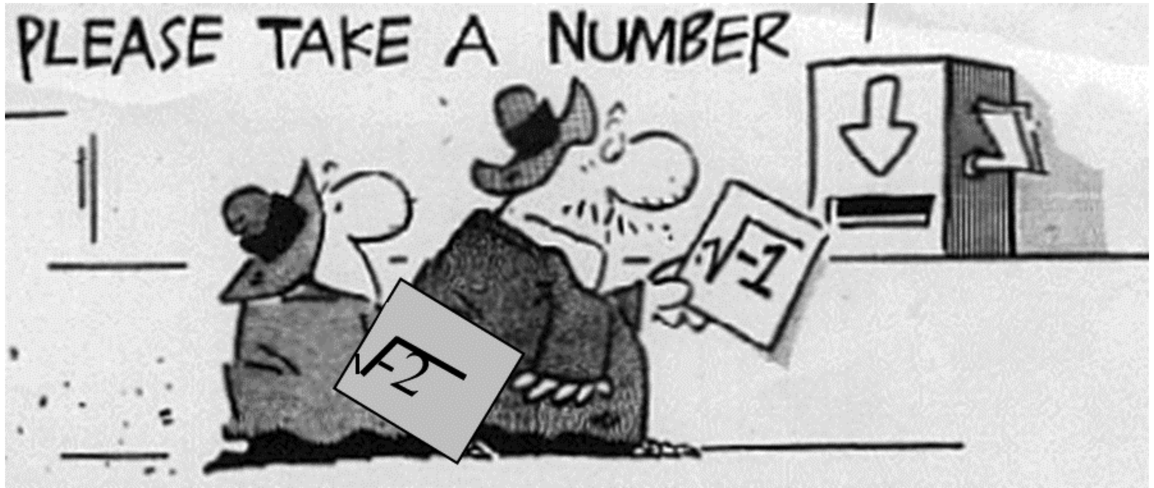
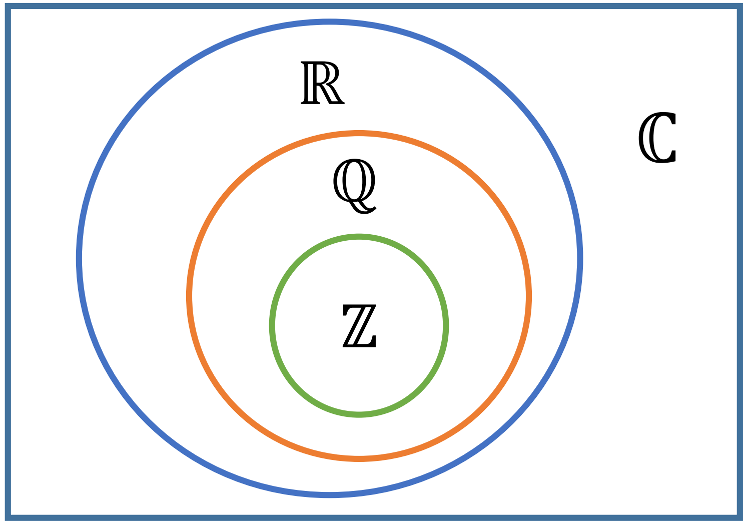
Complex Numbers



The Complex Number System

- In the beginning there were counting, or natural, numbers $\mathbb{N}$
- And then we needed integers $\mathbb{Z}$ (or $\mathbb{J}$)
- And rational numbers $\mathbb{Q}$
- Adding irrational numbers ($\pi, \sqrt{2}, \ldots$) gave the real numbers $\mathbb{R}$
- Including a number i with the property that $i^2 = -1$ gives the complex numbers $\mathbb{C}$

- This allows us to solve all quadratic equations with real coefficients.



Definitions:

A number such as $3i$ is a purely imaginary number

A number such as 6 is a purely real number

$6 + 3i$ is a complex number

$x + iy$ is the general form of a complex number

If $x + iy = 6 - 4i$ then $x = 6$ and $y = -4$

Let $z = 6 - 4i$. The real part of z, $\mathrm{Re}\, z$ is 6

The imaginary or unreal part of z, $\mathrm{Im}\, z$ is -4

Complex Arithmetic

Example 1. Solve

a) $x^2 + 25 = 0$

b) $x^2 + 2x + 4 = 0$

Example 2. Simplify

a) $\sqrt{-9}$

b) $2i \times -5i$

c) $7 + 3i - (4 - i)$

d) $(2 + i)(1 - i)$

Complex Conjugates

If $z = x + iy$ then the complex conjugate of z, is $\bar{z} = x - iy$.

Example 3. Find $\bar{z}$ and $z\bar{z}$ for each value of z.

a) $1 + 3i$

b) $4 - 2i$

c) $5i$

Notice that the product $z\bar{z}$ is always real. We can use this to divide by complex numbers.

Example 4. Write in the form $a + ib$

a) $\frac{1}{1+3i} \times \frac{1-3i}{1-3i} = \frac{1-3i}{1+9}$

$= \frac{1-3i}{10} = \frac{1}{10} - \frac{3}{10}i$

b) $\frac{2+3i}{1-4i}$

Square Roots of Complex Numbers

Example 5. Find the square roots of $12 - 5i$

Let $z = a + bi$ be a solution. Then $z^2 = a^2 + 2abi - b^2$.

Equating real and imaginary parts,

$12 = a^2 - b^2$ (1)

$5 = -2ab$ (2)

Rearranging (2), $b = -\frac{5}{2a}$

Substituting into (1) $12 = a^2 - (\frac{5}{2a})^2$

$$a^2 - \frac{25}{4a^2} - 12 = 0$$

$$a^4 - 12a^2 - \frac{25}{4} = 0$$

By the quadratic formula, $a^2 = \frac{12 \pm 13}{2}$

Since a is real, $a^2 > 0$, hence $a^2 = \frac{25}{2}$

$$a = \pm\frac{5\sqrt{2}}{2} \qquad b = \mp\frac{\sqrt{2}}{2}$$

$$z = \pm\left(\frac{5\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

Find the square roots of

a) $3 + 4i$,

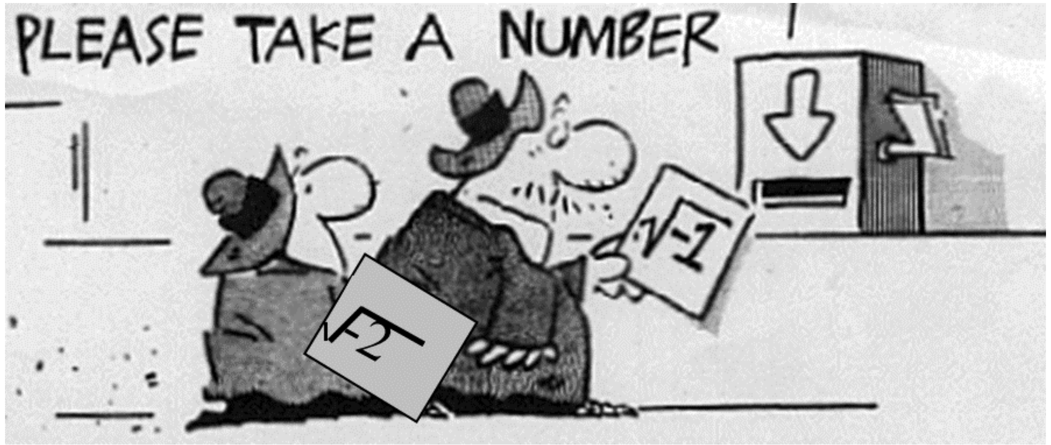
b) $8 - 6i$

Then…

Ex. 31(a) a, c, e, g
and 31(b) 1-26 odds
27-32 a, c, e…

Or: new book
Ex 1.1 (all)

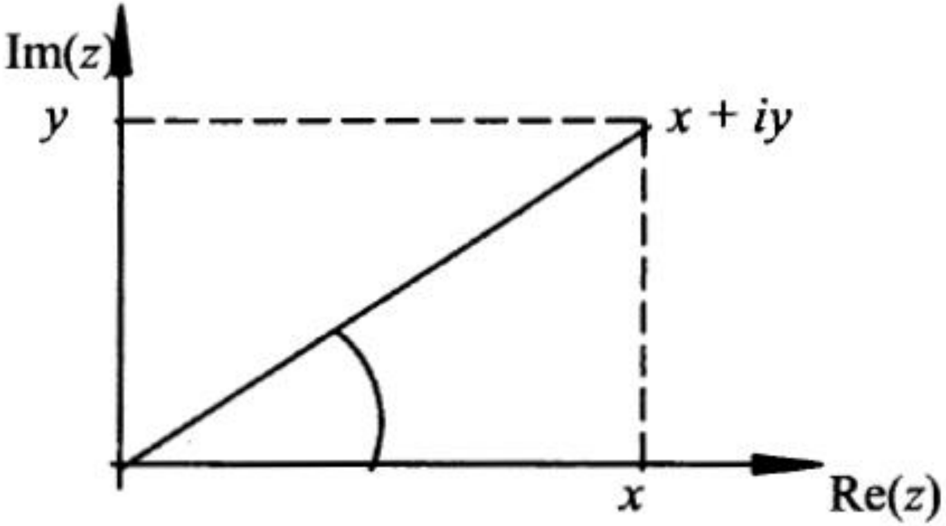
Argand Diagrams



Who goes first? …

Complex numbers do not have order.

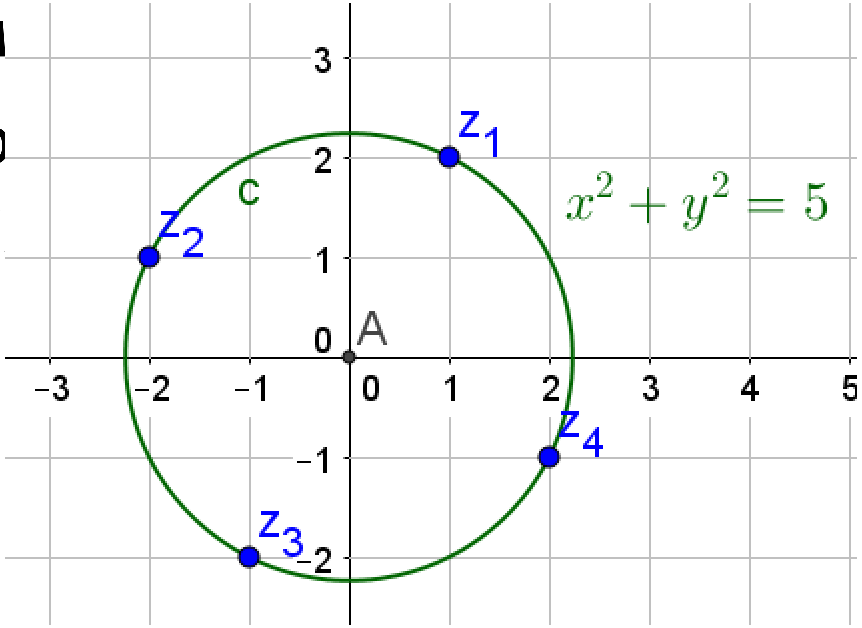
We cannot represent them as points on a line like the reals, however we can represent them as points on the complex number plane or Argand diagram, where the x-coordinate represents the real part of the complex number, and the y-coordinate represents the imaginary part.



Addition on the Argand Diagram

Example 1. Plot the points representing $z, w, -w, z+w, z-w$ on the Argand diagram where $z = 2 + i$ and w

What shap
$0, z, z + w$



Multiplication by i

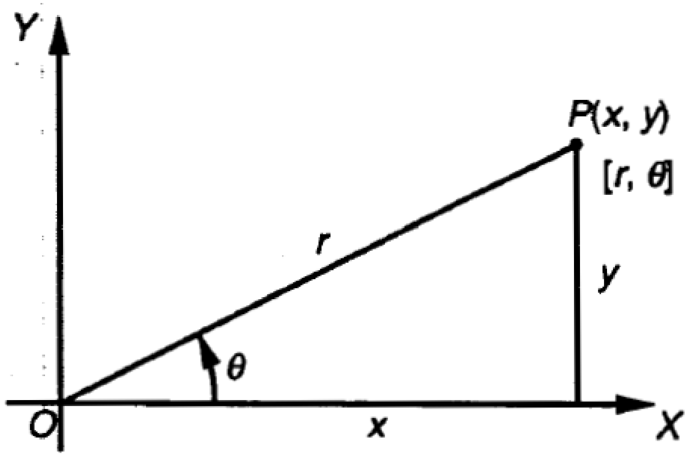
Multiplication by -1 rotates a point by $180°$ in the complex plane and multiplication by i rotates a point $90°$ anticlockwise.

Example 2. If $z = 1 + 2i$, plot the points iz, i^2z, i^3z, i^4z on a complex plane.

Polar Form (Modulus-Argument Form) of a Complex Number

A complex number $z = x + iy$ can be expressed in polar form as

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\ \mathrm{cis}\ \theta.$$



The modulus, or absolute value, of z is defined as

$$\mathrm{mod}\ z = |z| = \sqrt{x^2 + y^2} = r.$$

The argument, or phase, or angle, of z is defined as

$$\arg z = \theta \pm 2n\pi, n \text{ a positive integer.}$$

There are many possible arguments for any nonzero complex number, but the principal value of an argument of z, denoted $\mathrm{Arg}\, z$ satisfies

$$-\pi < \mathrm{Arg}\, z \leq \pi.$$

Example 3. Express in mod-arg form. a) $4 - 4i$ b) $2 + 3i$

Multiplication and Division in Polar Form

Let $z_1 = r_1 \text{cis}\,\theta_1$ and $z_2 = r_2 \text{cis}\,\theta_2$.

Then $z_1 z_2 = r_1 \text{cis}\,\theta_1 \cdot r_2 \text{cis}\,\theta_2$

$$= r_1 r_2 (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1))$$

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

$$= r_1 r_2 \text{cis}\,(\theta_1 + \theta_2)$$

Similarly,

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}\,(\theta_1 - \theta_2)}$$

De Moivre's Theorem

If $z = r \text{ cis } \theta$, then for any integer n,

$$\boxed{z^n = r^n \text{cis } n\theta}$$

Proof: by induction…will be done later

Example. a) Write in modulus-argument form $z = -1 + i$ and $w = 1 + \sqrt{3}i$.

b) Hence express in modulus-argument form zw and $\frac{z}{w}$.

Example. Express in Cartesian form

a) $(-1 + i)^5$

b) $(1 + \sqrt{3}i)^{-5}$

Old Fitzpatrick
Ex 31 (c) odds
Ex 31 (d) 1, 3, 4, 5, 7, 10, 13, and 14-18 a,c,e

OR: New Fitzpatrick:
Ex 1.2
Q1-14, 16, 18

Complex Numbers as Vectors

A vector quantity has both magnitude (size) and direction.

A complex number z can be represented as a vector with magnitude $\mathrm{mod}\, z$ and direction $\arg z$.

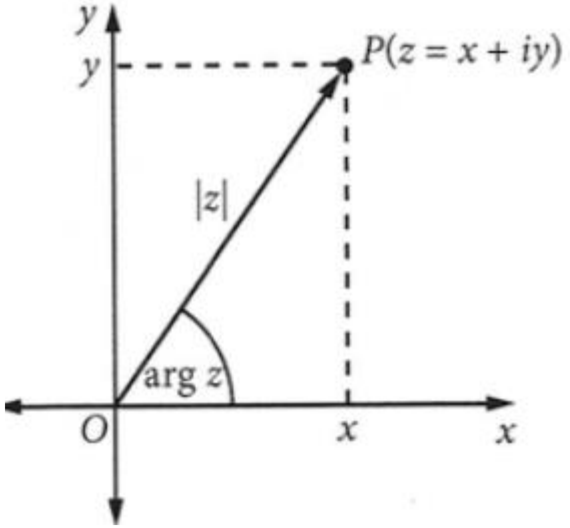
Let P be the point representing z on the Argand diagram.

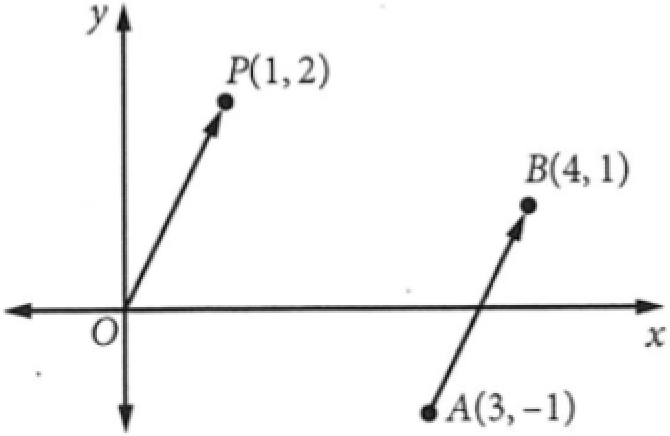
The position vector $\overrightarrow{OP}$ represents z, but there are many other free vectors (vectors that do not start at the origin) which also represent z.

For example, if $z = 1 + 2i$, then both $\overrightarrow{OP}$

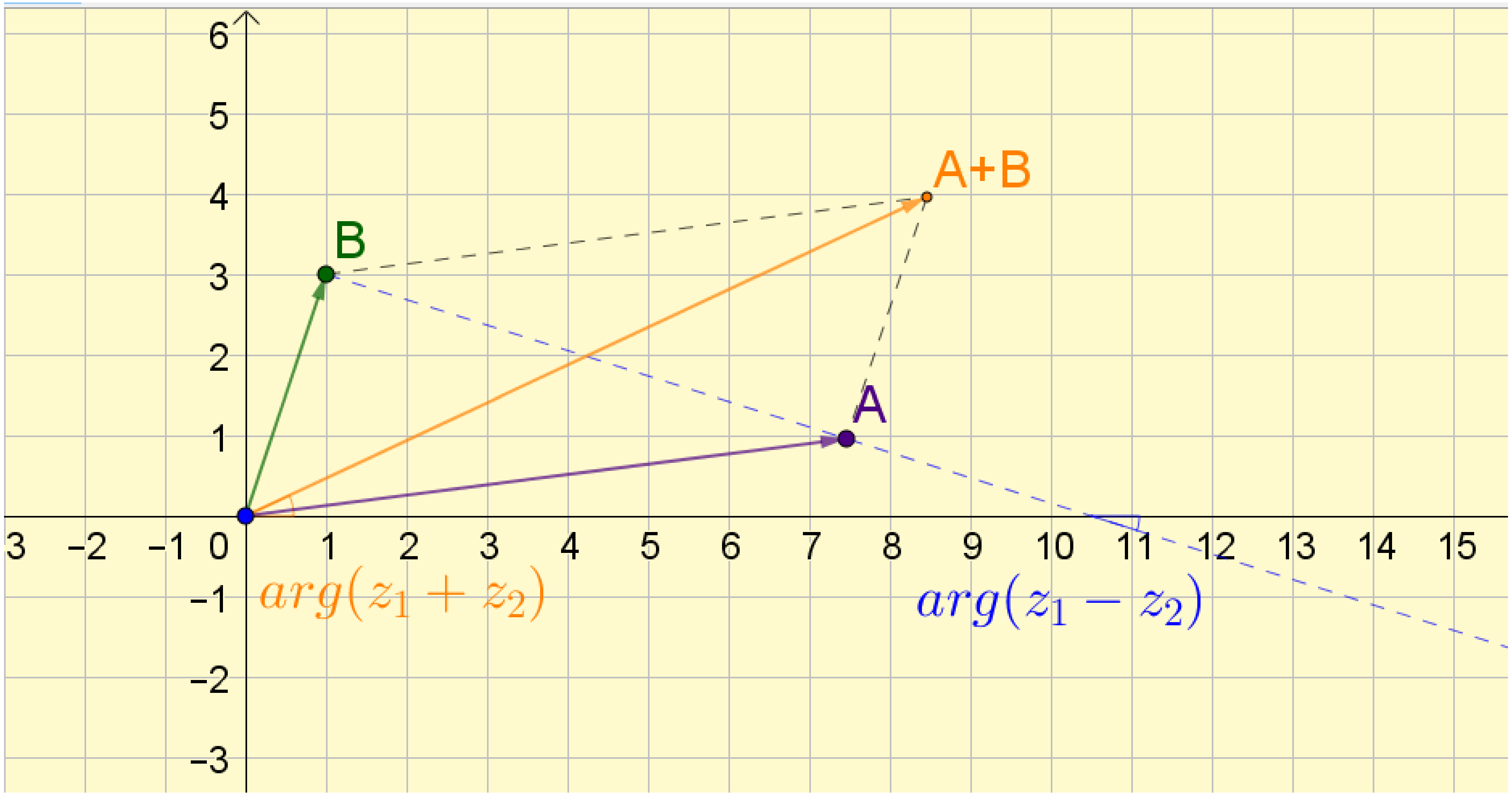
and $\overrightarrow{AB}$ represent z.

Note that order is important! $\overrightarrow{PO}$ and $\overrightarrow{BA}$ represent $-z$.





Let A represent z_1 and B represent z_2.



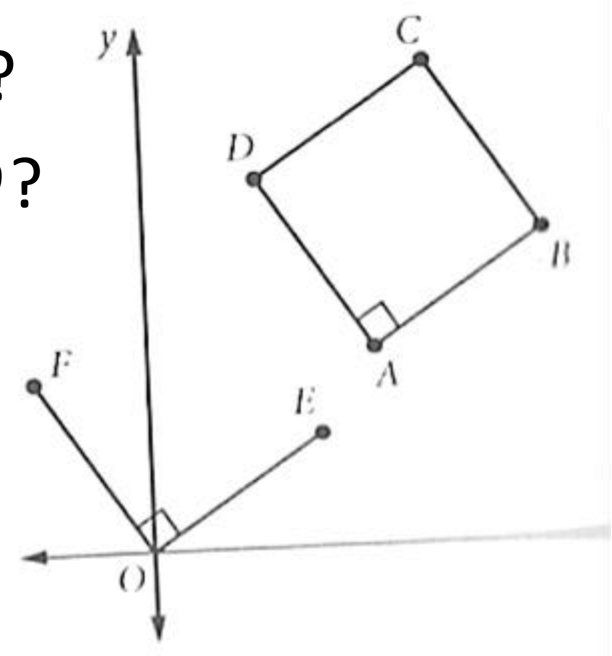
Example 1 (HSC '91 and Ex 1.3 Q10)

On an Argand diagram, $ABCD$ is a square. OE and OF are parallel to and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively.

a) Explain why the point E corresponds to $w_2 - w_1$.

b) What complex number corresponds to the point F?

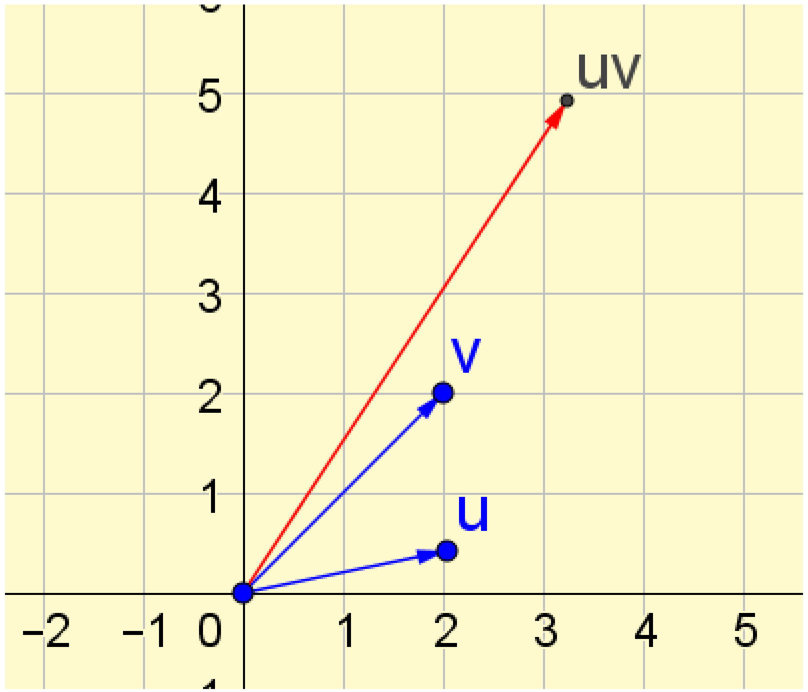
c) What complex number corresponds to the vertex D?



Geometrical Interpretation of Multiplication

- Multiplication by i rotates a vector by $\frac{\pi}{2}$ anticlockwise.
- Multiplication by ki ($k \in \mathbb{R}$) rotates a vector by $\frac{\pi}{2}$ and scales by k.
- Multiplication by $r \text{ cis } \theta$ rotates a vector by θ and scales by r.

This is most easily seen in polar form, since $\mathrm{mod}(uv) = \mathrm{mod}(u) \cdot \mathrm{mod}(v)$ and $\arg(uv) = \arg(u) + \arg(v)$.



Example 2. If $\frac{z_1 - z_2}{z_1 + z_2} = ki, k \in \mathbb{R}$, show that $|z_1| = |z_2|$.

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent z_1 and z_2.

Construct the parallelogram $OACB$.

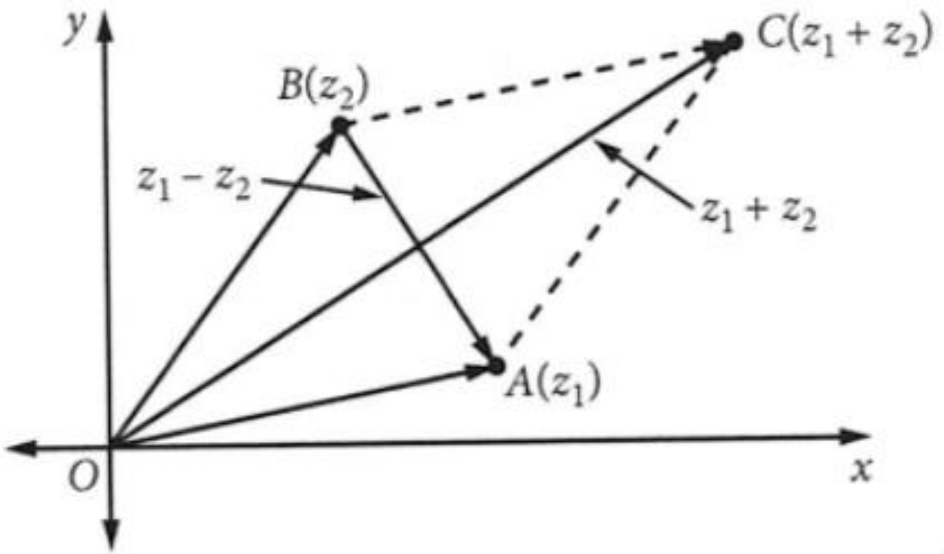
Since $z_1 - z_2 = ki(z_1 + z_2)$, the diagonals meet at 90°,

hence $OACB$ is a rhombus.

Therefore $OA = OB$ (sides of a rhombus are equal)

and $|z_1| = |z_2|$.

The Triangle Inequalities



In triangle OAC, $OC \le OA + AC$, with equality if O, A, C are collinear.

It follows that $|z_1 + z_2| \le |z_1| + |z_2|$ with equality if $z_2 = kz_1, k$ real, positive.

Similarly $|z_1 - z_2| \le |z_1| + |z_2|$ with equality if $z_2 = -kz_1$.

Example 3. If $z_1 = 4 + 3i$ and $|z_2| = 7$,

a) Find the maximum possible value of $|z_1 + z_2|$

b) If $|z_1 + z_2|$ takes its greatest value, express z_2 in the form $a + ib$.

c) When does $|z_1 + z_2|$ take its least value?

New Fitzpatrick:
Ex 1.3
Q1-9,11,14

Quadratic Equations with Complex Coefficients

Suppose that $az^2 + bz + c = 0$ where $a, b, c \in \mathbb{C}$.

To solve:

1. Find $\Delta = b^2 - 4ac$
2. Find $\sqrt{\Delta}$
3. By the quadratic formula, $z = \frac{-b \pm \sqrt{\Delta}}{2a}$

Example. Solve

a) $z^2 - (2 + 4i)z - 3 + 4i = 0$

b) $z^2 + (2 + 3i)z + (1 + 3i) = 0$

c) $z^2 - (4 + i)z + 5 - i = 0$

Roots of Complex Numbers

To find roots of complex numbers we usually use de Moivre's Theorem.

Example. Cube Roots of Unity

Solve the equation $z^3 = 1$.

Let $z = r \text{ cis } \theta$.

Then $r^3 \text{ cis } 3\theta = 1$.

$r^3 = 1$ and $3\theta = 2\pi n, n \in \mathbb{Z}$.

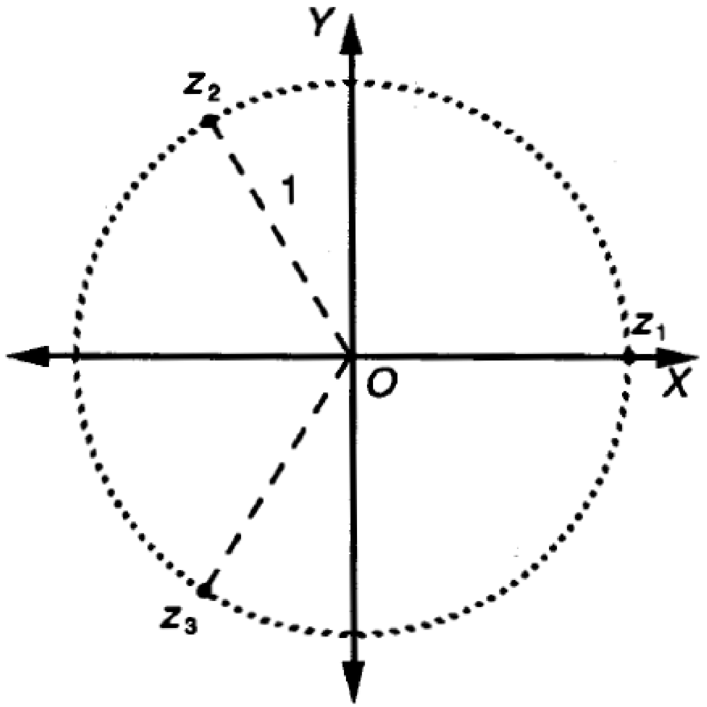
$r = 1$ and $\theta = \frac{2\pi n}{3}, n \in \mathbb{Z}$.

$r = 1$ and $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \ldots$

Taking principal arguments, $\theta = 0, \pm\frac{2\pi}{3}$

The solutions are $z_1 = \text{cis } 0 = 1$, $z_2 = \text{cis } \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$,

$z_3 = \text{cis}\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$.



Example ctd. Cube Roots of Unity

If $\omega = z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, then

$$\omega^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{2}i - \frac{3}{4}$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$= z_3.$$

Note that this is what we should expect since if $\omega^3 = 1$, then

$$(\omega^2)^3 = (\omega^3)^2 = 1^2 = 1.$$

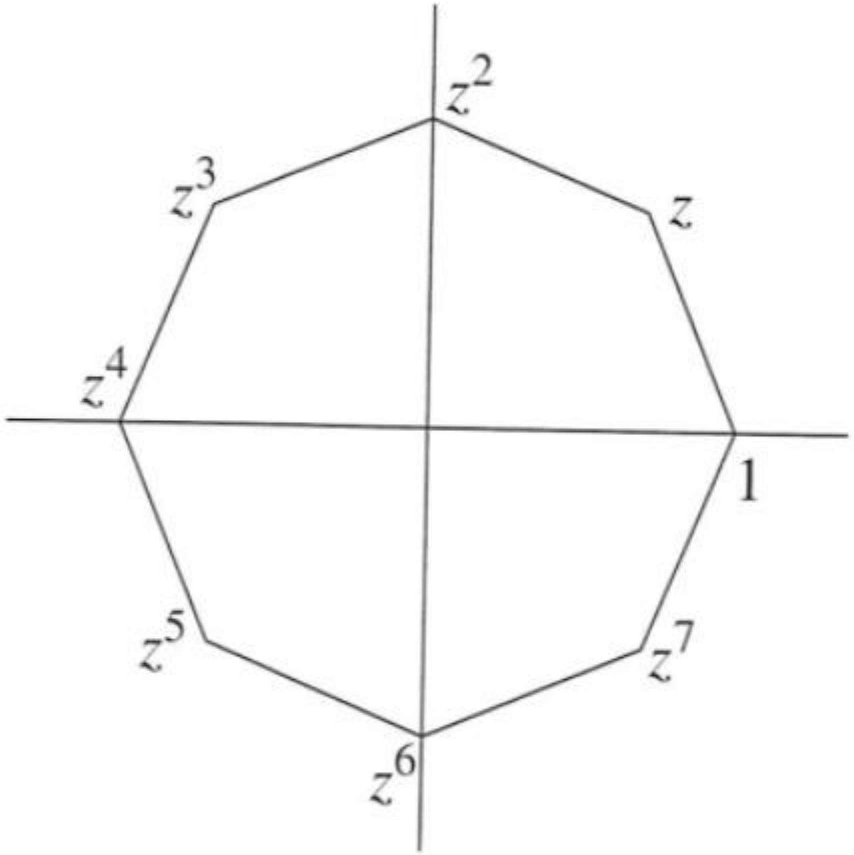
What is the sum of the roots $1 + \omega + \omega^2$?

Since $z^3 - 1 = 0$, sum of roots $= -\frac{b}{a} = 0$.

Also $\omega^2 = \overline{\omega}$

The complex number z is chosen so that the complex numbers $1, z, z^2, \dots, z^7$ form the vertices of the regular polygon shown.

Which polynomial has all these complex numbers as roots?



A. $x^7 - 1 = 0$

B. $x^7 + 1 = 0$

C. $x^8 - 1 = 0$

D. $x^8 + 1 = 0$

New Fitzpatrick:

Ex 1.5 Q1-8

Curves and Regions on the Argand Diagram

Example Sketch the locus of z for each of the following.

a) $|z| \leq 3$

b) $|z - 1 + i| > 2$

c) $1 \leq |z| \leq 4$

d) $Im\, \mathrm{z} > -1$

e) $z + \bar{z} = 4$

f) $\frac{\pi}{4} < \mathrm{Arg}\, z < \frac{3\pi}{4}$

g) $\arg\left(\frac{z-i}{z-2}\right) = 0$

Useful Circle Geo result: If an interval subtends equal angles at two points on the same side of it, then the two points and the endpoints of the intervals are concyclic.

Example 2. Sketch the locus of z such that

a) $\arg\left(\frac{z-i}{z+3i}\right) = \frac{\pi}{2}$ b) $\arg\left(\frac{z-i}{z+3i}\right) = \frac{\pi}{4}$

c) Find the Cartesian equation of (b)

Let $A = (0, 1), B = (-3, 0)$ and let P be the point representing z.

The interval AB subtends the same angle for all possible locations of P, so A, P, B are concyclic… i.e. P stands on the arc of a circle with endpoints A and B.

Example: Let $w \neq 1$ be an n-th root of unity.

Show that $1 + w + w^2 \dots + w^n = 0$.

$1 + z + z^2 \dots + z^n$ is a geometric series with $a = 1, r = z$

$$\therefore 1 + z + z^2 \dots + z^n = \frac{z^n - 1}{z - 1}$$

$$z^n - 1 = (z - 1)(1 + z + z^2 \dots + z^n)$$

Now $w^n - 1 = 0$, so either $w - 1 = 0$ or $1 + w + w^2 \dots + w^n = 0$.

Since $w \neq 1$, we must have $1 + w + w^2 \dots + w^n = 0$

Pascal’s Triangle and de Moivre’s Theorem

We can find the coefficients of $(a+b)^n$ by looking at the $(n+1)$th row of Pascal’s triangle:

Example 1. Use Pascal’s triangle to expand

a) $(a+b)^5$

b) $(\cos\theta + i\sin\theta)^5$

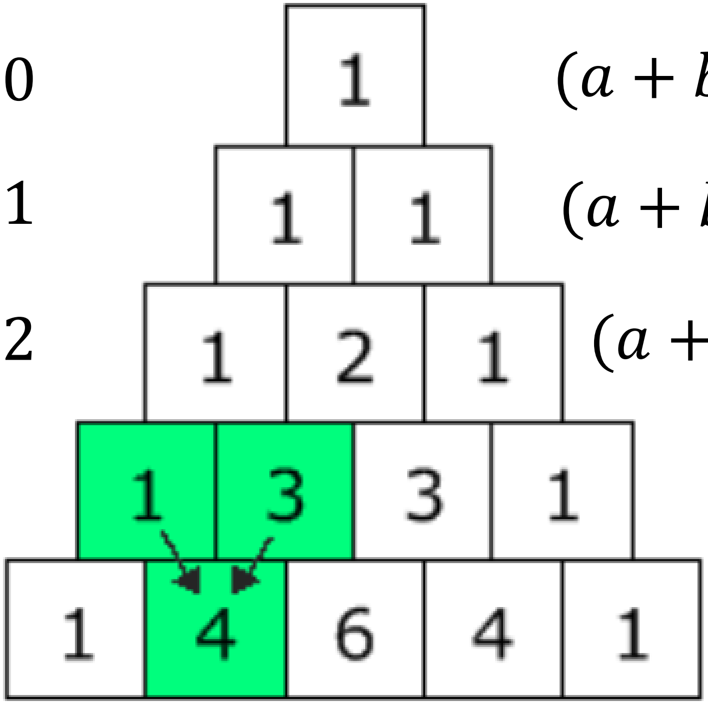
$n = 0$ $\quad (a+b)^0 = 1$

$n = 1$ $\quad (a+b)^1 = a + b$

$n = 2$ $\quad (a+b)^2 = a^2 + 2ab + b^2$

$n = 3$ $\quad (a+b)^3$
$= a^3 + 3a^2b + 3ab^2 + b^3$

$n = 4$



Comparing the expansion given by Pascal's triangle and by de Moivre's theorem allows us to prove trigonometric identities.

It may also give us a substitution which allows us to solve polynomial equations.

Example 2. a) Show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

b) Hence solve $16x^5 - 20x^3 + 5x - 1 = 0$.

New Fitzpatrick:
Ex 1.6 Q1-4