Intermediate Division

1. $1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + 9 + 10 =$

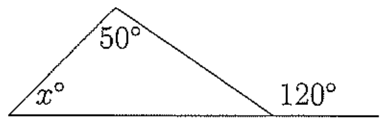
(A) 0 (B) 1 (C) 10 (D) 11 (E) 19

▶ $1 + (2 - 3) + (5 - 4) + (6 - 7) + (9 - 8) + 10 = 1 - 1 + 1 - 1 + 1 + 10 = 11$,

hence (D).

2. (Also J2)
In the diagram the value of x is

(A) 80 (B) 70 (C) 60 (D) 50 (E) 40



▶ *Alternative 1*
The three interior angles of the triangle are $x°$, $50°$ and $60°$. These add to $180°$, so $x + 110 = 180$ and $x = 70$,

hence (B).

Alternative 2
Using the exterior angle formula, $x + 50 = 120$, so that $x = 70$,

hence (B).

3. (Also S3)
If $p = 9$ and $q = -3$ then $p^2 - q^2$ is equal to

(A) 64 (B) 72 (C) 84 (D) 90 (E) 96

▶ Now $p^2 = 81$ and $q^2 = 9$, so $p^2 - q^2 = 81 - 9 = 72$,

hence (B).

4. What value can be placed in the shape to make this statement true?

$$2014 \div \boxed{} = 100$$

(A) 0.02014 (B) 0.2014 (C) 2.014 (D) 20.14 (E) 201.4

▶ For any a and b, $a \div b = 100$ only if $a = 100b$. So the number in the shape must be $2014 \div 100 = 20.14$,

hence (D).

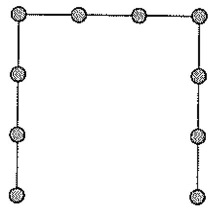
5. (Also J6)
If $\frac{5}{6}$ of a number is 30, what is $\frac{3}{4}$ of the number?

(A) 22.5 (B) 24 (C) 25 (D) 27 (E) 40

▸ One-sixth of the number is 6, so the number is 36. Then $\frac{1}{4}$ of the number is 9 and $\frac{3}{4}$ of the number is 27,

hence (D).

6. (Also J8)
This diagram is called an *open square* of order 4, since the three sides are all the same length and each side has four posts spaced evenly along it. The total number of posts which would be evenly spaced along an open square of order 10 would be

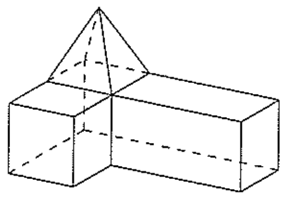


(A) 26 (B) 27 (C) 28 (D) 30 (E) 32

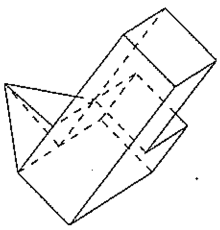
▸ There will be 10 posts on the left and on the right, and ignoring the corner posts, 8 posts along the top, giving a total of 28 posts,

hence (C).

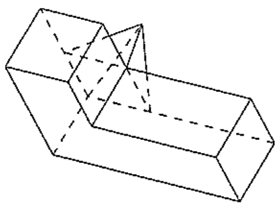
7. Which of the objects below is not the same as the one on the right?



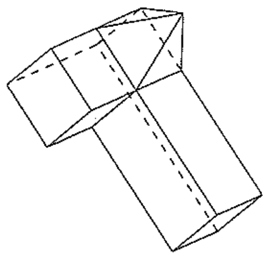
(A)



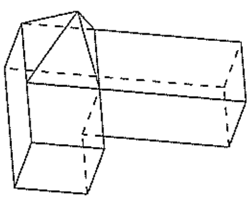
(B)



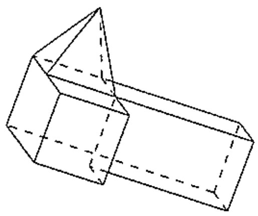
(C)

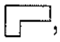


(D)



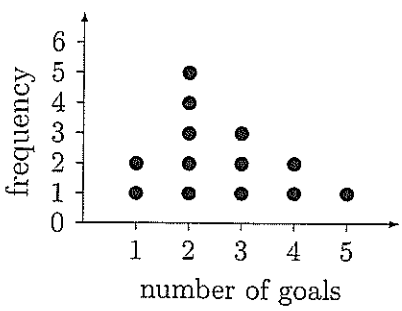
(E)



▸ If each of the shapes is oriented so that the L-shaped section is in this orientation, , then the sample shape as well as B, C, D and E have the pyramid pointing towards the viewer, whereas A has the pyramid pointing away,

hence (A).

8. This graph shows the number of goals scored by Ranjit's soccer team in each of the first thirteen matches. After the fourteenth match, the median has increased but the mode has remained the same. Which of the following best describes the team's score in this last game?



(A) Goals = 1 (B) Goals = 2 (C) Goals ≥ 2 (D) Goals < 3 (E) Goals ≥ 3

- Currently
  (i) The mode is 2.
  (ii) The median is 2 (there are 7 scores up to and including 2, and 6 scores 3 or more).

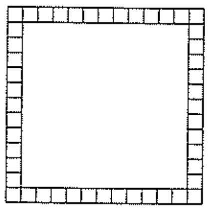
  The mode won't change with another match, no matter what the score, but the median will increase to 2.5 if there is another score of 3 or more,

  hence (E).

9. Forty-eight pavers, each $1\,\text{m} \times 1\,\text{m}$ in size, are used to form a path 1 metre wide around a square garden. What is the area, in square metres, inside this path?

(A) 100 (B) 110 (C) 121 (D) 132 (E) 144

- Four pavers are used on the corners, leaving $44 = 4 \times 11$ along the sides:



Then the central square is $11\,\text{m} \times 11\,\text{m} = 121\,\text{m}^2$,

hence (C).

10. Each May a farmer plants barley seed and then in October he harvests 12 times the weight of seed planted. From each harvest, he sells 50 tonnes and the rest he keeps as seed for the next year's crop. This year he has planted enough to harvest 120 tonnes. How many tonnes did he plant last year?

(A) 5 (B) 10 (C) 20 (D) 30 (E) 60

- This year he planted 10 tonnes. So last year he harvested 60 tonnes. Thus last year he planted 5 tonnes,

hence (A).

11. If x is an integer and $x < -1$, which of the following expressions has the greatest value?

(A) $\dfrac{1}{x}$ (B) $\dfrac{1}{x^2}$ (C) $x+1$ (D) $-\dfrac{1}{x^2}$ (E) $-\dfrac{1}{x}$

- Since $x < -1$, the only positive values are $\frac{1}{x^2}$ and $-\frac{1}{x}$. But then $0 < \frac{-1}{x} < 1$ and

$$0 < \frac{1}{x^2} = \frac{-1}{x}\,\frac{-1}{x} < \frac{-1}{x}$$

hence (E).

Comment
Evaluating the five expressions with the trial value $x = -2$ gives (A) $-\frac{1}{2}$, (B) $\frac{1}{4}$, (C) -1, (D) $-\frac{1}{4}$ and (E) $\frac{1}{2}$. So the only possible answer is (E).

12. The 11 boys in Tom's cricket team have a contest to see how far they can throw a cricket ball. Their results, to the nearest metre, are

$$19, 26, 31, 31, 31, 33, 37, 42, 42, 48, 56$$

Which of the following lists the statistical measures for these results in the correct ascending order?

(A) mean, median, mode (B) median, mean, mode (C) mode, mean, median (D) median, mode, mean (E) mode, median, mean

▶ We can calculate mode = 31, median = 33, and mean = $\frac{396}{11} = 36$,

hence (E).

Comment
Rather than calculating the mean, we can use an estimation strategy. For each of the 'nested' pairs $\{19, 56\}$, $\{26, 48\}$, $\{31, 42\}$, $\{31, 42\}$ and $\{31, 37\}$, the midpoint (mean) is greater than the median 33, so the overall mean must be greater than 33.

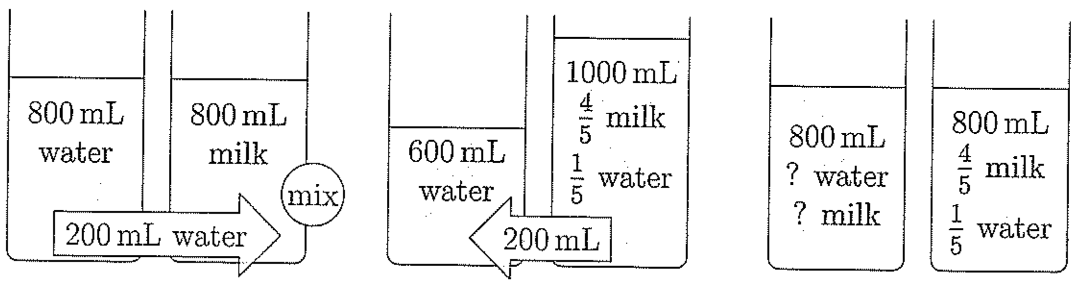
13. I have 800 mL of water in jug X and 800 mL of milk in jug Y. I pour 200 mL from jug X into jug Y and stir the mixture thoroughly. I then pour 200 mL of the resulting mixture from jug Y into jug X. What is the volume of milk that is now in jug X?

(A) 150 mL (B) 160 mL (C) 175 mL (D) 180 mL (E) 200 mL

▶ *Alternative 1*
After pouring 200 mL from jug X into jug Y, the mixture in jug Y contains 80% milk. So 80% of the 200 mL that is poured from jug Y into jug X is milk. Hence, the volume of milk that ends up in jug X is $0.8 \times 200 = 160$ mL,

hence (B).

Alternative 2



At the end, jug Y has $\frac{4}{5} \times 800$ mL = 640 mL of milk, so jug X has 160 mL of milk,

hence (B).

14. The women's world record for running 400 metres was set in Canberra at 47.60 seconds. Which of the following is closest to the runner's average speed, in kilometres per hour?

(A) 22 (B) 24 (C) 26 (D) 28 (E) 30

▶ Running 47.60 seconds for 400 metres represents $47.6 \times 2.5 = 47.6 + 47.6 + 23.8 = 119$ seconds for a kilometre. So the speed in kilometres per hour is $\frac{1}{119} \times 60 \times 60 = \frac{120}{119} \times 30$, which is slightly more than 30,

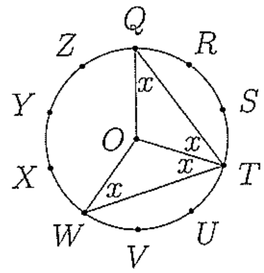
hence (E).

15. Ten points Q, R, S, T, U, V, W, X, Y and Z are equally and consecutively spaced on a circle. What is the size, in degrees, of the angle $\angle QTW$?

(A) 36 (B) 54 (C) 60 (D) 72 (E) 75

▶ *Alternative 1*

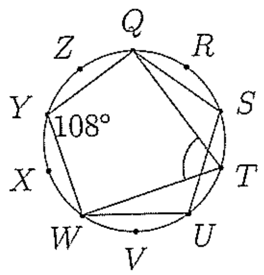
Let O be the centre of the circle and draw congruent isosceles triangles ΔQOT and ΔTOW, and let x be the acute angle in these two triangles.



Each of the ten angles $\angle QOR$, $\angle ROS$, ... are equal, so that $\angle QOT = \frac{3}{10} \times 360° = 108°$. In the triangle ΔQOT, $108° + 2x = 180°$, so that $x = 36°$. Therefore $\angle QTW = 2x = 72°$,

hence (D).

Alternative 2

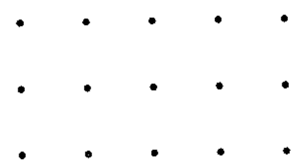


The pentagon $QSUWY$ is regular, so that $\angle QYW = 108°$. The quadrilateral $QTWY$ is cyclic, so opposite angles add to $180°$. Thus $\angle QTW = 180° - 108° = 72°$,

hence (D).

16. (Also UP26, J20)

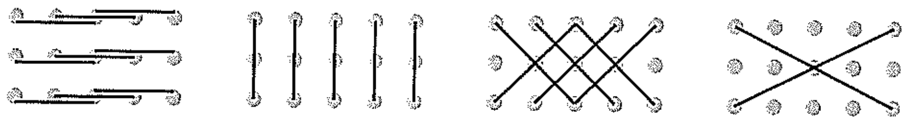
A 3 by 5 grid of dots is set out as shown. How many straight line segments can be drawn that join two of these dots and pass through exactly one other dot?



(A) 14 (B) 20 (C) 22 (D) 24 (E) 30

▸ *Alternative 1*

We draw all such line segments—horizontal, vertical, at 45°, and others:



$$9 \quad + \quad 5 \quad + \quad 6 \quad + \quad 2 \quad = \quad 22,$$

hence (C).

Alternative 2

The line segments can be classified by the midpoint dot, since then each line segment is counted only once. Also the number of line segments through each dot form a symmetric pattern of numbers:

$$\begin{array}{ccccc} \bullet & \overset{1}{\bullet} & \overset{1}{\bullet} & \overset{1}{\bullet} & \bullet \\ \overset{1}{\bullet} & \overset{4}{\bullet} & \overset{6}{\bullet} & \overset{4}{\bullet} & \overset{1}{\bullet} \\ \bullet & \overset{1}{\bullet} & \overset{1}{\bullet} & \overset{1}{\bullet} & \bullet \end{array}$$

There are 22 line segments,

hence (C).

17. (Also J22)

A hotel has rooms that can accommodate up to two people. Couples can share a room, but otherwise men will share only with men and women only with women. How many rooms are needed to guarantee that any group of 100 people can be accommodated?

(A) 50 (B) 51 (C) 67 (D) 98 (E) 99

▸ At worst, 51 rooms will be needed. There are an even number of people coming as couples. So if there are an even number of single men there must be an even number of single women too, so that everyone can be paired up and 50 rooms will do. If, on the other hand, there are an odd number of single men, there will be an odd number of single women. So everyone can be paired except for one man and one woman, with 98 pairs in 49 rooms, plus one man and one woman who need a room each,

hence (B).

18. Two rectangular prisms are constructed. One measures $4\,\text{cm} \times 6\,\text{cm} \times x\,\text{cm}$ and the other measures $3\,\text{cm} \times 8\,\text{cm} \times y\,\text{cm}$ where both x and y are integers. If they have equal surface area, what is the smallest possible value of $x + y$?

(A) 11 (B) 21 (C) 26 (D) 42 (E) 63

▸ *Alternative 1*

The surface areas are $8x + 12x + 48$ and $6y + 16y + 48$ so $20x + 48 = 22y + 48$ and $y = \frac{10}{11}x$. The smallest integer value of x which satisfies this equation is $x = 11$ when $y = 10$, so the smallest value of $x + y$ is 21,

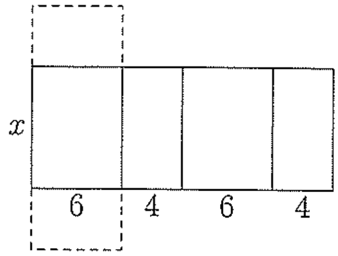
hence (B).

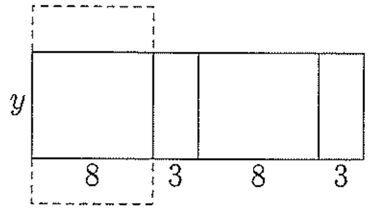
Comment

The other values are $20 + 22$, $30 + 33$ and so on.

Alternative 2

Each box has two $24\,\text{cm}^2$ faces, so these can be ignored. The remaining 4 faces on the first form a $20 \times x$ rectangle, and on the second a $22 \times y$ rectangle, shown here on nets:





The smallest integer solution to $20x = 22y$ has $x = 11$, $y = 10$ and $x + y = 21$,

hence (B).

19. A four-digit number *abcd* is called *cool* if *a* is divisible by 4, the two-digit number *ab* is divisible by 5, the three-digit number *abc* is divisible by 6 and *abcd* is divisible by 7. How many cool numbers are there where 8 is not one of the digits?

(A) 3 (B) 4 (C) 5 (D) 6 (E) more than 6

▶ The first digit has to be 4 (the alternative 8 is not allowed). The second digit can only be 0 or 5.

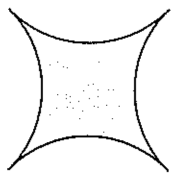
Consider $40cd$. For $40c$ to be divisible by 6, either $c = 2$ or $c = 8$, but 8 is not included. Now for $402d$ to be divisible by 7, try $4020 \div 7 = 574r2$. So 4025 is divisible by 7, and is cool.

Consider $45cd$. For $45c$ to be divisible by 6, either $c = 0$ or $c = 6$. Now $4500 \div 7 = 642r6$ and $4560 \div 7 = 651r3$ so that 4501 and 4564 are the only cool numbers of this form.

So there are three cool numbers without an 8: 4025, 4501 and 4564,

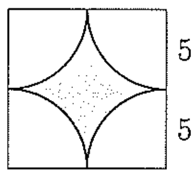
hence (A).

20. The shape shown is formed from four identical arcs, each a quarter of the circumference of a circle of radius 5 cm. What is the area of the shape, in square centimetres?



(A) $100 - 20\pi$ (B) 100 (C) $25\pi + 25$ (D) 25π (E) $100 - 25\pi$

▶ The shape is that left when four 90° sectors with radius 5 are removed from a square of side 10. Since the four sectors make a circle of radius 5, the shaded area is



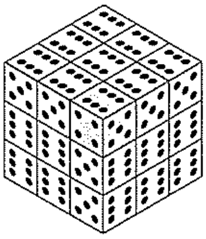
$$10^2 - \pi 5^2 = 100 - 25\pi\,,$$

hence (E).

21. Standard six-sided dice have their dots arranged so that the opposite faces add up to 7. If 27 standard dice are arranged in a $3 \times 3 \times 3$ cube on a solid table what is the maximum number of dots that can be seen from one position?

(A) 90 (B) 94 (C) 153 (D) 154 (E) 189

▶ There are at most three 3×3 faces of the cube visible, and the maximum will occur with exactly three cube faces visible. A total of 153 is possible.



dice face	count	pips
⚃	1	4
⚄	7	35
⚅	19	114
		153

To see that no greater total is possible, of the 19 visible dice, one has 3 faces visible, six have 2 faces visible and 12 have one face visible. So the sum of all visible faces cannot exceed $1 \times 15 + 6 \times 11 + 12 \times 6 = 153$,

hence (C).

22. There are 10 integers in a set. Some are odd and some are even. For each possible pair selected from the set, the sum is written down. Of these 45 numbers, exactly 20 are even. How many of the numbers in the original set are even?

(A) 0 (B) 3 (C) 5 (D) 8 (E) 10

▶ There are 25 odd numbers in the list, each obtained by adding even to odd from the original 10 numbers. This can only occur if there are 5 odd and 5 even in the set, giving $5 \times 5 = 25$ odd numbers in the list,

hence (C).

23. (Also S21)

Starting with $\frac{2}{3}$ of a tank of fuel, I set out to drive the 550 km from Scone to Canberra. At Morisset, 165 km from Scone, I have $\frac{1}{2}$ of a tank remaining. If I continue with the same fuel consumption per kilometre and without refuelling, what happens?

(A) I will arrive in Canberra with $\frac{1}{9}$ of a tank to spare.

(B) I will arrive in Canberra with $\frac{1}{20}$ of a tank to spare.

(C) I will run out of fuel precisely when I reach Canberra.

(D) I will run out of fuel 110 km from Canberra.

(E) I will run out of fuel 220 km from Canberra.

▶ Driving 165 km uses $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ of a tank, so on a full tank I can travel $6 \times 165 = 990$ km. Consequently the trip to Canberra uses $550 \div 990 = \frac{5}{9}$ of a tank. Since the car started with $\frac{2}{3} = \frac{6}{9}$ of a tank, I will make it to Canberra with $\frac{1}{9}$ of a tank remaining,

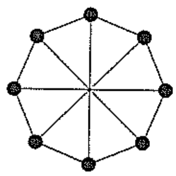
hence (A).

24. At a party, each person shakes hands with exactly three other people and no two people shake hands with each other more than once. If fewer than fifteen handshakes take place, what is the maximum number of people who can be at the party?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

▶ Let N be the number of people at the party and let H be the number of handshakes that take place. Since two people are involved in each handshake, we know that $3N = 2H$. We are told that $H < 15$, so $N = \frac{2}{3}H < 10$. The equation $3N = 2H$ implies that N must be even, so we have $N \leq 8$.

The answer is indeed 8, since the handshakes could have taken place as indicated in the following diagram. Here, the dots represent people at the party and the line segments represent handshakes.

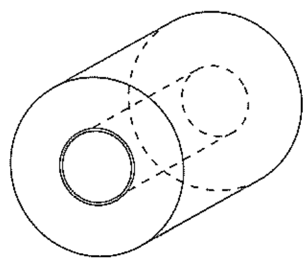


hence (C).

25. (Also S22)

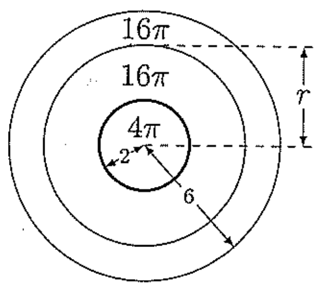
Thanom has a roll of paper consisting of a very long sheet of thin paper tightly rolled around a cylindrical tube, forming the shape indicated in the diagram.

Initially, the diameter of the roll is 12 cm and the diameter of the tube is 4 cm. After Thanom uses half of the paper, the diameter of the remaining roll is closest to



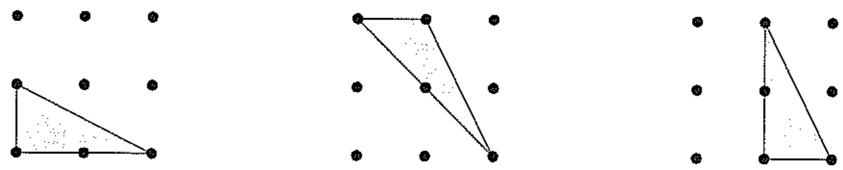
(A) 6 cm (B) 8 cm (C) 8.5 cm (D) 9 cm (E) 9.5 cm

▶ Working in centimetres, let the half-roll's radius be r. From end on, the full roll had area $\pi(6^2 - 2^2) = 32\pi$, so half the roll has area 16π. Including the tube, the end of the half-roll has area $20\pi = \pi r^2$.
Then $r^2 = 20$, but $4.5^2 = 20.25$ and $4.4^2 = 19.36$, so that $4.4 < r < 4.5$, and the diameter is twice that,

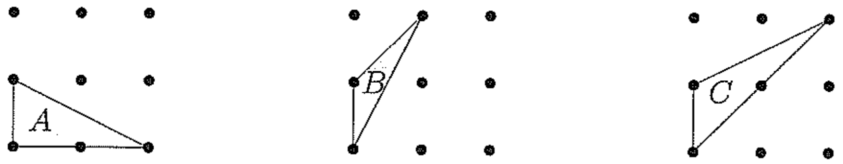


hence (D).

26. (Also J28)
In a 3×3 grid of points, many triangles can be formed using 3 of the points as vertices. Three such triangles are shown below. Of all these possible triangles, how many have all three sides of different lengths?



▶ Here are three non-congruent scalene triangles in the grid.



To see that there are no others, consider the longest side of such a triangle.

(i) If the longest side has length $\sqrt{2^2 + 2^2} = 2\sqrt{2}$, then it must be a diagonal of the grid. The third vertex cannot be a corner dot, so it must be an edge dot, giving triangle C.

(ii) If the longest side has length $\sqrt{2^2 + 1^2} = \sqrt{5}$ then the other sides can only be 1, $\sqrt{2}$ or 2. With 1 and $\sqrt{2}$ we get triangle B and with 1 and 2 we get triangle A. But with 2 and $\sqrt{2}$ we don't get a grid point.

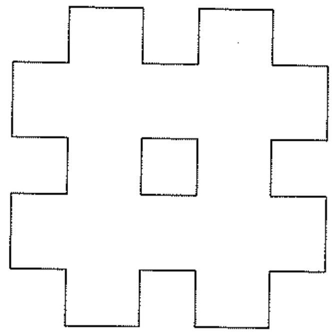
(iii) If the longest side has length 2 then the other sides must be 1 and $\sqrt{2}$, but there is no triangle with sides 1, $\sqrt{2}$ and 2 on the grid.

Triangle A can be in 8 different orientations—4 rotations and the reflections of these. Since it only occupies 2×1 cells on the grid, each of these orientations can be in 2 positions, giving 16 triangles. Triangle B is similar, also giving 16 triangles.
Triangle C can be in 8 different orientations, but it occupies 2×2 cells on the grid, and so each orientation can only be in one position, giving 8 triangles. So in total there are $16 + 16 + 8 = 40$ triangles,

hence (40).

27. Small squares of side x cm have been removed from the corners, sides and centre of a square of side y cm to form the gasket shown.

If x and y are prime numbers and the sum of the inside and outside perimeters of the gasket, in centimetres, is equal to the area of the gasket, in square centimetres, what is the smallest possible value of the area of the gasket?



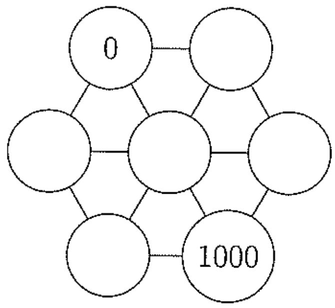
▶ The outer perimeter is $4y + 8x$, the inner perimeter is $4x$, and the area is $y^2 - 9x^2$. So $4y + 12x = (y + 3x)(y - 3x)$, which reduces to $y - 3x = 4$. So we need to minimise $4(y + 3x)$ subject to $y - 3x = 4$ and x, y both prime. But the area is $(4 + 3x)^2 - 9x^2 = 16 + 24x = 4(6x + 4)$.

Try $x = 2$, then $y = 10$, which is not prime.

Try $x = 3$, then $y = 13$, which is prime. So the minimum area is $88\,\text{cm}^2$,

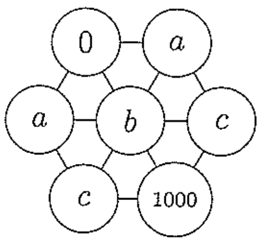
hence (88).

28. In the diagram on the right, each circle has three or six neighbours. Each circle will contain a number, and each of the five missing numbers is the average of its neighbours. What is the largest of the five missing numbers?



▶ *Alternative 1*

Labelling the unknown circles with a, b, c in a symmetric pattern:



For this arrangement, the requirement that circles with a, b and c are averages of their neighbours gives a system of three equations:

$$a = \frac{1}{3}(0 + b + c) \implies 3a = b + c \quad (1)$$

$$b = \frac{1}{6}(2a + 2c + 1000) \implies 3b = a + c + 500 \quad (2)$$

$$c = \frac{1}{3}(a + b + 1000) \implies 3c = a + b + 1000 \quad (3)$$

We solve simultaneously, first finding $a + b + c$,

$$3a + 3b + 3c = b + c + a + c + 500 + a + b + 1000 \quad (1)+(2)+(3)$$

$$3(a + b + c) = 2(a + b + c) + 1500$$

$$a + b + c = 1500 \quad (4)$$

Then substituting, in turn, each of equations (1), (2) and (3) into equation (4) to find a, b and c

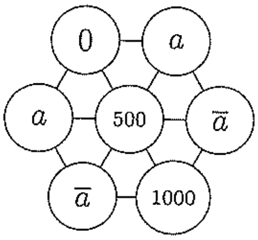
$$\begin{aligned} a + (3a) &= 1500 && \implies a = 375 \\ b + (3b - 500) &= 1500 && \implies b = 500 \\ (3c - 1000) + c &= 1500 && \implies c = 625 \end{aligned}$$

The largest of the missing numbers is 625,

hence (625).

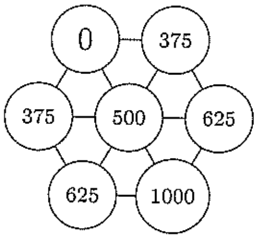
Alternative 2

We can try the following arrangment, as it has as much symmetry as possible. The full reasons for this symmetry are discussed below.



where $\overline{a} = 1000 - a$. Then $3a = 500 + \overline{a} = 1500 - a$ so that $a = 1500 \div 4 = 375$ and $\overline{a} = 625$.

The completed diagram can now be verified to be a solution to the original problem.

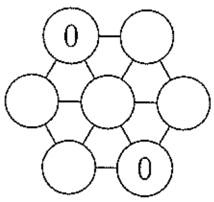


$$\begin{aligned} 0 + 500 + 625 &= 1125 = 3 \times 375 \quad \checkmark \\ 0 + 2 \times 375 + 2 \times 625 + 1000 &= 3000 = 6 \times 500 \quad \checkmark \\ 375 + 500 + 1000 &= 1875 = 3 \times 625 \quad \checkmark \end{aligned}$$

This motivation to start with an arrangement with the above symmetry follows from some manipulations of the diagram as a whole.

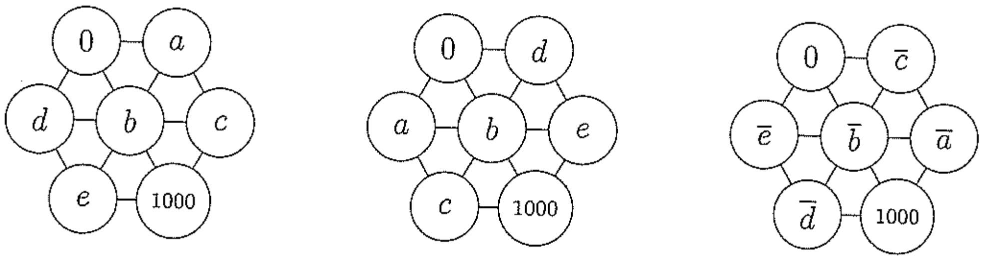
Claim: The solution is unique.

The difference between any two solutions will still have the average property, but will have



and the only way for the largest of the empty circles to equal the average of its neighbours is if all the circles are equal to 0. So there is no difference between the solutions.

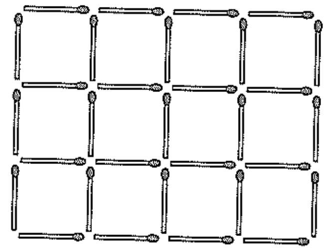
Consequently if there is a solution as in the first diagram below, then the 'reflected' second diagram (also a solution) must be equal to the first. The third diagram has a reflection along with the algebraic transformation $\overline{x} = 1000 - x$. This is also a solution, so it is also equal to the first.



Then $b = \overline{b} = 500$ and $a = d = \overline{c} = \overline{e}$, as was assumed above,

hence (625).

29. As shown in the diagram, you can create a grid of squares 3 units high and 4 units wide using 31 matches. I would like to make a grid of squares a units high and b units wide, where $a < b$ are positive integers. Determine the sum of the areas of all such rectangles that can be made, each using exactly 337 matches.



▶ *Alternative 1*

There are $a + 1$ rows of horizontal matches and each row contains b matches. There are $b + 1$ columns of vertical matches and each column contains a matches. So the total number of matches is $(a + 1)b + (b + 1)a = 2ab + a + b$.

We would like to solve the equation $2ab + a + b = 337$, where $a < b$ are positive integers. By multiplying the equation by 2 and adding 1 to both sides, we obtain

$$4ab + 2a + 2b + 1 = 675 \qquad \Rightarrow \qquad (2a + 1)(2b + 1) = 675$$

The only ways to factorise 675 into two positive integers are

$$1 \times 675, \quad 3 \times 225, \quad 5 \times 135, \quad 9 \times 75, \quad 15 \times 45, \quad 25 \times 27$$

We must have $2a+1$ correspond to the smaller factor and $2b+1$ to the larger factor. So the solutions we obtain for (a, b) are

$$(0, 337), \quad (1, 112), \quad (2, 67), \quad (4, 37), \quad (7, 22), \quad (12, 13)$$

We must disregard the first solution, but one can check that the remaining ones are all valid. So the sum of the areas of all such rectangles is

$$1 \times 112 + 2 \times 67 + 4 \times 37 + 7 \times 22 + 12 \times 13 = 704$$

hence (704).

Alternative 2

There are $(a + 1)b$ horizontal and $a(b + 1)$ vertical matches. Then

$$(a + 1)b + a(b + 1) = 2ab + a + b = 337 \qquad \Longrightarrow \qquad b = \frac{337 - a}{2a + 1}$$

In this table, for each a we work out b and the area ab, keeping only those where b is a whole number and $b > a$.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
b	$\frac{336}{3}$	$\frac{335}{5}$	$\frac{334}{7}$	$\frac{333}{9}$	$\frac{331}{11}$	$\frac{332}{13}$	$\frac{330}{15}$	$\frac{329}{17}$	$\frac{328}{19}$	$\frac{327}{21}$	$\frac{326}{23}$	$\frac{325}{25}$	$\frac{324}{27}$	
	112	67	$\times$	37	$\times$	$\times$	22	$\times$	$\times$	$\times$	$\times$	13	(12)	$\cdots$
ab	112	134		148			154					156		

Then the total area is $112 + 134 + 148 + 154 + 156 = 704$ square units,

hence (704).

30. (Also S28)
Consider the sequence a_1, a_2, a_3, a_4, $\ldots$ such that $a_1 = 2$ and for every positive integer n,

$$a_{n+1} = a_n + p_n, \qquad \text{where } p_n \text{ is the largest prime factor of } a_n.$$

The first few terms of the sequence are $2, 4, 6, 9, 12, 15, 20$. What is the largest value of n such that a_n is a four-digit number?

► *Alternative 1*
Let us write out some terms of the sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n	2	4	6	9	12	15	20	25	30	35	42	49
p_n	2	2	3	3	3	5	5	5	5	7	7	7

n	13	14	15	16	17	18	19	20	21	22	23	24
a_n	56	63	70	77	88	99	110	121	132	143	156	169
p_n	7	7	7	11	11	11	11	11	11	13	13	13

The crucial observation is that if p is a prime, then $a_{2p-2} = p^2$. If we assume that this is true, then we have $a_{192} = 97^2$. The next few terms can then be calculated as follows.

n	192	193	194	195	196	197	198	199
a_n	97^2	97×98	97×99	97×100	97×101	98×101	99×101	100×101
p_n	97	97	97	97	101	101	101	101

Since $a_{198} = 99 \times 101 = 9999$ and $a_{199} = 100 \times 101 = 10100$, the answer to the problem is 198.
In order to prove our crucial observation above, suppose that $a_{2p-2} = p^2$ for some prime p. Let q be the next largest prime after p. Then the next $q - p$ terms of the sequence after $a_{2p-2} = p^2$ are

$$p(p+1),\ p(p+2),\ p(p+3),\ \ldots,\ pq\,.$$

To see this, note that the difference between consecutive terms is p and that p divides each term. Furthermore, no prime larger than p can divide any term apart from the last. That is because each of those terms is of the form pk, where $p < k < q$. Since k

lies between the consecutive primes p and q, it cannot be divisible by a prime larger than p.

The next $q - p$ terms of the sequence are

$$(p+1)q,\ (p+2)q,\ (p+3)q,\ \ldots,\ q^2 .$$

To see this, note that the difference between consecutive terms is q and that q divides each term. Furthermore, no prime larger than q can divide any term. That is because each term is of the form kq, where $p < k \leq q$. Since k is at most q, it cannot be divisible by a prime larger than q.

We have shown that if $a_{2p-2} = p^2$ for a prime p, then $2q - 2p$ terms further along in the sequence we find q^2. In other words, $a_{2q-2} = q^2$, where q is the next largest prime after p. By induction on the prime numbers, this shows that $a_{2p-2} = p^2$ for all primes p,

hence (198).

Alternative 2

Let $q_1, q_2, \ldots$ be the primes in ascending order. In the sequence described, the difference between two terms is always a prime. It can be seen from examining a few terms that the change in difference occurs at a product of two successive primes. So the length of each sequence of common differences q_m is $q_{m+1} - q_{m-1}$.

(Note that these terms with common differences q_m are the elements of a multiplication table for q_m between $q_{m-1}q_m$ and $q_m q_{m+1}$ and so can never divide by a higher prime.)

So adding the lengths of sequences with a common difference we find:

Difference	Number of terms	Cumulative count
q_1	$q_2 - 1$	1
q_2	$q_3 - q_1$	$q_3 - q_1 - 1$
q_3	$q_4 - q_2$	$q_4 + q_3 - q_1 - 1$
q_4	$q_5 - q_3$	$q_5 + q_4 - q_1 - 1$
q_5	$q_6 - q_4$	$q_6 + q_5 - q_1 - 1$

and, in general, the common difference changes to q_m at term $q_m + q_{m+1} - 2$, so that

$$a_{q_m + q_{m+1} - 2} = q_m \times q_{m-1} .$$

Now, we find two consecutive primes with a product just less than 10000. $97 \times 101 = 9797$ and $101 \times 103 > 10000$ so $a_{97+101-2} = a_{196} = 9797$. From there, $a_{197} = 9898$, $a_{198} = 9999$,

hence (198).