CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Mathematics Extension 2

Assessment Task 1

2020

Time allowed: 1 hour 40 minutes

Student Number: ___________________________ **Teacher:** Ms Strilakos

Instructions:

- All questions should be attempted.
- Show ALL necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.

PROOF	Q1	Q2	Q3	Q4	Q5	Q6			TOTAL
	/3	/4	/5	/5	/4	/4			/25
COMPLEX NUMBERS	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	
	/4	/4	/4	/4	/5	/6	/8	/11	/46
							TOTAL		/71

Q.1 Consider the statement

For any integers a and b, $a + b \geq 15$ implies that $a \geq 8$ or $b \geq 8$,

(i) State the contrapositive of this statement

(ii) Hence prove this statement is true for the contrapositive of the statement.

[1+2=3]

Q.2 (i) Let $x \in Z$. Prove by contradiction that if $5x - 7$ is odd, then x is even.

(ii) Hence prove directly that if $5x - 7$ is odd, then $9x + 2$ is even.

[2+2=4]

Q.3 Let $x \in Z$. (i) Prove that if $3|x$, then $3|x^2$.

(ii) Prove that if $3 \nmid x$, then $3|(x^2-1)$, using cases.

[2+3=5]

Q.4 If $T(0) = 6$ and $T_n = 4T_{n-1} + 2^n$ for $n \geq 1$,

use Induction to prove that $T_n = 7 \cdot 4^n - 2^n$ [5]

Q.5 Use a calculus method to prove that if $x \in R,\ x > 0,$ then $x^4 + x^{-4} \geq 2.$

[4]

Q.6 The diagram below shows two right angled triangles.

The left one has sides a, b and c where c is the length of the hypotenuse.

The triangle on the right has sides of length a+1, b+1 and c+1, where c+1 is the length of the hypotenuse. Show that a, b and c cannot all be integers.

[4]

Q.7 Prove by contradiction, the proposition that:

For each real number x, if $0 < x < 1$, then

$$\frac{1}{x(1-x)} \geq 4$$

[4]

Q.8 (i) Show that $\frac{a}{b} + \frac{b}{a} \geq 2$ using the AM/GM inequality.

(ii) Hence show that, for a, b and c all positive reals, that

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$$

[2+2=4]

Q.9 (i) Find the square roots of $-8-6i$.

(ii) Hence or otherwise, solve the equation $2x^2 + (1+i)x + (1+i) = 0$

[4]

Q.10 A straight line L and a circle C are to be drawn on a standard Argand diagram.

The equation of L is $\arg z = \frac{\pi}{3}$.

The centre of C lies on L and its radius is 3 units. The line with equation $Im\, z = 0$ is tangent to C.

(i) Sketch L and C on the same diagram.

(ii) Determine an equation for C, giving the answer in the form $|z - \alpha| = k$, where α and k are constants.

The point that represents the complex number z_0 lies on C.

(iii) Determine the maximum value of $\arg z_0$, fully justifying the answer.

[4]

Q.11 (i) Express the solution in Cartesian form for the set of complex numbers described simultaneously below, using $z = x + iy$.

$Im\big(2z - \bar{z}(1+i)\big) = 0$ and $Re\big(2z - \bar{z}(1+i)\big) < 4,\ z \in C,$

(ii) Hence sketch the solution in the complex plane, labelling relevant points.

[4+1=5]

Q.12 Given the complex number $z^4 = -9i$,

(i) Determine the four fourth roots of z^4, giving answers in the form $re^{i\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$.

(ii) Plot the points represented by these roots on an Argand diagram and join them in order of increasing argument, labelled as A, B, C and D.

The midpoints of the sides of the quadrilateral $ABCD$ represent the four fourth roots of another complex number w.

(iii) Find the complex root w_1 of w, which represents the midpoint of the side AD, stating it in $re^{i\theta}$ form.

(iv) Hence find w in Cartesian form.

[2+1+2+1=6]

Q.13 Euler's formula states that for any real number θ,

$$e^{i\theta} = \cos\theta + i\,sin\,\theta$$

(i) Using Euler's formula, express $e^{-i\theta}$ in terms of $\sin\theta$ and $\cos\theta$,

and hence find expressions for $\sin\theta$ and $\cos\theta$ in terms of the complex exponential.

(ii) Using your results for part (i), express $sin^3\theta cos^2\theta$ in the form

$$a\sin\theta + b\ sin\ 3\theta + c\ sin\ 5\theta\ .$$

You may find the expansion of the Binomial Theorem helpful to use for this.

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

(iii) Hence, find the solutions of $\sin 5\theta - \sin 3\theta = 0$ in the interval $0 \leq \theta < \pi$.

Give your answers in exact form.

[2+2+4=8]

Q.14 (i) If $z = \cos\theta + i\sin\theta$, show that $1 + z = 2\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$.

(ii) z_1, z_2 are complex numbers such that $|z_1| = |z_2| = 1$.

If z_1, z_2 have arguments α, β respectively, where $-\pi < \alpha < \pi$ and $-\pi < \beta < \pi$,

show that $\frac{z_1+z_1z_2}{z_1+1}$ has modulus $\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$ and argument $\frac{\alpha+\beta}{2}$.

(iii) Given $|z_1| = |z_2| = 1$ and $\frac{z_1+z_1z_2}{z_1+1} = 2i$, find z_1 and z_2 in the form $x + iy$.

[1+2+8=11]

END OF PAPER