2016 MATHS EXTENSION 1 HSC AST 1 - SOLUTIONS

1a) $P(3) = 3^5 - 10(3^3) + 5(3)$

$= -12$ ✓ (1)

$\therefore$ The remainder is -12 (by the remainder theorem)

b) i)

$$\begin{array}{r l}
 & 8x^2 - 4x - 10 \\
x+2 \,\big) & 8x^3 + 12x^2 - 18x - 20 \\
 & 8x^3 + 16x^2 \\
 & \quad -4x^2 - 18x \\
 & \quad -4x^2 - 8x \\
 & \qquad\quad -10x - 20 \\
 & \qquad\quad -10x - 20 \\
 & \qquad\qquad\quad 0
\end{array}$$

$\therefore f(x) = (x+2)(8x^2 - 4x - 10)$ ✓ (1)

$= 2(x+2)(4x^2 - 2x - 5)$

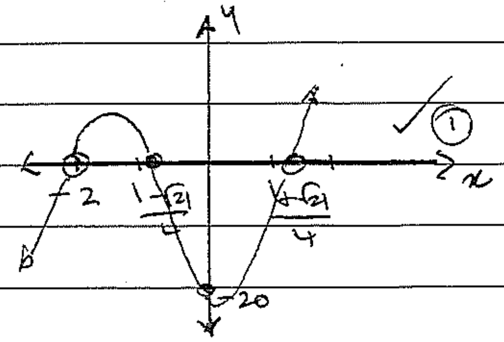
Let $4x^2 - 2x - 5 = 0$

$x = \dfrac{2 \pm \sqrt{4+80}}{8}$ (by the quadratic formula)

$= \dfrac{1 \pm \sqrt{21}}{4}$ ✓ (1)

$\therefore$ The roots are -2, $\dfrac{1+\sqrt{21}}{4}$, $\dfrac{1-\sqrt{21}}{4}$

(ii)



✓ (1)

Checking critical

$f(-3) = -74 < 0$

or $f(-1) = 2 > 0$

$f(0) = -20 < 0$

$f(2) = 56 > 0$

etc

$f(x) > 0$ for $-2 < x < \dfrac{1-\sqrt{21}}{4}$ (1)

or $x > \dfrac{1+\sqrt{21}}{4}$ (1) ✓✓

(iii) See working in (ii).

1 (c) $P(3) = 3^3 - 3k(3) + 6$

$= 33 - 9k$

$P(3) = 0$ (from the factor theorem)

$\therefore 33 - 9k = 0$

$k = \frac{11}{3}$ ✓ 1

d) i) $\alpha\beta\left(\frac{1}{\alpha}\right) = -2$

$\therefore \beta = -2$ ✓ 1

(ii) $\alpha + \beta + \frac{1}{\alpha} = 8$

$\alpha + \frac{1}{\alpha} - 2 = 8$

$\alpha + \frac{1}{\alpha} - 10 = 0$

$\alpha^2 - 10\alpha + 1 = 0$ ✓ 1

$\alpha = \frac{10 \pm \sqrt{100-4}}{2}$

$\alpha = 5 \pm 2\sqrt{6}$ ✓ 1

(iii) $\alpha\beta + \alpha\left(\frac{1}{\alpha}\right) + \frac{\beta}{\alpha} = k$

$(5+2\sqrt{6})(-2) + 1 - \frac{2}{5+2\sqrt{6}} = k$ ✓ 1

$k = -10 - 4\sqrt{6} + 1 - \frac{2(5-2\sqrt{6})}{1}$

$k = -19$ ✓ 1

$\cos t = \frac{x-3}{2}$ $\quad \cos^2 t = \frac{(x-3)^2}{4}$

$\sin t = \frac{y}{2}$ $\quad \sin^2 t = \frac{y}{4}$ $\quad \therefore (x-3)^2 + y^2 = 4$

2 a) $x = 3 + 2\cos t$

$2\cos t = x - 3$ ①

$2\sin t = y$ ②

$①^2 + ②^2$: $4\cos^2 t + 4\sin^2 t = (x-3)^2 + y^2$ ✓ ①

$4(\cos^2 t + \sin^2 t) = (x-3)^2 + y^2$

$4(1) = (x-3)^2 + y^2$

$\therefore (x-3)^2 + y^2 = 4.$ ✓ ①

$\boxed{3}$

This is a circle with centre $(3,0)$ and radius 2 ①.

b) i) $x^2 = -12y$ in form $x^2 = -4ay$

$\therefore a = 3.$ (or $a = -3$ from $x^2 = 4ay$)

Parametric equations: $x = 2at$, $y = -at^2$

$x = -6t$, $y = -3t^2$ ✓ ①

(ii) $P(2, -\frac{1}{3})$ lies on the parabola

$\therefore -6p = 2$ or $-3p^2 = -\frac{1}{3}$

$p = -\frac{1}{3}$ $\quad p^2 = \frac{1}{9}$

$p = \pm\frac{1}{3}.$

$p = -\frac{1}{3}$ is common. ✓ ①

$\boxed{2}$

PQ is a focal chord $\therefore pq = -1$

$\therefore -\frac{1}{3}q = -1$

$q = 3$

②.

$Q(-2aq, -aq^2) = (-2(3)3, -3(3^2))$

$\therefore Q = (-18, -27)$ ✓ ①

or $P(2, -\frac{1}{3})$, $S(0, -3)$

$\therefore$ Eqn. of PS: $y + 3 = \frac{-3+\frac{1}{3}}{2}(x - 0)$

$-2y - 6 = -\frac{8}{3}x$

$-6y - 18 = -8x$

$8x - 6y - 18 = 0$

$4x - 3y - 9 = 0$ ① ✓ ①

$x^2 = -12y$ ②

2b(ii) (cont)

$$② \rightarrow y = -\frac{x^2}{12} \rightarrow ①$$

$$4x - 3\left(-\frac{x^2}{12}\right) - 9 = 0$$

$$4x + \frac{x^2}{4} - 9 = 0$$

$$16x + x^2 - 36 = 0$$

$$x^2 + 16x - 36 = 0$$

$$(x+18)(x-2) = 0$$

$$x = -18, 2$$

when $x = -18$, $y = \frac{(-18)^2}{12}$

$$= -27 \quad ✓ \ 1$$

$$\therefore Q = (-18, -27)$$

c)(i) $x = 2ap$, $y = ap^2$

$$\frac{dx}{dp} = 2a, \quad \frac{dy}{dp} = 2ap.$$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

$$= 2ap \cdot \frac{1}{2a}$$

$$\therefore m_p = p. \quad ✓ \ 1$$

$$\boxed{2}$$

$$y - ap^2 = p(x - 2ap)$$

$$y - ap^2 = px - 2ap^2$$

$$y = px - ap^2 \quad ✓ \ 1$$

(ii) Q is the point $(2aq, aq^2)$

$\therefore$ T lies on the line $x = 2aq$

substitute $x = 2aq$ into $y = px - ap^2$

$$y = p(2aq) - ap^2 \quad ✓ \ 1$$

$$\boxed{2}$$

$$y = 2apq - ap^2$$

$$\therefore T = (2aq, 2apq - ap^2) \quad ✓ \ 1$$

(iii) $P(2ap, ap^2)$, $T(2aq, 2apq - ap^2)$

$$\therefore M_{PT} = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + 2apq - ap^2}{2}\right)$$

$$M_{PT} = (ap + aq, apq) \quad 1$$

(iv) $pq = 1$ $\qquad x = a(p+q)$ $\quad y = apq$ $\quad \boxed{\therefore y = -a.}$ $\quad 1$.

2(c)(iii) (cont) $\therefore M_{PT} = (ap+aq,\ apq)$ ✓ ① (1)

(iv) $x = ap+aq$, $y = apq$, $pq = -1$ (1)

$\therefore y = -a$ ✓ ①

3a) Geometric series $\therefore \frac{b}{2} = \frac{50}{b}$

$b^2 = 100$

$\therefore b = \pm 10$ ✓ ①

b)(i) $d = 8.5 - 12 = 5 - 8.5 = -3.5$, $a = 12$

$T_n = a + (n-1)d$

$T_n = 12 + (n-1)(-3.5)$

$T_n = 15.5 - 3.5n$. ✓ ①

(ii) $S_m = \frac{m}{2}(12 + 15.5 - 3.5m)$

$= \frac{m}{2}(27.5 - 3.5m)$ ✓ ①

$S_m = 0$ when $m = 0$, which is not possible.

or $27.5 - 3.5m = 0$

$3.5m = 27.5$

$m = 7\frac{6}{7}$

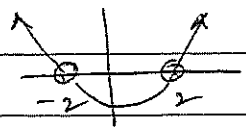
$\therefore m = 8$ ✓ ①

(c)(i) $r = \frac{\frac{3t^{3}}{4}}{3t} = \frac{t^2}{4}$ ✓ ①

The geometric series has a limiting sum when $\left|\frac{t^2}{4}\right|$

but $\frac{t^2}{4} > 0$ $\therefore$ when $\frac{t^2}{4} < 1$

i.e. $t^2 < 4$



$t^2 - 4 < 0$

$(t+2)(t-2) < 0$ ✓ ①

$\therefore -2 < t < 2$

(ii) $S_\infty = \frac{a}{1-r} = \frac{3t}{1-\frac{t^2}{4}} = \frac{3t}{\frac{4-t^2}{4}} = \frac{12t}{4-t^2}$ ✓ ①

3d)(i). $S_n = 1 + 2 + 3 + \dots + n$

This is an arithmetic series with $a = 1$ and $l = n$ or $d = 1$

$S_n = \frac{n}{2}(a+l)$

$= \frac{n}{2}(1+n)$ ✓ 1

(ii). $S_n = 1\times 6 + 2\times 7 + 3\times 8 + \dots + n\times(n+5)$

$= \sum_{r=1}^{n} r(r+5)$ ✓ 1

(iii). $S_n = \sum_{r=1}^{n} r^2 + 5r$

$= 1^2 + 5\times 1 + 2^2 + 5\times 2 + 3^2 + 5\times 3 + \dots + n^2 + 5n$

$= (1^2 + 2^2 + 3^2 + \dots + n^2) + (5\times 1 + 5\times 2 + 5\times 3 + \dots + 5n$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{n}{2}(5+5n)$ ✓ 1

$= \frac{1}{6}n(n+1)(2n+1) + \frac{5n}{2}(n+1)$.

$= \frac{1}{6}n(n+1)(2n+1) + \frac{15n}{6}(n+1)$

$= \frac{1}{6}n(n+1)(2n+1+15)$

$= \frac{1}{6}n(n+1)(2n+16)$

$= \frac{1}{3}n(n+1)(n+8)$ ✓ 1

(iv). Total number of bottles $= \frac{1}{3}n(n+1)(n+8)$ (from (iii))

Total number of bins $= \frac{n}{2}(n+1)$ (from (i))

$\therefore \dfrac{\frac{1}{3}n(n+1)(n+8)}{\frac{n}{2}(n+1)} = 10$ ✓ 1

$\frac{1}{3}n(n+1)(n+8) = 5n(n+1)$

$n(n+1)(n+8) = 15n(n+1)$

$n(n+1)(n+8) - 15n(n+1) = 0$

$n(n+1)(n+8-15) = 0$

$n(n+1)(n-7) = 0$ ✓ 1

$\therefore n = 0, -1, 7$ $\therefore$ There are 7 row