CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Extension 2 Mathematics
2015

Term 2 - Assessment Task 3

Time allowed: **1 hour 30 minutes**

Name: ______________________________ **Teacher: Ms Strilakos**

Instructions:

- All questions should be attempted on your own paper.
- Show ALL necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Please write on one side of each sheet of paper only.

	Q1	Q2	Q3	Total
E4	/4	/7	/8	/19
E7	/4			/4
E8	/10	/11	/9	/30
Total	/18	/18	/17	/53

QUESTION 1 **(18 marks)**

a. Use the substitution $u = x - 1$ to find $\int \frac{x}{\sqrt{x-1}}\,dx$ [2]

b. Find (i) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ [2]

(ii) $\int \frac{sin^3 x}{cos^2 x}\,dx$ [2]

c. Find constants c and d such that $\int_2^3 \frac{x^3-x+2}{x^2-1}\,dx = c + log_e d$ [4]

d. The region between the curve $x = 4y^2 + 2$ and the vertical line $x = 6$ is rotated about the y-axis. Find the volume of the solid generated by slicing perpendicular to the y-axis. [4]

.e. If α, β, γ are the roots of the equation $x^3 + 4x^2 + 3x - 5 = 0$, find a cubic equation whose roots are $\alpha\beta, \beta\gamma$ and $\gamma\alpha$. [4]

QUESTION 2 **(18 marks)**

a. If α, β are the roots of the equation $x^2 - bx + c = 0$,

and $S_n = \alpha^n + \beta^n$ where n is a positive integer,

(i) Show that $S_{n+2} - bS_{n+1} + cS_n = 0$ [4]

(ii) Hence or otherwise, find S_3 and S_4 in terms of b and c. [3]

b. (i) Use the substitution $u = x^2 - 4$ to show that [5]

$$\int \frac{x}{\sqrt{x^2-4}}\, dx = \sqrt{x^2-4} + c$$

(ii) Hence find the exact value of $\int_{\sqrt{5}}^{\sqrt{8}} \frac{x \ln(x^2-4)}{\sqrt{x^2-4}}\, dx$

c. (i) Use the substitution $t = tan\frac{x}{2}$ to show that $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, dx = ln3$ [6]

(ii) Use the substitution $u = \pi - x$ to show that

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}\, dx$$

(iii) Hence find the exact value of $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\, dx$

QUESTION 3 **(17 marks)**

a. $P(x) = 16x^4 - 32x^3 + 16x^2 + kx - 5$, where k is an integer. $P(x)$ has two rational roots which are opposites of each other, and two non-real roots. [8]

(i) If α is a non-real root of $P(x)$, show that $Re(\alpha) = 1$ and $|\alpha| > 1$.

(ii) If the rational roots are $\pm\beta$, deduce that $\beta^2 < \frac{5}{16}$.

(iii) Find the rational roots and the value of k.

(iv) Factor $P(x)$ into irreducible factors with integer coefficients.

b. (i) Show that $\int x \tan^{-1} x \, dx = \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + c$, c constant. [9]

(ii) $I_n = \int_0^1 x^n \tan^{-1} x \, dx, \quad n = 0, 1, 2, \ldots$

Show that $I_0 = \frac{\pi}{4} - \frac{1}{2}ln2,$

$$I_1 = \frac{\pi}{4} - \frac{1}{2}$$

and

$$I_n = \frac{1}{n+1} \cdot \frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} \cdot I_{n-2}, \quad n = 2, 3, 4, \ldots$$

END OF PAPER

YEAR 12
EXTENSION 2 MATHEMATICS
TERM 2 TASK 3 SOLUTIONS

QUESTION 1

a. Let $u = x-1$ $\therefore x = u+1$ $\quad \frac{dx}{du} = 1$

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{u^{1/2}} \cdot \frac{dx}{du}\,du \qquad ②$$

$$= \int \left(u^{1/2} + \frac{1}{u^{1/2}}\right) du \qquad 1$$

$$= \frac{2}{3}u^{3/2} + 2u^{1/2} + c$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + c, \quad x > 1 \qquad 1$$

b. i) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ $\qquad$ Let $u = e^x$ $\therefore u^2 = e^{2x}$

$\frac{du}{dx} = e^x$ $\qquad 1$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du \qquad ②$$

$$= \sin^{-1} u + c$$

$$= \sin^{-1}(e^x) + c \qquad 1$$

ii) $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx$ $\qquad$ Let $u = \cos x$

$\frac{du}{dx} = -\sin x$ $\qquad 1$

$$= -\int \frac{1-u^2}{u^2}\,du \qquad \sin^2 x = 1-\cos^2 x = 1-u^2 \qquad ②$$

$$= \int \left(\frac{u^2-1}{u^2}\right) du = \int \left(1 - \frac{1}{u^2}\right) du = u + \frac{1}{u} + c$$

$$= \cos x + \sec x + c \qquad 1$$

QUESTION 1

c. $$\int_2^3 \frac{x^3-x+2}{x^2-1}\,dx = c + \log_e d$$

$$\text{LHS} = \int_2^3 \frac{x(x^2-1)+2}{x^2-1}\,dx \qquad 1$$

$$= \int_2^3 \left(x + \frac{2}{x^2-1}\right) dx$$

$$\frac{2}{x^2-1} = \frac{a}{x-1} + \frac{b}{x+1}$$

$$2 = a(x+1) + b(x-1)$$

$x = 1$, $2 = 2a$, $a = 1$

$x = -1$, $2 = -2b$, $b = -1$

$$= \int_2^3 \left(x + \frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

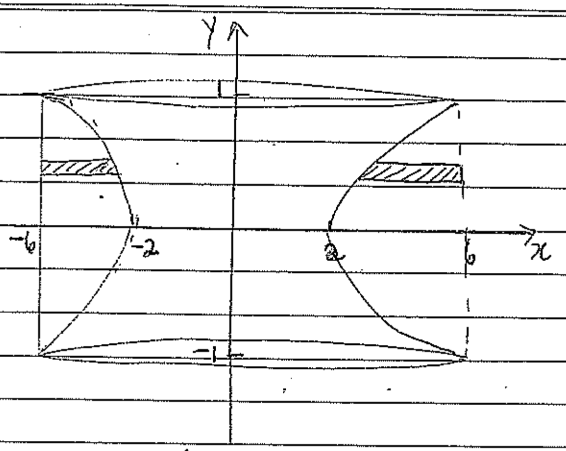
$$= \left[\frac{x^2}{2} + \ln\left(\frac{x-1}{x+1}\right)\right]_2^3 \qquad 1$$

④

$$= \frac{9}{2} + \ln\frac{2}{4} - 2 - \ln\frac{1}{3}$$

$$= \frac{5}{2} + \ln\frac{3}{2} \qquad \therefore c = \frac{5}{2} \quad d = \frac{3}{2} \qquad 1$$

d. $x = 4y^2 + 2$



$4y^2 = x - 2$

$y^2 = \frac{1}{4}(x-2)$

When $x = 6$,

$6 = 4y^2 + 2$

$y^2 = 1$, $y = \pm 1$ 1

The cross-sectional area A is given by $A = \pi(6^2 - x^2)$

$A(y) = \pi\left(6^2 - (4y^2+2)^2\right)$ 1

$= \pi\left(36 - (16y^4 + 16y^2 + 4)\right)$

$= \pi\left(32 - 16y^2 - 16y^4\right)$ (4)

$= 16\pi(2 - y^2 - y^4)$

The volume of a slice δV is given by

$\delta V = 16\pi(2 - y^2 - y^4)\,\delta y$ 1

The volume of the solid V is given by

$V = \lim_{\delta y \to 0} \sum_{y=-1}^{y=1} 16\pi(2 - y^2 - y^4)\,\delta y$

$= 32\pi \int_0^1 (2 - y^2 - y^4)\,dy$ 1

$= 32\pi\left[2y - \frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = 32\pi\left[2 - \frac{1}{3} - \frac{1}{5}\right] = \frac{704\pi}{15}\,u^3$.

QUESTION 1

e. $x^3 + 4x^2 + 3x - 5 = 0$.

roots are: $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$

Product of roots: $\alpha\beta\gamma = 5 \Rightarrow \alpha\beta = \frac{5}{\gamma}$, $\beta\gamma = \frac{5}{\alpha}$, $\gamma\alpha = \frac{5}{\beta}$.

the roots of the original equation are $x = \alpha, \beta, \gamma$

the roots of the new equation are $y = \frac{5}{\alpha}, \frac{5}{\beta}, \frac{5}{\gamma}$. $\therefore y = \frac{5}{x} \Rightarrow x = \frac{5}{y}$

$\therefore$ the new equation is: $\left(\frac{5}{y}\right)^3 + 4\left(\frac{5}{y}\right)^2 + 3\left(\frac{5}{y}\right) - 5 = 0$.

$\frac{125}{y^3} + \frac{100}{y^2} + \frac{15}{y} - 5 = 0$. (4)

$\times y^3$ $\quad 125 + 100y + 15y^2 - 5y^3 = 0$.

$\div 5$ $\quad 25 + 20y + 3y^2 - y^3 = 0$

or $\quad y^3 - 3y^2 - 20y - 25 = 0$.

QUESTION 2

a. $x^2 - bx + c = 0$. α, β roots. Sum of roots: $\alpha + \beta = b$
product of roots: $\alpha\beta = c$.

$S_n = \alpha^n + \beta^n$; n +ve integer.

i) To Show $S_{n+2} - bS_{n+1} + cS_n = 0$.

Proof: $S_{n+2} = \alpha^{n+2} + \beta^{n+2} = \alpha^2.\alpha^n + \beta^2.\beta^n$

$S_{n+1} = \alpha^{n+1} + \beta^{n+1} = \alpha.\alpha^n + \beta.\beta^n$

$S_n = \alpha^n + \beta^n$

(4)

$\therefore S_{n+2} - bS_{n+1} + cS_n = \alpha^2.\alpha^n + \beta^2.\beta^n - b(\alpha.\alpha^n + \beta.\beta^n) + c(\alpha^n + \beta^n)$ 1

$= \alpha^n(\alpha^2 - b\alpha + c) + \beta^n(\beta^2 - b\beta + c)$ 1

1

$= \alpha^n(0) + \beta^n(0)$ since α, β are roots of $x^2 - bx + c = 0$.

ii) $S_3 = \alpha^3 + \beta^3$ from $S_n = \alpha^n + \beta^n$.

Now, from $x^2 - bx + c = 0$ having roots α and β, 1

Σ roots: $b = \alpha + \beta$ and Product of Roots: $c = \alpha\beta$

$\therefore S_3 = \alpha^3 + \beta^3 = \underline{\underline{b^3 - 3cb}}$

Reason:

Since $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 \Rightarrow \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$. 1

Also, $S_4 = \alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2 = \left[(\alpha+\beta)^2 - 2\alpha\beta\right]^2 - 2(\alpha\beta)^2$

$\left[(\alpha^2+\beta^2)^2 = \alpha^4 + 2\alpha^2\beta^2 + \beta^4\right]$

$= (b^2 - 2c)^2 - 2c^2$

$= b^4 - 4b^2c + 4c^2 - 2c^2$

$= \underline{\underline{b^4 - 4b^2c + 2c^2}}$ 1

$S_{n+2} = bS_{n+1} - cS_n$

(3)

QUESTION 2

b. i) $u = x^2 - 4$ $\quad \frac{du}{dx} = 2x$

$$\int \frac{x}{\sqrt{x^2-4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2} \times 2u^{1/2} + c$$

$= \sqrt{x^2-4} + c$

(2)

ii) $$\int_{\sqrt{5}}^{\sqrt{8}} \frac{x\ln(x^2-4)}{\sqrt{x^2-4}}\,dx$$

Using Integration by Parts:

Let $u = \ln(x^2-4)$ $\quad \frac{dv}{dx} = \frac{x}{\sqrt{x^2-4}}$ $\quad v = \sqrt{x^2-4}$ (from

$\frac{du}{dx} = \frac{2x}{x^2-4}$ 1 $\quad v.\frac{du}{dx} = \sqrt{x^2-4}.\frac{2x}{x^2-4} = \frac{2x}{\sqrt{x^2-4}}$

Thus $$\int_{\sqrt{5}}^{\sqrt{8}} \frac{x\ln(x^2-4)}{\sqrt{x^2-4}}\,dx = \left[\sqrt{x^2-4}.\ln(x^2-4)\right]_{\sqrt{5}}^{\sqrt{8}} - \int_{\sqrt{5}}^{\sqrt{8}} \frac{2x}{\sqrt{x^2-4}}\,dx$$

$= 2\ln 4 - 1\ln 1 - 2\left[\sqrt{x^2-4}\right]_{\sqrt{5}}^{\sqrt{8}}$ (3)

$= 2\ln 4 - 2\left[2 - 1\right] = 2\ln 4 - 2$.

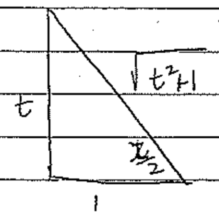
$= 4\ln 2 - 2$

(5)

c(i) $\quad t = \tan\frac{x}{2}$ $\quad \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ $\quad \sin\frac{x}{2} = \frac{t}{\sqrt{t^2+1}}$

$= \frac{1}{2\cos^2\frac{x}{2}}$

$= \frac{t^2+1}{2}$ $\quad \left(\cos\frac{x}{2} = \frac{1}{\sqrt{t^2+1}}\right)$



$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{t^2+1}$

For $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\,dx$: $\quad$ When $x = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad$ 1

When $x = \frac{2\pi}{3}$, $t = \tan\frac{\pi}{3} = \sqrt{3}$

Hence $I = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{t^2+1}{2t} \cdot \frac{dx}{dt} \cdot dt$

$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\cancel{t^2+1}}{2t} \cdot \frac{2}{\cancel{t^2+1}}\,dt$ $\quad$ 1

$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{t}\,dt = \Big[\ln t\Big]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}$

$= \ln 3^{\frac{1}{2}} - \ln 3^{-\frac{1}{2}}$

$= \frac{1}{2}\ln 3 + \frac{1}{2}\ln 3$

$= \underline{\underline{\ln 3}}$

as required

(2)

c(ii) $\quad u = \pi - x$ $\quad \frac{du}{dx} = -1$ $\quad$ and $x = \pi - u$. $\quad$ at $x = \frac{2\pi}{3}$, $u = \frac{\pi}{3}$

$x = \frac{\pi}{3}$, $u = \frac{2\pi}{3}$

To Show : $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}\,dx$

Proof : $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx = \int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} \frac{\pi - u}{\sin(\pi - u)} \cdot -du$ $\quad$ (2)

Note the change of terminals to absorb the -ve sign.

$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - u}{\sin u}\,du$ $\quad$ since $\sin(\pi - u) = \sin u$

$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}\,dx$ $\quad$ (Directly Changing variable from u to x ie using interchanged variables.)

iii) $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\,dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}\,dx$

$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left(\frac{\pi}{\sin x} - \frac{x}{\sin x}\right)dx$ $\quad$ 1

$\Rightarrow 2I = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi}{\sin x}\,dx = \pi\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\,dx$

$2I = \pi\ln 3$ $\quad$ (from (i)) $\quad$ 1

$\therefore \underline{\underline{I = \frac{\pi}{2}\ln 3}}$.

QUESTION 3

a. $P(x) = 16x^4 - 32x^3 + 16x^2 + kx - 5$, k an integer.

Let the two rational roots be β and $-\beta$.

Let the two non-real roots be α and $\bar{\alpha}$, since all coeffs are real.

i) To show $Re(\alpha) = 1$ and $|\alpha| > 1$,

Σ roots: $\alpha + \bar{\alpha} + \beta - \beta = \alpha + \bar{\alpha} = 2$.

If $\alpha = x + iy$ and $\bar{\alpha} = x - iy$ then $2x = 2$, $x = 1$

ie. $Re(\alpha) = 1$. $\therefore |\alpha|^2 = 1 + y^2 > 1$ $\therefore |\alpha| > 1$. 1

ii) If the rational roots are $\pm\beta$, deduce $\beta^2 < \frac{5}{16}$.

Product of Roots: $-\beta^2 . \alpha\bar{\alpha} = \frac{-5}{16}$

$\therefore |\alpha|^2 \beta^2 = \frac{5}{16}$.

Since $|\alpha|^2 > 1$, $\beta^2 < \frac{5}{16}$. 1

iii) To find the rational roots β and $-\beta$,

$\beta = \frac{p}{q}$, where p and q are integers with no common factors.

and p is a factor of 5 and q is a factor of 16. 1

Then $\left(x - \frac{p}{q}\right)\left(x + \frac{p}{q}\right)$ are the factors for β and $-\beta$, ie. $(qx-p)(qx+p)$ $= q^2x^2 - p^2$.

So $(q^2x^2 - p^2)$ is a factor of $P(x)$.

$\therefore q^2$ is a factor of 16 and p^2 is a factor of 5

$\therefore \beta^2 = \frac{p^2}{q^2} < \frac{5}{16}$ $\qquad P(x) = 16x^4 - 32x^3 + 16x^2 + kx - 5$.

Try $\beta = \frac{1}{4}$ $\quad P\left(\frac{1}{4}\right) = \frac{1}{16} - \frac{1}{2} + 1 + \frac{1}{4}k - 5 = 0 \Rightarrow \frac{1}{4}k = 4\frac{7}{16}$

$-\beta = -\frac{1}{4}$ $\quad P\left(-\frac{1}{4}\right) = \frac{1}{16} + \frac{1}{2} + 1 - \frac{1}{4}k - 5 = 0 \Rightarrow -\frac{1}{4}k = -3\frac{7}{16}$

Since the k values are different $\beta^2 \neq$

Try $\beta = \frac{1}{2}$ $\quad P\left(\frac{1}{2}\right) = 1 - 4 + 4 + \frac{1}{2}k - 5 = 0 \Rightarrow \frac{1}{2}k = 4$, $k =$

$P\left(-\frac{1}{2}\right) = 1 + 4 + 4 - \frac{1}{2}k - 5 = 0 \Rightarrow \frac{1}{2}k = 4$, $k =$

$\therefore$ Since we got the same k value for $\beta = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, these values satisfy $P(x)$, and the factors are $(2x-1)(2x+1)$

So, $\alpha\bar{\alpha}\beta(-\beta) = \alpha\bar{\alpha}\left(-\frac{1}{4}\right) = -\frac{5}{16} \Rightarrow \alpha\bar{\alpha} = \frac{5}{4}$

$\therefore (x-\alpha)(x-\bar{\alpha}) = x^2 - x(\alpha + \bar{\alpha}) + \alpha\bar{\alpha}$

$= x^2 - 2x + \frac{5}{4}$ is a factor of $P(x)$

from part (i)

Hence $P(x) = (2x+1)(2x-1)\left(x^2 - 2x + \frac{5}{4}\right) \times 4$

$= (2x+1)(2x-1)(4x^2 - 8x + 5)$. 1

another way to do it.

(iii) To find $\beta, -\beta$ and k:

$\sum$ products of pairs of roots $= \frac{c}{a} = 1 = -\beta^2 + \beta(x+iy) + \beta(x-iy)$
$\qquad - \beta(x+iy) - \beta(x-iy) + x^2+y^2$

i.e. $1 = -\beta^2 + x^2+y^2 \Rightarrow \beta^2 = y^2$ since $x=1$. from (i).

$\therefore \beta = \pm y$.

Now, product of roots is $\beta \times -\beta \times (1+iy)(1-iy)$

$= -\beta^2(1+y^2) = -\beta^2(1+\beta^2) = -\frac{5}{16}$.

i.e. $\beta^2 + \beta^4 = \frac{5}{16}$

$\Rightarrow 16\beta^4 + 16\beta^2 - 5 = 0$.

$\Rightarrow \beta^2 = \frac{-16 \pm \sqrt{16^2 - 4.16.(-5)}}{32}$

$= \frac{-16 \pm \sqrt{256+320}}{32}$

$= \frac{-16 \pm \sqrt{576}}{32} = \frac{-16 \pm 24}{32} = \frac{1}{4}$

(taking +ve sol^n for β^2 since β is rational)

$\therefore \beta = \pm\frac{1}{2}$ and $y = \pm\frac{1}{2}$.

$\sum$ products of triples is $\frac{-k}{16} = -\beta^2(x+iy) - \beta^2(x-iy) + \beta(x^2+y^2) - \beta(x^2+y^2)$

$\Rightarrow \frac{-k}{16} = -2\beta^2 x \Rightarrow \beta^2 = \frac{k}{32}$. Since $\beta^2 = 1/4 \Rightarrow k = 8$

QUESTION 3

b. i) To Show $\int x \tan^{-1}x \, dx = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + c$,

c constant.

Proof: $\int x \tan^{-1}x \, dx$

Let $x = \frac{dv}{dx}$ and $u = \tan^{-1}x$

$v = \frac{1}{2}x^2$ $\qquad \frac{du}{dx} = \frac{1}{1+x^2}$

$I = \frac{1}{2}x^2 . \tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$

$$\begin{array}{r|l} & 1 \\ 1+x^2 & x^2 \\ & x^2+1 \\ \hline & -1 \end{array}$$

$= \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$

2

$= \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right] + c$

$= \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + c$, c constant.

ii) $I_n = \int_0^1 x^n \tan^{-1}x \, dx \qquad n = 0, 1, 2, \ldots$

To show $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2$

$I_0 = \int_0^1 \underset{u}{\tan^{-1}x} \, dx$

Integrate using Integration by Parts:

Let $u = \tan^{-1}x \qquad \frac{du}{dx} = \frac{1}{1+x^2}$

$\frac{dv}{dx} = 1 \qquad v = x$.

$$\therefore I_0 = \int_0^1 \tan^{-1}x\,dx = \Big[x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \qquad 1$$

$$= \frac{\pi}{4} - \Big[\tfrac{1}{2}\ln(x^2+1)\Big]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\big[\ln 2 - \ln 1\big] \qquad (2)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2. \qquad 1$$

To Show $I_1 = \frac{\pi}{4} - \frac{1}{2}$,

$$I_1 = \int_0^1 x\tan^{-1}x\,dx = \Big[\tfrac{1}{2}(x^2+1)\tan^{-1}x - \tfrac{1}{2}x\Big]_0^1 \quad \text{from (i)}$$

$$= \frac{\pi}{4} - \frac{1}{2} \qquad (1)$$

To Show $I_n = \frac{1}{n+1}\cdot\frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}\cdot I_{n-2}$, $n = 2, 3, 4, \ldots$

$$I_n = \int_0^1 x^n \tan^{-1}x\,dx$$

$$= \int_0^1 \underbrace{x^{n-1}}_{u}\,\underbrace{x\tan^{-1}x\,dx}_{\frac{dv}{dx}} \qquad 1$$

$u = x^{n-1}$ $\quad \frac{du}{dx} = (n-1)x^{n-2}$

from (i) $\quad v = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x$

$$= x^{n-1}\cdot\Big[\tfrac{1}{2}(x^2+1)\tan^{-1}x - \tfrac{1}{2}x\Big]_0^1$$

$$- \int_0^1 (n-1)x^{n-2}\left\{\tfrac{1}{2}(x^2+1)\tan^{-1}x - \tfrac{x}{2}\right\}dx \qquad 1$$

$$= \frac{\pi}{4} - \frac{1}{2} - \frac{n-1}{2}\left\{I_n + I_{n-2}\right\} + \frac{n-1}{2n}\Big[x^n\Big]_0^1 \qquad 1$$

$$2I_n = \frac{\pi}{2} - 1 - (n-1)\left(I_n + I_{n-2}\right) + \frac{n-1}{n}$$

$$(n+1)I_n = \frac{\pi}{2} - 1 - (n-1)I_{n-2} + \frac{n-1}{n}$$

$$I_n = \frac{1}{n+1}\cdot\frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}\cdot I_{n-2} \qquad 1$$

Again, in more detail:

To show $I_n = \frac{1}{n+1}\cdot\frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}\cdot I_{n-2}, \quad n=2,3,4,\ldots$

$$I_n = \int_0^1 x^n \tan^{-1}x \, dx$$

We want a term containing $I_{n-2} = \int_0^1 x^{n-2}\tan^{-1}x \, dx$.

so we need to separate x^n in I_n into $x^{n-1}.x$ and differentiate the x^{n-1} term to reduce the power to x^{n-2}.

So, $$I_n = \int_0^1 \underbrace{x^{n-1}}_{u}.\underbrace{x\tan^{-1}x}_{\frac{dv}{dx}} \, dx$$

Let $u = x^{n-1}$ $\therefore$ $\frac{du}{dx} = (n-1)x^{n-2}$.

$\frac{dv}{dx} = x\tan^{-1}x$ $\therefore$ from (1) $v = \int x\tan^{-1}x \, dx$

$$= \tfrac{1}{2}(x^2+1)\tan^{-1}x - \tfrac{1}{2}x$$

(Ignore constant c as we are evaluating a definite integral)

Using Integration by Parts,

$$I_n = x^{n-1}\left[\tfrac{1}{2}(x^2+1)\tan^{-1}x - \tfrac{1}{2}x\right]_0^1$$

$$- \int_0^1 (n-1)x^{n-2}\left(\tfrac{1}{2}(x^2+1)\tan^{-1}x - \tfrac{1}{2}x\right) dx$$

$$= \frac{\pi}{4} - \frac{1}{2} - \int_0^1 \frac{n-1}{2}\left(x^{n-2}\left[(x^2+1)\tan^{-1}x - x\right]\right) dx$$

(cont.)

$$I_n = \frac{\pi}{4} - \frac{1}{2} - \int_0^1 \frac{n-1}{2}\Big[\underbrace{x^n\tan^{-1}x}_{I_n} + \underbrace{x^{n-2}\tan^{-1}x}_{I_{n-2}} - x^{n-1}\Big] dx$$

$$= \frac{\pi}{4} - \frac{1}{2} - \frac{n-1}{2}\left[I_n + I_{n-2}\right] + \frac{n-1}{2}\int_0^1 x^{n-1}\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2} - \frac{n-1}{2}\left[I_n + I_{n-2}\right] + \frac{n-1}{2}\left[\frac{x^n}{n}\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2} - \left(\frac{n-1}{2}\right)I_n - \frac{n-1}{2}I_{n-2} + \frac{n-1}{2n}.$$

$$2I_n = \frac{\pi}{2} - 1 - (n-1)I_n - (n-1)I_{n-2} + \frac{n-1}{n}.$$

$$2I_n + (n-1)I_n = \frac{\pi}{2} - 1 - (n-1)I_{n-2} + \frac{n-1}{n}.$$

i.e. $$(n+1)I_n = \frac{\pi}{2} - 1 - (n-1)I_{n-2} + \frac{n-1}{n}.$$

$\div(n+1)$ $$I_n = \frac{1}{n+1}\cdot\frac{\pi}{2} - \frac{1}{n+1} - \frac{n-1}{n+1}I_{n-2} + \frac{n-1}{n(n+1)}$$

$$= \frac{1}{n+1}\cdot\frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}I_{n-2}, \quad n=:$$

as required.