CARLINGFORD HIGH SCHOOL

2016

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Extension 1 Mathematics

Name: ______________________________ **Class: 12M1_**

Teacher: **Mr Gong** **Ms Kellahan** **Mrs Lobejko**

Circle your teacher's name

- **General Instructions**
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Only Board-approved calculators may be used
- A multiple choice answer sheet is provided at the back of this paper
- A Reference Sheet is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations
- Answer Question 11 – 14 in a separate answer booklet

Total Marks – 70

Section I Pages 3 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 13

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Outcome	MC	Q11	Q12	Q13	Q14	Total
Preliminary		/4	/4	/4		/12
Parametrics		/1	/5			/6
Calculus		/3	/4			/7
Mathematical Induction				/3	/5	/8
Applications of calculus to the Physical World		/4	/2	/5	/10	/21
Trigonometric Functions		/3		/3		/6
Multiple Choice	/10					/10
Total	/10	/15	/15	/15	/15	/70

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Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the **multiple-choice** answer sheet for Questions 1 – 10.

1. The netball coach at a school has decided that the junior netball team will consist of:

   4 students from Year Ten;

   3 students from Year Nine.

   After tryouts, there were 8 eligible Year Ten students and 9 eligible Year Nine students left to select from.

   In how many ways can the team be selected?

   (A) $^8\mathbf{C}_4 + {}^9\mathbf{C}_3$

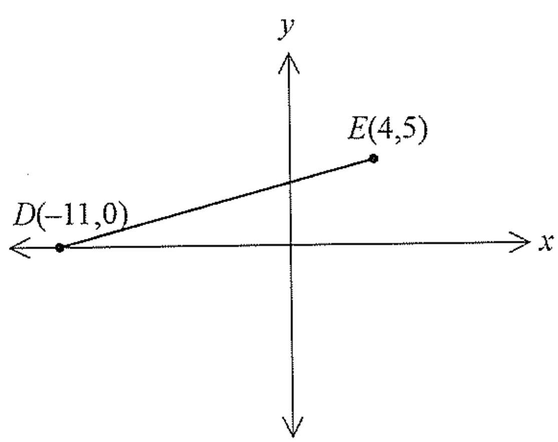
   (B) $^8\mathbf{P}_4 + {}^9\mathbf{P}_3$

   (C) $^{17}\mathbf{C}_7$

   (D) $^8\mathbf{C}_4 \times {}^9\mathbf{C}_3$

2. The interval DE is divided internally in the ratio 3:2 by the point F.

   Find the x-coordinate of F.



   (A) -5

   (B) $-\dfrac{7}{2}$

   (C) -2

   (D) $-\dfrac{3}{2}$

3. It is known that when the polynomial $P(x) = x^3 + 3x^2 + 2x$ is divided by $(x - a)$ the remainder is zero.

What values could a take?

(A) $-3, \ -2, \ 0$

(B) $-2, \ -1, \ 0$

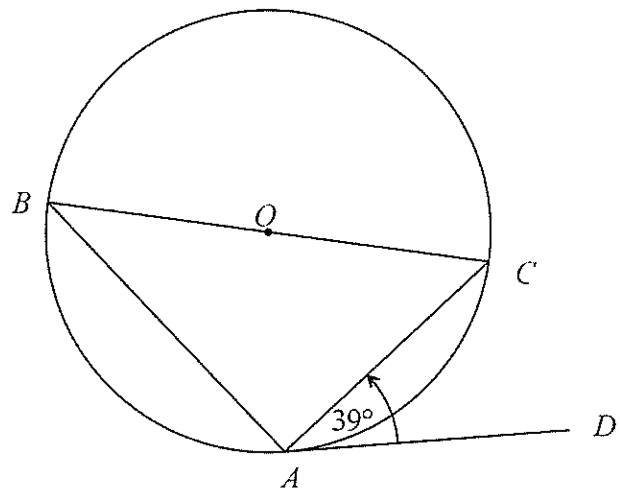
(C) $-2, \ 1, \ 3$

(D) $0, \ 1, \ 2$

4. In the circle, centre O, BC is a diameter.

AD is a tangent to the circle with A being the point of contact.

$\angle DAC = 39°$.



What is the size of $\angle BCA$?

(A) 39°

(B) 51°

(C) 78°

(D) 89°

5. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 4x}{3x}$.

(A) 0

(B) $\dfrac{3}{4}$

(C) $\dfrac{4}{3}$

(D) ∞

6. What is the domain of the function $y = \sin^{-1}(x + 5)$?

(A) $-6 \le x \le -4$

(B) $-5 \le x \le 5$

(C) $-\dfrac{\pi}{5} \le x \le \dfrac{\pi}{5}$

(D) $\dfrac{\pi}{6} \le x \le \dfrac{\pi}{4}$

7. Which of the following is an expression for $\dfrac{d}{dx}\sin^{-1}(2x-1)$.

(A) $\dfrac{-1}{\sqrt{x(x-1)}}$

(B) $\dfrac{-1}{2\sqrt{x(x-1)}}$

(C) $\dfrac{1}{2\sqrt{x(x-1)}}$

(D) $\dfrac{1}{\sqrt{x(x-1)}}$

8. Find the value of the constant term in the binomial expansion $\left(5x - \frac{3}{x^2}\right)^{12}$

(A) ${}^{12}C_6\, 5^6\, 3^6$

(B) ${}^{12}C_8\, 5^8\, 3^4$

(C) $-\ {}^{12}C_4\, 5^8\, 3^4$

(D) $-\ {}^{12}C_8\, 5^8\, 3^4$

9. A particle moves in simple harmonic motion such that $v^2 + 9x^2 = k$. What is the period of the particle's motion?

(A) $\frac{2\pi}{k}$

(B) 3π

(C) $\frac{3k}{2\pi}$

(D) $\frac{2\pi}{3}$

10. When the polynomial $P(x)$ is divided by $(x + 1)(x - 2)$ its remainder is $18x + 17$.
What is the remainder when $P(x)$ is divided by $(x - 2)$?

(A) $18x + 15$

(B) -19

(C) 35

(D) 53

End of Section I

Section II

60 marks

Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

Answer each question in a NEW writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW writing booklet.

(a) A curve is represented by the parametric equations $x = 3t$ and $y = 5t^2$. **1**

What is the Cartesian equation of the curve?

(b) If $(x^n)(x^5) = e^{11 \ln x}$ for all $x > 0$, find n **1**

(c) (i) Find the tangent of the acute angle between the lines $y = x$ and $y = 2x$. **1**

(ii) Hence show that the line y = 2x bisects the acute angle between the lines $y = x$ and $y = 7x$. **2**

(d) At 8:30 a.m. a sandwich which has an initial temperature $22°$ C, is placed in a refrigerator that is set to a constant temperature of $3°$ C.

The sandwich cools at a rate that is proportional to the difference between the temperature of the refrigerator and the temperature (T) of the sandwich.

The rate of temperature change can be expressed as:

$$\frac{dT}{dt} = -\mathrm{k}(T - 3),$$

where t is the number of minutes after the sandwich is placed in the refrigerator.

(i) Show that $T = 3 + Ae^{-kt}$ satisfies this equation. **1**

(ii) After 10 minutes in the refrigerator, the sandwich has a temperature of $12°$ C. **3**

To the nearest minute, at what time will the sandwich's temperature drop to $5°$ C?

Question 11 continues on page 8.

Question 11 continued

(e) Use the substitution $u = 2x + 1$ to evaluate $\displaystyle\int_0^2 \frac{x}{(2x+1)^2}\,dx$. **3**

(f) (i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $\alpha > 0$. **2**

(ii) Hence solve $\cos x - \sqrt{3}\sin x = -2$ for $0 \le x \le 2\pi$. **1**

End of Question 11.

Question 12 (15 marks) Start a NEW writing booklet.

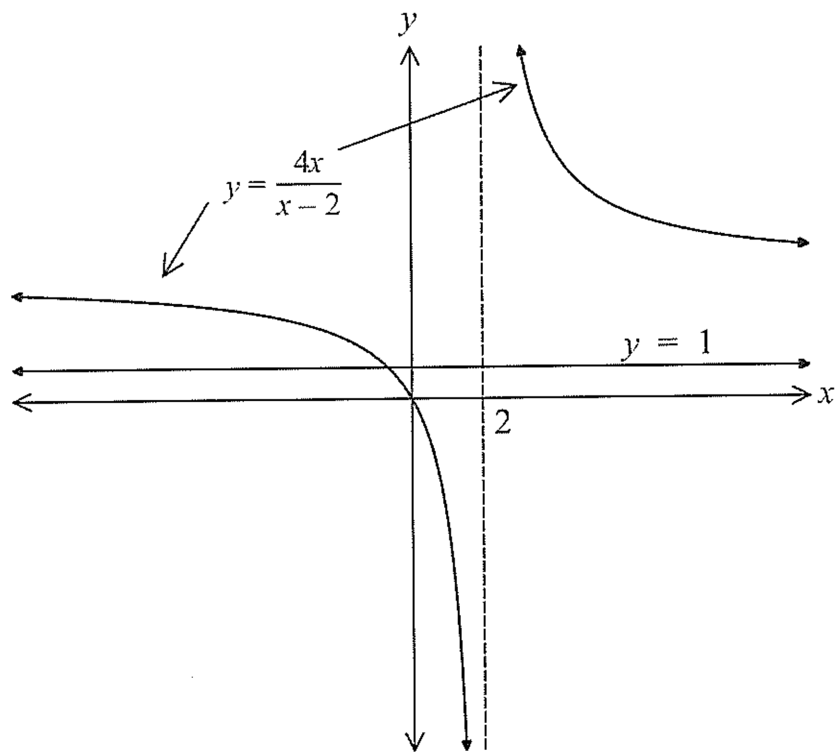
(a) The side x cm of a cube is decreasing in such a way that the V cm^3 is decreasing at a constant rate of 6 cm^3 per minute. **2**

What is the rate at which the side of the cube is decreasing when the side is 4 cm?

(b) (i) Solve the inequality $\dfrac{4x}{x-2} \leq 1$ by algebraic methods. **3**

(ii) The graph below shows the functions $y = 1$ and $y = \dfrac{4x}{x-2}$. **1**

Explain how the graph could be used to illustrate the solution found in part (i).



(c) (i) Show that the derivative of $y = \tan^{-1}\left(\dfrac{x^3}{2}\right)$ is $\dfrac{6x^2}{4+x^6}$. **3**

(ii) Hence find $\displaystyle\int \frac{x^2}{4+x^6}\,dx$. **1**

Question 12 continues on page 10.

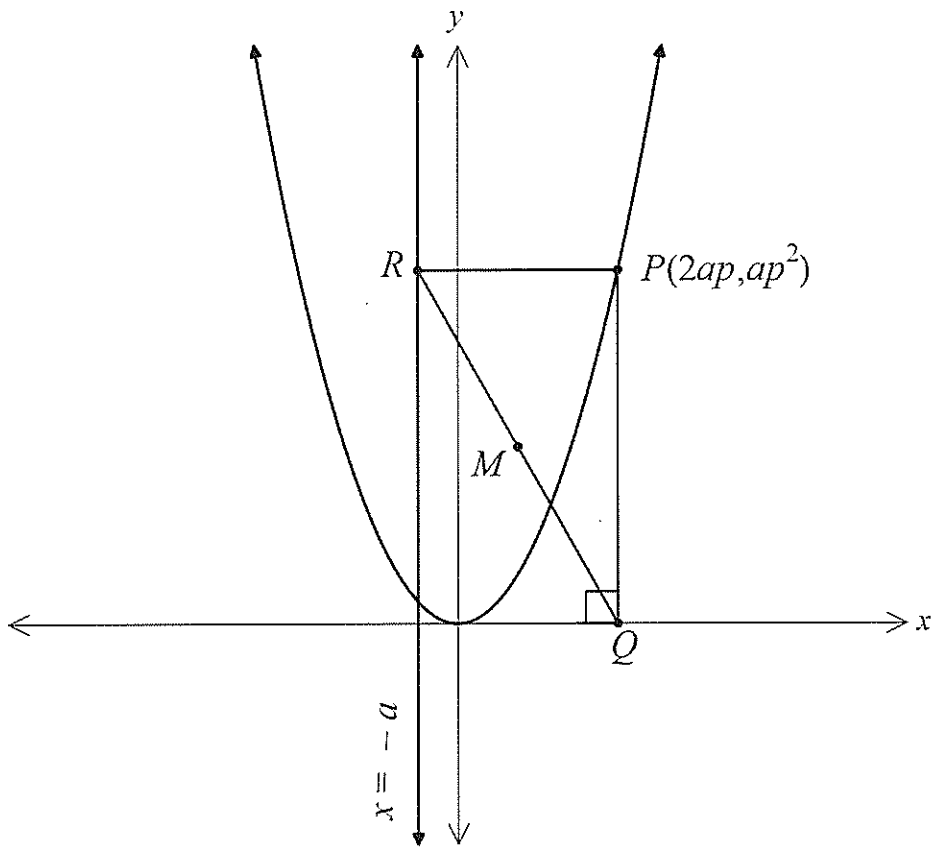
Question 12 continued

(d) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$.

The point Q is a point on the x-axis such that PQ is parallel to the y-axis.

The point R is a point on the line $x = -a$ such that RP is parallel to the x-axis.

M is the midpoint of interval RQ.



(i) Show that M has coordinates $\left(\dfrac{a(2p-1)}{2}, \dfrac{ap^2}{2}\right)$. **2**

(ii) Show that the locus of the point M is a parabola with equation $y = \dfrac{x^2}{2a} + \dfrac{x}{2} + \dfrac{a}{8}$. **2**

(iii) Find the equation for the axis of symmetry for the parabola which forms the locus of M. **1**

End of Question 12.

Question 13 (15 marks) Start a NEW writing booklet.

(a) Prove by mathematical induction that for all integers $n > 1$, **3**

$$12^n > 7^n + 5^n$$

(b) A particle is moving along the x-axis. Initially the particle is 1 metre to the right of the origin, travelling at a velocity of 3 metres per second and its acceleration is given by

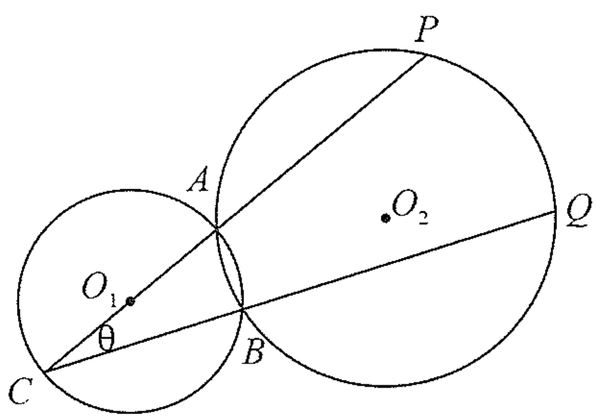
$$\ddot{x} = 2x^3 + 4x,$$

where x is the displacement of the particle at time t.

(i) Show that $\dot{x} = x^2 + 2$. **2**

(ii) Hence find an expression for x in terms of t. **3**

(c) Two circles with centres O_1 and O_2 intersect at points A and B as shown in the diagram.



AC is the diameter in circle centre O_1 and it intersects the other circle at A and P.

The chord CB produced intersects the second circle again at Q.

$\angle ACB = \theta$.

Copy or trace the diagram into your writing booklet.

(i) Prove that AQ is a diameter of the circle with centre O_2. **2**

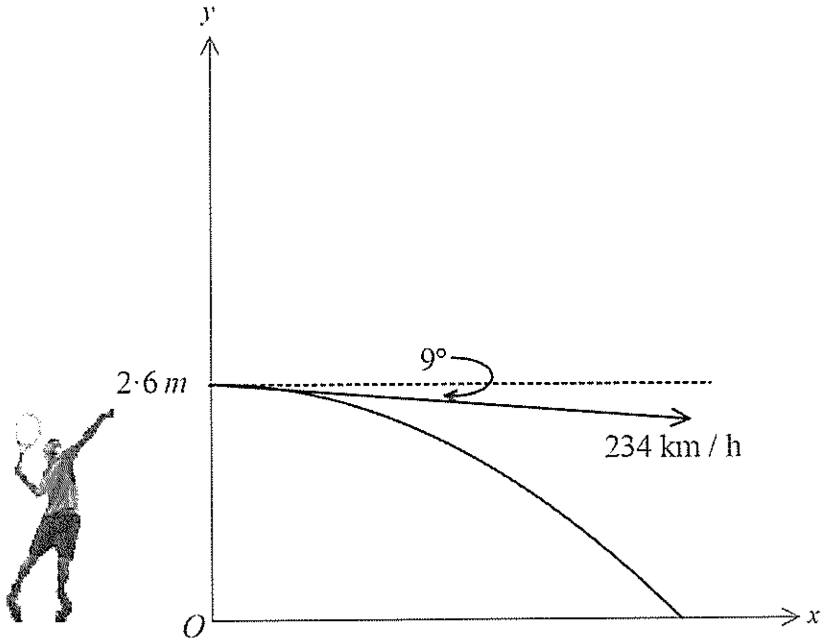
(ii) Show that $\angle ABO_1 = 90 - \theta$. **2**

(d) Using t- results solve the equation $\sin 2x = \tan x$ for $0 \leq x \leq \pi$ **3**

End of Question 13.

Question 14 (15 marks) Start a NEW writing booklet.

(a) Sam served a tennis ball to his opponent. The racquet hit the ball when the ball was 2.6 metres above the ground. The initial speed of the ball as 234 km/h at an angle of 9° below the horizontal.



Let the origin be a point on the ground, directly below where the racquet hit the ball.

Let gravity equal $10\ \mathrm{m/s}^2$.

(i) Show that the motion of the ball (in metres) can be expressed by the equations **2**

$$x = 65t\cos 9°$$

and $$y = 2.6 - 5t^2 - 65t\sin 9°.$$

(ii) The net at the centre of the court is 11.9 metres from the origin. The net is 91cm tall. Show that the ball will not make it over the net. **2**

(iii) What will the speed of the ball be when it hits the net? **2**

Question 14 continues on page 13.

Question 14 continued

(b) A particle is moving in simple harmonic motion with its acceleration given by

$$\ddot{x} = -12\sin 2t.$$

Initially, the particle is at the origin and has a positive velocity of 6 m/s.

(i) Show that the particle's velocity has equation $\dot{x} = -12\sin^2 t + 6$. **2**

(ii) Show that $\ddot{x} = -4x$. **2**

(c) Consider the equation $A_n = 8^n + 3^{n-2}$, $n \geq 2$.

(i) Show that $A_2 = 65$. **1**

(ii) Prove that A_3 is divisible by 5. **1**

(iii) Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 for all $n \geq 2$, where n is an integer **3**

End of Exam.

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

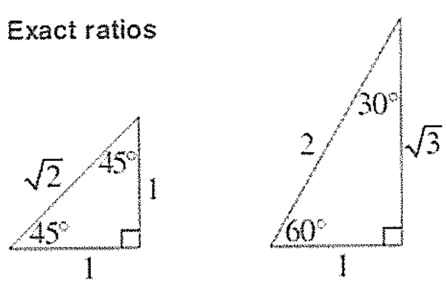
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$A = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\dfrac{dy}{dx} = F'(u)\dfrac{du}{dx}$

If $y = e^{f(x)}$, then $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\dfrac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\dfrac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

t **formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a,\quad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a,\quad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a,\quad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at, at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1, y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0, y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos(nt + \alpha)$$

$$\ddot{x} = -n^2(x - b)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Trial HSC Examination 2016

Mathematics Extension 1 Course

Name ______________________ Teacher __________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Trial HSC Examination 2016

Mathematics Extension 1 Course

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Section I – Multiple Choice Answer Sheet

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Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

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A ● B ~~●~~ C ○ D ○

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A ~~●~~ B ~~●~~ correct C ○ D ○

1.	A ○	B ○	C ○	D ●
2.	A ○	B ○	C ●	D ○
3.	A ○	B ●	C ○	D ○
4.	A ○	B ●	C ○	D ○
5.	A ○	B ○	C ●	D ○
6.	A ●	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ●
8.	A ○	B ●	C ○	D ○
9.	A ○	B ○	C ○	D ●
10.	A ○	B ○	C ○	D ●

2016 Extension One Mathematics

Question 11.

a) $x = 3t$

$t = \frac{x}{3}$

$y = 5t^2$

$= 5\left(\frac{x}{3}\right)^2$

$\therefore y = \frac{5x^2}{9}$ (1.) Parametrics

b) $x^n x^5 = \left(e^{\ln x}\right)^n \cdot \left(e^{\ln x}\right)^5$

$= \left(e^{\ln x}\right)^{n+5}$

$\therefore \left(e^{\ln x}\right)^{n+5} = e^{11 \ln x}$

$n + 5 = 11$

$\therefore n = 6$ (1.) Prelim.

c) i) $m_1 = 1$ $m_2 = 2$

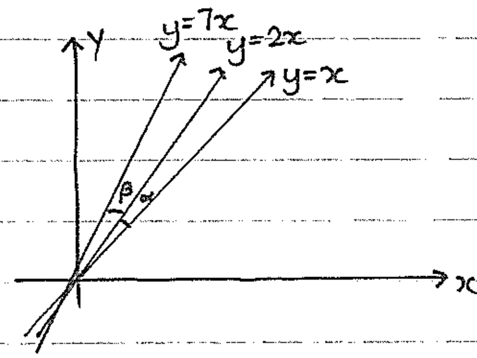
$\tan\alpha = \frac{2-1}{1+2\times 1}$

$= \frac{1}{3}$ (1.) prelim

ii) $\tan\beta = \frac{7-2}{1+7\times 2}$ $m_1 = 2$ $m_2 = 7$

$= \frac{1}{3}$ (1)

$\therefore \alpha = \beta$ and by position of lines then $y = 2x$ bisects $y = x$ and $y = 7x$



(1)

prelim.

d) i) $T = 3 + Ae^{-kt}$ (1)

$T - 3 = Ae^{-kt}$ (2)

From (1) $\frac{dT}{dt} = -kAe^{-kt}$ (3)

Sub (2) into (3)

$\frac{dT}{dt} = -k(T-3)$

$\therefore T = 3 + Ae^{-kt}$ satifies the equation. (1)

ACPW

Can also use substitution.

ii) $t = 0$ $T = 22$

$22 = 3 + Ae^0$

$19 = Ae^0$

$\therefore A = 19$ 1

$t = 10$ $T = 12$

$12 = 3 + 19e^{-10k}$

$9 = 19e^{-10k}$

$\frac{9}{19} = e^{-10k}$

$\ln 9/19 = -10k \ln e$

$\therefore k = \frac{\ln 9/19}{-10}$ 1

When $t = 5$

$5 = 3 + 19e^{-kt}$

$$\ln\frac{2}{19} = \frac{\ln\left(\frac{9}{19}\right)}{10}\, t$$

$$\therefore t = \frac{\ln 2/19}{\frac{\ln(9/19)}{10}}$$

$$= 30.129\ldots$$

$$\therefore t = 30 \text{ mins}$$

∴ Sandwich will have reached 5°C at 9am. (1)

AePW

e) $u = 2x+1 \Rightarrow x = \frac{u-1}{2}$

$x = 2 \quad u = 5$

$x = 0 \quad u = 1.$

$$\frac{du}{dx} = 2$$

$$du = 2dx$$

$$\frac{du}{2} = dx.$$

(1)

$$\int_0^2 \frac{x}{(2x+1)^2}\,dx = \int_1^5 \frac{\frac{u-1}{2}}{u^2}\,\frac{du}{2}$$

$$= \frac{1}{2}\int_1^5 \frac{u-1}{2u^2}\,du$$

$$= \frac{1}{4}\int_1^5 \frac{u}{u^2} - \frac{1}{u^2}\,du$$

$$= \frac{1}{4}\left[\ln u + u^{-1}\right]_1^5 \quad (1)$$

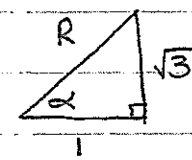
$$= \frac{1}{4}\left[\ln 5 + \frac{1}{5} - (\ln 1 + 1)\right]$$

f) i) $\cos x - \sqrt{3}\sin x = R\cos(x+\alpha)$

$R^2 = 1^2 + \sqrt{3}^2 \qquad \tan\alpha = \sqrt{3}$

$= 4 \qquad \alpha = \frac{\pi}{3}.$

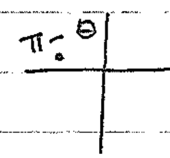
$\therefore R = 2$ (1) (1)



$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$

ii) $2\cos\left(x + \frac{\pi}{3}\right) = -2$

$\cos\left(x + \frac{\pi}{3}\right) = -1$



$x + \frac{\pi}{3} = 0$

$\therefore x = \pi - \frac{\pi}{3}$ 2nd quad

$= \frac{2\pi}{3}.$ (1)

trig

Question 12.

a) $V = x^3 \qquad x = 4 \qquad \frac{dV}{dt} = -6.$

$\frac{dV}{dx} = 3x^2$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$-6 = 3x^2 \times \frac{dx}{dt}$ 1

$-6 = 3 \times 4^2 \times \frac{dx}{dt}$

$-6 = 48 \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = -\frac{1}{8}$ 1

ACPW

b) i) $\frac{4x}{x-2} \leq 1 \qquad x \neq 2$

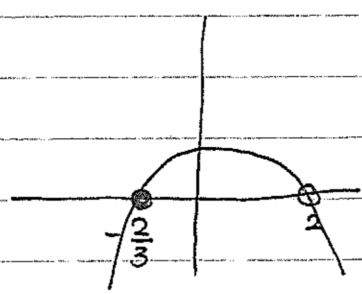
$(x-2)^2 \times \frac{4x}{x-2} \leq (x-2)^2$ 1

$(x-2)^2 - 4x(x-2) \geq 0$

$(x-2)\left[x - 2 - 4x\right] \geq 0$

$(x-2)(-3x-2) \geq 0$

$-(x-2)(3x+2) \geq 0.$ 1



$\therefore -\frac{2}{3} \leq x < 2.$ 1

Prelim.

ii) To solve $\frac{4x}{x-2} \leq 1$, we want values where the curve is below or equal to the line $y = 1$. This occurs between the asymptote $x = 2$ and the point of intersection from part i) is $x = -\frac{2}{3}$. 1

$\therefore -\frac{2}{3} \leq x \leq 2.$

Prelim.

c) i) $y = \tan^{-1}\left(\frac{x^3}{2}\right)$

$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x^3}{2}\right)^2} \times \frac{3x^2}{2}$ 1

$= \frac{1}{1 + \frac{x^6}{4}} \times \frac{3x^2}{2}$

$= \frac{3x^2}{2 + \frac{x^6}{2}} \times \frac{2}{2}$ 1

$\therefore \frac{dy}{dx} = \frac{6x^2}{4 + x^6}$ 1

ii) $\int \frac{x^2}{4 + x^6}\, dx = \frac{1}{6} \int \frac{6x^2}{4 + x^6}\, dx$

$= \frac{1}{6} \tan^{-1}\left(\frac{x^3}{2}\right) + c$

1

Calculus

d) i) $R(-a, ap^2)$ $Q(2ap, 0)$ ①

$$M = \left(\frac{-a+2ap}{2}, \frac{ap^2+0}{2}\right)$$

$$= \left(\frac{a(2p-1)}{2}, \frac{ap^2}{2}\right)$$ ①

ii) $$x = \frac{a(2p-1)}{2}$$

$$\frac{2x}{a} = 2p-1$$

$$\frac{2x}{a}+1 = 2p$$

$$\therefore p = \frac{x}{a} + \frac{1}{2}$$ ①

$$\therefore y = \frac{ap^2}{2}$$

$$= \frac{a\left(\frac{x}{a}+\frac{1}{2}\right)^2}{2}$$

$$= \frac{a\left(\frac{x^2}{a^2} + 2 \times \frac{1}{2} \times \frac{x}{a} + \frac{1}{4}\right)}{2}$$

$$= \frac{a}{2}\left(\frac{x^2}{a^2} + \frac{x}{a} + \frac{1}{4}\right)$$ ①

$$\therefore y = \frac{ax^2}{2a^2} + \frac{ax}{2a} + \frac{a}{8}$$

$$y = \frac{x^2}{2a} + \frac{x}{2} + \frac{a}{8}$$

parametrics

iii) $x = \frac{-b}{2a}$ $\quad b = \frac{1}{2}$ $\quad a = \frac{1}{2a}$

$$= \frac{-\frac{1}{2}}{2\left(\frac{1}{2a}\right)}$$

$$= -\frac{1}{2} \times a$$

$$= -\frac{a}{2}.$$

parametrics

Question 13.

a) $12^n > 7^n + 5^n \quad n > 1.$

i) Prove true for $n=2$.

$12^2 > 7^2 + 5^2$

$144 > 74$. (1)

$\therefore$ LHS > RHS

$\therefore$ true for $n=2$

ii) Assume true for $n=k$.

$12^k > 7^k + 5^k$

iii) Prove true for $n=k+1$

LHS $= 12^{k+1}$

$= 12^k \cdot 12$

$> 12(7^k + 5^k)$

$> 12 \cdot 7^k + 12 \cdot 5^k$ (1)

RHS $= 7^{k+1} + 5^{k+1}$

$= 7 \cdot 7^k + 5 \cdot 5^k$

Since k is a positive integer:

$12 \cdot 7^k + 12 \cdot 5^k > 7 \cdot 7^k + 5 \cdot 5^k$

$\therefore$ LHS > RHS. (1)

$\therefore$ true for $n=k+1$ when true for $n=k$.

$\therefore$ Since true for $n=2$ and proven true for $n=k+1$, when true for $n=k$, must

b) i) $\ddot{x} = \frac{d}{dx} \frac{1}{2} v^2$

$\frac{d}{dx} \frac{1}{2} v^2 = 2x^3 + 4x$

$\frac{1}{2} v^2 = \int 2x^3 + 4x \, dx$

$= \frac{2x^4}{4} + \frac{4x^2}{2} + c$

$= \frac{x^4}{2} + 2x^2 + c$ (1)

At $v = 3$ $x = 1$.

$\frac{1}{2} \times 3^2 = \frac{1}{2} + 2 + c$

$9 = 1 + 4 + c$

$\therefore c = 4$

$v^2 = x^4 + 4x^2 + 4$

$= (x^2 + 2)^2$

$v = \pm (x^2 + 2)$

$v = x^2 + 2$ since it satifies $x = 1, v = 3$. (1)

$\therefore \dot{x} = x^2 + 2$ #

ACPW

b ii) $\frac{dx}{dt} = x^2 + 2$

$$\frac{dt}{dx} = \frac{1}{x^2+2}$$

$$t = \int \frac{1}{x^2+2}\,dx$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c.$$

When $t=0$ $x=1$.

$$0 = \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} + c.$$

$$\therefore c = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$$

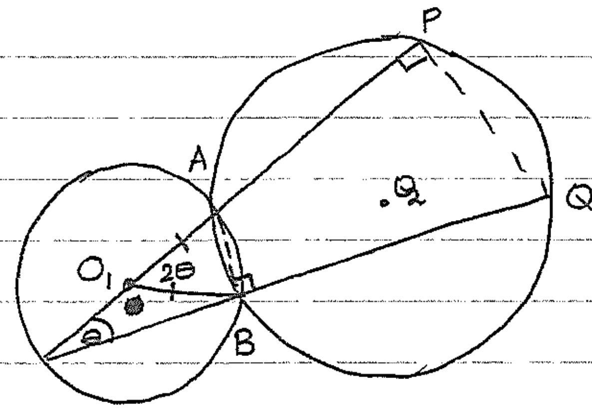
$$\therefore t = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}.$$

$$\sqrt{2}\left[t + \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right] = \tan^{-1}\frac{x}{\sqrt{2}}$$

$$\frac{x}{\sqrt{2}} = \tan\left(\sqrt{2}\left[t + \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right]\right)$$

$$x = \sqrt{2}\tan\left(\sqrt{2}\left[t + \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right]\right).$$

c)



i) $\angle CBA = 90°$ ($\angle$ in a semicircle)

$\angle ABQ = 90°$ (adj to $\angle CBA$ str. line) 1

$\therefore$ AQ is a diameter (angle formed in a semicircle) 1

ii) $\angle AO_1B = 2\angle ACB$ ($\angle$ at centre is twice $\angle$ at circum).

$= 2\theta$

$AO_1 = BO_1$ (equal radii)

$\therefore \angle BAO_1 = \angle ABO_1$ (base $\angle$'s isosceles triangle). 1

$\therefore \angle BAO_1 + \angle ABO_1 + \angle AO_1B = 180°$ ($\angle$ sum Δ)

$2\angle ABO_1 + 2\theta = 180$

$\angle ABO_1 = \frac{180 - 2\theta}{2}$ 1

$= 90 - \theta$ #

Prelim.

13d) $\sin 2x = \tan x$

$$\frac{2t}{1+t^2} = t \quad ①$$

$$2t = t(1+t^2)$$

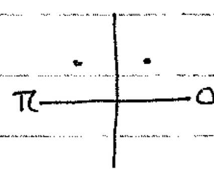
$$t^3 + t - 2t = 0$$

$$t^3 - t = 0$$

$$t(t^2-1) = 0$$

$\therefore t = 0, -1, +1.$ ①

$\tan x = 0, -1, 1.$



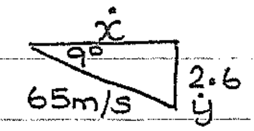
$$x = 0, \frac{\pi}{4}, \pi - \frac{\pi}{4}$$

$\therefore x = 0, \frac{\pi}{4}, \frac{3\pi}{4}$. $0 \leq x \leq \pi$

①

trig.

Question 14.



a) i) horizontal   vertical

$\ddot{x} = 0$

$\dot{x} = \int 0\,dt$

$\dot{x} = c$

At $t=0$ $\dot{x} = 65\cos 9^\circ$

$c = 65\cos 9^\circ$

$x = \int 65\cos 9^\circ\,dt$

$= 65t\cos 9^\circ + k$

At $t=0$ $x=0$.

$0 = 0 + k$

$\therefore k = 0$.

$\therefore x = 65t\cos 9^\circ$. 1

$\ddot{y} = -10$

$\dot{y} = \int -10\,dt$

$\dot{y} = -10t + c$

At $t=0$ $\dot{y} = -65\sin 9^\circ$

$\therefore -65\sin 9^\circ = -0 + c$

$\therefore c = -65\sin 9^\circ$

$\therefore \dot{y} = -10t - 65\sin 9^\circ$

$y = \int -10t - 65\sin 9^\circ\,dt$

$= \dfrac{-10t^2}{2} - 65t\sin 9^\circ + k$

At $t=0$ $y = 2.6$.

$2.6 = 0 - 0 + k$

$\therefore k = 2.6$. 1

$y = -5t^2 - 65t\sin 9^\circ + 2.6$.

ii) vertical displacement ≥ 0.91m   horizontal disp $= 11.9$

$x = 65t\cos 9^\circ$

$t = \dfrac{x}{65\cos 9^\circ}$

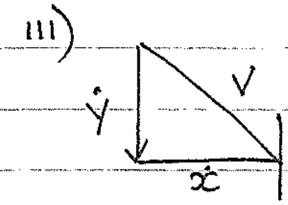
$= \dfrac{11.9}{65\cos 9^\circ}$. 1

$\therefore y = 2.6 - 5\times\left(\dfrac{11.9}{65\cos 9^\circ}\right)^2 - 65\left(\dfrac{11.9}{65\cos 9^\circ}\right)\sin 9^\circ$

$= 0.5434353658$

< 0.91. 1

So when $x = 11.9$ the ball will <u>not</u> clear the net iii)



$v^2 = \dot{x}^2 + \dot{y}^2 \qquad t = {}^{11.9}/_{65\cos 9^\circ}$

$v^2 = (65\cos 9^\circ)^2 + -10\left(\frac{11.9}{65\cos 9^\circ} - 65\sin 9^\circ\right)^2$ 1

$= 4266.13$

$V = 65.3156$

$\doteq 65.3.$ 1

$\therefore$ speed at which the ball hits the net is 65.3m/s.

---

b) i) $\ddot{x} = -12\sin 2t$

$t=0 \quad x=0 \quad \dot{x} = 6\text{ms}^{-1}$

$\dot{x} = \int -12\sin 2t \, dt$

$= -12\int 2\sin t\cos t \, dt$ 1

$= -12\sin^2 t + c$

$6 = -12\sin^2 0 + c$

$\therefore c = 6$

$\therefore \dot{x} = -12\sin^2 t + 6$ 1

ii) $x = \int -12\sin^2 t + 6 \, dt$

$= \int -12\left[\frac{1-\cos 2t}{2}\right] + 6 \, dt$

$= \int -6[1-\cos 2t] + 6 \, dt$

$= \int 6\cos 2t \, dt$

$= 3\sin 2t + k$

When $t=0 \quad x=0$

$0 = 0 + k$

iii) $x = 3\sin 2t$ 1

$\ddot{x} = -12\sin 2t$

$= -4(3\sin 2t)$ 1

$\ddot{x} = -4x.$

ACPW

c) i) $A_n = 8^n + 3^{n-2} \quad n \geq 2$

$A_2 = 8^2 + 3^{2-2}$

$= 64 + 1$

$= 65.$ 1

Must show

ii) $A_3 = 8^3 + 3^{3-2}$

$= 512 + 3$

$= 515$ 1

$515 \div 5 = 103$

Which is divisible by 5.

III) I. Prove true for $n=2$

$A_2 = 65$, which is divisible by 5.

$\therefore$ true for $n=2$.

II. Assume true for $n=k$

$A_k = 8^k + 3^{k-2} = 5M$

M an integer

III. Prove true for $n=k+1$.

$$A_{k+1} = 8^{k+1} + 3^{k+1-2}$$
$$= 8^{k+1} + 3^{k-1}$$
$$= 8 \cdot 8^k + 3^{-1} \cdot 3^k$$
$$= 8(8^k + 3^{k-2}) - 8 \times 3^{k-2} + 3^{k-1}$$

$$= 8(5M) - 3^{k-2}(8-3)$$
$$= 8(5M) - 5 \cdot 3^{k-2}$$
$$= 5[8M - 3^{k-2}]$$
$$= 5Q \quad Q \text{ an integer}$$

$\therefore$ divisible by five.

$\therefore$ if true for $n=k$ true $n=k+1$.

$\therefore$ Prove true for $n=2$, $n=k+1$...

$\therefore$ true for all integers $n \geq 2$.