CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Extension 1 Mathematics

Assessment Task 1

2018

Time allowed: 50 minutes

Student Number: ________________________________

Instructions:

- All questions should be attempted on your own paper.
- Show ALL necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Please write on one side of each sheet of paper only.

Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Q.7	Q.8	Q.9	Total
/4	/3	/4	/5	/3	/3	/3	/6	/6	/37

Q. 1 The first three terms of an arithmetic series are

$(m+1), (m^2+m)$ and $(3m^2-m-4)$, respectively,

where m is a constant.

(i) Show that the first three terms have the values 4, 12, and 20.

(ii) The sum of the first n terms of the series is denoted by S_n.

Show that S_n is always a square number.

2+2=4 marks

Q. 2 Joey is given \$50 as a present by her aunt on her first birthday. On every birthday after that, Joey's aunt gives her \$20 more than what she gave her on her previous birthday.

Find:

(i) How much money Joey's aunt gives her on her tenth birthday.

(ii) After receiving her gift of money from her aunt on her nth birthday, Joey has received a total of \$7800 from her aunt.

What is the value of n ?

1+2=3 marks

Q. 3 It is given that

$$\sum_{r=1}^{n} T_r = 128 - 2^{7-n}$$

where T_r is the r^{th} term of a geometric progression.

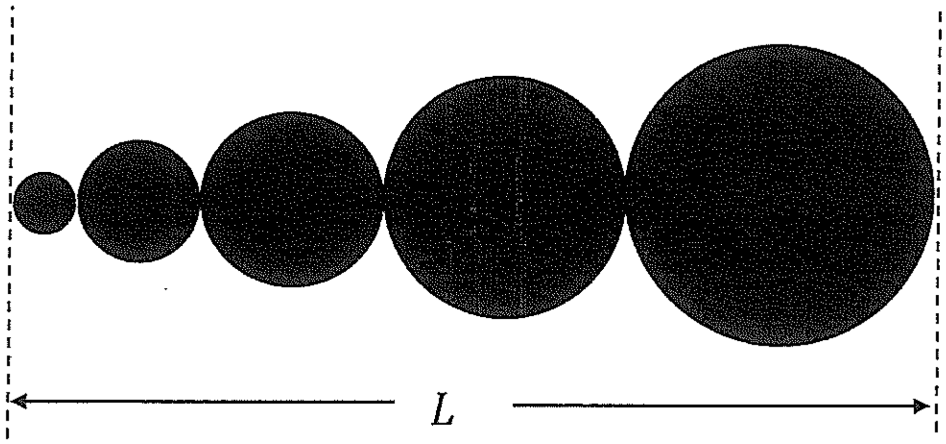
(i) Find the sum of the first 8 terms of the progression.

(ii) Determine the value of T_8 .

(iii) Find the common ratio of the progression.

1+1+2=4 marks

Q.4



The figure above shows a pattern of 5 circles, touching externally, whose centres lie on a straight line of length L units.

The radii of these circles form a geometric progression, where the radius of the smaller circle is 3 units and that of the fifth (larger) circle is 48 units.

(i) Find the common ratio of the geometric progression.

The pattern is extended by 5 more circles to 10 circles.

(ii) Determine the new value of L.

(iii) Calculate, in terms of π, the total area of the 10 circles of the new pattern.

1+2+2=5 marks

Q. 5 Prove by Induction that

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5), \qquad n \geq 1, \ n \text{ an integer.}$$

3 marks

Q. 6 Prove by Induction that

$$f(n) = 5^n + 8n + 3 \quad \text{is divisible by 4,} \qquad \text{for } n \geq 1, \ \ n \text{ an integer.}$$

3 marks

Q. 7 Prove by Induction that

$$3^n > (n+1)^2 \qquad \text{for } \ n \geq 3, \ \ n \ \text{ an integer.}$$

3 marks

Q. 8 A parabola C has parametric equations

$$x = 4t, \ y = -2t^2$$

(i) Determine the coordinates of the focus and equation of the directrix of C.

(ii) Show that the equation of the tangent to C at the general point $P(4t, -2t^2)$ is given by

$$xt + y = 2t^2$$

(iii) Show that any two tangents meeting on the directrix of C will be perpendicular.

3x2=6 marks

Q.9 A parabola has Cartesian equation

$$y = \frac{1}{2}x^2, \ \ x \in R$$

The points P and Q both lie on the parabola so that $\angle POQ$ is a right angle, where O is the origin.

The point M represents the midpoint of PQ.

(i) Using the parameters p and q for the points P and Q respectively, show that P and Q have coordinates $(p, \frac{1}{2}p^2)$ and $(q, \frac{1}{2}q^2)$.

(ii) Thus find the coordinates of the midpoint M in terms of p and q.

(iii) Show that $pq = -4$.

(iii) Using your results from (ii) and (iii) show that, as the position of P varies along the parabola, the locus of M is the curve with equation

$$y = x^2 + 2$$

2+1+1+2=6 marks

END OF TEST

SOLUTIONS

Q.1. $(m+1), (m^2+m)\ (3m^2-m-4)$.

(i) If sequence is arithmetic, then $(m^2+m)-(m+1) = (3m^2-m-4)-(m^2+m)$

ie $m^2-1 = 2m^2-2m-4$

$\Rightarrow m^2-2m-3=0$

$\Rightarrow (m-3)(m+1)=0$ (3)

Either $m=-1$ or $m=3$.

If $m=-1$, 0, 0, 0, which is not meaningful. - trivial sol^{n}. not a progression.

If $m=3$,
$T_1 = 4$, $T_2 = 12$, $T_4 = 20$.

$\therefore a=4$, $d=8$.

(ii) $S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[8+(n-1)\times 8] = \frac{n}{2}\times 8n = 4n^2$.

$=(2n)^2$.

thus S_n will always be a square number.

Q.2.

$\$50 + 70 + 90 + \cdots$ $a=50$ $d=20$.

T_1 T_2

(i)

$T_{10} = a+(n-1)d = 50+9\times 20 = \$230$.

(ii) $S_n = 7800 = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}\left[100+(n-1)\times 20\right]$

$7800 = \frac{n}{2}\left[80+20n\right]$.

$15600 = 80n + 20n^2$

$\div 20$

$n^2 + 4n - 780 = 0$

$(n+30)(n-26) = 0$

Since $n>0$, $n=26$

3

Q.3

$$\sum_{r=1}^{n} T_r = 128 - 2^{7-n}$$

(i) $\sum_{r=1}^{8} T_r = 128 - 2^{7-8} = 128 - \frac{1}{2} = 127{\cdot}5.$

(ii) $T_8 = S_8 - S_7 = \sum_{r=1}^{8} u_r - \sum_{r=1}^{7} u_r = 127{\cdot}5 - (128 - 2^{7-7})$

$= 127{\cdot}5 - 127$

$= 0{\cdot}5$

(iii) $S_1 = \sum_{r=1}^{1} T_r = 128 - 2^6 = 128 - 64 = 64 = u_1 \therefore a = 64.$

$S_2 \sum_{r=1}^{2} T_r = 128 - 2^{7-2} = 128 - 32 = 96$

$S_2 - S_1 = u_2 = 32 \qquad \frac{u_2}{u_1} = r = \frac{32}{64} = \frac{1}{2}$

4

Q.4.

$a=3$

(i) $a=3,\ ar,\ ar^2,\ ar^3,\ ar^4=48.$

$\frac{ar^4}{a}=\frac{48}{3}=16=r^4 \quad \therefore r=2$

(ii) new $L = 2\times\left[\underbrace{r_1+r_2+r_3+\dots+r_{10}}_{\text{G.P. with } a=3,\ r=2}\right]$

$S_{10}=\frac{a(r^n-1)}{r-1}=\frac{3(2^{10}-1)}{2-1}=3(2^{10}-1)=3069.$

new $L = 2\times 3069 = 6138$

(iii) Area $= A_1+A_2+A_3+\dots+A_{10}$

$= \pi\times 3^2+\pi\times 6^2+\pi\times 12^2+\pi\times 24^2$
$= \pi\times(3\times1)^2+\pi\times(3\times2)^2+\pi\times(3\times4)^2+\pi\times(3\times8)^2$
$= 9\pi+36\pi+144\pi+576\pi$

$= 9\pi\left(\underbrace{1+4+16+64+\dots}_{\text{GP with } a=1,\ r=4}\right)$

$S_{10}=\frac{1(4^{10}-1)}{4-1}=\frac{4^{10}-1}{3}$

Area $= 9\pi\times\left(\frac{4^{10}-1}{3}\right) = 3145725\pi$ units.

Prove

Q5. $$\sum_{r=1}^{n} r(r+3) = \frac{1}{3} n(n+1)(n+5) \quad , \quad n \geq 1.$$

Proof:

Show true for $n=1$

$$LHS = 1(4) = 4 \qquad RHS = \frac{1}{3} \times 1 \times (2) \times (6) = 4$$

$$LHS = RHS \quad \therefore \text{true}$$

Assume true for $n=k$, $k \geq 1$

ie $$\sum_{r=1}^{k} k(k+3) = \frac{1}{3} k(k+1)(k+5)$$

Prove true for $n=k+1$

ie prove $$\sum_{r=1}^{k+1} = \sum_{r=1}^{k} + (k+1)(k+4) = \frac{1}{3}(k+1)(k+2)(k+6) \quad , k \geq 1$$

$$LHS = \frac{1}{3} k(k+1)(k+5) + (k+1)(k+4) \qquad \text{from assumption}$$

$$= (k+1)\left[\frac{1}{3} k(k+5) + k+4\right]$$

$$= \frac{1}{3}(k+1)\left[k(k+5) + 3k+12\right]$$

$$= \frac{1}{3}(k+1)(k^2+8k+12)$$

$$= \frac{1}{3}(k+1)(k+6)(k+2) = RHS$$

thus it is true for $n=k+1$, if true for $n=k$ ...

Q.6.

To Prove: $f(n) = 5^n + 8n + 3$ is divisible by 4, for $n \geq 1$.

Proof: Show true for $n=1$ $\quad f(1) = 5 + 8 + 3 = 16$ which is div. by 4

$\therefore$ true for $n=1$.

Assume true for $n=k$. $\quad f(k) = 5^k + 8k + 3 = 4N$, N an integer.

Prove true for $n=k+1$ $\quad f(k+1) = 5^{k+1} + 8(k+1) + 3$

$$= 5.5^k + 8k + 8 + 3$$

$$= 5(5^k + 8k + 3) - 32k - 12$$

$$= 5.4N - 4(8k+3)$$

$$= 4(5N - 8k - 3)$$

Which is divisible by 4.

Thus it is true for $n=k+1$, if true for $n=k$, and ....

Q7. To Prove:

$$3^n > (n+1)^2 \quad , \; n \geq 3 \quad , \; n \text{ an integer.}$$

Proof: Show true for $n=3$.

$LHS = 3^3 = 27 \qquad RHS = 4^2 = 16$

Since $27 > 16$, it is true for $n=3$.

Assume true for $n=k$ ie $3^k > (k+1)^2$, $k \geq 3$, k an integer.

Prove true for $n=k+1$ ie that $3^{k+1} > (k+2)^2 = k^2+4k+4$

Now, $3^{k+1} = 3.3^k > 3.(k+1)^2$ from the assumption

$$= 3k^2+6k+3$$

$$> k^2+4k+\underbrace{2k+3}_{\text{we require } 2k+3>4}$$

Now $2k+3 \geq 9$ since $k \geq 3$ $\therefore$ $2k+3>4$

and thus $3^{k+1} > (k+2)^2$ for all $k \geq 3$.

Thus it is true for $n=k+1$ if true for $n=k$, and since it is ...

Q.8.

$x = 4t \qquad y = -2t^2 \qquad$

$t = \frac{x}{4} \qquad y = -\frac{x^2}{8} \qquad |a| = 2.$

(i) Focus $(0,-2)$

Directrix $y = 2$

(ii) For eq^n of tangent at $T(4t, -2t^2)$

$\frac{dy}{dx} = -\frac{x}{4} = -\frac{4t}{4} = -t$

$y + 2t^2 = -t(x - 4t) = -tx + 4t^2$

$\Rightarrow tx + y = 2t^2$

(iii) If meeting on directrix both pass through $(x, 2)$ when they meet.

$tx + y = 2t^2$

i.e. $tx + 2 = 2t^2$ for t values.

$2t^2 - tx - 2 = 0$

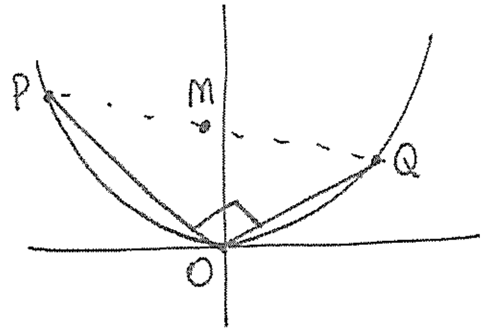
$t = \frac{x \pm \sqrt{x^2 + 16}}{4}$

$t_1 = \frac{x + \sqrt{x^2+16}}{4} \qquad t_2 = \frac{x - \sqrt{x^2+16}}{4}$ $\left\{ \text{where } t \text{ is the gradient of the tangent} \right\}$

$t_1 . t_2 = \frac{x^2 - (x^2+16)}{16} = \frac{-16}{16} = -1$ $\therefore$ the tangents must be perpendicular

Q.9. $y = \frac{1}{2}x^2$, $x \in \mathbb{R}$

P & Q lie on parabola, $\angle POQ = 90°$



(i) $2y = x^2$ $\therefore$ $4a = 2$ $a = \frac{1}{2}$

$\therefore$ Parametric eq^n are: $x = 2at = t$ $y = at^2 = \frac{1}{2}t^2$.

At $P(p, \frac{1}{2}p^2)$ & $Q(q, \frac{1}{2}q^2)$.

(ii) $M = \left(\frac{p+q}{2}, \frac{\frac{1}{2}p^2 + \frac{1}{2}q^2}{2}\right) = \left(\frac{p+q}{2}, \frac{p^2+q^2}{4}\right)$

(iii) $\angle POQ = 90°$

$\therefore$ $m_{OP} \cdot m_{OQ} = -1$ $m_{OP} = \frac{\frac{1}{2}p^2}{p} = \frac{p}{2}$ $\therefore$ $m_{OP} \cdot m_{OQ} = \frac{pq}{4} = -1$

$m_{OQ} = \frac{\frac{1}{2}q^2}{q} = \frac{q}{2}$ ie $pq = -4$

(iv) At M, $x = \frac{p+q}{2}$ $y = \frac{p^2+q^2}{4} = \left(\frac{p+q}{2}\right)^2 - \frac{pq}{2}$

But $pq = -4$

$\therefore$ $y = x^2 + 2$

as required.