Carlingford High School

Year 11 Mathematics Extension 2

HSC Assessment Task 1

Term 4 2017

Time allowed: 55 minutes

Name:________________________

Instructions:

- Use blue or black pen. Pencil may be used for diagrams **only**
- Board approved calculators may be used
- Show all necessary working
- Marks may be deducted for illegible or badly set out work
- No lending or borrowing

MC	Q6	Q7	Total
/5	/14	/16	/35

Multiple Choice Questions 1-5
Please circle the correct answer

1. What is the value of $(-i)^{-11}$?

(A) 1 (B) -1

(C) i (D) $-i$

2. What value of z satisfies the equation $z^2 = -7 - 24i$?

(A) $4 - 3i$ (B) $-4 - 3i$

(C) $3 - 4i$ (D) $-3 - 4i$

3. The complex number z satisfies $|z - 1| = 2$. What is the greatest distance z can be from the point $-i$ on the Argand diagram?

(A) 4 (B) 2

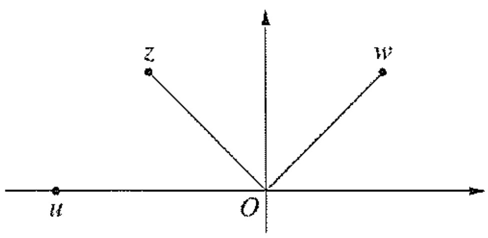
(C) $2 - \sqrt{2}$ (D) $2 + \sqrt{2}$

4. The principal argument of $(1 + i\sqrt{3})^4$ is

(A) $\frac{4\pi}{3}$ (B) $-\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$ (D) $-\frac{2\pi}{3}$

5. The Argand diagram shows the complex numbers w, z and u where w lies in the first quadrant, z lies in the second quadrant and u lies on the negative real axis.



Which of the following could be true?

(A) $u = zw$ and $u = z + w$ (B) $u = zw$ and $u = z - w$

(C) $z = uw$ and $u = z + w$ (D) $z = uw$ and $u = z - w$

Question 6 (14 marks) Start a new page

a) Let $z = 1 - 2i$ and $w = 3 + i$. Find, in the form $x + iy$, showing working,

i) $\overline{zw}$ **1**

ii) $\frac{10}{z}$ **2**

b) Let $\alpha = 1 + i$ and $\beta = \sqrt{3} + i$, and let $z = \frac{\alpha}{\beta}$.

i) Find z in the form $x + iy$. **1**

ii) Express β in modulus-argument form. **2**

iii) Given that α has the modulus-argument form $\alpha = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, find the modulus-argument form of z. **1**

iv) Hence or otherwise, find the exact value of $\sin\frac{\pi}{12}$. **1**

c) i) Find a pair of integers a and b such that $(a + ib)^2 = 5 - 12i$. **1**

ii) Hence, solve $z^2 + (1 + 4i)z - 5 + 5i = 0$. **2**

d) Sketch the region on the Argand diagram where the inequalities **3**

$$|z - \bar{z}| < 2 \text{ and } |z - 1| > 1$$

hold simultaneously.

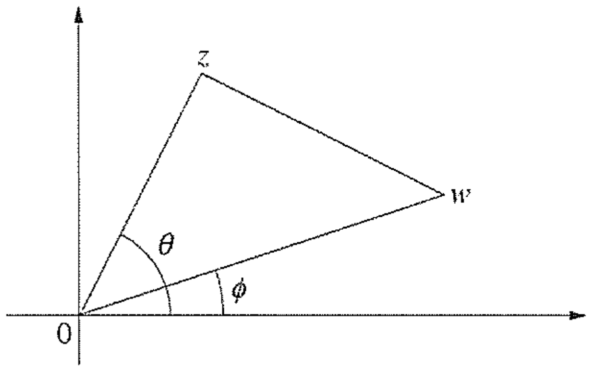
Question 7 (16 marks) Start a new page

a) Sketch the locus of the point $P(x, y)$ representing z in the complex plane, and find its Cartesian equation when

i) $\arg(z + 2i) = -\arg(z + 1)$ **2**

ii) $\arg(z - i + 3) - \arg(z - i - 5) = \frac{\pi}{2}$ **3**

b) The Argand diagram shows complex numbers w and z with arguments ϕ and θ respectively, where $\phi < \theta$. The area of the triangle formed by O, w and z is A.



i) Write an expression for $z\overline{w}$ in terms of $|z|$, $|w|, \phi$ and θ. **1**

ii) Show that $z\overline{w} - w\overline{z} = 4iA$ **3**

c) Let w be a root of $z^5 - 1, w \neq 1$. Find, in simplest form, the real quadratic equation whose roots are $w + w^4$ and $w^2 + w^3$. **2**

d) i) Use de Moivre's theorem to find an expression for $\cos 3\theta$ in terms of $\cos\theta$. **2**

ii) By considering the roots of $2\cos 3\theta = \sqrt{3}$, and your answer to part (i), show that **3**

$$\cos\frac{\pi}{18} + \cos\frac{11\pi}{18} + \cos\frac{13\pi}{18} = 0.$$

End of Exam – Please check your work

Extension 2 Solutions Assessment Task 1

Term 4 2017

Multiple Choice

1. D
2. C
3. D
4. D
5. B

6 a) i

$$\overline{zw} = (1+2i)(3-i)$$
$$= 3+2+6i-i$$
$$= 5+5i$$

ii

$$\frac{10}{z} = \frac{10(1+2i)}{(1-2i)(1+2i)}$$
$$= \frac{10(1+2i)}{5}$$
$$= 2+4i$$

b) i

$$z = \frac{1+i}{\sqrt{3}+i}$$
$$= \frac{(1+i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)}$$
$$= \frac{\sqrt{3}-i+\sqrt{3}i+1}{4}$$
$$= \frac{1}{4}\left[\sqrt{3}+1+(\sqrt{3}-1)i\right]$$

ii

$$\beta = \sqrt{3}+i$$

$|\beta| = \sqrt{3+1}$ $\qquad$ $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$= 2$ $\qquad$ $\arg\beta = \pi/6$

$$\beta = 2\left(\cos\pi/6 + i\sin\pi/6\right)$$

iii

$$z = \frac{\alpha}{\beta} = \frac{\sqrt{2}\operatorname{cis}\pi/4}{2\operatorname{cis}\pi/6}$$
$$= \sqrt{2}/2\operatorname{cis}(\pi/4 - \pi/6)$$
$$= \sqrt{2}/2\operatorname{cis}\pi/12$$

iv) equating imaginary parts from i) and iii) $\frac{\sqrt{2}}{2}\sin(\pi/12) = \frac{1}{4}(\sqrt{3}-1)$

$$\sin\pi/12 = \frac{\sqrt{2}}{4}(\sqrt{3}-1) \quad \text{or} \quad \frac{1}{4}(\sqrt{6}-\sqrt{2})$$

c) i $(a+ib)^2 = 5-12i$

$a^2-b^2 = 5$

$2ab = -12$

$b = -6/a$

$a^2 - 36/a^2 = 5$

$a^4-5a^2-36=0$

$(a^2-9)(a^2+4)$

$a^2>0 \therefore a^2=9$

$a+ib = \pm(3-2i)$

ii By the quadratic formula ← need to show that discriminant is $5-12i$

$$z = \frac{-(1+4i)\pm(3-2i)}{2}$$

$$= \frac{2-6i}{2}, \quad \frac{-4-2i}{2}$$

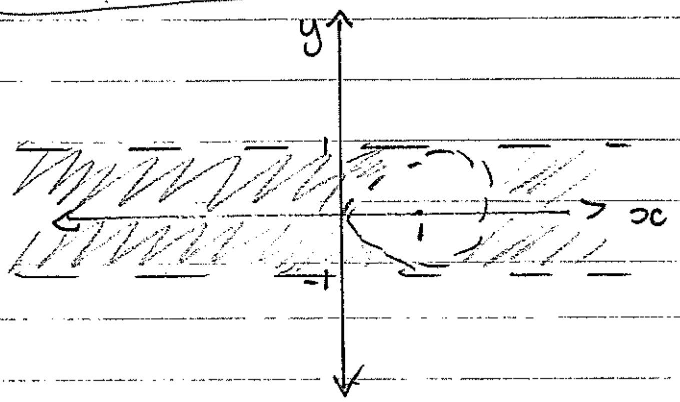
$$= 1-3i, \quad -2-i$$

$\Delta = (1+4i)^2 - 4(-5+5i)$

$= 1-16+8i+20-20i$

$= 5-12i$

d)

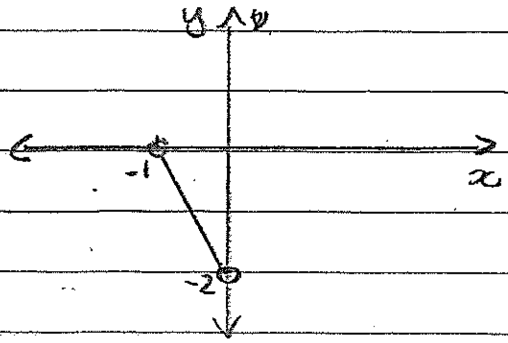


$|z-\bar{z}|<2$

$|2y|<2$

7a) i) $\arg(z+2i) = -\arg(z+1)$

$y = -2x - 2 \quad -1 < x < 0$



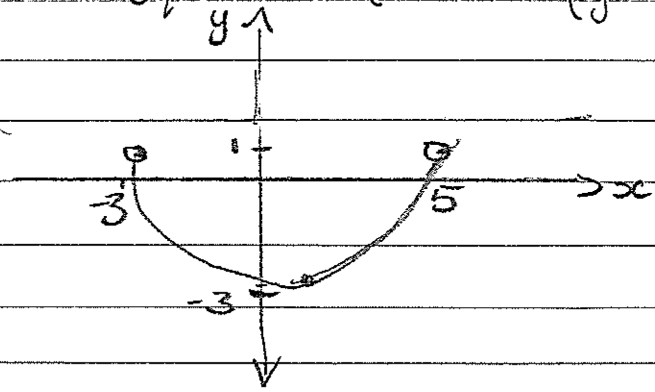
ii $\arg(z-i+3) - \arg(z-i-5) = \frac{\pi}{2}$

The locus of P is a semicircle

Let $A = (-3,1)$ and $B = (5,1)$

$\therefore$ centre is at $(1,1)$

radius = 4 equation $(x-1)^2+(y-1)^2 = 16$, $y<1$



b) i) $z = |z| \operatorname{cis} \theta$

$w = |w| \operatorname{cis} \phi \qquad \bar{w} = |w| \operatorname{cis}(-\phi)$

$z\bar{w} = |z||w| \operatorname{cis}(\theta - \phi)$

ii) $z\bar{w} - w\bar{z} = 2i \operatorname{Im}(z\bar{w})$

$= 2i \cdot \sin(\theta-\phi) \cdot |z||w|$

[or expand $z\bar{w} - w\bar{z} = |z||w|(\operatorname{cis}(\theta-\phi) - \operatorname{cis}(\cancel{\theta}\, \phi - \cancel{\theta}))$

$= \ldots$

~~Bu~~ $A = \frac{1}{2}|z||w| \sin(\theta - \phi)$

$4iA = 2i|z||w| \sin(\theta-\phi)$

$= z\bar{w} - w\bar{z}$

c) $(x-(\omega+\omega^4))(x-(\omega^2+\omega^3))$

$= x^2-(\omega+\omega^2+\omega^3+\omega^4)x+(\omega^3+\omega^4+\omega^6+\omega^7)$

Now $\omega+\omega^2+\omega^3+\omega^4+1=0$

$\therefore \omega+\omega^2+\omega^3+\omega^4=-1$

~~$P = x^2$~~ also $\omega^6=\omega$ and $\omega^7=\omega^2$ (since $\omega^5=1$)

$P(x) = x^2+x-1$

d) Let $z=\cos\theta+i\sin\theta$

$z^3=(\cos\theta+i\sin\theta)^3$

$= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

By de Moivre's theorem $z^3 = cis3\theta$

equating real parts $\cos3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$

$= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$.

ii) $2\cos3\theta=\sqrt{3}$

$\cos3\theta = \sqrt{3}/2$

$3\theta = \pi/6 \pm 2\pi n$

$\theta = \pi/18 \pm \frac{12\pi n}{18}$

take $n=0,1,-1$ $\theta = \pi/18, \frac{13\pi}{18}, -\frac{11\pi}{18}$

~~roots~~ $\therefore \cos(-x)=\cos x$: can take $\cos(\pi/18), (13\pi/18), (\frac{11\pi}{18})$

Let $x=\cos\theta$

$8x^3 - 6x - \sqrt{3} = 0$

If x_1, x_2, x_3 are roots then $x_1+x_2+x_3=0$

(sum of roots)

$\therefore \cos(\pi/18)+\cos(\frac{11\pi}{18})+\cos(\frac{13\pi}{18})=0$.