CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 11 Extension 1 Mathematics Examination

Term 4 Week 9A 2019

Time allowed : ***50 Minutes***

Student Number : ______________________________

Instructions

- Answer each question in the spaces provided
- Board approved calculators may be used
- Show all necessary working by using blue / black pen except graphs / diagrams
- Marks may be deducted for untidy setting out

Topics	Question 1	Question 2	Total
Further Trigonometric Equations	/ 29		**/ 29**
Mathematical Inductions		/ 12	**/ 12**
Total	**/ 29**	**/ 12**	**/41**

QUESTION 1 **(29 marks)**

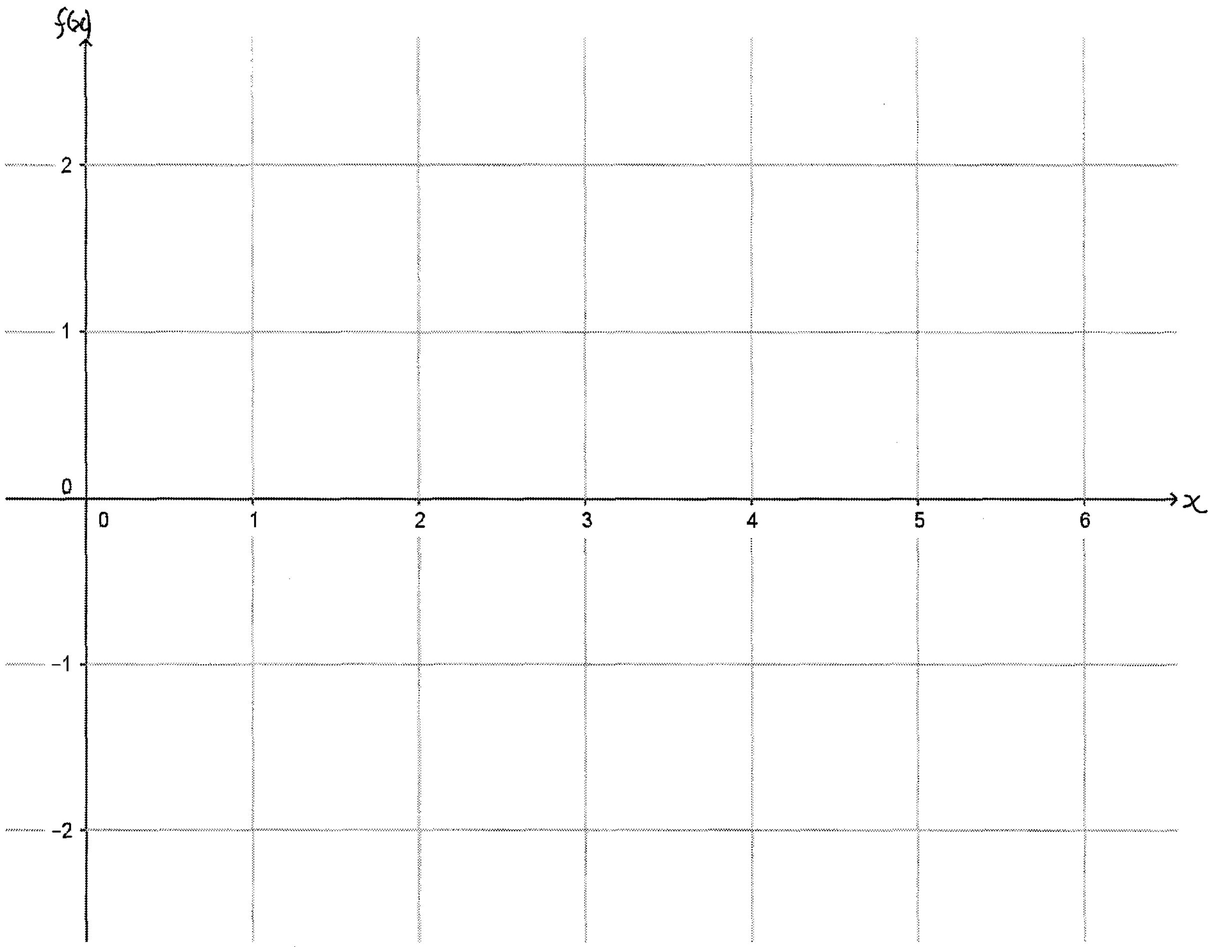
a).	Solve for θ the equation $\sqrt{3}\tan^2\theta + \tan\theta = 0$ for $0 \le \theta \le \pi$.	**[2]**
b).	Solve the equation $2\sin\left(x+150^\circ\right) = \sin x$ for $0^\circ \le x \le 360^\circ$, correct to nearest minute.	**[3]**
c).	Solve $2\cos^2 x + \sin x = 1$, for $0 \le x \le 2\pi$.	**[3]**

d). ***α*). By using the *t* - formula**, solve the equation $\sqrt{3}\sin\theta - \cos\theta = 1$ for $0^\circ \leq \theta \leq 2\pi$. **[2]**

***β*).** Is $\theta = \pi$ a solution of the equation? Justify with working. **[1]**

e). Solve the equation $5\cos\theta - 12\sin\theta = 2$ in the interval $0^\circ \leq \theta \leq 360^\circ$, correct to nearest minute. **[4]**
Show full working out, by using the Auxiliary angle method.

f). α). Sketch the graph of $f(x) = 2\cos(x + \frac{\pi}{6})$, for $0 \leq x \leq 2\pi$. Showing all intercepts, end points and turning points. **[3]**



β). From your sketch, find the values of x for which $f(x) = -1$, correct to 1 decimal place. **[1]**

γ). Hence, find the values of x for which $f(x) > -1$ within the given domain. **[1]**

g). Find the condition for the trigonometric equation $\boldsymbol{a}\mathbf{cos}\boldsymbol{x} + \boldsymbol{b}\mathbf{sin}\boldsymbol{x} = \boldsymbol{c}$ to have real solutions. **[3]**
[Hint: use t – formula]

h). By writing $\cos 3\theta = \cos(2\theta + \theta)$, express $\cos 3\theta$ in terms of $\cos\theta$. **[3]**

i). Prove $\dfrac{\cot^2\theta + 1}{\cot^2\theta - 1} = \sec 2\theta$. **[3]**

QUESTION 2 (12 marks)

a). Prove by mathematical induction, that for any positive integers n,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$ **[4]**

b). Prove by mathematical induction, that for all positive integer $n \geq 1$, **[4]**

$7^n(3n+1)-1$ is divisible by 9.

c). Let $f(n)$ be the statement: $n^2 - n$ is an odd integer.

α). Show that if $f(k)$ is true then $f(k+1)$ is true. **[2]**

β). Is $f(1)$ true ? **[1]**

γ). Is $f(n)$ true for all n ? Justify. **[1]**

END OF EXAM

CARLINGFORD HIGH SCHOOL

TERM 4 TEST

WEEK 9A 2019

Mathematics - Extension 1

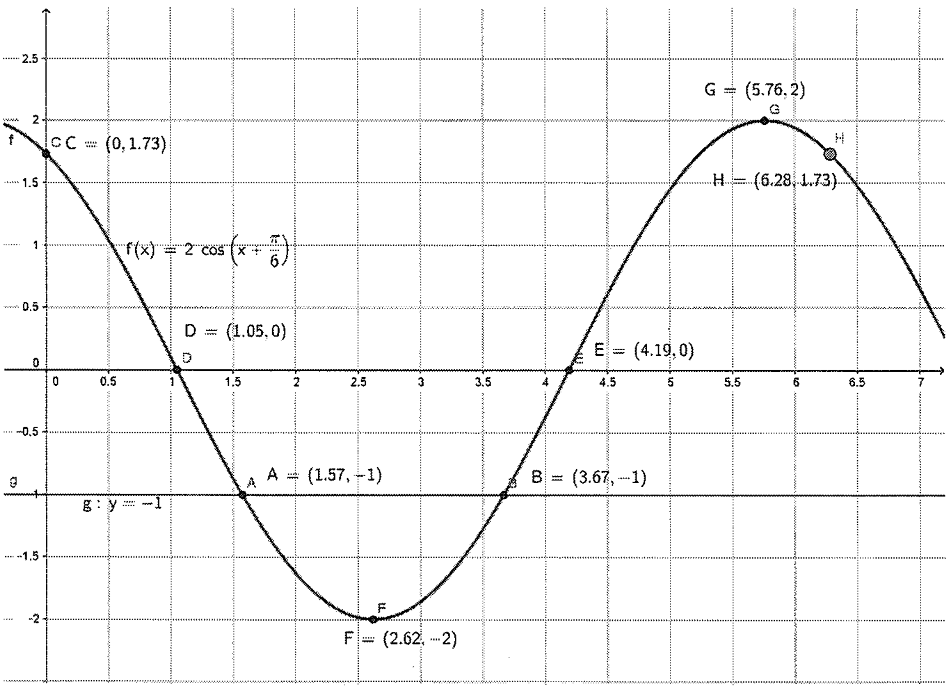
SOLUTIONS

AG mark Q1 a, b, c
Q2 a

Question 1	Mathematics Extension 1		
Part	Solution	Marks	Comment
a).	Solve for θ the equation $\sqrt{3}\tan^2\theta+\tan\theta=0$ for $0\le\theta\le\pi$. $\tan\theta\left(\sqrt{3}\tan\theta+1\right)=0$ $\tan\theta=0$ or $\sqrt{3}\tan\theta+1=0$ So $\theta=0,\ \pi$ or $\tan\theta=-\dfrac{1}{\sqrt{3}}$ $\theta=\pi-\dfrac{\pi}{6}=\dfrac{5\pi}{6}$ $\therefore\ \theta=0,\ \dfrac{5\pi}{6},\ \pi$	**2**	**1** mark for correct factorisation. **1** mark for correct answers.
b).	Solve the equation $2\sin\left(x+150^\circ\right)=\sin x$ for $0^\circ\le x\le 360^\circ$, correct to nearest minute. $2\sin\left(x+150^\circ\right)=\sin x$ $2\sin x\cos 150^\circ+2\cos x\sin 150^\circ=\sin x$ $2\sin x\times-\dfrac{\sqrt{3}}{2}+2\cos x\times\dfrac{1}{2}=\sin x$ $-\sqrt{3}\sin x+\cos x=\sin x$ $\cos x=\left(1+\sqrt{3}\right)\sin x$ $\tan x=\dfrac{1}{1+\sqrt{3}}$ $\therefore\ x=20^\circ 6'$ or $200^\circ 6'$	**3**	**1** mark for using compound formula. **1** mark for express as $\tan x$. **1** mark for correct answers.
c).	Solve $2\cos^2 x+\sin x=1$, for $0\le x\le 2\pi$. $2\cos^2 x+\sin x=1$ $2\left(1-\sin^2 x\right)+\sin x=1$ $2-2\sin^2 x+\sin x=1$ $2\sin^2 x-\sin x-1=0$ $\left(2\sin x+1\right)\left(\sin x-1\right)=0$ $2\sin x+1=0$ or $\sin x-1=0$ $\sin x=-\dfrac{1}{2}$ or $\sin x=1$ So $x=\dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$ or $x=\dfrac{\pi}{2}$ $\therefore\ x=\dfrac{\pi}{2},\ \dfrac{7\pi}{6},\ \dfrac{11\pi}{6}$	**3**	**1** mark for using square identity. **1** mark for correct factorisation. **1** mark for correct answers.

d).	**α). By using the *t* - formula**, solve the equation $\sqrt{3}\sin\theta - \cos\theta = 1$ for $0^\circ \le \theta \le 2\pi$. $\sqrt{3}\sin\theta - \cos\theta = 1$ $\sqrt{3}\left(\dfrac{2t}{1+t^2}\right) - \left(\dfrac{1-t^2}{1+t^2}\right) = 1$ $2\sqrt{3}\ t - 1 + t^2 = 1 + t^2$ $t = \dfrac{1}{\sqrt{3}}$ i.e. $\tan\dfrac{\theta}{2} = \dfrac{1}{\sqrt{3}}$ $\dfrac{\theta}{2} = \dfrac{\pi}{6}$ $\therefore\quad \theta = \dfrac{\pi}{3}$ **β).** Is $\theta = \pi$ a solution of the equation? Justify with working. When $t = \pi$, then LHS $= \sqrt{3}\sin\pi - \cos\pi$ $= \sqrt{3} \times 0 - (-1)$ $= 1$ So LHS = RHS $\therefore\quad \theta = \pi$ is a solution	**3**	**1** mark for using the correct *t*-formula. **1** mark for correct answer. **1** mark for correct testing and concluding.
e).	Solve the equation $5\cos\theta - 12\sin\theta = 2$ in the interval $0^\circ \le \theta \le 360^\circ$, correct to nearest minute. Show full working out, by using the Auxiliary angle method. $5\cos\theta - 12\sin\theta = R\cos(\theta + \alpha)$ $= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$ By comparing LHS & RHS we get $R\cos\alpha = 5$ .... [1] & $R\sin\alpha = 12$ .... [2] From $[1]^2 + [2]^2$ get $R^2(\cos^2\alpha + \sin^2\alpha) = 169$ $\therefore\quad R = 13$ From [2] ÷ [1] get $\tan\alpha = \frac{12}{5}$ So $\alpha = 67^\circ 23'$ Now solve $5\cos\theta - 12\sin\theta = 2$ $13\cos(\theta + 67^\circ 23') = 2$ $\cos(\theta + 67^\circ 23') = \frac{2}{13}$ $\theta + 67^\circ 23' = 81^\circ 9'$ or $278^\circ 51'$ or $-81^\circ 9'$ $\therefore\quad \theta = 13^\circ 46'$ or $211^\circ 28'$ or $-13^\circ 46'$ (rejected)	**4**	**1** mark for correct compound identity. **1** mark for correct R & α values. **1** mark for correct transformation. **1** mark for correct answers.

<table>
<tr>
<td rowspan="3">f).</td>
<td>α). Sketch the graph of $f(x) = 2\cos(x + \frac{\pi}{6})$, for $0^{\circ} \le x \le 2\pi$.
Showing all intercepts, end points and turning points.



</td>
<td rowspan="3">3</td>
<td>1 mark for correct sketch.

1 mark for correct x,y intercepts.

1 mark for correct turning points.</td>
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<td>β). From your sketch, find the values of x for which $f(x) = -1$, correct to 1 decimal place.

From the sketch the values of x for which $f(x) = -1$, are **1.6 & 3.7**.</td>
<td>1 mark for the correct values.</td>
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<tr>
<td>γ). Hence, find the values of x for which $f(x) > -1$ within the given domain.

The values of x for which $f(x) > -1$ within the given domain are

$\mathbf{0 \le x < 1.6}$ **or** $\mathbf{3.7 < x \le 2\pi}$</td>
<td>1 mark for correct intervals.</td>
</tr>
</table>

g).	Find the condition for the trigonometric equation $a\cos x + b\sin x = c$ to have real solutions. [Hint: use t – formula] $a\cos x + b\sin x = c$ $a\left(\frac{1-t^2}{1+t^2}\right)+b\left(\frac{2t}{1+t^2}\right)=c$ $a-at^2+2bt=c+ct^2$ $(a+c)t^2-2bt+(c-a)=0$ Now $\Delta=(-2b)^2-4(a+c)(c-a)$ $=4b^2-4(c^2-a^2)$ $=4(b^2+a^2-c^2)$ To have real solutions: $\Delta \geq 0$ i.e. $b^2+a^2-c^2 \geq 0$ $\therefore$ $a^2+b^2 \geq c^2$	**3**	**1** mark for correct t-formula. **1** mark for correct discriminant. **1** mark for correct final condition.
h).	By writing $\cos 3\theta = \cos(2\theta+\theta)$, express $\cos 3\theta$ in terms of $\cos\theta$. Now $\cos 3\theta = \cos(2\theta+\theta)$ $= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$ $= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$ $= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$ $= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$ $\therefore$ $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$	**3**	**1** mark for correct compound identity. **1** mark for correct double angle formulae. **1** mark for correct answer.
i).	Prove $\frac{\cot^2\theta+1}{\cot^2\theta-1}=\sec 2\theta$. Now LHS $=\frac{\cot^2\theta+1}{\cot^2\theta-1}$ $=\left(\frac{\cos^2\theta}{\sin^2\theta}+1\right)\div\left(\frac{\cos^2\theta}{\sin^2\theta}-1\right)$ $=\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}$ $=\frac{1}{\cos 2\theta}$ $=\sec 2\theta$ $\therefore$ LHS = RHS	**3**	**1** mark for correct replace $\cot^2\theta$ with $\cos^2\theta$ & $\sin^2\theta$. **1** mark for correct manipulation. **1** mark for correct answer.

Question 2	Mathematics Extension 1		
Part	Solution	Marks	Comment
a).	Prove by mathematical induction, that for any positive integers n, $1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2$. **Step 1.** For $n=1$ LHS $=1^3$ RHS $=\left[\frac{1(1+1)}{2}\right]^2=1$ So LHS = RHS $\therefore$ It is true for $n=1$. **Step 2.** Assume it is true for $n=k$ i.e. $1^3+2^3+3^3+\ldots+k^3=\left[\frac{k(k+1)}{2}\right]^2$ Now prove $n=k+1$ is true i.e. $1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\left[\frac{(k+1)(k+2)}{2}\right]^2$ Thus using the assumption we get $\text{LHS}=\left[\frac{k(k+1)}{2}\right]^2+(k+1)^3$ $=\frac{k^2(k+1)^2+4(k+1)^3}{4}$ $=\frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$ $=\frac{(k+1)^2(k+2)^2}{4}$ $=\left[\frac{(k+1)(k+2)}{2}\right]^2$ $\therefore$ LHS = RHS Thus it is true for $n=k+1$. **Step 3.** It is true for $n=1$ and it is true for $n=k+1$ if it is true for $n=k$. $\therefore$ By mathematical induction, it is true for any positive integers n.	**4**	**1** mark for $n=1$. **1** mark for correct assumption. **1** mark for using the assumption correctly. **1** mark for correct manipulation.

b).	Prove by mathematical induction, that for all positive integer $n \geq 1$, $7^n(3n+1)-1$ is divisible by 9. **Step 1.** Let $f(n) = 7^n(3n+1)-1$ Then $f(1) = 7^1(3\times1+1)-1$ $= 27$ $\therefore f(1)$ is divisible by 9 **Step 2.** Assume $f(k)$ is divisible by 9 i.e. $7^k(3k+1)-1 = 9M$ where M is an integer Now prove $f(k+1)$ is divisible by 9 i.e. $7^{k+1}\left[3(k+1)+1\right]-1 = 7^{k+1}(3k+4)-1$ $= 7^k \cdot 7(3k+4)-1$ $= 7^k(21k+28)-1$ $= 7^k(3k+1)-1+7^k(18k+27)$ Thus using the assumption we have $= 9M+7^k(18k+27)$ $= 9M+9\times7^k(2k+3)$ $= 9\left[M+7^k(2k+3)\right]$ $\therefore f(k+1)$ is divisible by 9 **Step 3.** It is true for $f(1)$ and it is true for $f(k+1)$ if it is true for $f(k)$. $\therefore$ By mathematical induction, $f(n)$ is true for all positive integer $n \geq 1$.	**4**	**1** mark for $f(1)$. **1** mark for assumption. **1** mark for correct manipulation. **1** mark for using the assumption.

c).	Let $f(n)$ be the statement: $n^2 - n$ is an odd integer. **α).** Show that if $f(k)$ is true then $f(k+1)$ is true. Now assume $f(k) = k^2 - k$ is an odd integer Show that $f(k+1) = (k+1)^2 - (k+1)$ is an odd integer $= (k+1)(k+1-1)$ $= k(k+1)$ $= k^2 + k$ $= k^2 - k + 2k$ = odd integer + even integer = odd integer $\therefore$ $f(k+1)$ is true if it is true for $f(k)$. **β).** Is $f(1)$ true ? Now $f(1) = 1^2 - 1$ $= 0$ This is not odd integer $\therefore$ $f(1)$ is not true **γ).** Is $f(n)$ true for all n ? Justify. When n is odd: $\text{odd}^2 - \text{odd} = \text{odd} - \text{odd}$ $= \text{even}$ When n is even: $\text{even}^2 - \text{even} = \text{even}$ By mathematical induction, this can not be proved, since for $n = 1$ is not true, hence, it is not going to be true for all $n \geq 1$.	**4**	**1** mark for correct assumption. **1** mark for correct working. **1** mark for proof $f(1)$. **1** mark for justify.