Solutions – Intermediate Division

1. $2013 + 2014 + 2015 = 6042$,

hence (E).

2. Angle $x°$ is the sum of the two opposite interior angles, so $x = 85 + 37 = 122$,

hence (C).

3. (Also J7)
If every digit of a whole number is either a 3 or a 5, the number must be odd. Note that 33 is not prime, even or divisible by 5 and 35 is not divisible by 3, showing that odd is the only consistent descriptor,

hence (E).

4. Let x be the smaller number. Then $\dfrac{x+12}{2} = 2x \rightarrow x + 12 = 4x \rightarrow 12 = 3x \rightarrow x = 4$,

hence (C).

5. For any triangle, $A = \frac{1}{2}bh$. Here $b = 3h$, so $A = 24 = \frac{3}{2}h^2 \rightarrow 3h^2 = 48 \rightarrow h^2 = 16 \rightarrow h = 4$. Thus base + height $= 4 + 12 = 16$ centimetres,

hence (E).

6. Each pair of opposite faces adds to 21 and $8 + 13 = 21$,

hence (C).

7. (Also S5)
When $b = 1, p = 30$ (which is divisible by 2, 5 and 6). When $b = 4, p = 42$ (which is divisible by 7). This eliminates alternatives A, C, D and E. However, $4p + 26 = 4(p + 6) + 2$ so p cannot be divisible by 4,

hence (B).

8. The faster dog runs 100 m while the slower dog runs 50 m, so the latter is still 20 m away when the former reaches me,

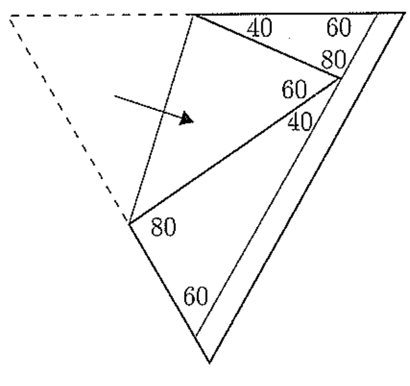
hence (B).

9. If $x = \dfrac{2}{3}$, then $x^2 = \dfrac{4}{9}$ and $\dfrac{1}{x^2} = \dfrac{9}{4}$. Adding, we obtain $\dfrac{9}{4} + \dfrac{4}{9} = \dfrac{97}{36} = 2\frac{25}{36}$, which is between 2.5 and 3,

hence (D).

10. (This is a slight modification of J13)
Alternative 1
In the following diagram, a line is drawn parallel to one side of the triangle, which passes through the folded corner. The smaller triangle formed is still equilateral. Using the facts that each corner of an equilateral triangle is 60°and the angles of any triangle add up to 180°, x is equal to 80,

hence (C).



Alternative 2
The sum of the interior angles of a pentagon (emphasised in the diagram) is 540°. Hence $60 + 60 + 40 + 300 + x = 540$ and so $x = 80$,

hence (C).

11. The last digits follow the cycle $2, 4, 8, 6, 2, 4, 8, 6, \ldots$ and so the sequence goes

$$\begin{gathered} 1; 2, 4, 8, 16; 22, 24, 28, 36; \ldots \\ \ldots; (20n+2), (20n+4), (20n+8), (20n+16); \ldots \\ \ldots; 982, 984, 988, 996 \end{gathered}$$

where the first quadruplet is at $n = 0$ and the last quadruplet less than 1000 is at $n = 49$. So including the initial '1' there are $1 + 50 \times 4 = 201$ terms,

hence (E).

12. The possible outcomes can be represented in a table:

	1	2	2	3	3	3
1		O	O			
2	O			O	O	O
2	O			O	O	O
3		O	O			
3		O	O			
3		O	O			

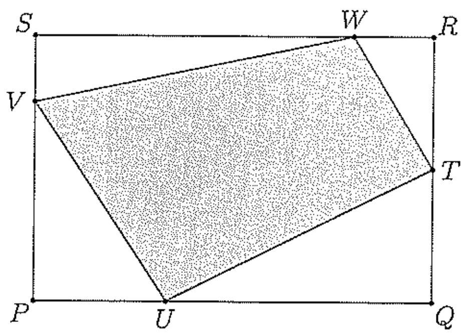
So the probability of an odd total is $16/36 = 4/9$,

hence (D).

13. $x^3 = x.x^2 = x(x+3) = x^2 + 3x = x + 3 + 3x = 4x + 3$,

hence (D).

14.



Alternative 1

If the rectangle has height h and width w, then $hw = 120$ and the areas of triangles are:

$$\begin{aligned} \Delta PVU &= \frac{1}{2}\frac{w}{3}\frac{3h}{4} = \frac{hw}{8} = 15 \\ \Delta QUT &= \frac{1}{2}\frac{2w}{3}\frac{h}{2} = \frac{hw}{6} = 20 \\ \Delta RTW &= \frac{1}{2}\frac{w}{5}\frac{h}{2} = \frac{hw}{20} = 6 \\ \Delta SVW &= \frac{1}{2}\frac{4w}{5}\frac{h}{4} = \frac{hw}{10} = 12 \end{aligned}$$

So the sum of the areas of the triangles is 53 and the area of $TUVW$ is 67,

hence (A).

Alternative 2

As the dimensions of the triangle are not given, they can be arbitrarily assigned (provided that they give an area of 120). If we let the rectangle be 15×8, then the ratios divide SR and RQ into intervals with integer lengths. The areas of the four triangles can then be easily calculated as follows: $\Delta RTW = 6, \Delta UTQ = 20, \Delta PVU = 15, \Delta SVW = 12$. Adding these gives 53, which when subtracted from 120 gives the required answer of 67,

hence (A).

15. This question is based on the triangle inequality which says that no side of a triangle can be longer than the sum of the other two sides. Clearly, a cannot be the longest side, so $2a < a + 1$ and $1 < 2a + a$, which when combined give $\frac{1}{3} < a < 1$,

hence (A).

16. The area of the quadrant is $\dfrac{\pi r^2}{4}$ and the area of the right-angled triangle is $\dfrac{r^2}{2}$. Subtracting gives the expression for the shaded area:

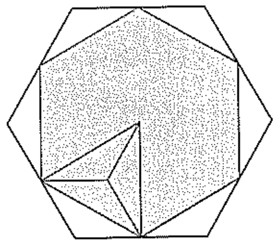
$$\begin{aligned}\frac{\pi r^2}{4} - \frac{r^2}{2} &= 1\\ \pi r^2 - 2r^2 &= 4\\ r^2 &= \frac{4}{\pi - 2}\\ r &= \sqrt{\frac{4}{\pi - 2}}\end{aligned}$$

hence (C).

17. For every metre Dan reads on his tape, he actually measures 1.04 m, whilst for every metre Jane reads, she actually measures 0.95 m. Hence when Dan reads 23.75, the correct distance is 23.75×1.04 and Jane will measure this amount divided by 0.95.
Observing that $23.75 = 23\frac{3}{4} = \dfrac{95}{4}$, the length is $\dfrac{95}{4} \times \dfrac{1.04}{0.95} = \dfrac{104}{4} = 26$ m,

hence (D).

18.

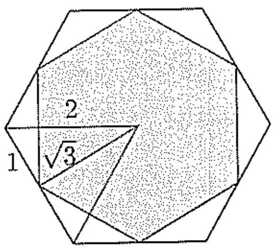


Alternative 1

The diagram above demonstrates that each of the six equilateral triangles which make up the shaded hexagon can be divided into three smaller triangles, each congruent to the triangle formed between the inner and outer hexagon. So for every three triangles in the small hexagon, there are four in the larger hexagon. Hence the areas of the two hexagons are in the ratio 3 : 4,

hence (A).

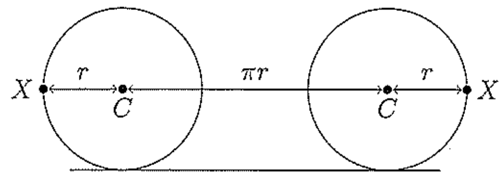
Alternative 2



Let a side of the larger hexagon have length 2. Then, from consideration of the equilateral triangle in the diagram above, we can show by Pythagoras that the distances from the centre of both hexagons to a side of each of the hexagons are in the ratio $\sqrt{3} : 2$ and hence the areas will be in the ratio 3 : 4,

hence (A).

19.



The distance between the centres of the circles is πr (half the circumference), then we add another r on each side to find the distance between the two positions of X,

hence (B).

20. The loser can win the most points in comparison to the winner by winning 5 rounds 4–0 and losing 6 rounds 3–4. Then the loser has won 38 points and the winner has won 24 points, a difference in the loser's favour of 14,

hence (E).

21. *Alternative 1*

Order the three even numbers A, B, C first: there are 6 ways of doing this.

There can't be two odd numbers together, so there are only four places for the three odd numbers — before A, between A and B, between B and C, and after C. There are four possible ways to choose three out of the four places (that is, any one of the four can be left out).

Finally, put the three odd numbers into these three places, which can be done in six ways. The total number of ways of doing this is $6 \times 4 \times 6 = 144$,

hence (D).

Alternative 2

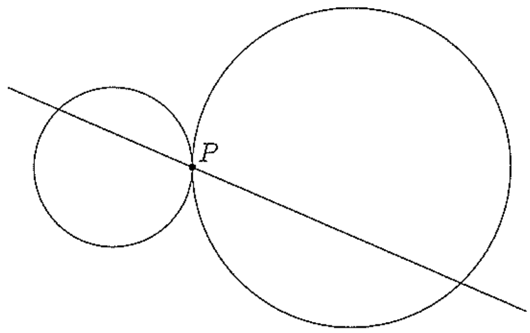
To ensure any adjacent pair of numbers has an even product, no two odd numbers can be together. This gives the following possibilities for placing odd and even numbers:

OEOEOE OEOEEO OEEOEO EOEOEO

For each, the odds and evens can be entered in $3! \times 3!$ ways, so total is $4 \times 6 \times 6 = 144$,

hence (D).

22.



Assume the area above is twice the area below the line. If the left circle is divided above : below in the ratio $a : b$, then the right circle is divided in the ratio $b : a$. Also the area of the right circle is four times the area of the left circle. Hence the total ratio above : below is $(a + 4b) : (b + 4a)$, which is $2 : 1$. So

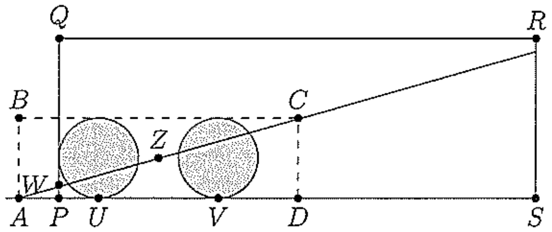
$$\begin{aligned} a + 4b &= 2(b + 4a) \\ 2b &= 7a \\ \frac{a}{b} &= \frac{2}{7} \end{aligned}$$

hence (D).

23. If $2n + 1$ is a divisor of $8n + 46$, it is also a divisor of $8n + 46 - 4(2n + 1) = 42$. Odd factors of 42 are $1, 3, 7, 21$, which have $n = 0, 1, 3, 10$. But $n > 0$ so there are three solutions,

hence (D).

24. (Also S22)



Since it passes through C, the centre of $PQRS$, the line will bisect the area of rectangle $PQRS$ as well as the area of the cut rectangle (that is, the area of the rectangle $PQRS$ minus the area of the two circles). So the line will bisect the combined area of the two circles, which means that by symmetry it will pass through the point Z, halfway between the centres of the two circles. Z will be the centre of the constructed rectangle $ABCD$ and the circles are symmetrically placed within this rectangle. $ABCD$ has length 7 and height 2, so diagonal AC has a gradient of $\frac{2}{7}$. AP is of length 1, so PW is $\frac{2}{7}$,

hence (A).

25. The table below shows which earls are satisfied with each choice of king and treasurer.

King \ Treasurer	A	B	C	D	E
A		D	BD	ACD	AD
B	DE		BE	ACE	AE
C	DE	E		ACE	AE
D	CDE	CE	BCE		ACE
E	DE	E	BE	ACE	

There are only seven out of the twenty cases where three of the earls are satisfied. Only when B is king and D is treasurer are the other three earls satisfied,

hence (B).

26. We have q, r, s single digits with

$$8000 + 400q + 40r + 4s = 1000s + 100r + 10q + 2$$

so that either $s = 8$ or $s = 9$. But $4s$ must end in $p = 2$ so that $s = 8$. Then

$$\begin{aligned} 8000 + 400q + 40r + 32 &= 8000 + 100r + 10q + 2 \\ 390q + 30 &= 60r \\ 13q + 1 &= 2r \end{aligned}$$

which must be even and less than 20. So the only solution is $q = 1$, $r = 7$. The 4-digit number is 2178 and qrs is 178,

hence (178).

27. (Also J30)
Each digit appears twice in the hundreds, the tens and the units column amongst the six numbers which can be formed. If the digits are a, b, c and the missing number is x, then $3231 + x = 222(a + b + c)$. Then $x = 222m - 3231$ must have three distinct digits that add to m. So $m > 3231/222 = 15\frac{1}{222}$. Try $m = 16, 17, \ldots$:

$$\begin{aligned} 222 \times 16 - 3231 = 3552 - 3231 = 321, &\quad 3 + 2 + 1 = 6 \quad \neq 16 \\ 222 \times 17 - 3231 = 3774 - 3231 = 543, &\quad 5 + 4 + 3 = 12 \quad \neq 17 \\ 222 \times 18 - 3231 = 3996 - 3231 = 765, &\quad 7 + 6 + 5 = 18 \end{aligned}$$

So the sixth number is 765,

hence (765).

28. (Also UP30 & S26)

Alternative 1

We tabulate the number of ways of getting a 'relatively close' sequence after n goals, according to the number of goals by which team A leads team B.

n	-2	-1	0	1	2
0			1		
1		1		1	
2	1		2		1
3		3		3	
4	3		6		3
5		9		9	
6	9		18		9
7		27		27	
8	27		54		27
9		81		81	
10	81		162		81
11		243		243	
12	243		486		243

There is a pattern that after $n = 2m$ goals, the numbers are $3^{m-1} + 2 \times 3^{m-1} + 3^{m-1} = 4 \times 3^{m-1}$. The total number of ways after 12 goals is $4 \times 3^5 = 972$,

hence (972).

Alternative 2

After every odd number of goals has been scored, the scores must differ by one. The leading team at this point cannot score both of the next two goals, so there are three different ways for the next two goals to be scored: AB, BA and BB (where team A is leading initially). We can then independently repeat this process for the next pair of goals, so that there are 3^n ways of scoring $2n+1$ goals after the first goal has been scored. Obviously there are two ways in which the first goal can be scored and there are also two ways in which the last goal can be scored if we finish with an even number of goals. So for 12 goals we have a first goal, followed by 5 pairs of goals, followed by the last goal, giving $2 \times 3^5 \times 2 = 972$ ways in which the goals can be scored,

hence (972).

29. We have $a = 2^m 3^n$ and $b = 2^p 3^q$ where $0 \le m \le p \le 6$ and $0 \le n \le q \le 6$. There are $\binom{8}{2} = 28$ ways of choosing m and p and, independent of that, $\binom{8}{2} = 28$ ways of choosing n and q. That is, if $m = 0$, p can have any value from 0 to 6; if $m = 1$, p can be 1 to 6, and so on. There are $7+6+5+4+3+2+1 = 28$ ways in which m and p can be chosen. A similar argument applies for b and q. Hence there are $28^2 = 784$ pairs in all,

hence (784).

30. (Also S28)

The number is a sum of different powers of 10, and the remainders on dividing powers of 10 by 37 are:

$$\begin{array}{rrll} & 1 = & 10^0 & = 10^3 - 27 \times 37 = 10^6 - 27027 \times 37 = \cdots \\ \text{or} & 10 = & 10^1 & = 10^4 - 270 \times 37 = 10^7 - 270270 \times 37 = \cdots \\ \text{or} & 26 = & 10^2 - 2 \times 37 & = 10^5 - 2702 \times 37 = 10^8 - 2702702 \times 37 = \cdots \end{array}$$

To make up a remainder of 18 on dividing by 37, we need combinations of 1, 10, 26 that add to one of 18, 55, 92, 129, etc. In particular we want $a + 10b + 26c = 18 + 37m$, where $a+b+c$ is as small as possible.

When $m = 0$, the smallest $a+b+c$ is 9, where $8 + 10 \times 1 = 18$.

When $m = 1$, the smallest $a+b+c$ is 5, where $3 \times 1 + 2 \times 26 = 55$.

When $m = 2$, the smallest $a+b+c$ is 6, where $4 \times 10 + 2 \times 26 = 92$.

When $m \ge 3$, $18 + 37m \ge 129 > 4 \times 26$ and so $a+b+c \ge 5$.

Consequently the smallest number of 1s possible is 5 (when $m = 1$), and an example of such a number is $1101101 = 18 + 29759 \times 37$,

hence (5).