Sandra Q.11 & Q14(b)

Graham Q12 & Q14(c)

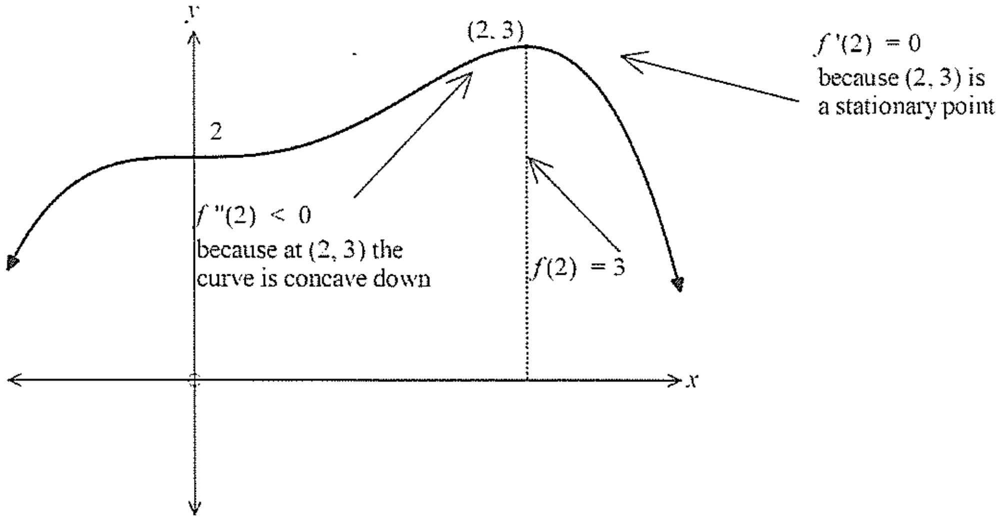
Ken Q13 & Q14(a)

ACE Examination 2019
Year 12 Mathematics Extension 1 Yearly Examination
Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1.	$u = 2 - x^2$ $\frac{du}{dx} = -2x$ $-\frac{1}{2}du = xdx$ $\int \frac{x}{(2-x^2)^3}dx = -\frac{1}{2}\int \frac{1}{u^3}\ du$ $= -\frac{1}{2} \times -\frac{1}{2}\ u^{-2} + c$ $= \frac{1}{4(2-x^2)^2}\ + c$	1 Mark: B
2.	$\int \sin^2 2x dx = \frac{1}{2}\int (1 - \cos 4x)dx$ $= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + c$ $= \frac{1}{8}(4x - \sin 4x) + c$	1 Mark: B
3.	Remainder when $P(x) = x^3 - 10x$ is divided by $x + 5$ $P(-5) = (-5)^3 - 10 \times (-5)$ $= -75$	1 Mark: A
4.	$x = \sin\theta$ 1 $y = \cos^2\theta - 3$ 2 $y = (1 - \sin^2\theta) - 3$ 3 using $\sin^2\theta + \cos^2\theta = 1$ sub 1 into 3 : $y = 1 - x^2 - 3$ $y = -2 - x^2$	1 Mark: A
5.	$x = t - 3$ then $\frac{dx}{dt} = 1$ $y = t^2 + 2$ then $\frac{dy}{dt} = 2t$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times 1 = 2t$ Now at $t = -3$ then $m = -6$ Equation of the tangent at $t = -3$ $(-6, 11)$ $y - y_1 = m(x - x_1)$ $y - 11 = -6 \times (x + 6)$ $6x + y + 25 = 0$	1 Mark: C

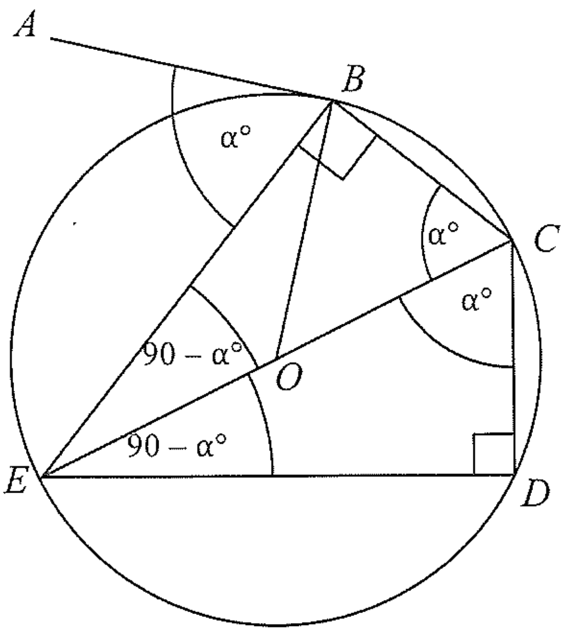
m.c. B B A A C

C D A D D

	Solution	Criteria
6.	 $\therefore f''(2) < f'(2) < 2 < f(2)$	1 Mark: C
7.	Solving the two equations simultaneously $(x+2)^2 + 4 = x^2 - 4$ $x^2 + 4x + 8 = x^2 - 4$ $4x = -12$ $x = -3$ Gradient of the curves at $x = -3$ $f(x) = (x+2)^2 + 4$ $f'(x) = 2(x+2)$ $f'(-3) = 2 \times -1 = -2 = m_1$ $g(x) = x^2 - 4$ $g'(x) = 2x$ $g'(-3) = -2 \times -3 = -6 = m_2$ $\tan\theta = \left\lvert\frac{m_1 - m_2}{1 + m_1 m_2}\right\rvert$ $= \left\lvert\frac{(-2) - (-6)}{1 + (-2) \times (-6)}\right\rvert$ $\theta = 17.1027\ldots \approx 17°$ $\therefore$ Obtuse angle is 163°	1 Mark: D
8.	$x = 4\sin 2t + 3\cos 2t$ $\dot{x} = 8\cos 2t - 6\sin 2t$ $\ddot{x} = -16\sin 2t - 12\cos 2t$ $= -4(4\sin 2t + 3\cos 2t)$ $= -4x$	1 Mark: A
9.	$-1 \leq 4x \leq 1$ $-\frac{1}{4} \leq x \leq \frac{1}{4}$	1 Mark: D
10.	$x = 6\cos^2 t - 1$ $= 3(1 + \cos 2t) - 1$ $= 2 + 3\cos 2t$ $\therefore$ Centre of motion is $x = 2$	1 Mark: D

Section II		
11(a)	$$f(x) = x^4 - 5x^3 + 11x^2 - 12x + 4$$ $$f'(x) = 4x^3 - 15x^2 + 22x - 12$$ Use $x_0 = 0.4$ $$\begin{aligned} x_1 &= x_0 - \frac{f(x)}{f'(x)} \\ &= 0.4 - \frac{f(0.4)}{f'(0.4)} \\ &= 0.4 - \frac{0.4^4 - 5\times 0.4^3 + 11\times 0.4^2 - 12\times 0.4 + 4}{4\times 0.4^3 - 15\times 0.4^2 + 22\times 0.4 - 12} \\ &= 0.5245\ldots \\ &\approx 0.52 \end{aligned}$$ pdp.	2 marks: Correct answer. 1 mark: Finds $f(0.4)$ and $f'(0.4)$ or shows some understanding of Newton's method.

11(b) (i)	$f(x) = \frac{5x+1}{(x+2)(x-1)} - \frac{3}{x+2}$ $= \frac{5x+1-3(x-1)}{(x+2)(x-1)}$ $= \frac{2x+4}{(x+2)(x-1)}$ $= \frac{2(x+2)}{(x+2)(x-1)}$ $= \frac{2}{x-1}$	1 mark: Correct answer.
11(b) (ii)	Making x the subject of the equation $y = \frac{2}{x-1}$ $xy - y = 2$ $xy = 2 + y$ $x = \frac{2+y}{y}$ $f^{-1}(x) = \frac{2+x}{x}$	2 marks: Correct answer. 1 mark: Shows some understanding.
11(b) (iii)	$\frac{2}{(x^2+5)-1} = \frac{1}{4}$ $\frac{2}{x^2+4} = \frac{1}{4}$ $x^2 + 4 = 8$ $x^2 = 4$ $x = \pm 2$ Functions	2 marks: Correct answer. 1 mark: Uses function notation correctly.
11(c)(i)	$\frac{10!}{2!3!2!} = 151200$	1 mark for correct answer
(c)(ii)	$\frac{9!}{2!3!2!} = 15120$	1 mark for correct answer
(c) (iii)	If the two C's are placed together then we are arranging 9 things with 3 U's and 2 R's repeated. This can be done $\frac{9!}{3!\,2!}$ ways. So the arrangements with the C's not together are P+C. $\frac{10!}{2!3!2!} - \frac{9!}{3!2!} = 120\,960$	1 mark for correct answer
11 (d)	$A(-4,1)$ and $B(x,y)$ $P(-2,5)$ divides AB in ratio 2 : 3 coord geo $\frac{2x + 3(-4)}{5} = -2$ and $\frac{2y + 3(1)}{5} = 5$ $2x - 12 = -10$ $2y + 3 = 25$ $2x = 2$ $2y = 22$ $x = 1$ $y = 11$ $\therefore B(1,11)$	2 marks for correct point 1 mark for setting up the correct equations to solve or equivalent merit

11(e)	 EC is a diameter $\angle BCE = \angle ABE = \alpha°$ ($\angle$ in alt segment) $\angle EBC = 90°$ ($\angle$ in semicircle) $\angle BEC = (90 - \alpha)°$ ($\angle$ sum Δ BEC) $\angle CED = (90 - \alpha)°$ (CE bisects $\angle$ BED) $\angle CDE = 90°$ (angle in semi circle) $\angle ECD = \alpha° = \angle ABE$ (angle sum Δ ECD) circle geo.	3 marks for a complete and valid proof 2 marks for a valid proof which is incomplete or has minor errors of logic 1 mark for a proof which includes some correct and relevant statements
12(a)	Let $u = 7x^2$ then $du = 14xdx$ $\int \frac{x}{\sqrt{1-49x^4}}dx = \frac{1}{14}\int \frac{du}{\sqrt{1-u^2}}$ $= \frac{1}{14}\sin^{-1} u + c$ $= \frac{1}{14}\sin^{-1} 7x^2 + c$	2 marks: Correct answer. 1 mark: Sets up substitution or shows some understanding.
12(b)	See separate solution attached.	3 marks: Correct answer. 2 marks: Makes significant progress. 1 mark: Finds one correct term.

HSC TRIAL EXTENSION 1 MATHEMATICS SOLUTIONS

Q.12(b)

$$(2+ax)^2(1+bx)^6 = 4+44x+85x^2+\dots$$

$$\text{LHS} = \left[4+4ax+a^2x^2\right]\left[\binom{6}{0}+\binom{6}{1}bx+\binom{6}{2}b^2x^2+\binom{6}{3}b^3x^3+\dots\right]$$

LHS Coeff of x: $4\times\binom{6}{1}b + 4a\binom{6}{0} = 44$ = RHS coeff of x. — ①

LHS Coeff of x^2: $4\times\binom{6}{2}b^2 + 4a\binom{6}{1}b + a^2\binom{6}{0} = 85$ = RHS coeff of x^2. — ②

thus we have from ① $4\times 6b + 4a = 44$ — ①′

and from ② $4\times 15b^2 + 24ab + a^2 = 85$ — ②′

ie $4a+24b = 44$ ie $a+6b=11$ — ①″

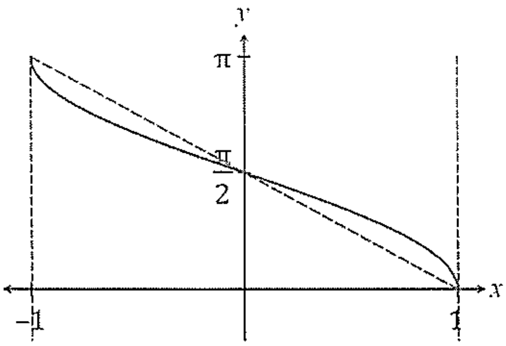
and $60b^2 + 24ab + a^2 = 85$ — ②″

from ①″ $a = 11-6b$ substitute into ②″ to obtain

$$60b^2 + 24(11-6b)b + (11-6b)^2 = 85$$

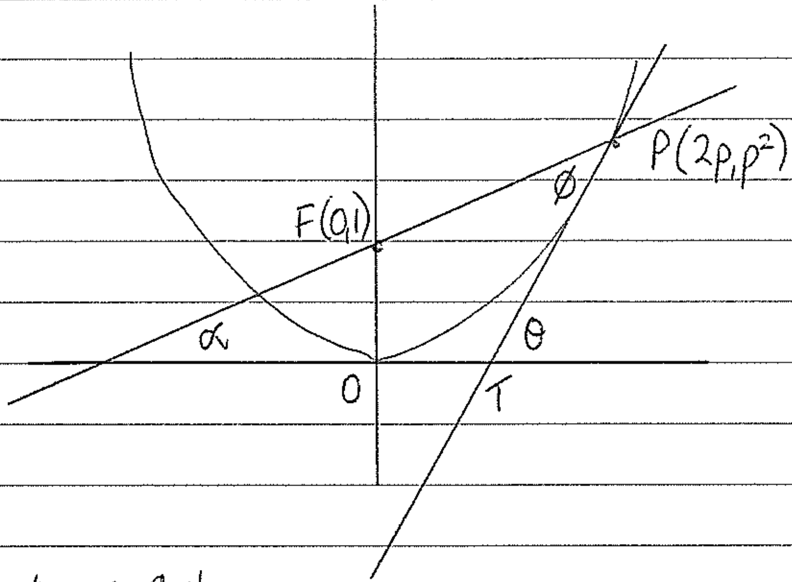
ie. $60b^2 + 264b - 144b^2 + 121 - 132b + 36b^2 = 85$

ie. $-48b^2 + 132b + 36 = 0$	If $b = -\frac{1}{4}$, $a = 12\frac{1}{2}$
$\div 12$ $4b^2 - 11b - 3 = 0$	If $b = 3$, $a = -7$
$(4b+1)(b-3) = 0$	
$b = -\frac{1}{4}, 3.$	

12(c) (i)	$y = \cos^{-1}x$ is symmetrical about the red dotted line shown opposite. $A = \frac{1}{2}\ bh$ $= \frac{1}{2}\ \times 2 \times \pi$ $= \pi$ square units 	1 mark: Correct answer.

12(c) (ii)	$$y = \cos^{-1} x$$ $$x = \cos y$$ $$x^2 = \cos^2 y$$ $$= \frac{1}{2}[\cos 2y + 1]$$ $$V = 2 \times \int_0^{\frac{\pi}{2}} \pi x^2 \ dy$$ $$= \int_0^{\frac{\pi}{2}} \pi[\cos 2y + 1] \ dy$$ $$= \pi \left[\frac{1}{2} \sin 2y + y\right]_0^{\frac{\pi}{2}}$$ $$= \pi \times \left[\frac{1}{2} \times 0 + \frac{\pi}{2}\right]$$ $$= \frac{\pi^2}{2} \text{ cubic units}$$	3 marks: Correct answer. 2 marks: Makes significant progress. 1 mark: Sets up the integral using the double angle.
12(d)	$$x = u^2 - 1$$ $$\frac{dx}{du} = 2\text{u}$$ $$dx = 2udu$$ When $x = 0$ then $u = 1$ and when $x = 15$ then $u = 4$ $$\int_0^{15} \frac{x}{\sqrt{x+1}} \ dx = \int_1^4 \frac{u^2 \ - 1}{u} \times 2udu$$ $$= 2 \times \int_1^4 u^2 - 1 \text{ du}$$ $$= 2\left[\frac{1}{3} u^3 - u\right]_1^4$$ $$= 2\left[\left(\frac{64}{3} - 4\right) - \left(\frac{1}{3} - 1\right)\right.$$ $$= 36$$	2 marks: Correct answer. 2 marks: Sets up the integral in terms of u.
12(e) (i)	See separate solution attached.	1 mark: Correct answer.
12(e) (ii)	See separate solution attached.	2 marks: Correct answer. 1 mark: Finds the period of the amplitude.

Q.12(e)



$x^2 = 4y \quad \therefore \ a = 1$

(i) $m_{TP} = \dfrac{dy}{dx} = \tan\theta$ $\qquad\qquad m_{FP} = \dfrac{\Delta y}{\Delta x} = \dfrac{p^2-1}{2p} = \tan\alpha$

$\dfrac{dy}{dx} = \dfrac{x}{2} = p$

$\therefore \tan\theta = p$

Now, $\theta - \alpha = \phi$ (exterior angle of Δ equals sum of opp interior angles)

$\therefore \tan(\theta - \alpha) = \tan\phi$

ie $\dfrac{\tan\theta - \tan\alpha}{1 + \tan\theta \tan\alpha} = \tan\phi$

ie $\dfrac{p - \left(\dfrac{p^2-1}{2p}\right)}{1 + \dfrac{p(p^2-1)}{2p}} = \tan\phi$

ie $\dfrac{\dfrac{p^2+1}{\cancel{2p}}}{\dfrac{2p + p^3 - p}{\cancel{2p}}} = \tan\phi$

$\Rightarrow \dfrac{\cancel{p^2+1}}{p(\cancel{p^2+1})} = \tan\phi$

$\therefore \tan\phi = \dfrac{1}{p}$.

Now,

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

$$= \frac{p + \frac{1}{p}}{1 - p \times \frac{1}{p}}$$

$$= \frac{p+\frac{1}{p}}{0}$$ ← This implies that $\tan(\theta+\phi)$ is undefined

$\therefore \theta + \phi$ must equal $\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}$ etc.

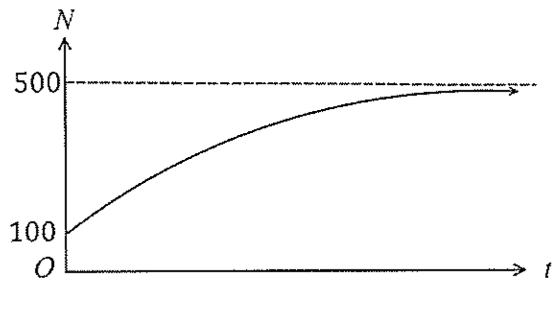
Take $\tan(\theta+\phi) = \tan\frac{\pi}{2}$ in this context.

$\therefore \theta + \phi = \frac{\pi}{2}$ as required.

(ii) If $\theta = \phi$, then $\theta = \phi = \frac{\pi}{4}$

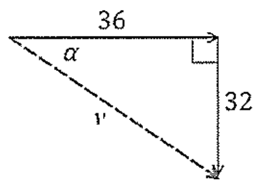
$\therefore \tan\theta = 1 = p$. $\therefore$ P is the point $(2, 1)$.

12(e) (iii)	See separate solution attached.	1 mark: Correct answer.
13(d) (i)	$\text{LHS} = 2\sin 2x - 3\cos 2x - 3\sin x + 3$ $= 2 \times (2\sin x\cos x) - 3(1 - 2\sin^2 x) - 3\sin x + 3$ $= 4\sin x\cos x + 6\sin^2 x - 3\sin x$ $= \sin x(6\sin x + 4\cos x - 3)$ $= \text{RHS}$	2 marks: Correct answer. 1 mark: Used the double angle formulas.
13(d) (ii)	$R\sin x\cos\alpha + R\cos x\sin\alpha = R\sin(x+\alpha)$ $6\sin x + 4\cos x = R\sin(x+\alpha)$ Hence $R\cos\alpha = 6$ and $R\sin\alpha = 4$ Dividing these equations $\tan\alpha = \frac{4}{6} = \frac{2}{3}$ $\alpha = 0.5880 \ldots.$ Squaring and adding the equations $R^2 = 6^2 + 4^2$ or $R = \sqrt{52}$ $\therefore 6\sin x + 4\cos x = \sqrt{52}\sin(x + 0.5880\ldots)$	2 marks: Correct answer. 1 mark: Finds the value of R or α
13(d) (iii)	$\sin x(6\sin x + 4\cos x - 3) = 0, \quad 0 \le x < \pi$ $\sin x \times (\sqrt{52}\sin(x + 0.5880\ldots) - 3) = 0, \quad 0 \le x < \pi$ Therefore $\sin x = 0$ $x = 0$ or $\sqrt{52}(\sin(x + 0.5880\ldots) - 3 = 0$, $\sin(x + 0.5880\ldots) = \frac{3}{\sqrt{52}}$ $x + 0.5880\ldots = 0.4290\ldots$ or. $2.7125\ldots.$ $x = 2.1245\ldots$ ≈ 2.12 $\therefore x = 0$ or $x = 2.12$	2 marks: Correct answer. 1 mark: Finds one of the answers or shows some understanding.

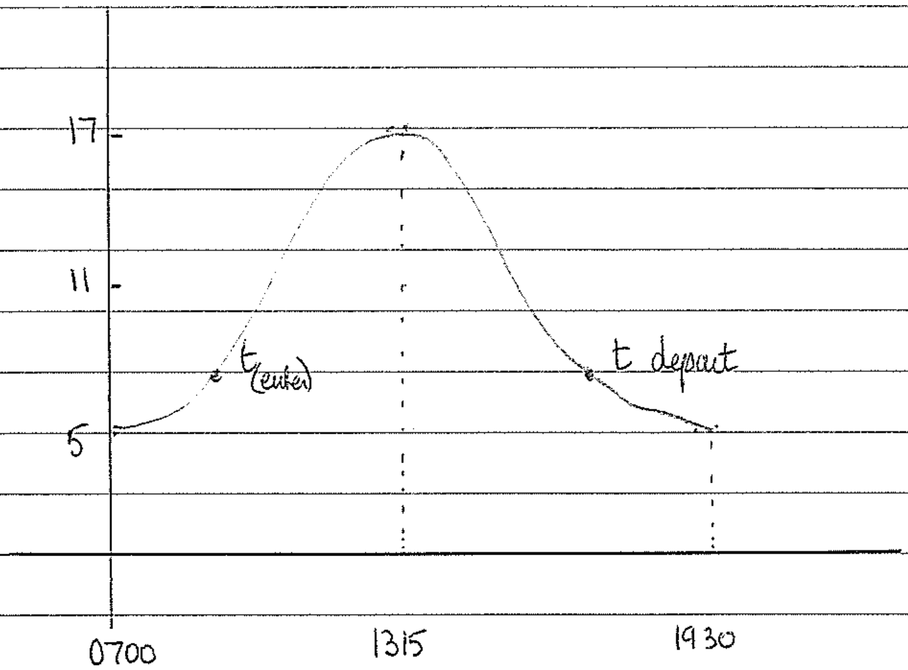
13 (b)	We want $\frac{dr}{dt}$ and have equations for V and for $\frac{dV}{dt}$ We can use $\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$ We know V in terms of r $V=\frac{9\pi r^4}{16}$ $\frac{dV}{dr}=\frac{9\pi r^3}{4}$ $\frac{dr}{dV}=\frac{4}{9\pi r^3}$ Rate at which V is increasing is $\frac{dV}{dt}=\frac{45}{h}$ $=45\div\frac{9r^2}{8}$ $=45\times\frac{8}{9r^2}$ $=\frac{40}{r^2}$ $\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$ $=\frac{4}{9\pi r^3}\times\frac{40}{r^2}$ $=\frac{160}{9\pi r^5}$ $h=8$ $\frac{9r^2}{8}=8$ $9r^2=64$ $r^2=\frac{64}{9}$ $r=\frac{8}{3}$ $\frac{dr}{dt}=\frac{160}{9\pi\left(\frac{8}{3}\right)^5}$ $=0.04196468226$ $=0.042$ cm/s	3 marks for correct answer 2 marks for correct use of related rates with a minor logical, numerical or algebraic error 1 mark for working which makes some use of related rates with errors or which is incomplete
13(c) (i)		1 mark: Correct answer.

13(c) (ii)	Rate of growth is the derivative $N = 500 - 400e^{-0.1t}$ $\frac{dN}{dt} = -400 \times -0.1e^{-0.1t}$ $= 0.1 \times 400e^{-0.1t}$ Initial rate of growth $\frac{dN}{dt} = 0.1 \times 400e^{-0.1\times 0} = 40$ To find N when $\frac{dN}{dt} = 20$ $\frac{dN}{dt} = 0.1(500 - N)$ $20 = 0.1(500 - N)$ $200 = 500 - N$ $N = 300$ $\therefore$ Population size is 300.	2 mark: Correct answer. 1 mark: Finds the rate of growth.
13(a) (i)	$x^3 - 5x^2 + 7x + 5 = 0$ $\alpha + \beta + \gamma = -\frac{b}{a}$ $= -\frac{-5}{1}$ $= 5$	1 mark: Correct answer.
13(a) (ii)	$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ $= \frac{7}{1}$ $= 7$	1 mark: Correct answer.
13(a) (iii)	$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ $5^2 = \alpha^2 + \beta^2 + \gamma^2 + 2 \times 7$ $\alpha^2 + \beta^2 + \gamma^2 = 11$	1 mark: Correct answer.

14(a)	Step 1: To prove the statement true for $n = 1$ $$\text{LHS} = \frac{1}{3\times4\times5} = \frac{1}{60}$$ $$\text{RHS} = \frac{1}{6} - \frac{1}{1+3} + \frac{2}{(1+3)(1+4)} = \frac{1}{60}$$ $\therefore$ Result true for $n = 1$ Step 2: Assume the result true for $n = k$ $$\frac{1}{3\times4\times5} + \frac{2}{4\times5\times6} + \cdots + \frac{k}{(k+2)(k+3)(k+4)} = \frac{1}{6} - \frac{1}{k+3} + \frac{2}{(k+3)(k+4)}$$ Step 3: To prove the result true for $n = k + 1$ $$\frac{1}{3\times4\times5} + \cdots + \frac{k}{(k+2)(k+3)(k+4)} + \frac{k+1}{(k+3)(k+4)(k+5)} = \frac{1}{6} - \frac{1}{k+4} + \frac{2}{(k+4)(k+5)}$$ $$\text{LHS} = \left[\frac{1}{6} - \frac{1}{k+3} + \frac{2}{(k+3)(k+4)}\right] + \frac{k+1}{(k+3)(k+4)(k+5)}$$ $$= \frac{1}{6} - \left[\frac{(k+4)(k+5) - 2(k+5) - (k+1)}{(k+3)(k+4)(k+5)}\right]$$ $$= \frac{1}{6} - \left[\frac{k^2 + 6k + 9}{(k+3)(k+4)(k+5)}\right]$$ $$= \frac{1}{6} - \left[\frac{(k+3)(k+5-2)}{(k+3)(k+4)(k+5)}\right]$$ $$= \frac{1}{6} - \left[\frac{(k+3)(k+5) - 2(k+3)}{(k+3)(k+4)(k+5)}\right]$$ $$= \frac{1}{6} - \frac{1}{(k+4)} + \frac{2}{(k+4)(k+5)}$$ $$= \text{RHS}$$ Result is true for $n = k + 1$ Step 4: Result true by the principle of mathematical induction.	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$ 1 mark: Proves the result true for $n = 1$.
14(b) (i)	Horizontal $$\ddot{x} = 0$$ $$\dot{x} = c_1$$ $$\dot{x} = V\cos\theta$$ $$x = Vt\cos\theta + c_2$$ When $t = 0$ then $x = 0$ $$x = Vt\cos\theta$$ Vertical $$\ddot{y} = -10$$ $$\dot{y} = -10t + c_3$$ When $t = 0$ then $\dot{y} = V\sin\theta$ $$\dot{y} = -10t + V\sin\theta$$ $$y = -5t^2 + Vt\sin\theta + c_4$$ When $t = 0$ then $y = 0$ $$\dot{y} = -10t + V\sin\theta$$ $$y = -5t^2 + Vt\sin\theta$$	2 marks: Correct answer. 1 mark: Finds the horizontal or the vertical displacements.

14(b) (ii)	After 8 seconds $x = 288$ and $y = 64$ $288 = 8V\cos\theta$ $V\cos\theta = 36$ ① $64 = -5\times 8^2 + 8V\sin\theta$ $384 = 8V\sin\theta$ $V\sin\theta = 48$ ② Solving the two equations simultaneously $V^2(\cos^2\theta + \sin^2\theta) = 36^2 + 48^2$ $V^2 = 3600$ or $V = 60$ $\dfrac{V\sin\theta}{V\cos\theta} = \dfrac{48}{36}$ $\tan\theta = \dfrac{4}{3}$ $\theta = \tan^{-1}\dfrac{4}{3}$ $\therefore V = 60$ and $\theta = \tan^{-1}\frac{4}{3}$	3 marks: Correct answer. 2 marks: Finds the value of V or θ. 1 mark: Sets up the two simultaneous equations or shows some understanding.
14(b) (iii)	To find the speed after 8 seconds $\theta = \tan^{-1}\frac{4}{3}$ then $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$ $\dot{x} = 60\times\dfrac{3}{5} = 36$ $\dot{y} = -10\times 8 + 60\times\dfrac{4}{5} = -32$ $v = \sqrt{36^2+32^2}$ $= \sqrt{2320}$ $= 48.1663\ldots.$ $\approx 48\ \text{ms}^{-2}$ $\tan\alpha = \dfrac{32}{36}$ $\alpha = 41.6335\ldots$ $\approx 42°$ 	3 marks: Correct answer. 2 marks: Finds the speed or angle to the horizontal. 1 mark: Finds the vertical and horizontal velocity just before impact.
14(c)	See separate solution attached.	4 marks

Q. 14(c)



(i) Between 5 metres & 17 metres we have twice the amplitude ie. $2a = 12 \therefore a = 6$.

$\therefore$ Centre of motion is 11.

$\frac{1}{2}T = (1315 - 0700)\text{hrs} = 6\cdot25 \quad \therefore T = 12\cdot5$ hrs.

Now, $T = \frac{2\pi}{n} = 12\cdot5 = \frac{25}{2} \quad \therefore n = \frac{4\pi}{25}$

$\therefore$ Equation is: $y = 11 - 6\cos\frac{4\pi}{25}t$.

(ii) At $y = 6\cdot5$, $6\cdot5 - 11 = -6\cos\frac{4\pi}{25}t$

$\frac{-4\cdot5}{-6} = \cos\frac{4\pi}{25}t$

ie. $0\cdot75 = \cos\frac{4\pi}{25}t$

$\frac{4\pi}{25}t = \cos^{-1} 0\cdot75$

$\frac{4\cancel{\pi}}{25}t = 41\cdot41^\circ \times \frac{\cancel{\pi}}{180}$

$4t = 25 \times \frac{41\cdot41}{180}$

$\Rightarrow t = 1\cdot438$ hrs. = 1hr 26min

Thus, the earliest the ship can enter is at $\underline{8:26 \text{ am}}$.

The latest the ship can leave will be 19:30 − 1 hr 26 mins

which is $\underline{18:04 \text{ hrs}}$.