

# Carlingford High School Mathematics Extension 2 Year 12

HSC ASSESSMENT TASK 3

Term 2 2016

Student Name:	Teacher: Mr Gor	ıG

#### • Time allowed 55 minutes.

- Start each question on a new page.
- Write on **ONE SIDE** of the paper only.
- Do not work in columns.
- Marks may be deducted for careless or badly arranged work.
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- All answers are to be completed in blue or black pen except graphs and diagrams.
- There is to be NO LENDING OR BORROWING.

	MC	Q6	Q7	Total	
Integration	/3	/16		/19	
Volumes	/2		/13	/15	
Total	/5	/16	/13	/34	

Multiple Choice - Start a new page (5 marks)

1. Evaluate 
$$\int \frac{dx}{x^2 - 4x + 13}$$

$$\mathbf{A.} \quad \frac{1}{3} \tan^{-1} \left( \frac{2x-4}{3} \right) + C$$

**B.** 
$$\frac{2}{3} \tan^{-1} \left( \frac{x-2}{3} \right) + C$$

C. 
$$\frac{1}{3} \tan^{-1} \left( \frac{x-2}{3} \right) + C$$

**D.** 
$$\frac{2}{3} \tan^{-1} \left( \frac{2x-4}{3} \right) + C$$

The region bounded by the curves  $y = x^2$  and  $y = x^3$  in the first quadrant is rotated about the y – axis. The volume of the solid of revolution formed can be found using:

**A.** 
$$V = \pi \int_0^1 \left( y^{\frac{1}{3}} - y^{\frac{1}{2}} \right) dy$$

**B.** 
$$V = \pi \int_0^1 \left( y^{\frac{1}{2}} - y^{\frac{1}{3}} \right) dy$$

C. 
$$V = \pi \int_0^1 (x^4 - x^6) dx$$

**D.** 
$$V = \pi \int_0^1 \left( y^{\frac{2}{3}} - y \right) dy$$

Which of the following is an expression for  $\int \frac{1}{1+\sin x + \cos x} dx$ ?

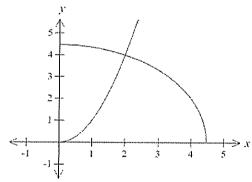
**A.** 
$$\ln |t-1| + C$$

**B.** 
$$\ln |t+1| + C$$

**C.** 
$$\ln |t^2 - 1| + C$$

**D.** 
$$\ln |t^2 + 1| + C$$

A solid is formed when the region bounded by the curves  $y = x^2$ ,  $y = \sqrt{20 - x^2}$  and the y – axis is rotated about the y – axis.



What is the correct expression for the volume of this solid using the method of cylindrical shells?

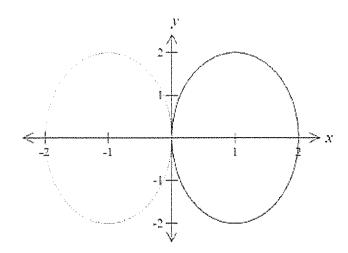
**A.** 
$$V = \int_{0}^{2} 2\pi \left( \sqrt{20 - x^2} - x^2 \right) dx$$

**B.** 
$$V = \int_{0}^{2} 2\pi \left(x^2 - \sqrt{20 - x^2}\right) dx$$

C. 
$$V = \int_{0}^{2} 2\pi x \left( \sqrt{20 - x^2} - x^2 \right) dx$$

C. 
$$V = \int_{0}^{2} 2\pi x \left(\sqrt{20 - x^2} - x^2\right) dx$$
 D.  $V = \int_{0}^{2} 2\pi x \left(x^2 - \sqrt{20 - x^2}\right) dx$ 

The region enclosed by the ellipse  $(x-1)^2 + \frac{y^2}{4} = 1$  is rotated about the y-axis to form a solid.



If the slices are taken perpendicular to the axis of rotation, what is the correct expression for the volume?

**A.** 
$$V = \int_{-2}^{2} 2\pi \sqrt{4 - y^2} \, dy$$

**B.** 
$$V = \int_{-2}^{2} 2\pi \sqrt{1 - y^2} dy$$
**D.**  $V = \int_{-2}^{2} \pi \sqrt{1 - y^2} dy$ 

C. 
$$V = \int_{-2}^{2} \pi \sqrt{4 - y^2} \, dy$$

**D.** 
$$V = \int_{-2}^{2} \pi \sqrt{1 - y^2} dy$$

## Question 6 - Start a new page - (16 marks)

Marks

a) Find 
$$\int \frac{dx}{\sqrt{9+16x-4x^2}}.$$

3

b) Find 
$$\int \frac{5x^2 - 3x + 13}{(x-1)(x^2 + 4)} dx$$
.

3

c) Evaluate 
$$\int_0^{\frac{\sqrt{\pi}}{2}} 3x \sin(x^2) dx.$$

3

d) Evaluate 
$$\int_1^e x^7 \ln x \ dx$$
.

3

e) i) Derive the reduction formula: 
$$\int x^n e^{-x^2} dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx.$$
 2

ii) Use this reduction formula to evaluate 
$$\int_0^1 x^5 e^{-x^2} dx$$
.

2

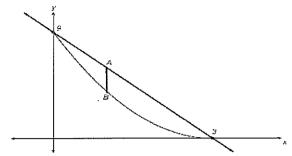
## <u>Question 7</u> - Start a new page – (13 marks)

Marks

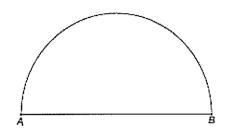
a) Use the method of cylindrical shells to find the volume of the solid generated by revolving the region enclosed by  $y = 3x^2 - x^3$  and the x-axis around the y-axis.

4

b)



Cross-section with base on AB



The diagram above shows the region enclosed by the parabola  $y = (x-3)^2$  and the line 3x + y = 9. The region forms the base of a solid.

When the solid is sliced perpendicular to the x-axis, each cross-section is a semi-circle with diameter across the region. A typical cross-section is shown above.

i) If the solid is sliced along the line x = a, show that the area of the cross-section is  $A = \frac{\pi}{8} a^2 (3-a)^2$ , where  $0 \le a \le 3$ .

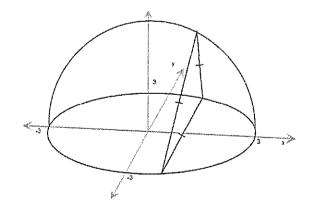
2

ii) Find the volume of the solid.

3

c) The diagram shows a solid which has the circle  $x^2 + y^2 = 9$  as its base.

The cross-section perpendicular to the x-axis is an equilateral triangle.



i) Show that the area of a triangle is given by:  $Area = \sqrt{3}(9-x^2)$ .

2

ii) Hence or otherwise find the volume of the solid.

2

## EnD of ExaM



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Term 2 2016

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**D.** 
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2. The region bounded by the curves  $y = x^2$  and  $y = x^3$  in the first quadrant is rotated about the y – axis. The volume of the solid of revolution formed can be found using:

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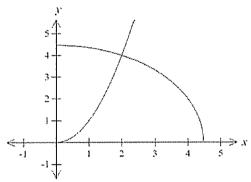
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4. A solid is formed when the region bounded by the curves  $y = x^2$ ,  $y = \sqrt{20 - x^2}$  and the y – axis is rotated about the y – axis.



What is the correct expression for the volume of this solid using the method of cylindrical shells?

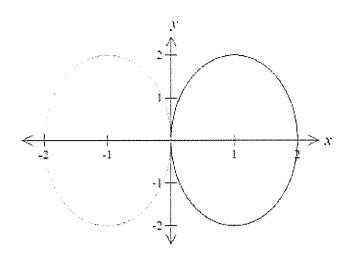
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C. 
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$$\mathbf{D.} \quad V = \int_{2}^{2} \pi \sqrt{1 - y^2} \, dy$$

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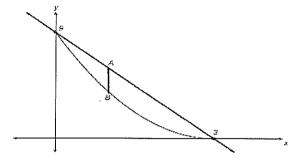
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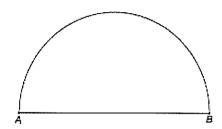
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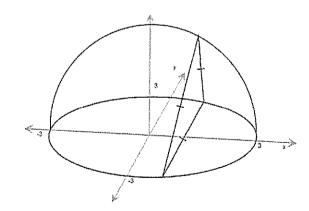
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2

ii) Hence or otherwise find the volume of the solid.

2

## END OF EXAM



# **Term 2 HSC Task 3 Examination**

# **Ext 2 Mathematics**

Solutions

## HSC Task 3 Term 2 Examination – Ext 2 Mathematics 2016

## Section I Multiple Choice Answer 1 Mark each

1. A ○ B○ C ● D○

2. A O BO CO D

3. A ○ B ● C ○ D ○

4. A ○ B○ C ● D○

5. A **B** BO CO DO

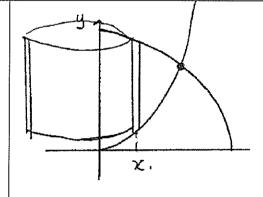
## Multiple Choice Working Out

Now 
$$\int \frac{dx}{x^2 - 4x + 13} = \int \frac{dx}{x^2 - 4x + 4 + 9}$$
$$= \int \frac{dx}{(x - 2)^2 + 9}$$
$$= \frac{1}{3} \tan^{-1} \left(\frac{x - 2}{3}\right) + C \implies C$$

 $\int \frac{1}{1+\sin x + \cos x} dx = \int \frac{1}{1+\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$   $= \int \frac{2}{1+t^2 + 2t + 1 - t^2} dt$   $= \int \frac{1}{1+t} dt$   $= \ln|t+1| + C \implies B$ 

 $y = x^2$  0.5  $y = x^3$  0.5 0.5 1

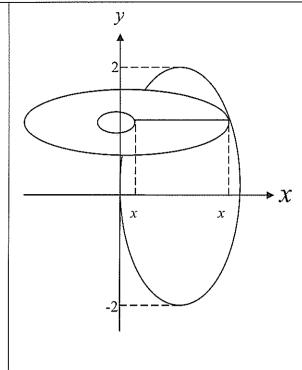
Now  $y = x^3$  then  $x = y^{\frac{1}{3}}$ and  $y = x^2$  then  $x = y^{\frac{1}{2}}$ So  $V = \pi \int_0^1 \left[ \left( y^{\frac{1}{3}} \right)^2 - \left( y^{\frac{1}{2}} \right)^2 \right] dy$  $= \pi \int_0^1 \left( y^{\frac{2}{3}} - y \right) dy \implies D$ 



 $2\pi r = 2 \pi x$   $y_1 - y_2 = \sqrt{20 - x^2} - x^2$ 

$$A = 2\pi x \left(\sqrt{20 - x^2} - x^2\right)$$

$$V = \int_0^2 2\pi x \left(\sqrt{20 - x^2} - x^2\right) dx \implies C$$



$$\delta V = \pi (R+r)(R-r)\delta y$$
 where  $R = x_2, r = x_1$ 

Taking  $x_2$ ,  $x_1$  as roots of equation  $(x-1)^2 + \frac{y^2}{4} = 1$ 

at some fixed height y above x – axis:

$$4(x^2 - 2x + 1) + y^2 - 4 = 0$$

$$4x^2 - 8x + y^2 = 0$$
 has roots  $x_1, x_2$ 

Sum of roots: 
$$x_1 + x_2 = 2$$

Product of roots: 
$$x_1 x_2 = \frac{y^2}{4}$$

Now 
$$(x_2 - x_1)^2 = (x_1 + x_2)^2 - 4x_1x_2$$
 where  $x_2 - x_1 > 0$ 

So 
$$x_2 - x_1 = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$$
  
=  $\sqrt{4 - 4 \cdot \frac{y^2}{4}}$   
=  $\sqrt{4 - y^2}$ 

$$\therefore V = 2\pi \int_{-2}^{2} \sqrt{4 - y^2} \, dy \implies A$$

#### **Section II Solutions**

## Ouestion 6 $\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$ a) $9 + 16x - 4x^2 = 9 - 4(x^2 - 4x)$ $=9-4(x^2-4x+4)+16$ $\int \frac{dx}{\sqrt{9+16x-4x^2}} = \int \frac{dx}{\sqrt{25-4(x-2)^2}}$ 1 for correct $= \frac{1}{5} \int \frac{dx}{\sqrt{1 - \frac{4}{25}(x - 2)^2}}$ manipulation $u = \frac{2(x-2)}{5} + c$ $du = \frac{2}{5}dx$ $dx = \frac{5}{2} du$ 1 for correct $=\frac{1}{5}\int \frac{\frac{5}{2}du}{\sqrt{1-u^2}}$ substitution $= \frac{1}{2} \int \frac{du}{\sqrt{1-y^2}}$ $= \frac{1}{2} \sin^{-1} u$ 1 mark for correct $=\frac{1}{2}\sin^{-1}\left(\frac{2(x-2)}{5}\right)$ answer b) Let $\frac{5x^2 - 3x + 13}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}$ $\equiv \frac{A(x^2+4) + (Bx+C)(x-1)}{(x-1)(x^2+4)}$ $\therefore 5x^2 - 3x + 13 = A(x^2 + 4) + (Bx + C)(x - 1) \blacktriangleleft$ 1 mark for this line When x = 1 then 15 = 5AA = 3When x = 0 then 13 = 12 - CC = -11 mark for the values of When x = -1 then 21 = 15 + (-B - 1)(-2)A, B & C6 = 2B + 2B = 2

$I = \int \left(\frac{3}{x-1} + \frac{2x-1}{x^2+4}\right) dx$ $= 3\ln x-1  + \int \left(\frac{2x}{x^2+4}\right) dx - \int \left(\frac{1}{x^2+4}\right) dx$	
$= 3 \ln x-1  + \ln x^2+4  - \frac{1}{2} \tan^{-1} \left(\frac{x}{2}\right) + C$	1 mark for correct answer
c) <u>J</u>	
$\begin{vmatrix} \mathbf{c} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 3x \sin(x^2) dx \end{vmatrix}$	
Substitute $u = x^2$ $\frac{du}{dx} = 2x$	
$du = 2x dx$ $\frac{3}{2} du = 3x dx$ $x = 0 \implies u = 0^2 = 0$	Using a Substitution
$x = 0 \Rightarrow u = 0 = 0$ $x = \frac{\sqrt{\pi}}{2} \Rightarrow u = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}$	1 for changing limits & variable
$\int_{0}^{\frac{\sqrt{\pi}}{2}} 3x \sin(x^{2}) dx = \int_{0}^{\frac{\pi}{4}} \frac{3}{2} \sin u du$	
$=\frac{3}{2}\int_{0}^{\frac{\pi}{4}}\sin udu$	1 for integral including correct limits
$= \frac{3}{2} \left[ -\cos u \right]_0^{\frac{\pi}{4}}$ $= -\frac{3}{2} \left( \cos \left( \frac{\pi}{4} \right) - \cos(0) \right)$	
$=-\frac{3}{2}\left(\frac{1}{12}-1\right)$	
$= -\frac{3}{2} \left( \frac{1 - \sqrt{2}}{\sqrt{2}} \right)$ $= \frac{3\sqrt{2} - 3}{2\sqrt{2}}$ $= \frac{6 - 3\sqrt{2}}{2\sqrt{2}}$	
$= \frac{2\sqrt{2}}{6-3\sqrt{2}} \blacktriangleleft$	1 for substitution & simplification to get answer
	OR

$\frac{\sqrt{z}}{2}$	Without a Substitution
$\int_{0}^{\frac{\sqrt{x}}{2}} 3x \sin(x^{2}) dx$	1 mark for correct integration
$= -\frac{3}{2} \left[ \cos(x^2) \right]_0^{\frac{\sqrt{\pi}}{2}}$ $= -\frac{3}{2} \left[ \cos\frac{\pi}{4} - \cos\theta \right]$	1 mark for correct working
$= \frac{3}{2} \left( \frac{1}{\sqrt{2} - 1} \right)$	1 mark for correct answer
d) $I = \int_{1}^{e} x^{7} \ln x dx$ Let $u = \ln x$ then $u' = \frac{1}{x}$ & $v' = x^{7}$ then $v = \frac{x^{8}}{8}$	1 mark for correct u' &
$= \left[ \frac{x^8}{8} \ln x \right]_1^e - \frac{1}{8} \int_1^e x^7 dx$ $= \frac{e^8}{8} - \frac{1}{8} \left[ \frac{x^8}{8} \right]_1^e$ $= \frac{e^8}{8} \cdot \frac{1}{8} \left( e^8 - 1 \right)$	1 mark for correct substitution
$= \frac{e^8}{8} - \frac{1}{8} \left( \frac{e^8}{8} - \frac{1}{8} \right)$ $= \frac{7e^8 + 1}{64} \blacktriangleleft$	1 mark for correct answer
e) i) $\int x^n e^{-x^2} dx$ Let $u = x^{n-1}$ $v' = xe^{-x^2}$ $u' = (n-1)x^{n-2}$ $v = -\frac{1}{2}e^{-x^2}$	- 1 mark for correct set up
$\int x^n e^{-x^2} dx = uv - \int vu'$ $= -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx$	1 mark for correct answer
ii) $\int_0^1 x^5 e^{-x^2} dx = \left[ -\frac{1}{2} x^4 e^{-x^2} \right]_0^1 + \frac{4}{2} \int_0^1 x^3 e^{-x^2} dx$	
$= \frac{-1}{2e} + 2 \int_0^1 x^3 e^{-x^2} dx$ $= \frac{-1}{2e} + 2 \left\{ \left[ -\frac{1}{2} x^2 e^{-x^2} \right]_0^1 + 1 \int_0^1 x e^{-x^2} dx \right\}$	1 mark for correct
$= \frac{-1}{2e} - 2\left(\frac{1}{2e}\right) + 2\left[-\frac{1}{2}x^{0}e^{-x^{2}}\right]_{0}^{1} + \frac{1-1}{2}\int x^{1-2}e^{-x^{2}}dx$ $= \frac{-1}{2e} - \frac{1}{e} + \left[-e^{-x^{2}}\right]_{0}^{1}$	
$= \frac{-1}{2e} - \frac{1}{e} - \frac{1}{e} + 1$ $= \frac{-1}{2e} - \frac{2}{2e} - \frac{2}{2e} + 1$	1 mark for correct
$=1-\frac{5}{2\epsilon}$	

Question 7		
a) $\partial V = 2\pi xy \partial x$ $= 2\pi x (3x^2 - x^3) \partial x$		
$= 2\pi x (3x^2 - x^3) \hat{c}$ $V = \lim_{\partial x \to 0} \sum_{0}^{3} 2\pi x (3x^2)$ $V = \int_{0}^{b} 2\pi x y dx$	$(x^2-x^3)\partial x$	— 2 marks for correct set up
$= \int_0^3 2\pi x (3x^2 - x^3)$ $= 2\pi \int_0^3 (3x^3 - x^4) dx$	$dx = -\frac{1}{2} \int dx$	— 1 mark correct integral
$= 2\pi \left[ \frac{3x^4}{4} - \frac{x^5}{5} \right]_0^3$ $= \frac{243\pi}{10} \text{ units}^3 \blacktriangleleft$		- 1 mark for correct answer
<b>b) i)</b> $A = \frac{\pi}{2}r^2$ $= \frac{\pi}{2} \left[ \frac{(9-3x)-(9-3x)-(9-3x)-(9-3x)}{2} \right]$ $= \frac{\pi}{2} \left[ \frac{(9-3x)-(9-3x)-(9-3x)}{2} \right]$ $= \frac{\pi}{2} \left[ \frac{3x-x^2}{4} \right]^2$	$\left((x-3)^2\right]^2$	— 1 mark for correct <i>r</i>
$= \frac{\pi}{8} [x(3-x)]^2$ When $x = a$ then $A = \frac{\pi}{8} a^2 (3-a)^2$		1 mark for correct expression for A
ii) $V = \int_0^3 \frac{\pi}{8} (9a^2 - \frac{\pi}{8})^3 = \frac{\pi}{8} \left[ 3a^3 - \frac{3a}{2} \right]$ $= \frac{\pi}{8} \left[ \frac{81}{10} \right]$	$6a^3 + a^4 da $ $\frac{a^4}{2} + \frac{a^5}{5} \Big]_0^3 $	<ul> <li>1 mark for correct integral for V</li> <li>1 mark for correct integration</li> </ul>
$\therefore V = \frac{8 \lfloor 10 \rfloor}{80} \text{ units}^3$	•	— 1 mark for correct answer

c) i)	)		
		$x^{2} + y^{2} = 9$ $y = \sqrt{9 - x^{2}}$ $sin 60 = \frac{h}{l}$ $h = l sin 60$	
The state of the s		``	1 mark for correct <i>h</i>
	A PROPERTY AND A PARTY AND A P	$A(x) = \frac{1}{2}bh$ $= \frac{1}{2}(2\sqrt{9 - x^2})(\sqrt{3}\sqrt{9 - x^2})$ $= \sqrt{3}(9 - x^2)$ $OR  A = = \frac{1}{2}\hat{f} \sin 60^{\circ}$ $= \frac{\sqrt{3}}{4}\hat{f}$	1 mark for correct area
	i)	$V = \int_{-3}^{3} \sqrt{3} (9 - x^2)  dx$	
		- Valia/ - 2/ - 2/ - 2/ - 2/ - 2/ - 1	1 mark for correct integral
		$= 2\sqrt{3} \left[ 9x - \frac{x^3}{3} \right]_0^3$ $= 2\sqrt{3} \left[ (27 - 9) - 0 \right]$	1 mark for correct answer
		$= 2\sqrt{3} \times 18$ $= 36\sqrt{3} \text{ units}^3$	



# **Term 2 HSC Task 3 Examination**

# **Ext 2 Mathematics**

Solutions

### HSC Task 3 Term 2 Examination – Ext 2 Mathematics 2016

3

## Section I Multiple Choice Answer 1 Mark each

1. A ○ B○ C ● D○

2. A O BO CO D

3. A O B C DO

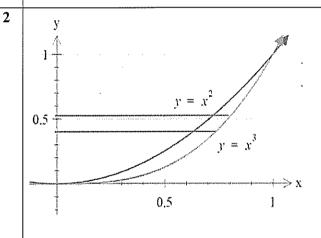
4. A O B O C D O

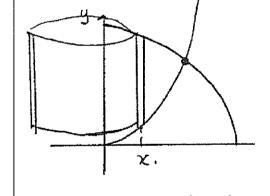
5. A **B**O CO DO

### Multiple Choice Working Out

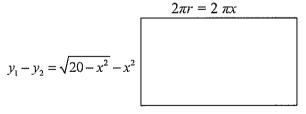
1	Now $\int \frac{dx}{x^2 - 4x + 13} = \int \frac{dx}{x^2 - 4x + 4 + 9}$
!	$=\int \frac{dx}{(x-2)^2+9}$
	$= \frac{1}{3} \tan^{-1} \left( \frac{x-2}{3} \right) + C \implies C$
	$= \frac{1}{3} \tan \left( \frac{1}{3} \right) + C \Rightarrow C$

$\int \frac{1}{1} dx = \int \frac{1}{1}$	1	2dt
$\int \frac{1}{1+\sin x + \cos x} dx = \int \frac{1}{1-x} dx$	$2t$ $1-t^2$	$1+t^2$
1-	$+\frac{1}{1+t^2}+\frac{1}{1+t^2}$	
$=\int \frac{1}{1+1}$	$\frac{2}{-t^2 + 2t + 1 - t^2}$	dt
$=\int \frac{1}{1+}$	$-\frac{dt}{dt}$	
$=\ln  t $	$+1 +C \Rightarrow B$	



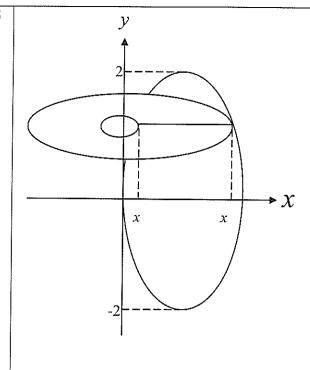


Now  $y = x^3$  then  $x = y^{\frac{1}{3}}$ and  $y = x^2$  then  $x = y^{\frac{1}{2}}$ So  $V = \pi \int_0^1 \left[ \left( y^{\frac{1}{3}} \right)^2 - \left( y^{\frac{1}{2}} \right)^2 \right] dy$  $= \pi \int_0^1 \left( y^{\frac{2}{3}} - y \right) dy \implies D$ 



$$A = 2\pi x \left(\sqrt{20 - x^2} - x^2\right)$$

$$V = \int_0^2 2\pi x \left(\sqrt{20 - x^2} - x^2\right) dx \implies C$$



$$\delta V = \pi (R+r)(R-r)\delta y$$
 where  $R = x_2, r = x_1$ 

Taking 
$$x_2$$
,  $x_1$  as roots of equation  $(x-1)^2 + \frac{y^2}{4} = 1$ 

at some fixed height y above x – axis:

$$4(x^2 - 2x + 1) + y^2 - 4 = 0$$

$$4x^2 - 8x + y^2 = 0$$
 has roots  $x_1, x_2$ 

Sum of roots: 
$$x_1 + x_2 = 2$$

Product of roots: 
$$x_1 x_2 = \frac{y^2}{4}$$

Now 
$$(x_2 - x_1)^2 = (x_1 + x_2)^2 - 4x_1x_2$$
 where  $x_2 - x_1 > 0$   
So  $x_2 - x_1 = \sqrt{(x_1 + x_2)^2 - 4x_1x_2}$ 

So 
$$x_2 - x_1 = \sqrt{(x_1 + x_2)^2 - 4x}$$
  
=  $\sqrt{4 - 4 \cdot \frac{y^2}{4}}$ 

$$\therefore V = 2\pi \int_{-2}^{2} \sqrt{4 - y^2} dy \implies A$$

#### Section II Solutions

# Ouestion 6 $\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$ a) $9 \div 16x - 4x^2 = 9 - 4(x^2 - 4x)$ $=9-4(x^2-4x+4)+16$ $\int \frac{dx}{\sqrt{9+16x-4x^2}} = \int \frac{dx}{\sqrt{25-4(x-2)^2}}$ 1 for correct $= \frac{1}{5} \int \frac{dx}{\sqrt{1 - \frac{4}{25}(x - 2)^2}}$ manipulation $u = \frac{2(x-2)}{5} + c$ $du = \frac{2}{5}dx$ $dx = \frac{5}{2} du$ 1 for correct $=\frac{1}{5}\int \frac{\frac{5}{2}du}{\sqrt{1-u^2}}$ substitution $=\frac{1}{2}\int \frac{du}{\sqrt{1-2t^2}}$ $=\frac{1}{2}\sin^{-1}u$ 1 mark for correct $=\frac{1}{2}\sin^{-1}\left(\frac{2(x-2)}{5}\right)$ answer b) Let $\frac{5x^2 - 3x + 13}{(x - 1)(x^2 + 4)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4}$ $\equiv \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$ $\therefore 5x^2 - 3x + 13 = A(x^2 + 4) + (Bx + C)(x - 1) \blacktriangleleft$ 1 mark for this line When x = 1 then 15 = 5AA = 3When x = 0 then 13 = 12 - CC = -11 mark for the values of When x = -1 then 21 = 15 + (-B - 1)(-2)A, B & C6 = 2B + 2B = 2

$\therefore I = \int \left(\frac{3}{x-1} + \frac{2x-1}{x^2+4}\right) dx$	
$= 3\ln x-1  + \int \left(\frac{2x}{x^2+4}\right) dx - \int \left(\frac{1}{x^2+4}\right) dx$	
$= 3 \ln  x - 1  + \ln  x^2 + 4  - \frac{1}{2} \tan^{-1} \left(\frac{x}{2}\right) + C$	1 mark for correct answer
$ \begin{array}{c c} c) & \frac{\sqrt{\pi}}{2} \\ \int_{-2}^{2} 3x \sin(x^2) dx \end{array} $	
$\int_{0}^{\infty} 3x \sin(x^{2}) dx$	
Substitute $u = x^2$	
$\frac{du}{dx} = 2x$ $du = 2x dx$	
$du = 2x dx$ $\frac{3}{2} du = 3x dx$	
	Using a Substitution
$x = \frac{\sqrt{\pi}}{2} \Rightarrow u = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}$	<ul> <li>1 for changing limits &amp; variable</li> </ul>
$\left  \begin{array}{c} x = \frac{\sqrt{n}}{2} \Rightarrow u = \left( \frac{\sqrt{n}}{2} \right) = \frac{\pi}{4} \end{array} \right $	
$\sqrt{2\pi}$ $\pi$	
$\int_{0}^{\frac{\sqrt{\pi}}{2}} 3x \sin(x^{2}) dx = \int_{0}^{\frac{\pi}{4}} \frac{3}{2} \sin u  du$	
<u>#</u> 3 (*4	_ 1 for integral including
$=\frac{3}{2}\int_{0}^{\infty}\sin udu$	correct limits
$=\frac{3}{2}\left[-\cos u\right]_0^{\xi}$	
$= -\frac{3}{2} \left( \cos \left( \frac{\pi}{4} \right) - \cos(0) \right)$	
$= -\frac{3}{2} \left( \frac{1}{\sqrt{2}} - 1 \right)$	
$= -\frac{3}{2} \left( \frac{1 - \sqrt{2}}{\sqrt{2}} \right)$ $= \frac{3\sqrt{2} - 3}{2\sqrt{2}}$ $= \frac{6 - 3\sqrt{2}}{\sqrt{2}}$	
$=\frac{3\sqrt{2}-3}{2\sqrt{2}}$	1 for substitution &
$= \frac{6-3\sqrt{2}}{4}$	simplification to get answer
	3
	OR

	$\frac{\sqrt{\pi}}{2}$	Without a Substitution
	$\int_{0}^{\sqrt{\pi}} 3x \sin(x^{2}) dx$	1 mark for correct integration
- Webster	$= -\frac{3}{2} \left[ \cos\left(x^2\right) \right]_0^{\frac{\sqrt{\pi}}{2}}$ $= -\frac{3}{2} \left[ \cos\frac{\pi}{4} - \cos 0 \right]$	1 mark for correct working
	$= -\frac{3}{2} \left[ \cos \frac{1}{4} - \cos \theta \right]$ $= -\frac{3}{2} \left( \frac{1}{\sqrt{2} - 1} \right)$	1 mark for correct
4)	-	
d)	$I = \int_1^1 x' \ln x dx$ Let $u = \ln x$ then $u' = \frac{1}{x}$ & $v' = x^7$ then $v = \frac{x}{8}$	1 mark for correct u' &
my particular and the second	$= \left[\frac{x^8}{8} \ln x\right]_1^e - \frac{1}{8} \int_1^e x^7 dx  \blacktriangleleft$	
	$=\frac{e^8}{8}-\frac{1}{8}\left[\frac{x^8}{8}\right]_1^e$	1 mark for correct substitution
	$=\frac{e^8}{8}-\frac{1}{8}\left(\frac{e^8}{8}-\frac{1}{8}\right)$	
	$=\frac{7e^8+1}{64}$	1 mark for correct answer
e) i	$\int x^n e^{-x^2} dx$	
, , , , , , , , , , , , , , , , , , ,	$\begin{cases} \int x^n e^{-x^2} dx \\ \text{Let } u = x^{n-1} & v' = xe^{-x^2} \\ u' = (n-1)x^{n-2} & v = -\frac{1}{2} e^{-x^2} \end{cases}$	1 mark for correct set up
	$\int x^n e^{-x^2} dx = uv - \int vu'$	1 mark for correct
	$= -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx$	answer
1	i) $\int_0^1 x^5 e^{-x^2} dx = \left[ -\frac{1}{2} x^4 e^{-x^2} \right]_0^1 + \frac{4}{2} \int_0^1 x^3 e^{-x^2} dx$	
	$= \frac{-1}{2e} + 2 \int_0^1 x^3 e^{-x^2} dx$	
	$= \frac{-1}{2e} + 2\left\{ \left[ -\frac{1}{2} x^2 e^{-x^2} \right]_0^1 + 1 \int_0^1 x e^{-x^2} dx \right\}$	1 mark for correct reduction
***************************************	$= \frac{-1}{2e} - 2\left(\frac{1}{2e}\right) + 2\left[-\frac{1}{2}x^{0}e^{-x^{2}}\right]_{0}^{1} + \frac{1-1}{2}\int x^{1-2}e^{-x^{2}}dx$ $= \frac{-1}{2e} - \frac{1}{e} + \left[-e^{-x^{2}}\right]_{0}^{1}$	
	$= \frac{-1}{2e} - \frac{1}{e} - \frac{1}{e} + 1$ $= \frac{-1}{2e} - \frac{2}{2e} - \frac{2}{2e} + 1$	
		1 mark for correct answer
	$=1-\frac{5}{2\epsilon}$	

Question 7				
a)				
$\partial V = 2\pi x y \partial x$ $= 2\pi x (3x^2 - x^3) \partial x$ $V = \lim_{\partial x \to 0} \sum_{0}^{3} 2\pi x (3x^2 - x^3) \partial x$	—— 2 marks for correct set up			
$V = \int_{a}^{b} 2\pi xy dx$ $= \int_{0}^{3} 2\pi x (3x^{2} - x^{3}) dx \blacktriangleleft$ $= 2\pi \int_{0}^{3} (3x^{3} - x^{4}) dx$ $= 3x^{4} - x^{5} \int_{0}^{3} (3x^{3} - x^{4}) dx$	1 mark correct integral			
$= 2\pi \left[ \frac{3x^4}{4} - \frac{x^5}{5} \right]_0^3$ $= \frac{243\pi}{10} \text{ units}^3$	— 1 mark for correct answer			
$A = \frac{\pi}{2}r^{2}$ $= \frac{\pi}{2} \left[ \frac{(9-3x)-(x-3)^{2}}{2} \right]^{2}$ $= \frac{\pi}{2} \left[ \frac{(9-3x)-(x-3)^{2}}{2} \right]^{2}$ $= \frac{\pi}{2} \left[ \frac{3x-x^{2}}{4} \right]^{2}$	1 mark for correct <i>r</i>			
$= \frac{\pi}{8} [x(3-x)]^2$ When $x = a$ then $A = \frac{\pi}{8} a^2 (3-a)^2$	1 mark for correct expression for A			
ii) $V = \int_0^3 \frac{\pi}{8} (9a^2 - 6a^3 + a^4) da$ $= \frac{\pi}{8} \left[ 3a^3 - \frac{3a^4}{2} + \frac{a^5}{5} \right]_0^3$ $= \frac{\pi}{8} \left[ \frac{81}{10} \right]$	1 mark for correct integral for V  1 mark for correct integration			
$\therefore V = \frac{81\pi}{80} \text{ units}^3$	—— 1 mark for correct answer			

c)	i)		
	444Ville de Aldrich de	$x^{2} + y^{2} = 9$ $y = \sqrt{9 - x^{2}}$ $sin 60 = \frac{h}{l}$ $h = l sin 60$	
Abbenishing the formula in the second	n mada A A dini a a A A A A dini a A A A A A A A A A A A A A A A A A A	$h = \frac{\sqrt{3}}{2}l$ $h = \sqrt{3}\sqrt{9 - x^2}$ $h = \sqrt{3}\sqrt{9 - x^2}$	1 mark for correct <i>h</i>
черовория венения однижения однижения однижения одна одна одна одна одна одна одна одна		$A(x) = \frac{1}{2}bh$ $= \frac{1}{2}(2\sqrt{9-x^2})(\sqrt{3}\sqrt{9-x^2})$ $= \sqrt{3}(9-x^2)$ $OR  A = = \frac{1}{2}\vec{l} \sin 60^{\circ}$ $= \frac{\sqrt{3}}{4}\vec{l}$ $= \frac{1}{2}(2\sqrt{9-x^2})(\sqrt{3}\sqrt{9-x^2})$ $= \frac{\sqrt{3}}{4}\vec{l}$	1 mark for correct area
	ii)	$V = \int_{-3}^{3} \sqrt{3} (9 - x^2)  dx$	
AND THE PROPERTY OF THE PROPER		$= \sqrt{3} \left[ 9x - \frac{x^{5}}{3} \right]_{-3}^{3}$ $= \sqrt{3} [(27 - 9) - (-27 + 9)]$ $= \sqrt{3} [18 + 18]$ $= 36\sqrt{3}$	1 mark for correct integral
		OR $V = 2\int_0^3 \sqrt{3} (9 - x^2) dx$ $= 2\sqrt{3} \left[ 9x - \frac{x^3}{3} \right]_0^3$ $= 2\sqrt{3} \left[ (27 - 9) - 0 \right]$	1 mark for correct answer
		$= 2\sqrt{3} \times 18$ $= 36\sqrt{3} \text{ units}^3$	

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