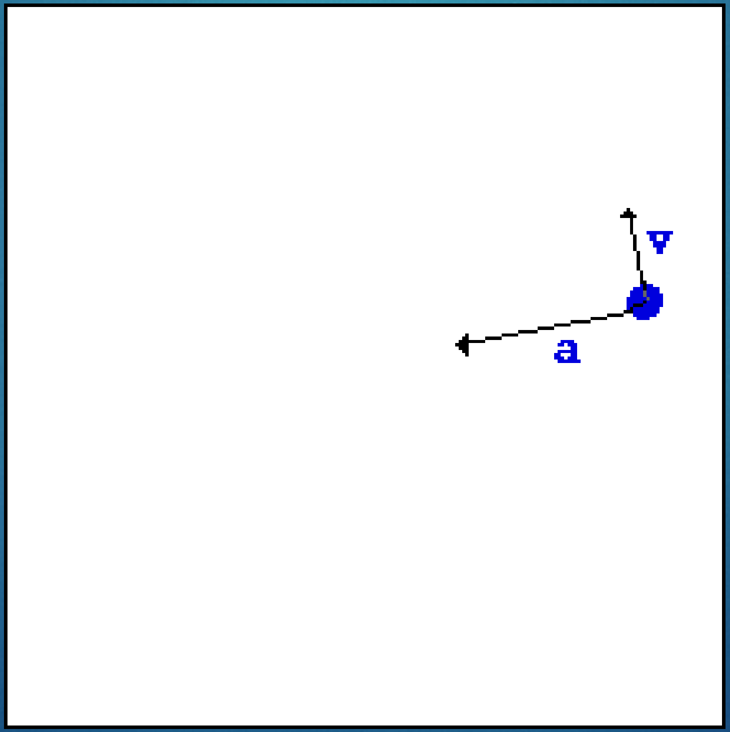
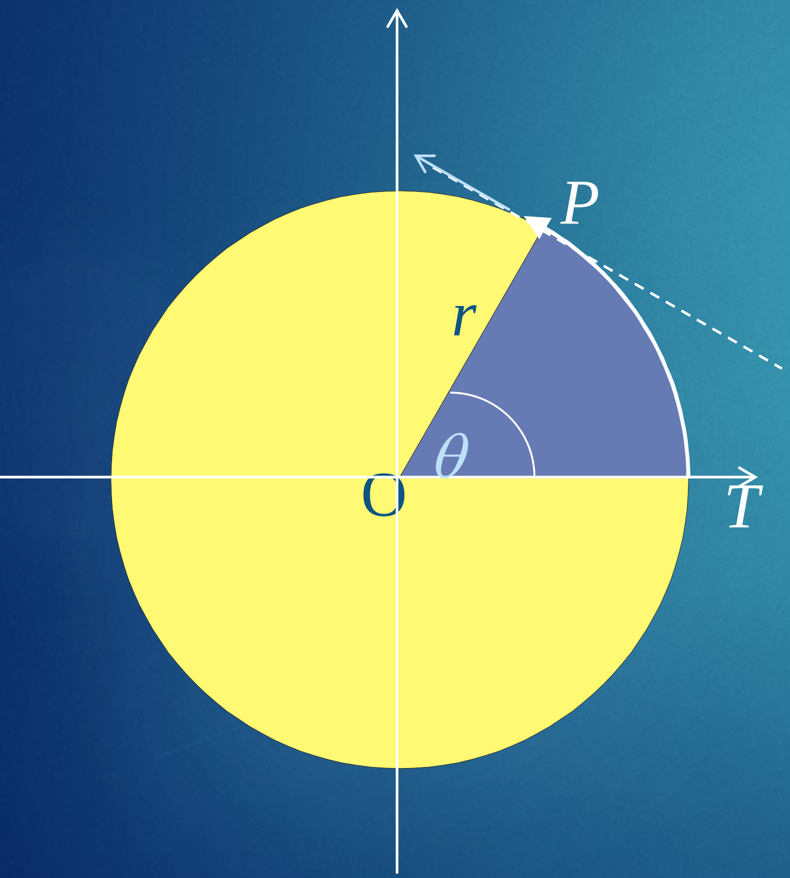
Mechanics

Circular Motion



Circular Motion in a horizontal plane



- *P* moves around a circle of radius r.
- As P moves both the arc length *PT* change and the angle θ changes
- The angular velocity of *P* is given by

$$\boxed{\omega = \frac{d\theta}{dt} = \dot{\theta}}$$

- Force = mass × acceleration

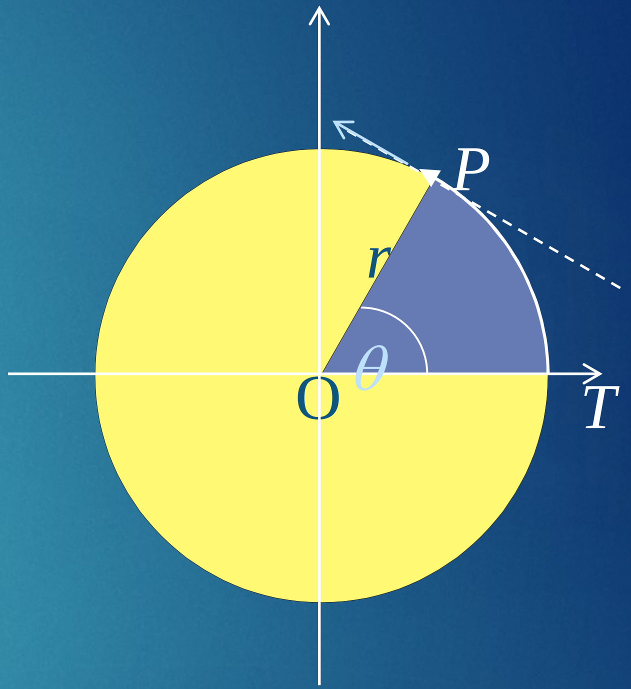
$$\boxed{\omega = \frac{d\theta}{dt} = \dot{\theta}}$$

$$Let\ PT = x$$

$$x = r\theta$$

$$\frac{dx}{dt} = r\frac{d\theta}{dt}$$

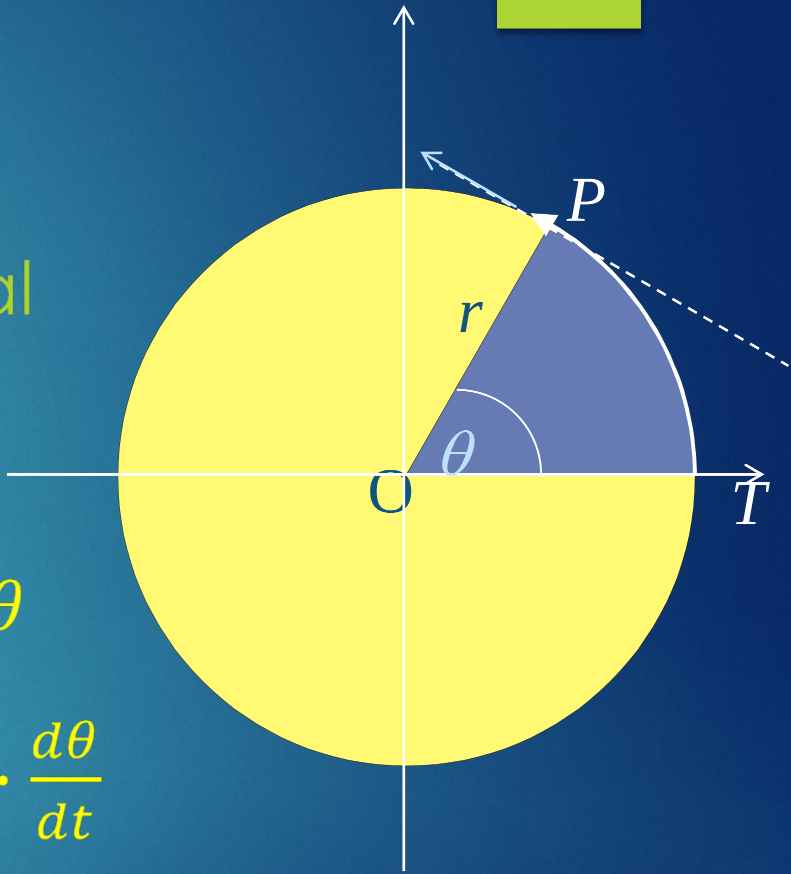
$$\boxed{v = r\omega}$$



This equation is important since it links angular and linear velocity

$$\boxed{v = r\omega}$$

To discuss acceleration we should consider the motion in terms of horizontal and vertical components.



$$x = r\cos\theta\,,\, y = r\sin\theta$$

$$\frac{dx}{dt} = r\frac{d}{dt}\cos\theta\,,\frac{dy}{dt} = r\frac{d}{dt}\sin\theta$$

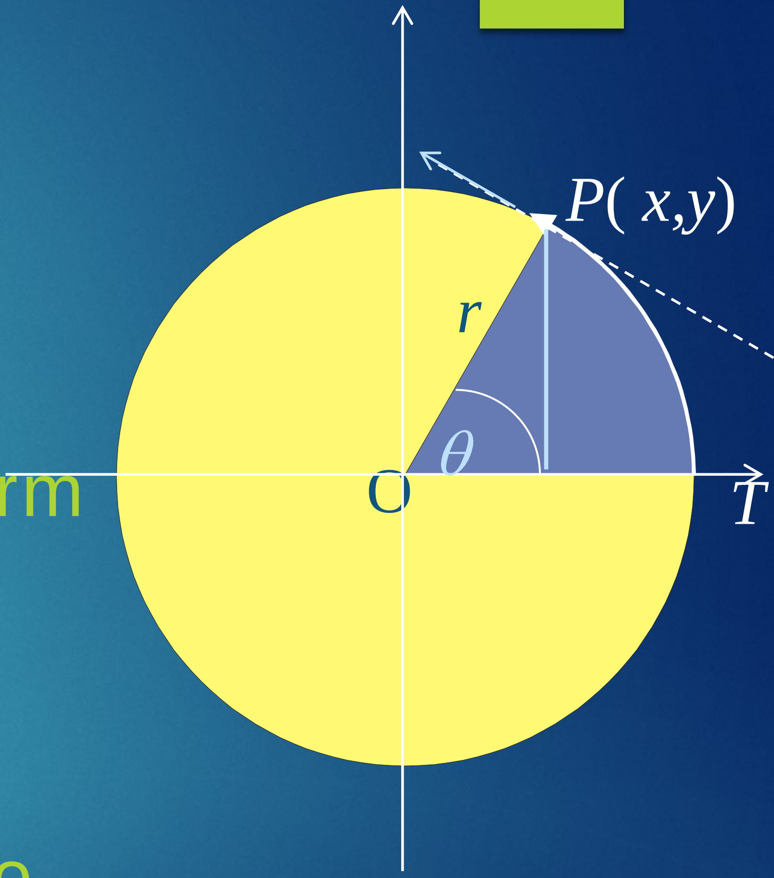
$$\dot{x} = r\frac{d}{d\theta}\cos\theta \cdot \frac{d\theta}{dt},\quad \dot{y} = r\frac{d}{d\theta}\sin\theta \cdot \frac{d\theta}{dt}$$

$$\boxed{\dot{x} = -r\omega\sin\theta\,,\dot{y} = r\omega\cos\theta}$$

$$\boxed{v = r\omega}$$

$$\boxed{\dot{x} = -r\omega\sin\theta, \dot{y} = r\omega\cos\theta}$$

- To simplify life we are going to consider that the angular velocity remains constant throughout the motion (uniform circular motion).
- This will be the case in any problem you do , but you should be able to prove these results for variable angular velocity.



$$\boxed{v = r\omega}$$

$$\boxed{\dot{x} = -r\omega\sin\theta, \dot{y} = r\omega\cos\theta}$$

$$\ddot{x} = -r\omega\frac{d}{dt}\sin\theta, \dot{y} = r\omega\frac{d}{dt}\cos\theta$$

$$\ddot{x} = -r\omega\frac{d}{d\theta}\sin\theta\frac{d\theta}{dt}, \ddot{y} = r\omega\frac{d}{d\theta}\cos\theta\frac{d\theta}{dt}$$

$$\ddot{x} = -r\omega\cos\theta\omega, \ddot{y} = r\omega(-\sin\theta)\omega$$

$$\boxed{\ddot{x} = -r\omega^2\cos\theta, \ddot{y} = -r\omega^2\sin\theta}$$

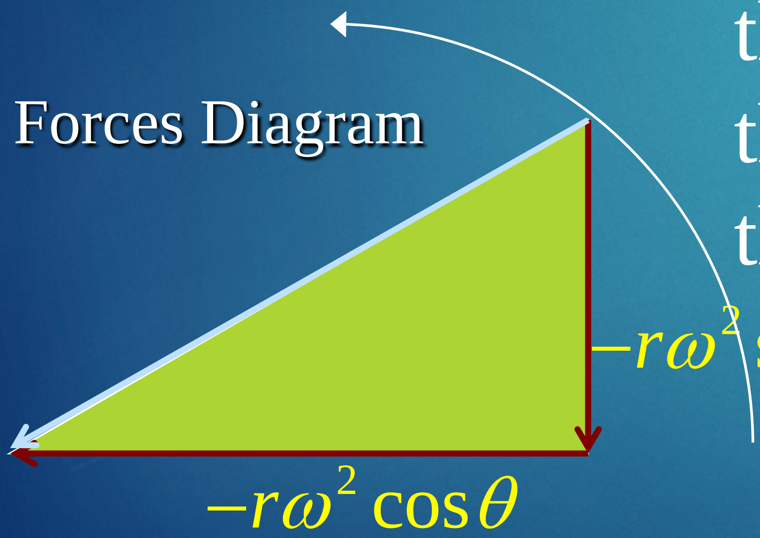
$$\boxed{v = r\omega}$$

$$\boxed{\dot{x} = -r\omega\sin\theta, \dot{y} = r\omega\cos\theta}$$

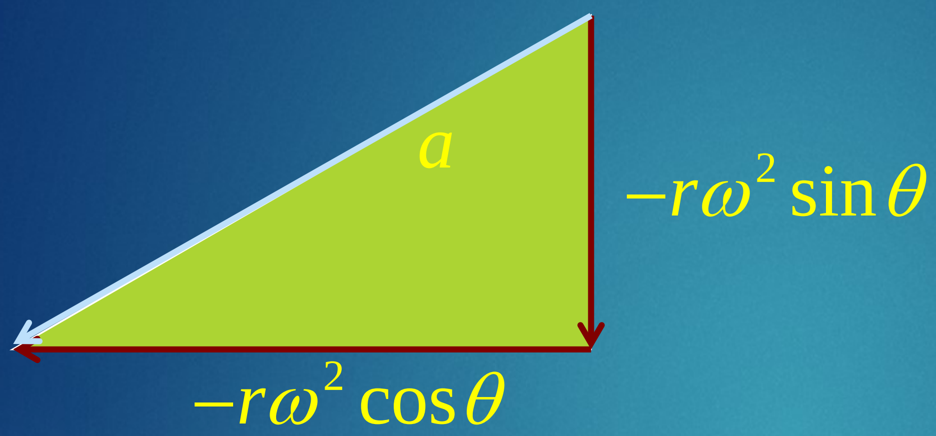
$$\boxed{\ddot{x} = -r\omega^2\cos\theta, \ddot{y} = -r\omega^2\sin\theta}$$

It is important to note that the vectors demonstrate that the force is directed along the radius towards the centre of the circle.

Forces Diagram



$$\boxed{\ddot{x} = -r\omega^2 \cos\theta, \ddot{y} = -r\omega^2 \sin\theta}$$



$$a^2 = \left(-r\omega^2 \cos\theta\right)^2 + \left(-r\omega^2 \sin\theta\right)^2$$

$$a = \sqrt{r^2\omega^4 \cos^2\theta + r^2\omega^4 \sin^2\theta}$$

$$a = \sqrt{r^2\omega^4 \left(\cos^2\theta + \sin^2\theta\right)}$$

$$a = \sqrt{r^2\omega^4}$$

$$\boxed{a = r\omega^2}$$

$$a = r\omega^2 \text{ but } v = r\omega \rightarrow \omega = \frac{v}{r}$$

$$a = r\left(\frac{v}{r}\right)^2$$

$$\boxed{a = \frac{v^2}{r}}$$

Summary

$$\boxed{\omega = \frac{d\theta}{dt} = \dot{\theta}}$$

(angular velocity)

$$\boxed{v = r\omega}$$

(links angular velocity and linear velocity)

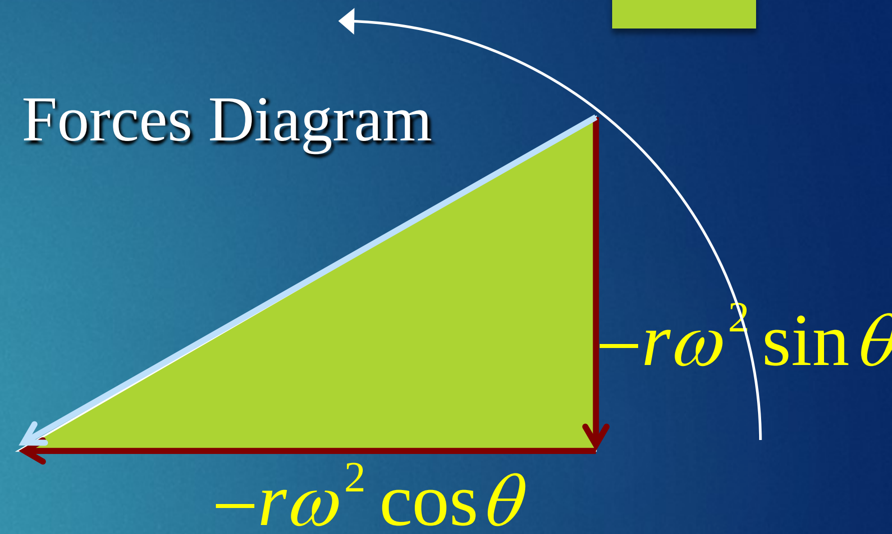
$$\boxed{\dot{x} = -r\omega\sin\theta, \dot{y} = r\omega\cos\theta}$$

(horizontal and vertical components of acceleration)

$$\boxed{a = r\omega^2} \qquad \boxed{a = \frac{v^2}{r}}$$

(gives the size of the force towards the centre)

Forces Diagram



force is directed along the radius towards the centre of the circle.

1. A body of mass 2kg is revolving at the end of a light string 3m long, on a smooth horizontal table with uniform angular speed of 1 revolution per second.

Draw a neat diagram to represent the forces.

(a) Find the tension in the string.

(b) If the string would break under a tension of equal to the weight of 20 kg, find the greatest possible speed of the mass.

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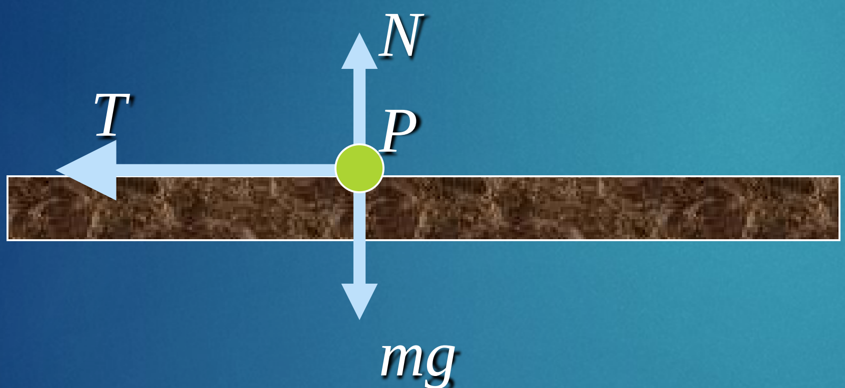


$$T = F = ma$$

$$T = mr\omega^2$$

$$= 2 \times 3 \times (2\pi)^2$$

$$= 24\pi^2 N$$

The tension in the string is the resultant of the forces acting on the body.

(b) If the string would break under a tension of equal to the weight of 20 kg, find the greatest possible speed of the mass.

$$T \leq 20g$$

$$T = \frac{mv^2}{r}$$

$$\frac{mv^2}{r} \leq 20g$$

$$v^2 \leq 20g \times \frac{r}{m}$$

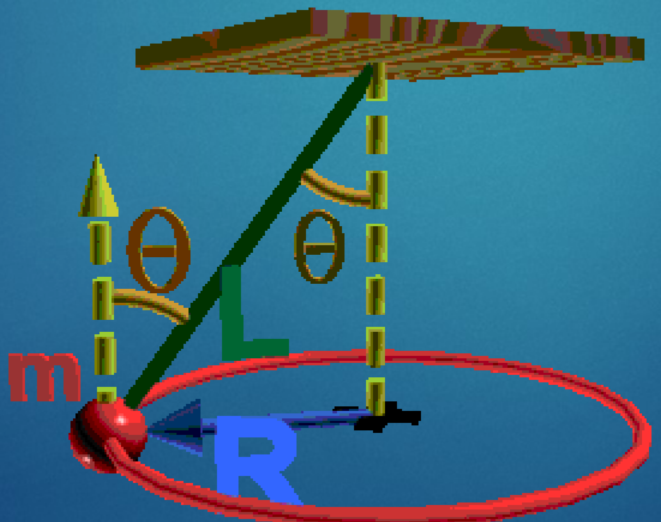
$$v^2 \leq 20g \times \frac{3}{2}$$

$$v \leq \sqrt{30g} \ m/s$$

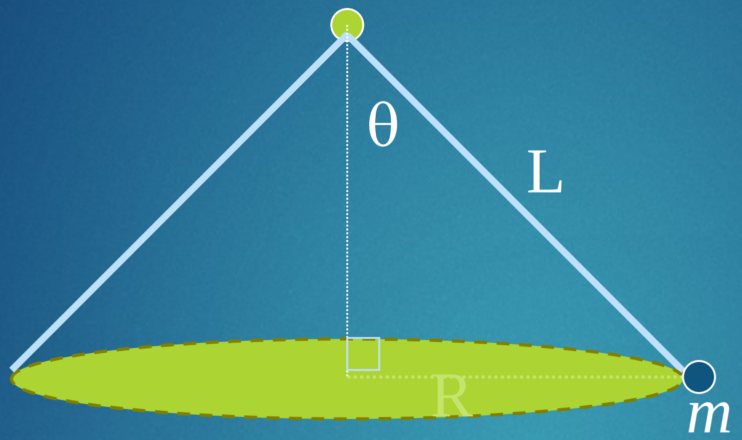
Conical Pendulum

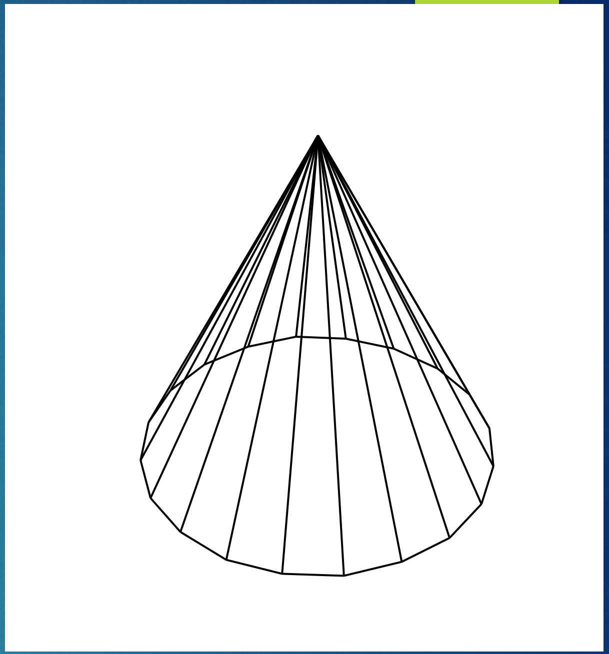
If a particle tied by a string to a fixed point moves so that the string describes a cone, and the mass at the end of the string describes a horizontal circle, then the string and the mass describe a conical pendulum.

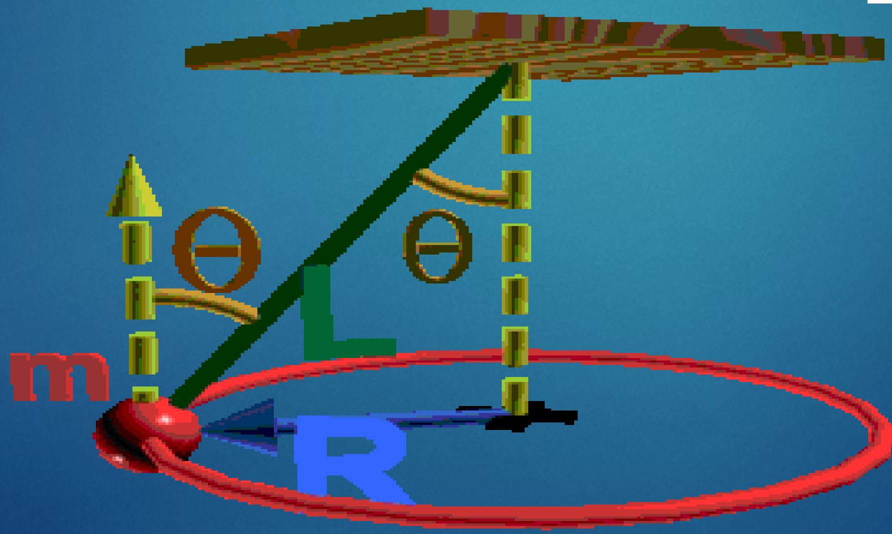
https://www.youtube.com/watch?v=5C4RJlFABic



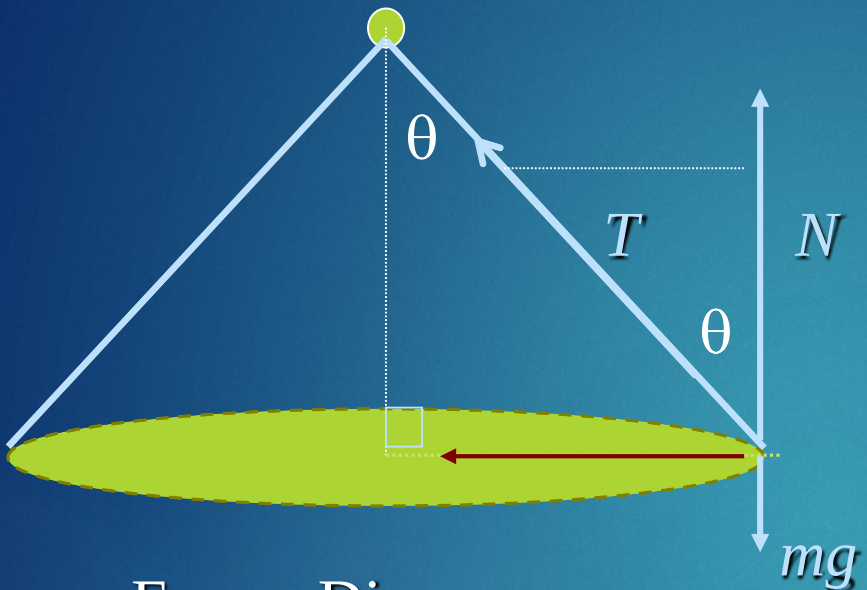
Conical Pendulum



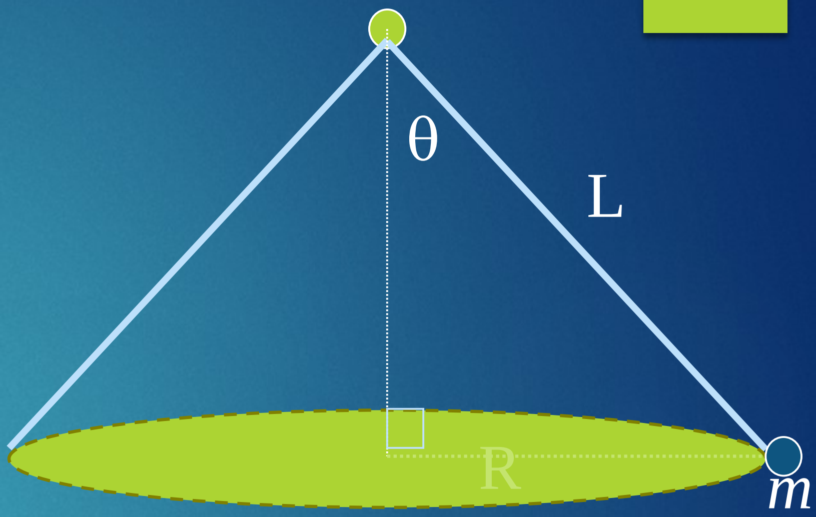




Conical Pendulum



Forces Diagram



Dimensions Diagram

Vertically

$N + (-mg) = 0$

$N = mg$

but $\cos\,\theta = {}^{N}/_{T}$

so $N = T\cos\,\theta$

$\therefore\ T\cos\,\theta = mg$

Horizontally

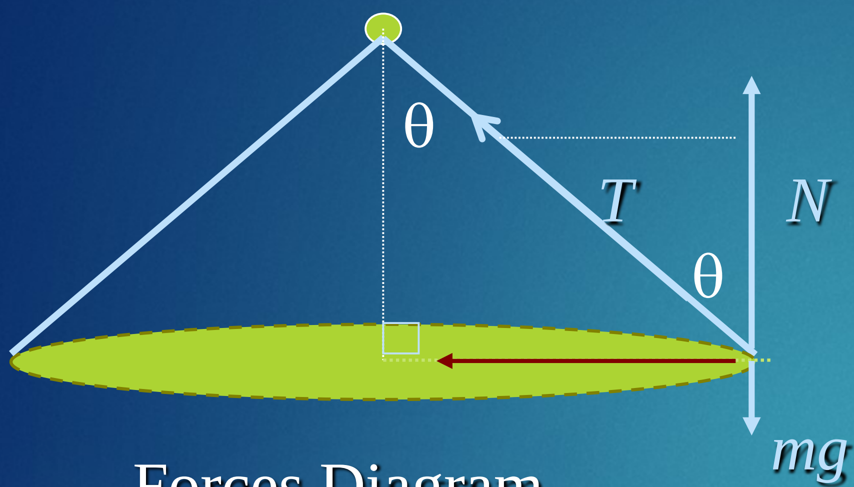
$T\sin\,\theta = mr\omega^2$

Since the only horizontal force is directed along the radius towards the centre

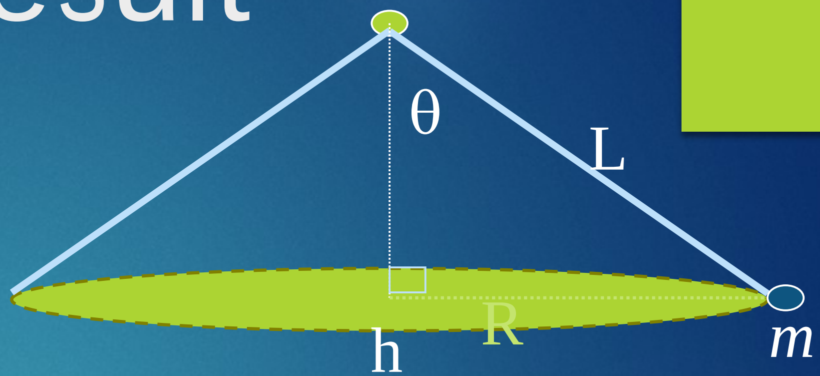
$T\sin\,\theta = mr\omega^2$

$T\cos\,\theta = mg$

An important result



Forces Diagram



$$T\sin\theta = mr\omega^2 \ldots\ldots 1$$
$$T\cos\theta = mg \ldots\ \ldots\ 2$$
$$v = r\omega$$

Now dividing equation 1 by equation 2..

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg}$$

$$\tan\theta = \frac{r\omega^2}{g}$$

$$\tan\theta = \frac{r}{h}$$

$$\frac{r\omega^2}{g} = \frac{r}{h}$$

$$h = \frac{g}{\omega^2}$$

Example 1

A string of length 2 m, fixed at one end *A* carries at the other end a particle of mass 6 kg rotating in a horizontal circle whose centre is 1m vertically below A. Find the tension in the string and the angular velocity of the particle.

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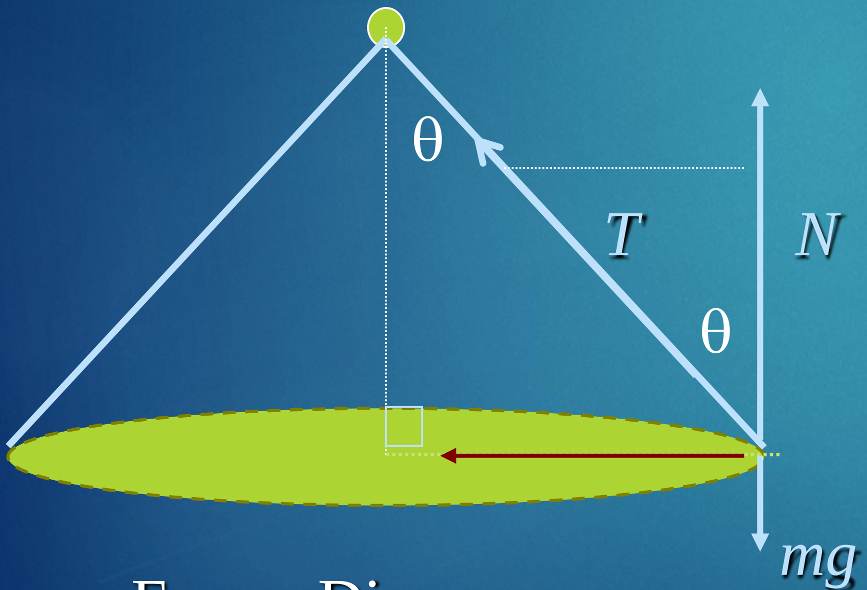

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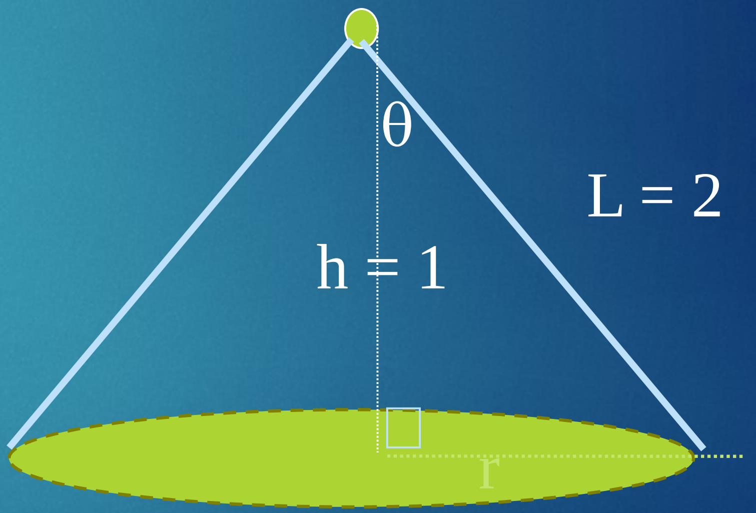
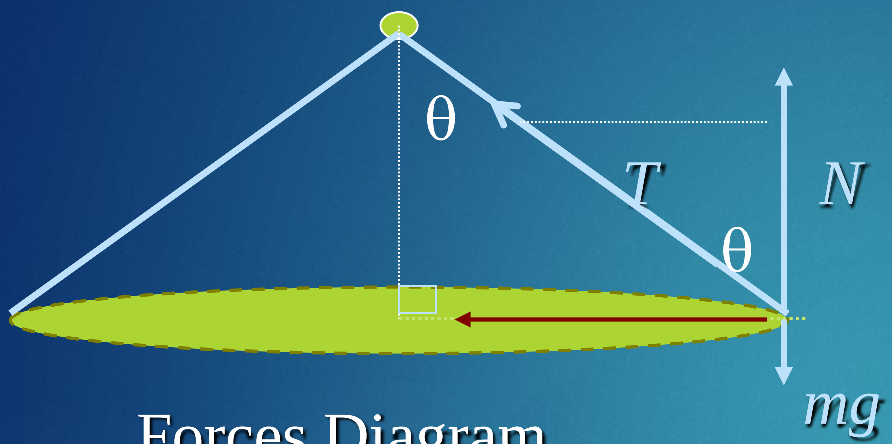


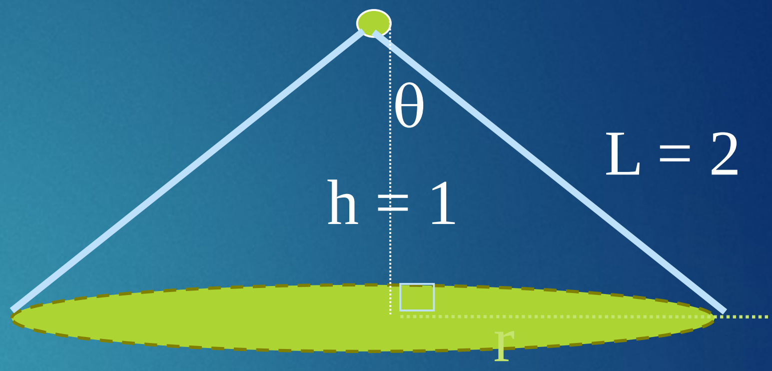
Dimensions Diagram

EXAMPLE 1

FIND THE TENSION IN THE STRING AND THE ANGULAR VELOCITY OF THE PARTICLE.



Forces Diagram



Dimensions Diagram

Vertically

Horizontally

$$T\cos\theta = mg$$

$$T = \frac{mg}{\cos\theta}$$

$$T = \frac{6g}{\frac{1}{2}}$$

$$T = 12g$$

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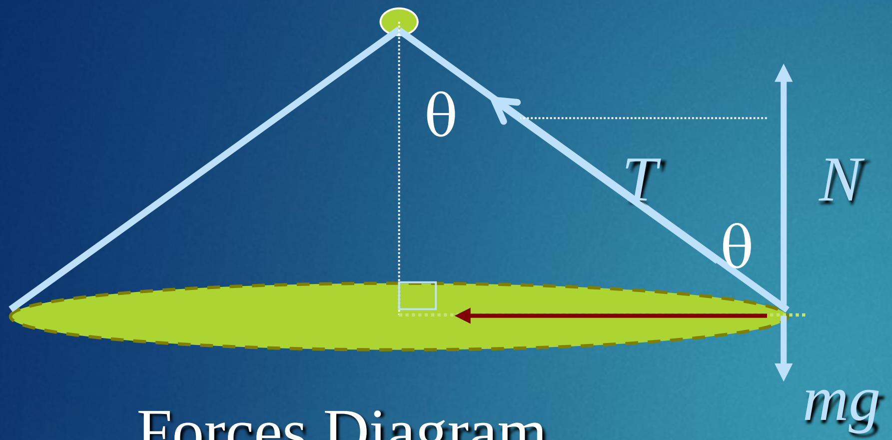

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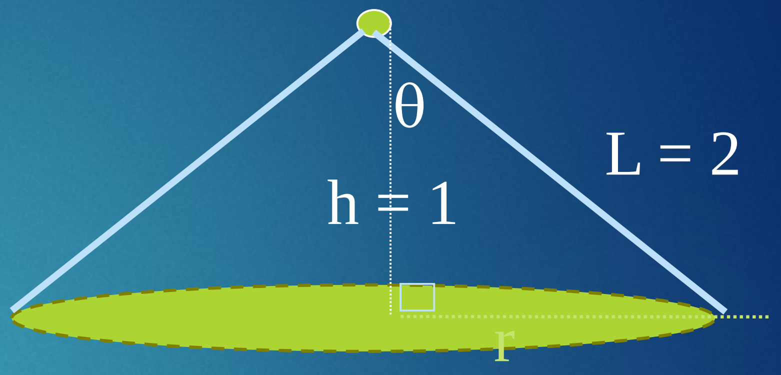


Dimensions Diagram

Vertically	Horizontally
$T\cos\theta = mg$	$T\sin\theta = mr\omega^2$
$T = \dfrac{mg}{\cos\theta}$	$T\sin\theta = mL\sin\theta\omega^2$
$T = \dfrac{6g}{\frac{1}{2}}$	$\omega^2 = \dfrac{T}{mL}$
$T = 12g$	$\omega = \sqrt{\dfrac{12g}{12}} = \sqrt{g}$

Example 2.

A light inextensible string OP is fixed at one end O. A particle P is attached to the other end and moves uniformly in a horizontal circle whose centre is vertically below and at a distance h cm from O.

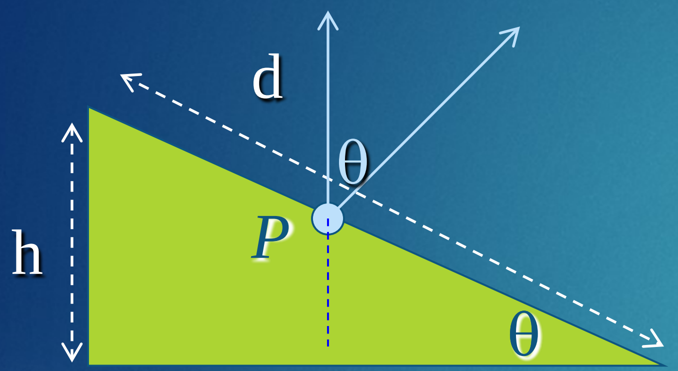
a) Show that the period of the motion is given by $2\pi\sqrt{\frac{h}{g}}$.

b) If the frequency of the rotating particle is decreased from 2 revolutions per second to 1 rev/s, find the distance by which the level of the circle is lowered.

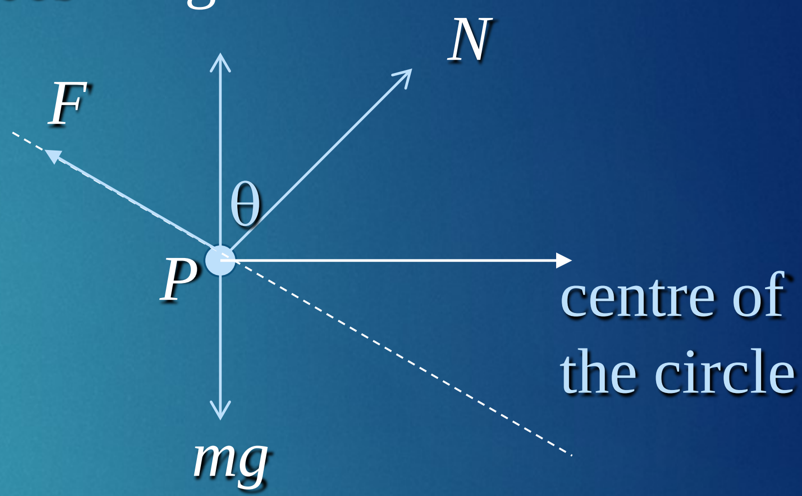
Motion on a banked track

MOTION ON A BANKED TRACK

Dimensions diagram



Forces diagram



Vertical Forces

$$N\cos\theta + F\sin\theta = mg$$

Horizontal Forces

$$N\sin\theta - F\cos\theta = \frac{mv^2}{r}$$

If there is no tendency to slip then $F = 0$

If there is no tendency to slip at $v = v_0$ then $F = 0$ and the equations are …

$$N \sin\theta = \frac{mv_0^{\;2}}{r}$$

$$N \cos\theta = mg$$

$$\tan\theta = \frac{\frac{mv_0^{\;2}}{r}}{mg}$$

$$\tan\theta = \frac{v_0^{\;2}}{rg}$$

$$v_0 = \sqrt{rg\tan\theta}$$

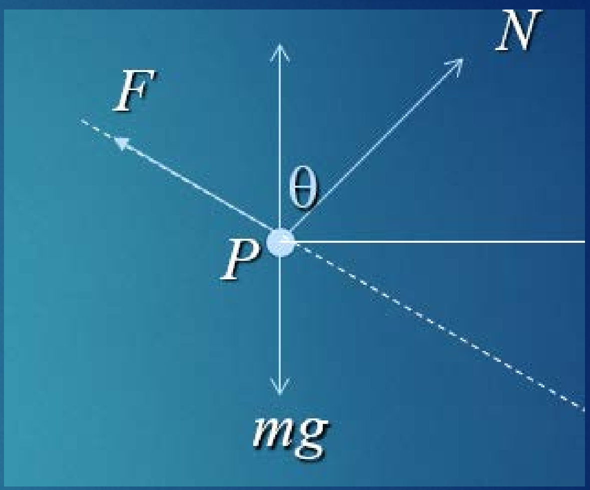
This is the method used by engineers to measure the camber of a road.

Suppose the body is travelling at a speed $v \neq v_0$.

Then there is a friction force F opposing a slide up or down the slope.

$$N\cos\theta + F\sin\theta = mg \qquad (1)$$

$$N\sin\theta - F\cos\theta = \frac{mv^2}{r} \qquad (2)$$



$\sin\theta \times (1) - \cos\theta \times (2)$ ...

$$F = m(g\sin\theta - \frac{v^2}{r}\cos\theta)$$

$$= m\cos\theta\,(\frac{{v_0}^2 - v^2}{r}),$$ since ${v_0}^2 = rg\tan\theta$.

Therefore if $v < v_0$, F is up the slope, opposing a tendency to slide down.

If $v > v_0$, F is down the slope, opposing a tendency to slide upwards.

In the case of a train,

For $v < v_0$ (tendency to slip down) lateral force is exerted on the inner rail.

For $v > v_0$, lateral force is exerted on the outer rail.

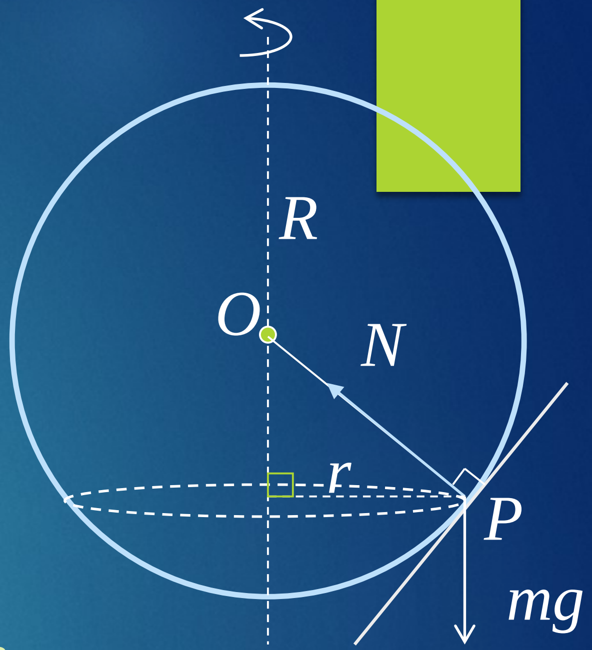
See for example Cambridge Example 13 and Ex 7.6 Q7.

Example. An engine of mass 50 tonnes travels around a track of 400m radius at 50 km/h.

a) Find the lateral force on the wheels if the track is horizontal.
b) At what angle should the track be tilted to eliminate this lateral force?
c) By how much should the outer rail be raised above the inner rail if the gauge (the distance between the rails) is 1.575m?

2004 HSC question 6(c)

A smooth sphere with centre O and radius R is rotating about the vertical diameter at a uniform angular velocity ω radians per second. A marble is free to roll around the inside of the sphere.



Assume that the marble can be considered as a point P which is acted upon by gravity and the normal reaction force N from the sphere. The marble describes a horizontal circle of radius r with the same uniform angular velocity ω radians per second. Let the angle between OP and the vertical diameter be θ.

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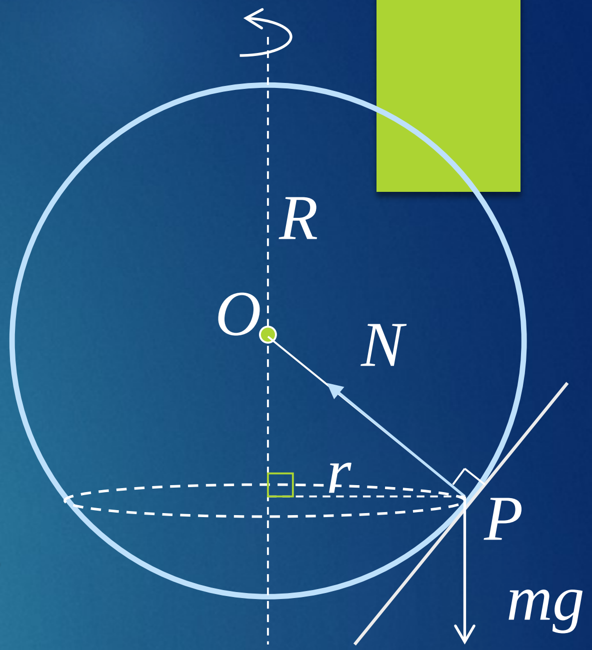

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i) Explain why $mr\omega^2 = N\sin\theta$ and $mg = N\cos\theta$

ii) Show that either $\theta = 0$ or $\cos\theta = \dfrac{g}{r\omega^2}$

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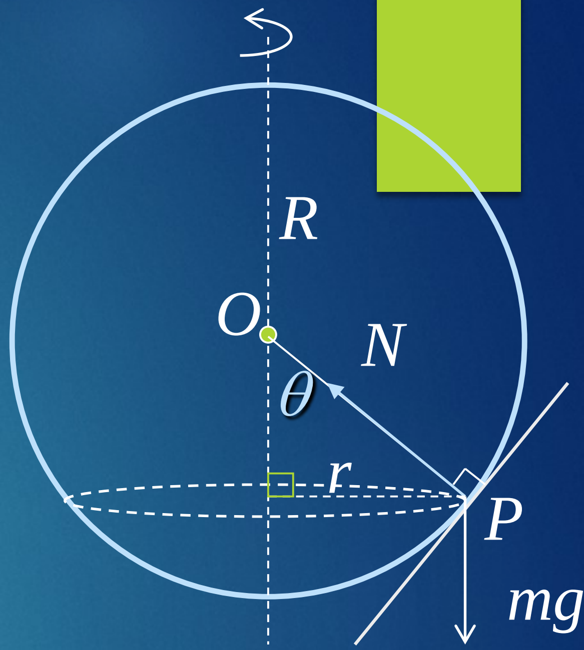
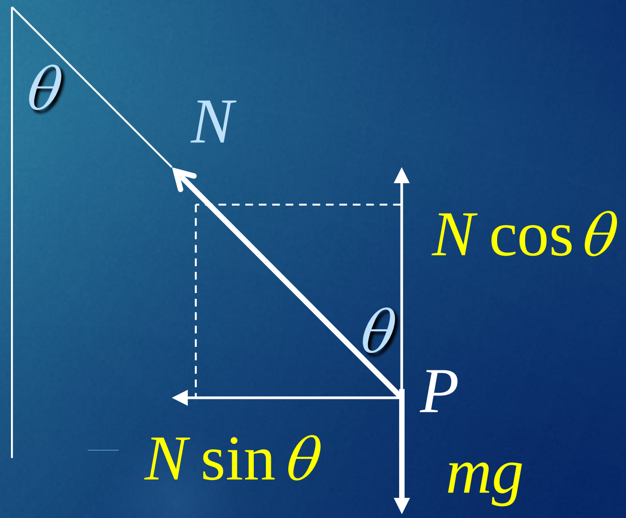


(i) Explain why $mr\omega^2 = N\sin\theta$ and $mg = N\cos\theta$

Net vertical force is 0 $\rightarrow mg = N\cos\theta$

Net horizontal force $= mr\omega^2$

$$N\sin\theta = mr\omega^2$$

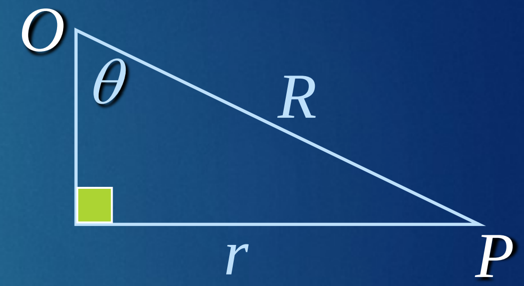


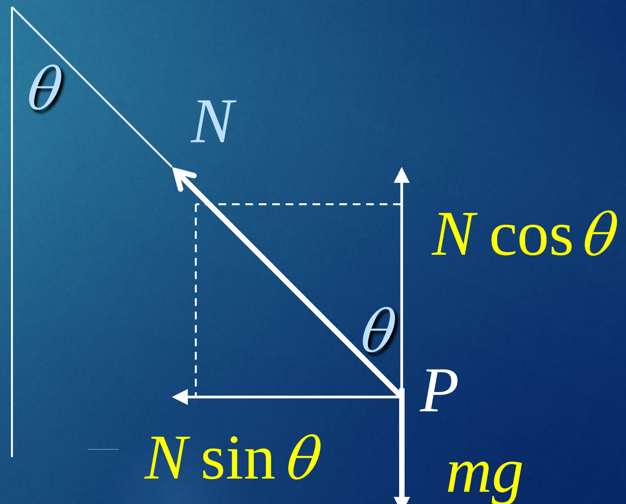
2004 HSC question 6(c)

(ii) Show that either $\theta = 0$ or $\cos\theta = \dfrac{g}{r\omega^2}$

$$mr\omega^2 = N\sin\theta$$

$$mg = N\cos\theta$$





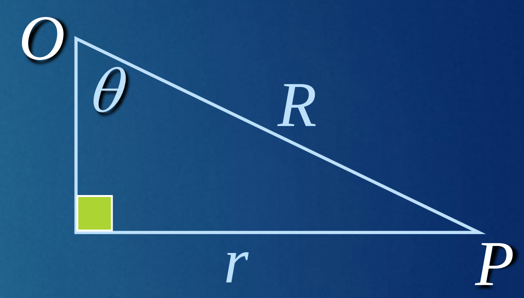
2004 HSC question 6(c)

(ii) Show that either $\theta = 0$ or $\cos\theta = \dfrac{g}{R\omega^2}$

$$mr\omega^2 = N\sin\theta$$

$$mg = N\cos\theta$$

$$\sin\theta = \frac{r}{R}$$



$$N\sin\theta = mr\omega^2 \rightarrow N\frac{r}{R} = mr\omega^2$$

Now either $r = 0$ and the marble is stationary,

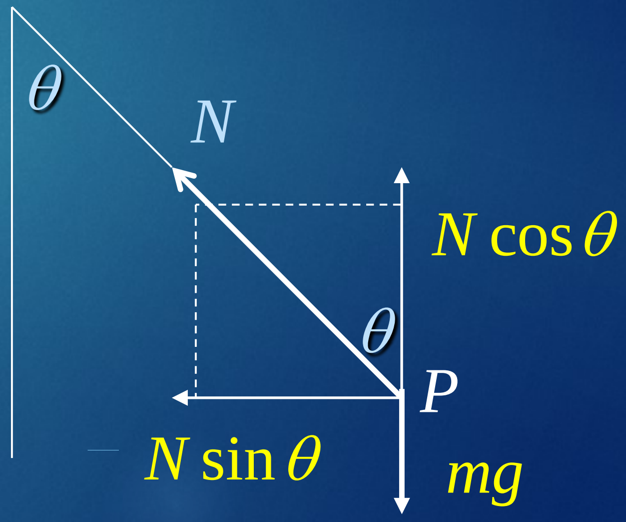
or $r \neq 0$ and….

$$\frac{N}{R} = m\omega^2 \rightarrow N = mR\omega^2$$

$$N\cos\theta = mg$$

$$mR\omega^2\cos\theta = mg$$

$$\cos\theta = \frac{g}{R\omega^2}$$



Two identical particles are attached to the ends X and Y of a string which passes through a small hole at the apex Z of a hollow cone which is fixed with the axis vertical and the apex uppermost.

Let θ be the semi vertical angle.

The particle at X moves in a horizontal circle with constant angular velocity ω rad/s on the surface of the cone, while the particle attached to Y hangs at rest inside the cone.

Assume there is no friction between the particle at X and the surface of the cone. Let $ZX = \ell$.

i) Draw a diagram to represent this information, showing the relevant forces.

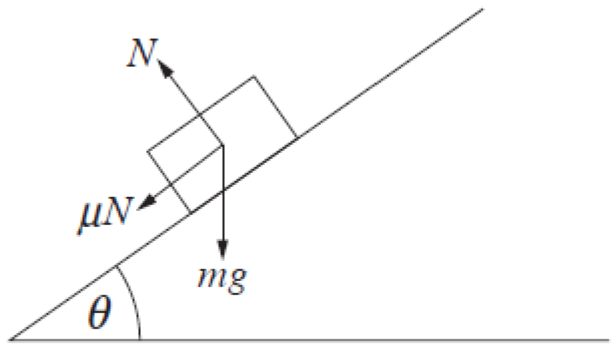
The particle at X moves in a horizontal circle with constant angular velocity ω rad/s on the surface of the cone, while the particle attached to Y hangs at rest inside the cone.

Assume there is no friction between the particle at X and the surface of the cone. Let $ZX = \ell$.

ii) Show that $\omega^2 = \dfrac{g}{\ell(1+\cos\theta)}$

iii) Deduce that $\dfrac{g}{2\omega^2} < \ell < \dfrac{g}{\omega^2}$.

A car of mass m is driven at speed v around a circular track of radius r. The track is banked at a constant angle θ to the horizontal, where $0 < \theta < \frac{\pi}{2}$. At the speed v there is a tendency for the car to slide up the track. This is opposed by a frictional force μN, where N is the normal reaction between the car and the track, and $\mu > 0$. The acceleration due to gravity is g.



(i) Show that $v^2 = rg\left(\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)$. **3**

(ii) At the particular speed V, where $V^2 = rg$, there is still a tendency for the car to slide up the track. **2**

Using the result from part (i), or otherwise, show that $\mu < 1$.