ACE
EXAM
PAPER

Student name: ____________________

PAPER 4

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 1

General Instructions

- Working time - 120 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt questions 11-14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. The angle between the vectors $\underset{\sim}{u} = -2\underset{\sim}{i} + 6\underset{\sim}{j}$ and $\underset{\sim}{v} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$ is closest to:

   (A) 45°

   (B) 60°

   (C) 120°

   (D) 135°

2. A helicopter travelling at a speed of 70 ms^{-1} is approaching a sinking boat. The helicopter is to drop a survival package to the boat. The package will be released to fall under gravity from the helicopter as it approaches the boat. The equations of motion of the package as it falls (using gravity as 10 ms^{-2}) are $x = 70t$ and $y = -5t^2$.

   What is the equation of the trajectory of the package?

   (A) $x^2 = 24\,500y$

   (B) $x^2 = 980y$

   (C) $x^2 = -24\,500y$

   (D) $x^2 = -980y$

3. What is the area between the curves $y = x^3$ and $y = x^2$ and the *x*-axis from *x* = 1 to *x* = 3?

   (A) 11 square units

   (B) $11\frac{1}{3}$ square units

   (C) 12 square units

   (D) $12\frac{2}{3}$ square units

4. Thomas made an error proving that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for all $n \geq 1$ using mathematical induction. Part of the proof is shown below.

   Step 2: Assume the result true for $n = k$

   $1^3 + 2^3 + \ldots + k^3 = (1 + 2 + \ldots + k)^2$ Line 1

   Step 3: To prove the result true for $n = k + 1$

   $1^3 + 2^3 + \cdots + k^3 = \left(1 + 2 + \cdots + k + (k+1)\right)^2$ Line 2

   $\text{LHS} = (1 + 2 + \cdots + k)^2 + (k+1)^3$

   $= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$ Line 3

   $= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right]$ Line 4

   $= (k+1)^2\left[\frac{k^2 + 4k + 4}{4}\right]$

   $= \frac{1}{4}(k+1)^2(k+2)^2$

   $= [1 + 2 + \cdots + k + (k+1)]^2$

   $= \text{RHS}$

   Which line did Thomas make an error?

   (A) Line 1
   (B) Line 2
   (C) Line 3
   (D) Line 4

5. If $\sin A = t$ and $\cos B = t$ with $180° < A < 270°$ and $90° < B < 180°$ then $\sin(A + B)$ is:

   (A) $2t^2 - 1$
   (B) $1 - 2t^2$
   (C) -1
   (D) 14

6. Mia knows that each ticket has a probability of 0.2 of winning a prize in a lucky ticket competition. She buys 25 tickets. What is a general rule for the probability distribution of the number of winning tickets?

   (A) $P(X = x) = {}^{20}C_x\ 0.25^x 0.75^{20-x}$
   (B) $P(X = x) = {}^{20}C_x\ 0.25^x 0.75^{20-x}$
   (C) $P(X = x) = {}^{25}C_x\ 0.2^x 0.8^{25-x}$
   (D) $P(X = x) = {}^{25}C_x\ 0.8^x 0.2^{25-x}$

7. What is the value of $\int_0^1 \frac{4x}{2x+1}dx$? Use the subsitution $u = 2x + 1$.

(A) $2 - \ln 2$

(B) $2 - \ln 3$

(C) $4 - 2\ln 2$

(D) $4 - 2\ln 3$

8. $3\sin^2 x - 4\cos x + 1 = 0$

What is the solution of the trigonometric equation in the domain $0 \le x \le \pi$?

(A) $x = \frac{2}{3}$

(B) $x = \frac{2\pi}{3}$

(C) $x \approx 0.841$

(D) $x \approx 1.969$

9. If $\frac{dy}{dx} = \frac{2x+1}{4}$ and $y = 0$ when $x = 2$ then:

(A) $y = \frac{1}{4}(x^2 + x - 6)$

(B) $y = \frac{x(x+1)}{4}$

(C) $y = \frac{1}{4}(x^2 + x) + \frac{1}{2}$

(D) $y = \frac{1}{4}(x^2 + x - 1)$

10. What is the correct expression for the indefinite integral $\int (\cos^2 x + 2\sec^2 x)dx$?

(A) $\frac{1}{2}x + \frac{1}{4}\sin 2x + \tan x + C$

(B) $\frac{1}{2}x - \frac{1}{4}\sin 2x + \tan x + C$

(C) $\frac{1}{2}x + \frac{1}{4}\sin 2x + 2\tan x + C$

(D) $\frac{1}{2}x - \frac{1}{4}\sin 2x + 2\tan x + C$

Section II

60 marks
Attempt questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (11 marks) **Marks**

(a) Two standard dice are rolled together, and the sum of the numbers rolled is noted. The result is recorded as either a 6 or not a 6.

(i) If p is the probability of a sum of 6, find the value of p. **1**

(ii) Find $\text{Var}(X)$. **1**

(b) Solve the equation $\sin^2\theta = \cos\theta$ for $0° \leq \theta \leq 360°$. **2**

(c) Evaluate $\displaystyle\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4dx}{\sqrt{9 - x^2}}$ **2**

(d) (i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ **2**

(ii) Hence, solve $\cos x - \sqrt{3}\sin x = -2$ for $0 \leq x \leq 2\pi$ **1**

(e) The area bounded by $y^2 = 3 - 2x - x^2$ $(y \geq 0)$ and between $x = -3$ and $x = 1$ is rotated about the x-axis. Calculate the exact volume of the solid revolution. **2**

Question 12 (13 marks) **Marks**

(a) A population of koalas has an initial population of 500. Birth rates and the amount of food affect the population of the koalas. The change in the population, P, is given by the formula:

$$P = \frac{500}{1 + ke^{-1\cdot 5t}}$$ where k is a constant and t is in months

(i) Explain why the population will eventually die out. **1**

(ii) Show that the value of k is 499, if at $t = 0$ the change in the population is 1. **1**

(iii) How long will it take for only 100 koalas to remain? Give your answer to the nearest month. **3**

(iv) Show that $\frac{dP}{dt} = \frac{3P}{1000}(500 - P)$ **3**

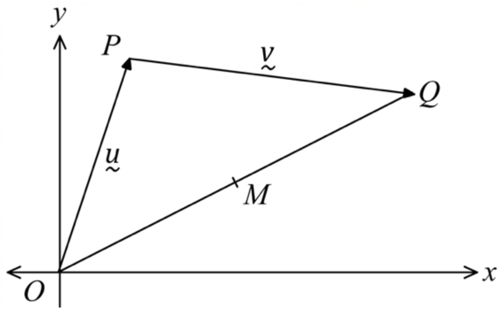
(b) (i) Show that $\frac{\sec^2\theta}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$ **1**

(ii) Find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin\theta\cos\theta}\,d\theta$. Use the substitution $u = \tan\theta$. **2**

(c) What is the exact value of $\cos(75°)$? **2**

Question 13 (17 marks) **Marks**

(a) In ΔOPQ, $\overrightarrow{OP} = \underset{\sim}{u}$ and $\overrightarrow{PQ} = \underset{\sim}{v}$. Point M is the midpoint of $\overrightarrow{OQ}$.



Find the following vectors in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

(i) $\overrightarrow{OQ}$ **1**

(ii) $\overrightarrow{QO}$ **1**

(iii) $\overrightarrow{OM}$ **1**

(iv) $\overrightarrow{PM}$ **1**

(v) $\overrightarrow{QM}$ **1**

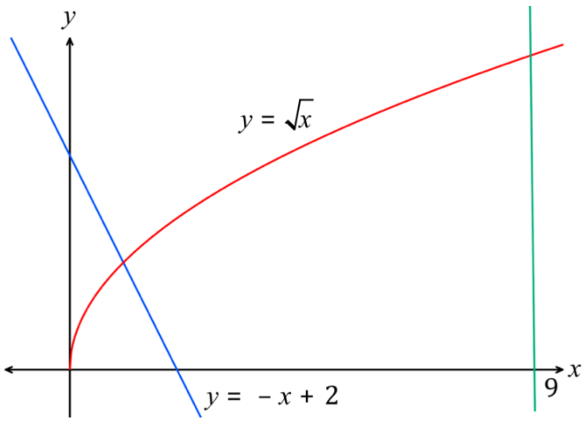
(vi) $\overrightarrow{PM} + \overrightarrow{MQ}$ **1**

(b) Prove by mathematical induction that $7^n - 1$ is divisible by 6 for all positive integers n $(n \geq 1)$. **3**

(c) Using the substitution $u = 3x^3 + 1$ or otherwise, evaluate the integral: **2**

$$\int_0^1 x^2\sqrt{3x^3 + 1}dx$$

(d) The area enclosed by the curve $y = \sqrt{x}$, and the lines $y = -x + 2$, $x = 9$ and x-axis is rotated about the x-axis. **4**



Not to scale

Find the volume of the solid of revolution.

(e) What is the value of $\sin 2\theta$ if $\cos\theta = -\dfrac{2}{3}$ and $\tan\theta > 0$? **2**

Question 14 (19 marks) **Marks**

(a) A stone is projected horizontally from the top of a 80 metre high vertical cliff. The stone is thrown with an initial velocity of V ms^{-1} at an angle of projection of θ. It reaches its greatest height after 3 seconds and hits the ground at a horizontal distance of 320 m from the foot of the cliff. Assume g = 10 ms^{-2}.

(i) Determine the parametric equations of the path. (Take the origin at the top of the vertical cliff.) **2**

(ii) Show that $V\sin\theta = 30$. **1**

(iii) Show that the stone reaches the ground after 8 seconds. **1**

(iv) Show that $V\cos\theta = 40$. **1**

(v) Find the value of V and the angle of projection θ. **2**

(b) What is the exact value of the definite integral $\int_0^{\frac{\pi}{12}} 2\sin^2 x dx$? **2**

(c) Use the principle of mathematical induction to prove that for all positive integers n: **3**

$1 + 5 + 9 + \ldots + 4n - 3 = 2n^2 - n$

(d) A fair die is rolled twelve times. What is the expected value for the number of sixes obtained? **2**

(e) What is the value of $|\overrightarrow{PQ}|$ given $\overrightarrow{OP} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{OQ} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$? **2**

(f) Five different fair dice are thrown together. What is the probability the five scores include at most one six? **3**

End of paper

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

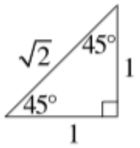
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

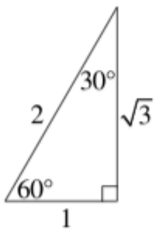
$A = \dfrac{1}{2}ab\sin C$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

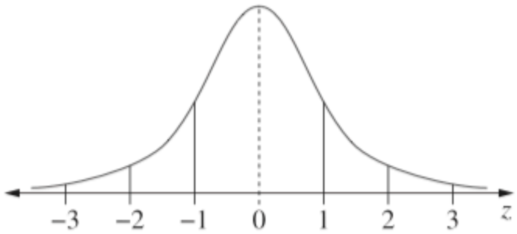
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$