \geq 1$.

6 Assume $T_n = \left(\dfrac{1+\sqrt{5}}{2}\right)^n+\left(\dfrac{1-\sqrt{5}}{2}\right)^n$ is true.

When $n = 1$, $T_1 = \dfrac{1+\sqrt{5}}{2}+\dfrac{1-\sqrt{5}}{2} = \dfrac{1}{2}+\dfrac{\sqrt{5}}{2}+\dfrac{1}{2}-\dfrac{\sqrt{5}}{2} = 1$

When $n = 2$, $T_2 = \left(\dfrac{1+\sqrt{5}}{2}\right)^2+\left(\dfrac{1-\sqrt{5}}{2}\right)^2 = \dfrac{1+2\sqrt{5}+5}{4}+\dfrac{1-2\sqrt{5}+5}{4} = 3$

$\therefore$ T_n is true for $n = 1, 2$

Assume $T_k = T_{k-1}+T_{k-2}$ is true for $n = k$, $n = 3, 4, \ldots$

$$\begin{aligned}
\therefore\quad T_{k+1} &= T_k+T_{k-1}\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^k+\left(\frac{1-\sqrt{5}}{2}\right)^k+\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}+\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^k+\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}+\left(\frac{1-\sqrt{5}}{2}\right)^k+\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}+1\right)+\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3+\sqrt{5}}{2}\right)+\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{6+2\sqrt{5}}{4}\right)+\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{6-2\sqrt{5}}{4}\right)
\end{aligned}$$

$$\boxed{\begin{aligned}6+2\sqrt{5} &= 1+2\sqrt{5}+5\\ &= 1^2+2\sqrt{5}+\left(\sqrt{5}\right)^2\\ &= \left(1+\sqrt{5}\right)^2\end{aligned}}$$

continued . . .

$$\begin{aligned}
&= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^2+\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}
\end{aligned}$$

$\therefore$ T_n is true for $n = k+1$

Since T_1, T_2 are true for $n = 1, 2$, and T_n is true for $n = k+1$, it is true for $n = 2+1 = 3$ and so on.

$\therefore$ T_n is true for all $n = 1, 2, 3, \ldots$

7 Assume $S_n = 1.1!+2.2!+3.3!+\cdots+nn! = (n+1)!-1$ is true.

When $n = 1$, LHS $= 1.1! = 1$

RHS $= (1+1)!-1 = 2!-1 = 2-1 = 1$

$\therefore$ S_n is true for $n = 1$

Assume S_n is true for $n = k$

$$\begin{aligned}
\therefore\quad S_k &= 1.1!+2.2!+\cdots+kk! = (k+1)!-1\\
S_{k+1} &= 1.1!+2.2!+\cdots+kk!+(k+1)(k+1)!\\
&= (k+1)!-1+(k+1)(k+1)!\\
&= (k+1)!+(k+1)(k+1)!-1\\
&= (k+1)!\left[1+(k+1)\right]-1\\
&= (k+2)(k+1)!-1\\
&= (k+2)!-1\\
&= \left[(k+1)+1\right]!-1
\end{aligned}$$

$\therefore$ S_n is true for $n = k+1$

Since S_n is true for $n = 1$, it is true for $n = 1+1 = 2$.
Since it is true for $n = 2$, it is true for $n = 2+1 = 3$, and so on.
i.e. $1.1!+2.2!+3.3!+\cdots+n(n+1)! = (n+1)!-1$ is true for all $n \geq 1$

8 Assume the statement S_n: $\sin(n\pi+x) = (-1)^n\sin x$ is true.

For $n = 1$, LHS $= \sin(\pi+x) = -\sin x$

RHS $= (-1)^1\sin x = -\sin x$

$\therefore$ true for $n = 1$.

Assume S_n is true for all $n = k$

$$\begin{aligned}
\therefore\ S_k:\quad \sin(k\pi+x) &= (-1)^k\sin x\\
S_{k+1}:\quad \sin\left[(k+1)\pi+x\right] &= \sin(k\pi+\pi+x)\\
&= \sin\left[(k\pi+x)+\pi\right]\\
&= \sin(k\pi+x)\cos\pi+\cos(k\pi+x)\sin\pi\\
&= \left[(-1)^k\sin x\right](-1)+0\\
&= (-1)^{k+1}\sin x
\end{aligned}$$

$\boxed{\sin(A+B) = \sin A\cos B+\cos B\sin A}$

$\boxed{\sin\pi = 0}$

$\therefore$ S_n is true for $n = k+1$

Since S_n is true for $n = 1$, it is true for $n = 1+1 = 2$.
Since it is true for $n = 2$, it is true for $n = 2+1 = 3$, and so on.
i.e. $\sin(n\pi+x) = (-1)^n\sin x$ for all integral x

9 (i) $\alpha_{n+1} = \alpha_n^2+\alpha_n$, with $\alpha_1 = 1$

$$\begin{aligned}
\alpha_2 &= \alpha_1^2+\alpha_1 = 1^2+1 = 2\\
\alpha_3 &= \alpha_2^2+\alpha_2 = 2^2+2 = 6\\
\alpha_4 &= \alpha_3^2+\alpha_3 = 6^2+6 = 42
\end{aligned}$$

(ii) Assume $\alpha_{n+1}-1 = \sum_{k=1}^{n}\alpha_k^2$ is true for all $n \ge 1$

When $n = 1$, LHS $= \alpha_2 - 1 = 2-1 = 1$

RHS $= \sum_{k=1}^{1}\alpha_1^2 = \alpha_1^2 = 1$

$\therefore$ the statement is true for $n = 1$.

Assume $\alpha_{k+1}-1 = \sum_{k=1}^{n}\alpha_k^2$ is true for $n = k$

i.e. $\alpha_1^2+\alpha_2^2+\cdots+\alpha_k^2 = \alpha_{k+1}-1$

When $n = k+1$,
$$\begin{aligned}\alpha_1^2+\alpha_2^2+\cdots+\alpha_k^2+\alpha_{k+1}^2 &= \alpha_{k+1}-1+\alpha_{k+1}^2\\ &= \left(\alpha_{k+1}^2+\alpha_{k+1}\right)-1\\ &= \alpha_{k+2}-1, \quad \text{using } \alpha_{n+1} = \alpha_n^2+\alpha_n\end{aligned}$$

$\therefore$ $\alpha_{(k+1)+1}-1 = \sum_{k=1}^{n}\alpha_{k+1}^2$ which is true for $n = k+1$

Since the statement is true for $n = 1$, it is true for $n = 1+1$, i.e. $n = 2$.
Since it is true for $n = 2$, it is true for $n = 3$, and so on.

$\therefore$ $\alpha_{n+1}-1 = \sum_{k=1}^{n}\alpha_k^2$ for all $n \ge 1$

(iii)
$$\begin{aligned}\left(2\alpha_{n+1}+1\right)^2 &= 4\alpha_{n+1}^2+4\alpha_{n+1}+1\\ &= 4\alpha_{n+2}+1\end{aligned}$$

$$\begin{aligned}\left(2\alpha_n+1\right)^2+\left(2\alpha_{n+1}\right)^2 &= 4\alpha_n^2+4\alpha_n+1+4\alpha_{n+1}^2\\ &= 4\alpha_{n+1}+1+4\alpha_{n+1}^2\\ &= 4\alpha_{n+1}^2+4\alpha_n+1\\ &= 4\alpha_{n+2}+1\end{aligned}$$

$$\begin{aligned}&\underbrace{\left(2\alpha_1+1\right)^2+\left(2\alpha_2\right)^2}+\left(2\alpha_3\right)^2+\cdots+\left(2\alpha_n\right)^2+\left(2\alpha_{n+1}\right)^2\\ =\ &\underbrace{\left(2\alpha_2+1\right)^2+\left(2\alpha_3\right)^2}+\cdots+\left(2\alpha_n\right)^2+\left(2\alpha_{n+1}\right)^2\\ =\ &\underbrace{\left(2\alpha_3+1\right)^2+\left(2\alpha_4\right)^2}+\cdots+\left(2\alpha_n\right)^2+\left(2\alpha_{n+1}\right)^2\\ =\ &\left(2\alpha_4+1\right)^2+\cdots+\left(2\alpha_n\right)^2+\left(2\alpha_{n+1}\right)^2\\ =\ &\text{etc.}\\ =\ &\underbrace{\left(2\alpha_n\right)^2+\left(2\alpha_{n+1}\right)^2}\\ =\ &\left(2\alpha_{n+1}+1\right)^2\end{aligned}$$

$\therefore$ $\left(2\alpha_{n+1}+1\right)^2 = \left(2\alpha_n+1\right)^2+\sum_{k=2}^{n}\left(2\alpha_k\right)^2$

(v) $\alpha_5 = \alpha_4^2+\alpha_4 = \alpha_4\left(\alpha_4+1\right) = 42\times 43 = 1806$

Also, $\alpha_5 = 1+\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2$
$= 1^2+1^2+2^2+6^2+42^2$
$= 1806$

$\boxed{\alpha_5 - 1 = \sum_{k=1}^{4}\alpha_k^2}$

10 (i)
$$\begin{aligned}\tan\left(A+\frac{\pi}{2}\right) &= \tan\left[\pi-\left(\frac{\pi}{2}-A\right)\right]\\ &= -\tan\left(\frac{\pi}{2}-A\right)\\ &= -\frac{\sin\left(\frac{\pi}{2}-A\right)}{\cos\left(\frac{\pi}{2}-A\right)}\\ &= -\frac{\cos A}{\sin A}\\ &= -\cot A\end{aligned}$$

$\boxed{\text{Complements of sine, cosine}}$

(ii) Assume $\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n$ for $n \ge 1$

For $n = 1$, LHS $= \tan\left(\frac{3\pi}{4}\right) = -1$

RHS $= (-1)^1 = -1$

$\therefore$ the assumption is true for $n = 1$

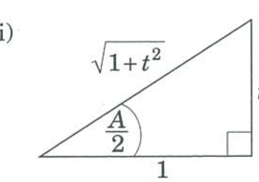
Assume $\tan\left[(2k+1)\frac{\pi}{4}\right] = (-1)^k$ is true for $n = k$

For $n = k+1$,
$$\begin{aligned}\tan\left\{\left[2(k+1)+1\right]\frac{\pi}{4}\right\} &= \tan\left[(2k+3)\frac{\pi}{4}\right]\\ &= \tan\left[(2k+1)\frac{\pi}{4}+2\cdot\frac{\pi}{4}\right]\\ &= \tan\left[(2k+1)\frac{\pi}{4}+\frac{\pi}{2}\right]\\ &= -\cot\left[(2k+1)\frac{\pi}{4}\right]\\ &= -1(-1)^k, \quad \text{using } \tan\left[(2k+1)\frac{\pi}{4}\right] = (-1)^k\\ &= (-1)^{k+1}\end{aligned}$$

$\therefore$ The assumption is true for $n = k+1$.
Since the assumption is true for $n = 1$, it is true for $n = 1+1$
i.e. $n = 2$. Since it is true for $n = 2$, it is true for $n = 2+1$
i.e. $n = 3$, and so on.

$\therefore$ $\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n$ for all $n \ge 1$

11 (i)



$\tan\frac{A}{2} = t = \frac{t}{1}$

$\cos\frac{A}{2} = \frac{1}{\sqrt{1+t^2}}$

$$\begin{aligned}\cos A = \cos 2\left(\frac{A}{2}\right) &= 2\cos^2\frac{A}{2}-1\\ &= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2-1\\ &= \frac{2}{1+t^2}-1\\ &= \frac{2-\left(1+t^2\right)}{1+t^2}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

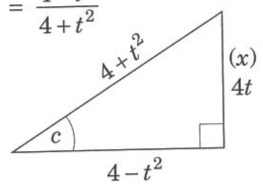
(ii) $\cos C = \dfrac{5\cos A+3}{3\cos A+5}$

$$= \frac{5\left(\frac{1-t^2}{1+t^2}\right)+3}{3\left(\frac{1-t^2}{1+t^2}\right)+5}$$

$$= \frac{5-5t^2+3+3t^2}{3-3t^2+5+5t^2}$$

$$= \frac{8-2t^2}{8+2t^2}$$

$$= \frac{4-t^2}{4+t^2}$$



$$x^2+\left(4-t^2\right)^2 = \left(4+t^2\right)^2$$
$$x^2+16-8t^2+t^4 = 16+8t^2+t^4$$
$$x^2 = 16t^2$$
$$x = 4t$$

$$\tan C = \frac{4t}{4-t^2}$$

Also, $\tan 2B = \dfrac{2\tan B}{1-\tan^2 B}$

$$= \frac{2\left(\frac{t}{2}\right)}{1-\left(\frac{t}{2}\right)^2}$$

$$= \frac{4t}{4-t^2}$$

$$= \tan C \qquad \therefore\ \angle C = 2\angle B$$

$$\boxed{\dfrac{\dfrac{5\left(1-t^2\right)+3\left(1+t^2\right)}{1+t^2}}{\dfrac{3\left(1-t^2\right)+5\left(1+t^2\right)}{1+t^2}} = \text{etc.}}$$

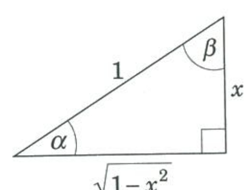
$$\boxed{\dfrac{\frac{2t}{2}}{1-\frac{t^2}{4}} = \dfrac{t}{\frac{4-t^2}{4}} = \dfrac{4t}{4-t^2}}$$

12 (i) Let $\sin^{-1}x = \alpha$

$\cos^{-1}x = \beta$

$\therefore\ x = \sin\alpha,\ x = \cos\beta$

and $\sin\beta = \cos\alpha = \sqrt{1-x^2}$



$$\sin\left(\sin^{-1}x-\cos^{-1}x\right) = \sin(\alpha-\beta)$$
$$= \sin x\cos\beta-\cos\alpha\sin\beta$$
$$= x\,.\,x-\sqrt{1-x^2}\sqrt{1-x^2}$$
$$= x^2-\left(1-x^2\right)$$
$$= 2x^2-1$$

(ii) $\sin^{-1}x-\cos^{-1}x = \sin^{-1}(1-x)$

$\therefore\ \sin\left(\sin^{-1}x-\cos^{-1}x\right) = 1-x$

$2x^2-1 = 1-x$ from (i)

$2x^2-x-2 = 0$

$$\therefore\quad x = \frac{-(-1)\pm\sqrt{(-1)^2-4(2)(-2)}}{2(1)}$$
$$= \frac{1\pm\sqrt{1+16}}{2}$$
$$= \frac{1\pm\sqrt{17}}{2}$$

13 (i) $\tan[(A+B)+C] = \dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$

$$= \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left(\dfrac{\tan A+\tan B}{1-\tan A\tan B}\right)\tan C}$$

$$= \frac{\dfrac{\tan A+\tan B+\tan C(1-\tan A\tan B}{1-\tan A\tan B}}{\dfrac{(1-\tan A\tan B)-(\tan A+\tan B)\tan C}{1-\tan A\tan B}}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-(\tan A\tan B+\tan B\tan C+\tan A\tan C)}$$

(ii) Let $A = B = C$

$$\therefore\ \tan(A+A+A) = \frac{\tan A+\tan A+\tan A-\tan A\tan A\tan A}{1-(\tan A\tan A+\tan A\tan A+\tan A\tan A)}$$

i.e. $\tan 3A = \dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

(iii) If A, B and C form the angles in a triangle, then

$\tan(A+B+C) = \tan 180° = 0$

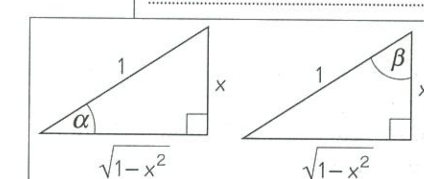
$$\therefore\ \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-(\tan A\tan B+\tan B\tan C+\tan A\tan C)} = 0$$

$\Rightarrow\ \tan A+\tan B+\tan C-\tan A\tan B\tan C = 0$

$\therefore\ \tan A+\tan B+\tan C = \tan A\tan B\tan C$

14 Let $\sin^{-1}x = \alpha,\ \cos^{-1}\alpha = \beta\ \rightarrow\ \sin\alpha = x,\ \cos\beta = x$

$\sin^{-1}x-\cos^{-1}x = \alpha-\beta = \sin^{-1}(3x-2)$



$$\therefore\quad \sin(\alpha-\beta) = \sin\left[\sin^{-1}(3x-2)\right]$$
$$\sin\alpha\cos\beta-\cos\alpha\sin\beta = 3x-2$$
$$x\,.\,x-\left(\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}\right) = 3x-2$$
$$x^2-\left(1-x^2\right) = 3x-2$$
$$\therefore\quad 2x^2-1 = 3x-2$$
$$2x^2-3x+1 = 0$$
$$(2x-1)(x-1) = 0$$
$$\therefore\ x = \frac{1}{2},\ 1$$

15 (i) $\operatorname{cosec} 2\theta+\cot 2\theta = \dfrac{1}{\sin 2\theta}+\dfrac{\cos 2\theta}{\sin 2\theta}$

$$= \frac{1+\cos 2\theta}{\sin 2\theta}$$
$$= \frac{1+\left(2\cos^2\theta-1\right)}{2\sin\theta\cos\theta}$$
$$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$$
$$= \frac{\cos\theta}{\sin\theta}$$
$$= \cot\theta$$

(ii) $\cot\theta = \operatorname{cosec} 2\theta + \cot 2\theta$

If $\theta = \frac{\pi}{8}$, $2\theta = \frac{\pi}{4}$:

$$\cot\frac{\pi}{8} = \operatorname{cosec}\frac{\pi}{4} + \cot\frac{\pi}{4} = \frac{1}{\sin\frac{\pi}{4}} + \frac{1}{\tan\frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} + \frac{1}{1} = \sqrt{2}+1$$

If $\theta = \frac{\pi}{12}$, $2\theta = \frac{\pi}{6}$:

$$\cot\frac{\pi}{12} = \frac{1}{\sin\frac{\pi}{6}} + \frac{1}{\tan\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{\sqrt{3}}} = 2+\sqrt{3}$$

$$\operatorname{cosec}\frac{2\pi}{15} = \cot\frac{\pi}{15} - \cot\frac{2\pi}{15}$$
$$\operatorname{cosec}\frac{4\pi}{15} = \cot\frac{2\pi}{15} - \cot\frac{4\pi}{15}$$
$$\operatorname{cosec}\frac{8\pi}{15} = \cot\frac{4\pi}{15} - \cot\frac{8\pi}{15}$$
$$\operatorname{cosec}\frac{16\pi}{15} = \cot\frac{8\pi}{15} - \cot\frac{16\pi}{15}$$

Adding both sides:
$$\text{LHS} = \cot\frac{\pi}{15} - \cot\frac{16\pi}{15} = \cot\frac{\pi}{15} - \cot\left(\pi + \frac{\pi}{15}\right) = \cot\frac{\pi}{15} - \cot\frac{\pi}{15} = 0$$

$\tan\theta = \tan(\pi+\theta)$
$\therefore\ \cot\theta = \cot(\pi+\theta)$

$$\therefore\ \operatorname{cosec}\frac{2\pi}{15} + \operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} = 0$$

16 To score 85% or better, the contestant must obtain at least 17 correct responses. If the contestant knows 10 correct answers, then he must obtain 7, 8, 9 or 10 of the remaining answers by tossing a coin.

$\therefore\ P(\text{at least } 85\%)$
$$= {}^{10}C_7\left(\tfrac{1}{2}\right)^7\left(\tfrac{1}{2}\right)^3 + {}^{10}C_8\left(\tfrac{1}{2}\right)^8\left(\tfrac{1}{2}\right)^2 + {}^{10}C_9\left(\tfrac{1}{2}\right)^9\left(\tfrac{1}{2}\right)^1 + {}^{10}C_{10}\left(\tfrac{1}{2}\right)^{10}\left(\tfrac{1}{2}\right)^0$$
$$= \left(\tfrac{1}{2}\right)^{10}\left({}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right)$$
$$= \left(\tfrac{1}{2}\right)^{10}(120+45+10+1)$$
$$= \frac{176}{1024}$$
$$= \frac{11}{64}$$

$P(\text{correct}) = P(\text{incorrect}) = P(\text{head}) = \frac{1}{2}$

17 (i) Consider the 3 people as a single unit.
Number of ways of arranging 3 people: $3! = 6$
Since the 3 people are treated as 1 unit, there are $n-3$ people still to be arranged.
$\therefore$ No. of ways of arranging $(n-3)$ people $= (n-3)!$
$\therefore$ Total no. of arrangements $= 6(n-3)!$

(ii) Since the 3 people are treated as 1 unit, there are $(n-1)$ 'people' to be arranged.
No. of ways of arranging $(n-1) = (n-1)!$

$$\therefore\ P(\text{3 sit together}) = \frac{6(n-3)!}{(n-1)!} = \frac{6(n-3)!}{(n-1)(n-2)(n-3)!} = \frac{6}{(n-1)(n-2)}$$

18 (i) Face value of 7 $\Rightarrow \{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)\}$
i.e. 6 possible outcomes.
Face value of 8 $\Rightarrow \{(2,6)(3,5)(4,4)(5,3)(6,2)\}$
i.e. 5 possible outcomes.

$P(A \text{ wins}) = \frac{5}{36}$, $P(B \text{ wins}) = \frac{6}{36} = \frac{1}{6}$

$$P(\text{neither wins}) = 1 - P(\text{both win}) = 1 - \left(\frac{5+6}{36}\right) = \frac{25}{36}$$

$\therefore\ P(A \text{ wins on first throw}) = \frac{5}{36}$

(ii) $P(A \text{ wins before 6th throw})$
$$= P\{A(1)+A(2)+\cdots+A(5)\}$$
$$= P(A)+P(\bar{A}A)+P(\bar{A}\bar{A}A)+P(\bar{A}\bar{A}\bar{A}A)+P(\bar{A}\bar{A}\bar{A}\bar{A}A)$$
$$= \frac{5}{36} + \left(\frac{25}{36}\right)\left(\frac{5}{36}\right) + \left(\frac{25}{36}\right)^2\left(\frac{5}{36}\right) + \left(\frac{25}{36}\right)^3\left(\frac{5}{36}\right) + \left(\frac{25}{36}\right)^4\left(\frac{5}{36}\right)$$
$$= \frac{5}{36}\left\{1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \left(\frac{25}{36}\right)^3 + \left(\frac{25}{36}\right)^4\right\}$$
$$= \frac{5}{36} \times \frac{36}{11}\left[1 - \left(\frac{25}{36}\right)^5\right]$$
$$= 0{\cdot}38 \text{ (2 d.p.)}$$

$A(1)$ – A wins on 1st throw
$A(2)$ – A wins on 2nd throw
$\vdots$
$A(5)$ – A wins on 5th throw

a GP with $S_4 = \left[\dfrac{1-\left(\frac{25}{36}\right)^5}{1-\frac{25}{36}}\right] = \dfrac{1-\left(\frac{25}{36}\right)^5}{\frac{11}{36}}$

(iii) $P(A \text{ wins eventually})$
$$= P(A)+P(\bar{A}A)+P(\bar{A}\bar{A}A)+\cdots+P(\bar{A}\bar{A}\ldots\bar{A}A)$$
$$= \frac{5}{36} + \left(\frac{25}{36}\right)\left(\frac{5}{6}\right) + \cdots + \left(\frac{25}{36}\right)^{n-1}\left(\frac{5}{6}\right), \quad n\to\infty$$
$$= \frac{5}{36}\left[1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \cdots + \left(\frac{25}{36}\right)^{n-1}\right], \quad n\to\infty$$
$$= \frac{5}{36}\left(\frac{1}{1-\frac{25}{36}}\right)$$
$$= \frac{5}{36} \times \frac{36}{11}$$
$$= \frac{5}{11}$$
$$= 0{\cdot}45$$

Sum to infinity

19 (i) Let team players be identified as A_1, A_2, A_3, A_4, and B_1, B_2, B_3, B_4

A_1 can pair off against 4 players from team B
A_2 can pair off against 3 remaining players from team B
A_3 can pair off against 2 remaining players from team B
A_4 can pair off against 1 remaining player from team B

$\therefore$ No. of pairings: $4\times3\times2\times1 = 24$

(ii) Each game has 3 outcomes — win for A, win for B, draw for A and B.

$\therefore$ In a particular arrangement of pairs, there are 3 possible outcomes for each pair, hence the four matches can have $3\times3\times3\times3 = 81$ outcomes.

(iii) Each team obtains 2 points if the 4 games give results as follows:

2 wins for A, 2 wins for B $\rightarrow$ ${}^4C_2 = 6$ ways
1 win A, 1 win B, 2 draws $\rightarrow$ ${}^4C_2\times 2 = 12$ ways
4 draws $\rightarrow$ 1 way

No. ways of equal points $= 6+12+1 = 19$ ways

$\therefore$ $P(\text{drawn tournament}) = \dfrac{19}{81}$

20 (i) $P(\text{success on 3rd attempt})$

$$= P(\text{fail 1st})\times P(\text{fail 2nd})\times P(\text{successful 3rd})$$
$$= \frac{1}{2}\times\frac{2}{3}\times\frac{1}{4}$$
$$= \frac{1}{12}$$

Let L = light, D = dark

1st attempt	1 L 1 D
2nd attempt	2 L 1 D
3rd attempt	3 L 1 D

(ii) $P(\text{success on } n\text{th attempt})$

$$= P(\text{fail 1st})\times P(\text{fail 2nd})\times \cdots \times P(\text{success } n\text{th})$$
$$= \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\cdots\times\frac{n-1}{n}\times\frac{1}{n+1}$$
$$= \frac{1}{n(n+1)}$$

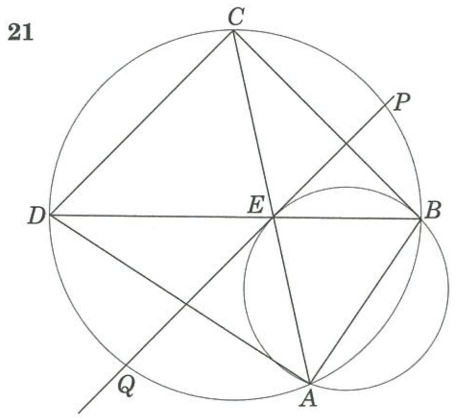
(iii) $P(\text{success before } (n+1)\text{th attempt})$

$$= P(\text{success 1st}) + P(\text{success 2nd}) + \cdots + P(\text{success } (n+1)\text{th})$$
$$= \frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{n(n+1)}$$
$$= \frac{n}{n+1}, \quad \text{using } \sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$$

(iv) As $n\to\infty$, $\quad \dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}} \to 1$

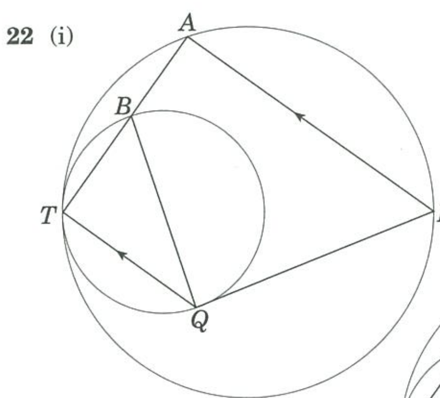
Certainty of success increases with each additional attempt.

21



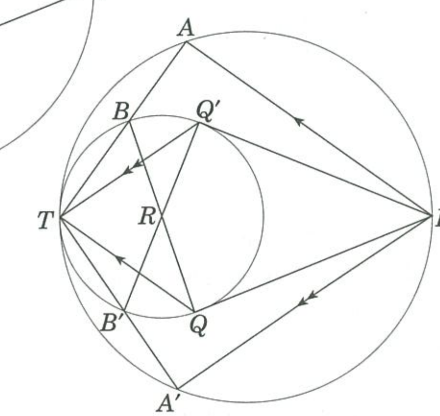
$\angle PEB = \angle BAE$ (alt. segment)
$\angle BAE = \angle EDC$ ($\angle$s on the same arc BC)
$\therefore$ $\angle PEB = \angle EDC$
$\therefore$ $PQ \parallel CD$ (equal corresp. $\angle$s)

22 (i)



$\angle BQP = \angle BTQ$ ($\angle$ in alt. segment)
$\angle BTQ + \angle PAT = 180°$ (coint. $\angle$s, $TQ \parallel AP$)
$\angle BQP + \angle PAT = 180°$
$\therefore$ $PABQ$ is a cyclic quadrilateral

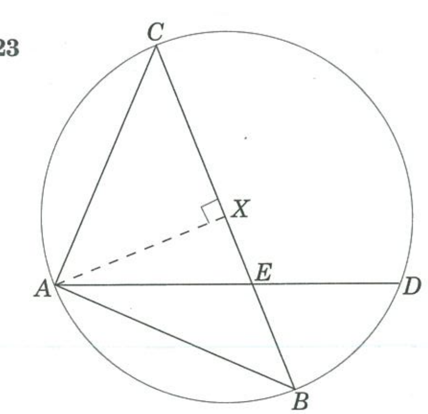
(ii)



(iii) $\angle PAT = 180° - \angle PQR$ (cyclic quad. $PABQ$)
$\angle PA'T = 180° - \angle PAT$ (cyclic quad. $PATA'$)
$= 180° - (180° - \angle PQR)$
$= \angle PQR$
$\angle PQ'R = 180° - \angle PA'B'$ (cyclic quad. $PA'B'Q'$)
$\therefore$ $PQRQ'$ is cyclic (opp. $\angle$s supplementary)

$\boxed{\angle PA'B' \equiv \angle PA'T}$

23



Let AX be a perpendicular from A to side CB.

Since ΔABC is isosceles ($AB = AC$), then AX is the perpendicular bisector.

$\therefore$ $BX = CX$

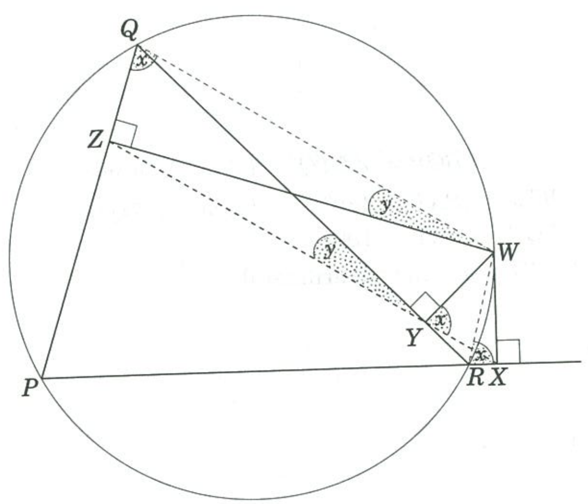
continued . . .

By Pythagoras' theorem: $AB^2 = BX^2 + AX^2$
$AE^2 = EX^2 + AX^2$

$$\begin{aligned} \therefore \quad AB^2 - AE^2 &= BX^2 - EX^2 \\ &= (BX+EX)(BX-EX) \\ &= (CX+EX)(BE), \quad \text{since } BX = CX \\ &= CE \cdot BE \end{aligned}$$

i.e. $AB^2 - AE^2 = BE \cdot EC$

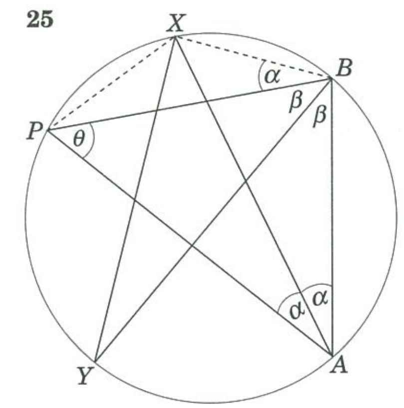
24



(i) $\angle WYR + \angle WXR = 180°$ (both 90°, given)
$\therefore$ $WYRX$ is a cyclic quadrilateral (opp. $\angle$s are supplementary)
$\angle QZW = \angle QYW = 90°$ (given)
Since they stand on a common chord QW, then $QZYW$ is a cyclic quadrilateral. (right $\angle$s in a semicircle, on diameter QW)

(ii) In the cyclic quadrilateral $XRYW$,
$\angle XRW = \angle XYW$ ($\angle$s on same chord XY)
In the cyclic quadrilateral $RPQW$,
$\angle XRW = \angle PQW$ (ext. $\angle$ of cyclic quad.)
In the cyclic quadrilateral $YZQW$,
$\angle ZYQ = \angle ZWQ$ ($\angle$s on same chord ZQ)
In the ΔZQW,
$\angle PQW + \angle ZWQ = 90°$ (since $\angle WZQ = 90°$)
$\therefore$ $\angle XYW + \angle ZYQ = 90°$
$\therefore$ $(\angle XYW + \angle ZYQ) + \angle WYQ = 90° + 90° = 180°$
$\therefore$ $\angle ZYX$ forms a straight angle
$\therefore$ X, Y, and Z are collinear.

25

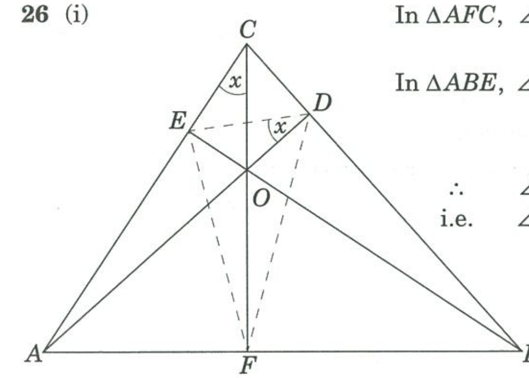


Let $\angle PAB = 2\alpha$ and $\angle PBA = 2\beta$.
Since AB is a constant length, all angles standing on AB and at the circumference will be constant.
Let $\angle APB = \theta$, a constant.
Join PX, BX.
In the ΔPAB,

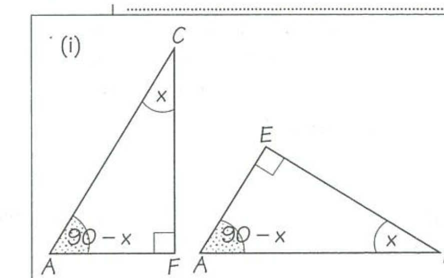
$$\begin{aligned} \angle PAB + \angle PBA + \angle APB &= 180° \\ \text{i.e.} \quad 2\alpha + 2\beta + \theta &= 180° \\ \therefore \quad \alpha + \beta &= \frac{180° - \theta}{2} \\ &= 90° - \frac{\theta}{2} \\ &= \text{a constant} \end{aligned}$$

Since $\angle YBX = \alpha + \beta$ is a constant, and an angle at the circumference, it is subtended by a chord (XY) which is constant in length.
$\therefore$ XY is a fixed length

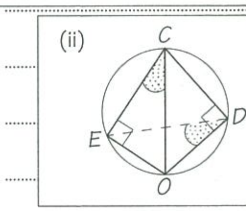
26 (i)



In ΔAFC, $\angle FAC = 90° - \angle FCE$ ($\angle$ sum of right Δ)
$= 90° - x$
In ΔABE, $\angle ABE = 90° - \angle BAE$ ($\angle$ sum of right Δ)
$= 90° - (90 - x)$ ($\angle BAE = \angle FAC$)
$= x$
$\therefore$ $\angle FCE = \angle ABE$
i.e. $\angle ECO = \angle FBO$



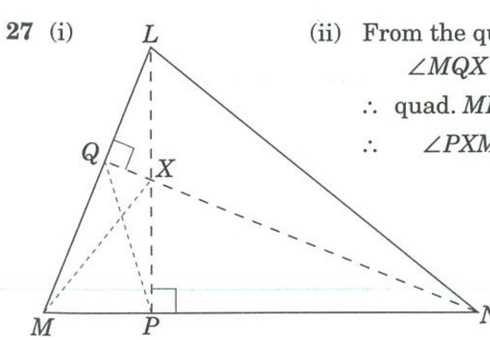
(ii) $\angle ODC = \angle CEO = 90°$ ($AD \perp CB$, $BE \perp AC$)
$\therefore$ C, E, O, D are concyclic (opp. $\angle$s supplementary)



(iii) $\angle ECO = \angle EDO$ ($\angle$s on the same arc EO in a cyclic quad.)

(iv) The points B, D, O, F are concyclic (opp. $\angle$s supplementary)
$\therefore$ $\angle ODF = \angle FBO$
$= \angle ECO$, from (i)
$= \angle EDO$, from (ii)
$\therefore$ OD bisects $\angle EDF$
i.e. AD bisects $\angle EDF$

27 (i)



(ii) From the quadrilateral $MPXQ$,
$\angle MQX = \angle MPX = 90°$ (given)
$\therefore$ quad. $MPXQ$ is cyclic (opp. $\angle$s supplementary)
$\therefore$ $\angle PXM + \angle PQM = 180°$ ($\angle$s at the circumference on the same chord MP)

(iii) From the quadrilateral $LQPN$,

$\angle LQN = \angle LPN = 90°$ (given)

$\therefore$ $LQPN$ is a cyclic quad. ($\angle$ in a semicircle, diameter LN)

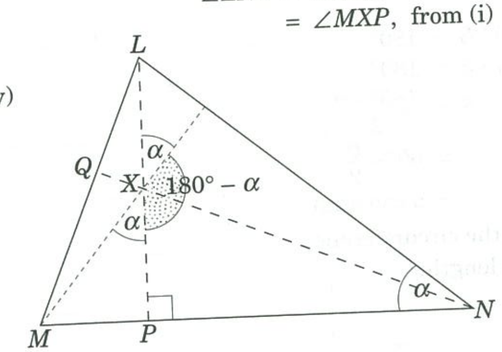
$\therefore$ $\angle LNP + \angle LQP = 180°$ (opp. $\angle$s of cyclic quad. are supplementary)

Also, $\angle MQP + \angle LQP = 180°$ (straight $\angle$)

$\angle LNP = \angle MQP$

$= \angle MXP$, from (i)

(iv)



$\angle LXR = \angle PXM$ (vert. opp. $\angle$s)

$\angle PXR = 180 - \angle LXR$ (straight $\angle$)

$\therefore$ $\angle PXR + \angle PNR = 180 - \angle LXR + \angle PNR$

$= 180°$, since $\angle LXR = \angle PXM = \angle PNR$

$\therefore$ $PXRN$ is a cyclic quadrilateral (opp. $\angle$s supplementary)

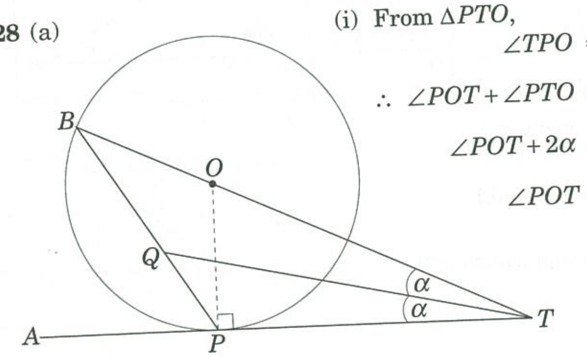
$\therefore$ $\angle XRN + \angle XPN = 180°$ (opp. $\angle$s of cyclic quad. are supplementary)

$\angle XRN + 90° = 180°$

$\angle XRN = 90°$

i.e. $MR \perp LN$

28 (a)



(i) From ΔPTO,

$\angle TPO = \frac{\pi}{2}$ (radius $OP \perp$ tangent PT)

$\therefore$ $\angle POT + \angle PTO = \frac{\pi}{2}$

$\angle POT + 2\alpha = \frac{\pi}{2}$

$\angle POT = \frac{\pi}{2} - 2\alpha$

(ii) From ΔPBT,

$\angle PBT + \angle BTP + \angle TPB = \pi$ ($\angle$ sum of a Δ)

$\angle PBT + 2\alpha + \left(\frac{\pi}{2} + \angle BPO\right) = \pi$

$\therefore$ $2\angle PBT + 2\alpha + \frac{\pi}{2} = \pi$, since $\angle BPO = \angle PBO$ (eq. radius of isos. ΔOBP)

$2\angle PBT = \pi - \frac{\pi}{2} - 2\alpha$

$2\angle PBT = \frac{\pi}{2} - 2\alpha$

$\angle PBT = \frac{\pi}{4} - \alpha$

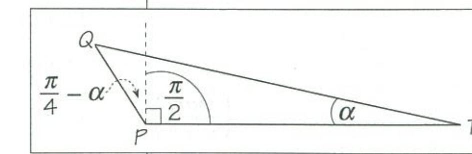
(iii) From ΔPQT, $\angle PQT + \angle QPT + \angle PTQ = \pi$

$\angle PQT + (\angle QPO + \angle OPT) + \alpha = \pi$

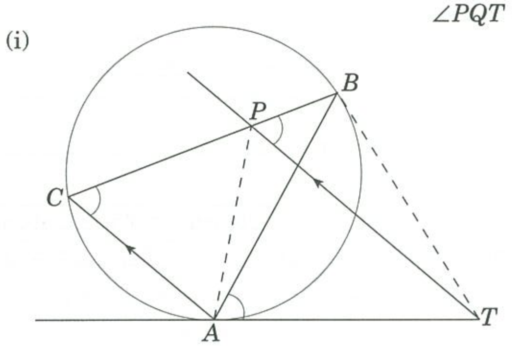
$\angle PQT + \left(\frac{\pi}{4} - \alpha + \frac{\pi}{2}\right) + \alpha = \pi$

$\angle PQT + \frac{3\pi}{4} = \pi$

$\angle PQT = \frac{\pi}{4}$



(b) (i)



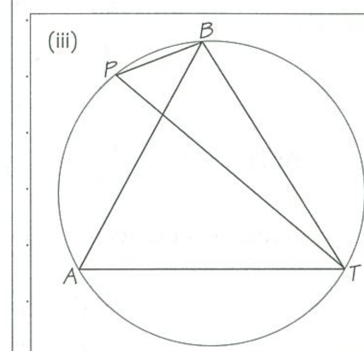
(ii) $\angle BAT = \angle ACB$ ($\angle$ in alt. segment)

But $\angle BPT = \angle ACB$ (corresp. $\angle$s, $PT \| CA$)

$\angle BAT = \angle BPT$

(iii) Since $\angle BAT = \angle BPT$, and are subtended by the same interval BT, then $BPAT$ is a cyclic quadrilateral (equal angles BAT and BPT on the same arc BT)

$\therefore$ $\angle ATB = \angle APC$ (ext. $\angle$ of cyclic quad. equals int. opp. $\angle$)



29 (i) Consider $\frac{1}{5}, \frac{1}{8}, \frac{1}{20}$: $T_2 - T_1 = \frac{1}{8} - \frac{1}{5} = -\frac{3}{40}$

$T_3 - T_2 = \frac{1}{20} - \frac{1}{8} = -\frac{3}{40}$

Since the common difference indicates $\frac{1}{5}, \frac{1}{8}, \frac{1}{20}$ form an AP, then 5, 8, 20 are in harmonic progression.

(ii) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ $\Rightarrow$ $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

$\therefore$ $\frac{2}{b} = \frac{1}{c} + \frac{1}{a}$

$\frac{2}{b} = \frac{c+a}{ac}$

$\therefore$ $b = \frac{2ac}{a+c}$

(iii) Consider $\left(\frac{2ac}{a+c}\right)^2 - \left(\sqrt{ac}\right)^2 = \frac{4a^2c^2}{a^2+2ac+c^2} - ac$

$= \frac{4a^2c^2 - ac\left(a^2+2ac+c^2\right)}{a^2+2ac+c^2}$

$= \frac{2a^2c^2 - a^3c - ac^3}{(a+c)^2}$

$= \frac{-ac\left(a^2 - 2ac + c^2\right)}{(a+c)^2}$

continued . . .

$$= \frac{-ac(a-c)^2}{(a+c)^2}$$

≤ 0 since $a,\ c > 0,\ (a-c)^2,\ (a+c)^2 \ge 0$

$$\therefore \quad \left(\frac{2ac}{a+c}\right)^2 < \left(\sqrt{ac}\right)^2$$

$$\text{i.e.} \quad \frac{2ac}{a+c} < \sqrt{ac}$$

30 (i)

$$\sqrt{x}+\sqrt{y} = \sqrt{a}$$

$$x^{\frac{1}{2}}+y^{\frac{1}{2}} = a^{\frac{1}{2}}$$

$$\therefore \quad \frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

Use implicit differentiation

$$\therefore \quad \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$\text{At } (x_1, y_1) \quad \frac{dy}{dx} = -\frac{y_1^{1/2}}{x_1^{1/2}}$$

$$\text{Equation of tangent:} \quad \frac{y-y_1}{x-x_1} = -\frac{y_1^{1/2}}{x_1^{1/2}}$$

$$\begin{aligned}
\therefore \quad (y-y_1)x_1^{1/2} &= -(x-x_1)y_1^{1/2} \\
yx_1^{1/2}-y_1x_1^{1/2} &= -xy_1^{1/2}+x_1y_1^{1/2} \\
xy_1^{1/2}+yx_1^{1/2} &= x_1y_1^{1/2}+y_1x_1^{1/2} \\
&= x_1^{1/2}y_1^{1/2}\left(x_1^{1/2}+y_1^{1/2}\right) \\
&= a^{1/2}x_1^{1/2}y_1^{1/2} \\
&= (ax_1y_1)^{1/2}
\end{aligned}$$

$\sqrt{x_1}+\sqrt{y_1} = \sqrt{a}$
i.e. $x_1^{1/2}+y_1^{1/2} = a^{1/2}$

$$\text{i.e.} \quad \sqrt{y_1}\,x+\sqrt{x_1}\,y = \sqrt{ax_1y_1}$$

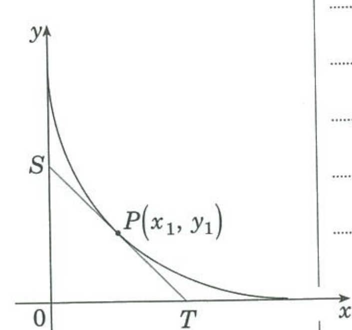
(ii) At $x = 0$ $\quad y = \sqrt{ay_1}$

At $y = 0$ $\quad x = \sqrt{ax_1}$

Coordinates of axis intersection:

$S\left(0, \sqrt{ay_1}\right)$, $T\left(\sqrt{ax_1}, 0\right)$

$$\begin{aligned}
\therefore \quad OS+OT &= \sqrt{ay_1}+\sqrt{ax_1} \\
&= \sqrt{a}\left(\sqrt{x_1}+\sqrt{y_1}\right) \\
&= \sqrt{a}\sqrt{a} \\
&= a
\end{aligned}$$



31 (i) $\quad y = 2\cos^{-1}\frac{x}{\sqrt{2}}-\sin^{-1}\left(1-x^2\right)$

$$\frac{dy}{dx} = 2\cdot\frac{-1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{1-\left(1-x^2\right)^2}}\times(-2x)$$

continued . . .

$$\begin{aligned}
&= \frac{-2}{\sqrt{2}\cdot\sqrt{1-\frac{x^2}{2}}}+\frac{2x}{\sqrt{1-\left(1-2x^2+x^4\right)}} \\
&= \frac{-2}{\sqrt{2-x^2}}+\frac{2x}{\sqrt{2x^2-x^4}} \\
&= \frac{-2}{\sqrt{2-x^2}}+\frac{2x}{x\sqrt{2-x^2}} \\
&= \frac{-2}{\sqrt{2-x^2}}+\frac{2}{\sqrt{2-x^2}} \\
&= 0
\end{aligned}$$

i.e. $f'(x) = 0$

(ii) Since $f'(x) = 0$ $\quad \therefore\ f(x) = k$, a constant

Since $f(x)$ a constant, any value of x for $0 \le x \le 1$ will give $y = f(x)$ its value. Choose any suitable value of x.

At $x = 0$ (say): $\quad f(x) = 2\cos^{-1}0-\sin^{-1}1 = 2\left(\frac{\pi}{2}\right)-\frac{\pi}{2} = \frac{\pi}{2}$

At $x = 1$ (say): $\quad f(x) = 2\cos^{-1}\frac{1}{\sqrt{2}}-\sin^{-1}(0) = 2\left(\frac{\pi}{4}\right)-0 = \frac{\pi}{2}$

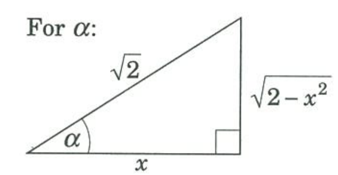
Alternative approach:

$$f(x) = 2\cos^{-1}\frac{x}{\sqrt{2}}-\sin^{-1}\left(1-x^2\right)$$

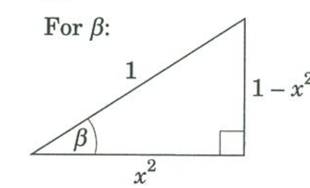
$$= 2\alpha-\beta \quad \text{where } \alpha = \cos^{-1}\frac{x}{\sqrt{2}},\ \beta = \sin^{-1}\left(1-x^2\right)$$

'Triangulate' for α, β in terms of x.

For α:



For β:



$$\frac{x}{\sqrt{2}} = \cos\alpha \qquad \frac{x^2}{1} = \cos\beta \text{ and } 1-x^2 = \sin\beta$$

$$\therefore \quad x^2 = 2\cos^2\alpha \qquad \therefore\ x^2 = \cos\beta \text{ and } \quad x^2 = 1-\sin\beta$$

$$\therefore \quad 2\cos^2\alpha = \cos\beta \text{ or } 2\cos^2\alpha = 1-\sin\beta$$

$$\therefore \quad 1+\cos 2\alpha = 1-\sin\beta$$

$\cos 2A = 2\cos^2 A-1$

$$\cos 2\alpha = -\sin\beta = \sin(-\beta)$$

$\sin A = \cos(90°-A)$

$$\begin{aligned}
\cos 2\alpha &= \cos\left[\frac{\pi}{2}-(-\beta)\right] \\
&= \cos\left(\frac{\pi}{2}+\beta\right)
\end{aligned}$$

$$\therefore \quad 2\alpha = \frac{\pi}{2}+\beta$$

$$2\alpha-\beta = \frac{\pi}{2}$$

$$\text{i.e.} \quad 2\cos^{-1}\frac{x}{\sqrt{2}}-\sin^{-1}\left(1-x^2\right) = \frac{\pi}{2}$$

32 $\quad f(x) = \log_e\left(x+\sqrt{x^2+1}\right)$

$$\begin{aligned}
f(-x) &= \log_e\left(-x+\sqrt{(-x)^2+1}\right) \\
&= \log_e\left(-x+\sqrt{x^2+1}\right)
\end{aligned}$$

continued . . .

$$-f(x) = (-1)\log_e\left(x+\sqrt{x^2+1}\right)$$
$$= \log_e\left(x+\sqrt{x^2+1}\right)^{-1}$$
$$= \log_e\left(\frac{1}{x+\sqrt{x^2+1}}\right)$$
$$= \log_e\left(\frac{1}{x+\sqrt{x^2+1}}\times\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right)$$

Rationalise the denominator

$$= \log_e\left[\frac{x-\sqrt{x^2+1}}{x^2-\left(x^2+1\right)}\right]$$
$$= \log_e\left(\frac{x-\sqrt{x^2+1}}{-1}\right)$$
$$= \log_e\left(-x+\sqrt{x^2+1}\right)$$
$$= f(-x)$$

$\therefore f(x) = \log_e\left(x+\sqrt{x^2+1}\right)$ is odd.

33 $S = \frac{1}{5}+\frac{2}{5^2}+\frac{3}{5^3}+\cdots$ can be written as a 'triangular sum' involving rows of limiting sums.

$$S = \frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\cdots \qquad \cdots S_1$$
$$\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\cdots \qquad \cdots S_2$$
$$\frac{1}{5^3}+\frac{1}{5^4}+\cdots \qquad \cdots S_3$$
$$\frac{1}{5^4}+\cdots \qquad \cdots S_4$$

etc.

$$S_n = \frac{a}{1-r}$$
$$S_1 = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}$$
$$S_2 = \frac{\frac{1}{5^2}}{1-\frac{1}{5}} = \frac{\frac{1}{5^2}}{\frac{4}{5}} = \frac{1}{4}\cdot\frac{1}{5}$$
$$S_3 = \frac{\frac{1}{5^3}}{1-\frac{1}{5}} = \frac{\frac{1}{5^3}}{\frac{4}{5}} = \frac{1}{4}\cdot\frac{1}{5^2}$$
$$\therefore\quad S_n = \frac{1}{4}\cdot\frac{1}{5^{n-1}}$$
$$\therefore\quad S = \frac{1}{4}+\frac{1}{4}\cdot\frac{1}{5}+\frac{1}{4}\cdot\frac{1}{5^2}+\cdots$$
$$= \frac{1}{4}\left(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots\right)$$
$$= \frac{1}{4}\cdot\frac{1}{1-\frac{1}{5}} = \frac{1}{4}\cdot\frac{1}{\frac{4}{5}} = \frac{1}{4}\cdot\frac{5}{4} = \frac{5}{16}$$

i.e. $S = \frac{5}{16} = 0{\cdot}3125$

34 (i) (a) A function $y = f(x)$ is decreasing if $f'(x) < 0$ for $x > 0$.

$$f(x) = \frac{1}{1+x^2} = \left(1+x^2\right)^{-1}$$
$$f'(x) = -\left(1+x^2\right)^{-2}(2x)$$
$$= -\frac{2x}{\left(1+x^2\right)^2} > 0 \quad \text{for } x > 0$$

$\therefore\ y = f(x)$ is a decreasing function for $x > 0$.

(b) For $x > 0$, $\quad 1+x^2 > 1$

$$\therefore\quad \frac{1}{1+x^2} < 1$$

For $x < 1$, $\quad 1+x^2 < 2$

$$\therefore\quad \frac{1}{1+x^2} > \frac{1}{2}$$
$$\text{i.e.}\quad \frac{1}{2} < \frac{1}{1+x^2}$$
$$\therefore\ \frac{1}{2} < \frac{1}{1+x^2} < 1 \quad \text{for } 0 < x < 1$$

(ii) $$x^4(1-x)^4 = x^4\left(1-4x+6x^2-4x^3+x^4\right)$$
$$= x^8-4x^7+6x^6-4x^5+x^4$$
$$= \left(1+x^2\right)Q(x)+K$$

Divide $x^4(1-x)^4$ by $1+x^2$:

$$\begin{array}{rl}
 & x^6-4x^5\qquad\qquad +4x^2\qquad -4\ \ +\text{ Rem. }4\\
x^2+1 \big) & x^8-4x^7+6x^6-4x^5+x^4+0x^3+0x^2+0x+0\\
 & x^8\qquad +6x^6\\
\hline
 & -4x^7\qquad -4x^5\\
 & -4x^7\qquad -4x^5\\
\hline
 & 4x^4+0x^3+0x^2\\
 & 4x^4\qquad +4x^2\\
\hline
 & -4x^2\qquad +0\\
 & -4x^2\qquad -4\\
\hline
 & 4
\end{array}$$

i.e. $x^4(1-x)^4 = \left(x^2+1\right)\left(x^6-4x^5+4x^2-4\right)+4$

$\therefore\quad K = 4$

(iii) $$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx$$
$$= \left[\frac{x^7}{7}-\frac{4x^6}{6}+\frac{5x^5}{5}-\frac{4x^3}{3}+4x-4\tan^{-1}x\right]_0^1$$
$$= \left(\frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-4\tan^{-1}1\right)-0$$
$$= \left(5+\frac{1}{7}-\frac{6}{3}-4\cdot\frac{\pi}{4}\right)$$
$$= 3\frac{1}{7}-\pi$$
$$= \frac{22}{7}-\pi$$

(iv) $\frac{1}{2} < \frac{1}{1+x^2} < 1$ from (i) (b)

$$\frac{x^4(1-x)^4}{2} < \frac{x^4(1-x)^4}{1+x^2} < x^4(1-x)^4$$

$$\therefore \int_0^1 \frac{x^4(1-x)^4}{2}\,dx < \int_0^1 x^4(1-x)^4\,dx < \int_0^1 x^4(1-x)^4\,dx$$

Now, $\int_0^1 x^4(1-x)^4\,dx = \int_0^1 \left(x^8-4x^7+6x^6-4x^5+x^4\right)dx$

$$= \left[\frac{x^9}{9}-\frac{4x^8}{8}+\frac{6x^7}{7}-\frac{4x^6}{6}+\frac{x^5}{5}\right]_0^1$$

$$= \left(\frac{1}{9}-\frac{1}{2}+\frac{6}{7}-\frac{2}{3}+\frac{1}{5}\right)-0$$

$$= \frac{70-315+540-420+126}{2.5.7.9}$$

$$= \frac{1}{630}$$

$$\therefore \frac{1}{2}\cdot\frac{1}{630} < \frac{22}{7}-\pi < \frac{1}{630}$$

$$-\frac{1}{630} < \pi-\frac{22}{7} < -\frac{1}{1260}$$

$$\frac{22}{7}-\frac{1}{630} < \pi < \frac{22}{7}-\frac{1}{1260}$$

$$a < b < c$$
$$\Rightarrow c > b > a$$
$$\Rightarrow -c < -b < -a$$

35 (i) $f(x) = \sqrt{2-\sqrt{x}}$

The following conditions apply for real $f(x)$:

$2-\sqrt{x} \ge 0$ and $x \ge 0$

$2 \ge \sqrt{x}$

$4 \ge x$

$\therefore$ Domain $\{0 \le x \le 4\}$

(ii) $f(x) = \left(2-x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}\left(2-x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \times -\frac{1}{2}x^{-\frac{1}{2}}$$

$$= -\frac{1}{4}\cdot\frac{1}{\left[\left(2-x^{\frac{1}{2}}\right)x\right]^{\frac{1}{2}}}$$

$$= -\frac{1}{4\sqrt{x}\sqrt{2-\sqrt{x}}} < 0 \quad \text{for } 0 \le x \le 4$$

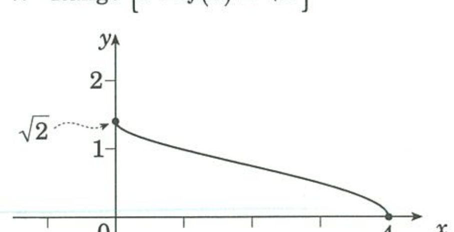
$\therefore$ $f'(x)$ is decreasing since $f'(x) < 0$.

At $x = 0$, $f(0) = \sqrt{2}$

At $x = 4$, $f(4) = \sqrt{2-\sqrt{4}} = 0$

$\therefore$ Range $\left\{0 \le f(x) \le \sqrt{2}\right\}$

(iii)



(iv) Let $u = 2-\sqrt{x}$ When $x = 0$, $u = 2$

$\sqrt{x} = 2-u$ When $x = 4$, $u = 0$

$x = (2-u)^2$

$\therefore$ $dx = -2(2-u)\,du$

$$\int_0^4 \sqrt{2-\sqrt{x}}\,dx = \int_2^0 \sqrt{u}(-2)(2-u)\,dx$$

$$= 2\int_0^2 \left(2u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)dx$$

$$= 2\left[\frac{2u^{\frac{3}{2}}}{\frac{3}{2}}-\frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$= 2\left[\frac{4}{3}u\sqrt{u}-\frac{2}{5}u^2\sqrt{u}\right]_0^2$$

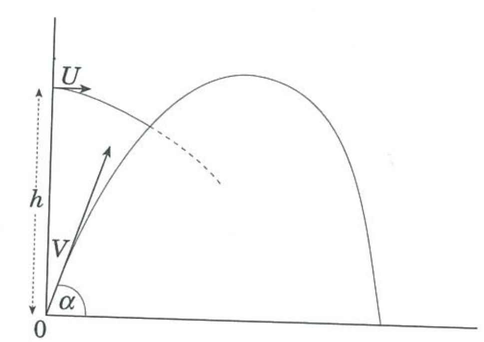
$$= 2\left(\frac{4}{3}\cdot 2\sqrt{2}-\frac{2}{5}\cdot 4\sqrt{2}\right)$$

$$= 2\sqrt{2}\left(\frac{8}{3}-\frac{8}{5}\right)$$

$$= 2\sqrt{2}\cdot\frac{16}{15}$$

$$= \frac{32\sqrt{2}}{15}$$

36



(a) For particle P:

$\ddot{x} = 0$ $\ddot{y} = -g$

$\dot{x} = V\cos\alpha$ $\dot{y} = -gt + V\sin\alpha$

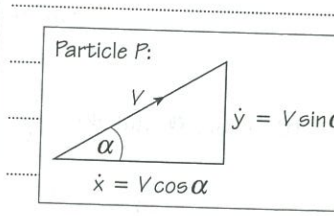
$x_P = Vt\cos\alpha$ $y_P = -\frac{1}{2}gt^2 + Vt\sin\alpha$

For particle Q:

$\ddot{x} = 0$ $\ddot{y} = -g$

$\dot{x} = U$ $\dot{y} = -gt$

$x_Q = Ut$ $y_Q = -\frac{1}{2}gt^2 + h$



(b) For P and Q to collide, $x_P = x_Q$

$\therefore$ $Vt\cos\alpha = Ut$

$V\cos\alpha = U$

$\cos\alpha = \frac{U}{V}$

$\therefore$ $0 < \frac{U}{V} < 1$ since $0 < \cos\alpha < 1$

$0 < U < V$

$\therefore$ $V > U$

continued . . .

Particles P and Q collide when $y_P = y_Q$

$$-\frac{1}{2}gt^2 + Vt\sin\alpha = -\frac{1}{2}gt^2 + h$$
$$\therefore \quad Vt\sin\alpha = h$$
$$t = \frac{h}{V\sin\alpha} = \frac{h}{V\sqrt{1-\frac{U^2}{V^2}}} = \frac{h}{\sqrt{V^2-U^2}}$$

$$\cos\alpha = \frac{U}{V}$$
$$\cos^2\alpha = \frac{U^2}{V^2}$$
$$\sin^2\alpha = 1-\frac{U^2}{V^2}$$
$$\sin\alpha = \sqrt{1-\frac{U^2}{V^2}}$$

(c) Along the plane through 0, $y_Q = 0$

$$-\frac{1}{2}gt^2 + h = 0$$
$$gt^2 = 2h$$
$$t^2 = \frac{2h}{g}$$
$$\frac{h^2}{V^2-U^2} = \frac{2h}{g}$$
$$\frac{V^2-U^2}{h^2} = \frac{g}{2h}$$
$$V^2-U^2 = \frac{1}{2}gh$$

(d) At the maximum height of P's trajectory $\dot{y}_P = 0$

$$\therefore \quad -gt + V\sin\alpha = 0$$
$$t = \frac{V\sin\alpha}{g}$$
$$\therefore \quad X = Vt\cos\alpha = \frac{V^2\sin\alpha\cos\alpha}{g} \quad —❶$$

Using $\dot{y}_P = -\frac{1}{2}gt^2 + Vt\sin\alpha$

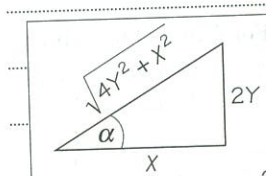
Also, $$Y = -\frac{1}{2}g\left(\frac{V\sin\alpha}{g}\right)^2 + V\cdot\frac{V\sin\alpha}{g}\cdot\sin\alpha = -\frac{V^2\sin^2\alpha}{2g} + \frac{V^2\sin^2\alpha}{g} = \frac{V^2\sin^2\alpha}{2g} \quad —❷$$

(i) From ❶ and ❷: $$\frac{Y}{X} = \frac{\frac{V^2\sin^2\alpha}{2g}}{\frac{V^2\sin\alpha\cos\alpha}{g}} = \frac{1}{2}\tan\alpha$$

$$\frac{Y}{X} = \frac{V^2\sin^2\alpha}{2g} \times \frac{g}{V^2\sin\alpha\cos\alpha} = \frac{\sin\alpha}{2\cos\alpha} = \frac{1}{2}\tan\alpha$$

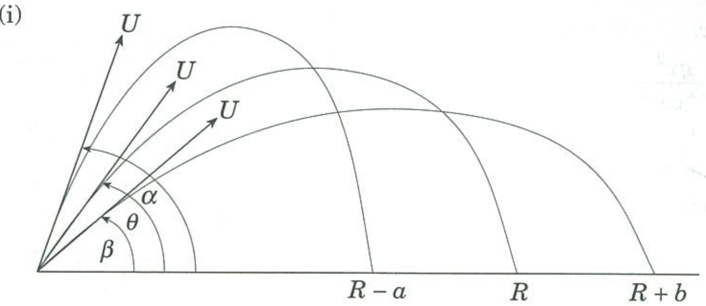
$$\therefore \quad \tan\alpha = \frac{2Y}{X}$$
$$\alpha = \tan^{-1}\left(\frac{2Y}{X}\right)$$

(ii) From ❷: $$V^2 = \frac{2gY}{\sin^2\alpha} = \frac{2gY}{\frac{4Y^2}{4Y^2+X^2}}, \quad \text{using } \tan\alpha = \frac{2Y}{X}$$
$$= \frac{2gY(4Y^2+X^2)}{4Y^2} = \frac{g}{2Y}(4Y^2+X^2)$$



$$\Rightarrow \sin\alpha = \frac{2Y}{\sqrt{4Y^2+X^2}}$$
$$\sin^2\alpha = \frac{4Y^2}{4Y^2+X^2}$$

37 (i)

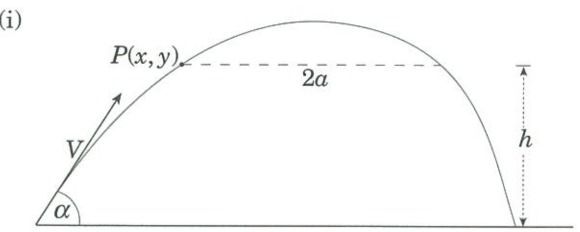


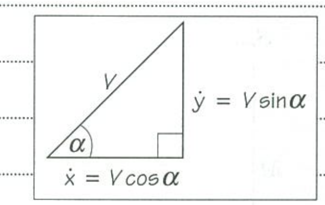
$$R = \frac{U^2}{g}\sin 2\theta$$
$$R+b = \frac{U^2}{g}\sin 2\beta \quad —❶$$
$$R-a = \frac{U^2}{g}\sin 2\alpha \quad —❷$$

Subtracting ❶ − ❷:
$$R+b-(R-a) = \frac{U^2}{g}\sin 2\beta - \frac{U^2}{g}\sin 2\alpha$$
$$R+b-R+a = \frac{U^2}{g}(\sin 2\beta - \sin 2\alpha)$$
i.e. $$a+b = \frac{U^2}{g}(\sin 2\beta - \sin 2\alpha)$$

(ii) Consider $$(a+b)R = aR+bR = aR+ab+bR-ab = a(R+b)+b(R-b)$$
$$\therefore (a+b)\frac{U^2}{g}\sin 2\theta = a\cdot\frac{U^2}{g}\sin 2\beta + b\cdot\frac{U^2}{g}\sin 2\alpha$$
i.e. $$(a+b)\sin 2\theta = a\sin 2\beta + b\sin 2\alpha$$

38 (i)





Horizontally:	Vertically:
$\dot{x} = V\cos\alpha$	$\ddot{y} = -g$
$x = Vt\cos\alpha$	$\dot{y} = -gt + V\sin\alpha$
	$y = Vt\sin\alpha - \frac{1}{2}gt^2$

(ii) The range, R, occurs when $y = 0$.

$$\therefore \quad Vt\sin\alpha = \frac{1}{2}gt^2$$
$$t = \frac{2V\sin\alpha}{g}$$
$$\therefore \quad R = V\cdot\frac{2V\sin\alpha\cos\alpha}{g} = \frac{2V^2\sin\alpha\cos\alpha}{g}$$

continued . . .

From (i), $x = Vt\cos\alpha \quad \therefore \quad t = \dfrac{x}{V\cos\alpha}$

$$\begin{aligned}
\therefore \quad y &= \frac{V\sin\alpha \cdot x}{V\cos\alpha} - \frac{gx^2}{2V^2\cos^2\alpha} \\
&= x\tan\alpha - \frac{x^2 g}{2V^2\cos^2\alpha} \\
&= x\tan\alpha - \frac{x^2\tan\alpha}{R} \\
&= x\tan\alpha\left(1 - \frac{x}{R}\right) \\
&= x\left(1 - \frac{x}{R}\right)\tan\alpha
\end{aligned}$$

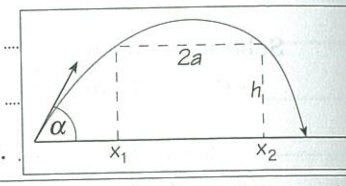
$$\begin{aligned}
R &= \frac{2V^2\sin\alpha\cos\alpha}{g} \\
\frac{g}{2V^2} &= \frac{\sin\alpha\cos\alpha}{R} \\
\frac{g}{2V^2\cos^2\alpha} &= \frac{\sin\alpha\cos\alpha}{R\cos^2\alpha} \\
&= \frac{\tan\alpha}{R}
\end{aligned}$$

(iii) At $y = h$,

$$\begin{aligned}
h &= x\left(1 - \frac{x}{R}\right)\tan\alpha \\
h &= x\tan\alpha - \frac{x^2}{R}\tan\alpha \\
Rh &= xR\tan\alpha - x^2\tan\alpha
\end{aligned}$$

$\therefore \ x^2\tan\alpha - (R\tan\alpha)x + Rh = 0$

The difference of the roots of $x^2\tan\alpha - (R\tan\alpha)x + Rh = 0$ is $2a$.

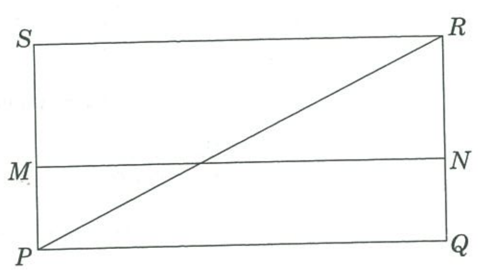


$$\begin{aligned}
x_2 - x_1 &= \frac{\left(-b + \sqrt{b^2 - 4ac}\right) - \left(-b - \sqrt{b^2 - 4ac}\right)}{2a} \\
&= \frac{2\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{\sqrt{b^2 - 4ac}}{a}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad x_2 - x_1 &= \frac{\sqrt{b^2 - 4ac}}{a} \\
2a &= \frac{\sqrt{(R\tan\alpha)^2 - 4(\tan\alpha)Rh}}{\tan\alpha} \\
2a\tan\alpha &= \sqrt{R^2\tan^2\alpha - 4Rh\tan\alpha} \\
4a^2\tan^2\alpha &= R^2\tan^2\alpha - 4Rh\tan\alpha \\
4a^2 &= R^2 - \frac{4Rh}{\tan\alpha}
\end{aligned}$$

i.e. $R(R - 4h\cot\alpha) = 4a^2$

39



Let the length PQ be ℓ and the breadth RQ be b.
Let L be the sum of MN and PR and A be the area of rectangle $PQRS$.

$MN = PQ \quad$ and $\quad PR = \sqrt{PQ^2 + RQ^2}$

$MN = \ell \quad$ and $\quad PR = \sqrt{\ell^2 + b^2}$

Also, $\quad L = MN + PR \quad$ and $\quad A = PQ \, . \, RQ$

i.e. $\quad L = \ell + \sqrt{\ell^2 + b^2} \quad$ and $\quad A = \ell b$

$$\begin{aligned}
&= \ell + \sqrt{\ell^2 + \frac{A^2}{\ell^2}} \\
&= \ell + \left(\ell^2 + A^2\ell^{-2}\right)^{\frac{1}{2}}
\end{aligned}$$

continued . . .

$$\begin{aligned}
\frac{dL}{d\ell} &= 1 + \frac{1}{2}\left(\ell^2 + A^2\ell^{-2}\right)^{-\frac{1}{2}} . \left(2\ell - 2A^2\ell^{-3}\right) \\
&= 1 + \frac{\ell - \dfrac{A^2}{\ell^3}}{\sqrt{\ell^2 + \dfrac{A^2}{\ell^2}}} = 0 \ \text{ for stationary points}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \frac{\ell - \dfrac{A^2}{\ell^3}}{\sqrt{\ell^2 + \dfrac{A^2}{\ell^2}}} &= -1 \\
\frac{\ell^4 - A^2}{\ell^3} &= -\sqrt{\frac{\ell^4 + A^2}{\ell^2}}
\end{aligned}$$

Squaring both sides:

$$\begin{aligned}
\frac{\ell^8 - 2\ell^4A^2 + A^4}{\ell^6} &= \frac{\ell^4 + A^2}{\ell^2} \\
\ell^8 - 2\ell^4A^2 + A^4 &= \frac{\ell^6\left(\ell^4 + A^2\right)}{\ell^2} \\
&= \ell^4\left(\ell^4 + A^2\right) \\
&= \ell^8 + \ell^4A^2
\end{aligned}$$

$$\begin{aligned}
\therefore \quad A^4 &= 3A^2\ell^4 \\
A^2 &= 3\ell^4 \\
\ell &= \sqrt[4]{\frac{A^2}{3}}
\end{aligned}$$

Since $A = \ell b$, then $\ell = \dfrac{A}{b}$.

From $A^2 = 3\ell^4$, $\ A^2 = 3\left(\dfrac{A}{b}\right)^4 = \dfrac{3A^4}{b^4}$

$\therefore \quad b^4 = \dfrac{3A^4}{A^2} = 3A^2$

$\therefore \quad b = \sqrt[4]{3A^2}$

Since it is assumed $\dfrac{d^2L}{d\ell^2} > 0$, then L is a minimum for $\ell = \sqrt[4]{\dfrac{A^2}{3}}$ and $b = \sqrt[4]{3A^2}$.

40 (i)

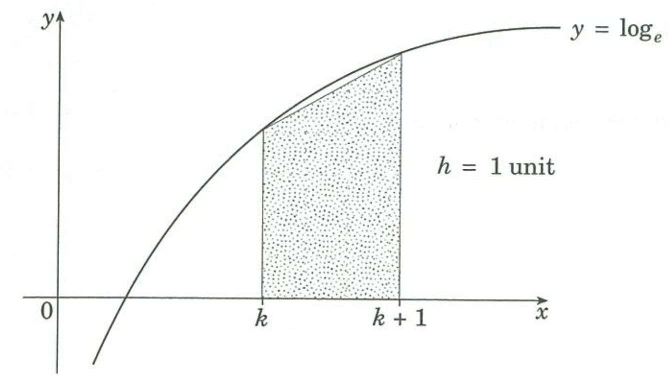
$$\begin{aligned}
y &= \log_e x, \quad x > 0 \\
y' &= \frac{1}{x} \\
y'' &= -\frac{1}{x^2} < 0 \quad \text{since } x^2 > 0
\end{aligned}$$

$\therefore \ y = \log_e x$ is concave down for all $x > 0$

(ii)

$$\begin{aligned}
\text{Area} &= \int_1^n \log_e x \, dx \\
&= \Big[x\log_e x - x \Big]_1^n \\
&= \left(n\log_e n - n\right) - \left(1\log 1 - 1\right) \\
&= n\log_e n - n - 0 + 1 \\
&= n\log_e n - n + 1
\end{aligned}$$

(iii)



$\boxed{\text{A trap} = \frac{h}{2}(a+b)}$

Area of a typical trapezium under $y = \log_e x$

$A_{k+1} = \frac{1}{2}(1)\left[\log_e k + \log_e(k+1)\right]$

$\therefore$ Area of trapezia

$$= \frac{1}{2}(\log_e 1 + \log_e 2) + \frac{1}{2}(\log_e 2 + \log_e 3) + \frac{1}{2}(\log_e 3 + \log_e 4) + \cdots + \frac{1}{2}\left[\log_e(n-1)\right] + \log_e n$$

$$= \frac{1}{2}\left[0 + 2\log_e 2 + 2\log_e 3 + \cdots + 2\log_e(n-1) + \log_e n\right]$$

$$= \log_e 2 + \log_e 3 + \cdots + \log_e(n-1) + \frac{1}{2}\log_e n$$

$$= \log_e\left[2 . 3 . 4 . \cdots . (n-1)\right] + \frac{1}{2}\log_e n$$

$$= \frac{1}{2}\log_e n + \log_e(n-1)!$$

$$= \log_e n^{\frac{1}{2}} + \log_e(n-1)!$$

$$= \log_e n^{\frac{1}{2}}(n-1)!$$

$$= \log_e\left[(n-1)!\sqrt{n}\right]$$

Area under trapezia < area under curve (from $x = 1$ to $x = n$).

$$\log_e(n-1)!\sqrt{n} < n\log_e n - n + 1$$

$$\log_e\left[(n-1)!\frac{\sqrt{n}}{\sqrt{n}}\sqrt{n}\right] < \log_e n^n - \log_e e^n + \log_e e$$

$\boxed{n = \log_n e^n \text{ and } 1 = \log_e e}$

$$< \log_e\left(\frac{n^n . e}{e^n}\right)$$

$$\therefore \quad \frac{n(n-1)!}{\sqrt{n}} < \frac{n^n e}{e^n}$$

$$n! < \frac{\sqrt{n} . n^n . e}{e^n}$$

$$= \frac{n^{\frac{1}{2}} n^n e}{e^n}$$

$$= \frac{n^{n+\frac{1}{2}} . e}{e^n}$$

i.e. $\quad n! < \dfrac{en^{n+\frac{1}{2}}}{e^n}$