CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Mathematics Extension 2

Assessment Task 3

2020

Time allowed: 1 hour 40 minutes

Student Number: ______________________________ **Teacher: Ms Strilakos**

Instructions:

- All questions should be attempted.
- Show ALL necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.

VECTORS									INT	TOTAL
Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Q.7	Q.8	Q.9	Q.10	
/2	/4	/5	/4	/7	/2	/5	/7	/10	/4	/50

Q.1 Find the obtuse angle between the vectors

$\underset{\sim}{u} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$

[2]

$$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

$$= \frac{4-9-4}{\sqrt{26}\cdot\sqrt{26}}$$

$$= \frac{-9}{26}$$

$$\therefore \theta = 110^\circ 15'$$

Q.2 Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

Find a unit vector $\underset{\sim}{u}$ which is perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

[4]

Let $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ $\qquad x^2+y^2+z^2=1$ (unit vector) $-(3)$

$\underset{\sim}{a}\cdot\underset{\sim}{u} = 3x+2y+2z \qquad -(1)$

$\underset{\sim}{b}\cdot\underset{\sim}{u} = -x+y+z \qquad -(2)$ $\qquad 1$

$3\times(2)+(1) \qquad 5y+5z=0 \qquad \therefore y=-z \qquad -(4)$ $\qquad 1$

$(1)-2\times(2) \qquad 5x=0 \qquad \therefore x=0 \qquad -(5)$

Subst. (4) & (5) in (3) $\qquad 2y^2=1$

$y^2=\frac{1}{2}, \; y=\pm\frac{1}{\sqrt{2}}$ $\qquad 1$

$\therefore z=\mp\frac{1}{\sqrt{2}}$

$x=0$

$\therefore \underset{\sim}{u} = \pm\frac{1}{\sqrt{2}}(\underset{\sim}{j}-\underset{\sim}{k})$ $\qquad 1$

Q.3 (i) Find a vector form of the equation of the plane which contains the points $P(4,-3,1)$, $Q(-3,-1,1)$ and $R(4,-2,8)$

(ii) Hence find the Cartesian equation of the plane.

[5]

(i) $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \begin{bmatrix}-4\\3\\-1\end{bmatrix} + \begin{bmatrix}-3\\-1\\1\end{bmatrix} = \begin{bmatrix}-7\\2\\0\end{bmatrix}$

$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = \begin{bmatrix}-4\\3\\-1\end{bmatrix} + \begin{bmatrix}4\\-2\\8\end{bmatrix} = \begin{bmatrix}0\\1\\7\end{bmatrix}$

$\therefore$ the equation of the plane is:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\-3\\1\end{bmatrix} + \lambda\begin{bmatrix}-7\\2\\0\end{bmatrix} + \mu\begin{bmatrix}0\\1\\7\end{bmatrix}$$

(ii) $x = 4 - 7\lambda$ — ①

$y = -3 + 2\lambda + \mu$ — ②

$z = 1 + 7\mu$ — ③

$7 \times ② - ③$ $\quad 7y - z = -22 + 14\lambda$ — ④

~~$2\times③ + ④ \quad 2z + 7y + z = 24$~~

~~$2z - 7y + z$~~

~~$2\times① + ④ \quad 2x + 7y - z = 8 - 14\lambda + 14\lambda - 22$~~

~~$2\times①$~~

~~$2x + 7y = -13 + 7\mu$ ⑤~~

~~$2\times⑤ - ④ \quad 4x + 14y - 7y + z = -26 + 22$~~

~~$4x + 7y + z = -14$~~

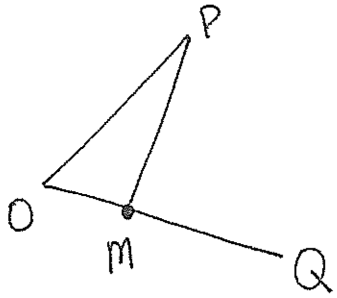
$2\times① + ④$

$2x + 7y - z = 8 - 22 = -14$

$2x + 7y - z = -14$

Q.4 The position vectors of the points P and Q are given by $(\overrightarrow{OP}) = 3\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ and $(\overrightarrow{OQ}) = 5\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$ respectively. Find the position vector of M on $\overrightarrow{OQ}$ such that $\overrightarrow{PM}$ is perpendicular to $\overrightarrow{OQ}$.

[4]



4

We require $\overrightarrow{OM}$ and $\overrightarrow{PM} \cdot \overrightarrow{OQ} = 0$

Now, M lies on $\overrightarrow{OQ}$ $\therefore$ $\overrightarrow{OM} = m(5\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$

$\overrightarrow{PM} = \overrightarrow{PO} + \overrightarrow{OM} = (-3\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}) + m(5\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$

$= (5m-3)\underset{\sim}{i} + (-4m-1)\underset{\sim}{j} + (3m-3)\underset{\sim}{k}$

Since $\overrightarrow{PM} \perp \overrightarrow{OQ}$, $\overrightarrow{PM} \cdot \overrightarrow{OQ} = 0$

i.e. $(5m-3) \cdot 5 + (-4m-1) \cdot -4 + (3m-3) \cdot 3 = 0$

i.e. $25m - 15 + 16m + 4 + 9m - 9 = 0$

i.e. $50m = +20$ $\therefore m = \frac{2}{5}$

$\therefore \overrightarrow{OM} = \frac{2}{5}(5\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$

$= 2\underset{\sim}{i} - \frac{8}{5}\underset{\sim}{j} + \frac{6}{5}\underset{\sim}{k}$

Alternatively,

$\overrightarrow{OM} = \text{proj}_{\overrightarrow{OQ}} \overrightarrow{OP}$

Q.5 Find a unit vector $\underset{\sim}{u}$ which makes an angle of $\frac{\pi}{4}$ with the Z-axis and is such that $\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{u}$ is a unit vector.

[7]

$$\underset{\sim}{u} = \cos\alpha\, \underset{\sim}{i} + \cos\beta\, \underset{\sim}{j} + \cos\gamma\, \underset{\sim}{k}$$

Making an angle $\frac{\pi}{4}$ with z-axis:

$$\underset{\sim}{u} = \cos\alpha\, \underset{\sim}{i} + \cos\beta\, \underset{\sim}{j} + \frac{1}{\sqrt{2}}\underset{\sim}{k}$$

Sum of squares of components is 1 i.e. $\cos^2\alpha + \cos^2\beta + \frac{1}{2} = 1$ 1

$\therefore \cos^2\alpha + \cos^2\beta = \frac{1}{2}$ — 1

Now, $\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{u}$ is also a unit vector.

$\therefore \underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{u} = \underset{\sim}{i}+\underset{\sim}{j}+\cos\alpha\, \underset{\sim}{i} + \cos\beta\, \underset{\sim}{j} + \frac{1}{\sqrt{2}}\underset{\sim}{k}$ 1

Where $(1+\cos\alpha)^2 + (1+\cos\beta)^2 + \frac{1}{2} = 1$ — 3

i.e. $1 + 2\cos\alpha + \cos^2\alpha + 1 + 2\cos\beta + \cos^2\beta = \frac{1}{2}$

i.e. $2\cos\alpha + 2\cos\beta + \cos^2\alpha + \cos^2\beta = -1\frac{1}{2}$ — 2

Substituting 1 into 2 $2\cos\alpha + 2\cos\beta + \frac{1}{2} = -1\frac{1}{2}$ 1

i.e. $2\cos\alpha + 2\cos\beta = -2$

$\cos\alpha + \cos\beta = -1$ i.e. $\cos\alpha = -(1+\cos\beta)$ — 4

Subst. 4 into 3

$(-\cos\beta)^2 + (1+\cos\beta)^2 = \frac{1}{2}$ 1

$\cos^2\beta + 1 + 2\cos\beta + \cos^2\beta = \frac{1}{2}$

$2\cos^2\beta + 2\cos\beta + \frac{1}{2} = 0$

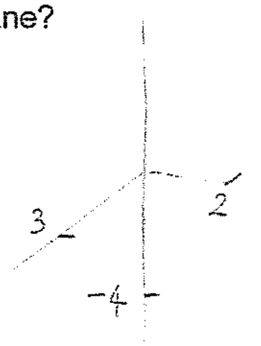
$\cos^2\beta + \cos\beta + \frac{1}{4} = 0$ 1 1

$(\cos\beta + \frac{1}{2})^2 = 0$ $\cos\beta = -\frac{1}{2}$ $\therefore \cos\alpha = -\frac{1}{2}$

$\therefore \underset{\sim}{u} = -\frac{1}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j} + \frac{1}{\sqrt{2}}\underset{\sim}{k}$ 1

Q.6 What would be the Cartesian equation of the set of all points common to both the sphere described by the equation $(x-3)^2 + (y-2)^2 + (z+4)^2 = 5^2$, and the $x - y$ plane?

Centre $(3,2,-4)$

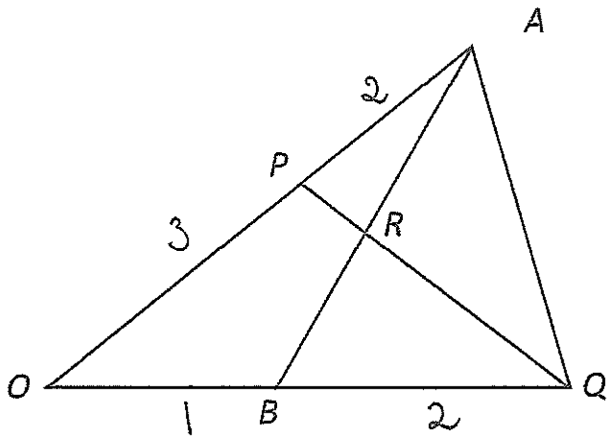


Meets x-y plane when $z=0$ [2]

ie. $(x-3)^2+(y-2)^2+4^2=5^2$

$(x-3)^2+(y-2)^2=3^2$

Q.7 The figure below shows a triangle OAQ.



- The point P lies on OA so that $OP{:}OA = 3{:}5$.
- The point B lies on OQ so that $OB{:}BQ = 1{:}2$.

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

(i) Given that $\overrightarrow{AR} = h\overrightarrow{AB}$, where h is a scalar parameter with $0 < h < 1$, show that

$$\overrightarrow{OR} = (1-h)\underset{\sim}{a} + h\underset{\sim}{b}.$$

(ii) Given further that $\overrightarrow{PR} = k\overrightarrow{PQ}$, where k is a scalar parameter with $0 < k < 1$, find a similar expression for $\overrightarrow{OR}$ in terms of k, $\underset{\sim}{a}$, and $\underset{\sim}{b}$.

(iii) Determine

(a) the value of k and the value of h.

(b) the ratio of $\overrightarrow{PR}$: $\overrightarrow{PQ}$ [5]

(i) $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$= \underset{\sim}{a} + h(\overrightarrow{AO} + \overrightarrow{OB})$

$= \underset{\sim}{a} + h(-\underset{\sim}{a} + \underset{\sim}{b})$

$= (1-h)\underset{\sim}{a} + h\underset{\sim}{b}$ — ①

(ii) $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= \frac{3}{5}\underset{\sim}{a} + k(\overrightarrow{PO} + \overrightarrow{OQ})$

$= \frac{3}{5}\underset{\sim}{a} + k(-\frac{3}{5}\underset{\sim}{a} + 3\underset{\sim}{b})$

$= \frac{3}{5}(1-k)\underset{\sim}{a} + 3k\underset{\sim}{b}$ — ②

(iii) (a) Equating ① & ② components :

$1-h = \frac{3}{5}(1-k)$ and $h = 3k$ — ③

$5-5h = 3-3k$

$5h-2 = 3k$. ie. $5h-2=h$ from ③

ie. $4h = 2$, $h = \frac{1}{2}$ $\therefore k = \frac{1}{6}$

(b) $\overrightarrow{PR} : \overrightarrow{PQ}$

Given $\overrightarrow{PR} = k\overrightarrow{PQ}$

$\overrightarrow{PR} = \frac{1}{6}\overrightarrow{PQ}$

5

$\therefore$ Ratio is $1:6$

....continued

Q.8 Relative to a fixed origin O, the points A, B and C have coordinates $(2,3,5)$, $(1,1,1)$ and $(4,3,1)$, respectively.

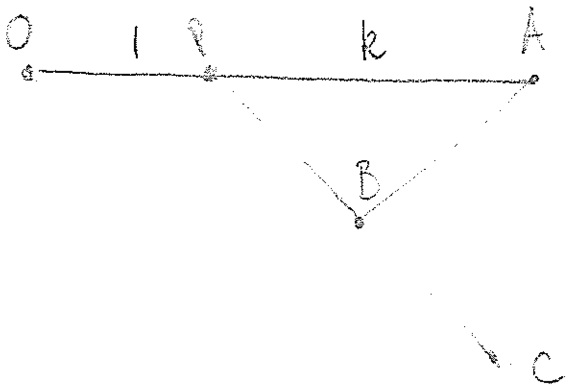
The line segment CB is extended to the point P so that $\overrightarrow{CP} = \mu\overrightarrow{CB}$

It is further given that P lies on the line segment OA so that $|OP|:|PA| = 1:k$.

(i) Find an expression for $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$.

(ii) Using all of the above information, determine the value of k.

[7]



$\overrightarrow{CP} = \mu\overrightarrow{CB}$

$|OP|:|PA| = 1:K$

(i) $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP} = \overrightarrow{OC} + \mu\overrightarrow{CB} = \overrightarrow{OC} + \mu(\overrightarrow{CO} + \overrightarrow{OB})$

$\overrightarrow{OC} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

In terms of $\overrightarrow{OA}$, since P divides OA in the ratio $1:K$

$\overrightarrow{OP} = \frac{1}{1+k}\overrightarrow{OA}$ 1

(ii) Using above information, $\overrightarrow{OP} = \overrightarrow{OC} + \mu(\overrightarrow{CB})$

$\overrightarrow{OP} = \frac{1}{1+k}\overrightarrow{OA} = \overrightarrow{OC} + \mu(-4\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k} + \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$. 1

$= 4\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k} + \mu(-3\underset{\sim}{i} - 2\underset{\sim}{j})$

$\frac{1}{1+K}(2\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}) = (4-3\mu)\underset{\sim}{i} + (3-2\mu)\underset{\sim}{j} + \underset{\sim}{k}$ 1

Equating components: $\frac{2}{1+K} = 4 - 3\mu$ —① , $\frac{3}{1+k} = 3 - 2\mu$ —② 1

$\frac{5}{1+k} = 1$ —③

$-③$

Solving for k from $\frac{5}{1+k} = 1$, $1+k=5$, $\underline{\underline{k=4}}$ 1

<u>CHECK:</u>

$\therefore \quad \frac{3}{1+4} = 3 - 2\mu$ from ②

$\frac{3}{5} = 3 - 2\mu$

$2\mu = 2\frac{2}{5} \quad \mu = 1\frac{1}{5}$. 1 for μ.

and $\frac{2}{5} = 4 - 3\mu$ from ①

$3\mu = 3\frac{3}{5} \quad \mu = 1\frac{1}{5}$ 1 for checks.

We have consistency for μ with $k=4$. in ① & ②

$\therefore \ \underline{\underline{k=4}}$ is a correct solution.

⑦.

....continued

Q.9 The straight lines l_1 and l_2 have the vector equations

$$r_1 = 4\underset{\sim}{i} + 7\underset{\sim}{j} + 4\underset{\sim}{k} + \lambda(\underset{\sim}{i} - \underset{\sim}{j}) \quad \text{and} \quad r_2 = 8\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k} + \mu(\underset{\sim}{i} - \underset{\sim}{k})$$

respectively, where λ and μ are scalar parameters.

(i) Show that l_1 and l_2 intersect at some point A and find the coordinates of this point A.

(ii) Calculate the acute angle between l_1 and l_2 .

The point $B(8,3,4)$ lies on l_1 and the point C lies on l_2 where $\mu = 4$.

(iii) Find the distance AB .

(iv) Find the area of the triangle ABC .

[10]

(i) For pt. of intersection, equate components of $\underset{\sim}{r_1}$ & $\underset{\sim}{r_2}$

$\underset{\sim}{r_1} = (4+\lambda)\underset{\sim}{i} + (7-\lambda)\underset{\sim}{j} + 4\underset{\sim}{k}$ $\qquad$ $\underset{\sim}{r_2} = (8+\mu)\underset{\sim}{i} + 5\underset{\sim}{j} + (2-\mu)\underset{\sim}{k}$

$$\left.\begin{aligned} &4+\lambda = 8+\mu \quad -① \\ &7-\lambda = 5 \quad \Rightarrow \lambda = 2 \\ &4 = 2-\mu \quad \Rightarrow \mu = -2 \end{aligned}\right\} \begin{aligned} &\text{check for consistency in ①} \\ &4+2 = 8-2 \ \checkmark \end{aligned}$$

Substitute λ or μ into $\underset{\sim}{r_1}$ or $\underset{\sim}{r_2}$ to get $6\underset{\sim}{i} + 5\underset{\sim}{j} + 4\underset{\sim}{k}$

$\therefore$ pt. of intersection is $(6,5,4)$.

....continued

(ii) ~~l_1 is $r_1 = 4i$~~

Using direction vectors:

For l_1, $\underset{\sim}{r_1} = \underset{\sim}{i} - \underset{\sim}{j}$ for l_2 $\underset{\sim}{r_2} = \underset{\sim}{i} - \underset{\sim}{k}$

$$\therefore \cos\theta = \frac{(\underset{\sim}{i} - \underset{\sim}{j})(\underset{\sim}{i} - \underset{\sim}{k})}{\sqrt{2}.\sqrt{2}} = \frac{1}{2} \qquad \therefore \theta = \frac{\pi}{3}^c \qquad 1$$

(iii) $\overrightarrow{OA} = 6\underset{\sim}{i} + 5\underset{\sim}{j} + 4\underset{\sim}{k}$ from part (i)

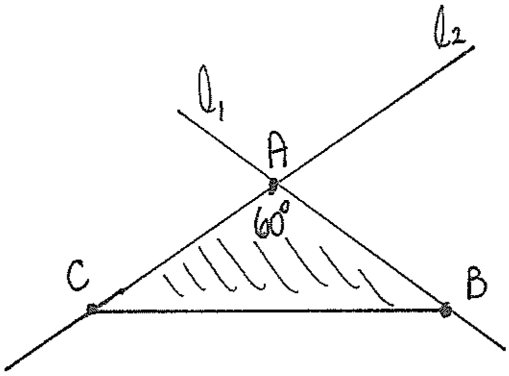
$\overrightarrow{OB} = 8\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$

$$\therefore \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = (-6+8)\underset{\sim}{i} + (-5+3)\underset{\sim}{j} + (-4+4)\underset{\sim}{k} \qquad 1$$

$$= 2\underset{\sim}{i} - 2\underset{\sim}{j}$$

$$|\overrightarrow{AB}| = \sqrt{8} = 2\sqrt{2} \qquad 1$$

(iv)



$\overrightarrow{OC} = 12\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$

with $\mu = 4$
on l_2

To find the area of ΔABC,

$$\text{Area} = \frac{1}{2}|\overrightarrow{AC}||\overrightarrow{AB}|\sin 60^\circ \qquad 1$$

Now, $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$$= -6\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k} + 12\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$= 6\underset{\sim}{i} - 6\underset{\sim}{k} \qquad 1$$

$$\therefore |\overrightarrow{AC}| = \sqrt{72} = 6\sqrt{2}$$

So, $\text{Area} = \frac{1}{2} \times 6\sqrt{2} \times 2\sqrt{2} \times \frac{\sqrt{3}}{2}$

$$= 6\sqrt{3} \text{ units}^2 \qquad 1$$

....continued

Q.10 Find $\int \frac{x^3}{\sqrt{4x^2-1}}\,dx$

[4]

Let $u = 4x^2-1$

$$\frac{du}{dx} = 8x$$

Also, $4x^2 = u+1$

$$x^2 = \frac{u+1}{4}$$

$$\therefore\ I = \int \frac{1}{u^{1/2}} \cdot \frac{u+1}{4} \cdot \frac{1}{8}\,du$$

$$= \frac{1}{32}\int \frac{u+1}{u^{1/2}}\,du$$

$$= \frac{1}{32}\int (u^{1/2} + u^{-1/2})\,du$$

$$= \frac{1}{32}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right] + c$$

$$= \frac{1}{48}(4x^2-1)^{3/2} + \frac{1}{16}(4x^2-1)^{1/2} + c$$

Can also be written as : $\frac{\sqrt{4x^2-1}\,(4x^2-1+3)}{48} + c$

$$= \frac{\sqrt{4x^2-1}\,(4x^2+2)}{48} + c = \frac{\sqrt{4x^2-1}\,(2x^2+1)}{24} + c$$

END OF PAPER