Carlingford High School

Mathematics Extension 1

Year 12 Half Yearly Exam

2017

Time allowed: 2 hours

Student Number: ______________________________

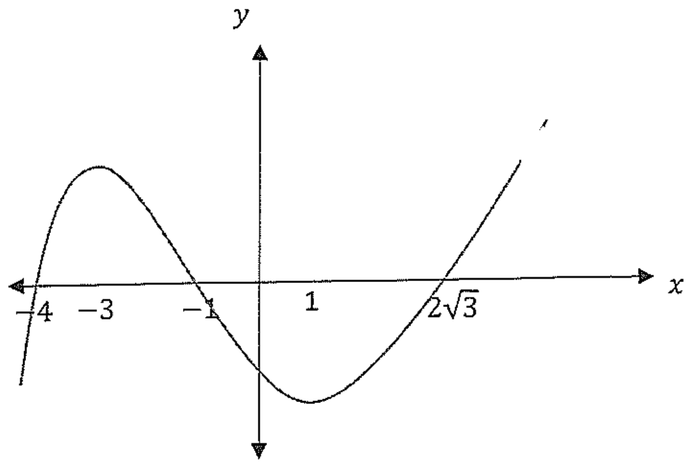
Instructions:

- Start a new booklet for each question
- Use black pen. Pencil may be used for graphs and diagrams.
- Board approved calculators may be used
- Show all necessary working
- Marks may be deducted for illegible or badly set out work

	Question 1	Question 2	Question 3	Question 4	Total
Geometrical Applications of Calculus	/20				/20
Integral Calculus		/17			/17
Logarithmic and Exponential Functions			/15		/15
Series and Applications				/16	/16
					/68

Question 1

(a)



This function has x-intercepts at -4, -1 and $2\sqrt{3}$, stationary points at $x = -3$ and $x = 1$, and a point of inflexion at $x = -1$. State the values of x for which:

(i) $f(x) > 0$ [1]

(ii) $f'(x) > 0$ [1]

(iii) $f''(x) > 0$ [1]

(b) For a given function $y = f(x)$, $f'(x) = \frac{(x+1)^2}{x-1}$, $x \neq 1$

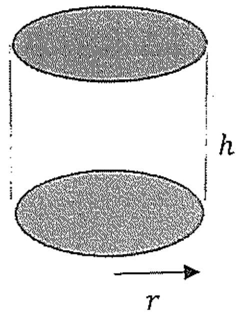
(i) Determine the nature of any stationary points. [3]

(ii) For what values of x is the function $y = f(x)$ increasing? [3]

(iii) Where are any points of inflexion located? [2]

(iv) Sketch the graph of $y = f(x)$ showing all relevant information. [2]

(c)



A closed cylindrical can is to have a volume of $32\pi cm^3$.

(i) Show that the surface area of the cylinder is given by $A = 2\pi r^2 + \frac{64\pi}{r}\ cm^2$ [2]

(ii) Find the radius of the can that can be made using the least amount of material. (Write your answer correct to the nearest mm) [3]

(iii) If the material used to make the can costs 3.2 cents/cm^2, calculate the minimum cost per can. [2]

Question 2 **(Please start a new booklet)**

(a) (i) $\int \left(\sqrt{x} + \frac{2}{x}\right)^2 dx$ [2]

(ii) $\int \frac{(x-2)(x^2+2x+4)}{8-x^3}\ dx$ [2]

(b) (i) Given $f(x) = \frac{x}{x^2+1}$, find $f'(x)$. (Simplify fully) [2]

(ii) Hence evaluate $\int_0^2 \frac{1-x^2}{(x^2+1)^2}\ dx$. (Write your answer correct to 2 decimal places) [2]

(iii) Evaluate $\int_0^2 \frac{1-x^2}{(x^2+1)^2}\ dx$, correct to 3 significant figures, using Simpson's Rule with 5 function values. [3]

(c) A region is bounded by $y^2 = x^3$, the y-axis and $y = 3\sqrt{3}$.

(i) Sketch $y^2 = x^3$. [2]

(ii) Find the area of the region, correct to 1 decimal place. [2]

(iii) Find the volume of the solid of revolution formed by rotating the region about the y-axis. [2]
(Write your answer correct to the nearest cubic unit)

<u>Question 3</u> (Please start a new booklet)

(a) Given $f'(x) = \frac{1}{2}xe^{2x^2}$, write an expression for $f(x)$, given that $f(1) = 0$. [2]

(b) Simplify $\log_p q \times \log_q p$ [2]

(c) Given $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, show that $\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}$ [2]

(d) Differentiate $\log_e(e^{2x})$ [2]

(e) The speed of signalling in a submerged telephone cable is $s = x^2 \log_e(\frac{1}{x})$, [3]
where x is the ratio of the radius of the copper wire to the thickness of the covering.
Show that for maximum speed $x = \frac{1}{\sqrt{e}}$.

(f) (i) State the domain and range of $y = \log_e|x|$ [2]

(ii) Draw the graph of $y = \log_e|x|$ [2]

Question 4 **(Please start a new booklet)**

(a) Prove by mathematical induction that $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$, for $n \geq 2$ [3]

(b) Prove by mathematical induction that $3^n > 2n^2$, for $n > 1$ [3]

(c) (i) Prove that the series $2-2(\sqrt{2}-1)+\frac{2}{(\sqrt{2}+1)^2}$ is geometric. [2]

(ii) Find the limiting sum of the series. [2]

(d) Scarlett sold her coffee shop for \$1.2 million and invested the full amount, earning an interest rate of 1.2% per quarter. She is to be repaid the money in 120 equal quarterly instalments, receiving the first payment 3 months after investing the money.
Given Q is the amount of each instalment, find:

(i) A_1, the amount owing after the first repayment. [1]

(ii) A_3, the amount owing after the third repayment. [1]

(iii) The amount of each instalment, correct to the nearest dollar [2]

(iv) Scarlett decided to withdraw the remainder of the investment when she was owed \$600 000. After how many repayments did she make this withdrawal? [2]

END OF TEST

MATHEMATICS EXTENSION 1 - YEAR 12 HALF YEARLY EXAM SOLUTIONS
2017

QUESTION 1

(a) (i) $-4 < x \leq 1, \quad x > 2\sqrt{3}$ (1)

(ii) $x < -3, \quad x > 1$ (1)

(iii) $x > -1$ (1)

(b) $f'(x) = \dfrac{(x+1)^2}{x-1}$

(i) Stationary points when $f'(x)=0$:

$$\frac{(x+1)^2}{x-1} = 0$$

$\therefore x = -1$ (1)

$$f''(x) = \frac{(x-1)2(x+1) - (x+1)^2(1)}{(x-1)^2}$$

$$= \frac{2x^2 - 2 - x^2 - 2x - 1}{(x-1)^2}$$

$$= \frac{x^2 - 2x - 3}{(x-1)^2}$$

$$= \frac{(x-3)(x+1)}{(x-1)^2}$$ (1)

or use $x / f'(x)$ table

$f''(-1) = 0$

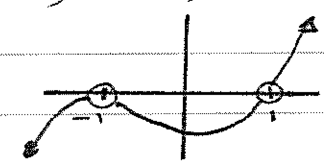
$\therefore$ Stationary point of inflexion at $x = -1$ (1)

(ii) $f(x)$ increasing when $f'(x) > 0$

i.e. $\dfrac{(x+1)^2}{x-1} > 0$ (1)

Multiply both sides by $(x-1)^2$:

$(x+1)^2(x-1) > 0$ (1)



or use critical pt. check.

$\therefore$ Increasing when $x > 1$

(1)

(iii) Let $f''(x) = 0$ for points of inflexion:

i.e. From (i) $\dfrac{(x-3)(x+1)}{(x-1)^2} = 0$

$x = 3, -1$ (1)

Stationary point of inflexion at $x = -1$

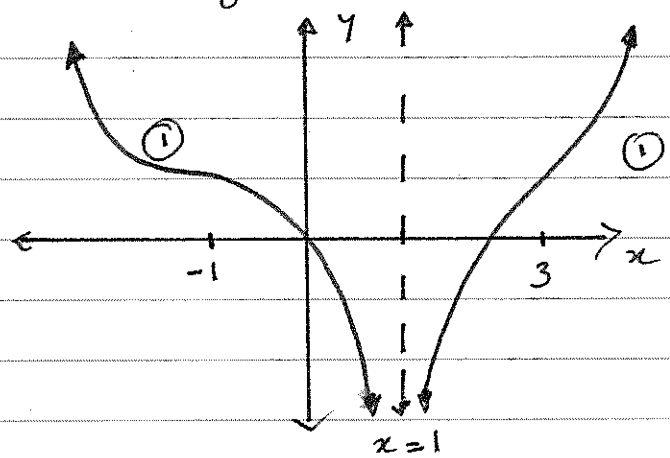
Point of inflexion at $x = 3$ (1)

(iv) Asymptote at $x = 1$

Stationary point of inflexion at $x = -1$

Point of inflexion at $x = 3$

Increasing when $x > 1$



(c) (i) $V = \pi r^2 h = 32\pi$

$h = \dfrac{32}{r^2}$ cm (1)

(ii) $A = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r\left(\dfrac{32}{r^2}\right)$ (1)

$= 2\pi r^2 + \dfrac{64\pi}{r}$ cm^2

(iii) $\dfrac{dA}{dr} = 4\pi r - \dfrac{64\pi}{r^2}$ (1)

Let $\dfrac{dA}{dr} = 0$,

i.e. $4\pi r - \dfrac{64\pi}{r^2} = 0$

$4\pi r = \dfrac{64\pi}{r^2}$

$4\pi r^3 = 64\pi$

QUESTION 11(c) CONTINUED

$r^3 = 16$

$r = \sqrt[3]{16}$

radius $\doteqdot$ 2.5 cm ①

$$\frac{d^2A}{dx^2} = 4\pi + \frac{128\pi}{r^3}$$

when $r = 2.5$,

$$\frac{d^2A}{dx^2} = 4\pi + \frac{128\pi}{(2.5)^3}$$

$$\therefore \frac{d^2A}{dx^2} > 0$$ ①

$\therefore$ minimum surface area when $r = 2.5$ cm.

(iii) Surface Area

$$= 2\pi(2.5)^2 + \frac{64\pi}{2.5}$$

$$= 119.7\text{ cm}^2$$ ①

Cost $= 119.7 \times 0.032$

$= \$3.83$ per can ①

QUESTION 2

(a) (i) $\int \left(x + \frac{4\sqrt{x}}{x} + \frac{4}{x^2} \right) dx$ (1)

$= \int \left(x + 4x^{-1/2} + 4x^{-2} \right) dx$

$= \frac{x^2}{2} + 8\sqrt{x} - \frac{4}{x} + c$ (1)

(ii) $\int \frac{x^3 - 8}{8 - x^3} \, dx$ (1)

$= \int -1 \, dx$

$= -x + c$ (1)

(b) (i) $f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$ (1)

$= \frac{x^2 + 1 - 2x^2}{(x^2+1)^2}$

$= \frac{1 - x^2}{(x^2+1)^2}$ (1)

(ii) $\int_0^2 \frac{1-x^2}{(x^2+1)^2} \, dx = \left[\frac{x}{x^2+1} \right]_0^2$ (1)

$= \frac{2}{5} - 0$

$= \frac{2}{5}$ (1)

(iii) $\frac{1}{6}\left[f(0) + 4f(\tfrac{1}{2}) + f(1) \right]$

$+ \frac{1}{6}\left[f(1) + 4f(1\tfrac{1}{2}) + f(2) \right]$

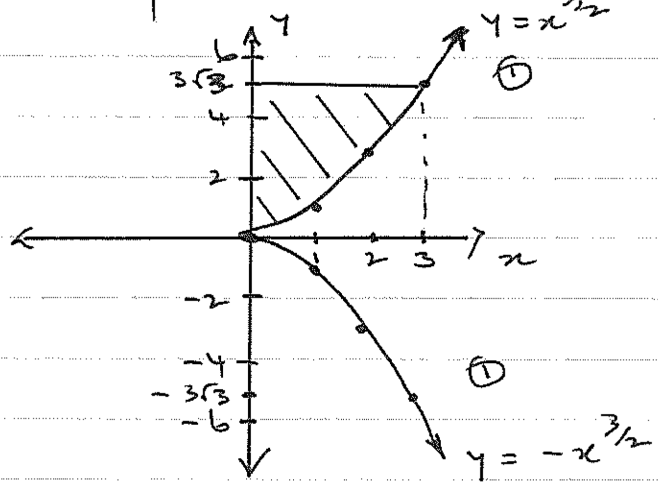
$= \frac{1}{6}\left[1 + 4(0.48) + 0 \right]$ (2)

$+ \frac{1}{6}\left[0 + 4(-0.1183) + \left(\frac{-3}{25}\right) \right]$

$= 0.388$ (1)

(c) (i) $y^2 = x^3$

$y = \pm x^{3/2}$



(ii) Area of shaded region

$= \int_0^3 3\sqrt{3} \, dx - \int_0^3 x^{3/2} \, dx$ (1)

$= \left[3\sqrt{3}x - \frac{2x^{5/2}}{5} \right]_0^3$

$= 3\sqrt{3}(3) - \frac{2}{5}(3)^{5/2}$

$= 9.3530\ldots$

$= 9.4$ units^2 (1)

OR

$= \int_0^{3\sqrt{3}} y^{2/3} \, dy$ (1)

$= \left[\frac{3y^{5/3}}{5} \right]_0^{3\sqrt{3}}$

$= \frac{3}{5}(3\sqrt{3})^{5/3}$

$= 9.3530\ldots$

$= 9.4$ units^2 (1)

QUESTION 20 continued.

(iii) $V = \pi \int_0^{3\sqrt{3}} y^{4/3}\, dy$

$= \pi \left[\frac{3}{7} y^{7/3} \right]_0^{3\sqrt{3}}$

$= \pi \left[\frac{3}{7} \left(3\sqrt{3}\right)^{7/3} \right]$

$= 62.964\ldots$

$= 63$ units^3

QUESTION 3

(a) $f(x) = \frac{1}{8}\int 4x\, e^{2x^2}\, dx$

$= \frac{1}{8} e^{2x^2} + c$ (1)

$f(1) = 0$:

$0 = \frac{1}{8}e^2 + c$

$\therefore c = -\frac{1}{8}e^2$ (1)

$\therefore f(x) = \frac{1}{8}e^{2x^2} - \frac{1}{8}e^2 + c$

(b) $\dfrac{\log_q q}{\log_q p} \times \log_q p$ (1)

$= \log_q q$

$= 1$ (1)

(c) $\dfrac{dy}{dx} = \dfrac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$ (1)

$= \dfrac{e^{2x} + 1 + 1 + e^{-2x} - e^{2x} + 1 + 1 - e^{-2x}}{(e^x + e^{-x})^2}$

$= \dfrac{4}{(e^x + e^{-x})^2}$ (1)

(d) $\log_e e^{2x} = 2x \log_e e$

$= 2x$ (1)

$\frac{d}{dx}(2x) = 2$ (1)

$\boxed{\text{or}}$

$\frac{d}{dx}(\log_e e^{2x}) = \dfrac{2e^{2x}}{e^{2x}}$

$= 2$

(e) $S = x^2 \log_e\left(\frac{1}{x}\right)$

$S = x^2 \log_e x^{-1}$

$S = -x^2 \log_e x$

$\frac{dS}{dx} = -x^2 \cdot \frac{1}{x} - 2x \log_e x$ (1)

$= -x - 2x \log_e x$

$\frac{dS}{dx} = 0$:

$2x \log_e x = -x$

$\log_e x = -\frac{1}{2}$

$x = e^{-1/2}$

$x = \frac{1}{\sqrt{e}}$ (1)

$\frac{d^2S}{dx^2} = -1 - 2x\left(\frac{1}{x}\right) - 2\log_e x$

$= -1 - 2 - 2\log_e x$

$= -3 - 2\log_e x$

$= -(3 + 2\log_e x)$

$f''(e^{-1/2}) = -(3 + 2\log_e e^{-1/2})$

$= -(3 - \log_e e)$

$= -(3 - 1)$

$= -2 < 0$ (1)

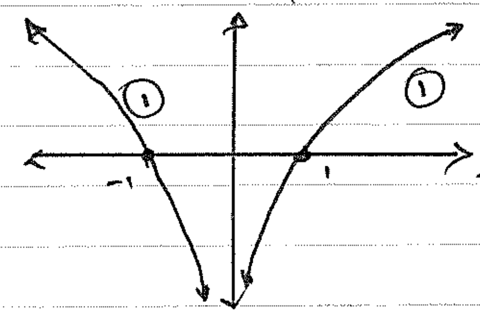
$\therefore$ maximum when $x = \frac{1}{\sqrt{e}}$

(f) (i) Domain: all real x, $x \neq 0$ (1)

Range: all real y (1)

(ii) $y = \log_e|-x| = \log_e|x|$

$\therefore$ even function.



QUESTION 4

(a) 1. Prove true for $n=2$:

$LHS = 1-\frac{1}{4} = \frac{3}{4}$

$RHS = \frac{2+1}{2\times 2} = \frac{3}{4}$

$\therefore$ True for $n=2$. (1)

2. Assume true for $n=k$:

ie. $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$

3. Hence prove true for $n=k+1$:

ie. $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$

From step 2.

ie $\frac{k+1}{2k}\left(\cancel{*}-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2k+2}$ (1)

$LHS = \frac{k+1}{2k} - \frac{1}{2k(k+1)}$

$= \frac{(k+1)^2 - 1}{2k(k+1)}$

$= \frac{k^2+2k+1-1}{2k(k+1)}$

$= \frac{k^2+2k}{2k(k+1)}$

$= \frac{k(k+2)}{2k(k+1)}$

$= \frac{k+2}{2(k+1)}$

$= RHS$ (1)

$\therefore$ The statement is true for $n=k+1$ if it is true for $n=k$.

$\therefore$ By the principle of mathematical induction, it is true for all integer $n \geq 2$

(b) 1. Prove true for $n=2$:

$LHS = 3^2 = 9$

$RHS = 2\times 2^2 = 8$

$\therefore$ True for $n=2$ (1)

2. Assume true for $n=k$:

ie $3^k > 2k^2$

3. Hence prove true for $n=k+1$:

ie. $3^{k+1} > 2(k+1)^2$

From step 2.

$3.3^k > 3.2k^2$ (1)

$3^{k+1} > 6k^2$

$> 2(k^2+2k+1)$

$> 2k^2+4k+2$

for $n>1$

$\therefore 3^{k+1} > 2(2k+1)^2$ (1)

$\therefore$ The statement is true for $n=k+1$ if it is true for $n=k$.

$\therefore$ By the process of mathematical induction, it is true for all integers $n>1$

QUESTION 4 (CONTINUED)

(c) (i) $\frac{T_2}{T_1} = -\frac{2(\sqrt{2}-1)}{2}$

$= -(\sqrt{2}-1)$
$= 1-\sqrt{2}$ ①

$\frac{T_3}{T_2} = \frac{\frac{2}{(\sqrt{2}+1)^2}}{-2(\sqrt{2}-1)}$

$= \frac{-1}{(\sqrt{2}+1)(\sqrt{2}+1)(\sqrt{2}-1)}$

$= \frac{-1}{(\sqrt{2}+1)(\sqrt{2}^2-1)}$

$= \frac{-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$= 1-\sqrt{2}$
$= -(\sqrt{2}-1)$ ①

Common ratio ∴ geometric.

(ii) $S_\infty = \frac{a}{1-r}$

$= \frac{2}{1+(\sqrt{2}-1)}$ ①

$= \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \sqrt{2}$ ①

(d) (i) $A_1 = 1200000 \times 1.012 - Q$ ①

(ii) $A_2 = A_1 \times 1.012 - Q$
$= 1200000 \times 1.012^2 - 1.012Q - Q$

$A_3 = A_2 \times 1.012 - Q$
$= 1200000 \times 1.012^3 - 1.012^2Q - 1.012Q - Q$ ①

(iii) $A_{120} = 1200000 \times 1.012^{120} - 1.012^{119}Q - \ldots - 1.012Q - Q$

$= 1200000 \times 1.012^{120} - Q(1 + 1.012 + \ldots + 1.012^{119})$

$= 1200000 \times 1.012^{120} - \frac{Q(1.012^{120}-1)}{1.012-1}$ ①

$A_{120} = 0$:

$Q(1.012^{120}-1) = 1200000 \times 1.012^{120} \times 0.012$

$Q = \frac{14400 \times 1.012^{120}}{1.012^{120}-1}$

$= \$18921.6576\ldots$
$= \$18922$ ①

(iv) $600000 = 120000 \times 1.012^n - \frac{18922(1.012^n-1)}{0.012}$ ①

$0.012 \times 600000 = 0.012 \times 120000 \times 1.012^n - 18922(1.012^n - 1)$

$7200 = 14400 \times 1.012^n - 18922 \times 1.012^n + 18922$

$-11722 = -4522 \times 1.012^n$

$1.012^n = 2.5922\ldots$

$n = \frac{\log_e 2.5922}{\log_e 1.012}$

$= 79.8508\ldots$

∴ 80 quarterly repayments ①
(accept 79)