Mechanics

EXTENSION 2

Newton’s Laws of Motion

Newton’s First Law of Motion

A body remains at rest or in uniform motion unless it is acted on by a non-zero resultant force.

Newton’s Laws of Motion

Newton’s Second Law of Motion

The rate of change of momentum is proportional to the applied force and occurs in the direction of the force.

The momentum ρ of a body is the product of its mass and velocity, $\rho = mv$.

The standard unit of mass is the kg, and of velocity is m/s.

We define one unit of force, one Newton, as the force which gives an acceleration of 1 $\mathrm{m/s^2}$ to a mass of 1 kg.

With these units, Newton’s second law says

$$F = ma.$$

Newton’s Laws of Motion

Newton’s Third Law of Motion

When two objects exert force on each other, the forces are equal in magnitude but opposite in direction

Example 1

A particle moves in a straight line away from a fixed point O such that

$$\frac{1}{v} = A + Bt$$

for positive constants A and B.

a) Show that the retardation of the particle is proportional to the square of the speed.
b) If the retardation is $1\ \mathrm{m/s^2}$ and the velocity is 80 m/s when $t = 0$, find the values of A and B.
c) Express x in terms of t and v in terms of x.

- Example 2.

The force of attraction experienced by a mass m at a distance $x \geq r$ from the centre of the earth is $\frac{mgr^2}{x^2}$, where r is the radius of the earth.

A particle of mass m leaves the surface of the earth with speed u away earth.

a) Find the speed of the particle at distance x away from the centre of the earth.

b) Deduce that the particle will escape if $u^2 > 2gr$.

- Example 3. (Assume that tidal motion is SHM.)

On a certain day, the depth of water at high tide at 5am is 9m. At the following low tide at 11:20 am the depth is 3m. Find the latest time before noon that a ship can enter the harbour if a minimum depth of 7.5 m of water is required.

Resisted Motion

Resisted Motion

Two important results for resisted motion from 3U

1. $$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

2 $$\ddot{x} = v\frac{dv}{dx}$$

Three types of resisted motion

- 1. Along a straight line
- 2. Going up
- 3. Coming down

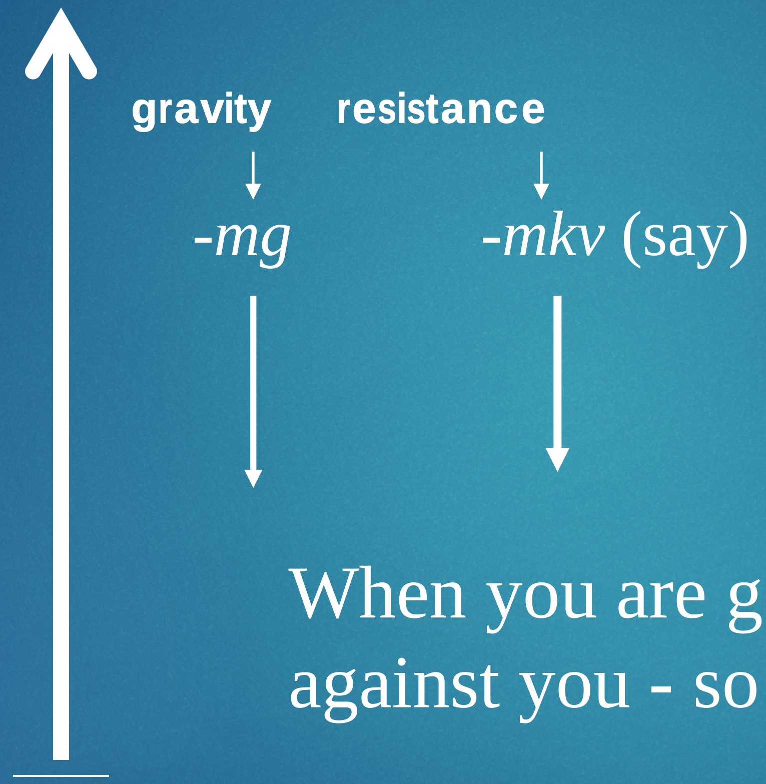
Type 1 - along a horizontal line

$$\boxed{m\ddot{x} = -mkv}$$

When you move on a surface you need not include gravity in your equation .

Resistance always acts against you so make it **negative**.

Type 2 - going up



$$\boxed{m\ddot{x} = -mg - mkv}$$

When you are going up **gravity** acts against you - so make it **negative**.

Resistance always acts against you so make it **negative** as well.

Type 3 - going down

gravity **resistance**

$+mg$ $-mkv$ (say)

$$\boxed{m\ddot{x} = mg - mkv}$$

When you are going down **gravity** acts with you - so make it **positive**.

Resistance always acts against you so make it **negative**.

How do you approach the problems ?

- Draw a forces diagram
- Understand that force = mass $\times$ acceleration
- Write down an initial equation

Along a horizontal line - what the syllabus says.......

- Derive from Newton’s Laws of motion the equation of motion of a particle moving in a single direction under a resistance proportional to a power of the speed
- Derive expressions for velocity as functions of time and displacement where possible
- Derive an expression for displacement as a function of time

Example- Horizontal Motion

A particle unit mass moves in a straight line against a resistance of size $|v + v^3|$ where v is the velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

- Show that v is related to the displacement x by the formula

$$x = \tan^{-1}\left[\frac{Q-v}{1+Qv}\right]$$

$$m\ddot{x} = -m\left(v + v^3\right) \text{ but } m = 1$$

$$\ddot{x} = -\left(v + v^3\right)$$

$$v\frac{dv}{dx} = -\left(v + v^3\right)$$

$$\frac{dv}{dx} = \frac{-v\left(1 + v^2\right)}{v}$$

$$\frac{dx}{dv} = \frac{-1}{\left(1 + v^2\right)}$$

$$x = -\tan^{-1} v + C, \text{ but when } x = 0, v = Q$$

$$0 = -\tan^{-1} Q + C \Rightarrow C = \tan^{-1} Q$$

$$x = \tan^{-1} Q - \tan^{-1} v$$

now......

$$\tan x = \tan\left(\tan^{-1} Q - \tan^{-1} v\right)$$

$$= \frac{\tan\left(\tan^{-1} Q\right) - \tan\left(\tan^{-1} v\right)}{1 + \tan\left(\tan^{-1} Q\right) \times \tan\left(\tan^{-1} v\right)}$$

$$= \frac{Q - v}{1 + Qv}$$

$$x = \tan^{-1}\left(\frac{Q - v}{1 + Qv}\right)$$

Motion upwards - what the syllabus says…….

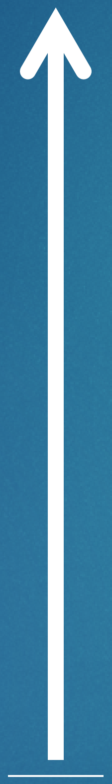
- Derive from Newton’s Laws of motion the equation of motion of a particle moving vertically upwards in a medium with resistance proportional to the first or second power of the speed

Derive expressions for velocity as functions of time and displacement where possible
Solve problems by using expressions derived for acc, vel and displacement.

Motion upwards - a problem…

A particle of unit mass is thrown vertically upwards with velocity of U into the air and encounters a resistance of kv^2. Find the greatest height H achieved by the particle and the corresponding time.

Forces diagram



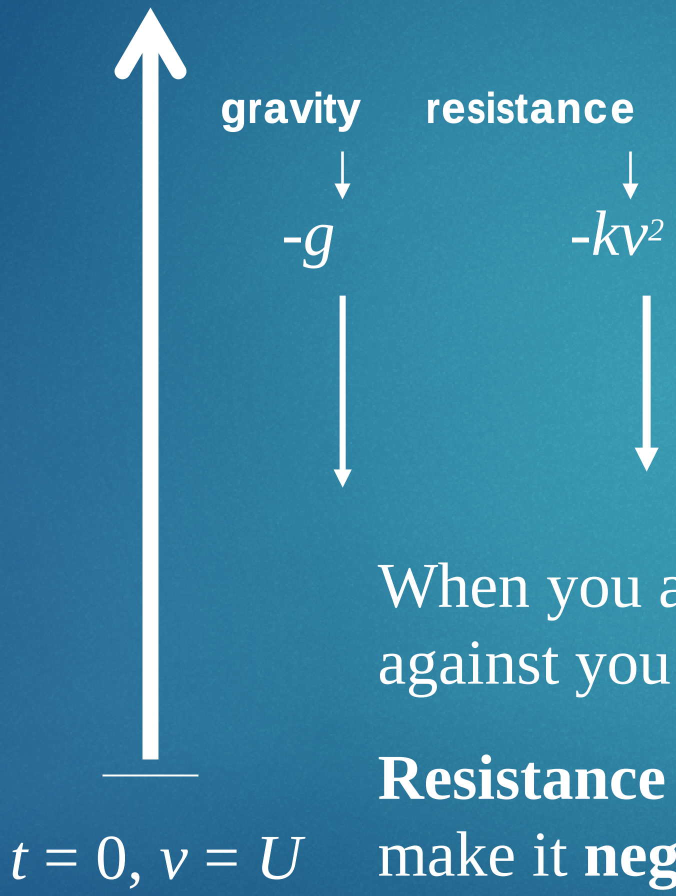
$t = 0,\ v = U$

Forces diagram

$t = 0, v = U$

When you are going up **gravity** acts against you - so make it **negative**.

Forces diagram



When you are going up **gravity** acts against you - so make it **negative**.

Resistance always acts against you so make it **negative** as well.

The equation of motion is given by…...

$$\ddot{x} = -g - kv^2$$

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = \frac{-g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{-v}{g + kv^2}$$

$$x = \int \frac{-v}{g + kv^2} dv$$

$$x = \frac{-1}{2k}\int \frac{2kv}{g + kv^2} dv$$

$$x = \frac{-1}{2k}\ln\left(g + kv^2\right) + C \Rightarrow x = 0, v = U \Rightarrow C = \frac{1}{2k}\ln\left(g + kU^2\right)$$

The equation of motion is given by…...

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$$x = \frac{-1}{2k}\ln\left(g+kv^2\right)+C \Rightarrow x=0, v=U \Rightarrow C = \frac{1}{2k}\ln\left(g+kU^2\right)$$

$$x = \frac{1}{2k}\left(\ln\left(g+kU^2\right)-\ln\left(g+kv^2\right)\right)$$

$$x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right) \qquad x = H, v = 0$$

$$H = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g}\right)$$

$$H = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g}\right)$$

$$H = \frac{1}{2k}\ln\left(1+\frac{k}{g}U^2\right)$$

we use $\frac{dv}{dt} = -g - kv^2$ to get an expression for t

$$\frac{dt}{dv} = \frac{-1}{g + kv^2}$$

$$t = \int \frac{-dv}{g + kv^2}$$

$$t = \frac{-1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}v\right) + C$$

using $v = U$ when $t = 0$ this becomes .....

$$\mathrm{C} = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}U\right)$$

$$t = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}U\right) - \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}v\right)$$

using $v = 0$ for greatest height...

$$t = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}U\right)$$

Motion downwards - what the syllabus says.......

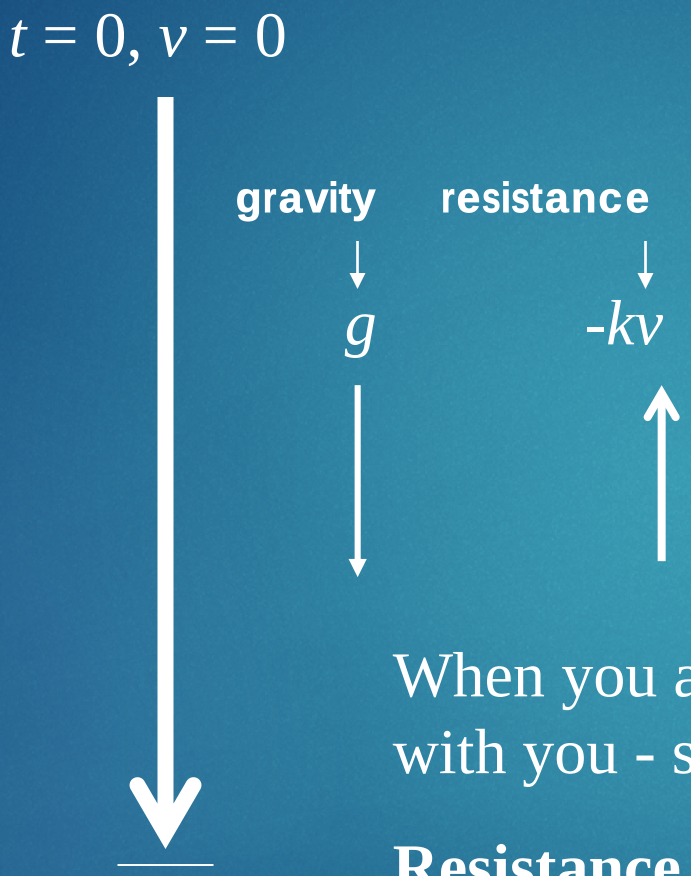
- Derive from Newton's Laws of motion the equation of motion of a particle falling in a medium with resistance proportional to the first or second power of the speed
- find terminal velocity

Derive expressions for velocity as functions of time and displacement where possible
Solve problems by using expressions derived for acc, vel and displacement.

Motion downwards - a problem…

- A particle of unit mass falls vertically from rest in a medium and encounters a resistance of kv.
- Find the velocity in terms of time and use two different methods to find the terminal velocity.

Forces diagram



When you are going down **gravity** acts with you - so make it **positive**.

Resistance always acts against you so make it **negative**.

$$\ddot{x} = g - kv$$

$$\frac{dv}{dt} = g - kv$$

$$\frac{dt}{dv} = \frac{1}{g - kv}$$

$$t = \int \frac{dv}{g - kv}$$

$$t = -\frac{1}{k} \int \frac{-kdv}{g - kv}$$

$$t = -\frac{1}{k} \ln(g - kv) + C, v = 0 \text{ when t} = 0....$$

$$\mathrm{C} = \frac{1}{k} \ln(g)$$

$$t = \frac{1}{k} \ln(g) - \frac{1}{k} \ln(g - kv)$$

$t = \frac{1}{k}\ln(g) - \frac{1}{k}\ln(g - kv)$ and making v the subject

$$kt = \ln\left(\frac{g}{g - kv}\right)$$

$$e^{kt} = \frac{g}{g - kv}$$

$$e^{-kt} = \frac{g - kv}{g}$$

$$ge^{-kt} = g - kv$$

$$kv = g - ge^{-kt}$$

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

……..

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

Now we can find the terminal velocity two ways:

1. Consider what happens to v as $t \to \infty$.

$$v \to \frac{g}{k}(1 - 0)$$

$$\therefore \text{ terminal velocity } = \frac{g}{k}$$

....

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

Now we can find the terminal velocity two ways:

1. Consider what happens to v as $t \to \infty$.

$$v \to \frac{g}{k}(1 - 0)$$

$$\therefore \text{ terminal velocity } = \frac{g}{k}$$

2. Or alternatively we can just let the acceleration equal zero

$$\ddot{x} = g - kv$$

$$0 = g - kv \Rightarrow v = \frac{g}{k}$$

- When a problem involves both motion upward and motion downward, restart the problem at the point where direction changes.
- The positive direction should always be that in which motion is occurring.
- The origin is the point from which motion begins.