CARLINGFORD HIGH SCHOOL

Year 12 Mathematics Extension 1

Term 2 Examination 2017

Time allowed: 60 minutes

Student Number: ______________________________

Instructions:

- All questions should be attempted
- Show ALL necessary working
- Marks may not be awarded for careless or badly arranged work
- Only board-approved calculators may be used
- Start each question on a new page

	Question 1	Question 2	Question 3	Total
Trigonometric Functions	/13			/13
Inverse Functions		/11		/11
Integration Techniques			/13	/33
	/13	/11	/13	/37

Question 1 (Trigonometric Functions)

(a) Find the exact value of θ, such that $cos2\theta = \frac{\sqrt{3}}{2}$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ [2 marks]

(b) Calculate the area of the minor segment that subtends an angle of 2.5^c at the centre of a circle of radius $2.5cm$. (Write your answer correct to 3 significant figures) [2 marks]

(c) (i) Draw a neat sketch of the function $y = sin\frac{2x}{3}$, where $0 \leq x \leq \pi$ [2 marks]

(ii) Find the equation of the tangent to the graph at the point $(\frac{\pi}{4}, \frac{1}{2})$ [2 marks]

(iii) Calculate the area of the region bounded by the graph of $y = sin\frac{2x}{3}$, the y-axis and the line $y = 1$ [3 marks]

(d) Use Simpson's rule with five function values to evaluate $\int_0^{\frac{\pi}{6}} tan^2x\, dx$, correct to 2 decimal places. [2 marks]

Question 2 (Inverse Functions)

(a) Consider the function $f(x) = cosec\, x$, where $0 < x \leq \frac{\pi}{2}$

(i) State the range of $f(x)$. [1 mark]

(ii) Explain why its inverse, $f^{-1}(x)$, exists. [1 mark]

(iii) Find $f^{-1}(x)$, and state its domain. [2 marks]

(iv) Show that $\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{-1}{x\sqrt{x^2-1}}$ [2 marks]

(b) Find the exact value of:

(i) $sin^{-1}(\sin(-600^\circ))$ [1 mark]

(ii) $\cos(cos^{-1}\sqrt{2})$ [1 mark]

(c) Consider the function $f(x) = sin^{-1}(\frac{x}{2})$

(i) Sketch the function for $-1 \leq x \leq 1$ [1 mark]

(ii) Find the exact volume of the solid formed when $y = f(x)$ is rotated around the y-axis between $y = 0$ and $y = \frac{\pi}{6}$ [2marks]

<u>Question 3</u> (Integration Techniques)

(a) Given $g'(x) = sec^2(2x - \frac{\pi}{4})$ and $g(-\pi) = 1$, use the substitution $u = 2x - \frac{\pi}{4}$ to find the function $g(x)$ [3 marks]

(b) Find $\int \frac{1}{\sqrt{1-9x^2}}\, dx$ [2 marks]

(c) Use the substitution $u = t + 1$ to evaluate $\int_0^1 \frac{t}{\sqrt{t+1}}\, dt$ [2 marks]

(d) (i) Show that $\frac{d}{dx}(tan^3 x) = 3sec^4 x - 3sec^2 x$ [2 marks]

(ii) Hence find $\int sec^4 x\, dx$ [2 marks]

(e) Find $\int \frac{(log_e 2x)^2}{x}\, dx$ [2 marks]

END OF TEST

YEAR 12 MATHS EXTENSION 1 TERM 2 EXAM

SOLUTIONS

QUESTION 1

(a) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\therefore -\pi \le 2\theta \le \pi$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\therefore 2\theta = \frac{\pi}{6}, -\frac{\pi}{6}$ ✓

$\theta = \frac{\pi}{12}, -\frac{\pi}{12}$ ✓

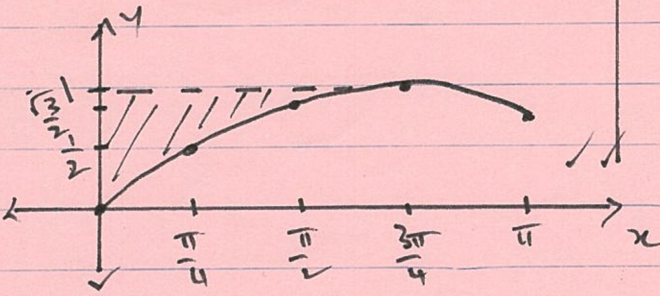
(b) $A = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$ ✓

$= \frac{1}{2}(2.5)^2 2.5 - \frac{1}{2}(2.5)^2 \sin 2.5$

$= 5.9422\ldots$

$= 5.94 \text{ cm}^2$ ✓

(c) (i)



✓✓ ✓

(ii) $\frac{dy}{dx} = \frac{2}{3}\cos\frac{2x}{3}$

at $\left(\frac{\pi}{4}, \frac{1}{2}\right)$,

$m = \frac{2}{3}\cos\frac{2}{3}\left(\frac{\pi}{4}\right)$

$= \frac{2}{3}\cdot\frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{3}$ ✓

$y - \frac{1}{2} = \frac{\sqrt{3}}{3}\left(x - \frac{\pi}{4}\right)$

$y = \frac{\sqrt{3}x}{3} - \frac{\sqrt{3}\pi}{12} + \frac{1}{2}$ ✓

(iii) See shaded area in (i).

$A = \frac{3\pi}{4}(1) - \int_0^{\frac{3\pi}{4}} \sin\frac{2x}{3}\,dx$ ✓

$= \frac{3\pi}{4} - \left[-\frac{3}{2}\cos\frac{2}{3}x\right]_0^{3\pi/4}$ ✓

$= \frac{3\pi}{4} - \left[-\frac{3}{2}\cos\frac{2}{3}\left(\frac{3\pi}{4}\right) + \frac{3}{2}\cos\frac{2}{3}(0)\right]$

$= \frac{3\pi}{4} - \left(-\frac{3}{2}\cos\frac{\pi}{2} + \frac{3}{2}\cos 0\right)$

$= \frac{3\pi}{4} - \frac{3}{2} \text{ units}^2$ ✓

(d)

x	0	$\frac{\pi}{24}$	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$
y	0	0.017	0.072	0.172	0.333

$\int_0^{\pi/6} \tan^2 x\,dx =$

$\frac{\frac{\pi}{12} - 0}{6}\left(0 + 4(0.017) + 0.072\right)$ ✓

$+ \frac{\frac{\pi}{6} - \frac{\pi}{12}}{6}\left(0.072 + 4(0.172) + 0.333\right)$

$= 0.0061 + 0.\cancel{1922}\ 0477$

$= \cancel{0.26}\ 0.054$ (2dp) ✓

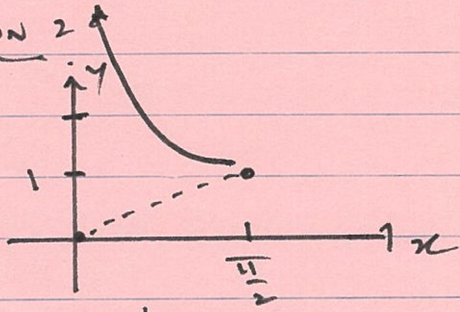
or

$\frac{\frac{\pi}{24}}{3}\left[0 + 0.333 + 4(0.017 + 1) + 2(0.072)\right]$ ✓

$= 0.20$ ✓

QUESTION 2

(a) (i).



Range: all real y, $y \geq 1$ ✓

(ii). horizontal line test
one-to-one
monotonic decreasing
(must have graph). ✓

(iii). $y = \text{cosec}\, x$

$f^{-1}: x = \text{cosec}\, y$

$x = \frac{1}{\sin y}$

$\sin y = \frac{1}{x}$

$y = \sin^{-1}\frac{1}{x}$ ✓

Domain: all real x, $x \geq 1$ ✓

(iv) $\frac{d}{dx}\left(\sin^{-1}\frac{1}{x}\right) = \frac{1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$ ✓

$= \frac{-1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$ ✓

$= \frac{-1}{x\sqrt{x^2-1}}$

(b)(i). $\sin^{-1}(-600)$

$= \sin(-600+720)$

$= \sin(120)$

$= \sin 60°$

$= \frac{\sqrt{3}}{2}$

$-1 \leq \frac{\sqrt{3}}{2} \leq 1$ ✓

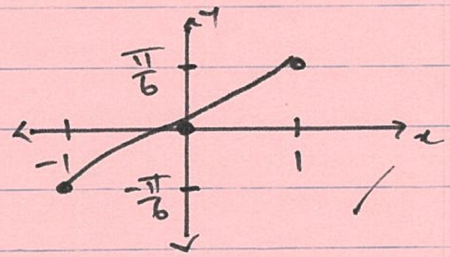
$\therefore \sin^{-1}(\sin(-600°)) = \frac{\pi}{3}$.

(ii). $\sqrt{2} > 1$

$\therefore \cos^{-1}\sqrt{2}$ does not exist

$\therefore \cos(\cos^{-1}\sqrt{2})$ does not exist ✓

(c) (i)



✓

(ii). $y = \sin^{-1}\left(\frac{x}{2}\right)$

$\frac{x}{2} = \sin y$

$x = 2\sin y$

$V = \pi\int_0^{\pi/6}(2\sin y)^2\,dy$ ✓

$= 4\pi\int_0^{\pi/6}\sin^2 y\,dy$

$= 4\pi\int_0^{\pi/6}\left(\frac{1}{2}-\frac{1}{2}\cos 2y\right)dy$

$= 4\pi\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_0^{\pi/6}$

PTO

QUESTION 2 (CONT.)

$$= 4\pi \left[\frac{1}{2}\left(\frac{\pi}{6}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{6}\right) - 0 \right]$$

$$= 4\pi \left(\frac{\pi}{12} - \frac{1}{4}\sin\frac{\pi}{3} \right)$$

$$= 4\pi \left(\frac{\pi}{12} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2} \right)$$

$$= 4\pi \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8} \right) \text{ units}^3.$$

$$= \frac{\pi^3}{3} - \frac{\sqrt{3}}{2}\pi \text{ units}^3$$

QUESTION 3

(a). $u = 2x - \frac{\pi}{4}$

$du = 2\,dx$

$dx = \frac{1}{2}\,du$

$\therefore \int \sec^2\left(2x - \frac{\pi}{4}\right) dx$

$= \frac{1}{2}\int (\sec^2 u)\,du$

$= \frac{1}{2}\tan u + c.$ ✓

$= \frac{1}{2}\tan\left(2x - \frac{\pi}{4}\right) + c$

$g(-\pi) = \frac{1}{2}\tan\left(-2\pi - \frac{\pi}{4}\right) + c$

$1 = \frac{1}{2}\tan\left(-\frac{\pi}{4}\right) + c$

$1 = -\frac{1}{2}\tan\frac{\pi}{4} + c$ ✓

$1 = -\frac{1}{2} + c$

$\therefore c = \frac{3}{2}$

$\therefore g(x) = \frac{1}{2}\tan\left(2x - \frac{\pi}{4}\right) + \frac{3}{2}$ ✓

(b). $\int \frac{1}{\sqrt{1-9x^2}}\,dx$

$= \int \frac{1}{\sqrt{9\left(\frac{1}{9} - x^2\right)}}\,dx$

$= \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}$ ✓

$= \frac{1}{3}\sin^{-1}\frac{x}{1/3} + c$

$= \frac{1}{3}\sin^{-1} 3x + c.$ ✓

(c). $u = t+1 \rightarrow t = u-1$

$du = dt$

when $t=0$, $u=1$

$t=1$, $u=2$

$\therefore \int_0^1 \frac{t}{\sqrt{t+1}}\,dt$

$= \int_1^2 \frac{u-1}{\sqrt{u}}\,du$

$= \int_1^2 \left(u^{1/2} - u^{-1/2}\right) du$ ✓

$= \left[\frac{2u^{3/2}}{3} - 2u^{1/2}\right]_1^2$

$= \frac{2}{3}(2)^{3/2} - 2(2)^{1/2} - \frac{2}{3} + 2$

$= \frac{4\sqrt{2}}{3} - 2\sqrt{2} + 1\frac{1}{3}.$

$= 0.39$ (2dp). ✓

$= \frac{4 - 2\sqrt{2}}{3}$

(d)(i) $\frac{d}{dx}\left(\tan^3 x\right)$

$= 3\tan^2 x \sec^2 x$ ✓

$\tan^2 x = \sec^2 x - 1$ ✓

$\therefore \frac{d}{dx}\left(\tan^3 x\right) = 3\left(\sec^2 x - 1\right).\sec^2 x$

$= 3\sec^4 x - 3\sec^2 x$

(ii). $\int 3\sec^4 x\,dx - \int 3\sec^2 x\,dx = \tan^3 x + c_1$ ✓

$\int 3\sec^4 x\,dx = \tan^3 x + 3\tan x + c_2$

$\therefore \int \sec^4 x\,dx = \frac{1}{3}\tan^3 x + \tan x + c.$ ✓

QUESTION 3 (CONT.)

(e). $\frac{d}{dx}(\log_e 2x) = \frac{2}{2x}$

$= \frac{1}{x}.$ ✓

$\therefore \int \frac{(\log_e 2x)^2}{x}\,dx$ is in the form $\int f'(x)\,(f(x))^2\,dx$

$\therefore \int \frac{(\log_e 2x)^2}{x}\,dx = \frac{(\log_e 2x)^3}{3} + c.$ ✓

[or use substitution $u = \log_e 2x$]