ACE
EXAM
PAPER

Student name: ____________________

PAPER 1

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 1

General Instructions

- Working time - 120 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt questions 11-14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. A coin is biased such that the probability of a head is 0.7. The probability that exactly three heads will be observed when the coin is tossed five times is:

   (A) $0.7^3 0.34^2$

   (B) $10 \times 0.7^3 0.3^2$

   (C) ${}^5C_3\ 0.7^3$

   (D) ${}^5C_3\ 0.7^5$

2. Which of the following is an expression for $\int \sin^2 2x dx$?

   (A) $\frac{x}{2} - \frac{1}{8}\sin 4x + C$

   (B) $\frac{x}{2} + \frac{1}{8}\sin 4x + C$

   (C) $x - \frac{1}{4}\sin 4x + C$

   (D) $x + \frac{1}{4}\sin 4x + C$

3. A stone is thrown at an angle of θ to the horizontal. The position of the stone at time t seconds is given by $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ where g m/s² is the acceleration due to gravity and v m/s is the initial velocity of projection.
   What is the maximum height reached by the stone?

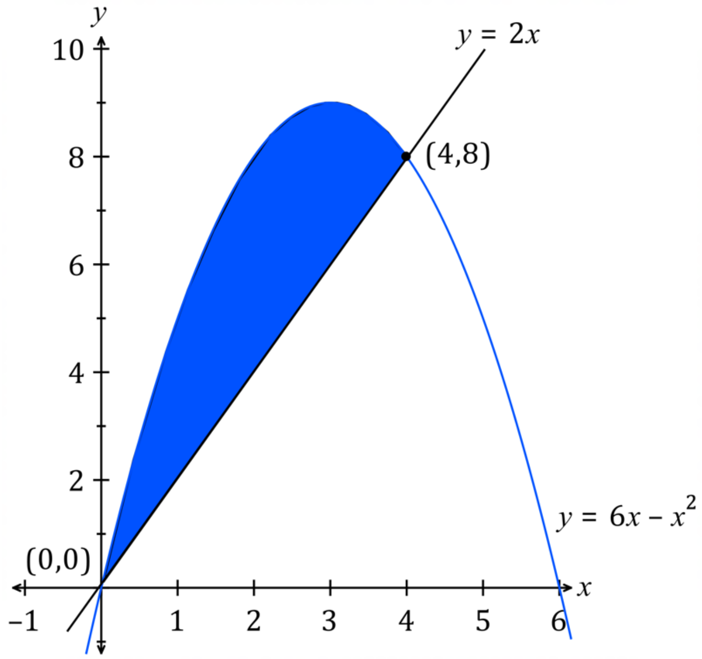
   (A) $\dfrac{V\sin\theta}{g}$

   (B) $\dfrac{g\sin\theta}{g}$

   (C) $\dfrac{V^2\sin^2\theta}{2g}$

   (D) $\dfrac{g\sin^2\theta}{2V^2}$

4. A parabola $y = 6x - x^2$ meets the line $y = 2x$ at $(0,0)$ and $(4,8)$.



Which expression gives the area of the shaded region bounded by the parabola and the line?

(A) $\displaystyle\int_0^4 x^2 - 4x\,d$

(B) $\displaystyle\int_0^4 4x - x^2\,dx$

(C) $\displaystyle\int_0^8 x^2 - 4x\,dx$

(D) $\displaystyle\int_0^8 4x - x^2\,dx$

5. The number, N, of koalas in a population at time t years is given by $N = 135 + Ae^{kt}$ for constants $A > 0$ and $k > 0$. Which of the following is the correct differential equation?

(A) $\dfrac{dN}{dt} = -k(N + 135)$

(B) $\dfrac{dN}{dt} = -k(N - 135)$

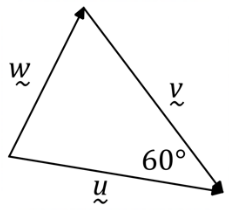
(C) $\dfrac{dN}{dt} = k(N + 135)$

(D) $\dfrac{dN}{dt} = k(N - 135)$

6. What is the value of $\int_e^{e^2} \frac{1}{x\ln x}\,dx$? Use the substitution $u = \ln x$.

   (A) $\ln 0.5$
   (B) $\ln 2$
   (C) $\ln 4$
   (D) 1

7. Vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are shown below.



   Which of the following statements is true?

   (A) $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2$
   (B) $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - |\underset{\sim}{u}\underset{\sim}{v}|$
   (C) $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 + |\underset{\sim}{u}\underset{\sim}{v}|$
   (D) $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - |\underset{\sim}{u}||\underset{\sim}{v}|$

8. Mathematical induction is used to prove for $n \geq 2$,

   $$2 \times 1 + 3 \times 2 + \ldots + n(n-1) = \frac{1}{3}n(n^2 - 1)$$

   Which of the following has an *incorrect* expression for part of the induction proof?

   (A) Step 1: To prove the statement true for $n = 1$
   LHS $= 2 \times 1 = 2$
   RHS $= \frac{1}{3} \times 1 \times (1^2 - 1) = 2$
   Result true for $n = 1$

   (B) Step 2: Assume the result true for $n = k$
   $2 \times 1 + 3 \times 2 + \ldots + k(k-1) = \frac{1}{3}k(k^2 - 1)$

   (C) To prove the result true for $n = k + 1$
   $2 \times 1 + \ldots + k(k-1) + (k+1)k = \frac{1}{3}(k+1)((k+1)^2 - 1)$

   (D) LHS $= 2 \times 1 + 3 \times 2 + \cdots + k(k-1) + (k+1)k$
   $= \frac{1}{3}k(k^2 - 1) + (k+1)k$
   $= \frac{1}{3}[k(k+1)(k-1) + 3(k+1)k]$
   $= \frac{1}{3}k(k+1)[(k-1) + 3]$
   $= \frac{1}{3}(k+1)[k^2 + 2k]$
   $= \frac{1}{3}(k+1)((k+1)^2 - 1)$
   $=$ RHS

9. What is $\sin x - \cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$?

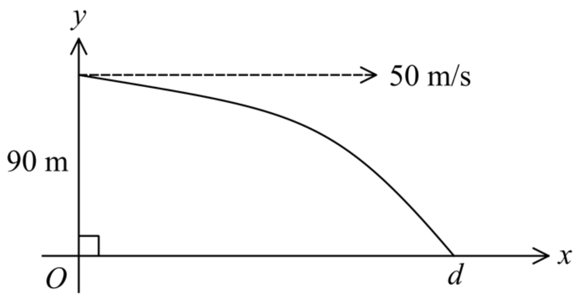
(A) $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

(B) $\sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$

(C) $\sqrt{2}\sin\left(x + \frac{5\pi}{4}\right)$

(D) $\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$

10. The diagram below shows the trajectory of a ball thrown horizontally, at a speed of 50 ms^{-1}, from the top of a tower 90 metres above ground level.



After what time does the ball strike the ground?

(A) $3\sqrt{\frac{5}{g}}$ seconds

(B) $3\sqrt{\frac{50}{g}}$ seconds

(C) $6\sqrt{\frac{5}{g}}$ seconds

(D) $6\sqrt{\frac{50}{g}}$ seconds

Section II

60 marks
Attempt questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) If $P = \binom{3}{4}$ and $Q = \binom{8}{0}$, what is vector $\overrightarrow{\mathrm{PQ}}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$? **2**

(b) (i) Show that $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sin x + \cos x}{\sqrt{2}}$ **2**

(ii) Hence or otherwise, solve $\frac{\sin x + \cos x}{\sqrt{2}} = \frac{\sqrt{3}}{2}$ for $0 \leq x \leq 2\,\pi$. **2**

(c) A box contains seven identical balls except for their colour. Four are blue, two are white and one is red. Two balls are selected at random. After replacing the two balls the selection is repeated several times.

(i) What is the probability of getting two blue balls on at least one occasion from five selections of two balls? Answer correct to three decimal places. **2**

(ii) What is the probability of getting two blue balls on exactly three occasions from five selections of two balls? Answer correct to three decimal places. **2**

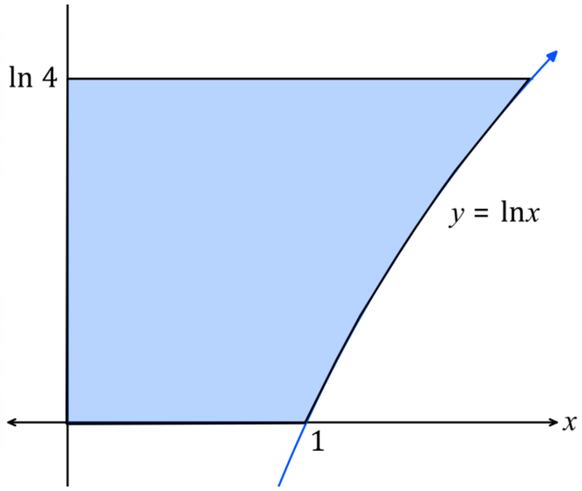
(d) The function $f(x)$ is given by $f(x) = 4\tan^{-1}x$. Find the slope of the tangent to the curve where the function $y = f(x)$ cuts the y-axis. **2**

(e) (i) Show that $\frac{1 + \cos 2x}{\sin 2x} = \cot x$ **2**

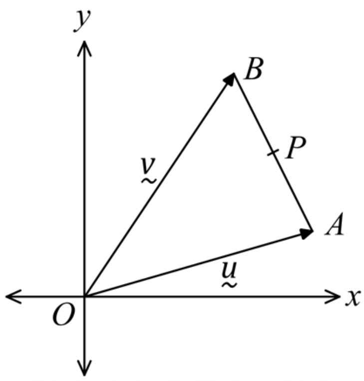
(ii) Hence find the exact value of $\cot 15°$. **1**

Question 12 (14 marks) **Marks**

(a) In the diagram, the shaded region bounded by the curve $y = \ln x$, x-axis, y-axis and the line $y = \ln 4$, is rotated about the y-axis. Find the exact volume of the solid of revolution. **2**



(b) In ΔOAB, $\overrightarrow{OA} = \underset{\sim}{u}$ and $\overrightarrow{OB} = \underset{\sim}{v}$. Point P is the midpoint of $\overrightarrow{AB}$.



Find the following vectors in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

(i) $\overrightarrow{AB}$ **1**

(ii) $\overrightarrow{OP}$ **2**

(iii) $\overrightarrow{AP}$ **1**

(iv) $\overrightarrow{BP}$ **1**

(c) Find $\displaystyle\int \frac{dx}{\sqrt{36 - x^2}}$ **2**

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 \frac{1}{2} x dx$ **2**

(e) Prove by mathematical induction that $3^n - 2n - 1$ is divisible by 4 for all positive integers greater than 1. **3**

Question 13 (15 marks) **Marks**

(a) A bottle of water has a temperature of 20˚C and is placed in a refrigerator whose temperature is 2˚C. The cooling rate of the bottle of water is proportional to the difference between the temperature of the refrigerator and the temperature T of the bottle of water. This is expressed by the equation:

$$\frac{dT}{dt} = -k(T - 2)$$

where k is a constant of proportionality and t is the number of minutes after the bottle of water is placed in the refrigerator.

(i) Show that $T = 2 + Ae^{-kt}$ satisfies the above equation. **1**

(ii) After 20 minutes in the refrigerator the temperature of the bottle of water is 10˚C. What is the value of A and k in the above equation? **3**

(iii) How long will it take for the bottle of water to cool down to 5˚C? **2**

(b) Use the substitution $u = x + 1$ to evaluate $\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$. **3**

(c) A binomial distribution is $X \sim \text{Bin}(40, p)$. If $E(X) = 5$ find:

(i) p **2**

(ii) $\text{Var}(X)$ **2**

(d) What is the unit vector in the direction $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$? **2**

Question 14 (16 marks) **Marks**

(a) A ball is thrown from the origin O with a velocity V and angle of elevation of θ, where $\theta \neq \frac{\pi}{2}$. You may assume that:

$$x = Vt\cos\theta, \qquad y = Vt\sin\theta - \frac{1}{2}gt^2$$

where x and y are the horizontal and vertical displacements of the ball in metres from O at time t seconds after being thrown.

(i) Let $h = \dfrac{V^2}{2g}$ and show that the equation of flight of the ball is: **3**

$$y = x\tan\theta - \frac{1}{4h}x^2(1 + \tan^2\theta)$$

(ii) The point of intersection when two balls are thrown with an angle of elevation of θ_1 and θ_2 is (a, b). Show that: **3**

$$a^2 > 4h(h - b)$$

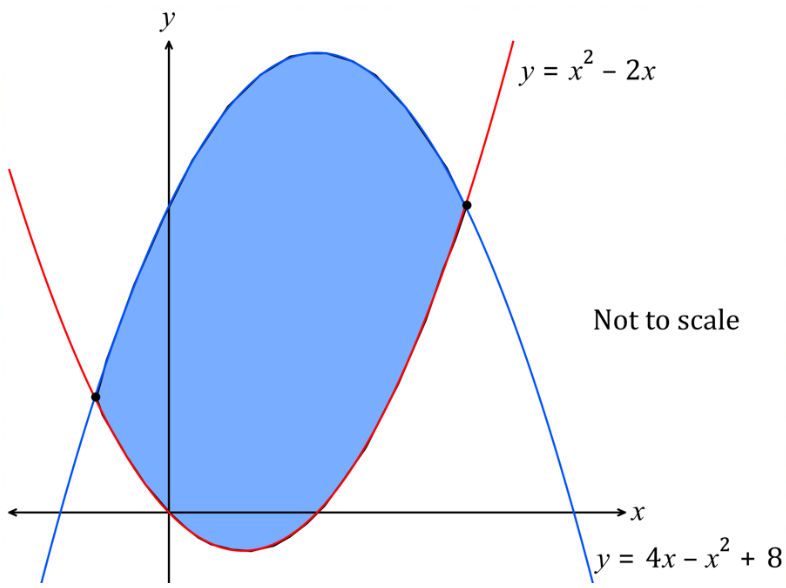
(b) Use mathematical induction to prove that: **3**

$$\frac{2}{1\times 2} + \frac{2}{2\times 3} + \frac{2}{3\times 4} + \ldots + \frac{2}{n\times(n+1)} = \frac{2n}{n+1}$$ for all positive integers n.

(c) Find $\displaystyle\int \frac{e^{3x}}{1+e^x}\,dx$ using the substitution $u = 1 + e^x$. **2**

(d) (i) Where do the curves $y = 4x - x^2 + 8$ and $y = x^2 - 2x$ intersect? **1**

(ii) Calculate the area between the two curves. **2**



(e) Solve the differential equation below using the method of separation of variables. **2**

$$\frac{dy}{dx} = e^{6x}(1 + y^2)$$

End of paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

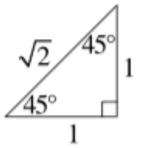
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

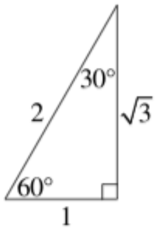


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

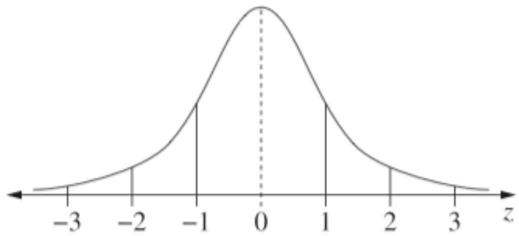
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$