Conics Revision Questions

(b) T is a variable point $\left(ct, \dfrac{c}{t}\right)$ on the hyperbola $xy = c^2$. **3**

The perpendicular from the centre O to the tangent at T meets it in N.

Find the coordinates of N and prove that N lies on the curve

$$\left(x^2 + y^2\right)^2 = 4c^2xy.$$

(c) The points P, Q with parameters θ, $\theta + \dfrac{1}{2}\pi$ both lie on the ellipse **2**

$$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad P(a\cos\theta,\ b\sin\theta)$$

Show that Q has coordinates $(-a\sin\theta,\ b\cos\theta)$ and prove that $OP^2 + OQ^2 = a^2 + b^2$ (O is the centre of E).

(i) Show that the midpoint M of PQ lies on the ellipse **1**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}.$$

(ii) State the equations of the tangents at P and Q to E and hence obtain the coordinates of T, their point of intersection. **2**

Show that T lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$.

(iii) If α is the angle between the tangents at P, Q prove that **3**

$$\tan\alpha = \frac{2\sqrt{1-e^2}}{e^2\sin 2\theta}. \qquad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Question 4 (15 marks)

a) Derive the equation of the tangent to the hyperbola **7**

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

at the point (x_1, y_1), and hence deduce that the equation of the chord of contact to this hyperbola from an external point $E(x_0, y_0)$ is $\dfrac{xx_0}{16} - \dfrac{yy_0}{9} = 1$.

If the chord of contact passes through a focus, show that E lies on a directrix.

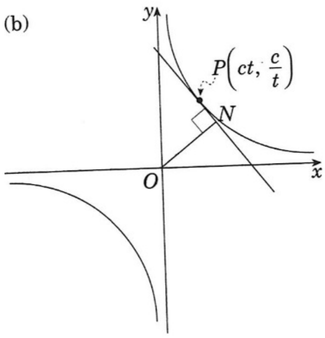
(b) Let $C_1 \equiv 3x^2 + y^2 - 1$ and $C_2 \equiv 7x^2 + 11y^2 - 3$ and let k be a real number. **8**

(i) Show that $C_1 + kC_2 = 0$ is the equation of a curve passing through the points of intersection of the ellipses $C_1 = 0$ and $C_2 = 0$.

(ii) Determine the values of k for which $C_1 + kC_2 = 0$ is the equation of an ellipse.

Solutions

(b)



$y = \frac{c^2}{x} \Rightarrow \frac{dy}{dx} = -\frac{c^2}{x^2}$

Gradient of the tangent at T is $\frac{dy}{dx} = -\frac{1}{t^2}$.

The tangent is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

i.e. $x + t^2 y = 2ct$

The gradient of ON is $m = t^2$ ($\perp$ to tangent)

$\therefore$ ON is $y = t^2 x$

N: $x + t^2 x = 2ct$

$$x = \frac{2ct}{1+t^4} \Rightarrow y = \frac{2ct^3}{1+t^4}$$

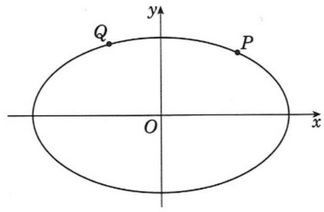
Show that the point lies on $\left(x^2+y^2\right)^2 = 4c^2xy$:

$$\begin{aligned}
\text{LHS} &= \left(x^2+y^2\right)^2 \\
&= \left(\frac{4c^2t^2}{\left(1+t^4\right)^2} + \frac{4c^2t^6}{\left(1+t^4\right)^2}\right)^2 \\
&= \left(\frac{4c^2t^2\left(1+t^4\right)}{\left(1+t^4\right)^2}\right)^2 \\
&= \frac{16c^4t^4}{\left(1+t^4\right)^2}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= 4c^2xy \\
&= 4c^2 \times \frac{2ct}{\left(1+t^4\right)} \times \frac{2ct^3}{\left(1+t^4\right)} \\
&= \frac{16c^4t^4}{\left(1+t^4\right)^2} = \text{LHS}
\end{aligned}$$

(c) $P(a\cos\theta, b\sin\theta)$

$$Q\left[a\cos\left(\theta+\frac{\pi}{2}\right), b\sin\left(\theta+\frac{\pi}{2}\right)\right] = Q(-a\sin\theta, b\cos\theta)$$



$$\begin{aligned}
OP^2 + OQ^2 &= a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta \\
&= a^2\left(\cos^2\theta+\sin^2\theta\right) + b^2\left(\cos^2\theta+\sin^2\theta\right) \\
&= a^2 + b^2
\end{aligned}$$

(i) M: $x = \frac{a(\cos\theta - \sin\theta)}{2}$, $y = \frac{b(\cos\theta+\sin\theta)}{2}$

$$\begin{aligned}
\text{LHS} &= \frac{x^2}{a^2} + \frac{y^2}{b^2} \\
&= \frac{1}{a^2} \times \frac{a^2}{4}(\cos\theta - \sin\theta)^2 + \frac{1}{b^2} \times \frac{b^2}{4}(\sin\theta+\cos\theta)^2 \\
&= \frac{1-2\cos\theta\sin\theta}{4} + \frac{1-2\cos\theta\sin\theta}{4} \\
&= \frac{1}{2} = \text{RHS}
\end{aligned}$$

(ii) At P, tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

At Q, tangent is $\frac{-x\sin\theta}{a} + \frac{y\cos\theta}{b} = 1$

Solving for x, y: $\frac{y}{b}\left(\sin^2\theta+\cos^2\theta\right) = \sin\theta+\cos\theta$

$\therefore$ $y = b(\sin\theta+\cos\theta)$

$\frac{x}{a}\left(\sin^2\theta+\cos^2\theta\right) = \cos\theta - \sin\theta$

$\therefore$ $x = a(\cos\theta - \sin\theta)$

$T[a(\cos\theta-\sin\theta), b(\cos\theta+\sin\theta)]$

$$\begin{aligned}
\text{LHS} &= \frac{x^2}{a^2} + \frac{y^2}{b^2} \\
&= \frac{a^2\left(\cos^2\theta+\sin^2\theta - 2\cos\theta\sin\theta\right)}{a^2} + b\left[\frac{a^2\left(\cos^2\theta+\sin^2\theta-2\cos\theta\sin\theta\right)}{b^2}\right] \\
&= 1 - 2\cos\theta\sin\theta + 1 + 2\cos\theta\sin\theta \\
&= 2 \\
&= \text{RHS}
\end{aligned}$$

(iii) Gradient of tangent at $P = \frac{-b\cos\theta}{a\sin\theta}$

Gradient of tangent at $Q = \frac{b\sin\theta}{a\cos\theta}$

$$\tan\alpha = \left|\frac{\frac{b\sin\theta}{a\cos\theta} + \frac{b\cos\theta}{a\sin\theta}}{1 - \frac{b\sin\theta}{a\cos\theta} \times \frac{b\cos\theta}{a\sin\theta}}\right|$$

continued . . .

$$\begin{aligned}
&= \left|\frac{\frac{b\cos^2\theta + b\sin^2\theta}{a\cos\theta\sin\theta}}{1 - \frac{b^2}{a^2}}\right| \\
&= \left|\frac{ab}{\left(a^2-b^2\right)\cos\theta\sin\theta}\right| \\
&= \left|\frac{2ab}{a^2e^2 2\sin\theta\cos\theta}\right| \\
&= \frac{2\frac{b}{a}}{e^2\sin 2\theta} \\
&= \frac{1\sqrt{1-e^2}}{e^2\sin 2\theta}
\end{aligned}$$

4 (a) $\dfrac{y^2}{16}-\dfrac{x^2}{9}=1.\quad \dfrac{2x}{16}-\dfrac{2y}{9}\cdot\dfrac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{9x}{16y}$$

At (x_1, y_1), $\dfrac{dy}{dx}=\dfrac{9x_1}{16y_1}$

The tangent is $y-y_1=\dfrac{9x_1}{16y_1}(x-x_1)$

$$\therefore\quad 16yy_1-16y_1^2=9xx_1-9x_1^2$$
$$9xx_1-16yy_1=9x_1^2-16y_1^2$$
$$\frac{xx_1}{16}-\frac{yy_1}{9}=\frac{x_1^2}{16}-\frac{y_1^2}{9}$$
$$\frac{xx_1}{16}-\frac{yy_1}{9}=1$$

The tangent at (x_2, y_2) is $\dfrac{xx_2}{16}-\dfrac{yy_2}{9}=1$

$E(x_0, y_0)$ lies on $\dfrac{xx_1}{16}-\dfrac{yy_1}{9}=1$

$$\therefore\quad \frac{x_0x_1}{16}-\frac{y_0y_1}{9}=1$$

$\therefore (x_1, y_1)$ lies on $\dfrac{xx_0}{16}-\dfrac{yy_0}{9}=1$

$E(x_0, y_0)$ lies on $\dfrac{xx_2}{16}-\dfrac{yy_2}{9}=1$

$$\therefore\quad \frac{x_0x_2}{16}-\frac{y_0y_2}{9}=1$$

$\therefore (x_2, y_2)$ lies on $\dfrac{xx_0}{16}-\dfrac{yy_0}{9}=1$

$\dfrac{xx_0}{16}-\dfrac{yy_0}{9}=1$ is the chord of contact, passing through $S(ae, 0)$

$$e^2=1+\frac{b^2}{a^2}=1+\frac{9}{16}=\frac{25}{16}$$

$e=\dfrac{5}{4}\Rightarrow ae=4\cdot\dfrac{5}{4}=5\qquad\therefore\ S$ is $(5, 0)$

$$\frac{5x_0}{16}=1$$
$$x_0=\frac{16}{5}=\frac{a}{e}$$

$\therefore\ E$ lies on the directrix

(b) (i) $C_1=0\cap C_2=0$

$$3x^2+y^2-1=0 \qquad \text{—❶}$$
$$7x^2+11y^2-3=0 \qquad \text{—❷}$$

❶ × 11: $33x^2+11y^2-33=0$ —❸

❸ − ❶: $26x^2-8=0$

$$x^2=\frac{4}{13}$$
$$x=\pm\frac{2}{\sqrt{13}}$$
$$3\cdot\frac{4}{13}+y^2=1$$
$$y^2=\frac{1}{13}$$
$$y=\pm\frac{1}{\sqrt{13}}$$

$C_1+kC_2=0$ is $3x^2+y^2-1+k\left(7x^2+11y^2-3\right)=0$

Test $\left(\pm\dfrac{2}{\sqrt{13}}, \pm\dfrac{1}{\sqrt{13}}\right)$:

$$3\cdot\frac{4}{13}+\frac{1}{13}-1+k\left(7\cdot\frac{4}{13}+11\cdot\frac{1}{13}-3\right)$$
$$=\frac{12}{13}+\frac{1}{13}-1+k\left(\frac{28}{13}+\frac{11}{13}-3\right)$$
$$=0$$

$\therefore$ the points $\left(\pm\dfrac{2}{\sqrt{13}}, \pm\dfrac{1}{\sqrt{13}}\right)$ lie on $C_1+kC_2=0$

$$\therefore\quad 16yy_1-16y_1^2=9xx_1-9x_1^2$$
$$9xx_1-16yy_1=9x_1^2-16y_1^2$$
$$\frac{xx_1}{16}-\frac{yy_1}{9}=\frac{x_1^2}{16}-\frac{y_1^2}{9}$$
$$\frac{xx_1}{16}-\frac{yy_1}{9}=1$$

The tangent at (x_2, y_2) is $\dfrac{xx_2}{16}-\dfrac{yy_2}{9}=1$

$E(x_0, y_0)$ lies on $\dfrac{xx_1}{16}-\dfrac{yy_1}{9}=1$

$$\therefore\quad \frac{x_0x_1}{16}-\frac{y_0y_1}{9}=1$$

$\therefore (x_1, y_1)$ lies on $\dfrac{xx_0}{16}-\dfrac{yy_0}{9}=1$

$E(x_0, y_0)$ lies on $\dfrac{xx_2}{16}-\dfrac{yy_2}{9}=1$

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$C_1+kC_2=0$ is $3x^2+y^2-1+k\left(7x^2+11y^2-3\right)=0$

Test $\left(\pm\dfrac{2}{\sqrt{13}}, \pm\dfrac{1}{\sqrt{13}}\right)$:

$$3\cdot\frac{4}{13}+\frac{1}{13}-1+k\left(7\cdot\frac{4}{13}+11\cdot\frac{1}{13}-3\right)$$
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$$=0$$

$\therefore$ the points $\left(\pm\dfrac{2}{\sqrt{13}}, \pm\dfrac{1}{\sqrt{13}}\right)$ lie on $C_1+kC_2=0$

(ii) $C_1+kC_2=0$ is $(3+7k)x^2+(1+11k)y^2=1+3k$

$$\frac{(3+7k)x^2}{1+3k}+\frac{(1+11k)y^2}{1+3k}=1$$
$$\frac{x^2}{\left(\frac{1+3k}{3+7k}\right)}+\frac{y^2}{\left(\frac{1+3k}{1+11k}\right)}=1$$

For $C_1+kC_2=0$ to be an ellipse,

$$\frac{1+3k}{3+7k}>0 \quad\text{and}\quad \frac{1+3k}{3+11k}>0$$
$$(1+3k)(3+7k)>0 \quad\text{and}\quad (1+3k)(3+11k)>0$$
$$\left\{x<-\frac{3}{7}\ \text{or}\ x>-\frac{1}{3}\right\}\quad\text{and}\quad\left\{x<-\frac{1}{3}\ \text{or}\ x>-\frac{1}{11}\right\}$$

$\therefore\ x<-\dfrac{3}{7}$ or $x>-\dfrac{1}{11},\qquad k\neq 0$