ACE
EXAM
PAPER

Student name: ____________________

PAPER 4

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 2

General Instructions

- Working time - 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Let $z = 3 - 4i$ and $w = \sqrt{3} + i$. What is the value of $z \div w$?

   (A) $\dfrac{3\sqrt{3}-4}{4} + \dfrac{\left(-4\sqrt{3}-3\right)i}{4}$

   (B) $\dfrac{3\sqrt{3}+4}{4} + \dfrac{\left(-4\sqrt{3}-3\right)i}{4}$

   (C) $\dfrac{3\sqrt{3}-4}{4} + \dfrac{\left(-4\sqrt{3}+3\right)i}{4}$

   (D) $\dfrac{3\sqrt{3}+4}{4} + \dfrac{\left(-4\sqrt{3}+3\right)i}{4}$

2. A particle is moving under SHM in a straight line with an acceleration of $\ddot{x} = 25 - 5x$, where x is the displacement after t seconds. What is the centre of motion?

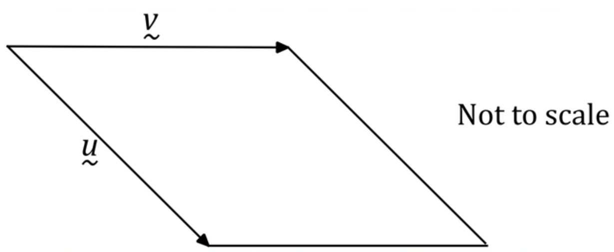
   (A) $x = 0$

   (B) $x = 5$

   (C) $x = 10$

   (D) $x = 15$

3.



   The diagram above shows a rhombus, spanned by two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.
   Which of the following statements is correct?

   (A) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

   (B) $\underset{\sim}{u} = \underset{\sim}{v}$

   (C) $(\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v}) = 0$

   (D) $|\underset{\sim}{u} + \underset{\sim}{v}| = |\underset{\sim}{u} - \underset{\sim}{v}|$

4. The contrapositive of $A \Rightarrow B$ is:

(A) $B \Rightarrow A$

(B) $B \Leftrightarrow A$

(C) $(\text{not } B) \Rightarrow (\text{not } A)$

(D) $(\text{not } A) \Rightarrow (\text{not } B)$

5. The velocity of a body moving in a straight line is given by $v = 2\sqrt{1-x^2}$ where x metres is the displacement from fixed point O and v is the velocity in metres per second. Initially the particle is at O. Let a be the acceleration in metres per second squared.
Which of the following is the correct expression for a in terms of x?

(A) $a = \dfrac{-2x}{\sqrt{1-x^2}}$

(B) $a = \dfrac{2}{\sqrt{1-x^2}}$

(C) $a = -2x$

(D) $a = -4x$

6. What is the value of the indefinite integral $\displaystyle\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx$?

(A) $\dfrac{x}{(1-x^2)^{\frac{1}{2}}} - \cos^{-1}x + C$

(B) $\dfrac{x}{(1-x^2)^{\frac{3}{2}}} - \cos^{-1}x + C$

(C) $\dfrac{x}{(1-x^2)^{\frac{1}{2}}} - \sin^{-1}x + C$

(D) $\dfrac{x}{(1-x^2)^{\frac{3}{2}}} - \sin^{-1}x + C$

7. What is the definite integral of $\displaystyle\int \frac{\sec^2(\ln x)}{x}\,dx$?

(A) $\tan(\ln x) + C$

(B) $\tan(\cos x) + C$

(C) $\sec(\ln x) + C$

(D) $\sec(\cos x) + C$

8. What is the square root of $8 + 6i$?

   (A) $-3 - i$

   (B) $-3 + i$

   (C) $3 - i$

   (D) $5 - 3i$

9. A particle is projected with a speed of 20 m/s and passes through a point P whose horizontal distance from the point of projection is 30 m and whose vertical height above the point of projection is 8.75 m. What is the angle of projection?

   (A) $\tan^{-1}\left(\frac{2}{3}\right)$

   (B) $\tan^{-1}\left(\frac{3}{2}\right)$

   (C) $\tan^{-1}\left(\frac{3}{4}\right)$

   (D) $\tan^{-1}\left(\frac{4}{3}\right)$

10. $(2 + 2i)z^2 + 8iz - 4(1 - i)$

   What is the value of the discriminant for the above quadratic expression?

   (A) –32

   (B) 0

   (C) 32

   (D) 64

Section II

90 marks
Attempt questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) If $z_1 = 2i$ and $z_2 = 1 + 3i$, express in the form $a + ib$ (where a and b are real)

(i) $z_1 + \overline{z_2}$ **1**

(ii) $z_1 \, z_2$ **1**

(iii) $\frac{1}{z_2}$ **1**

(b) Find $\int \frac{1}{\sqrt{5 + 4x - x^2}} \, dx$. **2**

(c) The points P and Q have position vectors of $p = \overrightarrow{OP}$ and $q = \overrightarrow{OQ}$. Express $\overrightarrow{OR}$ in terms of p and q , where R is:

(i) the midpoint of PQ . **2**

(ii) the point such that $\overrightarrow{PR} = -2\overrightarrow{PQ}$. **2**

(iii) the trisection of PQ and closer to P. **2**

(d) Find the exact value of $\int_2^3 \frac{x + 1}{\sqrt{x^2 + 2x + 5}} \, dx$. **2**

(e) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ for positive integers $n \geq 1$. **2**

Let $z = (\cos\theta + i\sin\theta)$ where $z \neq 0$.

Question 12 (15 marks) **Marks**

(a) If $|a| < 1$ and $|b| < 1$ prove that $|a + b| < |1 + ab|$ **2**

(b) $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$

(i) Find $z_1 \div z_2$ in the form $a + ib$ where a and b are real. **1**

(ii) Write z_1 and z_2 in modulus-argument form. **2**

(iii) Write $\cos\dfrac{5\pi}{12}$ as a surd by equating equivalent expressions for $z_1 \div z_2$. **2**

(c) Evaluate the integral $\displaystyle\int \frac{\ln x}{x^2}\,dx$. **2**

(d) Let $z = 1 - i$ be a root of the polynomial $z^3 + pz + q = 0$ where p and q are real numbers. Find the value of p and q. **2**

(e) (i) Find the values of A, B, and C such that: **2**

$$\frac{1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$$

(ii) Hence evaluate $\displaystyle\int \frac{dx}{(x+1)(x^2+2)}$. **2**

Question 13 (14 marks) **Marks**

(a) Prove by induction that $7^n + 15^n$ is divisible by 11 for odd $n \geq 1$. **4**

(b) Find $\displaystyle\int \frac{e^{3x}+1}{e^x+1}\,dx$. **2**

(c) A particle is moving in a straight line under SHM. At any time (t seconds) its displacement (x metres) from a fixed point O is given by:

$x = A\cos\left(\frac{\pi}{4}t + \alpha\right)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$

After 1 second the particle is 2 metres to the right of O and after 3 seconds the particle is 4 metres to the left of O.

(i) Show that $A\sin\alpha - A\cos\alpha = -2\sqrt{2}$ and $A\sin\alpha + A\cos\alpha = 4\sqrt{2}$. **2**

(ii) Show that $A = 2\sqrt{5}$ and $\alpha = \tan^{-1}\frac{1}{3}$ **2**

(iii) When does the particle first pass through O? **2**

(d) If $a > 0$, $b > 0$ and $a + b = t$ show that: **2**

$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{t}$

Question 14 (15 marks) **Marks**

(a) On an Argand diagram shade the region that satisfies $|z| \geq 1$ and $-\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}$. **2**

(b) A particle of mass m moves in a horizontal straight line. The particle is resisted by a constant force mk and a variable force mv^2, where k is a positive constant and v is the speed. Initially $v = u$ and $x = 0$.

(i) Show that the distance travelled is $-\frac{1}{2}\ln\frac{(k+v^2)}{(k+u^2)}$ **2**

(ii) Show that the time taken for the particle to be brought to rest is: **3**

$$t = \frac{1}{\sqrt{k}}\tan^{-1}\frac{u}{\sqrt{k}}$$

(c) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx \qquad n = 1,2,3,\ldots$

(i) Show that $I_{n+1} = \frac{2n-1}{2n}I_n + \frac{1}{n \times 2^{n+1}} \quad n = 1,2,3,\ldots$ **3**

(ii) Hence evaluate $\int_0^1 \frac{1}{(1+x^2)^3}\,dx$. **2**

(d) (i) If $z = \frac{1+i\sqrt{3}}{2}$ show that $z^3 = -1$. **2**

(ii) Hence or otherwise find the value of z^{10}. **1**

Question 15 (17 marks) **Marks**

(a) The point P has position vector $\overrightarrow{OP} = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, point Q has position vector $\overrightarrow{OQ} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and point R has position vector $\overrightarrow{OR} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$. Find the magnitude of $\angle PQR$. Answer correct to one decimal place. **4**

(b) (i) Show that $(1 - 3i)^2 = -8 - 6i$. **1**

(ii) Hence or otherwise solve the equation: $2z^2 - 8z + (12 + 3i) = 0$. **2**

(c) If $T_1 = 8, T_2 = 20$ and $T_n = 4T_{n-1} - 4T_{n-2}$ for $n \geq 3$ show that: **4**

$T_n = (n + 3)2^n$ for $n \geq 1$

(d) Find $|\underset{\sim}{u} + \underset{\sim}{v}|$ if $|\underset{\sim}{u}| = 6$, $|\underset{\sim}{v}| = 5$ and $|\underset{\sim}{u} \cdot \underset{\sim}{v}| = -4$. **2**

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate the following definite integral. **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{3 - \cos x - 2\sin x}\, dx$$

Question 16 (14 marks) **Marks**

(a) The point A has position vector $\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ relative to an origin O. **2**
Find a unit vector parallel to $\overrightarrow{OA}$.

(b) A particle of mass 30 kg is projected vertically upwards with an initial velocity of u metres per second. It experiences air resistance which is proportional to the square of its speed. Find the time taken for the particle to reach its maximum height. **4**

(c) Show that the inequality $x \geq \ln(1+x)$ is true for $x \geq -1$ using calculus. **3**

(d) Given vector $\underset{\sim}{u} = 3\underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$ where m is a real number.
Find the value(s) of m:

(i) if the length of vector $\underset{\sim}{u}$ is 10. **2**

(ii) if the vector $\underset{\sim}{u}$ makes an angle of $\cos^{-1}\left(\frac{1}{3}\right)$ with y axis. **3**

End of paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

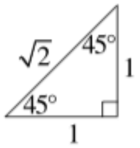
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

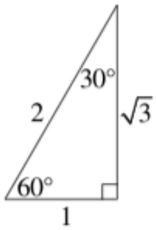
$A = \frac{1}{2}ab\sin C$



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \frac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

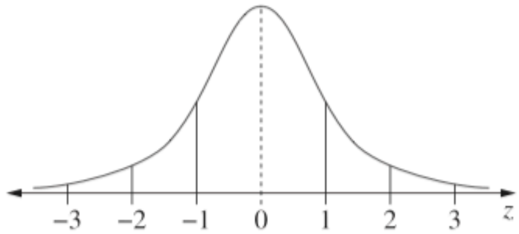
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$|\underset{\sim}{u}| = \left| x\underset{\sim}{i} + y\underset{\sim}{j} \right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$