Graphs Part Two

$y = f^n(x), n \text{ an integer} > 1$

The x-intercepts of $f(x)$ are intercepts of $f^n(x)$ and stationary points of $f^n(x)$. Proof…

- If n is even, they are also turning points of $f^n(x)$.
- The turning points of $f(x)$ are also turning points of $f^n(x)$.
- If $|f(x)| < 1$ then $|f^n(x)| < |f(x)|$.
- If $|f(x)| > 1$ then $|f^n(x)| > |f(x)|$.

Example 2. Sketch

a) $y = \sin^2 x$ b) $y = \sin^3 x$ c) $y = \ln^2 x$

Reciprocal Functions and Division of Ordinates

Suppose $f(x) = \frac{1}{h(x)}$. Then

when $h(x)$ is	$f(x)$ is/has
positive	positive
negative	negative
very small	very large
very large	very small
zero	vertical asymptotes
horizontal asymptote $y = a$	horizontal asymptote $y = \frac{1}{a}$

By the quotient rule, stationary points of $h(x)$, for which $h(x) \neq 0$ are also stationary points of $f(x)$.

Example. Let $f(x) = \frac{1}{(x-3)(x+1)}$.

By graphing $(x-3)(x+1)$, sketch $f(x)$.

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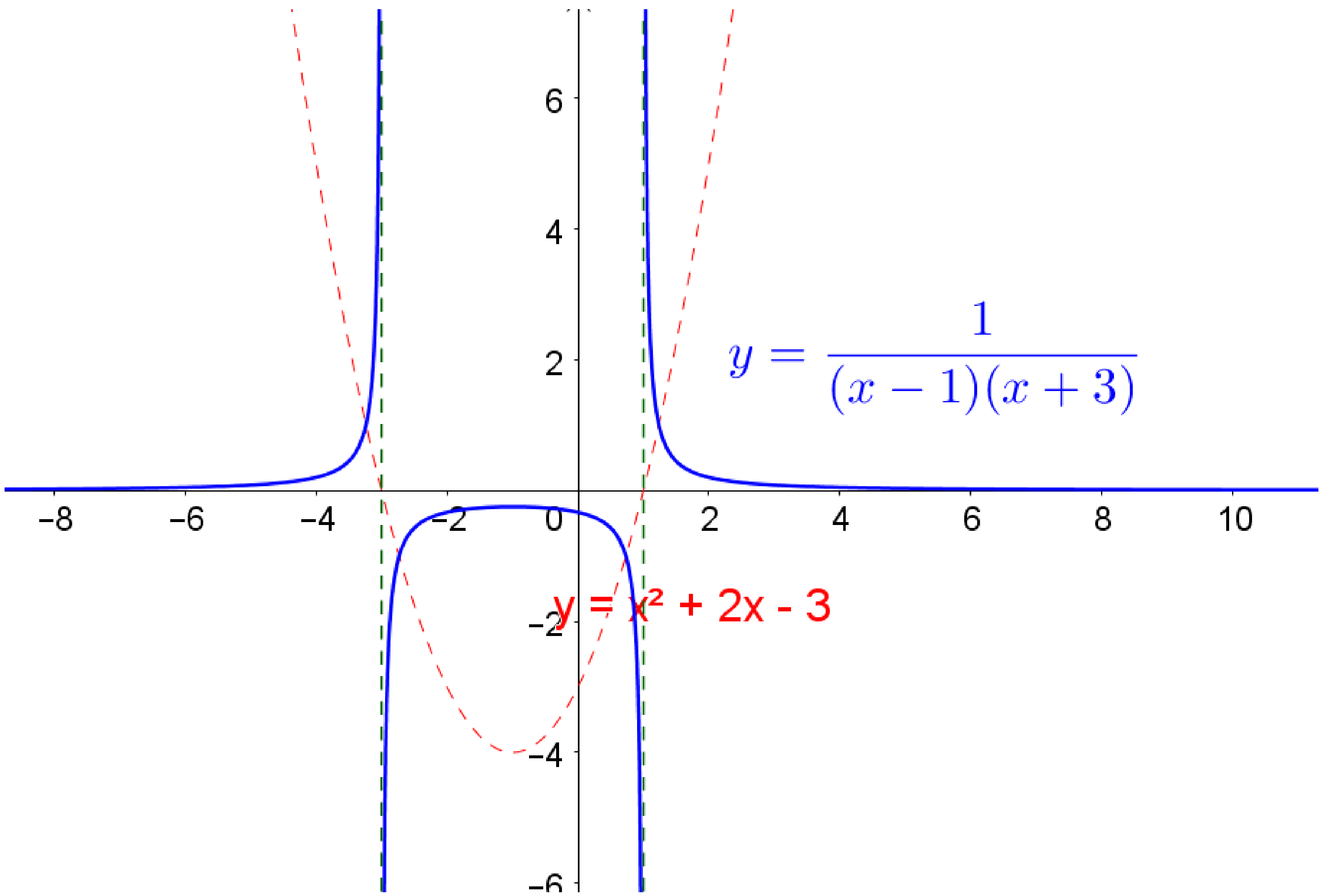
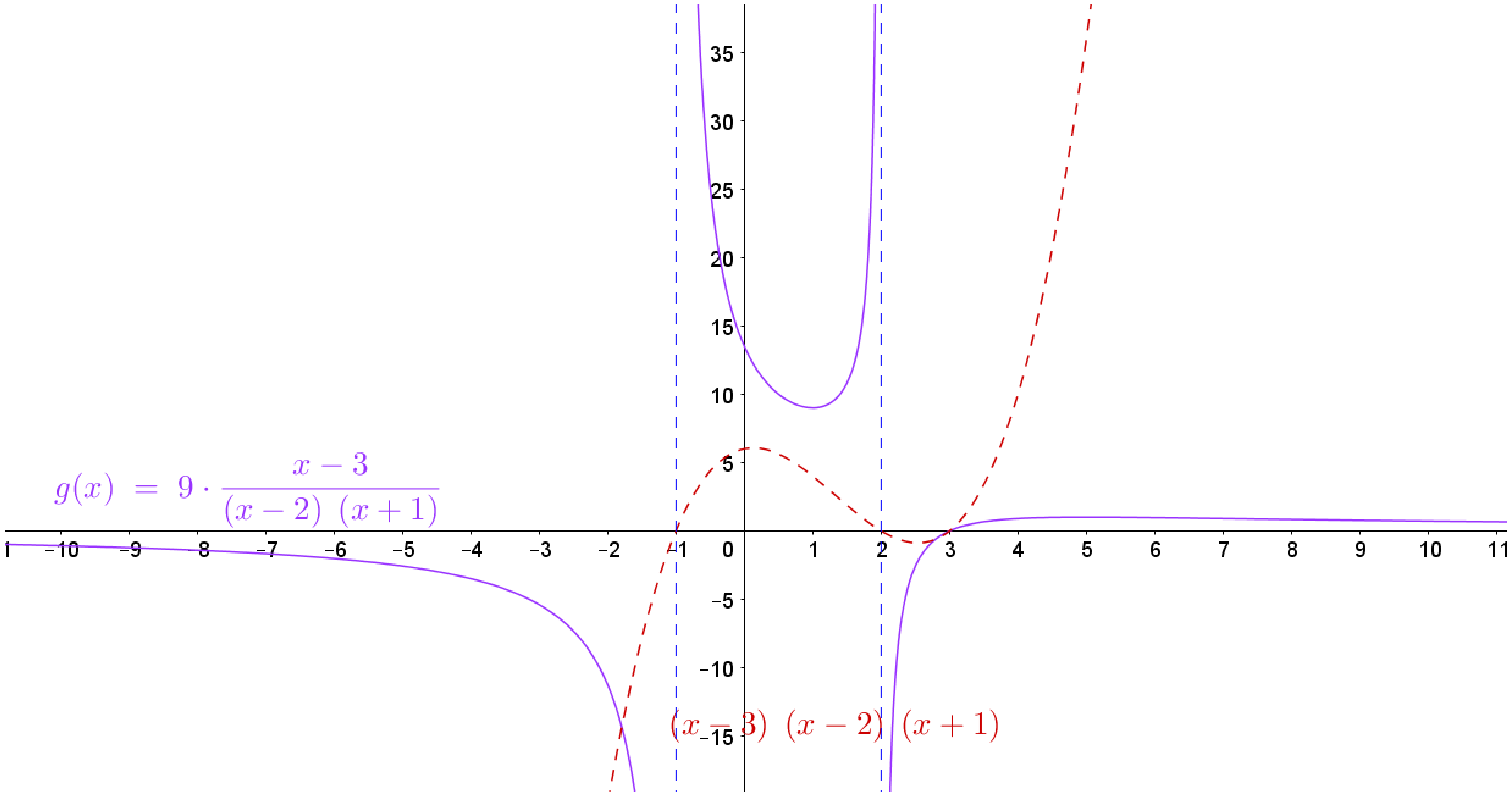


Example 2. $g(x) = \frac{9(x-3)}{(x-3)(x+1)}$



$y^2 = f(x)$

- Symmetrical about the x-axis, since $y = \pm\sqrt{f(x)}$
- Exists for $f(x) \geq 0$

Example. Sketch $y^2 = (x-2)^2(x-1)$, showing any stationary points.

Domain $x \geq 1$

x-intercepts: $x = 1, x = 2$

Differentiate implicitly, that is take the derivative of both sides of the equation with respect to x.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x-2)^2(x-1)$$

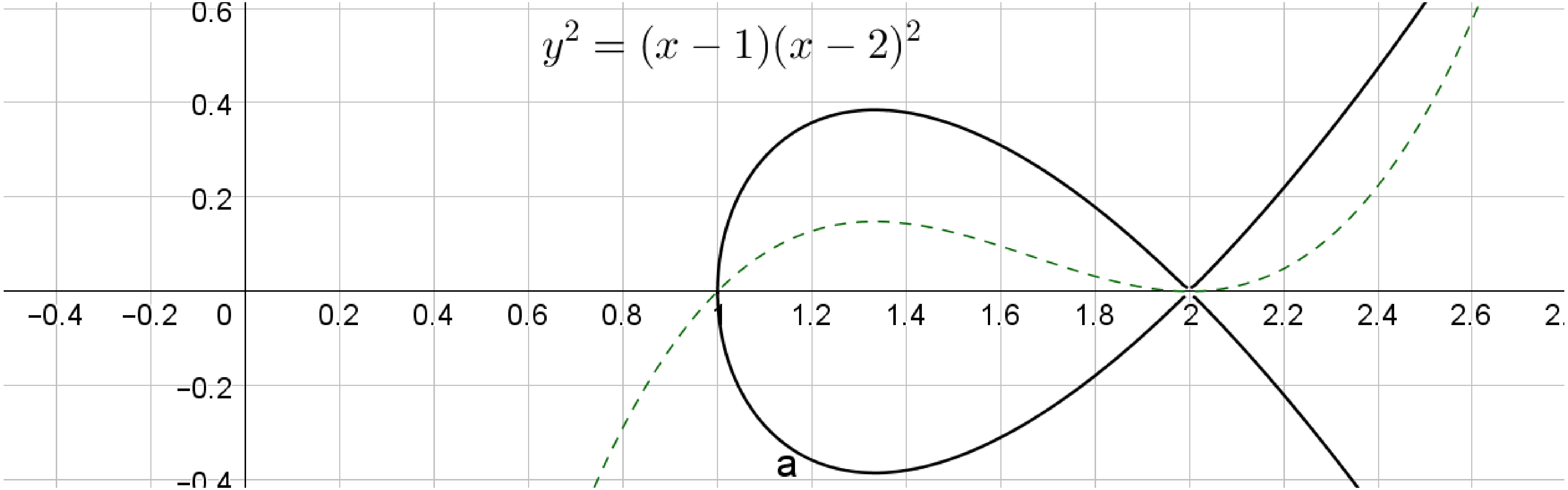
$$2y\frac{dy}{dx} = (x-2)^2 + 2(x-2)(x-1) \ldots$$

$\frac{dy}{dx} = \frac{3x-4}{\pm 2\sqrt{x-1}}, x \neq 2$

Stationary points: $\frac{dy}{dx} = 0$ at $\left({}^{4}/_{3}, \pm {}^{2\sqrt{3}}/_{9}\right)$

As $x \to 1$ from above $\frac{dy}{dx} \to \pm\infty$ and $\lim_{x\to 2} \frac{dy}{dx} = \pm 1$

To sketch, first sketch $y = (x-2)^2(x-1)$



Further examples using implicit differentiation

This technique uses the chain rule to allow us to find a derivative without making y the subject of the equation of the curve.

Example 1. Sketch $x^2 + y^2 = 4xy - 3$, showing any point with horizontal or vertical tangents.

Take the derivative of both sides with respect to x:

$$2x + 2y\frac{dy}{dx} = 4y + 4x\frac{dy}{dx}$$

$$2x - 4y = 4x\frac{dy}{dx} - 2y\frac{dy}{dx}$$

$$x - 2y = \frac{dy}{dx}(2x - y)$$

$$\frac{dy}{dx} = \frac{x-2y}{2x-y}$$

$\frac{dy}{dx} = 0$ (horizontal tangent) when $x = 2y$

Sub into equation of curve:

$$(2y)^2 + y^2 = 4(2y)y - 3$$

$$5y^2 = 8y^2 - 3$$

$$-3y^2 = -3$$

$$y = \pm 1$$

Horizontal tangents at (-2,-1) and (2,1)

Example 1 ctd: $x^2 + y^2 = 4xy - 3$

$\frac{dy}{dx} = \frac{x-2y}{2x-y}$

Vertical tangents when $2x = y$

$x^2 + (2x)^2 = 4x(2x) - 3$

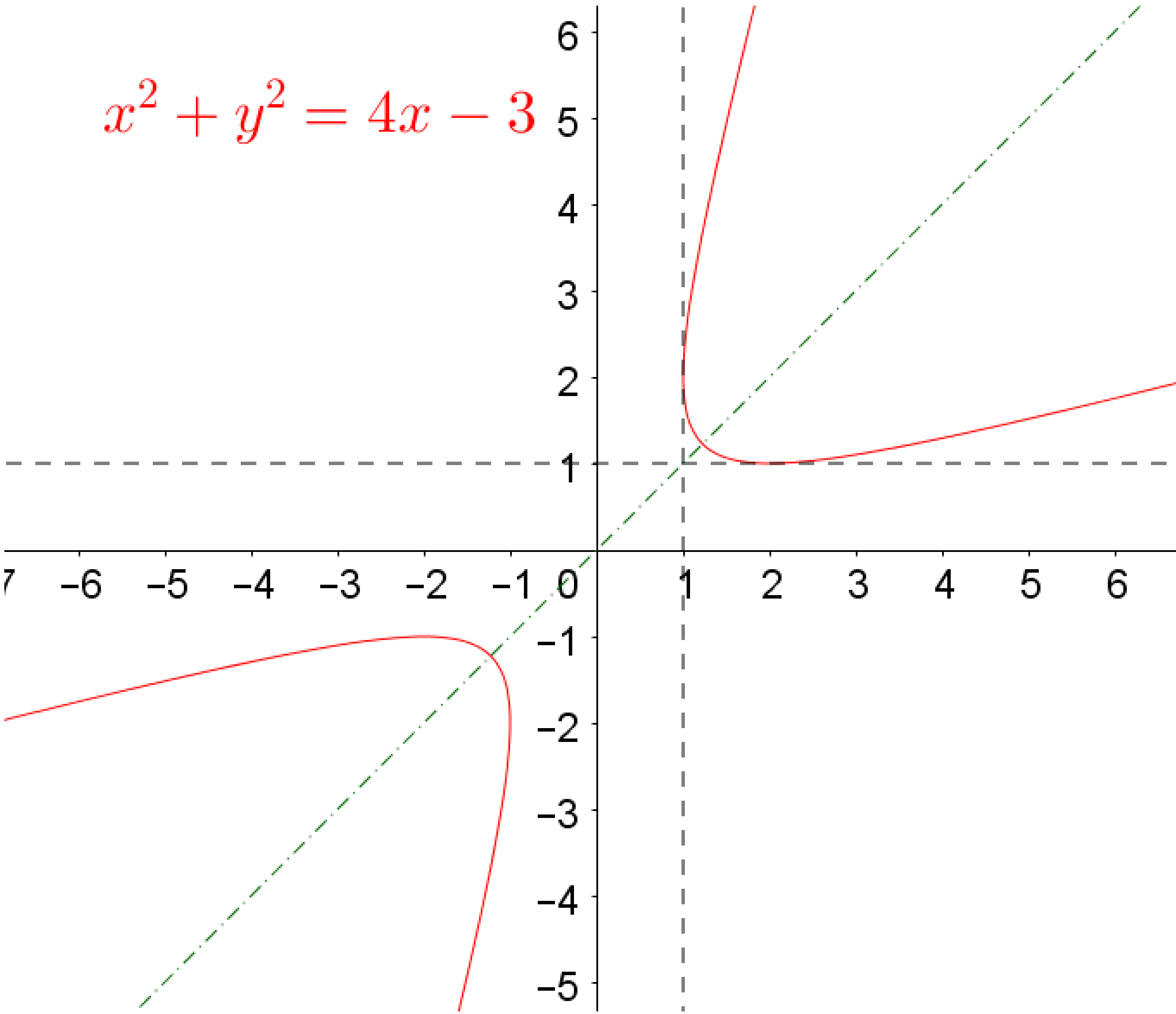
$5x^2 = 8x^2 - 3$

$x = \pm 1$

Vertical tangents at (-1, -2) and (1, 2)

Symmetry: Note that the curve is symmetric in $y = x$

No intercepts: must have $4xy \geq 3$



Using Graphs

Example 1. Sketch the graph of $y = |x-1| - |x-2|$. Hence solve the inequality $-1 < |x-1| - |x-2| < 1$.

Example 2. Sketch the graph of $y = x^5 - 5x^4$. Hence find the values of the real number k for which $x^5 - 5x^4 = kx$ has 3 distinct real roots.

Example 3. Sketch the graph of $y = \frac{x^2}{x^2-1}$. Hence find the values of x for which $\frac{x^2}{x^2-1} > 1$.

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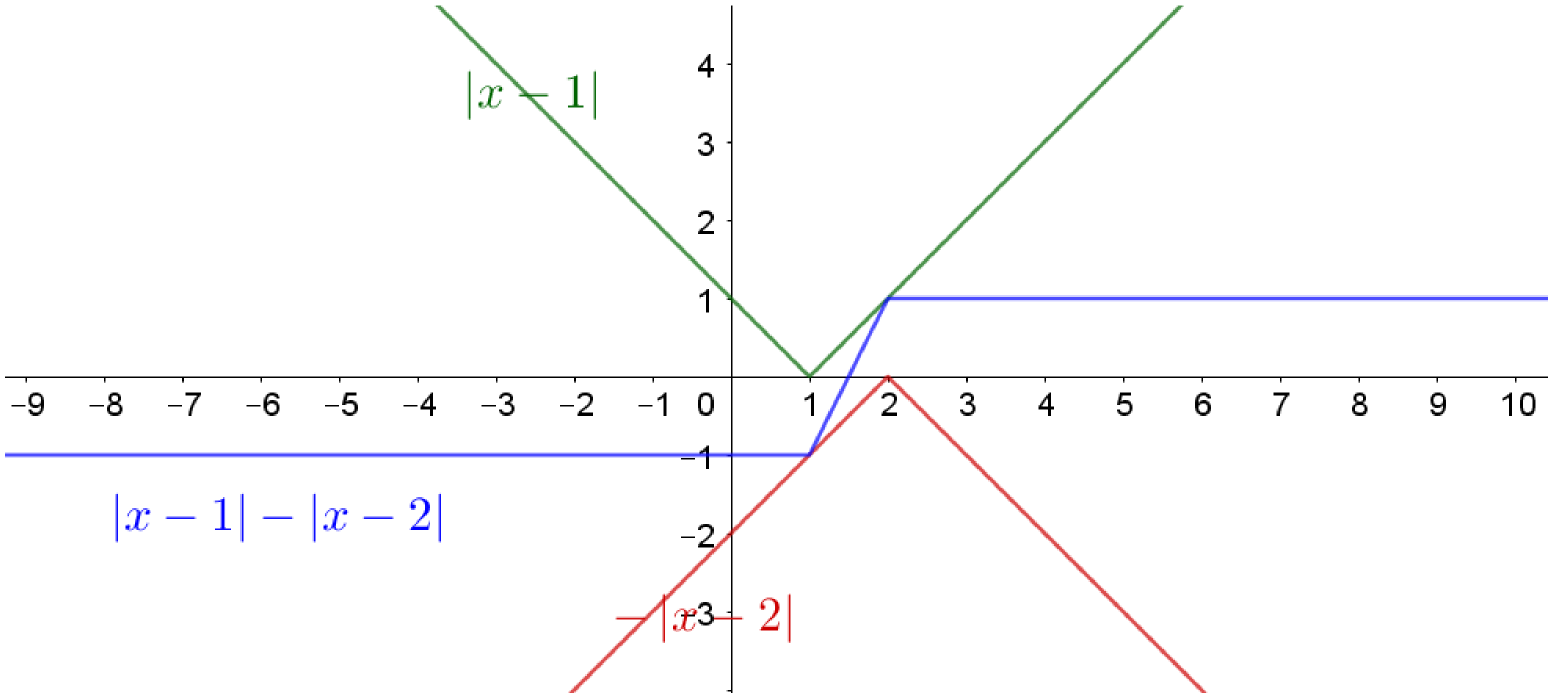
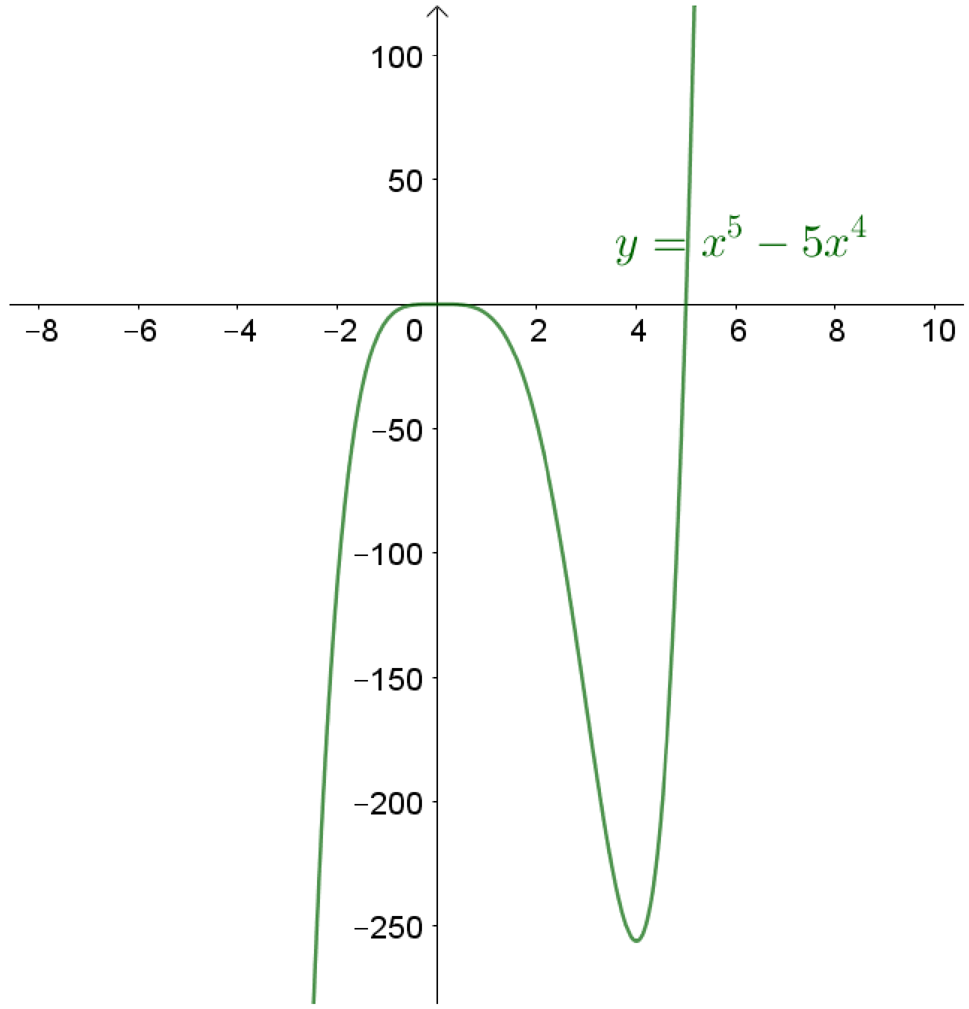


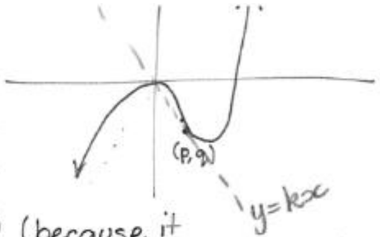
From the graph, $-1 < |x-1| - |x-2| < 1$ for $1 < x < 2$.

Example 2. Sketch the graph of $y = x^5 - 5x^4$. Hence find the values of the real number k for which $x^5 - 5x^4 = kx$ has 3 distinct real roots.



$y = x^5 - 5x^4$

suppose $y = kx$ is tangent at $y = (p, q)$



then $q = p^5 - 5p^4$ ① (because it is on $y = x^5 - 5x^4$

$\frac{dy}{dx} = 5x^4 - 20x^3$

$\therefore k = 5p^4 - 20p^3$

$q = 5p^4(p-4)$ ② – since it is on $y = kx$

① = ②

$\cancel{p^4}(p-5) = 5\cancel{p^4}(p-4)$

$p - 5 = 5p - 20$

$4p - 15 = 0$

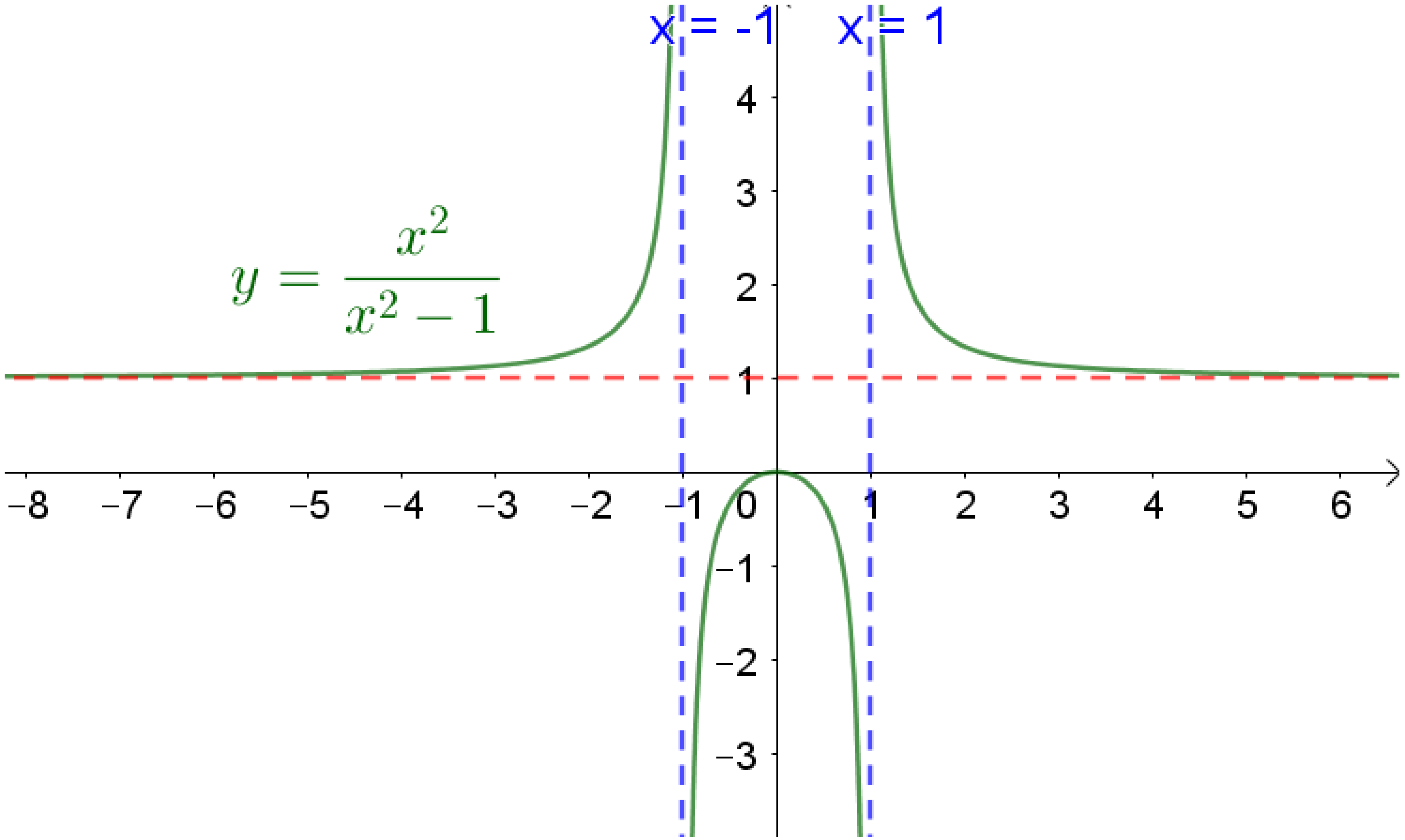
$p = 15/4$

$k = 5\left(\frac{15}{4}\right)^3\left(-\frac{1}{4}\right)$

$= -\frac{16875}{256}$

$x^5 - 5x^4 = kx$ has 3 distinct real roots for $k > 0$, and $\frac{-16875}{256} < k < 0$.

Example 3. Sketch the graph of $y = \frac{x^2}{x^2-1}$. Hence find the values of x for which $\frac{x^2}{x^2-1} > 1$.



$\frac{x^2}{x^2-1} > 1$ for $x < -1$ and $x > 1$.

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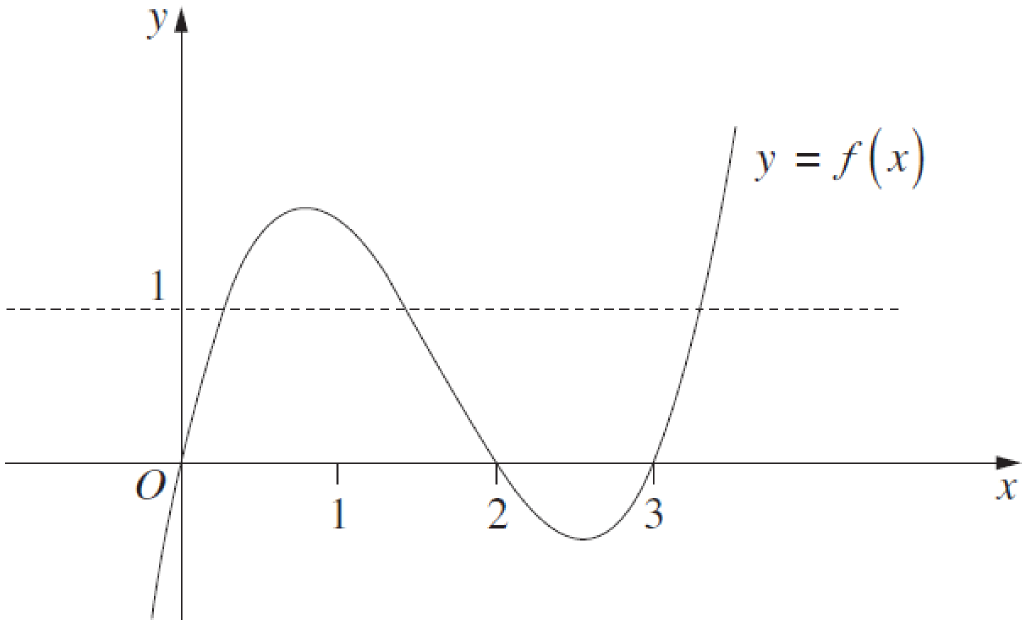
(b) The diagram shows the graph of a function $f(x)$.

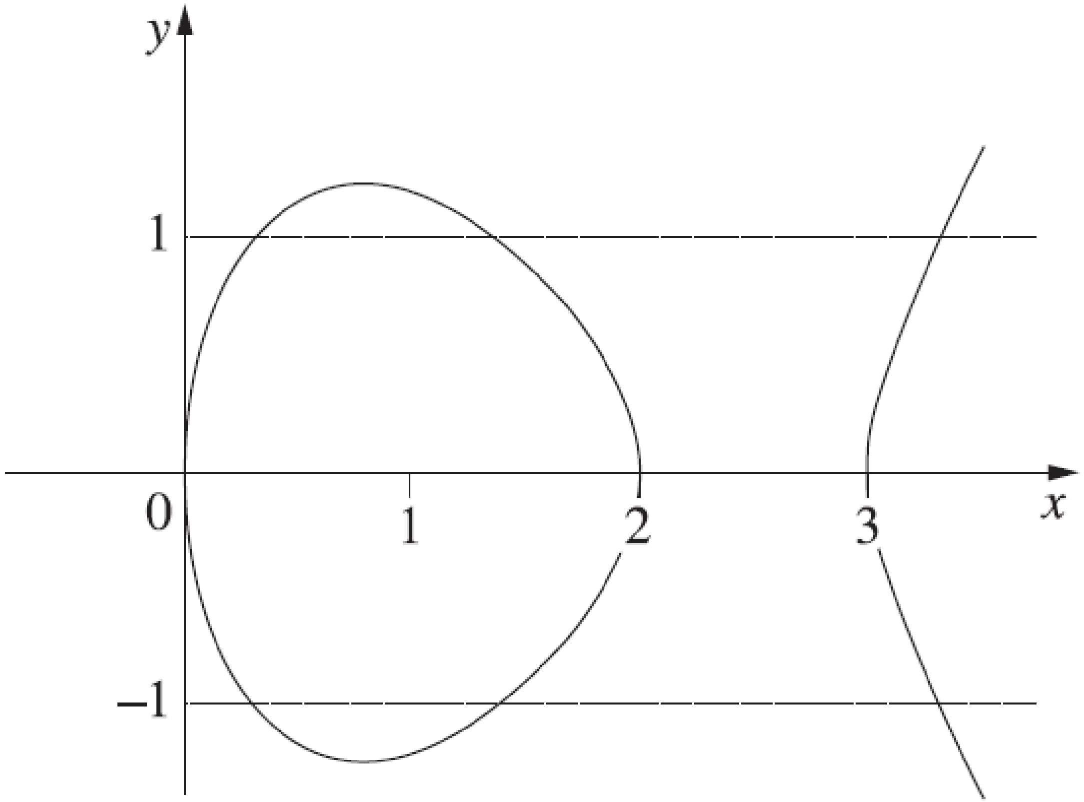


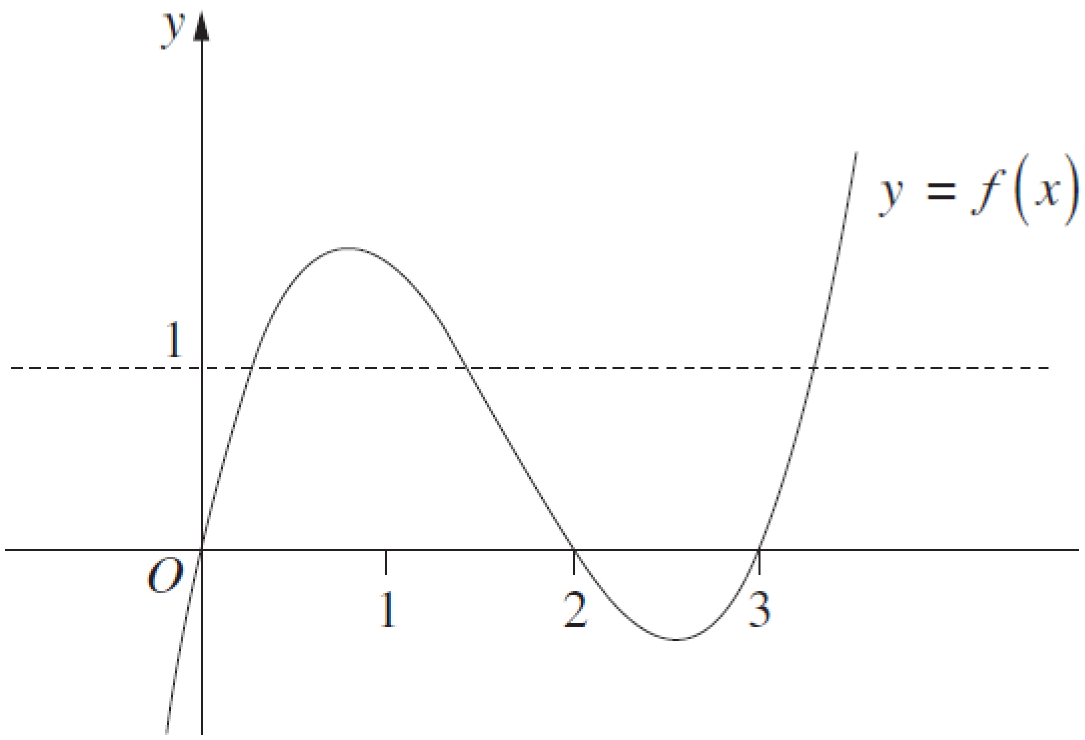
Sketch the following curves on separate half-page diagrams.

(i) $y^2 = f(x)$ **2**

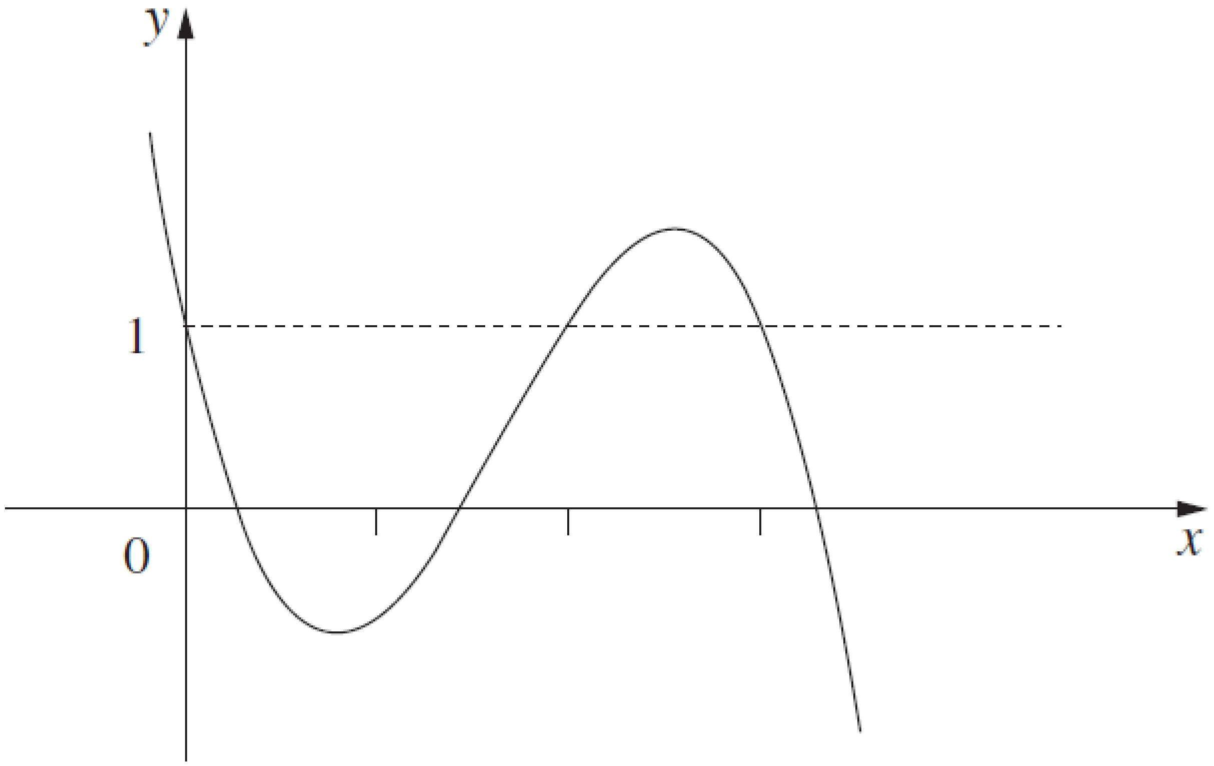
(ii) $y = \dfrac{1}{1 - f(x)}$ **3**

$$y^2 = f(x)$$





First sketch $1 - f(x)$



Now sketch $\dfrac{1}{1-f(x)}$