CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Extension 1 Mathematics

Half Yearly Examination 2018

Time allowed: 2 hours

Student Number: ______________________

Instructions:

- Start a new booklet for each question
- Use black pen. Pencil may be used for graphs and diagrams.
- Board approved calculators may be used
- Show all necessary working
- Marks may be deducted for illegible or badly set out work

	Question 1	Question 2	Question 3	Question 4	Total
Geometrical Applications of Calculus					/18
Integral Calculus					/18
Logarithmic & Exponential Functions					/20
Series & Applications					/14
Total	/18	/18	/20	/14	/70

QUESTION 1 (18 marks) – START A NEW BOOKLET –

a). Show by a sketch the shape of the graph of $y = f(x)$ if $f'(x) < 0$ for $-3 < x < 3$ **[3]**

$f'(x) > 0$ for $x < -3$ or $x > 3$

$f(-3) = 8$

$f(0) = 1$

$f(3) = -8$

$x \to \infty,\ y \to 0$

$x \to -\infty,\ y \to 0.$

b). For a given function $y = x^4 - 8x^2 + 7$. Find

i). the y–intercept. **[1]**

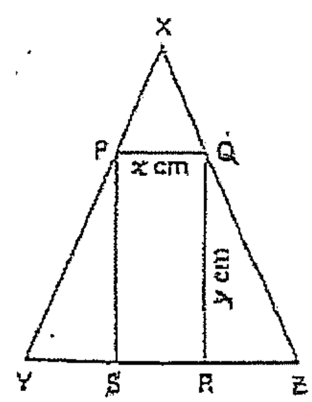
ii). the x–intercepts. **[2]**

iii). all the turning points and their nature. **[3]**

iv). the points of inflexion. **[2]**

v). On a half page, sketch the graph of the function, showing all the important information. **[2]**

c). In ΔXYZ, XY = XZ = 13 cm and YZ = 10 cm.
A rectangle PQRS is inscribed in the triangle with
PQ parallel to YZ. Given that PQ = x cm, QR = y cm.



i). Show that $y = 12 - \dfrac{6x}{5}$. **[2]**

ii). If the area of the rectangle is A cm^2, express A in terms of x. **[1]**

iii). Hence find the greatest area of the rectangle. **[2]**

QUESTION 2 (18 marks) – START A NEW BOOKLET –

a). i). $\int \left[5\left(\sqrt[3]{x^2}\right) + 2\sqrt{x} - \frac{3}{x^2} \right] dx.$ [2]

ii). $\int \frac{dx}{(2-5x)^3}.$ [2]

b). i). Evaluate $\int_1^2 \frac{dx}{x}$ in exact form. [2]

ii). Use Simpson's rule with 3 function values to approximate $\int_1^2 \frac{dx}{x}$ as a fraction. [2]

iii). Use your results to parts **i)** and **ii)** to obtain an approximation for e. Give your answer correct to 3 decimal places. [2]

c). A region is bounded by $y^2 = 8x$ and $y = x^2$.

i). Sketch the region and include the points of intersection. [2]

ii). Find the area of the region with respect to the x–axis. [3]

iii). Find the volume generated when the region bounded by the curves is rotated about the y–axis. [3]

QUESTION 3 (20 marks) – START A NEW BOOKLET –

a). Find $\int \frac{e^{2x}}{1-e^{2x}}dx$. **[2]**

b). Differentiate **i).** $3x^2\log_e x$ for $x > 0$. **[2]**

ii). $\frac{e^{2x}}{2x+1}$. **[2]**

c). If $\frac{1}{2}\log_e x + \log_e y = \log_e z$, express x in terms of y and z. **[2]**

d). Solve **i).** $\log_{10}(x-2)+\log_{10}(2x-3)=1$. **[3]**

ii). $e^{x+1}-e^{x-2}=24$, correct to 3 decimal places. **[2]**

e). If $f'(x)=(1+e^x)(1-e^x)$ and $f(0) = e$, find the value of $f(1)$. **[3]**

f). **i).** Show that $\frac{x+1}{(x-1)(x+5)}=\frac{1}{3(x-1)}+\frac{2}{3(x+5)}$. **[2]**

ii). Hence find $\int \frac{x+1}{(x-1)(x+5)}dx$. **[2]**

QUESTION 4 (14 marks) – START A NEW BOOKLET –

a). A new car, value \$35 000, is bought on a lease arrangement.
The interest is 13% per annum reducible, calculated fortnightly (assume 26 fortnights in a year). Repayments are made every fortnight. At the end of three years, there is still 40% of the original value of the car to be paid.

i). If the fortnightly repayments are \$M,
show that the amount owing after the second repayment is $\$35000(1.005)^2 - M(1.005) - M$. **[2]**

ii). Show that the amount owing at the end of three years is
$35000 \times 1.005^{78} - 200M(1.005^{78} - 1)$ dollars. **[3]**

iii). Hence find the fortnightly repayments correct to the nearest cent. **[3]**

b). Prove by induction that, for integers $n \geq 1$,

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \ldots + n(n+2) = \frac{n}{6}(n+1)(2n+7).$$ **[3]**

c). Prove by induction that $7^{2n-1} + 5$ is divisible by 12 for all integers $n \geq 1$. **[3]**

END OF EXAM

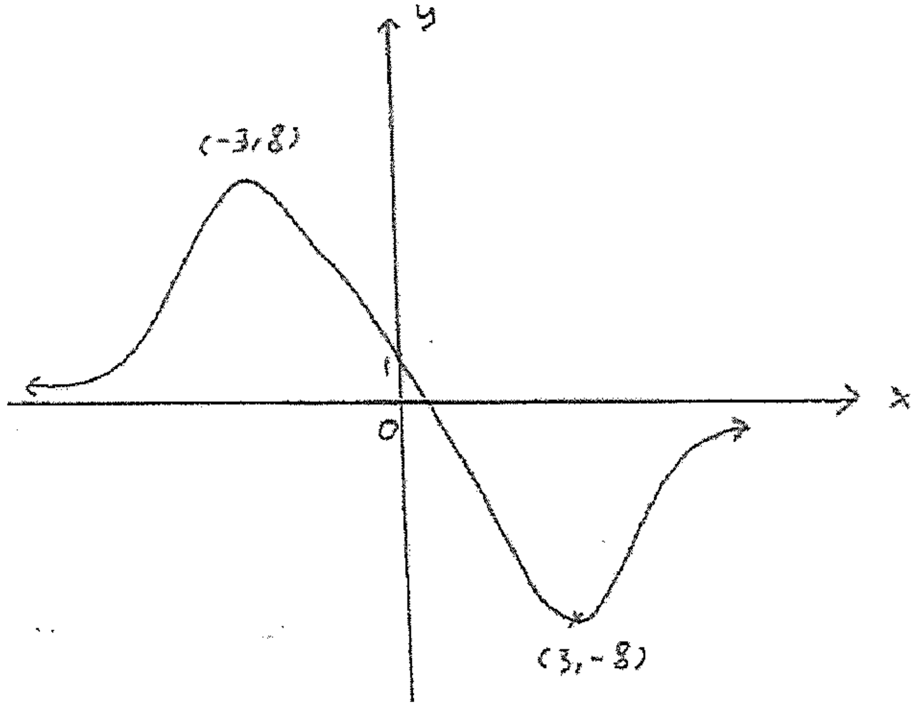
CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Extension 1 Mathematics

Half Yearly Examination 2018
Solutions

QUESTION 1 (18 marks)

a).	The graph of $y = f(x)$ should show these features. 	[3]
b).	For the given function $y = x^4 - 8x^2 + 7$. **i).** The y-intercept, when $x = 0$ i.e. $y = 0^4 - 8(0)^2 + 7$ $\therefore\ y = 7$	[1]
	ii). The x-intercepts, when $y = 0$ i.e. $x^4 - 8x^2 + 7 = 0$ $(x^2 - 7)(x^2 - 1) = 0$ $\therefore\ x = \pm\sqrt{7}$ or $x = \pm 1$	[2]
	iii). The turning points and their nature: $y' = 4x^3 - 16x$ and $y'' = 12x^2 - 16$ For stationary points, set $y' = 0$ i.e. $4x^3 - 16x = 0$ $4x(x^2 - 4) = 0$ $\therefore\ x = 0$ or $x = \pm 2$ and $y = 7$ or $y = -9$ Now test for nature: when $x = 0$ then $y'' = 12(0)^2 - 16$ $y'' < 0$, this implies maximum turning point So $(0, 7)$ is a maximum turning point. when $x = 2$ then $y'' = 12(2)^2 - 16$ $y'' > 0$, this implies minimum turning point So $(2, -9)$ is a minimum turning point. when $x = -2$ then $y'' = 12(-2)^2 - 16$ $y'' > 0$, this implies minimum turning point So $(-2, -9)$ is a minimum turning point.	[3]

iv). the points of inflexion, set $y'' = 0$ i.e. $12x^2 - 16 = 0$

$$\therefore \quad x = \pm\frac{2}{\sqrt{3}} \text{ and } y = -\frac{17}{9}$$

So $\left(\pm\frac{2}{\sqrt{3}}, -\frac{17}{9}\right)$ are possible inflexion points

[2]

Now test for concavity changes:

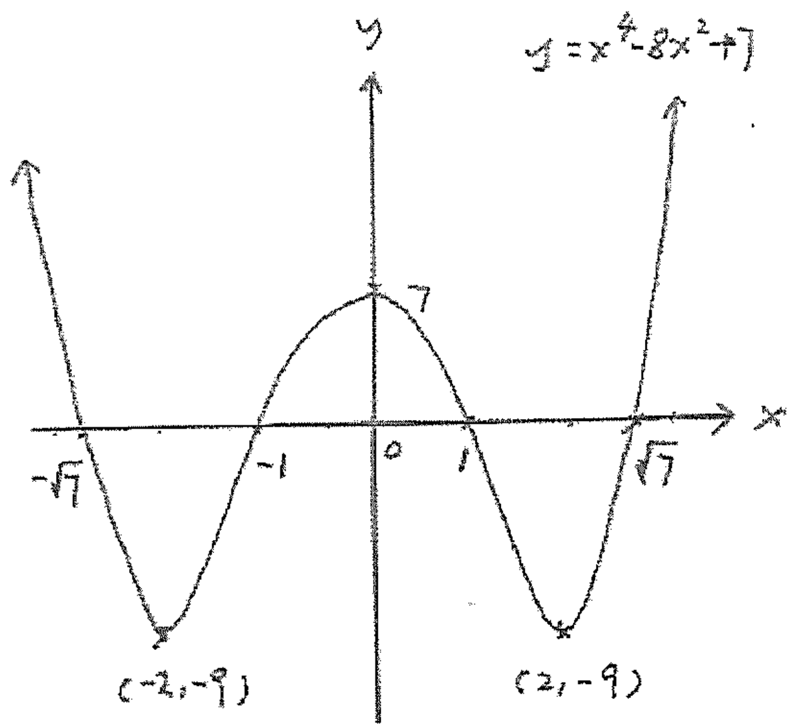
x	1.1	$\frac{2}{\sqrt{3}}$	1.2
y''	< 0	0	> 0

Since concavity changes then $\left(\frac{2}{\sqrt{3}}, -\frac{17}{9}\right)$ is a point of inflexion.

x	-1.2	$-\frac{2}{\sqrt{3}}$	-1.1
y''	> 0	0	< 0

Since concavity changes then $\left(-\frac{2}{\sqrt{3}}, -\frac{17}{9}\right)$ is a point of inflexion.

v). Now sketch the graph of the function.



[2]

c).

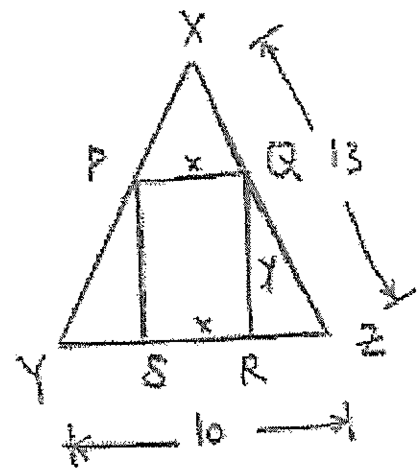
i). $\dfrac{y}{12}=\dfrac{10-x}{10}$ (Matching sides of similar Δs are in the same proportion.)

$10y = 12(10-x)$

$y=\dfrac{12}{10}(10-x)$

$\therefore\ y=12-\dfrac{6x}{5}$

[2]



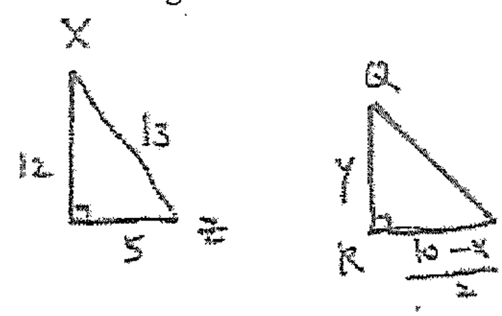
ii). $\because \mathrm{A}=x\cdot y$

$= x\cdot\left(12-\dfrac{6x}{5}\right)$

$= 12x-\dfrac{6}{5}x^2$

[1]

Similar triangles



iii). Now $\mathrm{A} = 12x-\dfrac{6}{5}x^2$ then

$\mathrm{A}' = 12-\dfrac{12}{5}x$ and $\mathrm{A}'' = -\dfrac{12}{5}$

For max / min put A' = 0

i.e. $12-\dfrac{12}{5}x=0$

$\therefore\ x=5$

$\because$ A'' < 0 this implies maximum

So $\mathrm{A}_{max} = 12(5)-\dfrac{6}{5}(5)^2$

$= 30$

Thus greatest area of the rectangle is 30 cm^2.

[2]

QUESTION 2 (18 marks)

a).

i) [2]

$$\int\left[5\left(\sqrt[3]{x^2}\right)+2\sqrt{x}-\frac{3}{x^2}\right]dx=\int\left(5x^{\frac{2}{3}}+2x^{\frac{1}{2}}-3x^{-2}\right)dx$$

$$=5\times\frac{3}{5}x^{\frac{5}{3}}+2\times\frac{2}{3}x^{\frac{3}{2}}-3\times\frac{1}{-1}x^{-1}+C$$

$$=3\left(\sqrt[3]{x^5}\right)+\frac{4}{3}\sqrt{x^3}+\frac{3}{x}+C$$

ii) [2]

$$\int\frac{dx}{(2-5x)^3}=\int(2-5x)^{-3}$$

$$=\frac{(2-5x)^{-2}}{(-5\times-2)}+C$$

$$=\frac{1}{10(2-5x)^2}+C$$

b).

i) [2]

$$\int_1^2\frac{dx}{x}=\left[\ln x\right]_1^2=\ln 2-\ln 1=\ln 2$$

ii) [2]

$$\int_1^2\frac{dx}{x}\approx\frac{2-1}{6}\left[\frac{1}{1}+4\times\frac{1}{1.5}+\frac{1}{2}\right]=\frac{25}{36}$$

iii). [2]

Now $\ln 2\approx\dfrac{25}{36}$

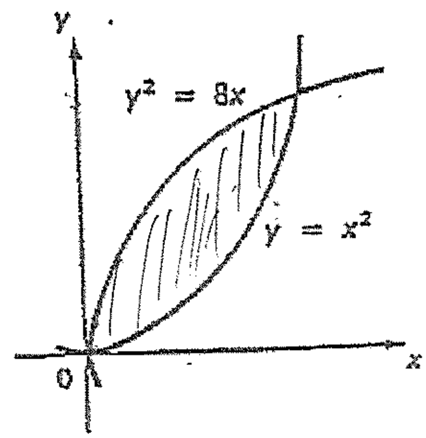
$$2\approx e^{\frac{25}{36}}$$

$$2^{\frac{36}{25}}\approx e$$

$$\therefore\quad e\approx 2.713$$

c). **i).** Let $y^2 = 8x$ ……….. [1]

The sketch of the region



[2]

and $y = x^2$ ……….. [2]

substitute [2] into [1] get $(x^2)^2 = 8x$

$$x^4 = 8x$$
$$x^4 - 8x = 0$$
$$x(x^3 - 8) = 0$$
$$\therefore\ x = 0 \text{ or } x = 2$$

ii). Now $A = \int_0^2 \left(\sqrt{8x} - x^2\right) dx$

$$= \int_0^2 \left(\sqrt{8}x^{\frac{1}{2}} - x^2\right) dx$$
$$= \left[\sqrt{8} \times \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^2$$
$$= \frac{2\sqrt{8}}{3} \times 2^{\frac{3}{2}} - \frac{1}{3} \times 2^3 - 0$$
$$= \frac{2^4}{3} - \frac{2^3}{3}$$
$$= \frac{8}{3} \text{ units}^2$$

[3]

iii). Now $V = \pi \int_0^4 \left(y - \frac{y^4}{64}\right) dy$

$$= \pi \left[\frac{1}{2}y^2 - \frac{y^5}{320}\right]_0^4$$
$$= \pi \left[\frac{1}{2} \times 4^2 - \frac{4^5}{320} - 0\right]$$
$$= \frac{24}{5}\pi \text{ units}^3$$

[3]

QUESTION 3 (20 marks)

a).	$\int \frac{e^{2x}}{1-e^{2x}}dx = -\frac{1}{2}\int \frac{-2e^{2x}}{1-e^{2x}}dx$ $= -\frac{1}{2}\ln\left(1-e^{2x}\right)+C$	[2]
b).	i). $\frac{d}{dx}\left(3x^2\log_e x\right) = 3x^2 \times \frac{1}{x} + \log_e x\left(3\times 2x\right)$ $= 3x + 6x\log_e x$	[2]
	ii). $\frac{d}{dx}\left(\frac{e^{2x}}{2x+1}\right) = \frac{\left(2x+1\right)\times 2e^{2x} - e^{2x}\left(2\right)}{\left(2x+1\right)^2}$ $= \frac{2e^{2x}\left(2x+1-1\right)}{\left(2x+1\right)^2}$ $= \frac{4xe^{2x}}{\left(2x+1\right)^2}$	[2]
c).	$\frac{1}{2}\log_e x + \log_e y = \log_e z$ $\log_e x^{\frac{1}{2}} + \log_e y = \log_e z$ $\log_e\left(x^{\frac{1}{2}}y\right) = \log_e z$ $x^{\frac{1}{2}}y = z$ $x^{\frac{1}{2}} = \frac{z}{y}$ $\therefore\ x = \left(\frac{z}{y}\right)^2$	[2]

d).	**i).** $\log_{10}(x-2)+\log_{10}(2x-3)=1$. $\log_{10}\left[(x-2)(2x-3)\right]=\log_{10}10$ $(x-2)(2x-3)=10$ $2x^2-7x-4=0$ $(2x+1)(x-4)=0$ $\therefore\ x=-\frac{1}{2}$ (rejected) or $x=4$ **ii).** $e^{x+1}-e^{x-2}=24$ $e\times e^x-\frac{1}{e^2}\times e^x=24$ $\left(e-\frac{1}{e^2}\right)e^x=24$ $e^x=24\div\left(e-\frac{1}{e^2}\right)$ $=24\div\left(\frac{e^3-1}{e^2}\right)$ $e^x=\frac{24e^2}{e^3-1}$ $x=\log_e\left(\frac{24e^2}{e^3-1}\right)$ $\therefore\ x\approx 2.229$	[3] [2]
e).	Now $f'(x)=(1+e^x)(1-e^x)$ $f(x)=\int(1+e^x)(1-e^x)dx$ $=\int(1-e^{2x})dx$ $=x-\frac{1}{2}e^{2x}+C$ When $x=0$, $y=e$ then $e=0-\frac{1}{2}e^{2(0)}+C$ So $C=e+\frac{1}{2}$ Thus $f(x)=x-\frac{1}{2}e^{2x}+e+\frac{1}{2}$ $\therefore\ f(1)=1-\frac{1}{2}e^2+e+\frac{1}{2}$ $=\frac{3}{2}-\frac{1}{2}e^2+e$	[3]

f).	i). Now $\text{RHS} = \dfrac{1}{3(x-1)} + \dfrac{2}{3(x+5)}$ $= \dfrac{x+5+2(x-1)}{3(x-1)(x+5)}$ $= \dfrac{x+5+2x-2}{3(x-1)(x+5)}$ $= \dfrac{3x+3}{3(x-1)(x+5)}$ $= \dfrac{x+1}{(x-1)(x+5)}$ Thus LHS = RHS	[2]
	ii). Now $\displaystyle\int \frac{x+1}{(x-1)(x+5)}\,dx = \int \left(\frac{1}{3(x-1)} + \frac{2}{3(x+5)} \right) dx$ $= \dfrac{1}{3}\log_e\lvert x-1\rvert + \dfrac{2}{3}\log_e\lvert x+5\rvert + C$	[2]

QUESTION 4 (14 marks)

a).	Let the amount owed after its repayment be A_i i). Now $A_1 = \$35000(1 + 0.5\%) - M$ $= \$35000(1.005) - M$ So $A_2 = A_1 \times (1.005) - M$ $= [\$35000(1.005) - M] \times (1.005) - M$ $= \$35000(1.005)^2 - M(1.005) - M$	[2]
	ii). Now $A_3 = A_2 \times (1.005) - M$ $= [\$35000(1.005)^2 - M(1.005) - M] \times (1.005) - M$ $= \$35000(1.005)^3 - M(1.005)^2 - M(1.005) - M$ . . . $A_{78} = \$35000(1.005)^{78} - M(1.005)^{77} - \ldots - M(1.005) - M$ $= \$35000(1.005)^{78} - M\left[\dfrac{1.005^{78} - 1}{0.005}\right]$ $= 35000 \times 1.005^{78} - 200M(1.005^{78} - 1)$	[3]

	iii). Now $A_{78} = \$35000 \times 40\%$ i.e. $35000 \times 1.005^{78} - 200M(1.005^{78} - 1) = \$35000 \times 0.4$ $$M = \frac{\$35000(1.005)^{78} - \$35000 \times 0.4}{200(1.005^{78} - 1)}$$ $= \$395.80$	**[3]**
b).	Let $T_n = n(n+2)$ and $S_n = \frac{n}{6}(n+1)(2n+7)$ <u>Proof</u> <u>Step 1</u> If $n = 1$ then LHS $= T_1$ $= 1(1+2)$ $= 3$ RHS $= S_1$ $= \frac{1}{6}(1+1)(2 \times 1 + 7)$ $= \frac{1}{6}(2)(9)$ $= 3$ So LHS = RHS $\therefore$ The statement is true for $n = 1$. <u>Step 2</u> Assume statement is true for $n = k$, i.e. $S_k = \frac{k}{6}(k+1)(2k+7)$ Now prove the statement is true for $n = k + 1$, i.e. $S_{k+1} = \frac{k+1}{6}(k+1+1)\left[2(k+1)+7\right]$ $= \frac{k+1}{6}(k+2)(2k+9)$ Now $S_{k+1} = S_k + T_{k+1}$ $= \frac{k}{6}(k+1)(2k+7) + (k+1)(k+3)$ $= \frac{(k+1)}{6}\left[k(2k+7) + 6(k+3)\right]$ $= \frac{(k+1)}{6}\left[2k^2 + 13k + 18\right]$ $= \frac{(k+1)}{6}(k+2)(2k+9)$ $\therefore$ The statement is true for $n = k + 1$ if it is true for $n = k$. <u>Step 3</u> $\therefore$ If the statement is true for $n = k$, it is also true for $n = k + 1$. But it is true for $n = 1$. Thus it is true for all integers $n \geq 1$.	**[3]**

c). Proof

Let $S(n)$ be the statement that $7^{2n-1}+5$ is divisible by 12.

Step 1

For $n=1$ then $7^{2\times 1-1}+5=12$, which is divisible by 12.

$\therefore$ $S(1)$ is true.

Step 2

Assume $S(k)$ is true,

i.e. $7^{2k-1}+5=12M$ where M is an integer.

So $7^{2k-1}=12M-5$ ....[1] **[3]**

Now required to prove $S(k+1)$ is true,

i.e. $7^{2(k+1)-1}+5=7^{2k+1}+5$ is divisible by 12.

$$\begin{aligned}&=7^2\times 7^{2k-1}+5 \quad \text{(sub [1] get)}\\&=49(12M-5)+5\\&=12\times 49M-49\times 5+5\\&=12(49M-20), \text{ which is divisible by 12 as } M \text{ is an integer.}\end{aligned}$$

$\therefore$ $S(k+1)$ is true when $S(k)$ is true.

Step 3

Since the result is true when $n=1$, hence it is true when $n=2$, and so by mathematical induction the result is true for all $n\geq 1$.

END OF SOLUTIONS