CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Extension 2 Mathematics

Assessment Task 1 2016

Time allowed: **1 hour 30 minutes**

Name: ____________________________________ **Teacher: Ms Strilakos**

Instructions:

- All questions should be attempted on your own paper.
- Show ALL necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Please write on one side of each sheet of paper only, and do not use multiple columns on the page.

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Total
E3	/4	/3	/6	/4	/3	/4	/4	/3	/10	/7	/7	/5	/60

Question 1

Let $z = 3 - i$ and $w = 2 + 4i$.

Find the following in the form $x + iy$:

(i) $z\overline{w}$ (ii) $\dfrac{z}{w}$

[4 marks]

Question 2

Use De Moivre's Theorem to evaluate:

(i) $\left(3cis\left(\frac{-3\pi}{4}\right)\right)^2$ (ii) $(cis\ 72°)^5$

[3 marks]

Question 3

Show on an Argand Diagram the region R containing all points representing the complex numbers z such that:

(i) $|z+2| \geq 2$ and $\frac{\pi}{3} \leq Arg(z+2) \leq \frac{\pi}{2}$

(ii) $|z+2i| < 2$ and $Im(z) > -1$

(iii) $0 < Arg(z+2+3i) \leq \frac{\pi}{6}$

[6 marks]

Question 4

If $z = r(cos\theta + isin\theta)$ find:

(i) $\left|7(1-2i)(\sqrt{3}+i)z^2\right|$ in terms of r

(ii) $arg\{7(-2i)(\sqrt{3}+i) \div z\}$ in terms of θ.

[4 marks]

Question 5

If $1, w$ and w^2 are the cube roots of unity, find the equation whose roots are w^4 and w^{-4}.

[3 marks]

Question 6

If $z = a + bi$ and $\bar{z}$ is the conjugate of z, find a and b if $\frac{2z+\bar{z}}{z.\bar{z}} = 1 + i$

[4 marks]

Question 7

(i) If $\frac{z-2i}{2z-4}$ is purely imaginary,

show that the locus of z on the complex plane is a circle.

(ii) Find its centre and radius and sketch its graph.

[4 marks]

Question 8

(i) State the Cartesian equation of the locus of $|z - 3 + 3i| = 2$ and sketch this locus on the complex number plane.

(ii) Hence find the greatest and least values of $|z + 1|$.

[3 marks]

Question 9

(i) Find the two square roots of $5 - 12i$ in the form $a + bi$ where a and b are real.

(ii) Show the points P and Q representing the square roots in part (i) on an Argand diagram.

(iii) Find the complex numbers represented by points R_1, R_2 such that the triangles PQR_1, PQR_2 are equilateral.

[10 marks]

Question 10

(i) Show that $1 + cos2\theta + isin2\theta = 2cos\theta(cos\theta + isin\theta)$

(ii) Hence prove that $\left(1 + cos\frac{2\pi}{n} + isin\frac{2\pi}{n}\right)^n = -2^n cos^n\frac{\pi}{n}$, n an integer.

(iii) Thus simplify the expression

$$\left(1 + cos\frac{2\pi}{n} + isin\frac{2\pi}{n}\right)^n - \left(1 + cos\frac{2\pi}{n} - isin\frac{2\pi}{n}\right)^n$$

[7 marks]

Question 11

(i) Find the locus of a point P representing the complex number z, where

$$|z - 1| = |z - 3i|$$

(ii) Sketch this locus on an Argand diagram and find z when $|z|$ has its least value on this locus.

[5 marks]

Question 12

Given that $z = cos\theta + isin\theta$,

(i) Show that $2\cos n\theta = z^n + z^{-n}$ and $2i\sin n\theta = z^n - z^{-n}$

(ii) Hence find an expression for $\cos\theta$ in terms of z.

(iii) Hence, find the values of A, B and C such that

$$32cos^6\theta = (cos6\theta + A\,cos4\theta + B\,cos2\theta + C)$$

You may find the following general expansion useful:

$$(x + a)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \binom{n}{3}x^{n-3}a^3 + \cdots \binom{n}{n}a^n$$

[7 marks]

END OF PAPER ☺

QUESTION 1

$z = 3 - i \qquad \omega = 2 + 4i$

(i) $z.\bar{\omega} = (3-i)(2-4i)$

$= 6 - 12i - 2i + 4i^2 \quad 1$

$= 2 - 14i \quad 1$

(ii) $\frac{z}{\omega} = \frac{3-i}{2+4i}$

$= \frac{3-i}{2+4i} \times \frac{2-4i}{2-4i} \quad 1$

$= \frac{2-14i}{4-16i^2} \ldots$

$= \frac{2-14i}{20} = \frac{1}{10} - \frac{7}{10}i \quad 1$

4

QUESTION 2

(i) $\left(3 \operatorname{cis}\left(\frac{-3\pi}{4}\right)\right)^2$

$= 3^2 \operatorname{cis}\left(-\frac{6\pi}{4}\right) \quad 1$

$= 9 \operatorname{cis} \frac{\pi}{2}$

$= 9i \quad 1$

(ii) $(\operatorname{cis} 72°)^5 = \operatorname{cis}(5 \times 72)°$

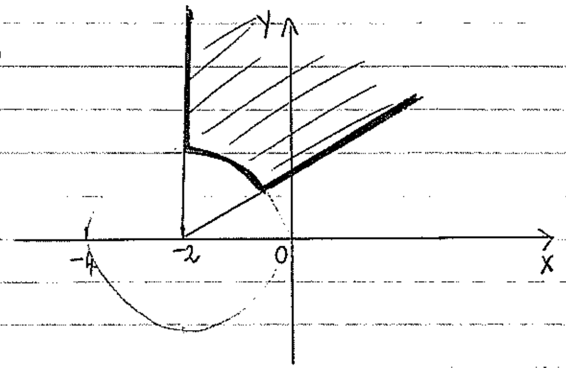
$= \operatorname{cis} 360° \quad 1$

$= 1 \quad 1$

3

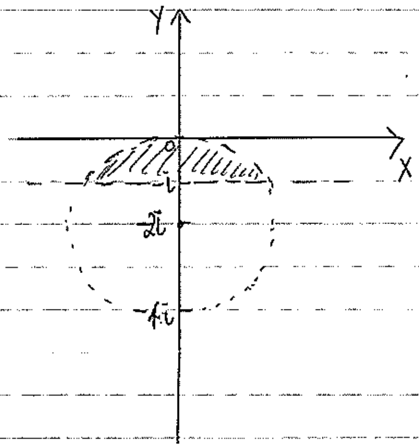
QUESTION 3

(i)



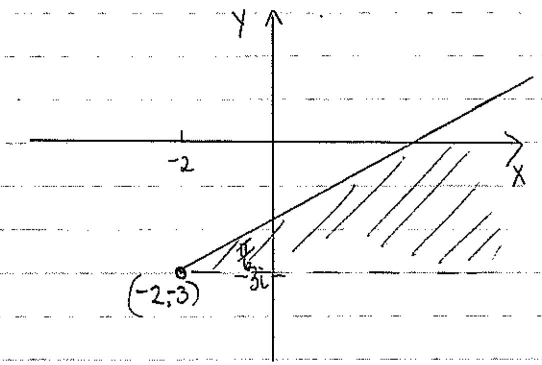
2 + 1 2

(ii)



2

(iii)



2

6

QUESTION 4

$z = r(\cos\theta + i\sin\theta) \qquad |z^2| = r^2 \qquad \arg z = \theta$

(i) $|7(1-2i)(\sqrt{3}+i)z^2| = |7|\,|1-2i|\,|\sqrt{3}+i|\,|z^2|$ 1

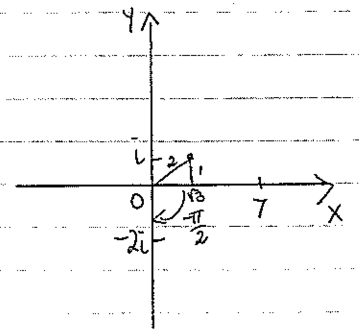
$= 7 \times \sqrt{5} \times \sqrt{4} \times r^2$ 1

$= 14\sqrt{5}\, r^2$ 1

(ii) $\arg\{7(-2i)(\sqrt{3}+i) \div z\} = \arg(7) + \arg(-2i) + \arg(\sqrt{3}+i) - \arg z$ 1

$= 0 + \left(-\frac{\pi}{2}\right) + \frac{\pi}{6} - \theta$ 1

$= -\frac{\pi}{3} - \theta$ 1



4

For $\arg(\sqrt{3}+i)$, $\frac{\pi}{6}$ ie $\sin^{-1}\frac{1}{2}$

QUESTION 5

$1+\omega+\omega^2=0 \qquad \omega^4 = \omega^3.\omega = \omega$

$\omega^3 = 1$

$\omega^{-4} = \omega^{-3}.\omega^{-1} = \omega^{-1}$ 1

Σ roots $= -\frac{b}{a} = \omega^4 + \omega^{-4} = \omega + \frac{1}{\omega} = \frac{\omega^2+1}{\omega}$ 1

Now, $\omega^2 + 1 = -\omega$ $\quad \therefore \Sigma$ roots $= \frac{-\omega}{\omega} = -1$. $\therefore -\frac{b}{a} = -1$ 1

If $a=1, b=1$.

Product of roots: $\omega^4.\omega^{-4} = \omega^0 = 1$ 1

$\therefore$ Eq^n is $z^2+z+1=0$. 1

3

QUESTION 6

$z = a+bi \qquad \bar{z} = a-bi$

$\frac{2z+\bar{z}}{z\bar{z}} = 1+i$

LHS $= \frac{2a+2bi+a-bi}{a^2+b^2} = \frac{3a+bi}{a^2+b^2} = 1+i$ 1

ie $\frac{3a}{a^2+b^2} = 1$ —① and $\frac{b}{a^2+b^2} = 1$ —②

4

Solving ① & ② $\quad 3a = b$ —③ 1

Subst. for b in ② yields $\frac{3a}{a^2+9a^2} = 1$ ie $3a = 10a^2$ 1

$a(10a-3)=0$

$a=0,\ a=3/10$

Disregard $a=0 \qquad a = 3/10,\ b = 9/10$. 1

QUESTION 7.

(i) $\dfrac{z-2i}{2z-4}$ purely imaginary

Let $z = x+iy$ $\qquad \dfrac{z-2i}{2z-4} = \dfrac{x+(y-2)i}{2x+2yi-4}$

$$= \frac{x+(y-2)i}{(2x-4)+2yi} \times \frac{(2x-4)-2yi}{(2x-4)-2yi}$$

$$= \frac{2x^2-4x-\cancel{2xyi}+2y^2-4y+\cancel{2xyi}-4xi-4yi+8i}{(2x-4)^2+4y^2}$$

Since it is purely imaginary, the real part equals zero

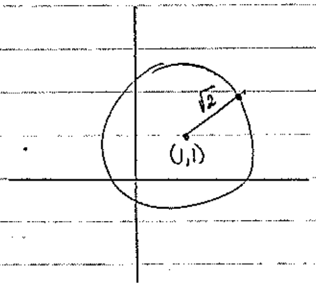
$\therefore \quad \dfrac{2x^2-4x+2y^2-4y}{(2x-4)^2+4y^2} = 0$

ie $\quad 2(x^2-2x+y^2-2y) = 0$

ie $\quad (x-1)^2-1+(y-1)^2-1 = 0$

ie $\quad (x-1)^2+(y-1)^2 = 2$ $\qquad$ This is the locus of a circle.

(ii) centre $(1,1)$ radius $\sqrt{2}$



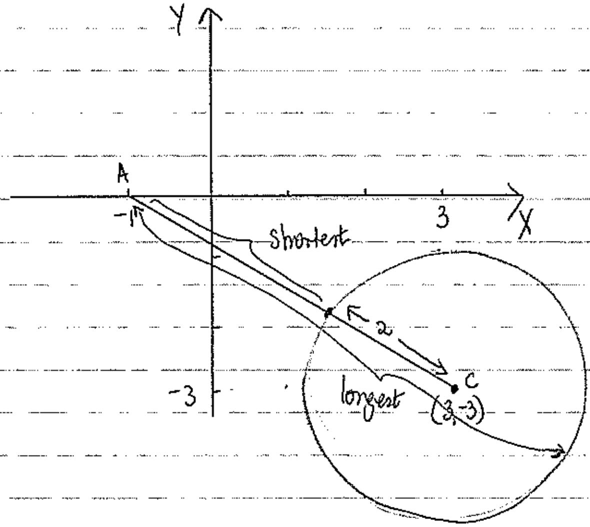
QUESTION 8

(i) $|z-3+3i| = 2$

Let $z = x+iy$ $\quad \therefore \sqrt{(x-3)^2+(y+3)^2} = 2$

Squarebs. $\quad (x-3)^2+(y+3)^2 = 4$

Circle centre $(3,-3)$ radius 2.



③.

Distance $AC = \sqrt{4^2+3^2} = 5$

$\therefore$ Shortest distance is $5-2 = 3$ units

Longest distance is $5+2 = 7$ units

QUESTION 9

(i) Let $z = x+iy$ and $z^2 = 5-12i$ $\quad x, y$, real

$\therefore \quad (x+yi)^2 = 5-12i$

ie. $x^2-y^2+2xyi = 5-12i$

ie. $x^2-y^2=5$ ① and $2xy=-12$, $xy=-6$, $y=\frac{-6}{x}$ ② $\quad$ 1

Substituting ② into ①

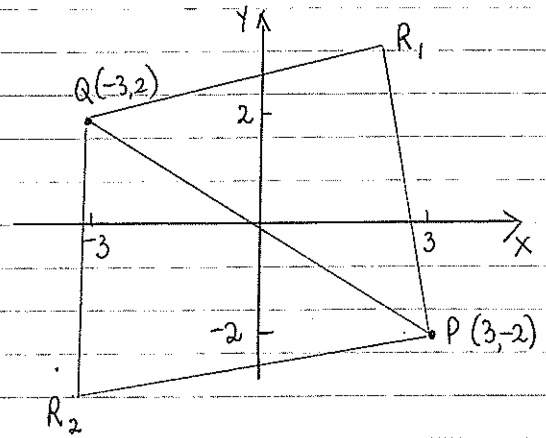
$$x^2 - \frac{36}{x^2} = 5$$

ie. $x^4-5x^2-36=0$ $\quad$ (3)

$(x^2-9)(x^2+4)=0$ $\quad$ 1

$x=\pm 3$ $\quad$ If $x=3, y=-2$ and if $x=-3, y=2$

$\therefore$ the roots are $3-2i$ and $-3+2i$ $\quad$ 1



(1)

Now, if the triangles are equilateral they contain $3 \times 60°$ angles, and sides are equal in length.

$d(PQ) = \sqrt{6^2+4^2} = \sqrt{52}$

Now, $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = (-3+2i)+(-3+2i) = -6+4i$ $\quad$ 1

and $\overrightarrow{PR_1}$ is a rotation clockwise through $\frac{\pi}{3}$ of PQ.

$\therefore \overrightarrow{PR_1} = (-6+4i) \times \text{cis}\left(-\frac{\pi}{3}\right)$

$= (-6+4i) \times \left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$= (-6+4i) \times \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ $\quad$ 1

$= -3 + 3\sqrt{3}i + 2i - 2\sqrt{3}i^2$

$= 2\sqrt{3} - 3 + i(3\sqrt{3}+2)$

To find $\overrightarrow{OR_1}$, $\quad \overrightarrow{PR_1} = \overrightarrow{PO} + \overrightarrow{OR_1}$

$\therefore \overrightarrow{OR_1} = \overrightarrow{PR_1} - \overrightarrow{PO} = \overrightarrow{PR_1} + \overrightarrow{OP} = 2\sqrt{3} \cancel{-3} + i(3\sqrt{3}+2) + \cancel{3} - 2i$

ie. $\overrightarrow{OR_1} = 2\sqrt{3} + 3\sqrt{3}i$ $\quad$ 1

Similarly, $\overrightarrow{PR_2}$ is a rotation of $\overrightarrow{PQ}$ anticlockwise through $\frac{\pi}{3}$.

$\therefore \overrightarrow{PR_2} = (-6+4i) \times \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= (-6+4i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -3 - 3\sqrt{3}i + 2i - 2\sqrt{3}$ $\quad$ 1

$\therefore \overrightarrow{PR_2} = -3-2\sqrt{3} + i(2-3\sqrt{3})$

$\overrightarrow{OR_2} = \overrightarrow{PR_2} + \overrightarrow{OP} = \cancel{-3}-2\sqrt{3} + i(2-3\sqrt{3}) + \cancel{3}-2i$ (6)

$= -2\sqrt{3} - 3\sqrt{3}i$ 1

Question 10

(i) Show that $1+\cos 2\theta + i\sin 2\theta = 2\cos\theta(\cos\theta + i\sin\theta)$

$LHS = 1 + 2\cos^2\theta - 1 + i \times 2\sin\theta\cos\theta$ 1

$= 2\cos\theta(\cos\theta + i\sin\theta) = RHS$ as required.

(ii) To prove that $\left(1+\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = -2^n\cos^n\frac{\pi}{n}$, $n \in J$

Proof: $LHS = 2^n\cos^n\theta\,(\cos\theta + i\sin\theta)^n$ from part (i), $\theta = \frac{\pi}{n}$

where $\theta = \frac{\pi}{n}$

$= 2^n\cos^n\frac{\pi}{n}\left(\cos\frac{n\pi}{n} + i\sin\frac{n\pi}{n}\right)$ 1

$= 2^n\cos^n\frac{\pi}{n}(\cos\pi + i\sin\pi)$ 1

$= -2^n\cos^n\frac{\pi}{n} = RHS$ as required. 1

(iii) $\left(1+\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n - \left(1+\cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)^n$

$= -2^n\cos^n\frac{\pi}{n} - \left(1+\cos\left(-\frac{2\pi}{n}\right) + i\sin\left(-\frac{2\pi}{n}\right)\right)^n$ 1

$= -2^n\cos^n\frac{\pi}{n} - \left[2^n\cos^n\left(-\frac{\pi}{n}\right)\cdot\left[\cos\left(-\frac{\pi}{n}\right) + i\sin\left(-\frac{\pi}{n}\right)\right]^n\right]$

$= -2^n\cos^n\frac{\pi}{n} - \left[2^n\cos^n\frac{\pi}{n}\left[\cos\frac{\cancel{n}\pi}{\cancel{n}} - i\sin\frac{\cancel{n}\pi}{\cancel{n}}\right]\right]$ 1

$= -2^n\cos^n\frac{\pi}{n} + 2^n\cos^n\frac{\pi}{n} = 0.$ 1

Question 12

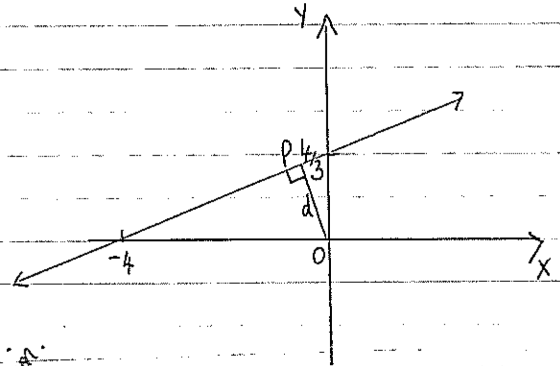
$|z-1| = |z-3i|$

$(x-1)^2 + y^2 = x^2 + (y-3)^2$

$\cancel{x^2} - 2x + 1 + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} - 6y + 9$

$\therefore\ 2x - 6y + 8 = 0$. i.e. $x - 3y + 4 = 0$.

$y = \frac{1}{3}x + \frac{4}{3}$



This part is not necessary to answer the question.

$|d|$ is when $|z|$ has its least value.

$d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ for $x - 3y + 4 = 0$, $a = 1, b = -3, c = 4$

and $(x_1, y_1) = (0,0)$

$= \dfrac{|4|}{\sqrt{10}}$

$\therefore$ this min^m distance is $\frac{4}{\sqrt{10}}$ or $\frac{4\sqrt{10}}{10}$ i.e. $\frac{2\sqrt{10}}{5}$

$m\ OP = -\frac{1}{m}$ for line. Line is $3y = x + 4$ $y = \frac{x}{3} + \frac{4}{3}$, $m = \frac{1}{3}$

$\therefore\ m\ OP = -3$ & its eq^n is $y = -3x$

The lines meet when we solve $x - 3y + 4 = 0$ with $y = -3x$

i.e. $x - 3\times(-3x) + 4 = 0$

i.e. $10x = -4$ $x = \frac{-2}{5}$, $y = \frac{6}{5}$.

$\therefore$ the point P is $\frac{-2}{5}, \frac{6i}{5}$

and it is on a complex no. plane.

$\therefore$ P is $\frac{1}{5}(-2 + 6i)$