# Solutions – Intermediate Division

2015

1. (Also J5)
The area is $\frac{1}{2} \times 12 \times 2 = 12\,\mathrm{cm}^2$,

hence (B).

2. (Also J7)
The movie lasts for $120 + 20 = 140$ minutes, so each half is $140 \div 2 = 70$ minutes,

hence (B).

3. $p^2 = 121$ and $q^2 = 16$, so $p^2 - q^2 = 105$,

hence (A).

4. $2015 - 20.15 = 1994.85$,

hence (C).

5. There are 5 twenty-cent coins for every dollar, and $2015 \div 5 = 403$,

hence (D).

6. (Also J10)
The table below shows that Ben leaves after calling out the number 7, Eve leaves after calling out the number 14, Ana leaves after calling out the number 21, and Con leaves after calling out the number 28. Therefore, Dan is the last person remaining at the table.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| A | B | C | D | E | A | B | C | D | E | A | C | D | E |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| A | C | D | A | C | D | A | C | D | C | D | C | D | C |

hence (D).

7. For every 4 cows there are 6 horses and 3 goats, so the ratio of goats to horses is $1 : 2$,

hence (E).

8. Since $141 \div 12 = 11\text{r}9$, Warren has completed 11 floors and 9 windows. Since he moves down after each completed floor, he has moved down 11 floors to floor number $38 - 11 = 27$,

hence (D).

9. The 20 blue and yellow lollies are $\frac{2}{3}$ of the packet, so there are 30 lollies in the packet. Of these 15 are yellow, which is $\frac{1}{2}$ of the total,

hence (C).

**10.** The $49\,\text{cm}^2$ of the large square has $1 + 4 = 5\,\text{cm}^2$ in the two small squares and then two equal pentagons. So the shaded pentagon has area $(49 - 5) \div 2 = 22\,\text{cm}^2$,

hence (C).

**11.** (Also J17, S6)

*Alternative 1*

Jenna must leave out a longer side and Dylan a shorter side, where the longer side is 8 cm longer than the shorter side. So the sides are $x$ cm and $(x + 8)$ cm. Then Jenna's measurement is $2x + x + 8 = 80$, so that $3x = 72$ and $x = 24$. The rectangle is 24 cm by 32 cm. The perimeter is then $2 \times 24 + 2 \times 32 = 112\,\text{cm}$,

hence (A).

*Alternative 2*

Suppose the rectangle has width $w$ and height $h$. Dylan's and Jenna's measurements are $2w + h$ and $2h + w$. Adding these, $3w + 3h = 80 + 88 = 168$ and so $w + h = 168 \div 3 = 56$. Then the perimeter is $2(w + h) = 112\,\text{cm}$,

hence (A).

**12.** (Also S10)

Eight days is $8 \times 24 = 192$ hours. Hence the speed in kilometres per hour is $\frac{11500}{192} \approx \frac{12000}{200} = 60\,\text{km/h}$,

hence (E).

**13.** (Also UP20, J14)

Faces C and T must be opposite, ruling out (B) and (C). With A upright, the face to the right is M, ruling out (D). With C upright, the face above is A, ruling out (E). (A) is possible with H opposite face A,

hence (A).

**14.** (Also J22, S7)

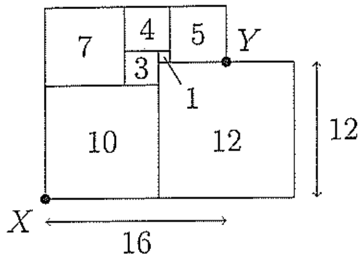
The score is a multiple of 6 if one of the dice is 6 or if one of the dice is even (2, 4 or 6) and the other is 3. We tabulate these possibilities amongst the 36 equally likely rolls.

Second dice

| First dice | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | | | | | ✓ |
| 2 | | | ✓ | | | ✓ |
| 3 | | ✓ | | ✓ | | ✓ |
| 4 | | | ✓ | | | ✓ |
| 5 | | | | | | ✓ |
| 6 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |

Then the probability that the score is a multiple of 6 is $\frac{15}{36} = \frac{5}{12}$,

hence (B).

**15.** 


The sides of the squares have length 1 cm, 3 cm, 4 cm, 5 cm, 7 cm, 10 cm and 12 cm, as shown, so that $XY$ is the hypotenuse of a $12 : 16 : 20$ triangle,

hence (E).

**16.** There are 9 possibilities for the hundreds digit.

For each of these 9 possibilities there are 9 possibilities for the tens digit, since the first digit is excluded, but 0 is possible. So there are 81 possibilities for the first two digits.

For each of these 81 possibilities, there are 8 possibilities for the units digit. Then there are $9 \times 9 \times 8 = 648$ possibilities in all,

hence (C).

**17.** *Alternative 1*

From the tilt of the two beams, these numbers are positive:

$$\begin{array}{rclcll} & & Y - X & > & 0 & \text{(lower beam)} \\ (2X + Y) - (2Y + Z) & = & 2X - Y - Z & > & 0 & \text{(upper beam)} \\ (Y - X) + (2X - Y - Z) & = & X - Z & > & 0 & \text{(sum of both)} \end{array}$$

Then $Z < X < Y$,

hence (E).

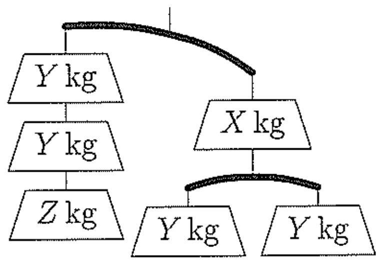
*Comment*

Although all solutions have $Z < X < Y$, not all situations with $Z < X < Y$ are solutions. For instance, if $Z = 1$, $X = 2$ and $Y = 4$ then $Z < X < Y$ but the upper beam has 9 kg on the left and 8 kg on the right, so this is not a solution. On the other hand, if $Z = 1$, $X = 3$ and $Y = 4$ then $Z < X < Y$, and now the lower beam has $4 > 3$ and the upper beam has $9 < 10$, so this is a solution.

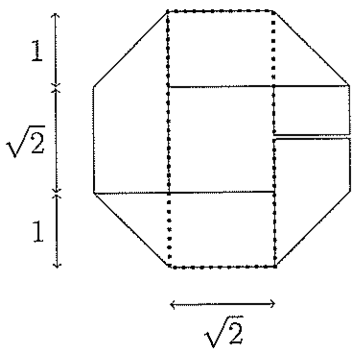
*Alternative 2*

Clearly $Y$ kg is heavier than $X$ kg. Replace the lower-right $X$ kg with $Y$ kg, and the top beam will still tilt down on the right. Then it is clear that $X$ kg is heavier than $Z$ kg. That is, $Z < X < Y$,

hence (E).




**18.** The side of the octagon is $\sqrt{2}$.



One edge of the strip becomes a rectangle of width $\sqrt{2}$ and height $2+\sqrt{2}$, and thus has perimeter $4+4\sqrt{2}$,

hence (C).

**19.** (Also J24)

We claim that it is possible to pay exactly for any amount up to 200 cents with the following 10 coins: one 1c coin, one 2c coin, two 4c coins, one 10c coin, one 20c coin, and four 40c coins.

It is easy to check that the three smaller denominations can be used to pay for any amount up to 10c. It is also easy to check that the three larger denominations can be used to pay for any multiple of 10c up to 190c. Combining these two facts, we see that it is possible to pay exactly for any amount up to 200c.

To see that it is not possible with just 9 coins, consider 199c. The smallest number of coins required to make 199c is 9, with $40+40+40+40+20+10+4+4+1=199$. However with these 9 coins, there are several amounts that are not possible, such as 2c or 6c. So 10 coins are required,

hence (B).

**20.** *Alternative 1*

Let the large triangle have height $h$ so that its area is $A=\frac{1}{2}\times 3\times h=\frac{3}{2}h$.

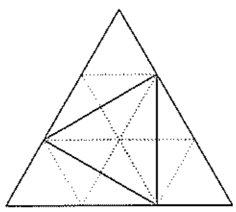
The lower-left triangle has height $\frac{h}{3}$ and area $A_1=\frac{1}{2}\times 2\times\frac{h}{3}=\frac{h}{3}$.

Consequently the centre triangle has area $A_2=A-3A_1=\frac{3}{2}h-h=\frac{1}{2}h$, which is $\frac{1}{3}$ of the area $A$,

hence (B).

*Alternative 2*

Draw a grid of unit equilateral triangles on the figure.



Measured in unit equilateral triangles, the large triangle has area 9 and the shaded triangle has area $6\times\frac{1}{2}=3$,

hence (B).

**21.** Suppose the mean is $x-1$, the median is $x$ and the mode is $x+1$. The numbers must be

| | | | | | |
|---|---|---|---|---|---|
| $a$ | $b$ | $x$ | $x+1$ | $x+1$ | $\rightarrow$ Sum $5x-5$ |

Then $a+b=5x-5-(3x+2)=2x-7$.

The largest $b$ can be is $x-1$ so the smallest $a$ can be is $x-6$.

So the greatest the range can be is $(x+1)-(x-6)=7$,

hence (D).

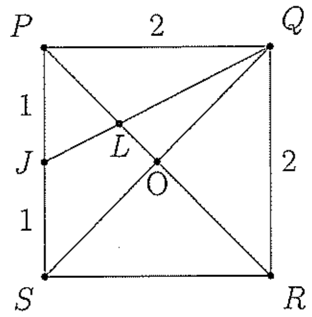
22. *Alternative 1*

In triangles $\triangle PJL$ and $\triangle RLQ$ there are corresponding angles equal: $\angle LPJ = \angle LRQ = 45°$ and $\angle PLJ = \angle RQL$. Thus $\triangle PJL$ and $\triangle RQL$ are similar. Then $\frac{RL}{PL} = \frac{RQ}{PJ} = 2$ so that $RL = 2PL$ and $PL = \frac{1}{3}PR = \frac{1}{3}\sqrt{8} = \frac{2}{3}\sqrt{2}$,

hence (E).

*Alternative 2*

Draw the other diagonal $SQ$ and the centre $O$.




Then $PO$ and $JQ$ are medians of $\triangle PQS$ and $PO = \sqrt{2}$. Since medians cut each other one-third of the way from the side to the vertex, $PL = \frac{2}{3}PO = \frac{2}{3}\sqrt{2}$,

hence (E).

23. *Alternative 1*

Each digit from 0 to 9 occurs in the hundreds position the same number of times, in the tens position the same number of times, and in the units position the same number of times. Given that there are 1000 integers, each digit must occur $1000 \div 10 = 100$ times in each position. Therefore, the sum of all of the digits in the integers from 0 to 999 is

$$3 \times 100 \times (0 + 1 + 2 + \cdots + 9) = 13\,500.$$

So the average of the 1000 numbers that André wrote down is $13\,500 \div 1000 = 13.5$,

hence (A).

*Alternative 2*

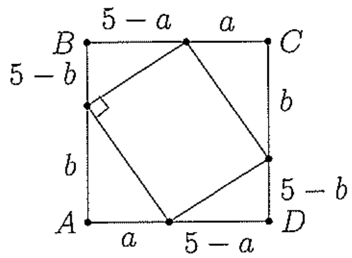
The integers can be arranged in pairs $(0, 999)$, $(1, 998)$, $\ldots$, so that $n$ is paired with $999 - n$ for $0 \leq n \leq 499$. In each pair, the digits of $n$ and $999 - n$ add to 9 in each column (ones, tens, hundreds) and so add to 27 in all. Hence the average digit sum in the pair is $27 \div 2 = 13.5$. Since all pairs have the same digit-sum average of 13.5, the complete list also has this average,

hence (A).

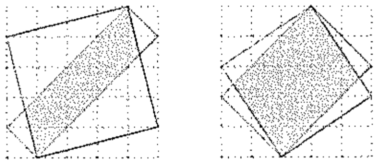
24. *Alternative 1*

There are 4 choices for each vertex, and so $4^4 = 256$ quadrilaterals. However this includes some rectangles (inclusive of the squares).

To count the rectangles, consider lengths as shown:



Due to similar triangles, $\frac{a}{b} = \frac{5-b}{5-a}$, so that $a(5-a) = b(5-b)$.
If $a = 1$ or $a = 4$, then $b^2 - 5b + 4 = (b-1)(b-4) = 0$ so $b = 1$ or $b = 4$.
If $a = 2$ or $a = 3$, then $b^2 - 5b + 6 = (b-2)(b-3) = 0$ so $b = 2$ or $b = 3$.
So there are 8 rectangles, the four shown below and their mirror images.

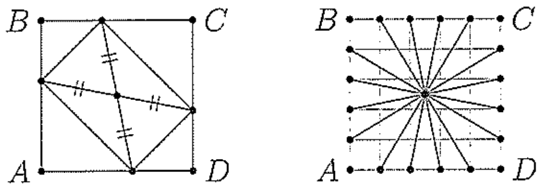


The number of quadrilaterals that are not rectangles is $256 - 8 = 248$,

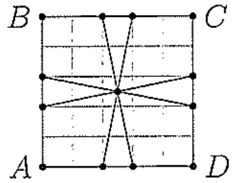
hence (D).

*Alternative 2*

As with the first solution, we count the rectangles in the set of $4^4 = 256$ quadrilaterals. The diagonals of a rectangle must be of equal length and bisect each other at the centre of $ABCD$. So the two diagonals are chosen from the lines shown on the right.




There are two different diagonal lengths: $\sqrt{26}$ and $\sqrt{34}$.
For diagonals of length $\sqrt{26}$, there are two possible left-to-right diagonals and two possible top-to-bottom diagonals, as shown.
So there are four rectangles with $\sqrt{26}$ length diagonals.
Similarly there are four rectangles with $\sqrt{34}$ length diagonals.




In all, there are $256 - 8 = 248$ quadrilaterals that are not rectangles,

hence (D).

**25.** Elena is twice as fast as Nicolai on the ceiling and 1.5 times as fast as Nicolai on the walls. So the most efficient plan will have Elena painting as much of the ceiling as possible.
If Elena starts on the ceiling and Nicolai starts on the walls, then after one hour, Elena will have completed the ceiling and Nicolai will have completed $\frac{2}{3}$ of the walls, leaving $\frac{1}{3}$ of the walls.
Then Elena helps Nicolai complete the wall painting, with the Elena : Nicolai ratio being 3 : 2. So Elena paints $\frac{3}{5}$ of what is left, which is $\frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$ of the walls, which takes her $\frac{1}{5}$ hours = 12 minutes. In the same time, Nicolai paints $\frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$ of the walls, and so all painting is done.
The painting takes $60 + 12 = 72$ minutes,

hence (A).

**26.** Suppose $n$ columns are made. Then the number of horizontal matches is

$$2+3+4+\cdots+(n+1)=\frac{n}{2}(n+3)$$

and the number of vertical matches is

$$1+2+\cdots+n+n=\frac{n}{2}(n+1)+n=\frac{n}{2}(n+3)\ .$$

Consequently the number of matches used is $n(n+3)=n^2+3n$.
To find the largest $n$ with $n^2+3n<2015$, we note that $44^2=1936$ and $45^2=2025$, so we check $44\times 47=2068$ and $43\times 46=1978$.
Hence 1978 matches are used, leaving $2015-1978=37$ matches,

hence (37).

**27.** (Also S26)
*Alternative 1*
Suppose

$$\frac{1}{3}+\frac{1}{n}=\frac{n+3}{3n}=\frac{a}{b}$$

where $b<n$ and $\gcd\{a,b\}=1$. Let $k=\gcd(n+3,3n)$, the factor that is cancelled. Then $k=\frac{3n}{b}>\frac{3b}{b}=3$.
Since $k$ divides $n+3$, it divides $3n+9$. However, $k$ also divides $3n$, so $k$ is a divisor of 9. Since $k>3$, we have $k=9$. Then $9a=n+3$ and $9b=3n$. Then $n=9a-3$ and $b=3a-1<n$.
So $n$ is one of $6,15,24,33,\ldots$. We can verify that all such $n$ work:

$$\frac{1}{3}+\frac{1}{n}=\frac{1}{3}+\frac{1}{9a-3}=\frac{3a-1}{9a-3}+\frac{1}{9a-3}=\frac{3a}{9a-3}=\frac{a}{3a-1}=\frac{a}{b}.$$

To count these, solve $n<2015$ where $n=9a-3$ and $a$ is a whole number.

$$9a-3<2015 \implies 9a<2018 \implies a<224\frac{2}{9} \implies a\le 224\ ,$$

hence (224).

*Alternative 2*
If $n$ is not a multiple of 3, then the common denominator is $3n$ which has no common factor with the numerator $n+3$. So we check where $n$ is a multiple of 3.

| $n$ | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{3}+\frac{1}{n}$ | $\frac{2}{3}$ | $\frac{3}{6}$ | $\frac{4}{9}$ | $\frac{5}{12}$ | $\frac{6}{15}$ | $\frac{7}{18}$ | $\frac{8}{21}$ | $\frac{9}{24}$ | $\frac{10}{27}$ | $\cdots$ |
| Simplification | | $\frac{1}{2}$ | | | $\frac{2}{5}$ | | | $\frac{3}{8}$ | | $\cdots$ |

The pattern for $n=6,15,24,\ldots$ continues, since cancellation occurs only when the numerator $\frac{n}{3}+1$ is a multiple of 3. In particular, $n$ is 3 less than a multiple of 9.
The last multiple of 3 before 2015 is 2013, which is 3 less than $2016=224\times 9$. So from $n=6=1\times 9-3$ up to $n=2013=224\times 9-3$ there are 224 values of $n$ where the simplification is possible,

hence (224).

**28.** Let the rectangle have sides $x$ and $y$.

$$\begin{aligned}(x+3)(y+2) &= 3xy\\ xy+3y+2x+6 &= 3xy\\ y(2x-3) &= 2x+6\\ y &= \frac{2x+6}{2x-3}\\ &= 1+\frac{9}{2x-3}\end{aligned}$$

So $2x-3=1,3$ or $9$ and $x=2,3$ or $6$.
So the three possible rectangles are $2\times 10$, $3\times 4$ and $6\times 2$ with area $20+12+12=44$ square units,

hence (44).

**29.** (Also S28)
In each fifteen-minute interval starting from noon, a train arrives after 0 minutes, 3 minutes, 5 minutes, 6 minutes, 9 minutes, 10 minutes, 12 minutes and 15 minutes. Between these arrivals, there are two 1-minute intervals, two 2-minute intervals, and three 3-minute intervals. Hence, we have the following facts.

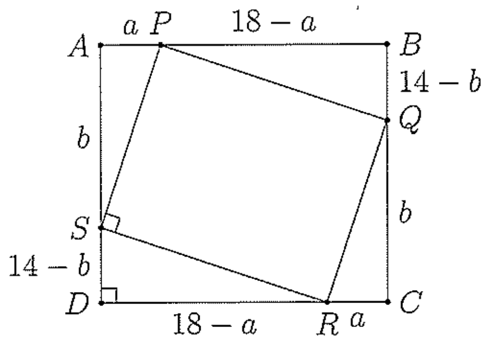
- The probability of a train arriving during a 1-minute interval is $\frac{2\times 1}{15}$ and the average wait is 30 seconds.
- The probability of a train arriving during a 2-minute interval is $\frac{2\times 2}{15}$ and the average wait is 60 seconds.
- The probability of a train arriving during a 3-minute interval is $\frac{3\times 3}{15}$ and the average wait is 90 seconds.

So on average, the number of seconds that I should expect to wait is

$$\frac{2}{15}\times 30+\frac{4}{15}\times 60+\frac{9}{15}\times 90=4+16+54=74,$$

hence (74).

**30.** *Alternative 1*
Label lengths and calculate the area, as shown.




$$\begin{aligned}\text{Area} &= 18\times 14-ab-(18-a)(14-b)\\ &= 18b+14a-2ab\end{aligned}$$

We can assume $1 \leq a \leq 9$. Since $\triangle APS$ and $\triangle BQP$ are similar, $\frac{a}{b} = \frac{14-b}{18-a}$ so $b^2 - 14b + a(18-a) = 0$. This quadratic in $b$ has solution

$$\begin{aligned} b &= \frac{14 \pm \sqrt{14^2 - 4a(18-a)}}{2} \\ &= 7 \pm \sqrt{49 - a(18-a)} \end{aligned}$$

and we check $a = 1, \ldots, 9$ for integer values of $b$:

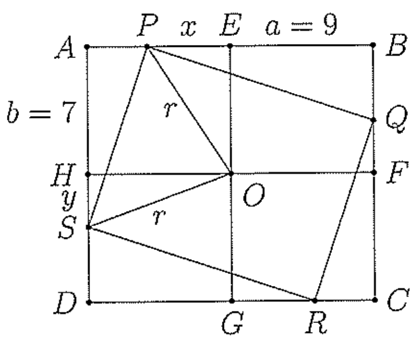
| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $49 - a(18-a)$ | 32 | 17 | 4 | $-7$ | $-16$ | $-23$ | $-28$ | $-31$ | $-32$ |
| $b$ | × | × | $7 \pm 2$ | × | × | × | × | × | × |

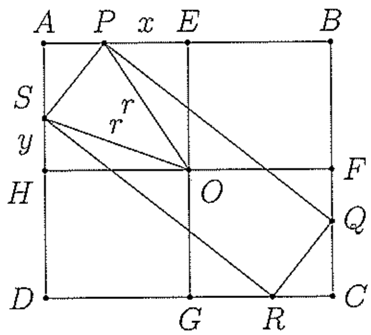
So there are two possibilities $(a, b) = (3, 5)$ and $(a, b) = (3, 9)$. The first gives area 102 and the second gives area 150,

hence (150).

*Alternative 2*

The two rectangles share a centre $O$ and since $PR = QS$, we can write $r = OP = OS$. Label midpoints $E, F, G, H$ and lengths as shown. For given values of $x$ and $y$, there are actually 4 configurations, the two shown and their mirror images.




The area of $PQRS$ in each case can be found by subtracting four triangles from $4ab$, the area of $ABCD$:

$$\begin{aligned} A_1 &= 4ab - (a-x)(b+y) - (a+x)(b-y) = 2ab + 2xy \\ A_2 &= 4ab - (a-x)(b-y) - (a+x)(b+y) = 2ab - 2xy \end{aligned}$$

Since $A_1 \geq A_2$ we consider the left configuration.
In $\triangle EOP$, $x^2 + b^2 = r^2$ and in $\triangle HOS$, $y^2 + a^2 = r^2$. Consequently

$$\begin{aligned} x^2 - y^2 &= a^2 - b^2 = 81 - 49 = 32 \\ (x+y)(x-y) &= 32 \end{aligned}$$

This factorisation of 32 is either $32 \times 1$, $16 \times 2$ or $8 \times 4$, but since $x < 9$ and $y < 7$, the only possibility is where $x + y = 8$ and $x - y = 4$. Then $x = 6$ and $y = 2$.
Consequently the area of $PQRS$ is $A_1 = 2ab + 2xy = 126 + 24 = 150$,

hence (150).