Carlingford High School

2018 HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

Student Number:______________

	1-10	11	12	13	14	15	16	Total
MC	/10							/10
Complex Numbers		/11			/4			/15
Graphs			/2	/7			/4	/13
Conics				/8		/6		/14
Polynomials					/11	/3		/14
Integration			/9			/3		/12
Volumes			/4			/3	/3	/10
Harder 3U		/4					/8	/12
Total	/10	/15	/15	/15	/15	/15	/15	/100

- **General Instructions**
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 11

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1. A curve has equation $x^2 - y^2 + x^3 \cos y - 6 = 0$.

   What is $\frac{dy}{dx}$ for the curve?

   (A) $\dfrac{2x - 3x^2 \cos y}{2y}$ (B) $\dfrac{2x + 3x^2 \cos y}{2y}$

   (C) $\dfrac{2x + 3x^2 \cos y}{2y - x^3 \sin y}$ (D) $\dfrac{2x + 3x^2 \cos y}{2y + x^3 \sin y}$

2. In which of the following pairs is $g(x) = f^{-1}(x)$?

   (A) $f(x) = 2x + 3,\quad g(x) = \frac{x+3}{2}$ (B) $f(x) = x^2 + 3,\quad g(x) = \sqrt{x+3}$

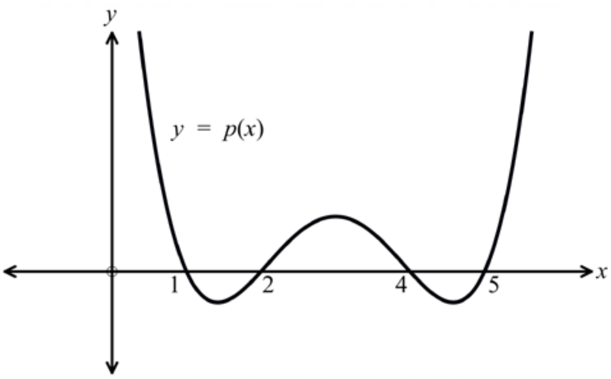
   (C) $f(x) = \frac{1}{\sqrt{x+1}},\quad g(x) = \frac{1-x^2}{x^2}$ (D) $f(x) = x^3 + 1,\quad g(x) = \sqrt[3]{x+1}$

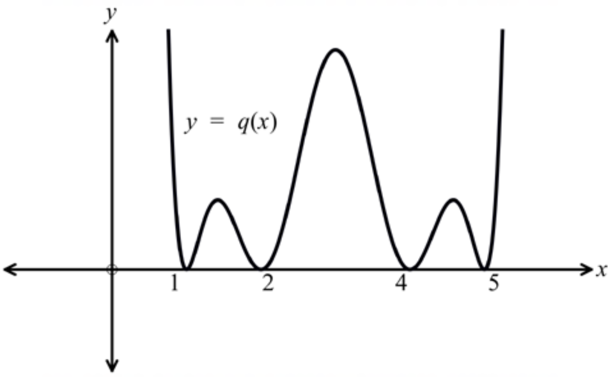
3. $\displaystyle\int \frac{dx}{x^2 + 4x + 9} =$

   (A) $\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$ (B) $\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x+2}{5}\right) + C$

   (C) $\frac{1}{5} \tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$ (D) $\frac{1}{5} \tan^{-1}\left(\frac{x+2}{5}\right) + C$

4. The graphs of two functions, $y = p(x)$ and $y = q(x)$ are drawn below.





Which of the following describes the relationship between the two functions?

(A) $q(x) = \frac{1}{p(x)}$

(B) $q(x) = [p(x)]^2$

(C) $p(x) = \frac{1}{q(x)}$

(D) $p(x) = [q(x)]^2$

5. An ellipse has cartesian equation $\frac{x^2}{4} + \frac{y^2}{2} = 1.$

What is the parametric equation of this ellipse?

(A) $x = 2\cos\theta, y = \sqrt{2}\sin\theta$

(B) $x = 4\cos\theta, y = 2\sin\theta$

(C) $x = \sqrt{2}\sin\theta, y = 2\cos\theta$

(D) $x = 2\sin\theta, y = 4\cos\theta$

6. What is the range for the curve $y = \frac{x^2-5x+1}{x-5}$?

(A) $-1 \leq y \leq 8;\ y \neq 5$

(B) $y \leq 1$ and $y > 5$

(C) $3 \leq y \leq 7$

(D) $y \leq 3$ and $y \geq 7$

7. What is the square root of $12 - 16i$?

(A) $\pm(2 - 4i)$

(B) $\pm(2\sqrt{3} - 4i)$

(C) $\pm(4 - 2i)$

(D) $\pm(4 - 2\sqrt{3}i)$

8. The polynomial $P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$ has a root of multiplicity 3.

Where is this root located?

(A) $x = -3$

(B) $x = -\frac{1}{2}$

(C) $x = \frac{1}{2}$

(D) $x = 3$

9. The region bounded by the curve $y = x^2$, the $x -$ axis, $x = 0$ and $x = 2$ is rotated around the line $x = 2$.

Which of the following gives the volume of the solid formed?

(A) $V = \pi \int_0^2 (2 - x)^2 \, dy$

(B) $V = \pi \int_0^4 (2 - x)^2 \, dy$

(C) $V = \pi \int_0^2 (2 - x^2)^2 \, dy$

(D) $V = \pi \int_0^4 (2 - x^2)^2 \, dy$

10. The equation $x^4 + px + q = 0$ where $p \neq 0$ and $q \neq 0$ has roots α, β, γ and δ.

What is $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$?

(A) $-4q$

(B) $p^2 - 2q$

(C) $p^4 - 2q$

(D) p^4

Section II

90 marks

Attempt Questions 11 – 16.

Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 writing booklet.

(a) z is a complex number such that $|z| = 2$ and $\arg z = \frac{\pi}{3}$.

(i) Evaluate z^5 **1**

(ii) Write down z in cartesian form. **1**

(iii) Find the value of $\frac{1}{z}$ in cartesian form. **2**

(iv) If $\omega = 2 - 3i$, find the value of $\omega^2 z$ **1**

(b) (i) Find the values of a, b, and c such that: **2**

$$\frac{x+1}{(x+3)(x+2)^2} = \frac{a}{(x+3)} + \frac{b}{(x+2)} + \frac{c}{(x+2)^2}$$

(ii) Hence evaluate $\int \frac{x+1}{(x+3)(x+2)^2}\, dx$. **2**

(c) (i) Show that $\cos x + \cos 3x = 4cos^3 x - 2\cos x$. **2**

(ii) Hence or otherwise solve $cos\, x + cos\, 3x = 0$ for $0 \le x \le 2\pi$. **2**

(d) Sketch the region in the Argand diagram where $1 < z\bar{z} \le 3$ and $Im\,(z) \ge 0$. **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 writing booklet.

(a) (i) Find $\int sec^4 x \tan x \; dx$. **2**

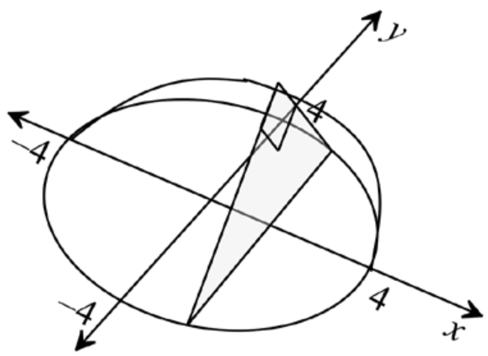
(ii) Find $\int \frac{dx}{\sqrt{7+4x-x^2}}$. **2**

(iii) Evaluate $\int \frac{dx}{x^2\sqrt{9+x^2}}$, using the trigonometric substitution $x = 3\,tan\,\theta$. **3**

(b) Find the exact value of $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}} dx$. **2**

(c) Find the equation of the normal to the curve $x^2 + xy + y^2 = 7$ at the point (1, 2). **2**

(d) Let *S* be the solid having for its base the region bounded by the circle $x^2 + y^2 = 16$. **4**



Every plane of the solid taken perpendicular to the *x* – axis is an isosceles right angled triangle with the hypotenuse in the plane of the base.

Find the volume of the solid *S*.

End of Question 12

Question 13 (15 marks) Use the Question 13 writing booklet.

(a) The points $P\left(3p, \frac{3}{p}\right), p \neq 0,$ and $Q\left(3q, \frac{3}{q}\right),\ q \neq 0$, are two points on the rectangular hyperbola $xy = 9$.

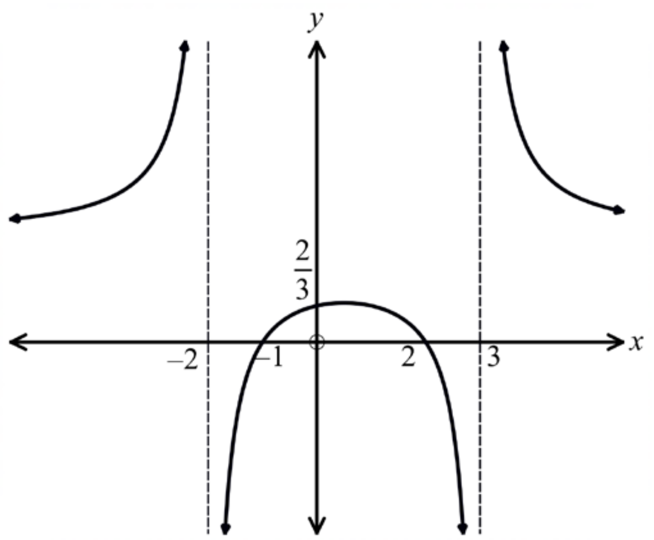
(i) Find the equation of the chord PQ. **2**

(ii) Prove that the tangent at P has equation $x + p^2y = 6p$. **2**

(iii) The tangents at P and Q intersect at T. Find the coordinates of T. **2**

(iv) The line through T, parallel to PQ passes through the point (0, 6). **2**
Show that $p + q = 2$.

(b) The graph of $y = f(x)$ is shown below.



Sketch the following curves on separate half page diagrams.

(i) $y = |f(x)|$ **1**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y = f'(x)$ **2**

(iv) $y^2 = f(x)$ **2**

End of Question 13

Question 14 (15 marks) Use the Question 14 writing booklet.

(a) The polynomial $x^3 - 3x^2 + 4x - 6 = 0$ has roots α, β and γ.

Calculate the value of:

(i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ **3**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(b) Suppose $p(x) = ax^3 + bx^2 + cx + d$ with a, b, c and d real, $a \neq 0$.

(i) Deduce that if $b^2 - 3ac < 0$ then $p(x)$ cuts the x-axis only once. **2**

(ii) If $b^2 - 3ac = 0$ and $p\left(-\frac{b}{3a}\right) = 0$, find the multiplicity of the root $x = -\frac{b}{3a}$. **2**

(c) (i) If $z^n = cos\, n\theta + isin\, n\theta$, use De Moivre's theorem to show that **1**

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(ii) Find the constants *A*, *B*, and *C* such that: **2**

$$cos^5\theta = Acos\, 5\theta + Bcos\, 3\theta + Ccos\, \theta.$$

(iii) Hence evaluate $\int cos^5\theta\, d\theta$. **1**

(d) One of the roots of the equation $x^2 - (6 - 2i)x + k = 0$ is $2 - 3i$.

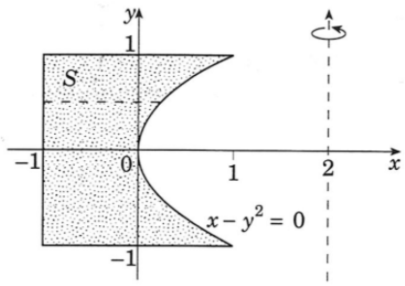
(i) State the other root. **1**

(ii) Find the value of *k*. **1**

End of Question 14

Question 15 (15 marks) Use the Question 15 writing booklet.

(a) The shaded region is bounded by the lines $x = -1$, $y = 1$ and $y = -1$ and by the curve $x - y^2 = 0$. The region is rotated $360°$ about the line $x = 2$ to form a solid.



When the region is rotated, the line segment S at height y sweeps out an annulus.

(i) Show that the area of the annulus at height y is equal to $\pi(5 + 4y^2 - y^4)$. **1**

(ii) Hence find the volume of the solid. **2**

(b) Find all solutions to the equation $x^4 - 5x^3 + 17x^2 + 37x - 50 = 0$, given that $x = 3 - 4i$ is one solution. **3**

(c) (i) Show that a reduction formula for $I_n = \int sec^n x \ dx$ is **2**

$$I_n = \frac{1}{n-1}\ sec^{n-2} x tan\, x + \frac{n-2}{n-1}\ I_{n-2}$$

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} sec^4 x\, dx$. **1**

(d) The hyperbola *H* has equation $9x^2 - 16y^2 = 144$.

(i) Write down the eccentricity for this hyperbola and find the coordinates of its foci S and S'. **2**

(ii) If $P(x_1, y_1)$ is an arbitrary point on *H*, prove that the equation of the tangent *T* at *P* is: $9xx_1 - 16yy_1 = 144$. **1**

(iii) Hence find the coordinates of the point *G* at which the tangent *T* cuts the *x* – axis. **1**

(iv) Hence prove that $\frac{SP}{S'P} = \frac{SG}{S'G}$. **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 writing booklet.

(a) If a, b and c are positive and unequal, prove that

(i) $a + b - 2\sqrt{ab} \geq 0$ **1**

(ii) $(a + b)(b + c)(c + a) > 8abc$ **2**

(b) The sequence $\{x_n\}$ is given by $x_1 = 1$ and $x_{n+1} = \frac{4+x_n}{1+x_n}$ for $n \geq 1$.

(i) Prove by induction that for $n \geq 1$, **4**

$$x_n = 2\left(\frac{1+\alpha^n}{1-\alpha^n}\right), \quad \text{where } \alpha = -\frac{1}{3}.$$

(ii) Hence find the limiting value of x_n as $n \to \infty$. **1**

(c) The region bounded by the curve $y = \sin x$, the $x -$ axis, $x = 0$ and $x = \pi$ is revolved around the y-axis **3**

Find the volume of the solid of revolution formed.

(d) Sketch the graph of $y = x + \frac{8x}{x^2-9}$, clearly indicating any asymptotes and any points where the graph meets the axes. **4**

End of Paper

Carlingford High School

2018

TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

REFERENCE SHEET

- Mathematics –

- Mathematics Extension 1–

- Mathematics Extension 2-

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

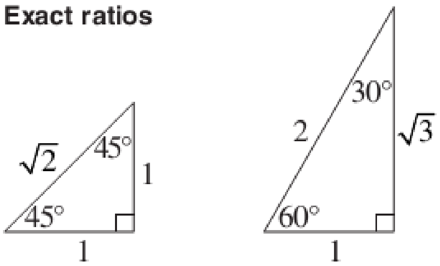
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\dfrac{dy}{dx} = F'(u)\dfrac{du}{dx}$

If $y = e^{f(x)}$, then $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\dfrac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\dfrac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$\alpha + \beta = -\dfrac{b}{a}$ $\qquad$ $\alpha\beta = \dfrac{c}{a}$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$\text{Area} = \dfrac{1}{2}r^2\theta$

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

t formulae

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \quad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \quad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \quad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $(2at, at^2)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At (x_1, y_1),

tangent: $xx_1 = 2a(y+y_1)$

normal: $y - y_1 = -\frac{2a}{x_1}(x - x_1)$

Chord of contact from (x_0, y_0): $xx_0 = 2a(y+y_0)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$x = b + a\cos(nt+\alpha)$

$\ddot{x} = -n^2(x-b)$

Further integrals

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha+\beta+\gamma = -\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Trial HSC Examination 2018
Mathematics Extension 2 Course

Student Number: ____________________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	$2 + 4 =$	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ●	B ●	C ○	D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ●	B ● correct	C ○	D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○