Solutions to Extension 2 Maths Test Ass^T Task 1 2020

Q1 Consider the statement

For any integers a and b, $a + b \geq 15$ implies that $a \geq 8$ or $b \geq 8$,

(i) State the contrapositive of this statement

(ii) Hence prove this statement is true using contrapositive.

[1+2=3]

(i) For any integers a and b, $a<8$ and $b<8$ implies that $a+b<15$

(ii) Proof: Suppose that a and b are integers such that $a<8$ and $b<8$.

Since they are integers this implies that $a \leq 7$ and $b \leq 7$.

$\therefore a+b \leq 14 \Rightarrow a+b < 15$ which is true.

Q2 (i) Let $x \in Z$. Prove by contradiction that if $5x - 7$ is odd, then x is even.

(ii) Hence prove directly that if $5x - 7$ is odd, then $9x + 2$ is even.

[2+~~3~~=~~5~~] 2 4 -1

(i) To Prove by contradiction:

Assume that if $5x-7$ is odd, then x is odd.

Let $x = 2y+1$, $y \in Z$

$\therefore$ $5x-7 = 5(2y+1) - 7 = 10y+5-7 = 10y-2 = 2(5y-1)$

Since $5y-1$ is an integer, $5x-7$ is even if x is odd.

thus by contradiction, $5x-7$ is odd if x is even.

(ii) If $5x-7$ is odd, then we have already proved that x is even.

$\therefore$ let $x = 2z$, $z \in Z$.

Thus, $9x+2 = 9 \times 2z + 2 = 18z+2 = 2(9z+1)$

Since $9z+1$ is an integer, $9x+2$ must be even.

So, if $5x-7$ is odd, $9x+2$ is Even $\square$.

Q3. Let $x \in Z$. (i) Prove that if $3|x$, then $3|x^2$.

(ii) Prove that if $3 \nmid x$, then $3|(x^2-1)$, using cases.

[2+~~4~~=~~6~~] 3 5 -1.

(i) Proof: Assume that 3 divides x. Then $x = 3q$, $q \in \mathbb{Z}$ 1

Hence $x^2 = 9q^2 = 3(3q^2)$. 1

Since $3q^2 \in \mathbb{Z}$, it follows that $3|x^2$.

(ii) To Prove that if $3 \nmid x$, then $3|(x^2-1)$:

Proof: If $3 \nmid x$, then $x = 3q+1$ or $x = 3q+2$, $q \in \mathbb{Z}$ 1

<u>Case 1:</u> If $x = 3q+1$, $q \in \mathbb{Z}$ then

$$x^2-1 = (3q+1)^2-1 = 9q^2+6q+1-1 = 3(3q^2+2q)$$ 1

Since $3q^2+2q$ is an integer, $3|(x^2-1)$.

<u>Case 2:</u> If $x = 3q+2$, $q \in \mathbb{Z}$ then

$$x^2-1 = (3q+2)^2-1 = 9q^2+12q+4-1 = 3(3q^2+4q+1)$$

Since $3q^2+4q+1$ is an integer, $3|(x^2-1)$ 1

$\therefore$ It is true, if $3 \nmid x$, then $3|(x^2-1)$

$\square$

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Q.4 If $T(0) = 6$ and $T_n = 4T_{n-1} + 2^n$ for $n \geq 1$,

use Induction to prove that $T_n = 7 \cdot 4^n - 2^n$ [5]

Proof: Given $T(0) = 6$ and $T_n = 4T_{n-1} + 2^n$, $n \geq 1$.

Show true for $n=1$. $T_1 = 4T_0 + 2^1 = 4 \times 6 + 2 = 26$

$T_1 = 7.4^1 - 2^1 = 28 - 2 = 26$ thus true for $n=1$. 1

Assume true for $n=k$. i.e. $T_k = 7.4^k - 2^k$

Prove true for $n=k+1$ i.e. $T_{k+1} = 7.4^{k+1} - 2^{k+1}$ 1

Now, from the recursion formula,

$$T_{k+1} = 4T_k + 2^{k+1}$$
$$= 4[7.4^k - 2^k] + 2^{k+1} \quad 1$$
$$= 7.4^{k+1} - 4.2^k + 2^{k+1}$$
$$= 7.4^{k+1} - 2^2.2^k + 2^{k+1} \quad 1$$
$$= 7.4^{k+1} - 2.2^{k+1} + 2^{k+1} \quad 1$$
$$= 7.4^{k+1} - 2^{k+1}$$

= RHS ∴ true using Induction.

more room

Q.5 Use a calculus method to prove that if $x \in R$, $x > 0$, then $x^4 + x^{-4} \geq 2$. [4]

Let $f(x) = x^4 + x^{-4}$

$f'(x) = 4x^3 - 4x^{-5} = 4x^{-5}(x^8 - 1) = 0$ for max/min/stat. points

Since $x > 0$, $4x^{-5} \neq 0$ ∴ $f'(x) = 0$ when $x^8 - 1 = 0$ i.e. $x = \pm 1$

But $x > 0$, ∴ disregard $x = -1$. 1 For $x = 1$, $f(x) = 1 + 1 = 2$. 1

Now, as $x \to +\infty$, $f(x) = x^4 + x^{-4} = x^4 + \frac{1}{x^4} \to +\infty$ 1

4

and there are no other t.p.s.

∴ $f(1) = 2$ is the minimum value of $f(x)$ and so

$x^4 + x^{-4} \geq 2 \quad \forall x$.

Q6 The diagram below shows two right angled triangles.

The left one has sides a, b and c where c is the length of the hypotenuse.

The triangle on the right has sides of length a+1, b+1 and c+1, where c+1 is the length of the hypotenuse. Show that a, b and c cannot all be integers.

[4]

Proof: For the 1st triangle, $a^2+b^2=c^2$

ie $a^2+b^2-c^2=0$ — ① 1

For the 2nd triangle,

$(b+1)^2+(a+1)^2=(c+1)^2$

ie $b^2+2b+1+a^2+2a+1=c^2+2c+1$

ie $\underbrace{a^2+b^2-c^2}_{=0 \text{ from ①}}+2(a+b)+1=2c$ 1

1

ie $2(a+b)+1=2c$

Now, since a, b, c are all positive, if a, b & c are all integers, then LHS is odd, and RHS is even, impossible! 1

∴ a, b, c cannot all be integers.

Q.7 Prove by contradiction, the proposition that:

For each real number x, if $0 < x < 1$, then

$$\frac{1}{x(1-x)} \geq 4$$

[4]

Proof: Using contradiction:

Assume there exists an x, $0<x<1$, such that

$$\frac{1}{x(1-x)} < 4 \quad -① $$

Now, since $0<x<1$, both x and $(1-x)$ are +ve, $\therefore x(1-x)>0$ 1

$\times$ b.s. of ① by $x(1-x)$ to obtain:

$1 < 4x(1-x)$ 1

i.e. $1 < 4x - 4x^2 \Rightarrow 4x^2-4x+1 < 0$

$(2x-1)^2 < 0$ 1

But since $(2x-1)$ is real, $(2x-1)^2 > 0$ which is a contradiction of the last statement. 1

Hence $\frac{1}{x(1-x)} \geqslant 4$ must be true.

4.

Q.8 (i) Show that $\frac{a}{b}+\frac{b}{a}\geq 2$ using the AM/GM inequality.

(ii) Hence show that, for a, b and c all positive reals, that

$$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$$

[2+2=4]

(i) Let $\frac{a}{b}=x$ and $\frac{b}{a}=y$.

$\therefore \frac{x+y}{2} \geqslant \sqrt{xy}$ using Am/Gm Inequality. 1

$\therefore \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right) \geqslant \sqrt{\frac{a}{b}\frac{b}{a}}$ i.e. $\frac{a}{b}+\frac{b}{a}\geqslant 2$. 1

(ii) $\because a^3+a^3+b^3 \geqslant 3\sqrt[3]{a^3a^3b^3} = 3a^2b$

$b^3+b^3+c^3 \geqslant 3\sqrt[3]{b^3b^3c^3} = 3b^2c$ 1

$c^3+c^3+a^3 \geqslant 3\sqrt[3]{c^3c^3a^3} = 3c^2a$

Adding, we get $3(a^3+b^3+c^3) \geq 3(a^2b+b^2c+c^2a)$ 1

$\therefore a^3+b^3+c^3 \geqslant a^2b+b^2c+c^2a$

$\underline{\underline{\text{or}}}$ see next sheet

Q8(ii) Alternative Solution.

$$a^3+b^3+c^3 \geqslant a^2b+b^2c+c^2a$$

ie. Prove that:

$$a^3+b^3+c^3 - (a^2b+b^2c+c^2a) \geqslant 0 \quad - ③$$

Letting $A=a^2b$, $B=b^2c$, $C=c^2a$

$$\frac{A+B+C}{3} \geqslant \sqrt[3]{a^2b.b^2c.c^2a} = \sqrt[3]{a^3b^3c^3}$$

$$\Rightarrow a^2b+b^2c+c^2a = A+B+C \geqslant 3\sqrt[3]{a^3b^3c^3} = 3abc \quad - ①$$

Also, $\frac{a^3+b^3+c^3}{3} \geqslant \sqrt[3]{a^3b^3c^3}$

ie $a^3+b^3+c^3 \geqslant 3\sqrt[3]{abc} = 3abc \quad - ②$

Subst. ① & ② into ③

$3abc - 3abc \geqslant 0$ as required.

Q. 9 (i) Find the square roots of $-8-6i$.

(ii) Hence or otherwise, solve the equation $2x^2+(1+i)x+(1+i)=0$

[4]

(i) Let $z = x+iy$ and $z^2 = -8-6i$ $\quad x,y \in \mathbb{R}$.

$\therefore\ z^2 = x^2-y^2+2xyi = -8-6i$

$\Rightarrow x^2-y^2=-8 \quad -①$ $\qquad 2xy=-6 \Rightarrow y = \frac{-3}{x} \quad -②$ $\quad 1$

② into ① yields $\quad x^2 - \frac{9}{x^2} = -8 \qquad$ i.e. $x^4+8x^2-9=0$

$(x^2-1)(x^2+9)=0$

Since x is real, $x=\pm 1 \qquad$ If $x=1, y=-3$

If $x=-1, y=3$

Thus the square roots are: $z_1 = 1-3i$, $z_2 = -1+3i$ $\quad 1$

(ii) To solve, $\quad x = \frac{-(1+i) \pm \sqrt{(1+i)^2-4.2(1+i)}}{4} = \frac{-1-i \pm \sqrt{1+2i-1-8-8i}}{4}$

$= \frac{-1-i \pm \sqrt{-8-6i}}{4}$ $\quad 1$

i.e. from (i) $\quad x = \frac{-1-i\pm(1-3i)}{4} = \frac{-1-i+1-3i}{4} = -i$

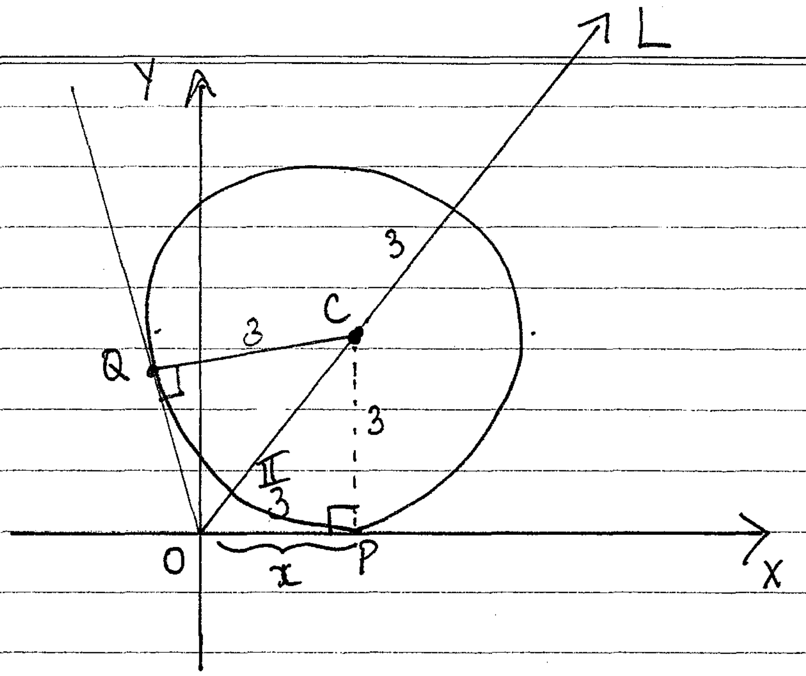
or $\frac{-1-i-1+3i}{4} = -\frac{1}{2}+\frac{i}{2}$

1

4

Q.10

(i)



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(ii) For coords of centre, $\tan\frac{\pi}{3} = \frac{3}{x}$ $\therefore x = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$.

$y = 3$ 2

$\therefore$ Circle has eqn $|z - \sqrt{3} - 3i| = 3$

(iii) $\triangle OCP \equiv \triangle OCQ$ $\therefore \angle COQ = \frac{\pi}{3}$. 1

$\therefore$ max $\arg z_0$ is $\frac{2\pi}{3}$

Q11. $\text{Im}\{(2z - \bar{z}(1+i)) = 0$ & $\text{Re}\,(2z - \bar{z}(1+i)) < 4$

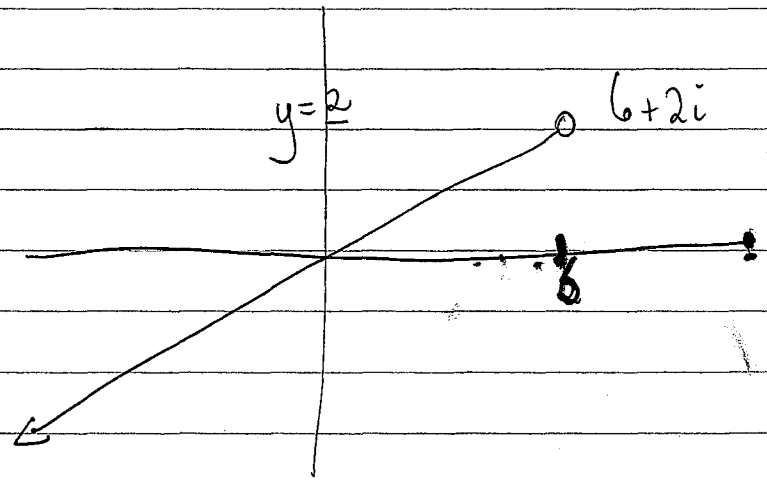
$$2z - \bar{z}(1+i) = 2(x+yi) - (x-yi)(1+i)$$
$$= 2x + 2yi - [x + xi - iy + y]$$
$$= x - y + 3yi - xi \quad 1$$
$$= (x-y) + i(3y-x)$$

(4)

$\text{Im}(\quad) = 0$ when $3y = x$ 1

$\text{Re}(\quad) < 4$ when $x - y < 4$ 1 ie. $3y - y < 4$, $2y < 4$, $y < 2$

So we are sketching $3y = x$ for $y < 2$. 1

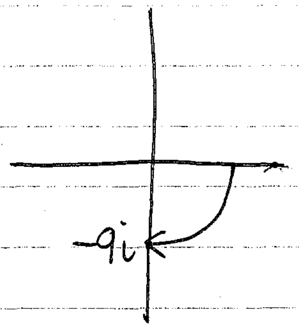


1 (5)

(1).

Q.12. $z^4 = -9i$

(i)



Let $z = r\,\text{cis}\,\theta$

$z^4 = r^4\,\text{cis}\,4\theta$

$= r^4(\cos 4\theta + i\sin 4\theta)$

$= -9i$

$\therefore \cos 4\theta = 0$ and $\sin 4\theta = -1$

$4\theta = \frac{3\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$

$\theta = \frac{3\pi}{8} + \frac{k\pi}{2} = \frac{3\pi + 4k\pi}{8}$

$= \frac{\pi}{8}(3+4k)$

$r^4 = 9$

$r = \sqrt{3}$ 2

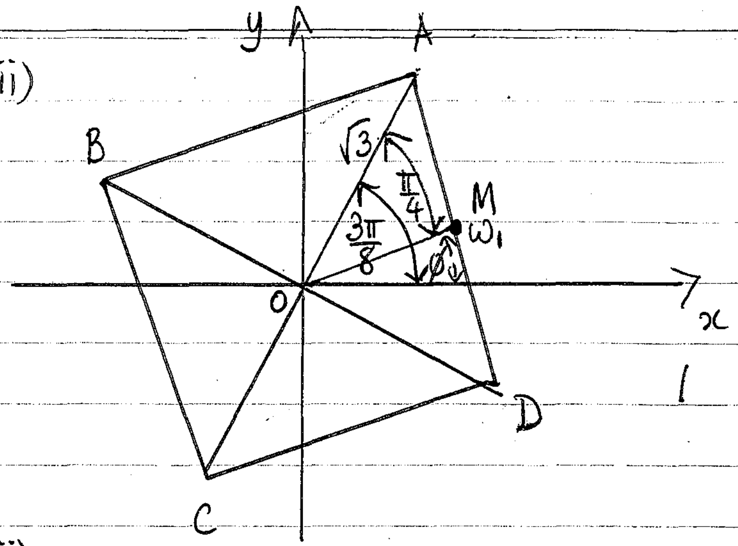
$\therefore z = \sqrt{3}\,e^{i\frac{\pi}{8}(3+4k)}$

$z_1 = \sqrt{3}\,e^{i\frac{3\pi}{8}}$ and $z_4 = \sqrt{3}\,e^{i\frac{15\pi}{8}}$

$z_2 = \sqrt{3}\,e^{i\frac{7\pi}{8}}$

$z_3 = \sqrt{3}\,e^{i\frac{11\pi}{8}}$

(ii)



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(iii)

$|OM| = \sqrt{3}\cos\frac{\pi}{4} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

$\omega_1 = |OM|\,\text{cis}\,\phi$

where $\phi = \frac{3\pi}{8} - \frac{\pi}{4} = \frac{\pi}{8}$ 2

$\therefore \omega_1 = \frac{\sqrt{6}}{2}e^{i\frac{\pi}{8}}$

(iv) $\omega = \left(\frac{\sqrt{6}}{2}e^{i\frac{\pi}{8}}\right)^4$

$= \frac{36}{16}e^{i\frac{\pi}{2}} = \frac{9}{4}i$ 1

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Q.13 (i) $e^{i\theta} = \cos\theta + i\sin\theta$ —①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$e^{-i\theta} = \cos\theta - i\sin\theta$ —②

① + ② yields $2\cos\theta = e^{i\theta} + e^{-i\theta}$ $\therefore \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ 1

① − ② yields $2i\sin\theta = e^{i\theta} - e^{-i\theta}$ $\therefore \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ 1

Q.13.

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \quad -① \qquad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

(ii) $$\sin^3\theta\cos^2\theta = -\frac{1}{8i}\left(e^{i\theta} - e^{-i\theta}\right)^3 \cdot \frac{1}{4}\left(e^{i\theta} + e^{-i\theta}\right)^2$$

$$= -\frac{1}{8i}\left(e^{i3\theta} - 3e^{i2\theta}.e^{-i\theta} + 3e^{i\theta}.e^{-i2\theta} - e^{-i3\theta}\right) \cdot \frac{1}{4}\left(e^{i2\theta} + 2e^{i\theta}.e^{-i\theta} + e^{-i2\theta}\right)$$

$$= -\frac{1}{32i}\left(e^{i3\theta} - e^{-i3\theta} - 3e^{i\theta} + 3e^{-i\theta}\right)\left(e^{i2\theta} + 2 + e^{-i2\theta}\right)$$

$$= -\frac{1}{32i}\left(e^{i5\theta} + 2e^{i3\theta} + e^{i\theta} - e^{-i\theta} - 2e^{-i3\theta} - e^{-i5\theta} - 3e^{i3\theta} - 6e^{i\theta} - 3e^{-i\theta} + 3e^{i\theta} + 6e^{-i\theta} + 3e^{-i3\theta}\right)$$

This is a bit tedious

$$= -\frac{1}{32i}\left[\left(e^{i5\theta} - e^{-i5\theta}\right) - \left(e^{i3\theta} - e^{-i3\theta}\right) - \left(2e^{i\theta} - 2e^{-i\theta}\right)\right]$$

$$= \frac{1}{32i}\left[2\left(e^{i\theta} - e^{-i\theta}\right) + \left(e^{i3\theta} - e^{-i3\theta}\right) - \left(e^{i5\theta} - e^{-i5\theta}\right)\right]$$

$$= \frac{1}{16}\left[\frac{2}{2i}\left(e^{i\theta} - e^{-i\theta}\right) + \frac{1}{2i}\left(e^{i3\theta} - e^{-i3\theta}\right) - \frac{1}{2i}\left(e^{i5\theta} - e^{-i5\theta}\right)\right]$$

$$= \frac{1}{16}\left[2\sin\theta + \sin 3\theta - \sin 5\theta\right] \quad \text{from ①}$$

(iii)

Now, to solve $\sin 5\theta - \sin 3\theta = 0$

$$\sin^3\theta\cos^2\theta = \frac{1}{16}\left[2\sin\theta + \sin 3\theta - \sin 5\theta\right]$$

$$\Rightarrow 16\sin^3\theta\cos^2\theta - 2\sin\theta = \sin 3\theta - \sin 5\theta$$

ie. $$\sin 5\theta - \sin 3\theta = 2\sin\theta - 16\sin^3\theta\cos^2\theta$$

$$= 2\sin\theta\left(1 - 8\sin^2\theta\cos^2\theta\right)$$

$$= 0$$

When either $\sin\theta = 0$ or

$$1 = 8\sin^2\theta\cos^2\theta$$

Using $\sin 2\theta = 2\sin\theta\cos\theta$,

$$1 = 2\left[\sin 2\theta\right]^2$$

$$1 = 2\sin^2 2\theta$$

$$\therefore \frac{1}{2} = \sin^2 2\theta$$

$$\sin 2\theta = \pm\frac{1}{\sqrt{2}}$$

Q.13

Now, for $\sin\theta = 0$ and $0 \leq \theta < \pi$

$\theta = 0$

and for $\sin 2\theta = \pm\frac{1}{\sqrt{2}}$, $2\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ 1

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If using $\sin^3\theta\cos^2\theta = \sin^3\theta(1-\sin^2\theta)$
$= \sin^3\theta - \sin^5\theta$

$$\sin^5\theta = \left(\frac{1}{2i}\right)^5 (e^{i\theta} - e^{-i\theta})^5$$

$$= -\frac{1}{32i}\left(e^{i5\theta} - 5e^{i3\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-i3\theta} - e^{-i5\theta}\right)$$

$$= -\frac{1}{32i}\left(2i\sin 5\theta + 5\times 2i\sin 3\theta + 10\times 2i\sin\theta\right)$$

$$= -\frac{1}{32i}\left(2i\sin 5\theta + 10i\sin 3\theta + 20i\sin\theta\right)$$

$\therefore \sin^3\theta - \sin^5\theta = \ldots$

Q.14 $z = \cos\theta + i\sin\theta$

(i) $1 + z = 1 + \cos\theta + i\sin\theta$ using $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$= 2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ $\Rightarrow 1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

(ii) $|z_1| = |z_2| = 1$ $\arg z_1 = \alpha$, $\arg z_2 = \beta$.

$$\frac{z_1 + z_1 z_2}{z_1 + 1} = \frac{z_1(1 + z_2)}{z_1 + 1}$$

Now, from part (i), $1 + z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

Since $\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$ has modulus 1, $1 + z$ has modulus $2\cos\frac{\theta}{2}$

$\therefore$ $1 + z_2$ has modulus $2\cos\frac{\beta}{2}$ and $1 + z_1$ has modulus $2\cos\frac{\alpha}{2}$

$\therefore \left|\frac{z_1(1+z_2)}{z_1+1}\right| = \frac{1 \times \cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$ since $|z_1| = 1$ also.

For $\arg\left(\frac{z_1(1+z_2)}{z_1+1}\right)$, $= \arg z_1 + \arg(1+z_2) - \arg(z_1+1)$

$= \alpha + \frac{\beta}{2} - \frac{\alpha}{2} = \frac{\alpha}{2} + \frac{\beta}{2} = \frac{\alpha+\beta}{2}$

Q14(iii) If $\dfrac{z_1 + z_1 z_2}{z_1 + 1} = 2i$,

then the real part $= 0$ and imaginary part $= 2$

Now, $\dfrac{z_1 + z_1 z_2}{z_1+1} = \dfrac{z_1(1+z_2)}{z_1+1} = \dfrac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}\left(\operatorname{cis}\dfrac{\alpha+\beta}{2}\right)$

$$= \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}\left(\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right)$$

$\operatorname{Re}\left(\dfrac{z_1+z_1z_2}{z_1+1}\right) = \dfrac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \cdot \cos\left(\dfrac{\alpha+\beta}{2}\right) = 0 \quad — ①$

and $\operatorname{Im}\left(\dfrac{z_1+z_1z_2}{z_1+1}\right) = \dfrac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \cdot \sin\left(\dfrac{\alpha+\beta}{2}\right) = 2. \quad — ②$

Now, assuming the number exists, $\dfrac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \neq 0$

$\therefore \cos\left(\dfrac{\alpha+\beta}{2}\right) = 0$ and $\sin\left(\dfrac{\alpha+\beta}{2}\right) > 0$

$\Rightarrow \dfrac{\alpha+\beta}{2} = \dfrac{\pi}{2} \quad —①'$ and $\sin\left(\dfrac{\alpha+\beta}{2}\right) = 1 \quad — ③$

from here

$\therefore \dfrac{\alpha}{2} + \dfrac{\beta}{2} = \dfrac{\pi}{2} \quad —①$ and $\sin\left(\dfrac{\alpha+\beta}{2}\right) = 1 \quad —②$, $\quad —①''$

$\Rightarrow \dfrac{\beta}{2} = \dfrac{\pi}{2} - \dfrac{\alpha}{2} \quad \therefore \cos\dfrac{\beta}{2} = \cos\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right) \Rightarrow \cos\dfrac{\beta}{2} = \sin\dfrac{\alpha}{2}$ from ①'

Q14 (continued)

Substituting ③ into ②.

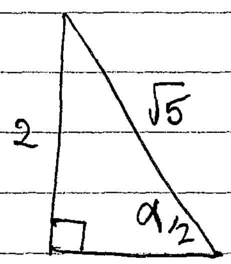
$$\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \cdot 1 = 2$$

$$\therefore 2\cos\frac{\alpha}{2} = \cos\frac{\beta}{2}.$$

But $\cos\frac{\beta}{2} = \sin\frac{\alpha}{2}$ from ①".

$$\therefore 2\cos\frac{\alpha}{2} = \sin\frac{\alpha}{2} \Rightarrow \tan\frac{\alpha}{2} = 2$$

Now $\tan\frac{\alpha}{2} = 2$ $\therefore$ $\sin\frac{\alpha}{2} = \frac{2}{\sqrt{5}}$, $\cos\frac{\alpha}{2} = \frac{1}{\sqrt{5}}$



$$\therefore \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{4}{5}.$$

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$

$$\therefore z_1 = -\frac{3}{5} + \frac{4}{5}i.$$

Also, since $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\pi}{2}$, $\alpha + \beta = \pi$ $\therefore \cos\beta = \cos(\pi - \alpha)$

$$= -\cos\alpha$$

$$= \frac{3}{5}$$

$$\sin\beta = \sin(\pi - \alpha)$$

$$= \sin\alpha = \frac{4}{5}$$

$$\therefore z_2 = \frac{3}{5} + \frac{4}{5}i.$$