CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Extension 2 Mathematics
Half Yearly
2017

Time allowed: 2 hours

Name: ______________________________ Teacher: Ms Strilakos

Instructions:

- All questions should be attempted on your own paper.
- Show ALL necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Please write on one side of each sheet of paper only, and do not use multiple columns on the page.

TOPIC	Complex Numbers	Graphs	Conics	TOTAL
MARKS	/25	/15	/32	/72

Question 1

(a) Given $1 - i$ is a root of the equation $x^2 + (a + i)x + (4 + bi) = 0$ where a and b are real, find the values of a and b.

[2 marks]

(b) (i) On the *same* diagram, draw a neat sketch of the locus specified by each of the following:

(α) $|z - (2 + i)| = 1$ (β) $|z + 1| = |z - 3|$

(ii) Hence write down all value(s) of z which satisfy simultaneously

$|z - (2 + i)| = 1$ and $|z + 1| = |z - 3|$

(iii) Use your diagram in (i) to determine the values of k for which the simultaneous equations

$|z - (2 + i)| = 1$ and $|z - i| = k$

have exactly one solution for z.

[5 marks]

(c) (α) Find algebraically the locus in the Argand plane represented by

$$|z^2 - \bar{z}^2| < 4$$

(β) Sketch this locus on an Argand diagram.

[3 marks]

Question 2

(i) Find the five fifth roots of $1 + \sqrt{3}i$.

(ii) Find the area of the pentagon formed by the five points representing these roots.

[4 marks]

Question 3

(i) Verify that $\propto = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of the equation $z^5 + z - 1 = 0$

(ii) Find the monic cubic equation with real coefficients whose roots are also roots of

$z^5 + z - 1 = 0$ but do not include $\propto$.

[6 marks]

Question 4

The complex numbers z_1 and z_2 are given by $z_1 = 1 + i\sqrt{3}$ and $z_2 = iz_1$.

(i) Label accurately the points representing z_1 and z_2 in an Argand diagram.

(ii) On the same Argand diagram, sketch the locus of the points z satisfying

(α) $|z - z_1| = |z - z_2|$

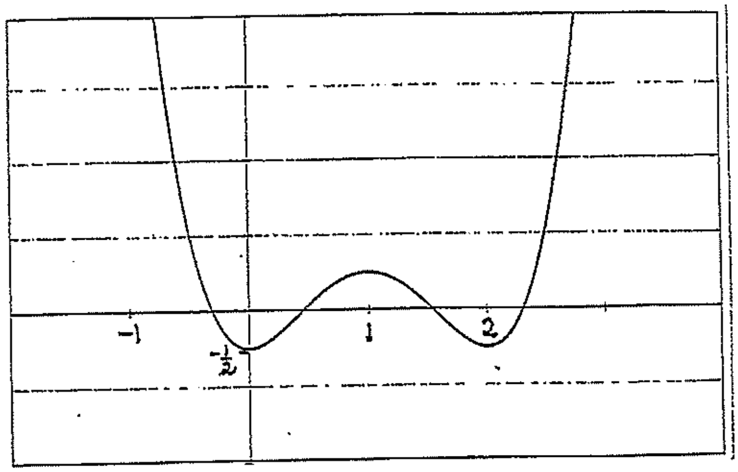
(β) $\arg(z - z_1) = \arg z_2$

(iii) Determine, in the form $x + iy$, the complex number z_3 represented by the intersection of the two loci of part (ii). (Show complete justification for your answer)

[5 marks]

Question 5

The graph of $g(x) = x^4 - 4x^3 + 4x^2 - \frac{1}{2}$ is shown below.



(i) Use this graph to sketch the graphs of each of the following on separate diagrams, clearly indicating the trends of the curves for all x, including when $g(x) = 0$.

(a) $y = |g(x)|$

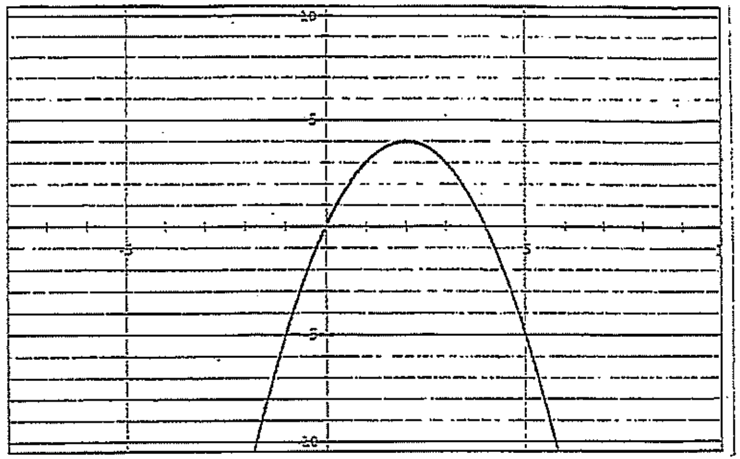
(b) $y^2 = g(x)$

(ii) Calculate the slope of the curve $y^2 = g(x)$ at any point x and describe the nature of the curve at a zero of $g(x)$.

[3+2=5 marks]

Question 6

The graph of the function $f(x) = 4x - x^2$ is shown below.



Sketch on separate sets of axes, **without using calculus,** the graph of each of the following, clearly showing any asymptotes, turning points and axes intercepts, and labelling where appropriate.

(i) $y = |f(x)|$

(ii) $y = \frac{1}{|f(x)|}$

(iii) $y = \sqrt{f(x)}$

(iv) $y = \log_e\{f(x)\}$

(v) $y = e^{f(x)}$

(vi) $y = f(x).\log_e x$

[1+2+1+2+2+2=10 marks]

Question 7

(i) Determine the real values of λ for which the equation

$$\frac{x^2}{4-\lambda}+\frac{y^2}{2-\lambda}=1$$

defines (α) an ellipse

(β) a hyperbola

(ii) Describe how the shape of this curve changes as λ increases from 1 towards 2.

(iii) What is the limiting position of the curve, as $\lambda \to 2^-$ {ie: what shape and dimensions does the curve approach as $\lambda \to 2^-$ }?

[2+1+1=4]

Question 8

(a) (i) Show that

$$4x^2+9y^2+24x-36y+36=0$$

represents an ellipse.

(ii) Find its centre, eccentricity, foci and directrices and sketch the graph.

[4 marks]

Question 9

An ellipse has Cartesian equation

$$\frac{x^2}{2} + y^2 = 1.$$

A straight line L has equation $y = mx + c$, where m and c are positive constants.

(i) Show that the x coordinates of the points of intersection between L and the ellipse satisfy the equation

$$(2m^2 + 1)x^2 + 4mcx + 2(c^2 - 1) = 0$$

(ii) Given that L is a tangent to the ellipse, show that $c^2 = 2m^2 + 1$.

The line L meets the negative x axis and the positive y axis at the points X and Y respectively. The point O is the origin.

(iii) Find the area of the triangle OXY, in terms of m.

(iv) Show that as m varies, the minimum area of the triangle OXY is $\sqrt{2}$.

(v) Find the x coordinate of the point of tangency between the line L and the ellipse when the area of the triangle is minimum.

[1+1+1+2+2=7]

Question 10

(b) The ellipse E with Cartesian equation

$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

has foci S and S' at $(\pm 1,0)$ and directrices with equations $x=\pm 4$

If $P(x_1,y_1)$ is a general point on this ellipse

(i) Prove that the sum of the distances SP and SP' is independent of P.

(ii) Find the gradient of the normal to E at $P(x_{1,}y_1)$.

(iii) Prove that the normal to E at P bisects $\angle SPS'$

NOTE: You may find the following formula useful:

$$\tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

[1+1+4=6 marks]

Question 11

(i) Show that, of the points $(r\ cos\theta, r\ sin\theta)$ and $\left(s\cos\left(\theta+\frac{\pi}{2}\right), s\sin\left(\theta+\frac{\pi}{2}\right)\right)$ lie on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre O, then $\frac{1}{r^2}+\frac{1}{s^2}=\frac{1}{a^2}-\frac{1}{b^2}$

(ii) Deduce that, if P and Q are points on the hyperbola such that OP is perpendicular to OQ, then

$$\frac{1}{OP^2}+\frac{1}{OQ^2}$$

is independent of the position of P and Q.

[5 marks]

Question 12

Let $P(cp,\frac{c}{p})$ and $Q(cq,\frac{c}{q})$ be the endpoints of a chord drawn on the rectangular hyperbola with equation $xy = c^2$.

If PQ has a constant length k, find the locus of R, the midpoint of PQ.

[6 marks]

END OF PAPER

YEAR 12 EXTENSION 2 MATHEMATICS

HALF-YEARLY 2017 SOLUTIONS

QUESTION 1

(a) $x^2 + (a+i)x + (4+bi) = 0 \quad a, b \in \mathbb{R}$

Let $x = 1-i$

$(1-i)^2 + (a+i)(1-i) + (4+bi) = 0$ 1

(2)

i.e. $1 - 2i + \cancel{i^2} + a - ai + i - \cancel{i^2} + 4 + bi = 0$

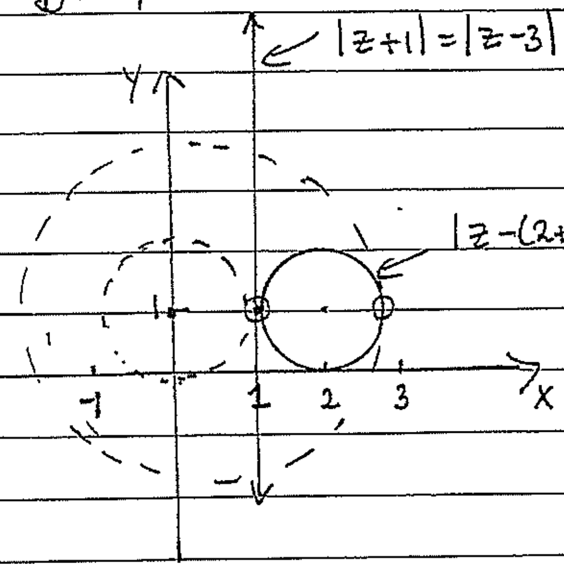
i.e. $1 + a + 4 + (-i - ai + bi) = 0$

Equating real and imaginary parts: $5 + a = 0$, $a = -5$ $\quad -1 - a + b = 0$ 1

Since $a = -5$, $-1 + 5 + b = 0$ i.e. $b = -4$

Solution: $a = -5$, $b = -4$

(b) (i) 2



(ii) They meet at only one point i.e. $z = 1+i$ 1

(iii) This is really asking where the circle will touch another circle with centre $(0,1)$ if the radius of this other circle is k. i.e. What possible values can k take?

Answer: k can be either 1 or 3 2

QUESTION 1 (cont.)

(c)

(α) $|z^2 - \bar{z}^2| < 4$

Let $z = x + iy$ $\quad \bar{z} = x - iy$

$\therefore z^2 = x^2 - y^2 + 2xyi$

and $\bar{z}^2 = x^2 - 2xyi - y^2$

$\therefore z^2 - \bar{z}^2 = 4xyi$

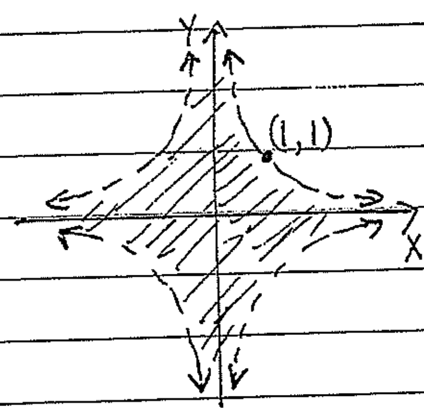
and $|z^2 - \bar{z}^2| = 4xy$ $\quad 1$

$|z^2 - \bar{z}^2| < 4$

means $|4xy| < 4$

ie. $|xy| < 1$ $\quad 1$

(β) $\quad 1$



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Question 2

(i)

Let $z^5 = 1+\sqrt{3}i \;=\; 2\,\text{cis}\left(\frac{\pi}{3} + 2k\pi\right)$

$k = 0, \pm 1, \pm 2, \ldots$

Let $z = r\,\text{cis}\,\theta$ be a root.

$r^5 = 2 \qquad \therefore r = 2^{1/5}$ 1

and $5\theta = \frac{\pi}{3} + 2k\pi$ (2)

$\therefore \theta = \frac{\pi}{15} + \frac{2k\pi}{5}$. 1

$\therefore$ For $k=0$, $z_1 = 2^{1/5}\,\text{cis}\,\frac{\pi}{15}$

For $k=1$, $z_2 = 2^{1/5}\,\text{cis}\left(\frac{\pi}{15} + \frac{2\pi}{5}\right)$

$= 2^{1/5}\,\text{cis}\,\frac{7\pi}{15}$

For $k=-1$, $z_3 = 2^{1/5}\,\text{cis}\left(\frac{\pi}{15} - \frac{2\pi}{5}\right)$

$= 2^{1/5}\,\text{cis}\left(\frac{-5\pi}{15}\right)$

$= 2^{1/5}\,\text{cis}\left(\frac{-\pi}{3}\right)$

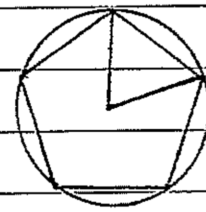
For $k=2$, $z_4 = 2^{1/5}\,\text{cis}\left(\frac{\pi}{15} + \frac{4\pi}{5}\right)$

$= 2^{1/5}\,\text{cis}\left(\frac{13\pi}{15}\right)$

For $k=-2$, $z_5 = 2^{1/5}\,\text{cis}\left(\frac{\pi}{15} - \frac{4\pi}{5}\right)$

$= 2^{1/5}\,\text{cis}\left(\frac{-11\pi}{\ }\right.$

(ii) The roots lie on a circle centre (0,0) radius $2^{1/5}$ equally spaced and separated by $\frac{2\pi}{5}^c$.



Using Area of $\Delta = \frac{1}{2}bc\sin A$ (2)

Total area of Pentagon

$= 5 \times \frac{1}{2} \times 2^{1/5} \times 2^{1/5} \sin\frac{2\pi}{5}$ 1

$= 3.14\ u^2$. (2.d.ps) 1

QUESTION 3

(i) $z^5+z-1=0.$

$\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\alpha = \frac{1}{2}+\frac{\sqrt{3}}{2}i$ 1

If α is a root then

$\alpha^5+\alpha-1=0.$ 2

$\alpha^5 = \cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$

$= \frac{1}{2}-\frac{\sqrt{3}}{2}i$

$\therefore \alpha^5+\alpha-1 = \frac{1}{2}-\frac{\sqrt{3}}{2}i+\frac{1}{2}+\frac{\sqrt{3}}{2}i-1$

$=0$

Thus α is a root of the eq^n. 1

$\therefore z^5+z-1 = (z^2-z+1)(z^3+az^2+bz-1)$

Now, we know its monic

$\therefore$ coeff. of $z^5=1$

Equating coeffs of z^4: 1

$0=-1+a \quad \therefore a=1$

Equating coeffs. of z^3:

$0=b-a+1 \quad \therefore b=-1+a \quad \therefore b=0.$

$\therefore$ the monic cubic eq^n is

$z^3+z^2-1=0.$ 1

4

(ii) Since α is complex, and the eq^n has real coeffs, $\bar{\alpha}$ is also a root.

$\alpha.\bar{\alpha} = \left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$ $\quad \alpha+\bar{\alpha} = \frac{1}{2}+\frac{\sqrt{3}}{2}i+\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$= \frac{1}{4}+\frac{3}{4}=1 \qquad\qquad =1$ 1

$\therefore (z-\alpha)(z-\bar{\alpha})=0$ is a factor

of the polynomial

ie. $z^2 - z(\alpha+\bar{\alpha})+\alpha.\bar{\alpha}=0$

ie. $z^2-z+1=0$ is a factor 1

Alternatively, using long division

$$\begin{array}{r|l} & z^3+z^2-1 \\ \hline z^2-z+1 & z^5+z^2-1 \\ & z^5-z^4+z^3 \\ \hline & z^4-z^3+z-1 \\ & z^4-z^3+z^2 \\ \hline & -z^2+z-1 \\ & -z^2+z-1 \\ \hline & 0 \end{array}$$

$\therefore z^5+z-1 = (z^2-z+1)(z^3+z^2-1)$

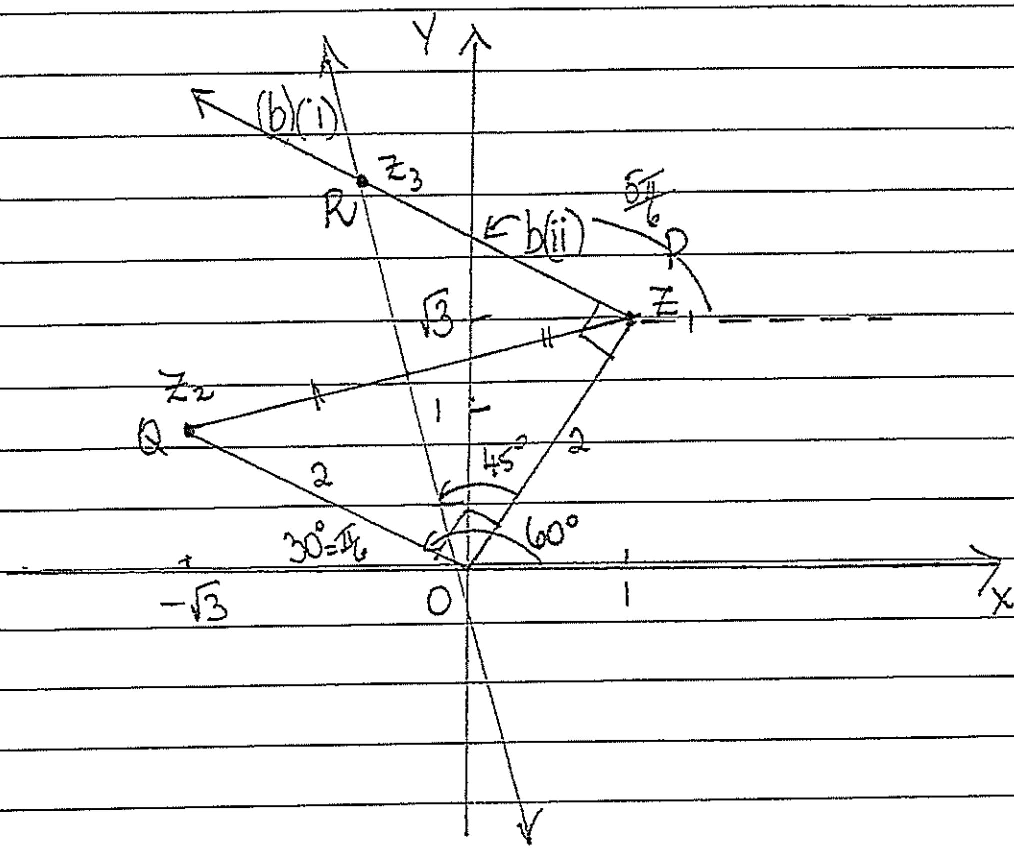
$\therefore$ the monic cubic is

$z^3+z^2-1=0$

QUESTION 4.

$z_1 = 1 + i\sqrt{3}$ $z_2 = iz_1$ i.e. $z_2 = i + i^2\sqrt{3} = -\sqrt{3} + i$

(i)



1

(ii) (α) $\perp$ bisector $y(-1+\sqrt{3}) = x(1+\sqrt{3})$ 1

(β) $\arg(z - z_1) = \arg z_2$ i.e. all points whose arg. in relation to z_1 is equal to the arg. z_2

For $\operatorname{Arg} z_2$, $z_2 = -\sqrt{3} + i$ $x = r\cos\theta$, $y = r\sin\theta$, $r = 2$

$\cos\theta = \frac{-\sqrt{3}}{2}$ $\sin\theta = \frac{1}{2}$

(iii) $|z_1| = |z_2| = 2$

$\tan\theta = \frac{-1}{\sqrt{3}}$ θ is in 2nd quadrant

$\triangle OPQ$ is isosceles $\therefore$ $\perp$ bisector must pass through Origin.

$\therefore \theta = \frac{5\pi}{6}$ 1

$\angle POQ = 90°$

0

$\therefore \angle RPO = 90°$

... $\angle POR = 45°$. $\therefore |PR| = 2$ also

QUESTION 2 (cont)

(ii) (α) $|z| \geq 1$.

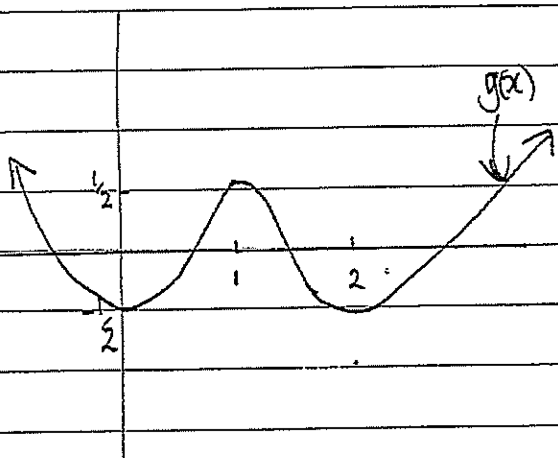
(β) $\arg z$ ...

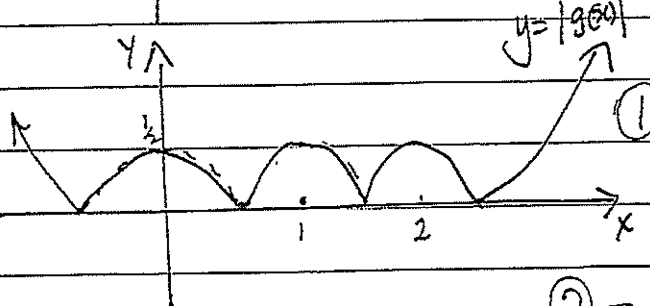
$\tan\theta = \pm\sqrt{3}$ $\quad \theta = \frac{\pi}{3}, -\frac{\pi}{3}$

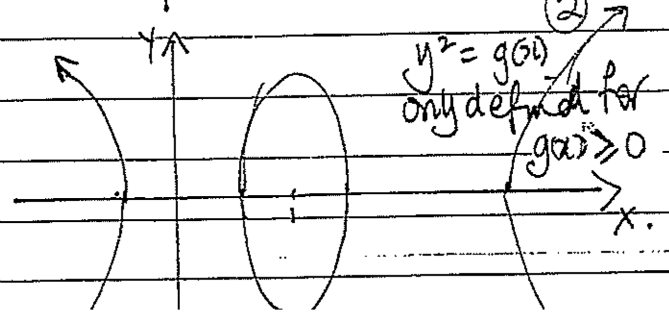
$\therefore -\frac{\pi}{3} < \arg z < \frac{\pi}{3}$

QUESTION 5

(i) (a) $y = |g(x)|$







(ii) For $y = |g(x)|$ at a zero,

the curve has sharp points, since it is reflected in the x-axis.

(ii) the slope of $y^2 = g(x)$ is given by $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = g'(x)$$

$$\therefore \frac{dy}{dx} = \frac{g'x}{2y} = \frac{g'(x)}{2\sqrt{g(x)}} \quad 1$$

This is undefined for $g(x) = 0$

$\therefore$ At $g(x) = 0$, $y^2 = g(x)$ must have vertical tangents. 1

$$g(x) = x^4 - 4x^3 + 4x^2 - \tfrac{1}{2}$$

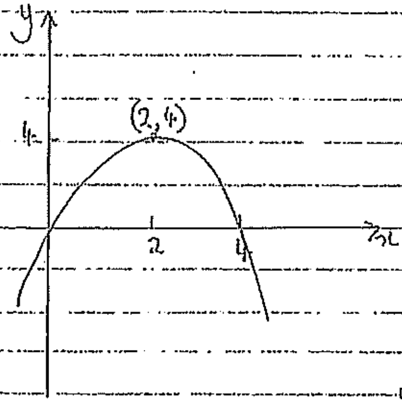
$$g'(x) = 4x^3 - 12x^2 + 8x$$

$$\frac{dy}{dx} = \frac{2x^3 - 6x^2 + 4x}{\sqrt{g(x)}} = \frac{2x(x^2 - 3x + 2)}{\sqrt{gx}}$$

$$= \frac{2x(x-2)(x-1)}{\sqrt{g(x)}}$$

QUESTION 6

1 $f(x) = 4x - x^2 = x(4-x)$



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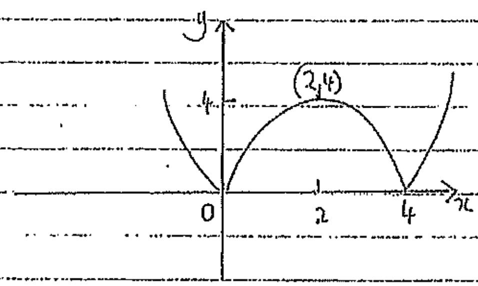
(i) $y = |f(x)|$



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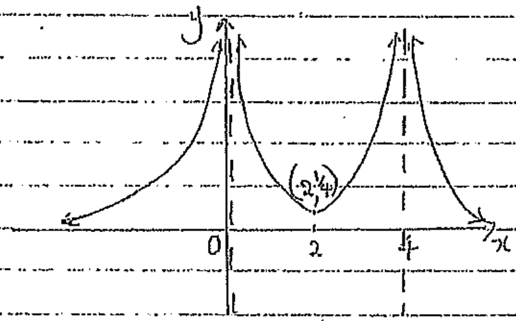
(ii) $y = \dfrac{1}{|f(x)|}$



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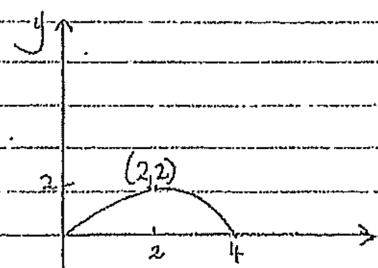
(iii) $y = \sqrt{f(x)}$

$f(x) \geq 0$



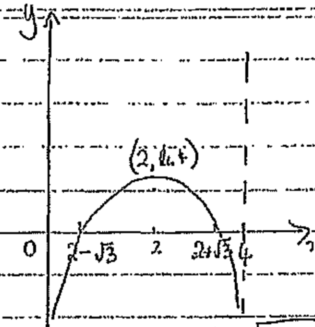
1

(iv) $y = \log_e f(x)$

$f(x) \geq 0$

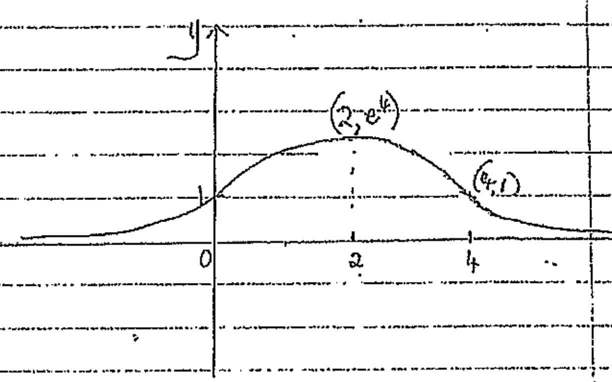
for $0 \leq f(x) < 1$, $\log_e f(x) < 0$

for $\log_e (4x - x^2)$



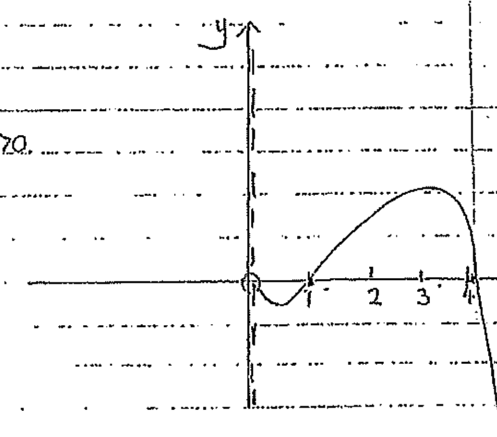
$\therefore 4x - x^2 = 1$ when $x^2 - 4x + 1 = 0$, $\therefore x = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

(v) $y = e^{f(x)}$



(vi) $y = f(x) \cdot \log_e x$

$f(x)$ overrides $\log_e x$, and $x > 0$



QUESTION 7

(i) $\dfrac{x^2}{4-\lambda} + \dfrac{y^2}{2-\lambda} = 1$

(α) For an ellipse $4-\lambda > 0$ and $2-\lambda > 0$

ie $\lambda < 4$ & $\lambda < 2$

$\therefore \lambda < 2$ (1)

(β) For a hyperbola $4-\lambda > 0$ and $2-\lambda < 0$

ie $\lambda < 4$ & $\lambda > 2$

$\therefore 2 < \lambda < 4$ (1).

(ii) As $\lambda \to 2^-$, the major axis gets shorter, $\therefore$ curve gets flatter. (1)

(iii) As $\lambda \to 2^-$, major axis approaches $2\sqrt{2}$, with endpoints at the foci. (1)

ie it approaches the interval joining the foci of the ellipse when $\lambda = 1$. ie SS'.

QUESTION 8

(a) (i) $4x^2 + 9y^2 + 24x - 36y + 36 = 0$

$4x^2 + 24x + 9y^2 - 36y + 36 = 0.$

$4(x^2 + 6x) + 9(y^2 - 4y) + 36 = 0$

$4\left[(x+3)^2 - 9\right] + 9\left[(y-2)^2 - 4\right] + 36 = 0$

ie $4(x+3)^2 + 9(y-2)^2 = 36$

ie $\dfrac{(x+3)^2}{9} + \dfrac{(y-2)^2}{4} = 1$ (1)

(ii) centre $\underline{(-3, 2)}$

$a = 3$, $b = 2$

$b^2 = a^2(1 - e^2)$ (1)

$4 = 9(1 - e^2)$

$1 - e^2 = \dfrac{4}{9}$ $\therefore e^2 = \dfrac{5}{9}$ $\underline{e = \dfrac{\sqrt{5}}{3}}$

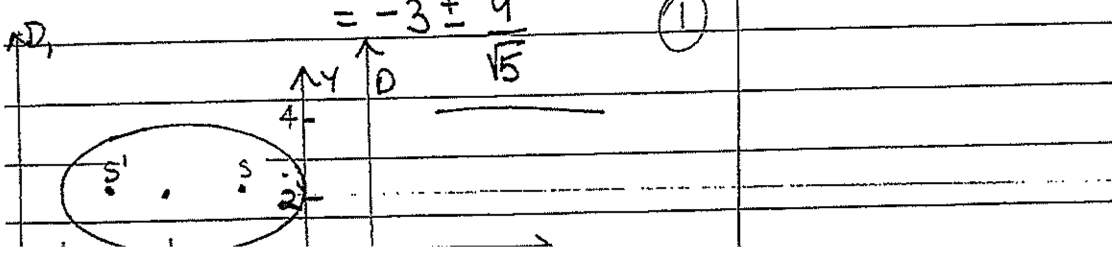
foci: $(h \pm ae, k)$

(1)

ie $\underline{(-3 \pm \sqrt{5},\ 2)}$

directrices: $x = h \pm \dfrac{a}{e}$

$\underline{= -3 \pm \dfrac{9}{\sqrt{5}}}$ (1)



Q.9(1) $\frac{x^2}{2} + y^2 = 1$ —①

L: $y = mx + c$. —②

(i) Substitute ② in ①.

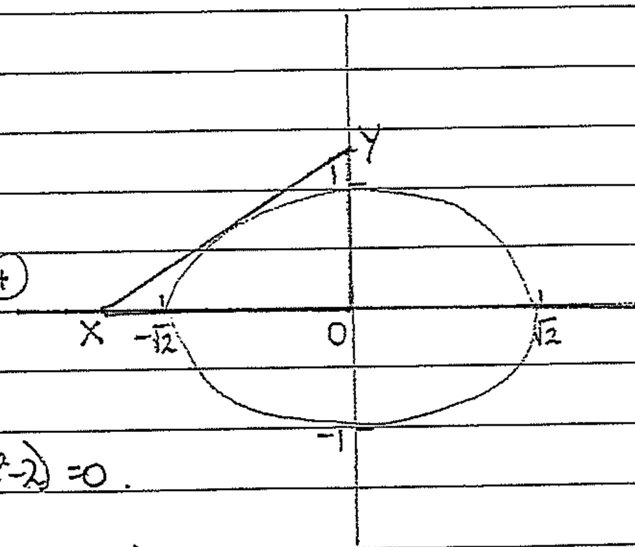
$\frac{x^2}{2} + (mx+c)^2 = 1$

$\frac{x^2}{2} + m^2x^2 + 2mcx + c^2 = 1$ —③

①

×2 $x^2(2m^2+1) + 4mcx + 2c^2 - 2 = 0$.

(ii) If L is a tangent, there is only one solution for ③



$\therefore$ $x = \frac{-b}{2a} = \frac{-4mc}{2(2m^2+1)}$ —④

and $\Delta = 0$ ie. $16m^2c^2 - 4(2m^2+1)(2c^2-2) = 0$.

ie. $16m^2c^2 - 4(4m^2c^2 - 4m^2 + 2c^2 - 2) = 0$.

ie. $16m^2 - 8c^2 + 8 = 0$.

①

$8c^2 = 8(2m^2+1)$

$\therefore c^2 = 2m^2+1$

~~$c^2 = \frac{-4(2m^2+1)}{4(2m^2-1)}$~~

(iii) L meets −ve x-axis ie at $y=0$, from $y=mx+c \Rightarrow x=\frac{-c}{m}$

Since $c^2=2m^2+1$, $x=\frac{-\sqrt{2m^2+1}}{m}$

1

L meets y-axis at $x=0$, $y=c=\sqrt{2m^2+1}$

$\therefore$ Area of $\Delta OXY = \frac{1}{2}\left(\frac{\sqrt{2m^2+1}}{m}\right)\times\sqrt{2m^2+1} = \frac{2m^2+1}{2m} = m+\frac{1}{2m}$ (u^2).

(iv) $\frac{dA}{dm} = 1-\frac{1}{2m^2}$

$=0$ for max/min.

When $m=\frac{1}{\sqrt{2}}$

Test: $\frac{d^2A}{dm^2} = \frac{4m}{4m^4} = \frac{1}{m^3}$

When $m=\frac{1}{\sqrt{2}}$ $\frac{d^2A}{dm^2}>0$ $\therefore$ min^m

or

$m<\frac{1}{\sqrt{2}}$, $\frac{dA}{dm}<0$ $m>\frac{1}{\sqrt{2}}$, $\frac{dA}{dm}>0$.

2

$\therefore$ min^m area when $m=\frac{1}{\sqrt{2}}$

Area $= m+\frac{1}{2m} = \frac{1}{\sqrt{2}}+\frac{1}{2/\sqrt{2}} = \frac{1}{\sqrt{2}}+\frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$ (u^2).

(v) At $m=\frac{1}{\sqrt{2}}$, $x=\frac{-b}{2a} = \frac{-4mc}{2(2m^2+1)}$ (since $\Delta=0$) see 4 in part (ii)

and $c=\sqrt{2m^2+1}$

$=\sqrt{\frac{2}{2}+1}$

$=\sqrt{2}$

$= \frac{-4\times\frac{1}{\sqrt{2}}\times\sqrt{2}}{2(2\times\frac{1}{2}+1)} = \frac{-4}{4} = -1$

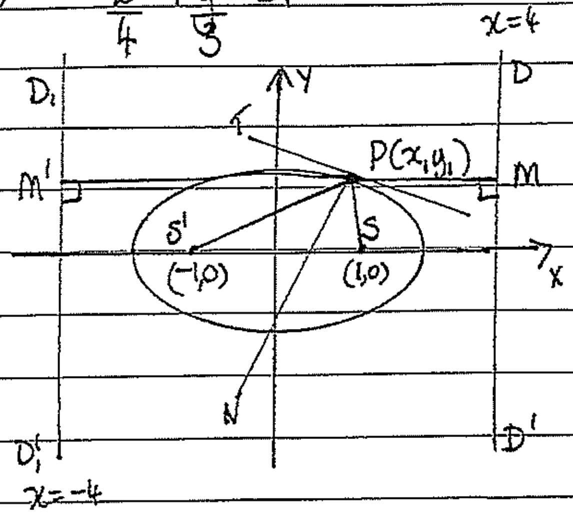
ie $x=-1$ at the point of tangency.

2

$\left(-1, \frac{1}{\sqrt{2}}\right)$.

Q10.

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$



(i) $d\,PS = \sqrt{(x_1-1)^2 + y_1^2}$

$d\,PS' = \sqrt{(x_1+1)^2 + y_1^2}$

$e = \dfrac{PS}{PM} = \dfrac{PS'}{PM'}$

$\therefore PS = e\,PM \quad \& \quad PS' = e\,PM'$

$= e(x_1+4)$

$= e(4 - x_1)$

$\therefore PS + PS' = 4e - ex_1 + ex_1 + 4e$

$= 8e$ 1

This is independent of P.

(ii) $\dfrac{2x}{4} + \dfrac{2y}{3}\dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{-\frac{x}{2}}{\frac{2y}{3}} = -\dfrac{3x}{4y}$ 1

$\therefore$ gradient of normal at P is $4y$...

(iii) $m\,PS = \dfrac{\Delta y}{\Delta x} = \dfrac{y_1}{x_1 - 1}$

1

$m\,PS' = \dfrac{y_1}{x_1+1}$

Using $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\angle SPN = \left|\dfrac{m_{PS} - m_{PN}}{1 + m_{PS}\cdot m_{PN}}\right|$ 2

$= \left|\dfrac{\dfrac{y_1}{x_1-1} - \dfrac{4y_1}{3x_1}}{1 + \dfrac{4y_1^2}{3x_1(x_1-1)}}\right|$

$= \left|\dfrac{3x_1y_1 - 4x_1y_1 + 4y_1}{3x_1^2 - 3x_1 + 4y_1^2}\right| = \left|\dfrac{4y_1 - x_1\ldots}{3x_1^2 - 3x_1 + 4\ldots}\right|$

1

$\tan\angle S'PN = \left|\dfrac{m_{PS'} - m_{PN}}{1 + m_{PS'}\cdot m_{PN}}\right|$

$= \left|\dfrac{\dfrac{y_1}{x_1+1} - \dfrac{4y_1}{3x_1}}{1 + \dfrac{4y_1^2}{3x_1(x_1+1)}}\right|$ 1

$= \left|\dfrac{3x_1y_1 - 4x_1y_1 - 4y_1}{3x_1^2 + 3x_1 + 4y_1^2}\right|$

$= \left|\dfrac{-x_1y_1 - 4y_1}{\ldots}\right|$

Now, $3x_1^2 + 4y_1^2 = 12$

is true since $P(x_1, y_1)$ lies

on the ellipse.

Subst. for $4y_1^2 = 12 - 3x_1^2$ in ① & ②.

$$\tan \angle SPN = \left| \frac{4y_1 - x_1 y_1}{12 - 3x_1} \right| = \left| \frac{y_1(4 - x_1)}{-3(x_1 - 4)} \right| = \left| \frac{y_1}{3} \right|$$

and $$\tan \angle S'PN = \left| \frac{-x_1 y_1 - 4y_1}{12 + 3x_1} \right| = \left| \frac{-y_1(4 + x_1)}{3(x_1 + 4)} \right| = \left| \frac{-y_1}{3} \right|$$

$\therefore \tan \angle SPN = \tan \angle S'PN$

$\therefore$ the normal bisects $\angle SPS'$.

QUESTION 11

(i) $(r\cos\theta, r\sin\theta)$ and $\left(s\cos\left(\theta+\frac{\pi}{2}\right), s\sin\left(\theta+\frac{\pi}{2}\right)\right)$ lie on

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If $(r\cos\theta, r\sin\theta)$ lies on the hyperbola then

$$\frac{r^2\cos^2\theta}{a^2} - \frac{r^2\sin^2\theta}{b^2} = 1 \quad — (1)$$

If $\left(s\cos\left(\theta+\frac{\pi}{2}\right), s\sin\left(\theta+\frac{\pi}{2}\right)\right)$ lies on the hyperbola

ie $(s(-\sin\theta), s\cos\theta)$ " " .. then,

$$\frac{s^2\sin^2\theta}{a^2} - \frac{s^2\cos^2\theta}{b^2} = 1 \quad — (2)$$

(1) becomes $r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1$ ie $\frac{1}{r^2} = \frac{b^2\cos^2\theta - a^2\sin^2\theta}{a^2b^2}$

and

(2) becomes $s^2\left(\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}\right) = 1$ ie $\frac{1}{s^2} = \frac{b^2\sin^2\theta - a^2\cos^2\theta}{a^2b^2}$

(2)

$$\therefore \frac{1}{r^2} + \frac{1}{s^2} = \frac{b^2(\sin^2\theta + \cos^2\theta) - a^2(\sin^2\theta + \cos^2\theta)}{a^2b^2}$$

$$= \frac{b^2 - a^2}{a^2b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2}$$ as required.

(ii) Now, $OP \perp OQ$. If P is $(r\cos\theta, r\sin\theta)$

and Q is $\left(s\cos\left(\theta+\frac{\pi}{2}\right), s\sin\left(\theta+\frac{\pi}{2}\right)\right)$

ie $(-s.\sin\theta, s\cos\theta)$

$m_{OP} = \dfrac{r\sin\theta}{r\cos\theta}$

$= \tan\theta$

$m_{OQ} = \dfrac{s\cos\theta}{-s\sin\theta} = -\cot\theta$

Since $m_{OP} \times m_{OQ} = \tan\theta . -\cot\theta = -1$. $\therefore$ $OP \perp OQ$.

Now $OP^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

(2)

and $OQ^2 = s^2\sin^2\theta + s^2\cos^2\theta = s^2$

$\therefore \dfrac{1}{OP^2} + \dfrac{1}{OQ^2} = \dfrac{1}{r^2} + \dfrac{1}{s^2}$

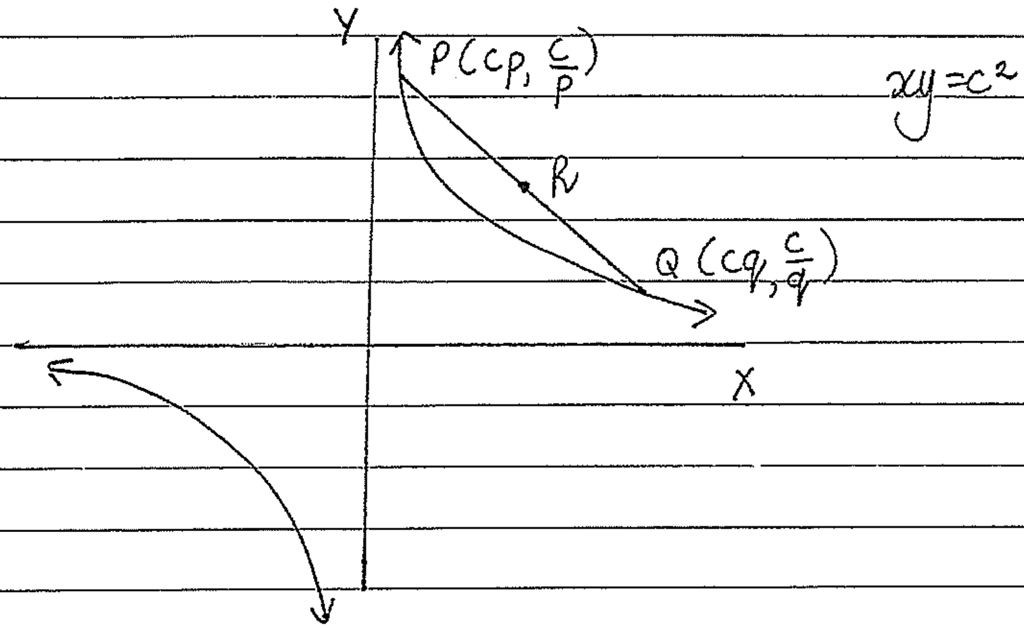
$= \dfrac{1}{a^2} - \dfrac{1}{b^2}$ from part (i).

Since a^2 and b^2 are constants independent of θ on the hyperbola,

$\therefore \dfrac{1}{OP^2} + \dfrac{1}{OQ^2}$ is independent of the position of P and Q.

as required.

Q12. $P(cp, \frac{c}{p})$ $Q(cq, \frac{c}{q})$



$d(PQ) = k$

$$[d(PQ)]^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$$

$$= (c(p-q))^2 + \left(c\left(\frac{1}{p} - \frac{1}{q}\right)\right)^2$$

$$= c^2(p-q)^2 + c^2\left(\frac{1}{p} - \frac{1}{q}\right)^2$$

$$= c^2(p^2 - 2pq + q^2) + c^2\left(\frac{1}{p^2} - \frac{2}{pq} + \frac{1}{q^2}\right) \quad — ①$$

$$= k^2$$

Now R, the midpoint of PQ, is: $\left(\frac{c(p+q)}{2}, \frac{c\left(\frac{1}{p} + \frac{1}{q}\right)}{2}\right)$

Thus, at R, $x = \dfrac{c(p+q)}{2}$, $y = \dfrac{c}{2}\left(\dfrac{1}{p}+\dfrac{1}{q}\right)$

Now, from (1), $PQ^2 = k^2$

$$= c^2\left((p^2+2pq+q^2)-4pq\right)$$

$$+c^2\left(\left(\frac{1}{p^2}+\frac{2}{pq}+\frac{1}{q^2}\right)-\frac{4}{pq}\right) \qquad 1$$

ie $$k^2 = c^2\left((p+q)^2-4pq\right)$$

$$+c^2\left(\left(\frac{1}{p}+\frac{1}{q}\right)^2-\frac{4}{pq}\right) \qquad -(2)$$

Now $x = \dfrac{c(p+q)}{2}$ and $y = \dfrac{c}{2}\left(\dfrac{1}{p}+\dfrac{1}{q}\right)$

$\therefore\ p+q = \dfrac{2x}{c}$ $-(3)$ $\therefore\ \dfrac{2y}{c} = \dfrac{1}{p}+\dfrac{1}{q}.$ $-(4)$ 1

Also, $$xy = \frac{c^2}{4}(p+q)\left(\frac{1}{p}+\frac{1}{q}\right)$$

$$= \frac{c^2}{4}\left(1+\frac{p}{q}+\frac{q}{p}+1\right) \qquad 1$$

$$= \frac{c^2}{4}\left(2+\frac{p^2+q^2}{pq}\right)$$

and $$\frac{y}{x} = \frac{\frac{1}{p}+\frac{1}{q}}{p+q} = \frac{\frac{p+q}{pq}}{p+q} = \frac{1}{pq}. \qquad -(5)$$

and $pq = \dfrac{x}{y}$ $-(5)'$

So, Substituting ③, ④ and ⑤ in ② yields

$$k^2 = c^2\left(\left(\frac{2x}{c}\right)^2 - \frac{4x}{y}\right) + c^2\left(\left(\frac{2y}{c}\right)^2 - \frac{4y}{x}\right)$$

i.e. $$k^2 = c^2\left(\frac{4x^2}{c^2} - \frac{4x}{y}\right) + c^2\left(\frac{4y^2}{c^2} - \frac{4y}{x}\right)$$ 1

$$k^2 = 4x^2 - \frac{4xc^2}{y} + 4y^2 - \frac{4c^2y}{x}$$

$\times\ xy$

$$k^2xy = 4x^3y - 4x^2c^2 + 4xy^3 - 4c^2y^2$$

$$= 4xy(x^2+y^2) - 4c^2(x^2+y^2)$$

$$k^2xy = (x^2+y^2)(4xy - 4c^2)$$

i.e. $$k^2xy = 4(xy-c^2)(x^2+y^2)$$ 1

which is the locus of R

⑥