CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Mathematics 2U

Term1 Half-Yearly HSC 2014

Time allowed: 2 hours

Name: ______________________________ **Class:** ________

Lobejko / White / Fardouly / Lego / Wilson

Instructions:

- All questions should be attempted.
- Show ALL necessary working on your own paper.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Start each question on a new page and only write on one side of each sheet of paper.

	MC	Q1	Q2	Q3	Q4	Q5	TOTAL
H3		/13	/6	/4	/5	/2	/30
H4			/7			/5	/12
H5				/9	/8	/6	/23
TOTAL	/10	/13	/13	/13	/13	/13	/75

Section A: Multiple Choice (10 marks)

	Questions
1.	Evaluate, correct to 3 significant figures, $\sqrt[3]{\frac{64.93 \times 13.4}{8.62+4.69}}$ A. 4.02 B. 4.027 C. 4.028 D. 4.03
2.	Solve for x, $\lvert 2x+1 \rvert = 7$ A. $x=-3, x=4$ B. $x=-3, x=-4$ C. $x=3,\ x=-4$ D. $x=3,\ x=4$
3.	Find the value of x if $\sqrt{75}+\sqrt{27}=\sqrt{x}$ A. 3 B. $\sqrt{3}$ C. 24 D. 192
4.	Solve for x, if $\frac{10x+1}{4} = x+1$ A. 0 B. $\frac{1}{2}$ C. $\frac{4}{9}$ D. 2
5.	The volume of a cylinder is given as $V=\pi r^2 h$. Find r, its radius, if the volume is $905 cm^3$ and the height is 18 cm. A. 4 cm B. 16 cm C. 72 cm D. 5185 cm
6.	The line l passes through the point $P(2,6)$ and has a gradient of $-\frac{1}{3}$. The equation of l is written in general form as: A. $3y = 20 - x$ B. $y = -\frac{1}{3}x + 6\frac{2}{3}$ C. $x + 3y - 16 = 0$ D. $x + 3y - 20 = 0$
7.	The domain and range for the function $y = \sqrt{7-x}$ is A. $x \leq 7$; $y \geq 0$ B. $0 \leq x \leq 7$; $y \geq 0$ C. $x \geq 0$; $y \geq 0$ D. All real x, All real y.
8.	What is the expression, $10x - 4x^2 + 24$, when fully factorised? A. $2(5x - 2x^2 + 12)$ B. $-2(-5x + 2x^2 - 12)$ C. $2(3-2x)(4-x)$ D. $2(3+2x)(4-x)$

	Questions
9.	Simplify $(cot\,\theta + cosec\,\theta)\,(cosec\,\theta - cot\,\theta)$. A. $\cos\theta - \sin\theta$ B. $\cos\theta + \sin\theta$ C. $cos^2\theta - sin^2\theta$ D. $cos^2\theta + sin^2\theta$
10.	The perpendicular distance from $(k, 1)$ to $3x + 4y = 1$ is 3 units. Find the possible value(s) of k. A. $k = 4$ B. $k = -6$ C. $k = 4$ or $k = -6$ D. $k = \frac{10}{3}$ or $k = -\frac{20}{3}$

End of Section A

Section B: Short responses

Question 1: Begin a new booklet (13 marks)

Questions	Marks
a) Given the parabola $x^2 - 6x = 8y + 23$	
(i) By completing the square, write the equation in the form $(x-h)^2 = 4a(y-k)$.	2
(ii) Find the coordinates of the vertex and focus.	2
(iii) Write the equation of the axis of symmetry of the parabola.	1
(iv) Draw a neat sketch of the parabola showing the above information.	2
b) The coordinates of E and F are $(-2, 1)$ and $(-3, -2)$ respectively. Find the equation of the locus of all points $P(x, y)$ such that $PF = 3 \times PE$.	2
c) $A\,(-1,3)$ and $B(3,1)$ are two points on the number plane.	
(i) Find the locus of a point $P(x, y)$ such that PA is perpendicular to PB.	2
(ii) Show algebraically that the locus of P is a circle and state its centre and radius.	2

Question 2: Begin a new booklet (13 marks)

Questions	Marks
a) Evaluate $\sum_{n=2}^{5} 3 + 2^{n-1}$	2
b) The third term of an arithmetic sequence is 8 and the sixteenth term is 47.	
(i) Find the first term and the common difference.	2
(ii) Find the sum of the first 40 terms of the series.	2

Questions	Marks
c) Gayle invests in a superannuation fund which pays 5% p.a. interest compounded annually. She pays \$12 000 into the fund on 1st July each year.	
(i) What is the value of Gayle's investment on 30th June, one year after she makes her first payment?	1
(ii) What is the value of the investment on the 30th June, ten years after she made her first payment?	3
(iii) After making her tenth payment, Gayle considers increasing her annual payment to M dollars each year. Show that if Gayle does this, the value of her investment twenty years after her first payment of \$12 000 was made would be approximately equal to $13.2068\,(12\,000 \times 1.05^{10} + M)$.	3

Question 3: Begin a new booklet (13 marks)

Questions	Marks
a) Differentiate the following :	
(i) $2x^3 - 5 + \frac{3}{x}$	1
(ii) $\sqrt{4x+1}$	1
(iii) $\frac{2x+4}{x+1}$	2
b) For the curve $y = 4x^3 - 12x + 2$	
(i) Find the coordinates of the two stationary points.	2
(ii) Determine the nature of the stationary points.	2
(iii) Determine any points of inflexion.	2
(iv) Sketch the curve, clearly marking all important features, for the domain $-3 \le x \le 3$.	2
(v) What is the minimum value of $4x^3 - 12x + 2$ in the domain $-3 \le x \le 3$	1

Question 4: Begin a new booklet (13 marks)

<table>
<tr><th>Questions</th><th>Marks</th></tr>
<tr><td>a) Find the primitive of

(i) $\int \sqrt{x}\ dx$

(ii) $\int \frac{3x^3-2x^2+x^{-1}}{x^2}\ dx$</td><td>

1

2</td></tr>
<tr><td>b) Evaluate $\int_{-1}^{2} (2x-2)^4\ dx$</td><td>2</td></tr>
<tr><td>c) Consider the function $y = \frac{x}{x+1}$

(i) Copy and complete the following table.

x	0	2	4	6	8
y					

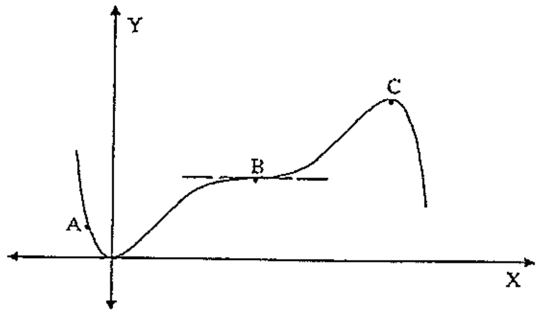
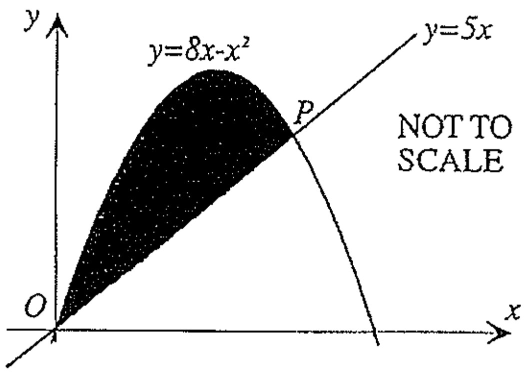
(ii) Apply Simpson's Rule with 5 function values to find an approximation for

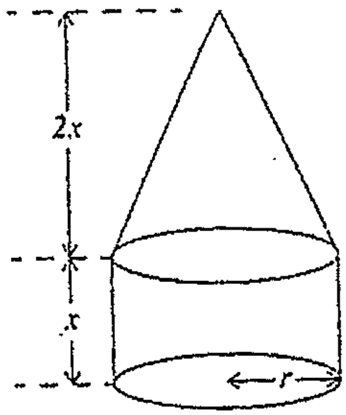
$$\int_0^8 \frac{x}{x+1}\ dx$$
Give your answer correct to one decimal place.</td><td>

2

3</td></tr>
<tr><td>d) Find y in terms of x if $\frac{d^2y}{dx^2} = 3x$ given $y = 12$ and $\frac{dy}{dx} = 9$ when $x = 2$.</td><td>3</td></tr>
</table>

Question 5: Begin a new booklet (13 marks)

Questions	Marks
a) 	
	$f'(x)$	$f''(x)$ | |---|---|---| | A | | | | B | | | | C | | |	
Copy the table onto your working page. Complete the table by indicating whether the derivatives are positive, negative or equal to zero at the points A, B and C.	2
b) 	
The graphs of $y = 5x$ and $y = 8x - x^2$ intersect at the origin and point P.	
(i) Show that the coordinates of P are (3, 15).	1
(ii) Find the area of the shaded region.	2
(iii) This region is rotated about the x −axis. Find the volume of the solid obtained. Answer to 2 decimal places.	2

Questions	Marks
c) A grain silo has a cylindrical shaped wall and a cone shaped roof as in the diagram. Let the radius of the base of the silo be r metres, the height of the cylinder be x metres and the height of the cone be $2x$ metres. NOT TO SCALE 	
(i) Show that if the length of the slant side of the cone is 20 metres, then $r^2 = 20^2 - 4x^2$.	1
(ii) Show that the volume, V, of the silo is given by $V = \frac{20}{3}\pi(100x - x^3)$	2
(iii) Find the exact height of the silo so that it holds the maximum amount of grain.	3

End of Paper

Name:____________________________

Class: _______________

Teacher : Mr White Mrs Lobejko Mr Fardouly Mrs Lego Mr Wilson

Year 12 Half Yearly 2014

Mathematics

Multiple Choice Sheet

Select the alternative A, B, C, or D that best answers the question. Fill in the response oval completely, using blue or black pen. Mark **only one** oval per question.

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

Y12 H/Y Sol'n.

MC (10 marks)

1. D
2. C
3. D
4. B
5. A
6. D
7. A
8. D
9. D
10. C

Section B

Q1 : 13 marks

1) (i) $x^2 - 6x = 8y + 23$

$(x^2 - \frac{6}{2}x + (-3)^2) = 8y + 23 + 9$ ①

$(x-3)^2 = 8y + 32$

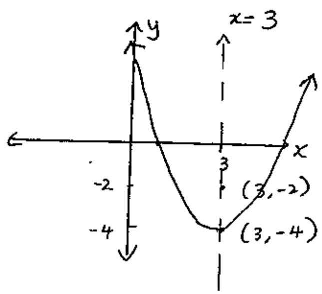
$(x-3)^2 = 8(y+4)$ ①

(ii) vertex $(3, -4)$ ①

focus $(3, -2)$ ①

(iii) $x = 3$ ①

(iv)



① concave up

① labelling pts correctly.

b) $PF = 3 \times PE$

$\sqrt{(x+3)^2 + (y+2)^2} = 3 \times \sqrt{(x+2)^2 + (y-1)^2}$

$(x+3)^2 + (y+2)^2 = 9\left[(x+2)^2 + (y-1)^2\right]$

$x^2 + 6x + 9 + y^2 + 4y + 4 = 9\left[x^2 + 4x + 4 + y^2 - 2y + 1\right]$

$x^2 + 6x + 9 + y^2 + 4y + 4 = 9x^2 + 36x + 9y^2 - 18y + 45$

$0 = 8x^2 + 8y^2 + 30x - 22y + 32$

$\therefore 8x^2 + 30x + 8y^2 - 22y + 32 = 0$

① correct expansion

① correct simplification.

c. (i) m of $PA = \frac{y-3}{x--1} = \frac{y-3}{x+1}$ $\qquad$ m of $PB = \frac{y-1}{x-3}$

$PA \perp PB$ & $\frac{y-3}{x+1} \times \frac{y-1}{x-3} = -1$

$\frac{y^2 - 4y + 3}{x^2 - 2x - 3} = -1$

$y^2 - 4y + 3 = -(x^2 - 2x - 3)$

$y^2 - 4y + 3 = -x^2 + 2x + 3$

$x^2 - 2x + y^2 - 4y = 0$

(ii) $(x^2 - \frac{2x}{2} + (-1)^2) + (y^2 - 4y + (-2)^2) = 0 + (-1)^2 + (-2)^2$

$(x-1)^2 + (y-2)^2 = 5$

$\therefore$ radius $= \sqrt{5}$ units.

centre $= (1, 2)$.

Question 2

a) $\sum_{2}^{5} 3 + 2^{n-1} = (3 + 2 + 3 + 2^2 + 3 + 2^3 + 3 + 2^4)$

$= 42$

b) (i) $T_3 = 8$ $\qquad$ $T_{16} = 47$

$a + 2d = 8$ $\qquad$ $a + 15d = 47$

$a + 15d = 47$

$a + 2d = 8$

$13d = 39$

$d = 3$

$a = 2$.

(ii) $S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{40}{2}(4 + 39.3)$

$= 2420$

c (i) $A = P(1+r)^n$

$= 12000(1 + 0.05)$

$= \$12600$

(ii) value after 2yrs: $12000(1.05)^2 + (2000(1.05)$

value after 3yrs: $12000(1.05)^3 + 12000(1.05)^2 + \ldots$

$\therefore$ after 10 yrs: $12000(1.05)^{10} + 12000(1.05)^9 + \ldots + 12000(1.05)$

$= 12000(1.05 + 1.05^2 + \ldots + 1.05^{10})$

$= 12000 \times 1.05 \left(\frac{1.05^{10} - 1}{1.05 - 1}\right)$

$= \$158\,481.45$.

(iii) At 20 yrs.

$A_{20} = 12000(1.05)^{20} + \ldots + 12000(1.05)^{11} + M(1.05)^{10} + \ldots + M(1.05)$

$= 12000 \times 1.05^{11}(1 + 1.05 + \ldots + 1.05^9) + M \times 1.05(1 + 1.05 + \ldots + 1.05^9)$

$= \left(\frac{1.05^{10} - 1}{1.05 - 1}\right)(12000 \times 1.05^{11} + 1.05M)$

$= 12.57789 \times 1.05\,(12000 \times 1.05^{10} + M)$

$= 13.2068\,(12000 \times 1.05^{10} + M)$

(ii) ~~$A_{10} = 158481.45$~~

(iii) $A_{11} = 158481.45(1.05) + m(1.05)$

$A_{12} = 158481.45(1.05)^2 + m(1.05)^2 + m(1.05)$

$A_{20} = 158481.45(1.05)^{10} + m(1.05 + 1.05^2 + \ldots 1.05^{10})$

$= 158481.45(1.05)^{10} + m\left(\frac{1.05(1.05^{10} - 1)}{0.05}\right)$

$= 158481.45(1.05)^{10} + M \times 13.20678$

$= 13.20678\,(12000(1.05)^{10} + m)$

Q3 :

a) (i) $\frac{d}{dx} = 6x^2 - 3x^{-2}$

(ii) $\frac{d}{dx} = \frac{1}{2}(4x+1)^{-\frac{1}{2}} \cdot 4$

$= \frac{2}{\sqrt{4x+1}}$

(iii) $\frac{d}{dx} = \frac{(x+1)\cdot 2 - (2x+4)\cdot 1}{(x+1)^2}$

$= \frac{2x+2-2x-4}{(x+1)^2}$

$= \frac{-2}{(x+1)^2}$

b) (i) $y = 4x^3 - 12x + 2$

$\frac{dy}{dx} = 12x^2 - 12$

stat when $\frac{dy}{dx} = 0$

$12x^2 - 12 = 0$

$12(x^2-1) = 0$

$12(x+1)(x-1) = 0$

$x = -1 \qquad x = 1$

$y = 10 \qquad y = -6$

$(-1, 10)$ and $(1, -6)$

(ii) $\frac{d^2y}{dx^2} = 24x$

at $x = 1$ $\quad \frac{d^2y}{dx^2} > 0$ min at $(1, -6)$

at $x = -1$ $\quad \frac{d^2y}{dx^2} < 0$ max at $(-1, 10)$.

(iii) pt of inf. when $\frac{d^2y}{dx^2} = 0$

$24x = 0$

$x = 0$

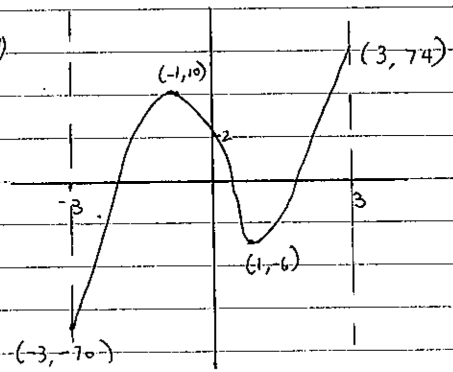
$y = 2$

test for concavity:

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{d^2y}{dx^2}$	-ve	0	+ve

change in sign, ∴ pt of inflexion at $(0, 2)$.

(iv)



(v) minimum value is -70 at $x = -3$.

Question 4

a) (i) $\int \sqrt{x}\, dx = \frac{x^{3/2}}{3/2} + c$

$= \frac{2\sqrt{x^3}}{3} + c$

(ii) $\int \frac{3x^3}{x^2} - \frac{2x^2}{x^2} + \frac{x^{-1}}{x^2}\, dx$

$= \int 3x - 2 + x^{-3}\, dx$

$= \frac{3x^2}{2} - 2x + \frac{x^{-2}}{-2} + c$

$= \frac{3x^2}{2} - 2x - \frac{1}{2x^2} + c$

b) $\int_{-1}^{2} (2x-2)^4\, dx$

$= \left[\frac{(2x-2)^5}{2 \cdot 5}\right]_{-1}^{2}$

$= \left(\frac{2^5}{10}\right) - \left(\frac{-4^5}{10}\right)$

$= 105\frac{3}{5}$

c) (i)

x	0	2	4	6	8
y	0	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{8}{9}$

(ii) $h = \frac{8-0}{4}$

$= 2$

$A \doteqdot \frac{2}{3}\left\{f(0) + f(8) + 4\left[f(2) + f(6)\right] + 2f(4)\right\}$

$\doteqdot \frac{2}{3}\left\{\left(0 + \frac{8}{9}\right) + 4\left(\frac{2}{3} + \frac{6}{7}\right) + 2\left(\frac{4}{5}\right)\right\}$

$\doteqdot 5.7$

d) $\frac{d^2y}{dx^2} = 3x$

$\frac{dy}{dx} = \frac{3x^2}{2} + c$

$9 = \frac{3(2)^2}{2} + c$

$9 = \frac{12}{2} + c$

$3 = c$

$\frac{dy}{dx} = \frac{3x^2}{2} + 3$

$y = \frac{3}{2} \cdot \frac{x^3}{3} + 3x + c$

$= \frac{3x^3}{6} + 3x + c$

$12 = \frac{3(2)^3}{6} + 3(2) + c$

$12 = 4 + 6 + c$

$2 = c$

$\therefore y = \frac{x^3}{2} + 3x + 2$

cylinder:

$V = \pi r^2 h$

$= \pi \times (20^2 - 4x^2) \times x$

$= x\pi (20^2 - 4x^2)$

$\therefore$ V of silo:

$= \frac{2x\pi}{3}(20^2 - 4x^2) + x\pi(20^2 - 4x^2)$

$= \frac{5x\pi}{3}(20^2 - 4x^2)$

$= \frac{5x\pi}{3}(400 - 4x^2)$

$= \frac{2000x\pi}{3} - \frac{20x^3\pi}{3}$

$V = \frac{20\pi}{3}(100x - x^3)$

(iii) $\frac{dV}{dx} = \frac{20\pi}{3}(100 - 3x^2)$

stat pt when $\frac{dV}{dx} = 0$

$0 = \frac{20\pi}{3}(100 - 3x^2)$

$100 - 3x^2 = 0$

$100 = 3x^2$

$\frac{100}{3} = x^2$

$\pm\sqrt{\frac{100}{3}} = x$

but x is height, > 0,

hence $x = \sqrt{\frac{100}{3}}$

$x = \frac{10}{\sqrt{3}}$

$\frac{d^2V}{dx^2} = \frac{20\pi}{3} \cdot -6x$

$= -40\pi x$

at $x = \frac{10}{\sqrt{3}}$, $\frac{d^2V}{dx^2} < 0$

$\therefore$ max.

Max. height is $(2x + x)$

$= 3x$

$= 3\left(\frac{10}{\sqrt{3}}\right)$

$= \frac{30}{\sqrt{3}}$

$= 10\sqrt{3}$ units.

Q5

a)

	$f'(x)$	$f''(x)$
A	$-$	$+$
B	0	0
C	0	$-$

b) (i) $5x = 8x - x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0 \qquad x = 3$

$y = 0 \qquad y = 15$

$\therefore P(3, 15)$

(ii) $A = \int_0^3 8x - x^2 - 5x \, dx$

$= \int_0^3 -x^2 + 3x \, dx$

$= \left[-\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3$

$= \frac{(-3)^3}{3} + \frac{3(3)^2}{2} - 0$

$= -9 + \frac{27}{2}$

$= 4\frac{1}{2}$ units^2

(iii) $V = \pi\int_0^3 (8x - x^2)^2 - \pi\int_0^3 (5x)^2 \, dx$

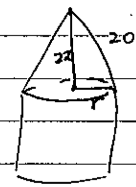
$= \pi\left[\frac{64x^3}{3} + \frac{x^5}{5} - \frac{16x^4}{4}\right]_0^3 - \pi\left[\frac{25x^3}{3}\right]_0^3$

$= 143\frac{3}{5} \times \pi$

$= 451.1327\ldots$

$= 451.13$ units^3

c) (i)



$(2x)^2 + r^2 = 20^2$

$4x^2 + r^2 = 20^2$

$r^2 = 20^2 - 4x^2$.

(ii) V = V of cone + V of cylinder

cone: $V = \frac{1}{3} \times \pi \times r^2 \times 2x$

$= \frac{1}{3} \times \pi \times (20^2 - 4x^2) \times 2x$

$= \frac{2x\pi}{3}(20^2 - 4x^2)$

V of cylinder $= \pi r^2 h$

$$= \pi(20^2 - 4x^2).x$$
$$= x\pi(20^2 - 4x^2)$$

V of silo $= \frac{2x\pi}{3}(20^2 - 4x^2) + x\pi(20^2 - 4x^2)$

$$= \frac{5x\pi}{3}(20^2 - 4x^2)$$
$$= \frac{5x\pi}{3}(400 - 4x^2)$$
$$= \frac{5x\pi}{3} \cdot 4(100 - x^2)$$
$$= \frac{20\pi}{3}(100x - x^3)$$

(iii) $\frac{dv}{dx} = \frac{20\pi}{3}(100 - 3x^2)$

stat when $\frac{dv}{dx} = 0$

$$\frac{20\pi}{3}(100 - 3x^2) = 0$$
$$100 - 3x^2 = 0$$
$$100 = 3x^2$$
$$\frac{100}{3} = x^2$$
$$\pm\sqrt{\frac{100}{3}} = x$$

but x is height, $\therefore$ positive. $\left[x = \sqrt{\frac{100}{3}} \right]$

at $x = \sqrt{\frac{100}{3}}$; $\therefore \frac{d^2v}{dx^2} = -6x$

$$= -6\sqrt{\frac{100}{3}} < 0$$

$\therefore$ maximum.

hence max height is $x + 2x = 3 \cdot \sqrt{\frac{100}{3}}$

$$= \frac{3 \cdot 10}{\sqrt{3}}$$
$$= \frac{30}{\sqrt{3}}$$
$$= \frac{30\sqrt{3}}{3}$$
$$= 10\sqrt{3} \text{ units.}$$