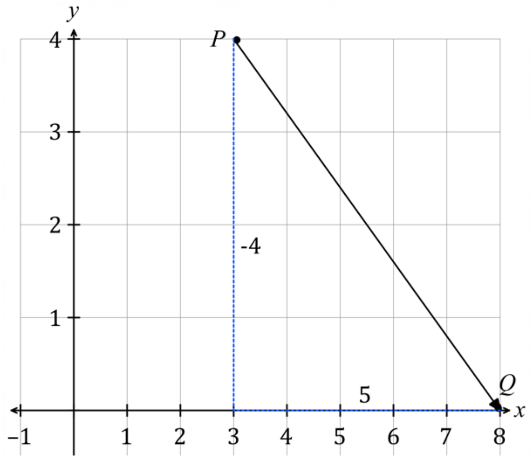
ACE Examination Paper 1
Year 12 Mathematics Extension 1 Yearly Examination
Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1.	Three heads with $n = 5$ $P(X=5) = {}^5C_3\, 0.7^3(1-0.7)^{5-3}$ $= 10 \times 0.7^3 0.3^2$	1 Mark: B
2.	$\int \sin^2 2x dx = \frac{1}{2}\int (1-\cos 4x)dx$ $= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + C$ $= \frac{x}{2} - \frac{1}{8}\sin 4x + C$	1 Mark: A
3.	The maximum height, for angle θ, occurs when $\dot{y} = 0$, that is $y = Vt\sin\theta - \frac{1}{2}gt^2$ $\dot{y} = V\sin\theta - gt$ $0 = V\sin\theta - gt$ $t = \frac{V\sin\theta}{g}$ Maximum height $y = Vt\sin\theta - \frac{1}{2}gt^2$ $y = \left(\frac{V^2\sin^2\theta}{g} - \frac{V^2 sin^2\theta}{2g}\right)$ $= \frac{V^2\sin^2\theta}{2g}$	1 Mark: C
4.	$A = \int_0^4 6x - x^2 - 2x\, dx$ $= \int_0^4 4x - x^2\, dx$	1 Mark: B
5.	$N = 135 + Ae^{kt}$ $\frac{dN}{dt} = k \times Ae^{kt}$ $= k(N-135)$	1 Mark: D
6.	$u = \ln x$ and $du = \frac{1}{x}dx$ $\quad u = \ln e = 1, u = \ln e^2 = 2$ $\int_e^{e^2} \frac{1}{x\ln x}dx = \int_1^2 \frac{1}{u}\mathrm{du}$ $= [\ln u]$ $= \ln 2 - \ln 1$ $= \ln 2$	1 Mark: B

	Solution	Criteria
7.	$\underset{\sim}{w} + \underset{\sim}{v} = \underset{\sim}{u}$ $\underset{\sim}{w} = \underset{\sim}{u} - \underset{\sim}{v}$ $\underset{\sim}{w} \cdot \underset{\sim}{w} = (\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v})$ $\quad = \underset{\sim}{u}^2 - 2\underset{\sim}{u}\underset{\sim}{v} + \underset{\sim}{v}^2$ $\lvert\underset{\sim}{w}\rvert^2 = \lvert\underset{\sim}{u}\rvert^2 + \lvert\underset{\sim}{v}\rvert^2 - 2\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert\cos 60$ $\therefore \lvert\underset{\sim}{w}\rvert^2 = \lvert\underset{\sim}{u}\rvert^2 + \lvert\underset{\sim}{v}\rvert^2 - \lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert$	1 Mark: D
8.	The correct step 1 is shown below Step 1: To prove the statement true for $n = 2$ $\text{LHS} = 2 \times 1 = 2$ $\text{RHS} = \frac{1}{3} \times 2 \times (2^2 - 1) = 2$ Result true for $n = 2$	1 Mark: A
9.	$\sin x - \cos x = R\sin(x + \alpha)$ $\quad = R\sin x\cos\alpha + R\cos x\sin\alpha$ $R\cos\alpha = 1$ ① $R\sin\alpha = -1$ ② Equation ② divided by equation ① $\tan\alpha = -1$ $\alpha = \frac{7\pi}{4}$ Squaring and adding the equations $R^2(\sin^2\alpha + \cos^2\alpha) = 1 + 1$ $R^2 = 2$ $R = \sqrt{2}\ (R > 0)$ $\therefore \sin x - \cos x = \sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$	1 Mark: D
10.	Particle reaches the ground when $y = 0$ $y = -\frac{1}{2}gt^2 + Vt\sin\theta + 90$ $0 = -\frac{1}{2}gt^2 + 50t\sin 0 + 90$ $\frac{1}{2}gt^2 = 90$ $t = \sqrt{\frac{180}{g}} = \sqrt{\frac{36 \times 5}{g}}$ $= 6\sqrt{\frac{5}{g}}$ seconds	1 Mark: C

Section II		
11(a)	$\overrightarrow{PQ} = 5\underset{\sim}{i} - 4\underset{\sim}{j}$ 	2 Marks: Correct answer. 1 Mark: Shows some understanding.
11(b)(i)	$\text{LHS} = \sin\left(x + \frac{\pi}{4}\right)$ $= \sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4}$ $= \sin x \times \frac{1}{\sqrt{2}} + \cos x \times \frac{1}{\sqrt{2}}$ $= \frac{\sin x + \cos x}{\sqrt{2}}$ $= \text{RHS}$	2 Marks: Correct answer. 1 Mark: Uses the sum of angles formula or exact values.
11(b)(ii)	$\frac{\sin x + \cos x}{\sqrt{2}} = \frac{\sqrt{3}}{2}$ $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$ $x + \frac{\pi}{4} = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$ $x = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$	2 Marks: Correct answer. 1 Mark: Finds one solution or shows some understanding.
11(c)(i)	$P(\text{Two blue balls at least once}) = 1 - P(\text{No blue ball 5 times})$ $= 1 - \left(1 - \frac{4}{7} \times \frac{3}{6}\right)^5$ $= 0.8140 \ldots$ ≈ 0.814	2 Marks: Correct answer. 1 Mark: Use of complementary event.
11(c)(ii)	Let p be the probability of getting two blue balls. $p = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7} \quad n = 5$ $P(X = x) = {}^5C_x \left(\frac{2}{7}\right)^x \left(\frac{5}{7}\right)^{50-x}$ $P(X = 3) = {}^5C_3 \left(\frac{2}{7}\right)^3 \left(\frac{5}{7}\right)^{5-3}$ $= \frac{2000}{16\,807} \approx 0.119$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
11(d)	$f(x) = 4\tan^{-1} x$ $f'(x) = \frac{4}{1 + x^2}$ Th curve cuts the y-axis when $x = 0$ $f'(0) = \frac{4}{1 + 0^2} = 4$ $\therefore$ Slope of the tangent is 4.	2 Marks: Correct answer. 1 Mark: Differentiates the inverse function.

11(e) (i)	$\text{LHS} = \dfrac{1+\cos 2x}{\sin 2x}$ $= \dfrac{2\cos^2 x}{2\sin x\cos x}$ $= \dfrac{\cos x}{\sin x}$ $= \cot x$ $= \text{RHS}$	2 Marks: Correct answer. 1 Mark: Uses a correct and appropriate trig identity.
11(e) (ii)	$\cot 15° = \dfrac{1+\cos(2\times 15°)}{\sin(2\times 15°)}$ $= \dfrac{1+\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}}$ $= 2+\sqrt{3}$	1 Mark: Correct answer.
12(a)	$V = \pi\int_0^{\ln 4} x^2\,dy$ $= \pi\int_0^{\ln 4} (e^y)^2\,dy$ $= \pi\left[\frac{1}{2}e^{2y}\right]_{10}^{\ln 4}$ $= \frac{\pi}{2}\left(e^{2\ln 4} - e^0\right)$ $= \dfrac{15\pi}{2}$ cubic units	2 Marks: Correct answer. 1 Mark: Uses volume formula with at least one correct value.
12(b) (i)	$\underset{\sim}{u} + \overrightarrow{AB} = \underset{\sim}{v}$ $\overrightarrow{AB} = \underset{\sim}{v} - \underset{\sim}{u}$	1 Mark: Correct answer.
12(b) (ii)	$\underset{\sim}{u} + \frac{1}{2}\overrightarrow{AB} = \overrightarrow{OP}$ $\overrightarrow{OP} = \underset{\sim}{u} + \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$ $= \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
12(b) (iii)	$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$ $= \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$	1 Mark: Correct answer.
12(b) (iv)	$\overrightarrow{BP} = -\overrightarrow{AP}$ $= -\frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$ $= \frac{1}{2}(\underset{\sim}{u} - \underset{\sim}{v})$	1 Mark: Correct answer.
12(c)	$\displaystyle\int \frac{dx}{\sqrt{36-x^2}} = \sin^{-1}\frac{x}{6} + C$	2 Marks: Correct answer. 1 Mark: Uses inverse sine fn.

12(d)	$\int_0^{\frac{\pi}{4}} \cos^2 \frac{1}{2}x dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos x)dx$ $= \frac{1}{2}[x+\sin x]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}\left[\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)-(0+0)\right]$ $= \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)$	2 Marks: Correct answer. 1 Mark: Uses double angle formula to simplify the integral.
12(e)	Step 1: To prove true for $n = 2$ $3^n - 2n - 1 = 3^2 - 2\times 2 - 1 = 4$ Result is true for $n = 2$ Step 2: Assume true for $n = k$ $3^k - 2k - 1 = 4m$ where m is an integer Step 3: To prove true for $n = k + 1$ $3^{k+1} - 2(k+1) - 1 = 4p$ where p is an integer LHS $= 3^{k+1} - 2(k+1) - 1$ $= 3(3^k) - 2k - 2 - 1$ $= 3(3^k - 2k - 1) + 4k$ $= 3(4m) + 4k$ $= 4(3m + k)$ $= 4p$ $=$ RHS Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$ 1 mark: Proves the result true for $n = 1$.
13(a) (i)	$T = 2 + Ae^{-kt}$ $(Ae^{-kt} = T - 2)$ $\frac{dT}{dt} = -k \times Ae^{-kt}$ $= -k(T-2)$	1 Mark: Correct answer.
13(a) (ii)	Initially $t = 0$ and $T = 20$ $T = 2 + Ae^{-kt}$ $20 = 2 + Ae^{-k\times 0}$ $A = 18$ Now $t = 20$ and $T = 10$ $T = 2 + 18e^{-kt}$ $10 = 2 + 18e^{-k\times 20}$ $e^{-20k} = \frac{8}{18} = \frac{4}{9}$ $-20k = \ln\frac{4}{9}$ $k = -\frac{1}{20}\ln\frac{4}{9}$ $= \frac{1}{20}\ln\frac{9}{4}$ $= 0.0405 \ldots$	3 Marks: Correct answer. 2 Marks: Finds the value of A and an expression for k. 1 Mark: Finds the value of A.

13(a) (iii)	We need to find t when $T = 5$ $T = 2 + 18e^{-kt}$ $5 = 2 + 18e^{-kt}$ $e^{-kt} = \frac{3}{18} = \frac{1}{6}$ $-kt = \ln\frac{1}{6}$ $t = -\frac{1}{k}\ln\frac{1}{6}$ $= \frac{1}{k}\ln 6$ $= 20\ \frac{\ln 6}{\ln\frac{9}{4}}$ $= 44.1902\ldots$ ≈ 44 minutes $\therefore$ It will take about 44 minutes for the bottle to cool to 5°C.	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
13(b)	$u = x + 1$ $du = dx$ when $x = 15$, $u = 16$ and $x = 0$, $u = 1$ $\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx = \int_1^{16} \frac{u-1}{\sqrt{u}}\,du$ $= \int_1^{16} u^{\frac{1}{2}} - u^{-\frac{1}{2}} du$ $= \left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right]_1^{16}$ $= \frac{2}{3}\left(16^{\frac{3}{2}} - 1\right) - 2\left(16^{\frac{1}{2}} - 1\right)$ $= 36$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up the integration using the substitution.
13(c) (i)	$E(X) = np$ $40 \times p = 5$ $p = 0.125$	2 Marks: Correct answer. 1 Mark: Makes some progress.
13(c) (ii)	$Var(X) = np(1-p)$ $= 40 \times 0.125(1 - 0.125)$ $= 4.375$	2 Marks: Correct answer. 1 Mark: Makes some progress.
13(d)	$\lvert \underset{\sim}{u} \rvert = \sqrt{2^2 + 3^2}$ $= \sqrt{13}$ $\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{\lvert \underset{\sim}{u} \rvert}$ $= \frac{1}{\sqrt{13}}(2\underset{\sim}{i} + 3\underset{\sim}{j})$	2 Marks: Correct answer. 1 Mark: Shows some understanding.

14(a) (i)	$x = Vt\cos\theta$ ① $y = Vt\sin\theta - \frac{1}{2}gt^2$ ② From equation ① $t = \frac{x}{V\cos\theta}$ substitute into equation ② $y = V \times \frac{x}{V\cos\theta} \times \sin\theta - \frac{1}{2}g \times \frac{x^2}{V^2\cos^2\theta}$ $= x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$ Given $h = \frac{V^2}{2g}$ and $\sec^2\theta = (1+\tan^2\theta)$ then $y = x\tan\theta - \frac{1}{4h}x^2(1+\tan^2\theta)$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Makes t the subject of equation ①.
14(a) (ii)	Now (a, b) satisfies the equation $y = x\tan\theta - \frac{1}{4h}x^2(1+\tan^2\theta)$ $b = a\tan\theta - \frac{1}{4h}a^2(1+\tan^2\theta)$ $4hb = 4ha\tan\theta - a^2(1+\tan^2\theta)$ $(1+\tan^2\theta)a^2 - 4ha\tan\theta + 4hb = 0$ $a^2\tan^2\theta - 4ha\tan\theta + 4hb + a^2 = 0$ Quadratic equation has 2 solutions if $\Delta > 0$ $b^2 - 4ac > 0$ $(-4ha)^2 - 4a^2(4hb + a^2) > 0$ $16h^2a^2 - 16a^2hb - 4a^4 > 0$ $4a^2(4h^2 - 4hb - a^2) > 0$ $4h^2 - 4hb - a^2 > 0$ $a^2 < 4h^2 - 4hb$ $a^2 < 4h(h-b)$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Substitutes (a, b). into equation of flight and simplifies.
14(b)	Step 1: To prove true for $n = 1$ $\text{LHS} = \frac{2}{1\times 2} = 1$ $\text{RHS} = \frac{2\times 1}{1+1} = 1$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $S_k = \frac{2k}{k+1}$ Step 3: To prove true for $n = k + 1$ $S_{k+1} = \frac{2(k+1)}{k+2}$ $S_k + T_{k+1} = S_{k+1}$ $\text{LHS} = \frac{2k}{k+1} + \frac{2}{(k+1)(k+2)}$ $= \frac{2k(k+2)+2}{(k+1)(k+2)}$ $= \frac{2(k^2+2k+1)}{(k+1)(k+2)} = \frac{2(k+1)(k+1)}{(k+1)(k+2)}$ $= \frac{2(k+1)}{k+2}$ $= \text{RHS}$ Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$ 1 mark: Proves the result true for $n = 1$.

14(c)	$u = 1 + e^x$ or $e^x = u - 1$ $\frac{du}{dx} = e^x$ or $du = e^x dx$ $\int \frac{e^{3x}}{1+e^x}dx = \int \frac{e^{2x} \times e^x}{1+e^x}dx$ $= \int \frac{(u-1)^2}{u}du$ $= \int \frac{u^2 - 2u + 1}{u}du$ $= \int (u - 2 + \frac{1}{u})du$ $= \frac{u^2}{2} - 2u + \ln u + C$ $= \frac{(1+e^x)^2}{2} - 2(1+e^x) + \ln(1+e^x) + C$	2 Marks: Correct answer. 1 Mark: Sets up the integral in terms of u.
14(d) (i)	$y = 4x - x^2 + 8$ ① $y = x^2 - 2x$ ② Substitute $x^2 - 2x$ for y into equation ① $x^2 - 2x = 4x - x^2 + 8$ $2x^2 - 6x - 8 = 0$ $x^2 - 3x - 4 = 0$ $(x-4)(x+1) = 0$ $x = 4$ and $y = 8$ $x = -1$ and $y = 3$ $\therefore$ Points of intersection are (4, 8) and (−1, 3)	1 Mark: Correct answer.
14(d) (ii)	$\int_{-1}^{4}(4x - x^2 + 8) - (x^2 - 2x)\,dx$ $= \int_{-1}^{4}(-2x^2 + 6x + 8)dx$ $= -2\int_{-1}^{4}(x^2 - 3x - 4)\,dx$ $= -2\left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_{-1}^{4}$ $= -2\left[\left(\frac{64}{3} - \frac{48}{2} - 16\right) - \left(-\frac{1}{3} - \frac{3}{2} + 4\right)\right]$ $= 41\frac{2}{3}$ square units	2 Marks: Correct answer. 1 Mark: Shows some understanding.
14(e)	$\frac{dy}{dx} = e^{6x}(1 + y^2)$ $\int e^{6x}\,dx = \int \frac{1}{1+y^2}dy$ $\frac{1}{6}e^{6x} + C = \tan^{-1} y$ $y = \tan\left(\frac{1}{6}e^{6x} + C\right)$	2 Marks: Correct answer. 1 Mark: Separates the variables and attempts to integrate.