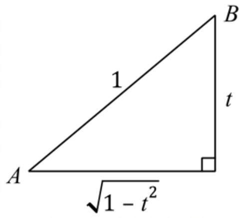
ACE Examination Paper 4
Year 12 Mathematics Extension 1 Yearly Examination
Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1.	$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert} = \frac{-2\times 4+6\times(-2)}{\sqrt{40}\times\sqrt{20}}$ $\cos\theta = -\frac{20}{20\sqrt{2}}$ $\theta = 135°$	1 Mark: D
2.	$x = 70t$ (1) $y = -5t^2$ (2) Making t the subject of equation (1) $t = \frac{x}{70}$ Substitute $\frac{x}{70}$ for t into equation (2) $y = -5\left(\frac{x}{70}\right)^2 = -\frac{x^2}{980}$ $x^2 = -980y$	1 Mark: D
3.	$A = \int_a^b y\,dx = \int_1^3 x^3 - x^2 dx$ $= [\frac{1}{4}x^4 - \frac{1}{3}x^3)]_1^3$ $= \left[\left(\frac{1}{4}3^4 - \frac{1}{3}3^3\right) - \left(\frac{1}{4}1^4 - \frac{1}{3}1^3\right)\right]$ $= 11\frac{1}{3}$ square units	1 Mark: B
4.	Line 2 $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(1+2+\cdots+k+(k+1)\right)^2$	1 Mark: B
5.	A – 3rd quadrant, B – 2nd quadrant $\sin(A+B) = \sin A\cos B + \cos A\sin B$ $= (-t)\times(-t) + (-\sqrt{1-t^2})\times\sqrt{1-t^2}$ $= t^2 - 1 + t^2$ $= 2t^2 - 1$  	1 Mark: A

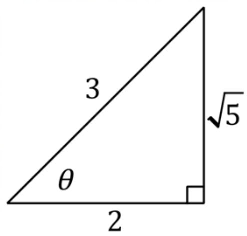
	Solution	**Criteria**
6.	Let p be the probability of winning. $p = 0.2$, $n = 25$ $P(X = x) = {}^{25}C_x\ 0.2^x 0.8^{25-x}$	1 Mark: C
7.	$u = 2x + 1$ $du = 2dx$ when $x = 1$, $u = 3$ and $x = 0$, $u = 1$ $\int_0^1 \frac{4x}{2x+1}dx = \int_1^3 \frac{2(u-1)}{u} \times \frac{1}{2}du = \int_1^3 1 - \frac{1}{u}du$ $= [u - \ln u\,]_1^3$ $= (3 - \ln 3) - (1 - \ln 1\,)$ $= 2 - \ln 3$	1 Mark: B
8.	$3\sin^2 x - 4\cos x + 1 = 0$ $3(1 - \cos^2 x) - 4\cos x + 1 = 0$ $3\cos^2 x + 4\cos x - 4 = 0$ $(3\cos x - 2)(\cos x + 2) = 0$ $3\cos x - 2 = 0$ or $\cos x + 2 = 0$ (No solution) $\cos x = \frac{2}{3}$ $x \approx 0.841$	1 Mark: C
9.	$\frac{dy}{dx} = \frac{2x+1}{4}$ $\int 4dy = \int 2x + 1\,dx$ $4y = x^2 + x + C$ Now $y = 0,\ x = 2 \Rightarrow C = -6$ $4y = x^2 + x - 6$ $y = \frac{1}{4}(x^2 + x - 6)$	1 Mark: A
10.	$\int (\cos^2 x + 2\sec^2 x)dx = \int \frac{1}{2}(1 + \cos 2x) + 2\sec^2 x dx$ $= \frac{1}{2}x + \frac{1}{4}\sin 2x + 2\tan x + C$	1 Mark: C

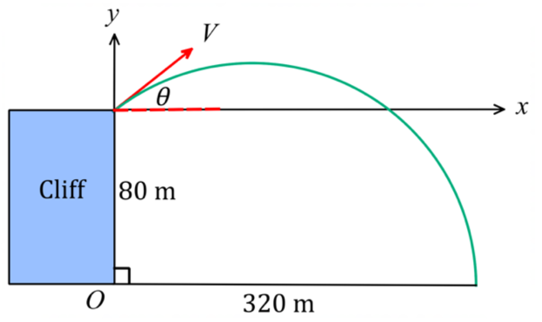
Section II		
11(a) (i)	Bernoulli trial. There are only two outcomes: either 6 or not 6 Total outcomes: $6 \times 6 = 36$ Successful outcomes: (1,5) (2,4) (3,3) (4,2) (5,1) $p = \frac{5}{36}$	1 Mark: Correct answer.:
11(a) (ii)	$Var(X) = p(1-p)$ $= \frac{5}{36} \times \left(1 - \frac{5}{36}\right)$ $= \frac{155}{1296}$	1 Mark: Correct answer.
11(b)	$\sin^2\theta = \cos\theta$ for $0° \leq \theta \leq 360°$ $2\sin\theta\cos\theta = \cos\theta$ $2\sin\theta\cos\theta - \cos\theta = 0$ $\cos\theta(2\sin\theta - 1) = 0$ $\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$ $\theta = 30°, 90°, 150°, 270°$	2 Marks: Correct answer. 1 Mark: Factorises the equation or finds one correct solution.
11(c)	$\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4dx}{\sqrt{9-x^2}} = 4\left[\sin^{-1}\frac{x}{3}\right]_{\frac{3}{\sqrt{2}}}^{3}$ $= 4\left[\left(\sin^{-1}\frac{3}{3}\right) - \left(\sin^{-1}\frac{1}{\sqrt{2}}\right)\right]$ $= 4\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$ $= \pi$	2 Marks: Correct answer. 1 Mark: Identifies the integration as a $\sin^{-1}x$ function.
11(d) (i)	$\cos x - \sqrt{3}\sin x = R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$ $R\cos\alpha = 1$ ① $R\sin\alpha = \sqrt{3}$ ② Equation ② divided by equation ① $\tan\alpha = \sqrt{3}$ $\alpha = \frac{\pi}{3}$ Squaring and adding the equations $R^2 = 1^2 + \left(\sqrt{3}\right)^2$ or $R = 2$ $\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$	2 Marks: Correct answer. 1 Mark: Finds either α or R.
11(d) (ii)	$\cos x - \sqrt{3}\sin x = -2$ for $0 \leq x \leq 2\pi$ $2\cos\left(x + \frac{\pi}{3}\right) = -2$ for $\frac{\pi}{3} \leq x + \frac{\pi}{3} \leq 2\pi + \frac{\pi}{3}$ $\cos\left(x + \frac{\pi}{3}\right) = -1$ $x + \frac{\pi}{3} = \pi$ $x = \frac{2\pi}{3}$	1 Mark: Correct answer.

11(e)	$V = \pi\int_b^b y^2 dx = \pi\int_{-3}^{1} 3 - 2x - x^2\, dx$ $= \pi\left[3x - x^2 - \frac{x^3}{3}\right]_{-3}^{1}$ $= \pi\left[\left(3 - 1^2 - \frac{1^3}{3}\right) - \left(3\times(-3) - (-3)^2 - \frac{(-3)^3}{3}\right)\right]$ $= \frac{32\pi}{3}$ cubic units	2 Marks: Correct answer. 1 Mark: Sets up the integral to determine the volume.
12(a) (i)	When $t \Rightarrow \infty$ $P \Rightarrow \frac{500}{1 + k\times 0} \Rightarrow 500$ The change in the population approaches the initial population of 500. Koalas eventually die out.	1 Mark: Correct answer.
12(a) (ii)	Initially t = 0 and P = 1 $1 = \frac{500}{1 + ke^{-1\cdot 5\times 0}}$ $1 + k = 500$ $k = 499$	1 Mark: Correct answer.
12(a) (iii)	When 100 koalas remain the change in population is 400. $400 = \frac{500}{1 + ke^{-1\cdot 5t}}$ $1 + 499e^{-1\cdot 5t} = 1.25$ $499e^{-1\cdot 5t} = 0.25$ $e^{-1\cdot 5t} = \frac{0.25}{499}$ $\ln e^{-1\cdot 5t} = \ln\frac{0.25}{499}$ $t = -\frac{1}{1.5}\times\ln\frac{0.25}{499}$ $= 5.0659\ldots$ ≈ 5 months	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Calculates P or shows some understanding.
12(a) (iv)	$P = \frac{500}{1 + ke^{-1\cdot 5t}} = 500\times(1 + ke^{-1\cdot 5t})^{-1}$ $\frac{dP}{dt} = 500\times -1(1 + ke^{-1\cdot 5t})^{-2}\times(-1.5ke^{-1\cdot 5t})$ $= \frac{1.5}{1 + ke^{-1\cdot 5t}}\times\frac{500ke^{-1\cdot 5t}}{1 + ke^{-1\cdot 5t}}$ $= \frac{1500}{1000\times(1 + ke^{-1\cdot 5t})}\times\frac{500ke^{-1\cdot 5t}}{1 + ke^{-1\cdot 5t}}$ $= \frac{3}{1000}\times\frac{500}{1 + ke^{-1\cdot 5t}}\times\frac{500\left(\frac{500}{P} - 1\right)}{1 + ke^{-1\cdot 5t}}$ $= \frac{3}{1000}\times P\times P\left(\frac{500}{P} - 1\right)$ $= \frac{3P}{1000}(500 - P)$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Finds $\frac{dP}{dt}$ or shows some understanding.

12(b) (i)	$\text{LHS} = \dfrac{\sec^2\theta}{\tan\theta}$ $= \dfrac{1}{\cos^2\theta} \div \dfrac{\sin\theta}{\cos\theta}$ $= \dfrac{1}{\cos^2\theta} \times \dfrac{\cos\theta}{\sin\theta}$ $= \dfrac{1}{\sin\theta\cos\theta}$ $= \text{RHS}$	1 Mark: Correct answer.
12(b) (ii)	$u = \tan\theta$ $du = \sec^2\theta d\theta$ When $\theta = \dfrac{\pi}{6}, u = \dfrac{1}{\sqrt{3}}$ and $\theta = \dfrac{\pi}{3}, u = \sqrt{3}$ $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin\theta\cos\theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\tan\theta} d\theta$ $\displaystyle= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\sec^2\theta}{u} \times \frac{1}{\sec^2\theta}\, du$ $\displaystyle= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{u} du$ $= [\ln u\,]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$ $= \ln\sqrt{3} - \ln\dfrac{1}{\sqrt{3}}$ $= \ln 3$	2 Marks: Correct answer. 1 Mark: Finds $\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}$ and changes the limits.
12(c)	$\cos(75°) = \cos(30° + 45°)$ $= \cos30°\cos45° - \sin30°\sin45°$ $= \left(\dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}}\right) - \left(\dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}\right)$ $= \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ $= \dfrac{\sqrt{6} - \sqrt{2}}{4}$	2 Marks: Correct answer. 1 Mark: Uses the compound angle formula with at least one correct value.
13(a) (i)	$\overrightarrow{OQ} = \underset{\sim}{u} + \underset{\sim}{v}$	1 Mark: Correct answer.
13(a) (ii)	$\overrightarrow{QO} = -\overrightarrow{OQ}$ $= -(\underset{\sim}{u} + \underset{\sim}{v})$	1 Mark: Correct answer.

13(c) (iii)	$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OQ}$ $= \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$	1 Mark: Correct answer.
13(a) (iv)	$\overrightarrow{PM} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v}) - \underset{\sim}{u}$ $= \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$	1 Mark: Correct answer.
13(a) (v)	$\overrightarrow{QM} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u}) - \underset{\sim}{v}$ $= -\frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$	1 Mark: Correct answer.
13(a) (iv)	$\overrightarrow{PM} + \overrightarrow{MQ} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u}) - \left[-\frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})\right]$ $= \underset{\sim}{v}$	1 Mark: Correct answer.
13(b)	Step 1: To prove true for $n = 1$ $7^1 - 1 = 6$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $7^k - 1 = 6m$ where m is an integer Step 3: To prove true for $n = k + 1$ $7^{k+1} - 1 = 6p$ where p is an integer LHS $= 7^{k+1} - 1$ $= 7(7^k) - 1$ $= 7(6m + 1) - 1$ $= 7(6m) + 7 - 1$ $= 6(7m + 1)$ $= 6p$ $=$ RHS Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$ 1 mark: Proves the result true for $n = 1$.
13(c)	$u = 3x^3 + 1$ $\frac{du}{dx} = 9x^2$ or $\frac{1}{9}du = x^2 dx$ $\int_0^1 x^2\sqrt{3x^3 + 1}dx = \int_1^4 \frac{1}{9}\sqrt{u}\,du = \frac{1}{9}\int_1^4 u^{\frac{1}{2}}\,du$ $= \frac{1}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^4$ $= \frac{2}{27}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$ $= \frac{2}{27} \times (8 - 1) = \frac{14}{27}$	2 Marks: Correct answer. 1 Mark: Sets up the integral using the substitution $u = 3x^3 + 1$.

13(d)	To find the point of intersection between $y = \sqrt{x}$, and $y = -x + 2$ $y = -x + 2$ ① $y = \sqrt{x}$ ② Substitute $-x + 2$ for y into equation ② $-x + 2 = \sqrt{x}$ $x^2 - 4x + 4 = x$ $x^2 - 3x + 4 = 0$ $(x - 1)(x - 4) = 0$ $x = 1$ and $y = 1$ (from the diagram $x = 4$ is not a solution) $\therefore$ Point of intersection is $(1, 1)$ $V = \pi \int_b^b y^2 dx$ $= \pi \int_1^2 \left(\sqrt{x}\right)^2 - (-x + 2)^2\, dx + \pi \int_2^9 \left(\sqrt{x}\right)^2 dx$ $= \pi \int_1^2 -x^2 + 5x - 4\, dx + \pi \int_2^9 x\, dx$ $= \pi \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^2 + \pi \left[\frac{x^2}{2}\right]_2^9$ $= \pi \left[\left(-\frac{8}{3} + \frac{20}{2} - 8\right) - \left(-\frac{1}{3} + \frac{5}{2} - 4\right) + \left(\frac{81}{2}\right) - \left(\frac{4}{2}\right)\right]$ $= \frac{119\pi}{3}$ cubic units	4 Marks: Correct answer. 3 Marks: Makes significant progress towards the solution. 2 Marks: Correctly sets up the integral. 1 Mark: Finds the point of intersection between $y = \sqrt{x}$, and $y = -x + 2$.
13(e)	$\cos\theta = -\frac{2}{3}$ and $\tan\theta > 0$ $\therefore$ Angle in the 3rd quadrant  $\sin 2\theta = 2\sin\theta\cos\theta$ $= 2 \times -\frac{\sqrt{5}}{3} \times -\frac{2}{3}$ $= \frac{4\sqrt{5}}{9}$	2 Marks: Correct answer. 1 Mark: Shows some understanding.

14(a) (i)	 Horizontally $a_x = \ddot{x} = 0$ $v_x = \dot{x} = c_1$ At $t = 0, v_x = V\cos\theta$ $\Rightarrow c_1 = V\cos\theta$ $v_x = V\cos\theta$ $x = Vt\cos\theta + c_2$ When $t = 0, x = 0 \Rightarrow c_2 = 0$ $x = Vt\cos\theta$ Vertically $a_y = \ddot{y} = -10$ $v_y = \dot{y} = -10t + c_3$ At $t = 0, v_y = V\sin\theta$ $\Rightarrow c_3 = 0$ $v_y = \dot{y} = -10t + V\sin\theta$ $y = -5t^2 + Vt\sin\theta + c_4$ When $t = 0$ $y = 0 \Rightarrow c_4 = 0$ $y = -5t^2 + Vt\sin\theta$	2 Marks: Correct answer. 1 Mark: Finds horizontal or vertical parametric equations or shows some understanding of the problem.
14(b) (ii)	Greatest height when $\dot{y} = 0$ at $t = 3$ $\dot{y} = -10t + V\sin\theta$ $0 = -10 \times 3 + V\sin\theta$ $V\sin\theta = 30$	1 Mark: Correct answer.
14(b) (iii)	Stone reaches the ground when $y = -80$ $y = -5t^2 + Vt\sin\theta$ $-80 = -5t^2 + 30t$ $t^2 - 6t - 16 = 0$ $(t-8)(t+2) = 0$ $\therefore t = 8$ (t must be positive)	1 Mark: Correct answer.
14(b) (iv)	Stone reaches the ground when $x = 320$ at $t = 8$ $x = Vt\cos\theta$ $320 = V \times 8 \times \cos\theta$ $V\cos\theta = 40$	1 Mark: Correct answer.
14(b) (v)	$V^2 = \dot{x}^2 + \dot{y}^2$ $= (V\cos\theta)^2 + (V\sin\theta)^2$ $= 40^2 + 30^2$ $V = 50$ $\tan\theta = \frac{V\sin\theta}{V\cos\theta}$ $= \frac{30}{40}$ $\theta = 36.8698\ldots$ $\approx 36°52'$	2 Marks: Correct answer. 1 Mark: Finds V or θ.
14(b)	$\int_0^{\frac{\pi}{12}} 2\sin^2 x\,dx = \int_0^{\frac{\pi}{12}} 2 \times \frac{1}{2}(1 - \cos 2x)dx$ $= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{12}}$ $= \left(\frac{\pi}{12} - \frac{1}{2}\sin\frac{\pi}{6}\right) - \left(0 - \frac{1}{4}\sin 0\right)$ $= \frac{\pi}{12} - \frac{1}{4}$ $= \frac{\pi - 3}{12}$	2 Marks: Correct answer. 1 Mark: Applies the double angle identity.

14(c)	Step 1: To prove true for $n = 1$ $\text{LHS} = 4 \times 1 - 3 = 1$ $\text{RHS} = 2(1)^2 - 1 = 1$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $S_k = 2k^2 - k$ Step 3: To prove true for $n = k + 1$ $S_{k+1} = 2(k+1)^2 - (k+1)$ $S_k + T_{k+1} = S_{k+1}$ $\text{LHS} = 2k^2 - k + 4(k+1) - 3$ $= 2k^2 + 4k + 2 - k - 1$ $= 2(k^2 + 2k + 1) - (k+1)$ $= 2(k+1)^2 - (k+1)$ $= \text{RHS}$ Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$ 1 mark: Proves the result true for $n = 1$.
14(d)	Let p be the probability of rolling a six. $p = \frac{1}{6}, n = 12$ $E(X) = np$ $= 12 \times \frac{1}{6} = 2$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
14(e)	$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ $= (-3\underset{\sim}{i} + 4\underset{\sim}{j}) - (4\underset{\sim}{i} + 3\underset{\sim}{j})$ $= -7\underset{\sim}{i} + \underset{\sim}{j}$ $\lvert\overrightarrow{PQ}\rvert = \sqrt{(-7)^2 + 1^2}$ $= \sqrt{50}$ $= 5\sqrt{2}$	2 Marks: Correct answer. 1 Mark: Finds $\overrightarrow{PQ}$.
14(f)	Let p be the probability of throwing a six. $p = \frac{1}{6}, n = 5$ $P(X = x) = {}^5C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{5-x}$ Require the probability of zero sixes or one six. $(X \leq 1) = {}^5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5 + {}^5C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4$ $= \frac{3125}{3888}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Finds the general rule for the probability distribution.