Year 12 E2 Half Yearly 2018

1 D

2 D

3 C

4 D

5 C

Q6.

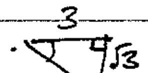
a) i) $\frac{w}{z} = \frac{-2+2i}{3+\sqrt{3}i} \times \frac{3-\sqrt{3}i}{3-\sqrt{3}i}$

$= \frac{-6+2\sqrt{3}i+6i+2\sqrt{3}}{12}$

$= \cancel{\frac{3+\sqrt{3}}{6}(-1+i)} \quad \frac{-3+\sqrt{3}}{6} + \frac{3+\sqrt{3}}{6}i$

ii) $|z| = \sqrt{9+3}$

$= 2\sqrt{3}$

$\arg(z)$ 

$\tan\theta = \frac{\sqrt{3}}{3}$

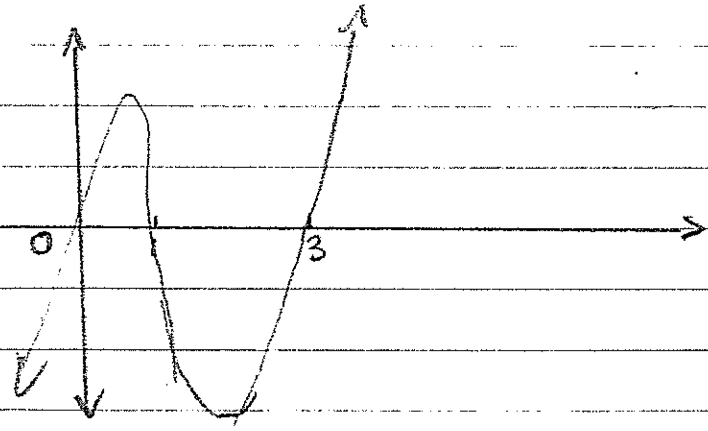
$\theta = -\frac{\pi}{6}$

iii) $z^4 = 2^4 \times 9 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

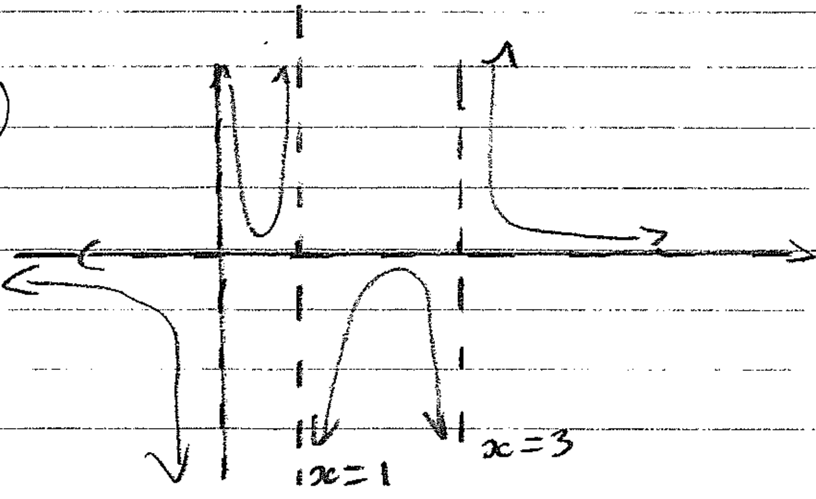
$= 144\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -72 - 72\sqrt{3}i$

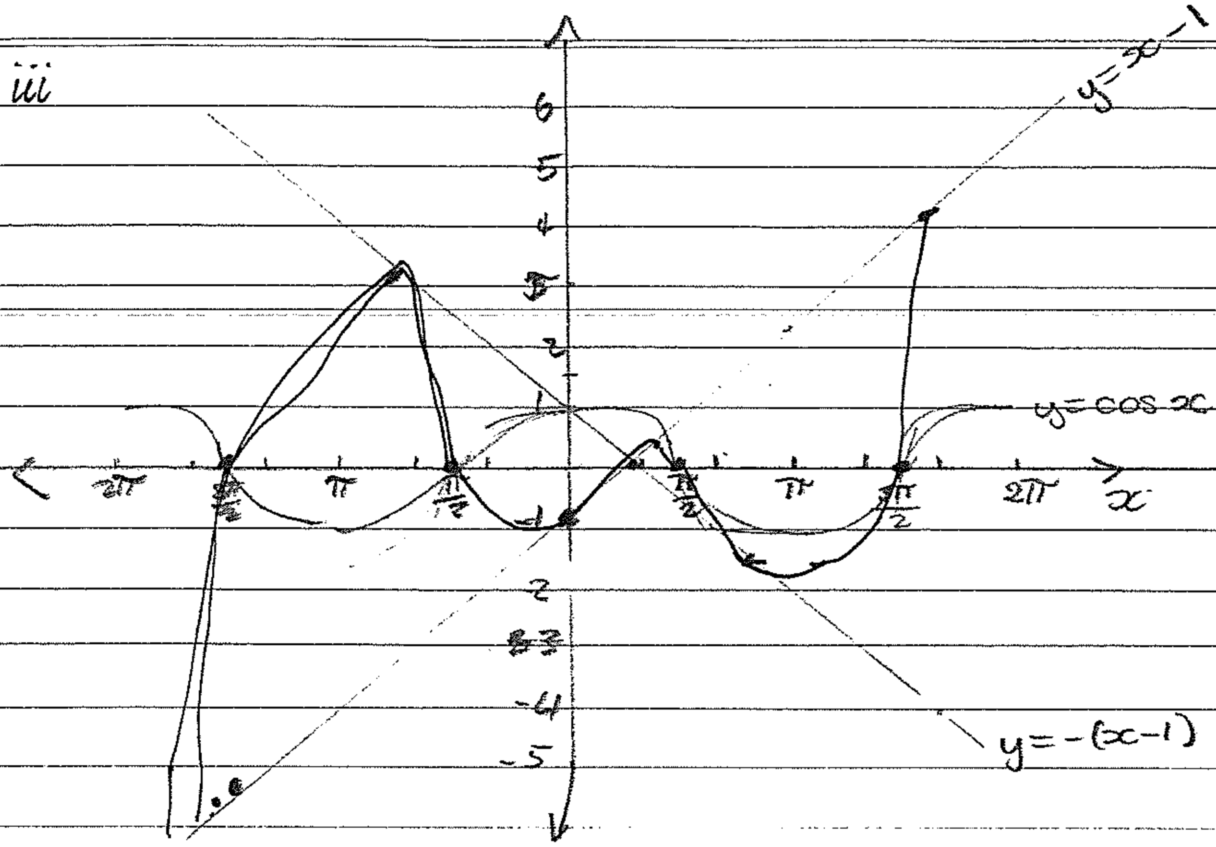
b) i)



ii)



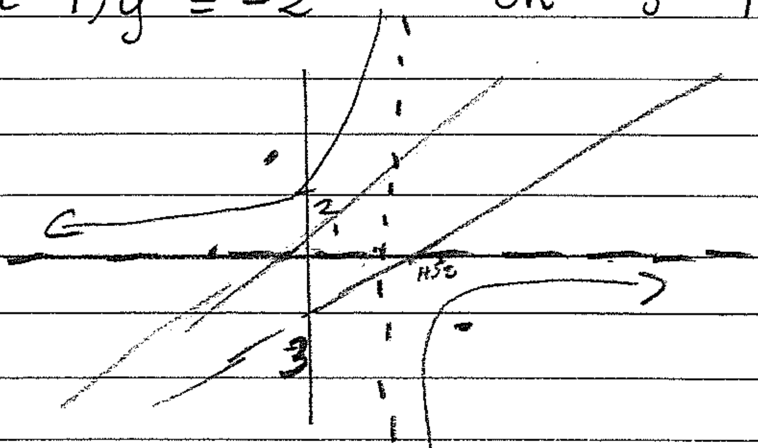
6 b iii



c) $x = 1 - \sqrt{2}t \quad y = \frac{\sqrt{2}}{t}$

i) $(x-1)y = -2$ OR $y = \frac{2}{1-x}$ or $(1-x)y = 2$

ii



Asymptotes $x = 1$ $\quad c^2 = 2 \therefore a^2 = 4$

$y = 0$

Directrices $(x-1) - y = \pm \cancel{2}$

$x - y = 1 \pm 2 \quad y = x+1, y = x-3$

Foci $(3, -2)$

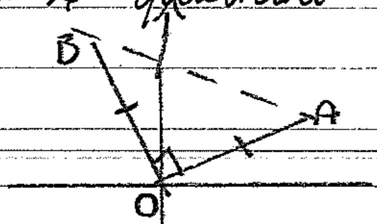
$(-1, 2)$

d) As e increases foci move further from vertices, directrices approach x-axis, asymptotes become steeper. As $e \to \infty$ asymptotes approach y axis.

Q7 B is in 2nd

a) i) ~~first~~ or 4^{th} quadrant

b ii)



OA = OB

iii) Since $|z_1| = |z_2|$ and since $OA \perp OB$

$z_2 = \pm i z_1$

$$\therefore z_1^2 + z_2^2 = z_1^2 + (\pm i)^2 z_1^2$$
$$= z_1^2 - z_1^2$$
$$= 0$$

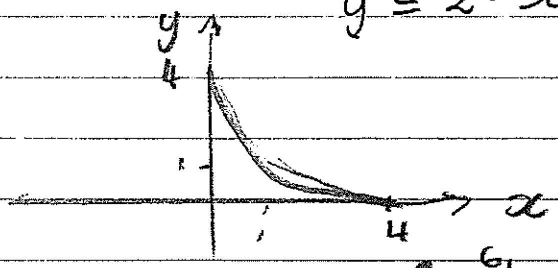
b) i) $x^{1/2} + y^{1/2} = 2$

$$\tfrac{1}{2} x^{-1/2} + \tfrac{1}{2} y^{-1/2} \frac{dy}{dx} = 0$$
$$y^{-1/2} \cdot \frac{dy}{dx} = -x^{-1/2}$$
$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

ii) $x = 1 \quad y = 1$, $\quad y - 1 = -(x-1)$

$$y = 2 - x$$

iii)



~~y = 0~~

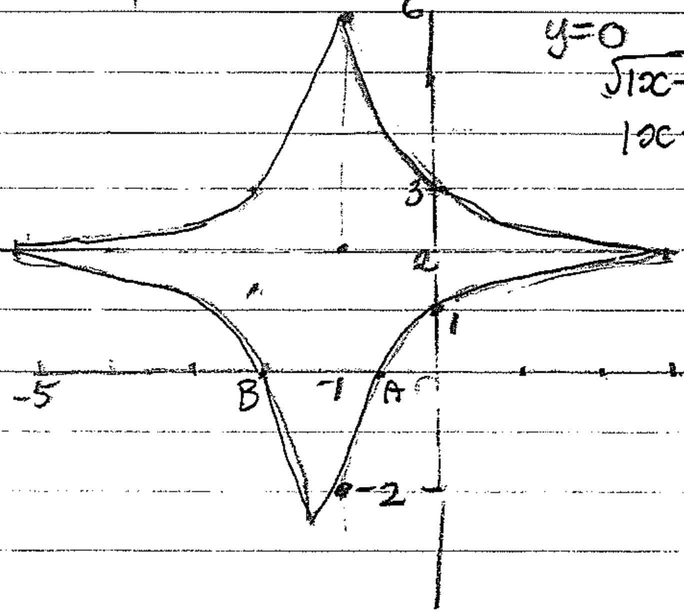
$x = 0 \quad \sqrt{|y-2|} = 1$

$y - 2 = \pm 1$

$y = 0$

$\sqrt{|x+1|} = 2 - \sqrt{2}$

$|x+1| = 6 - 4\sqrt{2}$



$A(5 - 4\sqrt{2}, 0)$

$B(-7 + 4\sqrt{2}, 0)$

7c) $1 \le |z-3| \le 2$ and $\frac{\pi}{3} < \arg z < \frac{2\pi}{3}$

~~no solution~~

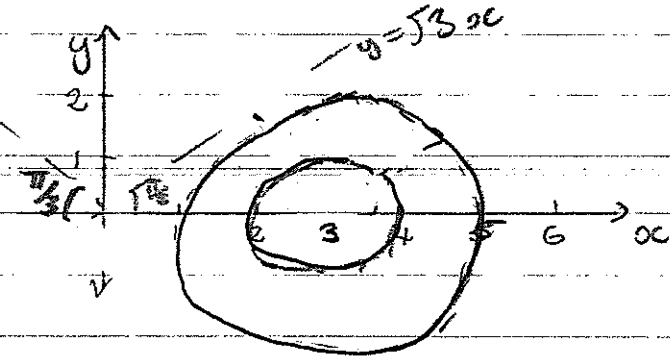
intersection $y = \sqrt{3}x$

$(x-3)^2 + (\sqrt{3}x)^2 = 4$

$4x^2 - 6x + 9 - 4 = 0$

$\Delta = 36 - 80 < 0$

no solution



1 mark if sketched with overlap

d) $z^2 - \bar{z}^2 = 36i$

Let $z = x + iy$ $\quad \bar{z} = x - iy$

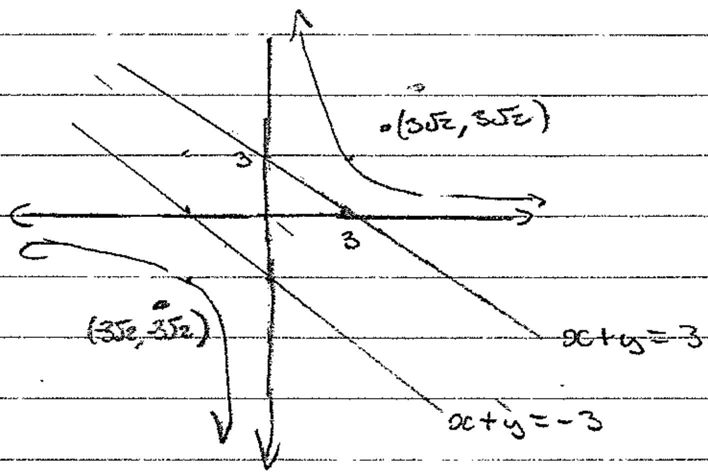
$z^2 = x^2 - y^2 + 2iy \quad \bar{z} = x^2 - y^2 - 2iy$

$z^2 - \bar{z}^2 = 4xyi$

$\therefore 4xyi = 36i$

$xy = 9$

The locus is a rectangular hyperbola with foci $(\pm 3\sqrt{2}, \pm 3\sqrt{2})$ and directrices $x + y = \pm 3$



Q8 a) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$4 = 9(1-e^2)$

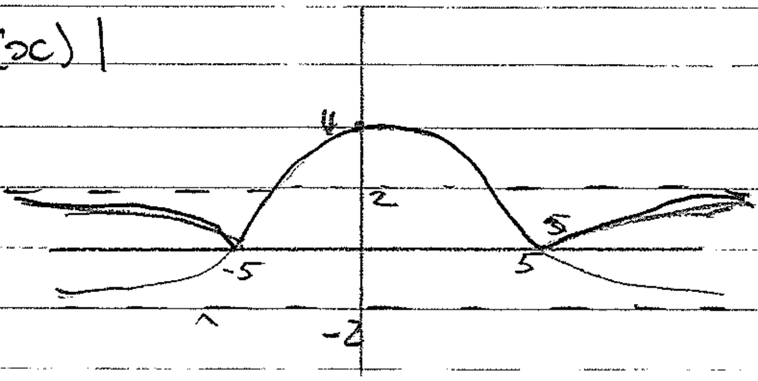
$e^2 = 5/9$

$e = \sqrt{5}/3$

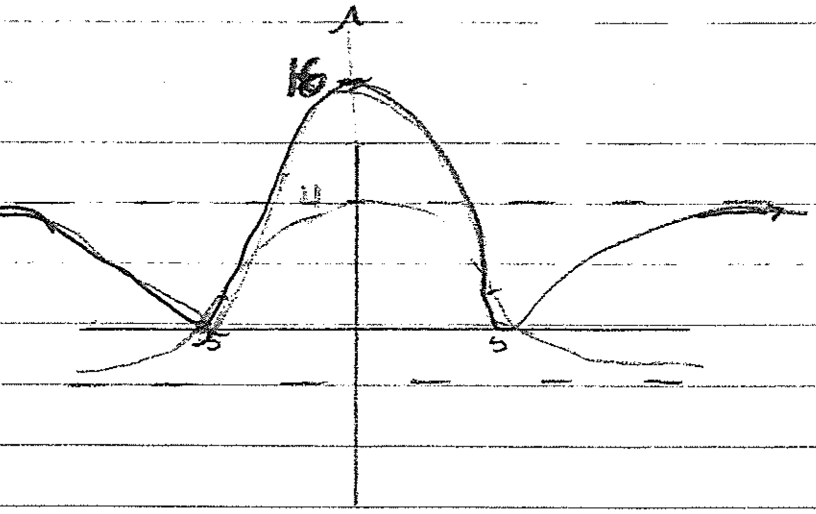
foci $(0 \pm\sqrt{5})$

directrices $y = \pm \frac{9}{\sqrt{5}}$

b) i) $y = |f(x)|$

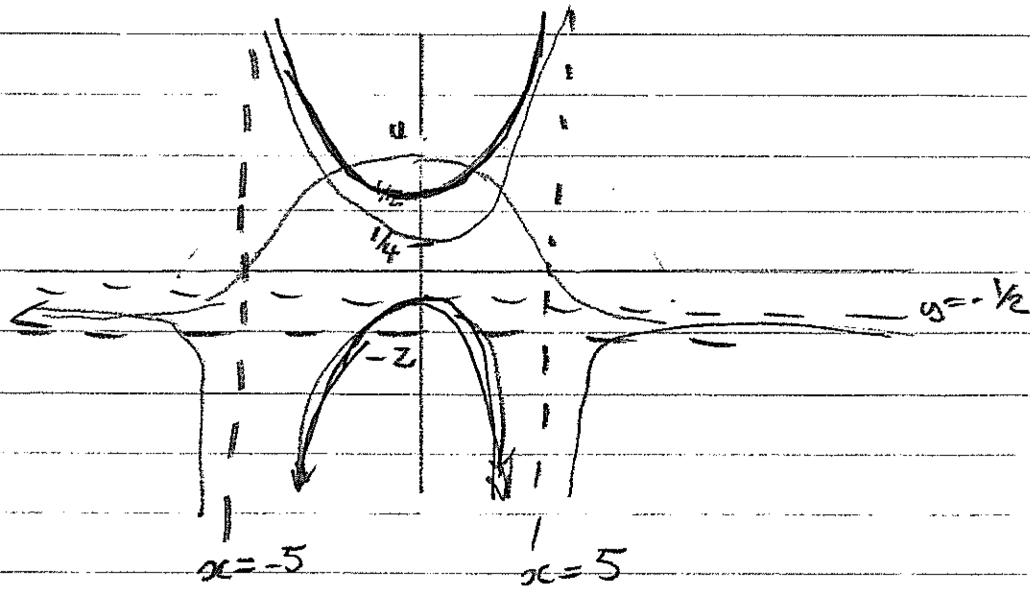


ii) $y = f^2(x)$

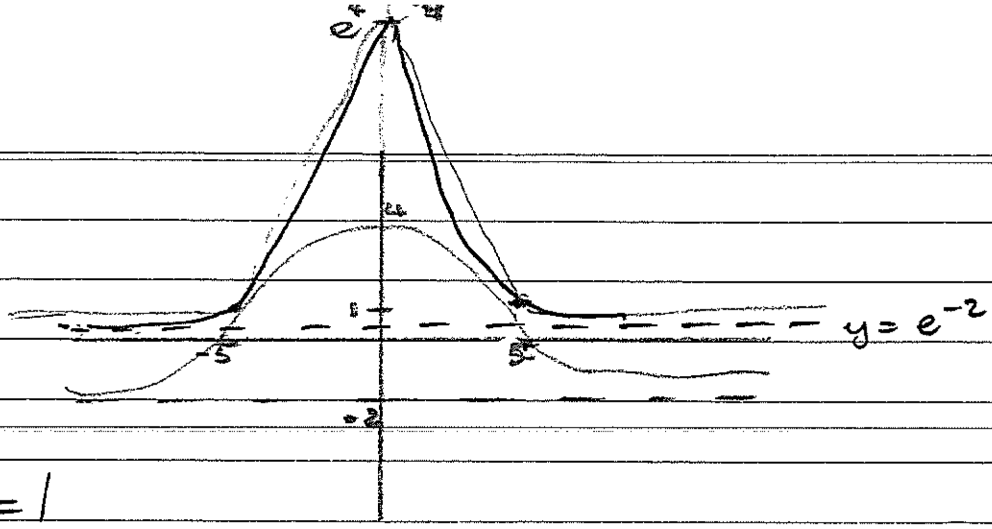


iii $y^2 = \frac{1}{f(x)}$

$y = \frac{1}{f(x)}$



Q8 b iv



$e^{f(5)} = e^{f(-5)} = 1$

As $x \to \pm\infty$

$e^{f(x)} \to e^{-2}$

c) Let $S(n)$ bee the statement that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$n=1$: $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$\therefore$ $S(1)$ holds

Suppose $S(k)$ is true for some integer $k \geq 1$

Then we need to prove that

$$(\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta)$$

$$\begin{aligned} LHS &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k \\ &= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta) \\ &= \cos\theta\cos k\theta - \sin\theta\sin k\theta + i(\cos\theta\sin k\theta + \sin\theta\cos k\theta) \\ &= \cos((k+1)\theta) + i\sin((k+1)\theta) \quad \text{by the compound angle formulae} \\ &= RHS \end{aligned}$$

Therefore since $S(1)$ holds and $S(k+1)$ holds whenever $S(k)$ holds, by induction $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all integers $n \geq 1$.

Let T be the point of intersection of QP and the y axis

Q9. a) i By the reflection property $\angle QPS = \angle TPS'$

But $\angle TPS' = \angle RPQ$ (vertically opposite angles)

$\therefore \angle QPS = \angle QPR$

QP is common and $\angle SQP = \angle RQP$ (both $90°$)

$\therefore \Delta QPS \equiv \Delta QPR$ (AAS)

$\therefore SQ = RQ$ (~~matches~~ corresponding sides of congruent Δs)

ii $S'P + PS = 2a$ (property of ellipse)

$PS = PR$ (from i)

$\therefore S'R = S'P + PR$

$= 2a$

iii) $S'R = S'S + SR$

$= 2OS + 2SQ$ (from i)

$\therefore 2a = 2(OS + SQ)$ (from ii)

$a = OQ$

Let $Q = Q(x, y)$

$OQ^2 = x^2 + y^2 = a^2$

$\therefore$ Q lies on the circle $x^2 + y^2 = a^2$

9 b) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating implicitly, $\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Tangent at $P(a\sec\theta, b\tan\theta)$:

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$-ay\tan\theta + b\sec\theta\, x = ab\sec^2\theta - ab\tan^2\theta$$

OR $$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

ii Asymptotes are $y = \pm \frac{b}{a} x$

$y = \frac{b}{a}x$: $$\frac{x\sec\theta}{a} - \frac{b}{a}\frac{x\tan\theta}{b} = 1$$

$$\frac{x}{a}(\sec\theta - \tan\theta) = 1$$

$$x = \frac{a}{\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}$$

$$x = \frac{a\cos\theta}{1 - \sin\theta}$$

$$y = \frac{b\cos\theta}{1 - \sin\theta}$$

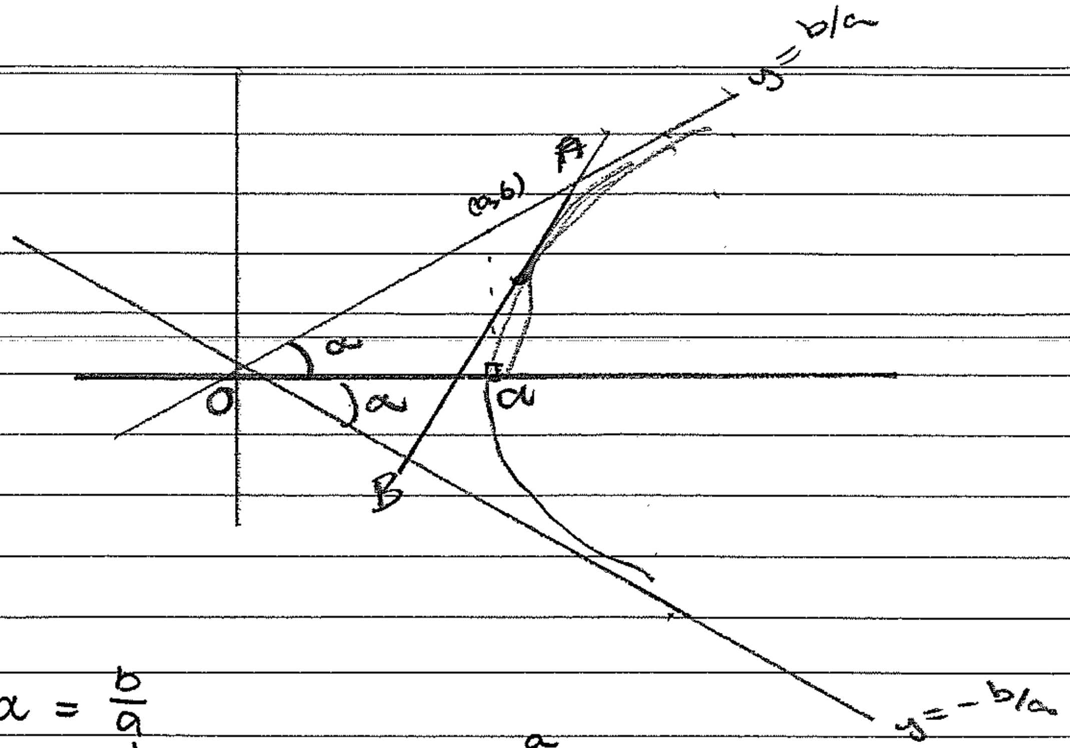
$y = -\frac{b}{a}x$: $$\frac{x\sec\theta}{a} + \frac{x\tan\theta}{a} = 1$$

$$x = \frac{a}{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{a\cos\theta}{1 + \sin\theta}$$

$$y = -\frac{b\cos\theta}{1 + \sin\theta}.$$

9 b)iii)



$\tan\alpha = \frac{b}{a}$

$\sin\alpha = \frac{b}{\sqrt{a^2+b^2}} \quad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$

$\angle AOB$

$= \sin(2\alpha) = 2\sin\alpha\cos\alpha$

$= \frac{2ab}{a^2+b^2}$

$\text{Area}(\Delta AOB) = \frac{1}{2} OA \cdot OB \cdot \sin(\angle AOB)$

$OA^2 = (a^2+b^2)\frac{\cos^2}{(1-\sin\theta)^2} \qquad OB^2 = (a^2+b^2)\frac{\cos^2\theta}{(1+\sin\theta)^2}$

$OAOB = a^2+b^2 \ \frac{\cos^2\theta}{1-\sin^2\theta}$

$= a^2+b^2$

$\therefore \text{Area}(\Delta AOB) = \frac{1}{2} \cdot (a^2+b^2) \cdot \frac{2ab}{a^2+b^2}$

$= ab$ square units as required.

9 b iii) Alternative method

$$AB = \sqrt{a^2\cos^2\theta\left(\frac{1}{1+\sin\theta} - \frac{1}{1-\sin\theta}\right)^2 + b^2\cos^2\theta\left(\frac{-1}{1+\sin\theta} - \frac{1}{1-\sin\theta}\right)^2}$$

$$= \sqrt{\frac{a^2\cancel{\cos^2\theta}\left[(1-\sin\theta)-(1+\sin\theta)\right]^2 + b^2\cancel{\cos^2\theta}\left(-1+\sin\theta-1-\ldots\right.}{(1-\sin^2\theta)^2}}$$

$$= \frac{\sqrt{a^2(4\sin^2\theta) + b^2 4}}{\cos\theta}$$

$$= \frac{2\sqrt{a^2\sin^2\theta + b^2}}{\cos\theta}$$

The line AB is the tangent

$$b\sec\theta\, x - a\tan\theta\, y - ab = 0$$

$\therefore$ perpendicular distance from O to AB is

$$d = \frac{|-ab|}{\sqrt{b\sec^2\theta + a^2\tan^2\theta}} = \frac{ab}{\frac{\sqrt{a^2\sin^2\theta + b^2}}{\cos\theta}}$$

$\therefore$ Area$(\Delta OAB) = \frac{1}{2} AB \cdot d$

$= ab\ u^2$.

Note: you can also break ΔAOB into two parts (along $y=0$ works well) or construct a trapezium and subtract triangles ...)