ACE
EXAM
PAPER

Student name: ________________

PAPER 1

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 2

General Instructions

- Working time - 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. What is $z = -\sqrt{2} + \sqrt{2}i$ in modulus-argument form?

   (A) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

   (B) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

   (C) $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

   (D) $2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

2. Which expression is equal to $\int 3\sqrt{x}\ln x\, dx$?

   (A) $2x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + C$

   (B) $2x\sqrt{x}\left(\ln x + \frac{2}{3}\right) + C$

   (C) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x - 1\right) + C$

   (D) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x + 1\right) + C$

3. What is the square of the magnitude of the vector $\underset{\sim}{u} = 5\underset{\sim}{i} - \underset{\sim}{j} + \sqrt{10}\underset{\sim}{k}$?

   (A) 5.8
   (B) 6
   (C) 34
   (D) 36

4. What is the value of $\dfrac{3}{iw}$ if $w = -1 + i$?

(A) $-3 + 3i$

(B) $3 + 3i$

(C) $\dfrac{-3 + 3i}{2}$

(D) $\dfrac{3 + 3i}{2}$

5. Let $I_n = \displaystyle\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

Which of the following is the correct expression for $\displaystyle\int \tan^6 x dx$?

(A) $\dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x - x + C$

(B) $\dfrac{\tan^6 x}{6} - \dfrac{\tan^4 x}{4} + \dfrac{\tan^2 x}{2} + C$

(C) $\dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x + C$

(D) $\dfrac{\tan^4 x}{4} - \dfrac{\tan^2 x}{2} + x + C$

6. A particle is moving in a straight line with $v^2 = 36 - 4x^2$ and undergoing simple harmonic motion. If the particle is initially at the origin, which of the following is the correct equation for its displacement in terms of t?

(A) $x = 2\sin(3t)$

(B) $x = 3\sin(2t)$

(C) $x = 2\sin(9t)$

(D) $x = 3\sin(4t)$

7. What is the exact value of $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x + 1} dx$? Use the subsitution $\mathrm{t} = \tan\frac{\mathrm{x}}{2}$.

(A) $2 - \sqrt{3}$

(B) $4 - 2\sqrt{3}$

(C) $2 + \sqrt{3}$

(D) $4 + 2\sqrt{3}$

8. What is the solution to the equation $(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}).(3\underset{\sim}{i} - x\underset{\sim}{j} + 2\underset{\sim}{k}) = 4$?

   (A) $x = -3$

   (B) $x = -2$

   (C) $x = -1$

   (D) $x = 1$

9. It is given that a, b are real and c, d are imaginary.
   Which pair of inequalities must always be true?

   (A) $a^2c^2 + b^2d^2 \leq 2abcd,\qquad a^2b^2 + c^2d^2 \leq 2abcd$

   (B) $a^2c^2 + b^2d^2 \leq 2abcd,\qquad a^2b^2 + c^2d^2 \geq 2abcd$

   (C) $a^2c^2 + b^2d^2 \geq 2abcd,\qquad a^2b^2 + c^2d^2 \leq 2abcd$

   (D) $a^2c^2 + b^2d^2 \geq 2abcd,\qquad a^2b^2 + c^2d^2 \geq 2abcd$

10. A particle moves in a straight line with a displacement of x and velocity of v. When $t = 0$ the acceleration is $3x^2$, velocity is $-\sqrt{2}$ and displacement is 1. Which of the following is the correct equation for x as a function of t?

    (A) $x = \dfrac{-2}{(t+\sqrt{2})^2}$

    (B) $x = \dfrac{-2}{(t-\sqrt{2})^2}$

    (C) $x = \dfrac{2}{(t+\sqrt{2})^2}$

    (D) $x = \dfrac{2}{(t-\sqrt{2})^2}$

Section II

90 marks
Attempt questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) **Marks**

(a) ΔOAB is isosceles with $\overrightarrow{OA} = \left(3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}\right)$, $\overrightarrow{OB} = \alpha\underset{\sim}{i}$ $(\alpha > 0)$ and $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right|$.

(i) Find the value of α. **1**

(ii) Find $\overrightarrow{OC}$, where C is the midpoint of the line segment AB. **2**

(iii) Show that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$. **3**

(b) Given that $|z| = 1$, show that $z^{-1} = \bar{z}$ **2**

(c) If $z_1 = 4 + i$ and $z_2 = 1 + 2i$ show geometrically how to construct the vectors representing :

(i) $z_1 + z_2$ **1**

(ii) $z_1 - z_2$ **1**

(d) Find $\displaystyle\int \frac{1}{\sqrt{12 + 4x - x^2}}\,dx$ **2**

(e) Find $\displaystyle\int \frac{x^2}{x^2 + 1}\,dx$ **2**

Question 12 (15 marks) **Marks**

(a) Express $\cos 5\theta$ as a polynomial in $\cos\theta$ by expanding $(\cos\theta + i\sin\theta)^5$ and applying De Moivre's theorem. **3**

(b) (i) Find real numbers A, B and C such that **2**

$$\frac{8-2x}{(1+x)(4+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{4+x^2}$$

(ii) Find real numbers A, B and C such that **2**

$$\frac{8-2x}{(1+x)(4+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{4+x^2}$$

(c) A particle moving in a straight line obeys $v^2 = -x^2 + 2x + 8$ where x is its displacement from the origin in metres and v is its velocity in ms^{-1}. Initially, the particle is 2.5 metres to the right of the origin.

(i) Prove that the motion is simple harmonic. **2**

(ii) Find the centre of motion, the period and the amplitude. **3**

(iii) The displacement of the particle at any t is given by the equation $x = a\cos(nt + \alpha) + b$. What is the value of b and α ? **1**

(d) Sketch the graph of $\arg\left(\frac{z-2}{z+2i}\right) = \frac{\pi}{2}$ **2**

Question 13 (14 marks) **Marks**

(a) Solve the equation $x^4 + x^2 + 6x + 4 = 0$ over the complex field given that it has a rational zero of multiplicity 2. **4**

(b) A body is moving in a straight line. Its velocity v ms^{-1} is given by $v = x$ when it is x metres from the origin at time t seconds. Find an expression for x in terms of t given $x = 1$ when $t = 3$. **2**

(c) Show that if $x \neq 1$ then $1 + x + x^2 + \ldots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$ for $n \geq 1$. **3**

(d) The straight line l_1 has the vector equation:

$\underset{\sim}{u} = 3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k})$

The straight line l_2 has the vector equation:

$\underset{\sim}{u} = 4\underset{\sim}{j} - 2\underset{\sim}{k} + \mu(\underset{\sim}{i} - \underset{\sim}{j})$

where λ and μ are parameters.

The lines l_1 and l_2 intersect at A and the acute angle between l_1 and l_2 is θ.

(i) What are the coordinates of A? **2**

(ii) Find the value of $\cos\theta$ giving answer as a simplified fraction. **3**

Question 14 (16 marks) **Marks**

(a) Find all real x such that $|4x - 1| > 2\sqrt{x}$ **3**

(b) Solve the equation $z^2 = i\bar{z}$ **3**

(c) Prove the following results related to the binomial theorem.

(i) $\dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r-1)!\,(n-r+1)!} = \dfrac{(n+1)!}{r!\,(n-r+1)!}$ **3**

(ii) ${}^nC_r = {}^nC_{n-k}$ **3**

(d) (i) Let $I_n = \displaystyle\int_1^e (\ln x)^n dx$ **2**

Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

(ii) Hence or otherwise, find the exact value of I_3. **2**

Question 15 (15 marks) **Marks**

(a) It is given that $(2+i)$ is a root of $P(z) = z^3 + az^2 + bz + 20$ where a and b are real numbers.

(i) State why $(2-i)$ is also a root of $P(z)$. **1**

(ii) Factorise $P(z)$ over the real numbers. **2**

(b) A particle is moving along the x axis. It starts from rest at the point $x = 1$. The acceleration of the particle is given by:

$$\ddot{x} = \frac{5}{x^3} - \frac{2}{x^2}$$

(i) Show that the particle starts moving in the positive x direction. **1**

(ii) Find the velocity v of the particle. **3**

(iii) Describe the behaviour of the velocity of the particle for $x > 2.5$ **2**

(c) The point A has position vector $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and point B has position vector $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$, relative to an origin O.

(i) Find position vector of point C, with position vector $\underset{\sim}{c}$ given by $\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b}$. **1**

(ii) Show that $OACB$ is a rectangle. **4**

(iii) Find the exact area of $OACB$. **1**

Question 16 (16 marks) **Marks**

(a) A particle of mass 40 kg experiences a force of 0.1 of the square of its velocity in metres per second when moving through the air. The particle is projected vertically upwards with an initial velocity of u metres per second. Assume g = 10 ms^{-2}.

(i) Find the time taken for the particle to reach its maximum height. **3**

(ii) Find the maximum height reached by the particle. **3**

(b) (i) Show that $(1-3i)^2 = -8-6i$ **1**

(ii) Hence solve the equation $2z^2 - 8z + (12+3i) = 0$ **2**

(c) Show that $1 + x + \dfrac{x^2e^x}{2} > e^x$ **3**

(d) Find $\displaystyle\int \frac{\sqrt{x^2-1}}{x^2}\,dx$ **4**

End of paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

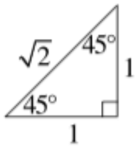
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

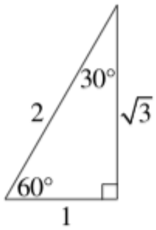


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

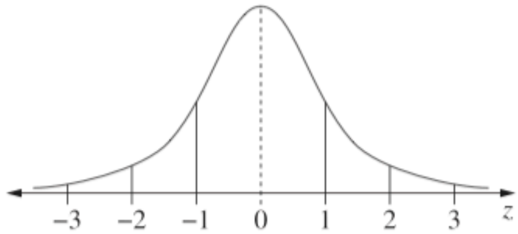
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \quad P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$