Volumes

Solids of revolution by slices

When the region under the curve $y = f(x)$ is rotated about an axis, the result is a solid with circular or annular cross sections.

By slicing the solid into n small slices perpendicular to the axis of revolution, we can obtain an expression for each slice which can be used to calculate the whole volume.

We have seen this method before, but in E2,

- The axis of revolution is not always the x or y axis
- The cross sectional shape may be an annulus, (not always a circle).

Example 1.

The area enclosed by the curve $y = x(4 - x)$, the x-axis and the ordinate $x = 2$ is rotated about

a) the x-axis

b) the y-axis

c) the ordinate $x = 2$

Find the volume so generated in each case

Example 1.

The area enclosed by the curve $y = x(4 - x)$, the x-axis and the ordinate $x = 2$ is rotated about

a) the x-axis

b) the y-axis

c) the ordinate $x = 2$

Find the volume so generated in each case

Example 2. A hole of radius a is drilled through the centre of a sphere with radius $2a$. Find the volume of the remaining solid.

Volumes by Cylindrical Shells

https://www.geogebra.org/m/TvkwtNqC

Rectangular strips **parallel** to the axis of rotation sweep out thin cylindrical shells.

Consider the region bounded by the curve $y = f(x)$ and the x-axis.

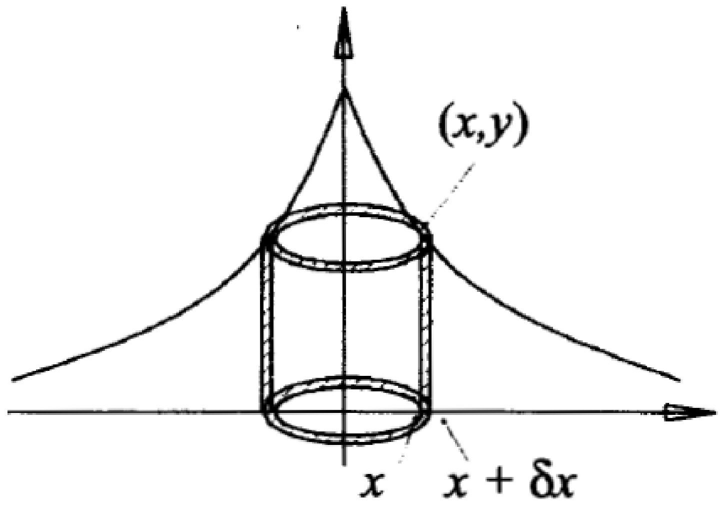
A shell has inner radius x, outer radius $x + \delta x$ and height y.

If ∂V is the volume of the shell,

$$\begin{aligned}\partial V &= \pi\{(x+\delta x)^2 - x^2\}y \\ &= \pi\{x^2 + 2x\partial x + \partial x^2 - x^2\}y \\ &= 2\pi xy\delta x + \pi y(\delta x)^2\end{aligned}$$

As $\delta x \to 0$, $(\delta x)^2 \to 0$ much more quickly, so we can ignore the terms in $(\delta x)^2$.

We write $\delta V = 2\pi xy\delta x$.



The volume of the solid is obtained by calculating the limiting sum

$$V = \lim_{\partial x \to 0} \sum_{x=a}^{b} 2\pi xy \, \delta x$$
$$= \int_a^b 2\pi xy \, dx$$

Do not memorise! You need to develop from first principles!

Example 1. The region bounded by the curve $y = x(4 - x)$ and the x-axis is rotated around

i) the y-axis.

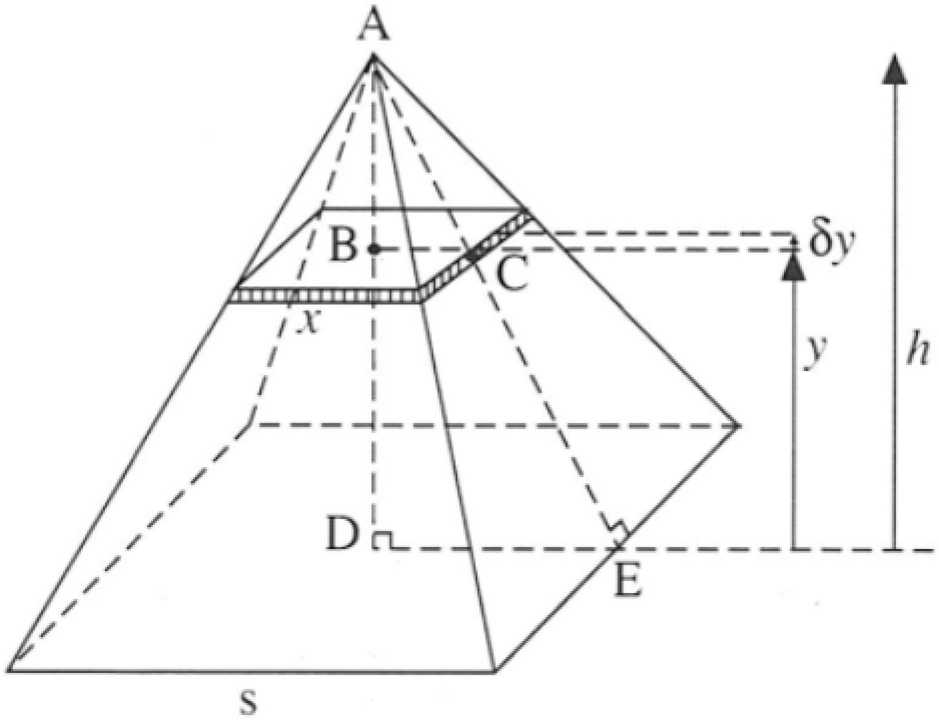
ii) the x-axis.

By considering the solid of revolution as a sum of cylindrical shells, find its volume.

Example 2. The region enclosed within the circle $x^2 + y^2 = 9$ is rotated around the line $x = 4$ to form a doughnut-shaped solid of revolution called a torus. Find its volume.

Volumes of Solids by Parallel Cross-sections of Similar Shape

Example 1. Show that the volume of a right square pyramid of height h on a base of side s is given by $V = \frac{1}{3}s^2h$.



Example 2. A solid is built on a circular base whose equation is $x^2 + y^2 = 1$. Find the volume of the solid if cross-sections perpendicular to the x-axis are

a) Isosceles right angled triangles with hypotenuse in the base.

b) Semicircles with diameter in the base.