# Carlingford High School

# Year 12 Mathematics Extension 2

## Task 3 Term 2 2018

*Time allowed: 70 minutes*

Student Number:______________________ Teacher: Mrs Wilson

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- Start each question on a new page.
- Do not work in columns.
- Marks may be deducted for careless or badly arranged work.
- Write in blue or black pen. Diagrams and graphs may be done in pencil.
- Only calculators approved by the Board of Studies may be used.
- There is to be NO LENDING OR BORROWING.
- Write your number on every page.

| | **MC** | **Q6** | **Q7** | **Mark** |
|---|---|---|---|---|
| **Polynomials** | **/2** | **/15** | | **/17** |
| **Integration** | **/3** | | **/14** | **/17** |
| **Total** | **/5** | **/15** | **/14** | **/34** |

# Section 1

## Multiple Choice – Complete on Multiple Choice Answer Sheet (5 marks)

**1.** Which of the following is an expression for $\int \frac{\sin x \cos x}{5+\sin x}dx$?

Use the substitution $u = 5+\sin x$.

A. $-5\ln|5+\sin x|+C$

B. $5\ln|5+\sin x|+C$

C. $-\sin x - 5\ln|5+\sin x|+C$

D. $\sin x - 5\ln|5+\sin x|+C$

**2.** What are the values of real numbers *p* and *q* such that $1-i$ is a root of the equation $z^3 + pz + q = 0$?

A. $p=-2$ and $q=-4$

B. $p=-2$ and $q=4$

C. $p=2$ and $q=4$

D. $p=2$ and $q=-4$

**3.** What is the value of $\int_0^{\frac{\pi}{6}} \sec 4x \tan 4x dx$?

A. $-\frac{3}{4}$

B. $-\frac{3}{8}$

C. $\frac{3}{8}$

D. $\frac{3}{4}$

**4.** Let $\alpha$, $\beta$ and $\gamma$ be the roots of the equation $x^3+2x^2+5=0$.

Which of the following polynomial equations has the roots $\alpha^2$, $\beta^2$ and $\gamma^2$?

A. $x^3-4x^2-20x-25=0$

B. $x^3-4x^2-10x-25=0$

C. $x^3-4x^2-20x-5=0$

D. $x^3-4x^2-10x-5=0$

**5.** Which of the following is an expression for $\int \frac{1}{x^2-6x+13}dx$?

A. $\frac{1}{2}\tan^{-1}\frac{(x-3)}{4}+C$

B. $\frac{1}{2}\tan^{-1}\frac{(x-3)}{2}+C$

C. $\frac{1}{4}\tan^{-1}\frac{(x-3)}{4}+C$

D. $\frac{1}{4}\tan^{-1}\frac{(x-3)}{2}+C$

**END OF SECTION 1**

# Section 2

**Question Six – Start a new page** **(15 marks)** **Marks**

**a)** Factorise the polynomial $P(x) = x^4 - 2x^2 - 15$ over:

i) the real field **R** **1**

ii) the complex field **C** **1**

**b)** i) The polynomial $P(x)$ has a double root at $x = \alpha$. **2**
Prove that $P'(x)$ has a root at $x = \alpha$.

ii) The polynomial $P(x) = x^5 - ax^2 + b$ has a double root at $x = \alpha$. **2**
Find the values of *a* and *b* in terms of ∝.

**c)** Let $z = 1 + i$ be a root of the polynomial $z^2 - biz + c = 0$ where *b* and *c* are real numbers. Find the values of *b* and *c*. **2**

**d)** If $\alpha$, $\beta$ and $\gamma$ are the roots of the equation $x^3 - 7x^2 + 18x - 7 = 0$ find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$. **3**

**e)** Let $x = \alpha$ be a root of the polynomial $P(x) = x^4 + bx^3 + cx^2 + bx + 1$
where $(2+c)^2 \neq 4b^2$

i) Show that $\alpha$ cannot be 0, 1 or –1. **1**

ii) Show that $x = \dfrac{1}{\alpha}$ is a root. **1**

iii) Deduce that if $\alpha$ is a multiple root then its multiplicity is 2 and $b^2 - 4c + 8 = 0$. **2**

**End of Question 6**

**Question Seven – Start a new page** **(14 marks)** **Marks**

**a)** Find $\int xe^x dx$ **2**

**b)** Evaluate in simplest exact form $\int_0^4 \frac{8-2x}{(x^2+4)(x+1)}dx$ **3**

**c)** Use the substitution $t=\tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos\theta+2\sin\theta}d\theta$ **4**

**d)** Let $I_n=\int_0^{\frac{\pi}{2}} \sin^n x dx$ , where *n* is a positive integer.

i) Show that $I_n=(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x dx$ when $n\geq 2$ . **2**

ii) Prove that $I_n=\frac{(n-1)}{n}I_{n-2}$ when $n\geq 2$ . **2**

iii) What is the value of $I_4$? **1**

**End of Question 7**

# END OF EXAM

## Year 12 Mathematics Section I - Answer Sheet

**Student Number** ________________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ● B ⊗ C ○ D ○

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1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○

### Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

### Angle sum of a polygon

$$S = (n-2) \times 180°$$

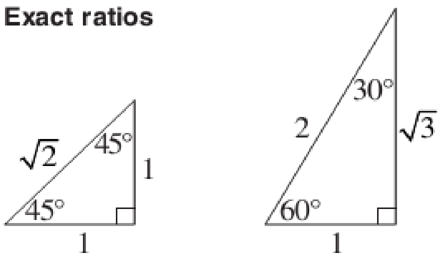
### Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

### Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$

### Exact ratios




### Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

### Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

### Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

### Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

### Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

### Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

### Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

### *n*th term of an arithmetic series

$$T_n = a + (n-1)d$$

### Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

### *n*th term of a geometric series

$$T_n = ar^{n-1}$$

### Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

### Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

### Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

**Differentiation from first principles**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**Derivatives**

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

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**Solution of a quadratic equation**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Sum and product of roots of a quadratic equation**

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

**Equation of a parabola**

$$(x-h)^2 = \pm 4a(y-k)$$

**Integrals**

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

**Trapezoidal rule (one application)**

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

**Simpson's rule (one application)**

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

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**Logarithms – change of base**

$$\log_a x = \frac{\log_b x}{\log_b a}$$

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**Angle measure**

$180° = \pi$ radians

**Length of an arc**

$l = r\theta$

**Area of a sector**

$$\text{Area} = \frac{1}{2}r^2\theta$$

**Angle sum identities**

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

**General solution of trigonometric equations**

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

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**Division of an interval in a given ratio**

$$\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right)$$

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**Parametric representation of a parabola**

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $(2at,\ at^2)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $(x_1,\ y_1)$,

tangent: $xx_1 = 2a(y+y_1)$

normal: $y - y_1 = -\frac{2a}{x_1}(x-x_1)$

Chord of contact from $(x_0,\ y_0)$: $xx_0 = 2a(y+y_0)$

**Acceleration**

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

**Simple harmonic motion**

$$x = b + a\cos(nt+\alpha)$$

$$\ddot{x} = -n^2(x-b)$$

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**Further integrals**

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

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**Sum and product of roots of a cubic equation**

$$\alpha+\beta+\gamma = -\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

**Estimation of roots of a polynomial equation**

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

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**Binomial theorem**

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

**Carlingford High School Mathematics Extension 2 Solutions Task 3 Term 2 2018**
**Section 1: Multiple Choice**

| | | |
|---|---|---|
| 1 | Let $u=5+\sin x$ then $\dfrac{du}{dx}=\cos x$<br>Now $\sin x=u-5$<br>$\int\dfrac{\sin x\cos x}{5+\sin x}dx=\int\dfrac{(u-5)\cos x}{u}\dfrac{du}{\cos x}$<br>$=\int 1-\dfrac{5}{u}\,du$<br>$=u-5\ln\lvert u\rvert+C$<br>$=5+\sin x-5\ln\lvert 5+\sin x\rvert+C$<br>$=\sin x-5\ln\lvert 5+\sin x\rvert+C$ | 1 Mark: D |
| 2 | Using the conjugate root theorem $1+i$ and $1-i$ are both roots of the equation $z^3+pz+q=0$.<br>$(1+i)+(1-i)+\alpha=0$ (sum of the roots)<br>$\alpha=-2$<br>$(1+i)\times(1-i)\times-2=-q$ (product of the roots)<br>$(1+1)\times-2=-q$<br>$q=4$<br>$(1+i)(1-i)+(1-i)(-2)+(1+i)(-2)=p$<br>$p=2$<br>Therefore $p=-2$ and $q=4$ | 1 Mark: B |
| 3 | $\int_0^{\frac{\pi}{6}}\sec 4x\tan 4x\,dx=\left[\dfrac{1}{4}\sec 4x\right]_0^{\frac{\pi}{6}}$<br>$=\dfrac{1}{4}\left[\sec\dfrac{2\pi}{3}-\sec 0\right]$<br>$=-\frac{3}{4}$ | 1 Mark: A |
| 4 | If $\alpha$, $\beta$ and $\gamma$ are zeros of $x^3+2x^2+5=0$ then the polynomial equation with roots $\alpha^2$, $\beta^2$ and $\gamma^2$ is:<br>$(\sqrt{x})^3+2(\sqrt{x})^2+5=0$<br>$(\sqrt{x})^3=-(2x+5)$<br>$x^3=4x^2+20x+25$<br>$x^3-4x^2-20x-25=0$ | 1 Mark: A |
| 5 | $\int\dfrac{1}{x^2-6x+13}dx=\int\dfrac{dx}{(x-3)^2+2^2}=\dfrac{1}{2}\tan^{-1}\dfrac{(x-3)}{2}+C$ | 1 Mark: B |

**Section 2**
**Question 6**

| | | |
|---|---|---|
| 6a (i) | $P(x) = x^4 - 2x^2 - 15$<br>$= (x^2 - 5)(x^2 + 3)$<br>$= (x - \sqrt{5})(x + \sqrt{5})(x^2 + 3)$ | 1 Mark: Correct answer. |
| 6a (ii) | $P(x) = (x - \sqrt{5})(x + \sqrt{5})(x - \sqrt{3}i)(x + \sqrt{3}i)$ | 1 Mark: Correct answer. |
| 6(b) (i) | $P(x) = (x-\alpha)^2 Q(x)$<br>$P'(x) = (x-\alpha)^2 Q'(x) + 2(x-\alpha)Q(x)$<br>$= (x-\alpha)\left[(x-\alpha)Q'(x) + 2Q(x)\right]$<br>Therefore $P'(\alpha) = 0$ and $x = \alpha$ is a root of $P'(x)$. | 2 Marks: Correct answer.<br><br>1 Mark: Finds $P'(x)$ |
| 6(b) (ii) | $P(x) = x^5 - ax^2 + b$ has a root $x = \alpha$<br>$P(\alpha) = \alpha^5 - a\alpha^2 + b = 0$ (1)<br>$P'(x) = 5x^4 - 2ax$<br>$P'(\alpha) = 5\alpha^4 - 2a\alpha = 0$ or $a = \frac{5}{2}\alpha^3 = 2.5\alpha^3$<br>Substituting $a = 2.5\alpha^3$ into equation (1)<br>$\alpha^5 - \frac{5}{2}\alpha^3 \times \alpha^2 + b = 0$<br>$b = \frac{3}{2}\alpha^5 = 1.5\alpha^5$<br>$\therefore a = 2.5\alpha^3$ and $b = 1.5\alpha^5$ | 2 Marks: Correct answer.<br><br>1 Mark: Shows some understanding of the problem. |
| 6(c) | $z = 1+i$ satisfies the polynomial $z^2 - biz + c = 0$<br>$(1+i)^2 - bi(1+i) + c = 0$<br>$2i - bi + b + c = 0$<br>Equating real and imaginary parts,<br>$2i - bi = 0$ and $b + c = 0$.<br>Therefore $b = 2$ and $c = -2$ | 2 Marks: Correct answer.<br><br>1 Mark: Uses the factor theorem. |
| 6(d) | $\alpha + \beta + \gamma = 7$<br>$\alpha\beta + \alpha\gamma + \beta\gamma = 18$<br>$\alpha\beta\gamma = 7$<br>$(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = (1+\alpha^2+\beta^2+\alpha^2\beta^2)(1+\gamma^2)$<br>$= 1+\alpha^2+\beta^2+\alpha^2\beta^2+\gamma^2+\alpha^2\gamma^2+\beta^2\gamma^2+\alpha^2\beta^2\gamma^2$<br>$= 1+(\alpha^2+\beta^2+\gamma^2)+(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)+\alpha^2\beta^2\gamma^2$<br>$= 1+13+226+7^2$<br>$= 289$<br>Note:<br>$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$<br>$= 7^2 - 2\times 18 = 13$ | 3 Marks: Correct answer.<br><br>2 Marks: Makes significant progress towards the solution.<br><br>1 Mark: Finds |

| | | |
|---|---|---|
| | $\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2$<br>$=(\alpha\beta+\beta\gamma+\alpha\gamma)^2-2(\alpha\beta.\beta\gamma+\beta\gamma.\alpha\gamma+\alpha\gamma.\alpha\beta)$<br>$=(\alpha\beta+\beta\gamma+\alpha\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$<br>$=18^2-2\times7\times7=226$ | $\alpha+\beta+\gamma$,<br>$\alpha\beta\gamma$ and<br>$\alpha\beta+\alpha\gamma+\beta\gamma$ |
| 6(e)<br>(i) | $x=\alpha$ is a root then $P(0)=0$ and $(2+c)^2\neq 4b^2$ or $2+c\neq\pm2b$<br>$P(0)=0^4+b\times0^3+c\times0^2+b\times0+1\neq0$<br>$P(1)=1^4+b\times1^3+c\times1^2+b\times1+1$<br>$=2b+2+c\neq0$ as $2+c\neq-2b$<br>$P(-1)=(-1)^4+b\times(-1)^3+c\times(-1)^2+b\times(-1)+1$<br>$=-2b+2+c\neq0$ as $2+c\neq2b$ | 1 Mark:<br>Correct answer. |
| 6(e)<br>(ii) | $P(\frac{1}{\alpha})=(\frac{1}{\alpha})^4+b(\frac{1}{\alpha})^3+c(\frac{1}{\alpha})^2+b(\frac{1}{\alpha})+1$<br>$=\frac{1}{\alpha^4}(1+b\alpha+c\alpha^2+b\alpha^3+\alpha^4)$<br>But $x=\alpha$ is a root then $P(\alpha)=\alpha^4+b\alpha^3+c\alpha^2+b\alpha+1=0$<br>$P(\frac{1}{\alpha})=\frac{1}{\alpha^4}\times0=0$.<br>Therefore $x=\frac{1}{\alpha}$ is a root. | 1 Mark:<br>Correct answer. |
| 6(e)<br>(iii) | Since $\alpha$ cannot be 0, 1 or –1 then $\alpha$ and $\frac{1}{\alpha}$ are distinct.<br>Now $\alpha$ is a multiple root then<br>$P(x)=x^4+bx^3+cx^2+bx+1=(x-\alpha)^2(x-\frac{1}{\alpha})(ex+f)$<br>To find *e* and *f*<br>Equating the coefficients of $x^4$ then *e* = 1<br>Equating the constants then $\alpha^2\times-\frac{1}{\alpha}\times f=1$ or $f=-\frac{1}{\alpha}$<br>$\therefore P(x)=(x-\alpha)^2(x-\frac{1}{\alpha})(x-\frac{1}{\alpha})=(x-\alpha)^2(x-\frac{1}{\alpha})^2$<br>$P(x)=\left[x^2-(\alpha+\frac{1}{\alpha})x+1\right]^2=x^4+bx^3+cx^2+bx+1$<br>Equating coefficients:<br>$b=-2\left(\propto+\frac{1}{\propto}\right)$<br>$c=(\frac{1}{\propto}+\propto)^2+2$<br>$b^2=4(\propto^2+2+\frac{1}{\propto^2})$ | 2 Marks:<br>Correct answer.<br><br>1 Mark: Makes some progress towards the solution. |

| | | |
|---|---|---|
| | $= \frac{1}{4}b^2 + 2$<br>$\therefore b^2 - 4c + 8 = 0$ | |

**Question 7**

| | | |
|---|---|---|
| 7(a) | Let $u = x$, $du = dx$ and $v = e^x$, $dv = e^x dx$<br>$\int xe^x dx = xe^x - \int e^x dx$<br>$= xe^x - e^x + C$ | 2 Marks: Correct answer.<br>1 Mark: Applies integration by parts. |
| 7(b) | $\frac{8-2x}{(x^2+4)(x+1)} = \frac{ax+b}{x^2+4} + \frac{c}{x+1}$<br>$8 - 2x = (ax+b)(x+1) + c(x^2+4)$<br>Let $x = -1$    Coefficients of $x^2$    $x = 0$<br>$10 = 5c$    $0 = a + c$    $8 = b + 4c$<br>$c = 2$    $a = -2$    $b = 0$<br>$\int_0^4 \frac{8-2x}{(x^2+4)(x+1)} dx = \int_0^4 \frac{-2x}{x^2+4} + \frac{2}{x+1} dx$<br>$= \left[-\ln(x^2+4) + 2\ln(x+1)\right]_0^4$<br>$= (-\ln 20 + 2\ln 5) - (-\ln 4 + 2\ln 1)$<br>$= -(\ln 20 - \ln 4) + 2(\ln 5 - \ln 1) = \ln 5$ | 3 Marks: Correct answer.<br>2 Mark: Correctly finds one of the integrals.<br>1 Mark: Makes some progress in finding *a*, *b* or *c*. |
| 7(c) | $t = \tan\frac{\theta}{2}$<br>$dt = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta$ or $dt = \frac{1}{2}(1+t^2)d\theta$ or $d\theta = \frac{2}{1+t^2}dt$<br>When $\theta = 0$ then $t = 0$ and when $\theta = \frac{\pi}{2}$ then $t = 1$ | 4 Marks: Correct answer.<br>3 Marks: Correct expression for the integral in terms of *t* using partial fractions |

| | | |
|---|---|---|
| | $2-\cos\theta+2\sin\theta=\frac{2(1+t^2)-(1-t^2)+2(2t)}{1+t^2}$<br>$=\frac{3t^2+4t+1}{1+t^2}=\frac{(3t+1)(t+1)}{1+t^2}$<br>$\int_0^{\frac{\pi}{2}}\frac{1}{2-\cos\theta+2\sin\theta}d\theta=\int_0^1\frac{1+t^2}{(3t+1)(t+1)}\times\frac{2}{1+t^2}dt$<br>$=\int_0^1\frac{2}{(3t+1)(t+1)}d$<br>$=\int_0^1\left\{\frac{3}{(3t+1)}-\frac{1}{(t+1)}\right\}dt$<br>$=\left[\ln(3t+1)-\ln(t+1)\right]_0^1$<br>$=2\ln 2-\ln 2=\ln 2$ | 2 Marks: Finds the value of $2-\cos\theta+2\sin\theta$ in terms of *t* and changes the limits.<br><br>1 Mark: Sets up the integration using *t* formulas |
| 7(d) (i) | Integration by parts<br>$I_n=\int_0^{\frac{\pi}{2}}\sin^{n-1}x\sin x dx$<br>$=-\left[\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x dx$<br>$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x dx$ | 2 Marks: Correct answer.<br>1 Mark: Sets up the integration and shows some understanding. |
| 7(d) (ii) | $I_n=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x dx$<br>$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)dx$<br>$=(n-1)\int_0^{\frac{\pi}{2}}(\sin^{n-2}x-\sin^n x)dx$<br>$=(n-1)\left[I_{n-2}-I_n\right]=(n-1)I_{n-2}-nI_n+I_n$<br>$nI_n=(n-1)I_{n-2}$<br>$I_n=\frac{(n-1)}{n}I_{n-2}$ | 2 Marks: Correct answer.<br><br>1 Mark: Makes some progress towards the solution. |
| 7d(iii) | $I_4=\frac{(4-1)}{4}I_2$<br>$=\frac{3}{4}\times\frac{(2-1)}{2}I_0$<br>$=\frac{3}{8}\times\int_0^{\frac{\pi}{2}}1dx=\frac{3\pi}{16}$ | 1 Mark: Correct answer. |