ACE
EXAM
PAPER

Student name: ____________________

PAPER 4

YEAR 12
YEARLY
EXAMINATION

Mathematics Advanced

General Instructions

- Working time - 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. If $y = 2x\sqrt{x}$, which of the following is an expression for $\frac{dy}{dx}$?

(A) $\dfrac{1}{\sqrt{x}}$

(B) $\dfrac{1}{2\sqrt{x}}$

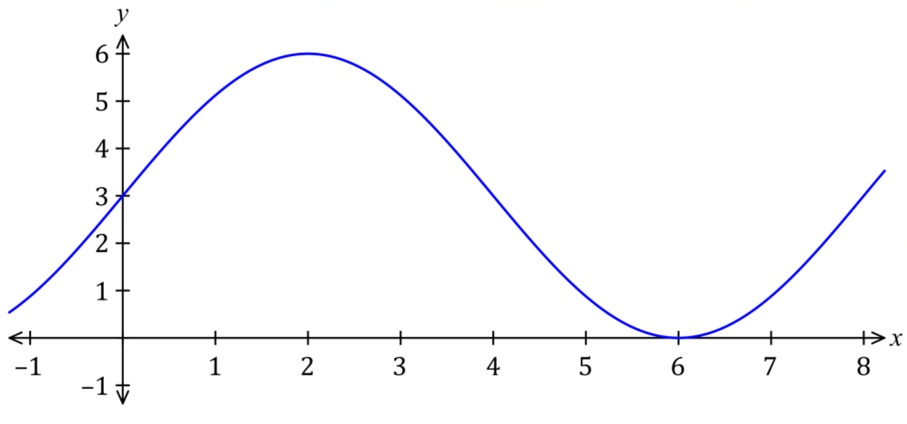
(C) $2\sqrt{x}$

(D) $3\sqrt{x}$

2. Evaluate $\displaystyle\int_0^5 dx$

(A) -5
(B) 0
(C) 5
(D) x

3.



Which of the following equations is likely to be the rule for the graph of the trigonometric function shown above?

(A) $y = 3 + 3\sin\left(\dfrac{\pi x}{4}\right)$

(B) $y = 3 + 3\cos\left(\dfrac{\pi x}{4}\right)$

(C) $y = 3 + 3\sin\left(\dfrac{x}{4}\right)$

(D) $y = 3 + 3\cos\left(\dfrac{x}{4}\right)$

4. What values of x is the curve $f(x) = x^3 + x^2$ concave down?

(A) $x > 3$

(B) $x < -3$

(C) $x > -\frac{1}{3}$

(D) $x < -\frac{1}{3}$

5. The heights of a group of friends are normally distributed with a mean of 160 cm and a standard deviation of 15 cm. What percentage of the group are more than 190 cm tall?

(A) 1%

(B) 2.5%

(C) 5%

(D) 95%

6. What is the domain and range of the function $y = \frac{1}{\sqrt{x-9}}$?

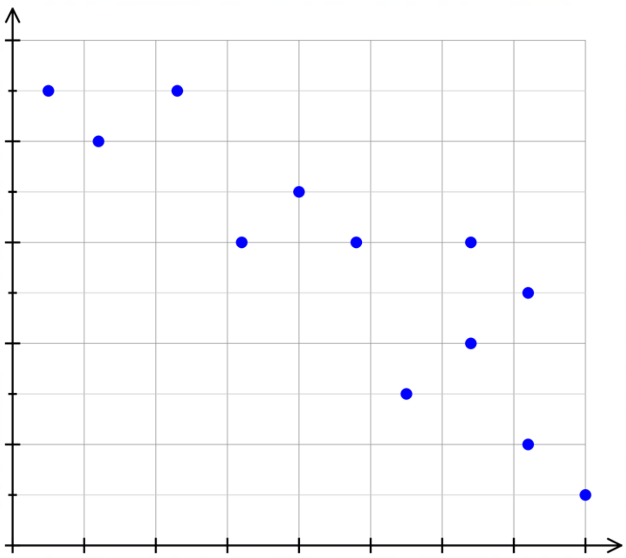
(A) $\{x : x \geq 9\}$ and $\{y : y > 0\}$

(B) $\{x : x > 9\}$ and $\{y : y > 0\}$

(C) $\{x : -\infty \leq x \leq \infty\}$ and $\{y : -\infty \leq y \leq \infty\}$

(D) $\{x : -3 \geq x \geq 3\}$ and $\{y : y < 0\}$

7.



What is the correlation between the variables in this scatterplot?

(A) Weak negative

(B) Weak Positive

(C) Moderate negative

(D) Moderate positive

8. The derivative of $e^{-4x}\cos 2x$ with respect to x is:

(A) $-2e^{-4x}(\sin 2x + 2\cos 2x)$

(B) $-e^{-4x}(\sin 2x - 2\cos 2x)$

(C) $2e^{-4x}(\sin 2x + 2\cos 2x)$

(D) $e^{-4x}(\sin 2x - 2\cos 2x)$

9. A runner started a new training program. On the first day he completed 30 laps of the running track. Every succeeding day he increased his training by 5 laps, until his daily schedule reached 105 laps. On which day did he complete 105 laps in a single day?

(A) 15^{th} day

(B) 16^{th} day

(C) 20^{th} day

(D) 21^{st} day

10. The probability density function for the continuous random variable X is:

$$f(x) = \begin{cases} \frac{3}{4}(x^2 - 1) & 1 < x < 2 \\ 0 & x \le 1 \text{ or } x \ge 2 \end{cases}$$

The value of $P(X \le 1.3)$ is closest to:

(A) 0.07

(B) 0.30

(C) 0.43

(D) 0.93

Section II

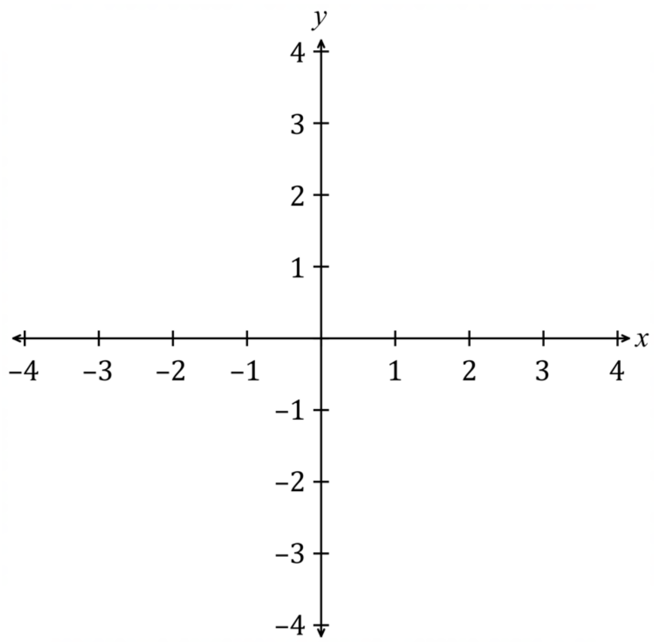
90 marks
Attempt questions 11 - 16
Allow about 2 hours and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (4 marks) **Marks**

The line $y = mx$ is a tangent to the curve $y = e^{0.5x}$ at a point A.

(a) Sketch the line and the curve on the diagram below. **1**



(b) Find the coordinates of A. **2**

(c) Find the value of m. **1**

Question 12 (2 marks) **Marks**

Solve the equation $(\cos x + 2)(2\cos x + 1) = 0$ in the domain $0 \leq x \leq 2\pi$. **2**

Question 13 (2 marks)

Simplify $\dfrac{5}{x-2} - \dfrac{2}{x-3}$ **2**

Question 14 (2 marks)

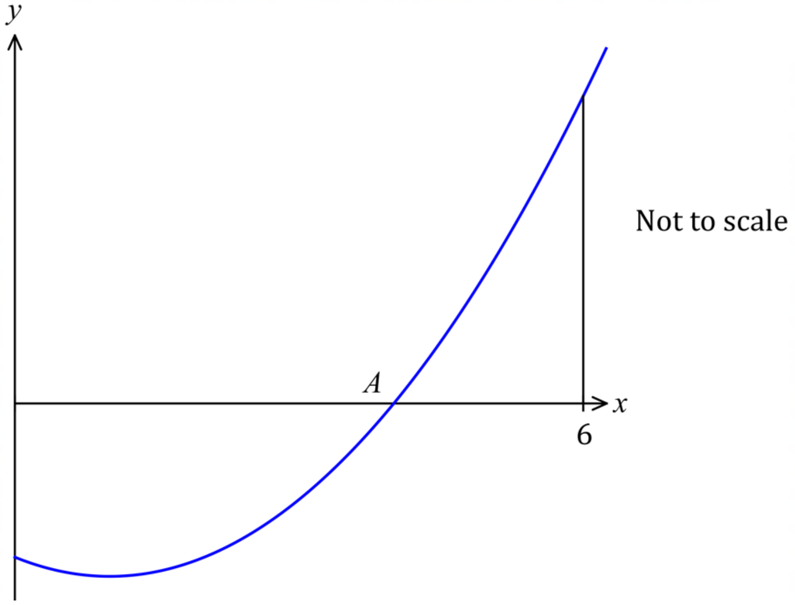
What is the derivative of $y = 2\sin 3x - 3\tan x$ at $x = 0$? **2**

Question 15 (2 marks)

A class compared their assessment results to their head circumference. The correlation coefficient for these quantities was 0.2. What is the meaning of this correlation? **2**

Question 16 (3 marks) **Marks**

The diagram below shows the graph of $y = x^2 - 2x - 8$



(a) What are the coordinates of A? **1**

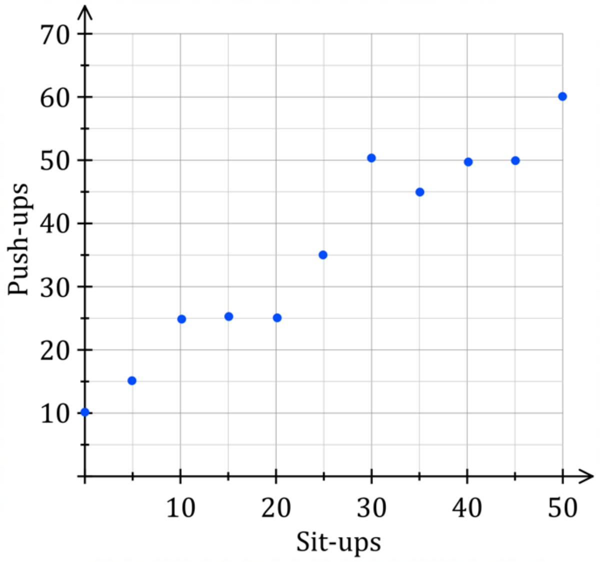
(b) Find the area bounded by the x-axis and the curve $y = x^2 - 2x - 8$ between $0 \leq x \leq 6$. **2**

Question 17 (3 marks)

The second term of an arithmetic series is 37 and the sixth term is 17.
What is the sum of the first ten terms? **3**

Question 18 (5 marks) **Marks**

The scatterplot shows the number of sit-ups (s) and the number of push-ups (p) performed by ten students during a fitness test.



(a) Draw a line of best fit on the scatterplot. Find the gradient of this line. **2**

(b) Alyssa was absent for the push-up test. Predict her push-up result if she scored 36 on the sit-up test. **1**

(c) Calculate the value of the Pearson's correlation coefficient. Answer correct to two decimal places. **2**

Question 19 (2 marks)

Find the period and amplitude for the graph $5y = \sin\left(2x - \frac{\pi}{3}\right)$. **2**

Question 20 (7 marks) **Marks**

A function $f(x)$ is defined by $f(x) = 4x^3 - 4x^2$.

(a) Find the stationary points for the curve $y = f(x)$ and determine their nature. **3**

(b) Sketch the graph of $y = f(x)$ showing the stationary points. **2**

(c) Show the point(s) at which $y = f(x)$ cuts the x-axis. **1**

(d) Determine the values of x for which $f(x)$ is positive. **1**

Question 21 (6 marks) **Marks**

A particle moves along a straight line about a fixed point O so that its acceleration, a ms^{-2}, at time t seconds, is given by:

$$a = 4\cos\left(2t + \frac{\pi}{6}\right)$$

Initially the particle is moving to the right with a velocity of 1 ms^{-1} from a position $0.5\sqrt{3}$ metres to the left of O.

(a) Find an expression for the velocity of the particle after t seconds. **2**

(b) Find an expression for the position of the particle after t seconds. **2**

(c) Show that the particle changes direction when $t = \frac{5\pi}{12}$ seconds. **2**

Question 22 (4 marks) **Marks**

The table below shows the present value of a $1 annuity.

Present value of $1						
Period	2%	4%	6%	8%	10%	12%
1	0.98	0.96	0.94	0.93	0.91	0.89
2	1.94	1.89	1.83	1.78	1.74	1.69
3	2.88	2.78	2.67	2.58	2.49	2.40
4	3.81	3.63	3.47	3.31	3.17	3.04

(a) What would be the present value of a $6 000 per year annuity at 4% per annum for 2 years, with interest compounding yearly? **1**

(b) What is the value of an annuity that would provide a present value of $47 988 after 3 years at 8% per annum compound interest? **2**

(c) An annuity of $1000 each six months is invested at 12% per annum, compounded biannually for 2 years. What is the present value of the annuity? **1**

Question 23 (2 marks)

The graph $y = f(x)$ passes through the point (1, 4) and $f'(x) = 3x^2 - 2$. **2**
Find the expression for $f(x)$.

Question 24 (6 marks) **Marks**

On 1 June 2010, Samuel invested $20,000 in a bank account that paid interest at a fixed rate of 7% per annum, compounded annually

(a) How much is in the account after the payment of interest on 1 June 2020 if no additional deposits were made? Answer to the nearest cent. **1**

(b) Samuel decided to deposit $2000 to his account on 1 June each year beginning on 1 June 2011. How much is in his account on 1 June 2020 after the payment of interest, his deposit and the original investment? Answer to the nearest cent. **3**

(c) Samuel's friend Maya invested $20,000 in an account at another bank on 1 June 2010 and made no further deposits. On 1 June 2020, the balance of Maya's account was $49,565. What was the annual rate of compound interest paid on Maya's account? Answer correct to one decimal place. **2**

Question 25 (3 marks) **Marks**

Find the solution to the equation $\cos 2x = 0.5$ in the domain $-\pi \leq x \leq \pi$ by sketching graphs. **3**

Question 26 (2 marks)

Lola gained a standardised score (z-score) of 2.5 for a class test out of 100.

(a) Describe Lola's result in terms of mean and standard deviation of the class test. **1**

(b) The class test has a mean of 56% and a standard deviation of 9.5. What is the actual mark scored by Lola? **1**

Question 27 (1 mark) **Marks**

The equation of least-squares line of best fit is given by $y = mx + c$ where

$$m = r\frac{S_y}{S_x} \text{ and } c = \bar{y} - m\bar{x}$$

What is the y-intercept of the least-squares line of best fit given $m = 0.6$, $\bar{x} = 50$ and $\bar{y} = 65$? **1**

Question 28 (3 marks)

Differentiate

(a) $(e^x - 2)^4$ **1**

(b) $\dfrac{3x}{\sin 2x}$ **2**

Question 29 (2 marks)

The ages of the residents who live in Hudson Creek are normally distributed. **2**
The mean age is 56 years and the standard deviation is 14. What percentage of the residents are younger than 70?

Question 30 (3 marks)

What is the equation of the normal to the curve $y = x\ln x$ at the point where $x = 1$? **3**

Question 31 (3 marks) **Marks**

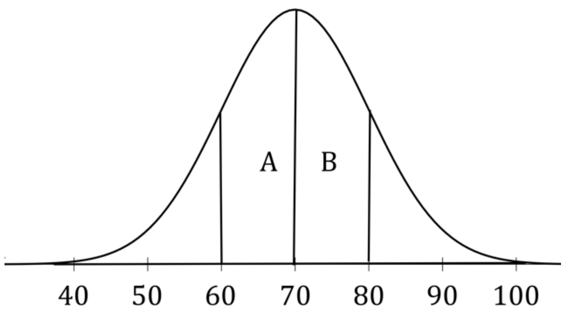
A continuous random variable X has a probability density function f given **3**

$$f(x) = \begin{cases} Ax + B & 1 \leq x \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

where A and B are constants. The median of X is 3. Find the values of A and B.

Question 32 (4 marks)

The normal distribution below represents the mass of 400 students. It has a standard deviation of 10 kg. All measurements are in kilograms.



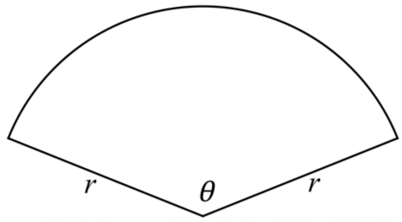
(a) What is the weight of a student with a z-score of -2? **1**

(b) How many students have a mass in the region marked with an A? **1**

(c) How many students will have a mass less than 100 kg? **2**

Question 33 (6 marks) **Marks**

Molly is designing a garden bed for her backyard in the shape of a sector with a radius r and sector angle θ. She has a total of 40 metres of garden edging materials to use as the perimeter of the garden bed.



Not to scale

(a) Show that the area A of the garden bed is given by the formula: **3**

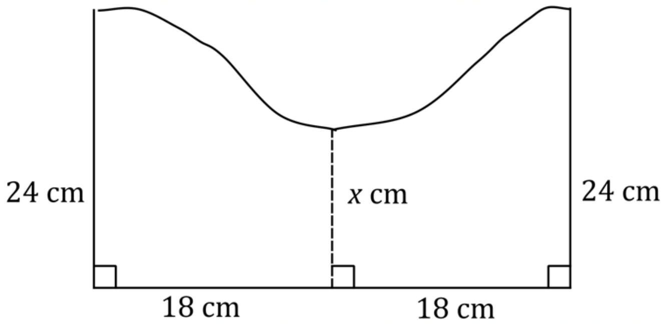
$A = \frac{r}{2}(40 - 2r)$

(b) What value of r gives the maximum value of A? Justify your answer. **2**

(c) Find the maximum possible area that can be made using the 40 metres of edging material. **1**

Question 34 (2 marks) **Marks**

The cross-section of a piece of artwork is shown below



Not to scale

The trapezoidal rule with 2 intervals was used to approximate the area of the artwork as 612 cm^2. What is the value of x? **2**

Question 35 (2 marks)

A radioactive isotope has a half-life of 163 days. Initially there was 10 mg of the isotope. The mass $M(t)$ in milligrams of isotope, after t days, is given by: **2**

$M(t) = 10e^{-kt}$ where k is a constant.

Given that after 163 days only 5 mg of isotope remain, find the value of k.
Answer correct to two significant figures.

Question 36 (5 marks) **Marks**

The rate of emission of carbon pollution C, in tonnes per year from a factory from 1st January 2014 is given by

$C = 500 - \left(\frac{10}{1+t}\right)^2$ where t is the time in years

(a) What was the rate of emission of carbon pollution C on 1st January 2014? **1**

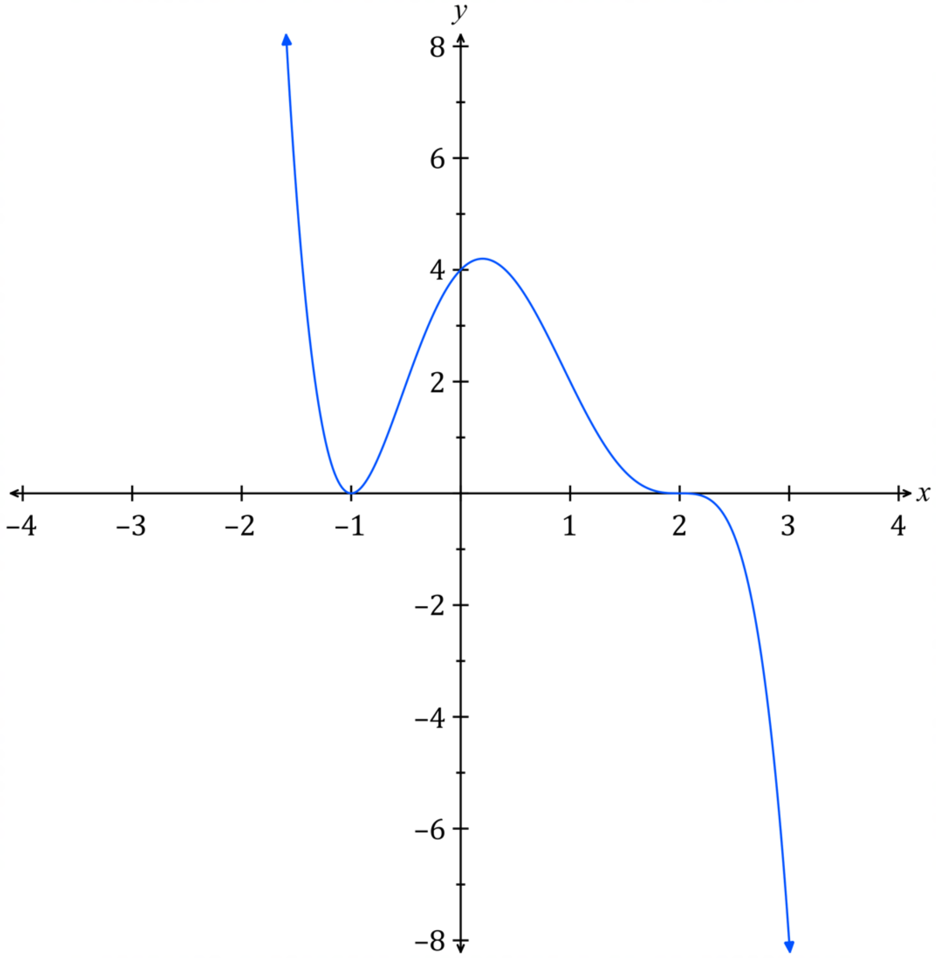
(b) What value does C approach as time passes? **1**

(c) Draw a sketch of C as a function of t? **1**

(d) Calculate the total amount of carbon pollution emitted from the factory from 1st January 2014 to 1st January 2020? Answer correct to the nearest tonne. **2**

Question 37 (4 marks) **Marks**

The graph of $y = f(x)$ is shown below.



Draw sketches of the following functions on the above number plane.
Clearly label each sketch. Indicate any asymptotes and intercepts with the axes.

(a) $y = f(x + 1)$ **2**

(b) $y = \sqrt{f(x)}$ **2**

End of paper

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Functions

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

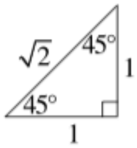
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

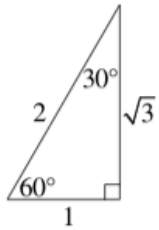


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

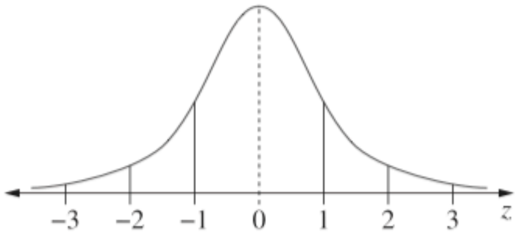
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$