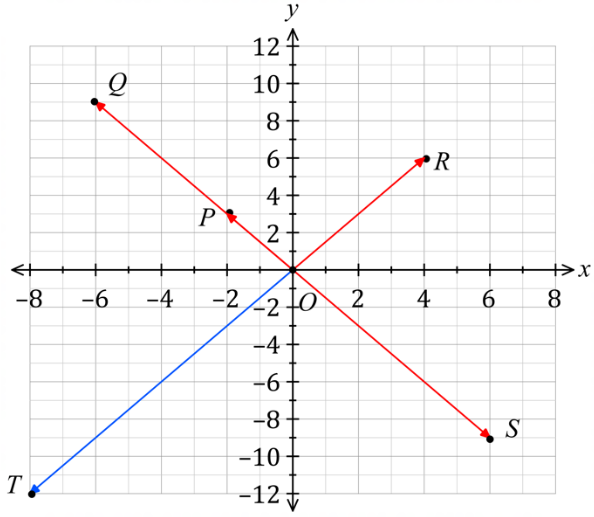
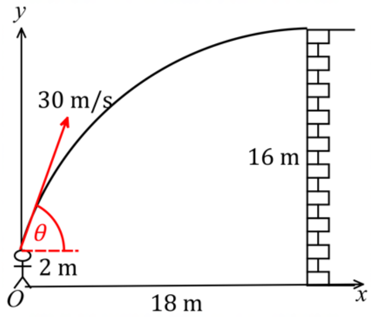
ACE Examination Paper 3
Year 12 Mathematics Extension 1 Yearly Examination
Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1.	$\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + C$	1 Mark: C
2.	Graphically vector $\overrightarrow{OT}$ and $\overrightarrow{OR}$ are parallel but in opposite directions.  	1 Mark: C
3.	$y = e^{mx}, \frac{dy}{dx} = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx}$ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$ $m^2e^{mx} - 2me^{mx} - 3e^{mx} = 0$ $(m^2 - 2m - 3)e^{mx} = 0$ $(m-3)(m+1)e^{mx} = 0$ $\therefore m = -1$ or $m = 3$	1 Mark: B
4.	$\cos x - \sin x = R\cos(x+\alpha)$ $= R\cos x\cos\alpha - R\sin x\sin\alpha$ $R\cos\alpha = 1$ (1) $R\sin\alpha = 1$ (2) Equation (2) divided by equation (1) $\tan\alpha = 1$ $\alpha = \frac{\pi}{4}$ Squaring and adding the equations $R^2 = 1^2 + 1^2$ or $R = \sqrt{2}$ $\therefore \cos x - \sin x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$	1 Mark: A

	Solution	Criteria
5.	 **Horizontally** $a_x = \ddot{x} = 0$ $v_x = \dot{x} = c_1$ At t = 0, $v_x = 30\cos\theta$ $\Rightarrow c_1 = 30\cos\theta$ $v_x = 30\cos\theta$ $x = 30t\cos\theta + c_2$ When $t = 0, x = 0 \Rightarrow\ c_2 = 0$ $x = 30t\cos\theta$ **Vertically** $a_y = \ddot{y} = -10$ $v_y = \dot{y} = -10t + c_3$ At $t = 0$, $v_y = 30\sin\theta$ $\Rightarrow c_3 = 30\sin\theta$ $v_y = \dot{y} = -10t + 30\sin\theta$ $y = -5t^2 + 30t\sin\theta + c_4$ When $t = 0\ y = 0 \Rightarrow\ c_4 = 2$ $y = -5t^2 + 30t\sin\theta + 2$	1 Mark: D
6.	$V = \pi \int_b^b x^2 dy = \pi \int_1^4 \frac{1}{y} dy$ $= \pi[\ln y]_1^4$ $= \pi[\ln 4 - \ln 1]$ $= \pi \ln 4$ cubic units	1 Mark: D
7.	Let p be the probability of selecting the correct answer. $p = \frac{1}{3}, n = 36$ $E(X) = np$ $= 36 \times \frac{1}{3}$ $= 12$	1 Mark: B
8.	$\overrightarrow{AB} = \overrightarrow{DC} = \underset{\sim}{u}$ Opposite sides of a parallelogram are equal. $\overrightarrow{BD} = \overrightarrow{BC} - \overrightarrow{DC}$ $= \underset{\sim}{v} - \underset{\sim}{u}$ Diagonals bisect each other in a parallelogram. $\overrightarrow{OD} = \frac{1}{2}\overrightarrow{BD}$ $\overrightarrow{OD} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$	1 Mark: B

	Solution	Criteria
9.	$\tan 2x + \tan x = 0$ $\frac{2\tan x}{1-\tan^2 x} + \tan x = 0$ $\tan x\left\{\frac{2}{1-\tan^2 x} + 1\right\} = 0$ $\therefore \tan x = 0$ or $\left(\frac{2}{1-\tan^2 x} + 1\right) = 0$ No solutions for $\tan x = 0$ in the domain $0° < x < 180°$ $\frac{2}{1-\tan^2 x} = -1$ $2 = \tan^2 x - 1$ $\tan^2 x = 3$ $\tan x = \pm\sqrt{3}$ $\therefore x = 60°$ or $120°$	1 Mark: C
10.	Step 2: Assume true for $n = k$ $S_k = \frac{1}{6}k(k+1)(2k+1)$ Step 3: To prove true for $n = k + 1$ $S_{k+1} = \frac{1}{6}(k+1)(k+2)(2k+3)$ $S_k + T_{k+1} = S_{k+1}$ $\text{LHS} = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ $= (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right]$ $= (k+1)\frac{1}{6}[k(2k+1) + 6(k+1)]$ $= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6)$ $= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$ $= \frac{1}{6}(k+1)(k+2)(2k+3)$ $= \text{RHS}$	1 Mark: A
Section II		
11(a)	$\text{LHS} = \tan^2 x - \tan^2 y$ $= (\sec^2 x - 1) - (\sec^2 y - 1)$ $= \sec^2 x - \sec^2 y$ $= \frac{1}{\cos^2 x} - \frac{1}{\cos^2 y}$ $= \frac{\cos^2 y - \cos^2 x}{\cos^2 x \cos^2 y}$ $= \text{RHS}$	2 Marks: Correct answer. 1 Mark: Uses a relevant trigonometric identity.
11(b)	$\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{DE}$ $= (\underset{\sim}{i} - 2\underset{\sim}{j}) + (2\underset{\sim}{i} + 6\underset{\sim}{j})$ $= 3\underset{\sim}{i} + 4\underset{\sim}{j}$ $\lvert\overrightarrow{CE}\rvert = \sqrt{3^2 + 4^2}$ $= 5$	2 Marks: Correct answer. 1 Mark: Adds the component form of each vector.

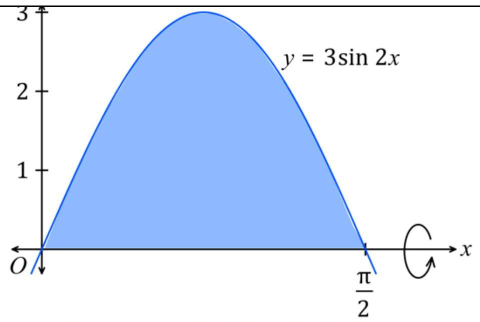
11(c)	$\int_0^{\frac{\pi}{3}} \sin^2 x\, dx = \int_0^{\frac{\pi}{3}} \frac{1}{2}(1-\cos 2x)dx$ $= \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{3}}$ $= \left(\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right) - \left(0 - \frac{1}{4}\sin 0\right)$ $= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$ $= \frac{4\pi - 3\sqrt{3}}{24}$	2 Marks: Correct answer. 1 Mark: Applies the double angle identity.
11(d)	$f(x) = 3x^2\cos^{-1}3x$ $f'(x) = 3x^2 \times \frac{-1}{\sqrt{\frac{1}{9} - x^2}} + 6x\cos^{-1}3x$ $= \frac{-9x^2}{\sqrt{1-9x^2}} + 6x\cos^{-1}3x$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
11(e)	$u = x^2 - 9$ $\frac{du}{dx} = 2x$ $\frac{1}{2}du = xdx$ $\int x\sqrt{x^2-9}dx = \int \frac{1}{2}\sqrt{u}du = \int \frac{1}{2}u^{\frac{1}{2}}du$ $= \frac{1}{2} \times \frac{2}{3}u^{\frac{3}{2}} + C$ $= \frac{(x^2-9)^{\frac{3}{2}}}{3} + C$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
11(f)	Step 1: To prove true for $n = 1$ $3^{2n} - 1 = 3^2 - 1 = 8$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $3^{2k} - 1 = 8\text{m}$ where m is an integer Step 3: To prove true for $n = k + 1$ $3^{2(k+1)} - 1 = 8\text{p}$ where p is an integer $\text{LHS} = 3^{2(k+1)} - 1$ $= 3^{2k+2} - 1$ $= 3^2 \times 3^{2k} - 1$ $= 3^2(8\text{m} + 1) - 1$ $= 3^2(8\text{m}) + 3^2 - 1$ $= 9(8\text{m}) + 8$ $= 8(9\text{m} + 1)$ $= 8p$ $= \text{RHS}$ Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$ 1 mark: Proves the result true for $n = 1$.

12(a) (i)	$T = T_0 + Ae^{-kt}$ $(Ae^{-kt} = T - T_0)$ $\frac{dT}{dt} = -k \times Ae^{-kt}$ $= -k(T - T_0)$	1 Mark: Correct answer.
12(a) (ii)	Initially $t = 0$ and $T = 24$, $T_0 = -40$ $T = T_0 + Ae^{-kt}$ $24 = -40 + Ae^{-k\times 0}$ $A = 64$ Now $t = 5$ and $T = 19$ $T = -40 + 64e^{-kt}$ $19 = -40 + 64e^{-k\times 5}$ $e^{-5k} = \frac{59}{64}$ $-5k = \ln\frac{59}{64}$ $k = -\frac{1}{5}\ln\frac{59}{64}$ $= -\frac{1}{5}\ln\frac{59}{64} = 0.0162 \ldots$ We need to find t when $T = 0$ $T = -40 + 64e^{-kt}$ $0 = -40 + 64e^{-kt}$ $e^{-kt} = \frac{40}{64} = \frac{5}{8}$ $-kt = \ln\frac{5}{8}$ $t = -\frac{1}{k}\ln\frac{5}{8} = \frac{1}{k}\ln\frac{8}{5}$ $= -5\frac{\ln\frac{8}{5}}{\ln\frac{59}{64}} = 28.8892\ldots$ ≈ 29 seconds $\therefore$ It will take about 29 seconds for the meal to cool to 0°C.	4 Marks: Correct answer. 3 Marks: Makes significant progress towards the solution. 2 Marks: Finds the value of A and an expression for k. 1 Mark: Finds the value of A.
12(b)	$x = u^2$ $dx = 2udu$ when $x = 1$, $u = 1$ and $x = 3$, $u = \sqrt{3}$ $\int_1^3 \frac{dx}{(x+1)\sqrt{x}} = \int_1^{\sqrt{3}} \frac{1}{(u^2+1)u} \times 2udu$ $= 2\int_1^{\sqrt{3}} \frac{1}{(u^2+1)}du$ $= 2[\tan^{-1}u]_1^{\sqrt{3}}$ $= 2\left(\tan^{-1}\sqrt{3} - \tan^{-1}1\right)$ $= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ $= \frac{\pi}{6}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Finds $dx = 2udu$ and changes the limits.

12(c)	$\text{LHS} = \dfrac{\cos A - \cos(A + 2\theta)}{2\sin\theta}$ $= \dfrac{\cos A - (\cos A\cos 2\theta - \sin 2\theta\sin A)}{2\sin\theta}$ $= \dfrac{\cos A - \cos A(1 - 2\sin^2\theta) + \sin A(2\sin\theta\cos\theta)}{2\sin\theta}$ $= \dfrac{\cos A - \cos A + 2\cos A\sin^2\theta + 2\sin A\sin\theta\cos\theta}{2\sin\theta}$ $= \cos A\sin\theta + \sin A\cos\theta$ $= \sin(A + \theta)$ $= \text{RHS}$	2 Marks: Correct answer. 1 Mark: Uses an appropriate trigonometric identity.
12(d)	$\int(\cos^2 3x)dx = \int \dfrac{1}{2}(1 + \cos 6x)\,dx$ $= \dfrac{1}{2}\left[x + \dfrac{1}{6}\sin 6x\right]$ $= \dfrac{1}{2}x + \dfrac{1}{12}\sin 6x + C$	2 Marks: Correct answer. 1 Mark: Applies the double angle identity.
12(e) (i)	$4\sin\theta + 3\cos\theta + 5 = 4 \times \dfrac{2t}{1+t^2} + 3 \times \dfrac{1-t^2}{1+t^2} + 5$ $= \dfrac{8t + 3 - 3t^2 + 5 + 5t^2}{1+t^2}$ $= \dfrac{2t^2 + 8t + 8}{1+t^2}$ $= \dfrac{2(t^2 + 4t + 4)}{1+t^2}$ $= \dfrac{2(t+2)^2}{1+t^2}$	2 Marks: Correct answer. 1 Mark: Substitutes the t-formula for $\sin\theta$ and $\cos\theta$.
12(e) (ii)	$4\sin\theta + 3\cos\theta + 5 = \dfrac{2(t+2)^2}{1+t^2} = 0$ $t = -2$ $t = \tan\dfrac{\theta}{2}$ with $0 \le \theta \le 360°$ $\tan\dfrac{\theta}{2} = -2$ $\dfrac{\theta}{2} = -63.4349\ldots$ $\dfrac{\theta}{2} = 116.5650\ldots$ for $0 \le \dfrac{\theta}{2} \le 180°$ $\theta = 233.1301\ldots$ $\therefore \theta \approx 233°$	2 Marks: Correct answer. 1 Mark: Finds $\tan\dfrac{\theta}{2} = -2$.

13(a)	Let p be the probability of throwing a four. $p = \frac{1}{6}, n = 6$ $P(X = x) = {}^6C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{6-x}$ $P(X = 3) = {}^6C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^3 = \frac{625}{11\,664}$	2 Marks: Correct answer. 1 Mark: Finds the correct expression for the probability distribution.
13(b) (i)	After 2 seconds $x = 90$ and $y = 4$ $90 = 2V\cos\theta$ $V\cos\theta = 45$ ① $4 = -5 \times 2^2 + 2V\sin\theta$ $24 = 2V\sin\theta$ $V\sin\theta = 12$ ② Dividing the two equations $\frac{V\sin\theta}{V\cos\theta} = \frac{12}{45}$ $\tan\theta = \frac{4}{15}$ $\theta = \tan^{-1}\frac{4}{15} = 14°56'$ ∴ Golf ball has an angle of projection of 14°56′.	3 marks: Correct answer. 2 marks: Makes significant progress. 1 mark: Sets up the two equations or shows some understanding.
13(b) (ii)	Particle reaches the ground when $y = 0$ $y = -5t^2 + Vt\sin\theta$ $0 = -5t^2 + Vt\sin\theta$ $0 = t(-5t + V\sin\theta)$ Hence $t = 0$ or $-5t + V\sin\theta = 0$ $t = \frac{V\sin\theta}{5}$ From equation ② $t = \frac{12}{5} = 2.4$ seconds Distance travelled by the ball $x = Vt\cos\theta$ Now using equation ① $= 45 \times 2.4$ $= 108$ metres ∴ Carter hits the ball 108 metres.	2 Marks: Correct answer. 1 Mark: Finds the time taken to reach the ground.
13(c) (i)	$y = 5 - x$ ① $y = \frac{4}{x}$ ② Substitute $5 - x$ for y into equation ② $5 - x = \frac{4}{x}$ $x^2 - 5x + 4 = 0$ $(x - 1)(x - 4) = 0$ $x = 1$ and $y = 4$ $x = 4$ and $y = 1$ ∴ Points of intersection are (1, 4) and (4, 1)	1 Mark: Correct answer.

13(c) (ii)	$A = \int_a^b y\,dx = \int_1^4 (5-x) - \left(\frac{4}{x}\right) dx$ $= \left[5x - \frac{x^2}{2} - 4\ln x\right]_1^4$ $= \left[\left(5 \times 4 - \frac{4^2}{2} - 4\ln 4\right) - \left(5 - \frac{1^2}{2} - 4\ln 1\right)\right]$ $= 7\frac{1}{2} - 4\ln 4 = 1.9548\ldots$ ≈ 1.95 square units	2 Marks: Correct answer. 1 Mark: Correctly sets up the integral.
13(d)	$\lvert \underset{\sim}{u} \rvert = \sqrt{2^2 + (-5)^2}$ $= \sqrt{29}$ $\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{\lvert \underset{\sim}{u} \rvert}$ $= \frac{1}{\sqrt{29}}(2\underset{\sim}{i} - 5\underset{\sim}{j})$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
13(e)	Step 1: To prove true for $n = 1$ LHS $= 1 \times 2^{1-1} = 1$ RHS $= 1 + (1-1)2^1 = 1$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $S_k = 1 + (k-1)\,2^k$ Step 3: To prove true for $n = k + 1$ $S_{k+1} = 1 + k\,2^{k+1}$ $S_k + T_{k+1} = S_{k+1}$ LHS $= 1 + (k-1)\,2^k + (k+1)\,2^k$ $= 1 + 2^k(k - 1 + k + 1)$ $= 1 + 2k \times 2^k$ $= 1 + k\,2^{k+1}$ $=$ RHS Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$ 1 mark: Proves the result true for $n = 1$.
14(a)	Let p be the probability of winning a prize in a game of chance. $p = 0.48$ $P(X = x) = {}^nC_x\, 0.48^x 0.52^{n-x}$ Probability of winning at least once is more than 0.95. Expression is: $P(X = x) = 1 - {}^nC_0\, 0.48^0 0.52^{n-0} > 0.95$ $1 - 0.52^n > 0.95$ $0.52^n < 0.05$ $n > \frac{\ln 0.05}{\ln 0.52}$ $> 4.5811\ldots$ $n = 5$ (n is a positive integer) $\therefore$ The least number of games is 5.	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Finds the general rule for probability distribution.

14(b)	 $V = \pi\int_b^b y^2dx = \pi\int_0^{\frac{\pi}{2}} 9\sin^2 2x\,dx =$ $= 9\pi\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 4x)dx$ $= \frac{9\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$ $= \frac{9\pi}{2}\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \left(0 - \frac{1}{4}\sin 0\right)$ $= \frac{9\pi^2}{4}$ cubic units	3 Marks: Correct answer. 2 Marks: Uses the double angle formula to simplify the integral. 1 Mark: Shows some understanding.
14(c)(i)	Let p be the probability of Ryan solving the problem. $p = 0.7$, $n = 7$ $P(X = x) = {}^7C_x\, 0.7^x 0.3^{7-x}$ $P(X = 5) = {}^7C_5(0.7)^5(0.3)^2$ $= 0.3176523$	2 Marks: Correct answer. 1 Mark: Finds the general rule for probability distribution
14(c)(ii)	$P(X \geq 5) = {}^7C_5(0.7)^5(0.3)^2$ $+{}^7C_6(0.7)^6(0.3)^1 + {}^7C_7(0.7)^7(0.3)^0$ $= 0.6470695$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
14(d)(i)	Now $\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b}$ $\lvert\underset{\sim}{c}\rvert^2 = \underset{\sim}{c}\cdot\underset{\sim}{c}$ $= (\underset{\sim}{a} + \underset{\sim}{b})\cdot(\underset{\sim}{a} + \underset{\sim}{b})$ $= \underset{\sim}{a}\cdot\underset{\sim}{a} + 2(\underset{\sim}{a}\cdot\underset{\sim}{b}) + \underset{\sim}{b}\cdot\underset{\sim}{b}$	2 Marks: Correct answer. 1 Mark: Makes some progress.
14(d)(ii)	Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular $\therefore\ \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ $\lvert\underset{\sim}{c}\rvert^2 = \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{b} + 2\times 0$ $= \lvert\underset{\sim}{a}\rvert^2 + \lvert\underset{\sim}{b}\rvert^2$	2 Marks: Correct answer. 1 Mark: Uses the result $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ for perpendicular vectors
14(e)	$\frac{dr}{dt} = 7, A = \pi r^2$ $\frac{dA}{dt} = \frac{dA}{dr}\times\frac{dr}{dt}$ $= 2\pi r \times 7$ $= 224\pi$ when $r = 16$ $\therefore$ The rate of change in the area is 224π.	2 Marks: Correct answer. 1 Mark: Shows some understanding of the rate of change.