CARLINGFORD HIGH SCHOOL

2014

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Name: ______________________ Teacher: Ms Kellahan

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Answer Section I on Multiple choice answer sheet at the back of this paper.
- Answer section II in answer booklets.
- Use a new answer booklet for each Question 11 to 16.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 6 – 11

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

	MC	11	12	13	14	15	16	Total
E2	/3			/4			/6	/13
E3	/2	/9				/11		/22
E4	/1	/6		/11				/18
E5							/9	/9
E6	/2				/7			/9
E7	/1				/8			/9
E8	/1		/15			/4		/20
Total	/10	/15	/15	/15	/15	/15	/15	%

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Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. The gradient of the function $x^3y^2 + x^3 + y = 6$ at the point (1, 1) is

   (A) -2 (B) $-\frac{3}{7}$ (C) $\frac{3}{7}$ (D) 2

2. Which of the following is an expression for the limiting sum of the geom etric series

   $1 + 2\cos^2\theta + 4\cos^4\theta + 8\cos^6\theta + \ldots$ whenever this limiting sum exists?

   (A) $-\cos\theta$ (B) $-\sec 2\theta$ (C) $\cos 2\theta$ (D) $\sec 2\theta$

3. $\int \frac{x\,dx}{\sqrt[3]{x^2+1}} = ?$

   (A) $\frac{3}{2}\sqrt[3]{x^2+1} + c$ (B) $\frac{1}{2}\sqrt[3]{(x^2+1)^2} + c$

   (C) $\frac{3}{4}\sqrt[3]{(x^2+1)^2} + c$ (D) $\frac{3}{2}\sqrt[3]{(x^2+1)^2} + c$

4. The eccentricity of the ellipse $3x^2 + 5y^2 - 12x + 30y + 42 = 0$ is:

   (A) $\sqrt{\frac{2}{5}}$ (B) $\sqrt{\frac{3}{5}}$ (C) $\sqrt{\frac{5}{3}}$ (D) $\sqrt{\frac{5}{2}}$

5. The region bounded by $y = x^2$, the x-axis, $x = 0$ and $x = 2$ is rotated about the line $x = 2$.

The volume of the resulting solid is:

(A) $\dfrac{3\pi}{8}$ (B) $\dfrac{8\pi}{3}$ (C) $\dfrac{56\pi}{15}$ (D) $\dfrac{2032\pi}{15}$

6. The polynomial equation $3x^3 - 2x^2 + x - 7 = 0$ has roots α, β and γ.

Which polynomial equation has roots $\dfrac{2}{\alpha}, \dfrac{2}{\beta}$ and $\dfrac{2}{\gamma}$?

(A) $3x^3 - 4x^2 + 4x - 56 = 0$

(B) $7x^3 - 2x^2 + 8x - 24 = 0$

(C) $9x^3 - 2x^2 - 27x - 49 = 09$

(D) $24x^3 - 8x^2 + 2x - 7 = 0$

7. Which of the following is the range of the function $f(x) = \sin^{-1} x + \tan^{-1} x$?

(A) $-\pi < y < \pi$

(B) $-\pi \le y \le \pi$

(C) $-\dfrac{3\pi}{4} \le y \le \dfrac{3\pi}{4}$

(D) $-\dfrac{3\pi}{4} \le y \le \dfrac{3\pi}{4}$

8. The number of ways that 6 items can be divided between 3 people so that each person receives 2 items is:

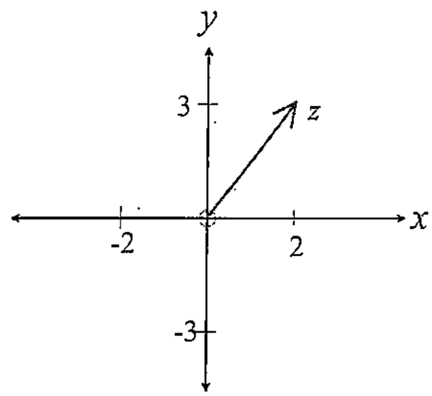
(A) 6

(B) 27

(C) 90

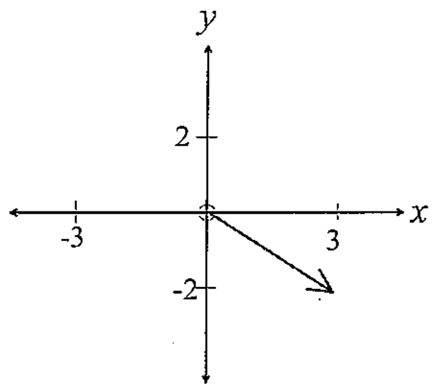
(D) 360

9. The Argand diagram below shows the complex number z.

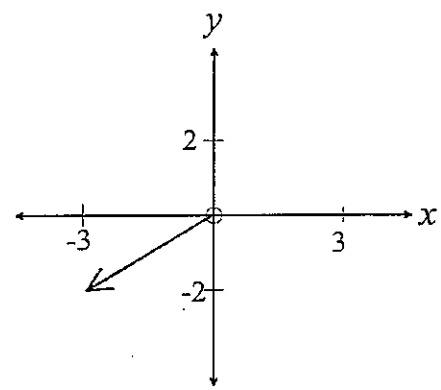


Which Argand diagram best represents $i\bar{z}$?

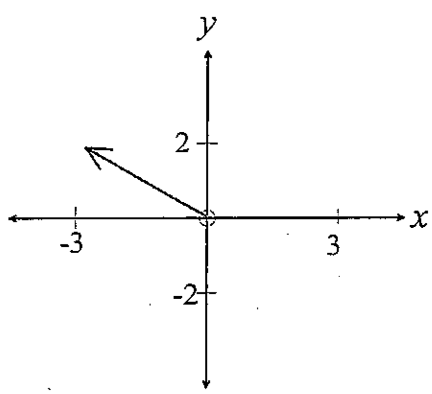
(A)



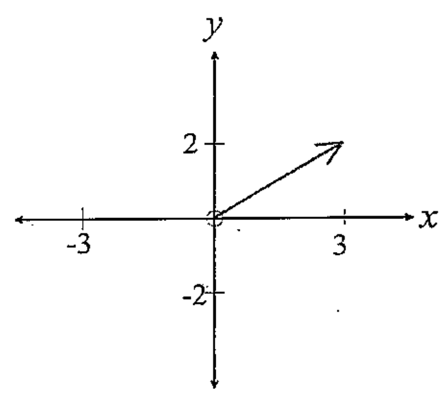
(B)



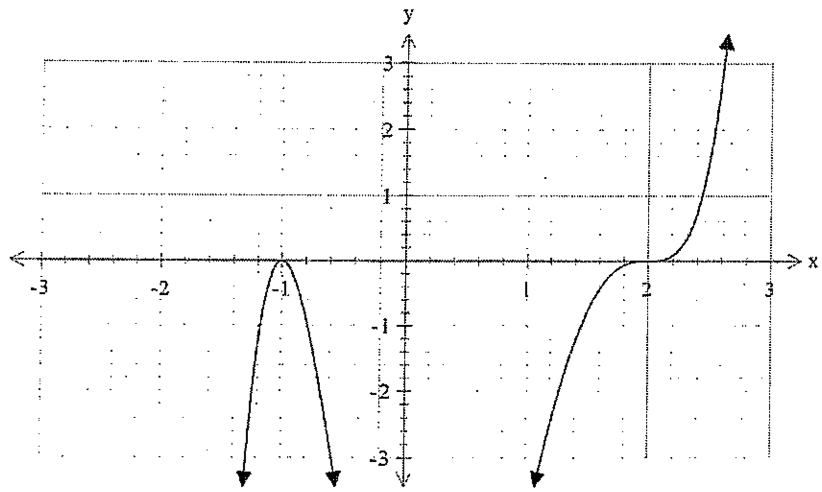
(C)



(D)

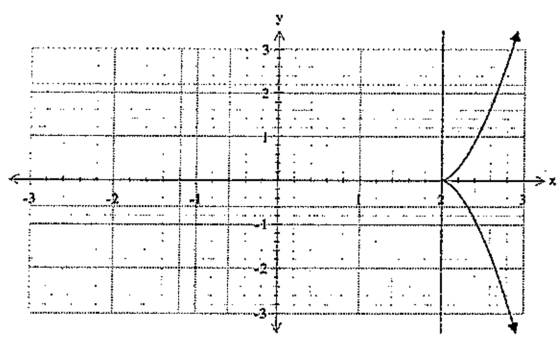


10. The graph of $y = f(x)$ is drawn below.

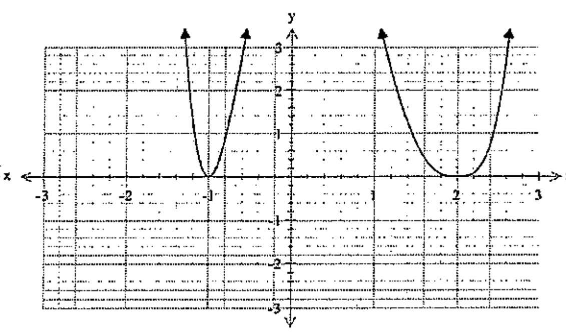


Which of the following graphs represents the graph of $y = [f(x)]^2$?

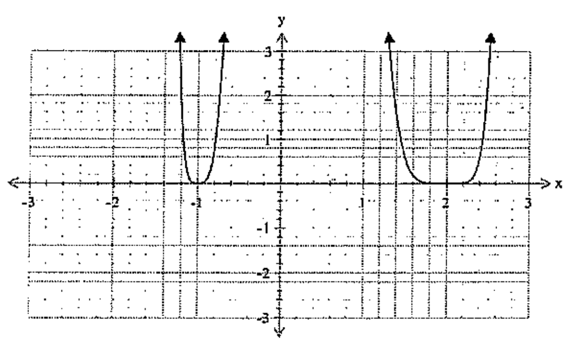
(A)



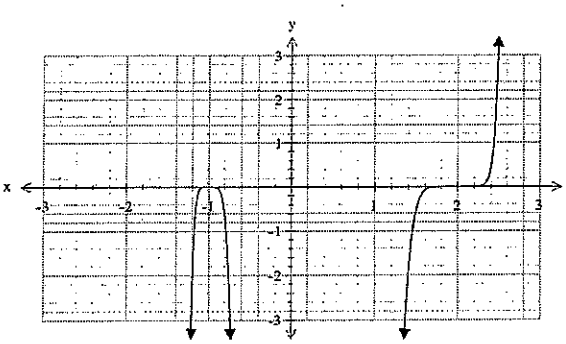
(B)



(C)



(D)



Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $w=\sqrt{3}+i$ and $z=3-\sqrt{3}i$.

(i) Find wz **1**

(ii) Express w in modulus-argument form. **2**

(iii) Write w^4 in simplest Cartesian form. **2**

(b) Find the Cartesian equation of the following curve and sketch it on an Argand Diagram. **2**

$$|z+3+2i|=|z-2+i|$$

(c) Solve the equation $x^4-2x^3+x^2-8x-12=0$, given that $(x-2i)$ is a root of the equation. **3**

(d) Prove that $cis(\alpha+\beta)=cis\,\alpha\ cis\,\beta$ **2**

(e) Given that the polynomial $x^3-px-q=0$ contains a double root, give an equation linking p and q. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{1}{\sqrt{3-(x^2-2x)}}\,dx$ **2**

(b) Find $\int \tan^{-1} x\, dx$ **2**

(c) (i) Find the values of A, B, C and D such that: **2**

$$\frac{5x^3-3x^2+2x-1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

(ii) Hence evaluate $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx$. **2**

(d) Evaluate $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$. **3**

(e) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x+3\cos x}\,dx$ **4**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

(i) Show that the point $P(4\cos\theta, 3\sin\theta)$ lies on the ellipse. **1**

(ii) Calculate the eccentricity of the ellipse and hence find the foci and the directrices of the ellipse. **3**

(iii) Find the equation of the tangent at $P(4\cos\theta, 3\sin\theta)$. **2**

(iv) Find the equation of the normal at $P(4\cos\theta, 3\sin\theta)$ **1**

(v) Show that the tangent at P cuts the positive directrix at $M\left(\frac{16\sqrt{7}}{7}, \frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}\right)$ **2**

(vi) Show that $\angle PSM = 90°$, if S is the positive focus. **2**

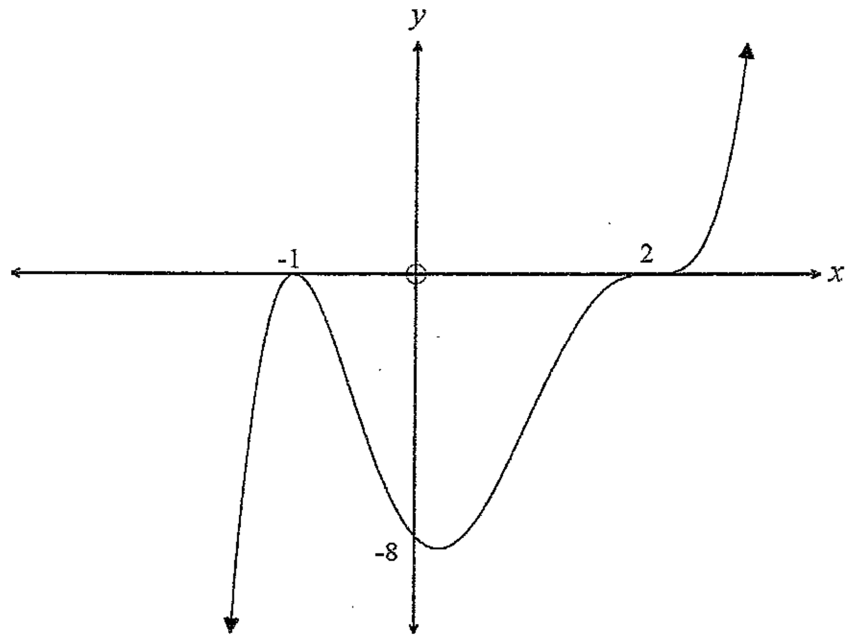
(b) Use logarithmic laws to find the first derivative of $y = \ln\left(\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}}\right)$ **2**

(c) Evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$ **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $y = f(x)$ is shown below.



Sketch the following curves on separate half page diagrams.

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

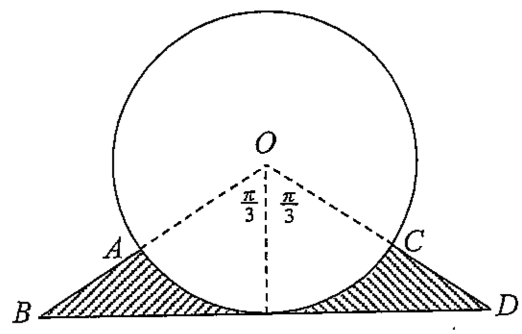
(iii) $y = \dfrac{d}{dx}[f(x)]$ **2**

(iv) $y^2 = f(x)$ **2**

(b) The parabola $y^2 = 4ax$ is rotated about the line $x = a$. By considering slices perpendicular to the axis of rotation, find the volume of the solid of revolution. **3**

Question 14 continues next page

(c) The diagram shows a two dimensional view of a trophy comprising a metal sphere of radius R cm with a centre O, mounted on a base (shaded) so that the sphere fits snugly in the indentation in the base. (The three-dimensional trophy is the rotation of his two-dimensional view about the vertical through O). In the diagram, OAB and OCD are straight lines, each making an angle of $\frac{\pi}{3}$ radians with the vertical through O.



(i) By taking annular, horizontal cross sections of thickness δy at a distance y cm below O, show that the volume V cm^3 of the solid base of the trophy (shaded) is given by $V = \pi\int_{\frac{1}{2}R}^{R} \left(4y^2 - R^2\right) dy$. **3**

(ii) Hence find the volume of V. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use De Moivre's Theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. **1**

(ii) Hence express $\tan 5\theta$ as a rational function of t, where $t = \tan\theta$. **1**

(iii) Deduce that: $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$ **2**

(b) Consider $I_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$ for $n = 0, 1, 2....$

(i) Show that $I_n = \frac{1}{n-1} - I_{n-2}$ for $n = 2, 3, 4...$ **2**

(ii) Hence find the value of I_4 **2**

(c) $P\left(cp, \frac{c}{p}\right)$, $Q\left(cq, \frac{c}{q}\right)$ are points on the rectangular hyperbola $xy = c^2$. Tangents to the rectangular hyperbola at P and Q intersect at the point $R(X, Y)$.

(i) Show that the tangent to the rectangular hyperbola at the point $\left(ct, \frac{c}{t}\right)$ has the equation $x + t^2 y = 2ct$. **2**

(ii) Show that $X = \frac{2cpq}{p+q}$, $Y = \frac{2c}{p+q}$. **2**

(iii) If P, Q are variable points on the rectangular hyperbola which move so that $p^2 + q^2 = 2$, find the equation of the locus R. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m kg is fired vertically upwards with speed 200ms^{-1} in a medium where resistance is $\frac{1}{10}mv$ Newtons when the speed is v ms^{-1}. Take $g = 10\text{ms}^{-2}$.

(i) For the upward journey, if x metres is the vertical displacement upwards from the point of projection, using the equation of motion $\ddot{x} = -\frac{1}{10}(100 + v)$, show that the maximum height attained above the point of projection is H metres where $H = 1000(2 - \ln 3)$. **3**

(ii) Show that the speed v of the particle on return to its point of projection satisfies $\frac{v}{100} + \ln\left(1 - \frac{v}{100}\right) + (2 - \ln 3) = 0$ **3**

(iii) Show that $\lambda + \ln(1 - \lambda) + (2 - \ln 3) = 0$ has a root between 0.8 and 0.9, and applying Newton's method once with 0.82 as a first approximation, find a second approximation for λ. **2**

(iv) What percentage of its terminal velocity has the particle attained on return to its point of projection? Explain your answer. **1**

(b) (i) Show that $\sin(2k+1)\theta - \sin(2k-1)\theta = 2\sin\theta\cos 2k\theta$ **1**

(ii) Hence show that $\sin\theta\sum_{k=1}^{n}\cos 2k\theta = \sin n\theta\cos(n+1)\theta$ **3**

(iii) Hence find the value of $\sum_{k=1}^{10}\sin^2\left(\frac{k\pi}{10}\right)$ **2**

End of Question 16
End of Exam

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CHS Trial HSC Examination 2014

Mathematics Extension 2

Name ____________________ Teacher ____________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ⊗ B ⊗ correct C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Trial HSC Examination 2014
Mathematics Course

Ext 2.

Name ____________________ Teacher ________________

Section I – Multiple Choice Answer Sheet

15 minutes for this section

rnative A, B, C or D that best answers the question. Fill in the response oval completely.

2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

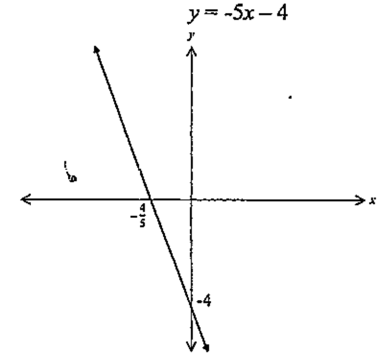
u have made a mistake, put a cross through the incorrect answer and fill in the new

A ● B ~~●~~ C ○ D ○

your mind and have crossed out what you consider to be the correct answer, then rrect answer by writing the word **correct** and drawing an arrow as follows.

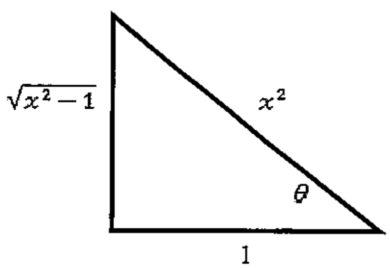
A ~~●~~ B ~~●~~ correct C ○ D ○

A ● B ○ C ○ D ○
A ○ B ● C ○ D ○ E2
A ○ B ○ C ● D ○
A ● B ○ C ○ D ○
A ○ B ● C ○ D ○ E7
A ○ B ● C ○ D ○ E3
A ○ B ○ C ● D ○ E2
A ○ B ○ C ● D ○ E2
A ○ B ○ C ○ D ●
A ○ B ○ C ● D ○ E6

Question 11		2014		
	Solution	Marks	Allocation of marks	O/C
(a)	$w = \sqrt{3} + i$ and $z = 3 - \sqrt{3}i$			E3
	(i) wz $= (\sqrt{3}+i)(3-\sqrt{3}i)$ $= 3\sqrt{3} - 3i + 3i + \sqrt{3}$ $= 3\sqrt{3} + \sqrt{3}$ $= 4\sqrt{3}$	1	Correct Answer	
	(ii) $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ $\tan\theta = \frac{1}{\sqrt{3}}$, $\theta = \frac{\pi}{6}$ $\therefore w = 2\,cis\,\frac{\pi}{6}$	2	1 for correct r 1 for correct θ	
	(iii) $w^4 = \left(2\,cis\,\frac{\pi}{6}\right)^4$ $= 2^4\,cis\,\frac{4\pi}{6}$ $= 16\,cis\,\frac{2\pi}{3}$ $= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $= 16\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$ $= -8 + 8\sqrt{3}i$	2	1 – Evaluating Power 1 Answer in Cartesian form	
(b)	$\lvert z+3+2i\rvert = \lvert z-2+i\rvert$ $(x+3)^2 + (y+2)^2 = (x-2)^2 + (y+1)^2$ $x^2 + 6x + 9 + y^2 + 4y + 4 = x^2 - 4x + 4 + y^2 + 2y + 1$ $10x + 2y + 8 = 0$ $5x + y + 4 = 0$ $y = -5x - 4$	1	1 – equation	E3
		1	1 - Graph	

	2014		
	Marks	Allocation of marks	O/C
$^3 + x^2 - 8x - 12$ $- 2i$ is one root then $x + 2i$ is another root. 4 is a factor. ion, $^3 + x^2 - 8x - 12 = (x^2 + 4)(x^2 - 2x - 3)$ $2i)(x - 2i)(x - 3)(x + 1)$ on is $x = \pm 2i, -1$ and 3	3	1 obtaining factor 1 – Correct division 1 all 4 roots	E4
$\beta) = cis\, \alpha\, cis\, \beta$ $os(\alpha + \beta) + i\sin(\alpha + \beta)$ $os\beta - \sin\alpha \sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$ $s\, \alpha\, cis\, \beta$ $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$ $os\alpha\cos\beta + i\sin\beta\cos\alpha + i\sin\alpha\cos\beta - \sin\alpha\sin\beta$ $\beta - \sin\alpha \sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$ HS	1 1	LHS RHS	E3
$- q = 0$ $x^3 - px - q$ $^2 - p$ oot if $y' = 0$ $- p$ ^{3} $- p\left(\sqrt{\frac{p}{3}}\right) - q = 0$ $- p\left(\sqrt{\frac{p}{3}}\right) = q$ $- p) = q$ $p)^2 = q^2$ $\frac{2p^2}{3} + p^2\Big) = 3q^2$ $\frac{p^3}{3} + p^3\Big) = 3q^2$ $^3 + 9p^3 = 27q^2$ $7q^2$	1 1 1	Double root value Substitution Relationship	E4

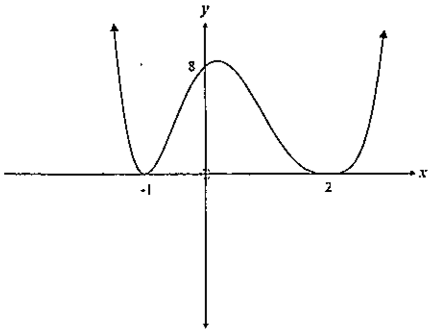
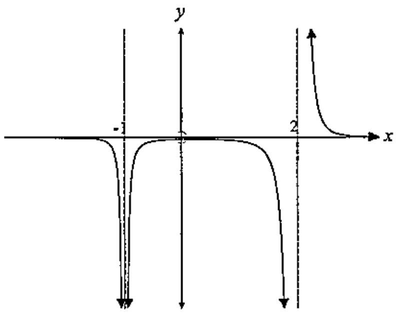
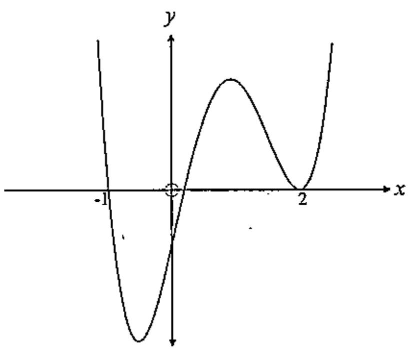
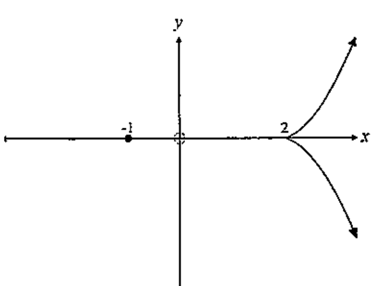
Question 12		2014		
	Solution	Marks	Allocation of marks	O/C
(a)	$\int \frac{1}{\sqrt{3-(x^2-2x)}}\,dx = \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$ $= \sin^{-1}\left(\frac{x-1}{2}\right) + c$	1 1	Complete the square Write the primitive function	E8
(b)	$\int \tan^{-1} x\,dx = \int 1\times\tan^{-1} x\,dx$ $= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$ $= x\tan^{-1} x - \frac{1}{2}\ln\left[1+x^2\right] + c$	1 1	Apply integration by parts Complete the primitive	E8
(c)	(i) $\frac{5x^3-3x^2+2x-1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$ $5x^3-3x^2+2x-1 \equiv Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$ $\equiv Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$ $(A+C)x^3 = 5x^3 \quad \therefore A + C = 5$ $(B+D)x^2 = -3x^2 \quad \therefore B + D = -3$ $Ax = 2 \quad \therefore A = 2 \quad \therefore C = 3$ $B = -1 \quad \therefore D = -2$ Hence, $A = 2, B = -1, C = 3, D = -2.$ (ii) $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3x-2}{x^2+1}\right)dx$ $= \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3x}{x^2+1} - \frac{2}{x^2+1}\right)dx$ $= 2\ln x + \frac{1}{x} + \frac{3}{2}\ln(x^2+1) - 2\tan^{-1}x + c$	2 2	2 – Correct A, B, C and D 1 – 3 correct 1 – Breakup of Integral 1 – Correct Answer	E8

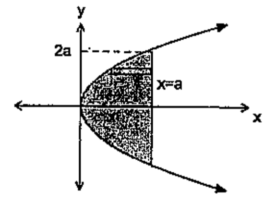
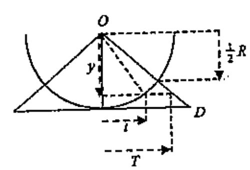
Question 12		2014		
	Solution	Marks	Allocation of marks	O/C
(d)	$\int \frac{\sqrt{x^2-1}}{x^2}\,dx$ Let $x = \sec\theta$ $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1}$ $dx = \tan\theta\,\sec\theta\,d\theta$ $\quad = \sqrt{\tan^2\theta}$ $\quad = \tan\theta$	3		E8
	$\int \frac{\sqrt{x^2-1}}{x^2}\,dx = \int \frac{\tan\theta\,\tan\theta\,\sec\theta\,d\theta}{\sec^2\theta}$ $= \int \frac{\tan^2\theta\,d\theta}{\sec\theta}$		1 – correct substitution	
	$= \int \left(\frac{\sin^2\theta}{\cos^2\theta} \times \frac{\cos\theta}{1}\right) d\theta$ $= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$ $= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$ $= \int (\sec\theta - \cos\theta)\,d\theta$ $= \ln(\sec\theta + \tan\theta) - \sin\theta + c$		1 – working	
				
	$\int \frac{\sqrt{x^2-1}}{x^2}\,dx = \ln\left(x + \sqrt{x^2-1}\right) - \frac{\sqrt{x^2-1}}{x} + c$		1 – correct answer	
(e)	$t = \tan\frac{x}{2}$ $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$ $dx = \frac{2}{1+t^2}\,dt$ $x = 0 \Rightarrow t = 0$ $x = \frac{\pi}{2} \Rightarrow t = 1$ $5 + 4\sin x + 3\cos x = \frac{5(1+t^2) + 8t + 3(1-t^2)}{1+t^2}$ $= \frac{2(t^2+4t+4)}{1+t^2}$ $= \frac{2(t+2)^2}{1+t^2}$	1	Express integrand in terms of t	E8
	$\int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x+3\cos x}\,dx = \int_0^1 \frac{(1+t^2)}{2(t+2)^2} \times \frac{2}{1+t^2}\,dt$	1	Write definite integral in terms of t	
	$= -\left[\frac{1}{t+2}\right]_0^1$	1	Find the primitive function	
	$= \frac{1}{6}$	1	Evaluate using t limits	

	2014		
	Marks	Allocation of marks	
$\frac{x^2}{16}+\frac{y^2}{9}=1$ P($4\cos\theta, 3\sin\theta$)			E4
$+\frac{(3\sin\theta)^2}{9}=1$			
$+\frac{9\sin^2\theta}{9}=1$	1	Working	
$\sin^2\theta=1$			
n the ellipse.			
$a^2(1-e^2)$ $4^2(1-e^2)$ $1-e^2$ $1-\frac{9}{16}$ $\frac{7}{16}$ $\frac{\sqrt{7}}{4}$	3	1 – eccentricity	
$\pm ae,0)=\left(\pm4\times\frac{\sqrt{7}}{4},0\right)=(\pm\sqrt{7},0)$		1 – foci	
es : $x=\pm\frac{a}{e}$ $x=\pm4\div\frac{\sqrt{7}}{4}$ $x=\pm\frac{16}{\sqrt{7}}=\pm\frac{16\sqrt{7}}{7}$		1 – directrices	
$\frac{x^2}{16}+\frac{y^2}{9}=1$ $+\frac{2y}{9}\frac{dy}{dx}=0$ $=-\frac{2x}{16}\div\frac{2y}{9}$ $=-\frac{2x}{16}\times\frac{9}{2y}$ $=-\frac{9x}{16y}$	2	1 – gradient	
s θ, 3 sin θ) $\frac{dy}{dx}=-\frac{36\cos\theta}{48\sin\theta}=-\frac{3\cos\theta}{4\sin\theta}$ $y-y_1=m(x-x_1)$ $y-3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}(x-4\cos\theta)$ $4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$ $3x\cos\theta+4y\sin\theta=12$		1 – Equation	
rmal $\frac{dy}{dx}=\frac{4\sin\theta}{3\cos\theta}$ $y-y_1=m(x-x_1)$ $y-3\sin\theta=\frac{4\sin\theta}{3\cos\theta}(x-4\cos\theta)$ $\cos\theta-9\sin\theta\cos\theta=4x\sin\theta-16\sin\theta\cos\theta$ $4x\sin\theta-3y\cos\theta-7\sin\theta\cos\theta=0$	1	1 – substitution	

Question 13

	Solution	2014 Marks	Allocation of marks
(a)	(v) $3x\cos\theta+4y\sin\theta=12$ $x=\frac{16\sqrt{7}}{7}$ $\frac{48\sqrt{7}}{7}\cos\theta+4y\sin\theta=12$ $4y\sin\theta=12-\frac{48\sqrt{7}}{7}\cos\theta$ $y=\frac{12-\frac{48\sqrt{7}}{7}\cos\theta}{4\sin\theta}$ $y=\frac{84-48\sqrt{7}\cos\theta}{28\sin\theta}=\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}$ $Therefore\ M=\left(\frac{16\sqrt{7}}{7},\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}\right)$	2	1 – substitution 1 – working
	(vi) Gradient PS $=\frac{3\sin\theta-0}{4\cos\theta-\sqrt{7}}=\frac{3\sin\theta}{4\cos\theta-\sqrt{7}}$ Gradient MS $=\frac{\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}-0}{\frac{16\sqrt{7}}{7}-\sqrt{7}}=\frac{\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}}{\frac{16\sqrt{7}-7\sqrt{7}}{7}}=\frac{\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}}{\frac{9\sqrt{7}}{7}}$ $=\frac{21-12\sqrt{7}\cos\theta}{9\sqrt{7}\sin\theta}$ m (PS). m (MS) $=\frac{3\sin\theta}{4\cos\theta-\sqrt{7}}\cdot\frac{21-12\sqrt{7}\cos\theta}{9\sqrt{7}\sin\theta}$ $=\frac{7-4\sqrt{7}\cos\theta}{(4\cos\theta-\sqrt{7})\sqrt{7}}$ $=\frac{7-4\sqrt{7}\cos\theta}{4\sqrt{7}\cos\theta-7}$ $=\frac{7-4\sqrt{7}\cos\theta}{-(7-4\sqrt{7}\cos\theta)}$ $=-1$ $\therefore MS\perp PS\ \ and\ \angle PSM=90°$	2	1 – Both Gradients 1 – proving perpendicular
(b)	$y=\ln\left(\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}}\right)$ $y=\ln\sqrt{x^2+1}-\ln\sqrt[3]{x^3+1}$ $y=\ln(x^2+1)^{\frac{1}{2}}-\ln(x^3+1)^{\frac{1}{3}}$ $y=\frac{1}{2}\ln(x^2+1)-\frac{1}{3}(x^3+1)$ $\frac{dy}{dx}=\frac{1}{2}\frac{2x}{x^2+1}-\frac{1}{3}\frac{3x^2}{x^3+1}$ $=\frac{x}{x^2+1}-\frac{x^2}{x^3+1}$	1 1	Use of logarithmic rules Answer simplified
(c)	$\lim_{\theta\to0}\frac{1-\cos\theta}{\theta}=\lim_{\theta\to0}\frac{1-\cos\theta}{\theta}\times\frac{1+\cos\theta}{1+\cos\theta}$ $=\lim_{\theta\to0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$ $=\lim_{\theta\to0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}$ $=\lim_{\theta\to0}\frac{\sin\theta}{\theta}\times\lim_{\theta\to0}\frac{\sin\theta}{(1+\cos\theta)}$ $=1\times0$	1 1	

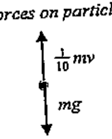
14	M		O/C
	1	Correct Graph	
	2	2- Shape of Graph 1 – Accuracy of critical points	
	2	2- Shape of Graph 1 – Accuracy of critical points	
		2- Shape of Graph 1 – Accuracy of critical points	

Question 14		2014		
	Solution	Marks	Allocation of marks	O/C
(b)				E7
	Consider a slice perpendicular to $x=a$ Slice rotated about $x=a$; radius of slice $a-x$, thickness of slice δy			
	Volume of slice $\delta V \approx \pi(a-x)^2\delta y$ $\delta V = \pi(a^2-2ax+x^2)\delta y$ $=\pi\left(a^2-\frac{y^2}{2}+\frac{y^4}{16a^2}\right)\delta y$	1	Correct expression	
	$\therefore V=\lim_{\delta y\to 0}\sum_{-2a}^{2a}\pi\left(a^2-\frac{y^2}{2}+\frac{y^4}{16a^2}\right)\delta y$ $=2\pi\int_0^{2a}\left(a^2-\frac{y^2}{2}+\frac{y^4}{16a^2}\right)dy$	1	correct $\int$ & values.	
	$=2\pi\left[a^2y-\frac{y^3}{6}+\frac{y^5}{80a^2}\right]_0^{2a}$ $=2\pi\left(2a^3-\frac{8a^3}{6}+\frac{32a^3}{80}-(0-0+0)\right)$ $=\frac{32a^3}{15}\pi$ cubic units	1	correct substitution & simplified.	
(c)	i. Cross section y below O is an annulus with inner radius t and outer radius T, where $t^2=R^2-y^2$ and $T=y\tan\frac{\pi}{3}=y\sqrt{3}$	1	radius of annular cross section	E7
	(since vertical through O makes angle $\frac{\pi}{3}$ with OD) Hence area of cross section is $\pi\left\{3y^2-\left(R^2-y^2\right)\right\}$	1	other radius & area	
	$\therefore V=\lim_{\delta y\to 0}\sum_{y=\frac{1}{2}R}^{R}\pi\left(4y^2-R^2\right)\delta y=\pi\int_{\frac{1}{2}R}^{R}\left(4y^2-R^2\right)dy$	1	Express V as limiting sum & hence definite integral	
	ii. $V=\pi\left[\frac{4}{3}y^3-R^2y\right]_{\frac{1}{2}R}^{R}=\pi\left\{\frac{4}{3}\left(R^3-(\frac{1}{2}R)^3\right)-R^2\left(R-\frac{1}{2}R\right)\right\}$	1	find primitive function	
	$\therefore V=\pi R^3\left(\frac{4}{3}\times\frac{7}{8}-\frac{1}{2}\right)=\frac{2}{3}\pi R^3$	1	Evaluate in terms of R.	
				

	2014		
	M	Allocation of marks	O/C
$)^5 = (\cos\theta + i\sin\theta)^5$ $\cos^5\theta + 5\cos^4\theta i\sin\theta + 10\cos^3\theta i^2\sin^2\theta + 10\cos^2\theta i^3\sin^3\theta + 5\cos\theta i^4\sin^4\theta + i^5\sin^5\theta$ $5\cos^4\theta i\sin\theta - 10\cos^3\theta\sin^2\theta - 10\cos^2\theta i\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$ ting real and imaginary parts $\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$	1	$Cos\,5\theta$ and $Sin\,5\theta$	E3
$\tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$ $\frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ $\frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$, where $t = \tan\theta$	1	$Tan\,5\theta$	
$\tan 5\theta = 0, \ \therefore 5\theta = 0, \pi, 2\pi, 3\pi, 4\pi \ldots\ldots$ $i.e\ \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5} \ldots$ $)t^3 + t^5$ / $t^2 + 5t^4$ $= 0$, then $5t - 10t^3 + t^5 = 0$	1	Solutions	
$.0t^2 + 5) = 0$ has roots $0, \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$ $t^2 + 5 = 0$ has roots $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$ e product of the roots: $\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5.$	1	Product of roots	
$\frac{x^n}{+x^2}dx = \int_0^1 \frac{\{(1+x^2)-1\}x^{n-2}}{1+x^2}dx$, $n = 2, 3, 4, \ldots$	1	rearrange integrand	E8
$I_n = \int_0^1 x^{n-2}dx - I_{n-2}$ $= \frac{1}{n-1}\left[x^{n-1}\right]_0^1 - I_{n-2}$ $= \frac{1}{n-1} - I_{n-2}$	1	Evaluate to obtain reduction formula	
$I_0 = \int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$	1	Evaluate I_0	
$I_4 = \frac{1}{3} - I_2$ $= \frac{1}{3} - (1 - I_0)$ $= \frac{\pi}{4} - \frac{2}{3}$	1	reduce & find I_4.	

Question 15

	Solution	2014 M	Allocation of marks	O/C
(c)	(i) $x = ct \Rightarrow \frac{dx}{dt} = c$ $y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = -\frac{c}{t^2}$ $\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{1}{t^2}$	1	Gradient terms of t	E3
	Hence tangent at $\left(ct, \frac{c}{t}\right)$ has gradient $-\frac{1}{t^2}$ and equation $x + t^2y = k$ for some constant k. $\left(ct, \frac{c}{t}\right)$ lies on the tangent $\Rightarrow ct + t^2\frac{c}{t} = k$ $\therefore k = 2ct$ and tangent has equation $x + t^2y = 2ct$.	1	Eqn of tangent	
	(ii) Where tangents at P, Q intersect $x + p^2y = 2cp$ $x + q^2y = 2cq$ $(p^2 - q^2)y = 2c(p-q)$ $(p-q)(p+q)y = 2c(p-q)$			
	Also $(p^2 - q^2)x = 2cpq(p-q)$ $(p-q)(p+q)x = 2cpq(p-q)$	1 1	Find X Find Y	
	$\therefore p \neq q \Rightarrow X = \frac{2cpq}{p+q}$, $Y = \frac{2c}{p+q}$		in terms of p & q	
	(iii) $p^2 + q^2 = (p+q)^2 - 2pq$ $\therefore p^2 + q^2 = 2 \Rightarrow (p+q)^2 = 2(1+pq)$	1	$p^2+q^2=2$ in terms of p+q, pq	
	Hence at $R(X, Y)$ $\frac{X}{Y} = pq$ and $\frac{2c}{Y} = p+q$	1	p+q, pq in terms of X & Y	
	Hence the locus of R has equation $\frac{4c^2}{y^2} = 2\left(1 + \frac{x}{y}\right)$ $y^2 + xy = 2c^2$	1	equation of locus	

5	2014		
n	Marks	Allocation of marks	
$\ddot{x}=-\frac{1}{10}(100+v)$ $v\frac{dv}{dx}=-\frac{1}{10}(100+v)$ $-\frac{1}{10}\frac{dx}{dv}=\frac{v}{100+v}$ $-\frac{1}{10}\frac{dx}{dv}=1-\frac{100}{100+v}$	1	$\frac{dx}{dv}$ in terms of v	E5
$-\frac{1}{10}x=v-100\ln(100+v)+c$ $t=0$; $x=0$, $v=200$ $\Big\}$ $0=200-100\ln 300+c$ $\therefore -\frac{1}{10}x=(200-v)-100\ln\left(\frac{300}{100+v}\right)$	1	use of initial conditions to find x in terms of v	
$x=H,\ v=0\Rightarrow$ $\frac{1}{10}H=200-100\ln 3$ $\therefore\ H=1000\,(2-\ln 3)$	1	find expression H by find x when v=0	
or the downward journey, let x be the distance llen below the position of maximum height, th initial conditions $x=0,\ v=0$. y Newton's 2nd Law $m\ddot{x}=mg-\frac{1}{10}mv$ $\ddot{x}=\frac{1}{10}(100-v)$ $v\frac{dv}{dx}=\frac{1}{10}(100-v)$ $\frac{1}{10}\frac{dx}{dv}=\frac{v}{100-v}$ $-\frac{1}{10}\frac{dx}{dv}=1-\frac{100}{100-v}$ *Forces on particle* 	1	eqn of motion downward journey	
$-\frac{1}{10}x=v+100\ln(100-v)+c$ $t=0, x=0, v=0\Rightarrow$ $0=\ 100\ln 100\ +c$ $-\frac{1}{10}x=v+100\ln\left(\frac{100-v}{100}\right)$	1	distance fallen in terms of v	
$x=H\Rightarrow$ $-100(2-\ln 3)=v+100\ln\left(1-\frac{v}{100}\right)$ $\frac{v}{100}+\ln\left(1-\frac{v}{100}\right)+(2-\ln 3)=0$	1	Use expression for max. height to est. required equation.	

Question 16		2014		
a)	iii. Let $f(\lambda)=\lambda+\ln(1-\lambda)+(2-\ln 3)$. $f'(\lambda)=1-\frac{1}{1-\lambda}=\frac{-\lambda}{1-\lambda}$			E5
	Then $f(\lambda)$ is continuous for $0<\lambda<1$ and $f(0\cdot 8)\approx 0\cdot 09>0$, $f(0\cdot 9)\approx -0\cdot 50<0$ Hence $f(\lambda)=0$ for some $0\cdot 8<\lambda<0\cdot 9$.	1	Note continuity & establish change in sign	
	Using $\lambda_0=0\cdot 82$, $\lambda_1=0\cdot 82-\frac{0\cdot 82+\ln(1-0\cdot 82)+2-\ln 3}{\left(\frac{-0\cdot 82}{1-0\cdot 82}\right)}$ $\approx 0\cdot 82$	1	Apply Newton's Method	
	iv. Since Newton's method returned the same approximate root to 2 decimal places, $\frac{v}{100}\approx 0\cdot 82$ gives the speed v on return to projection point as 82 ms^{-1} (to nearest 1). For the downward journey, $\ddot{x}\to 0$ as $v\to 100$. Hence the terminal velocity is 100 ms^{-1} . Hence particle has attained 82% of its terminal velocity on return to its point of projection.	1	Find terminal velocity & use λ to obtain %.	

Question 16		2014		
	Solution	Marks	Allocation of marks	
(b)				E2
	i. $\sin(2k+1)\theta = \sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta$			
	and $\sin(2k-1)\theta = \sin 2k\theta\cos\theta - \cos 2k\theta\sin\theta$			
	$\therefore \sin(2k+1)\theta - \sin(2k-1)\theta = 2\sin\theta\cos 2k\theta$	1	Expand & simplify	
	ii. $2\sin\theta\sum_{k=1}^{n}\cos 2k\theta = \sum_{k=1}^{n}\{\sin(2k+1)\theta - \sin(2k-1)\theta\}$		use identity from (i) to simplify sum	
	$= \sin(2n+1)\theta - \sin\theta$	1		
	Using $\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$.	1	use trig identity converting sum to difference	
	, $A = 2(n+1)\theta,\ B = \theta$:			
	$\sin\theta\sum_{k=1}^{n}\cos 2k\theta = \sin n\theta\cos(n+1)\theta$	1	Simplify obtained result.	
	iii. $\sum_{k=1}^{10}\sin^2\frac{k\pi}{10} = \frac{1}{2}\sum_{k=1}^{10}\left(1-\cos\frac{2k\pi}{10}\right)$	1	Use appropriate trig identity	
	$= \frac{1}{2}\left\{10 - \frac{\sin\frac{10\pi}{10}\cos\frac{11\pi}{10}}{\sin\frac{\pi}{10}}\right\} = 5$	1	Use (ii) to evaluate.	