MY COPY WITH SOLUTIONS

CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Extension 2 Mathematics

Half Yearly 2015

Time allowed: 2 hours

Name: ______________________________ Teacher: Ms Strilakos

Instructions:

- All questions should be attempted.
- Use Multiple Choice Answer Sheet for Section I.
- Use separate Answer Booklet(s) for each question.
- Put your name on every booklet.
- Show ALL necessary working in Section 2
- Do not work in columns.
- Please write on one side of each page only.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.

	Multiple Choice	Q6	Q7	Q8	Total
E3	/5	/14	/13	/5	/37
E4				/9	/9
E6		/5	/10	/6	/21
Total	/5	/19	/23	/20	/67

Section I

Multiple Choice - 5 marks

Use the multiple-choice answer sheet for Questions 1 – 5

1. The eccentricity of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 3$ is:

A $\frac{\sqrt{13}}{3}$ B $\frac{\sqrt{13}}{2}$ C $\frac{\sqrt{5}}{2}$ D $\frac{\sqrt{5}}{3}$

2. Given $z = 3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, which expression is equal to $(\bar{z})^{-1}$?

A $\frac{1}{3}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$ B $3(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$

C $\frac{1}{3}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ D $3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

3. The equation of the tangent to the curve $x^2y + 2x - 4xy = 0$

at the point (1, 2) is:

A $y = 2x + 4$ B $x + y - 3 = 0$

C $y = 0$ D $2x + 3y - 8 = 0$

4. The Cartesian equation of the curve whose parametric equations are

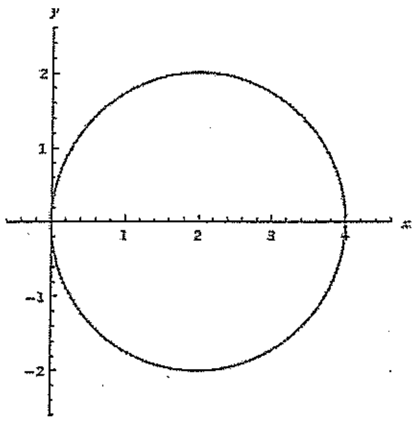
$x = 4cos\theta$ and $y = 6sin\theta$ is:

A $\frac{x^2}{36} + \frac{y^2}{16} = 1$ B $\frac{x^2}{16} - \frac{y^2}{36} = 1$

C $\frac{x^2}{4} + \frac{y^2}{6} = 1$ D $\frac{x^2}{4} + \frac{y^2}{9} = 4$

5. Which graph best represents the curve $y^2 = x(x-4)$?

A



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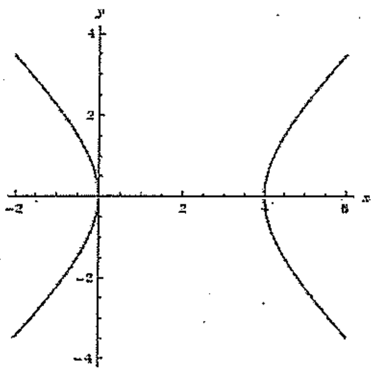

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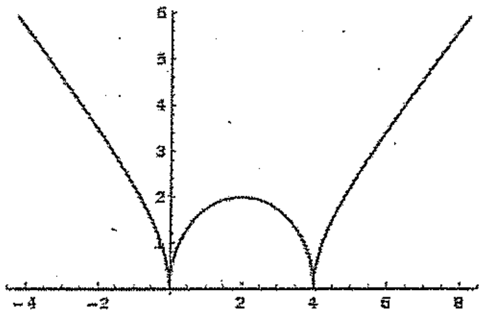

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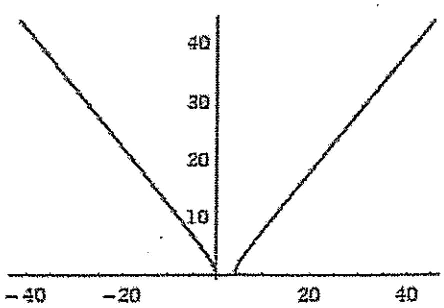


Section II

62 marks

Attempt Questions 6 – 8

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 6 – 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (19 marks) Use a SEPARATE writing booklet.

(a) (i) Find the Cartesian equation of the locus represented by $2|z| = 3(z + \bar{z})$ **2**

(ii) Sketch the locus on an Argand diagram. **1**

(b) (i) Show that $z = 1 + i$ is a root of the equation $z^2 - (3 - 2i)z + (5 - i) = 0$ **1**

(ii) Find the other root of the equation. **2**

(c) (i) Show that the x coordinates of the stationary points on the graph $y = \{f(x)\}^2$ are the same as the x coordinates of the stationary points or intercepts on the x axis of the graph $y = f(x)$. **5**

(ii) Given that $f(x) = sin^2x - \frac{1}{2}$, $0 \leq x \leq \pi$, on the same set of axes, and without using further calculus, sketch the graphs of $y = f(x)$ and $y = \{f(x)\}^2$.

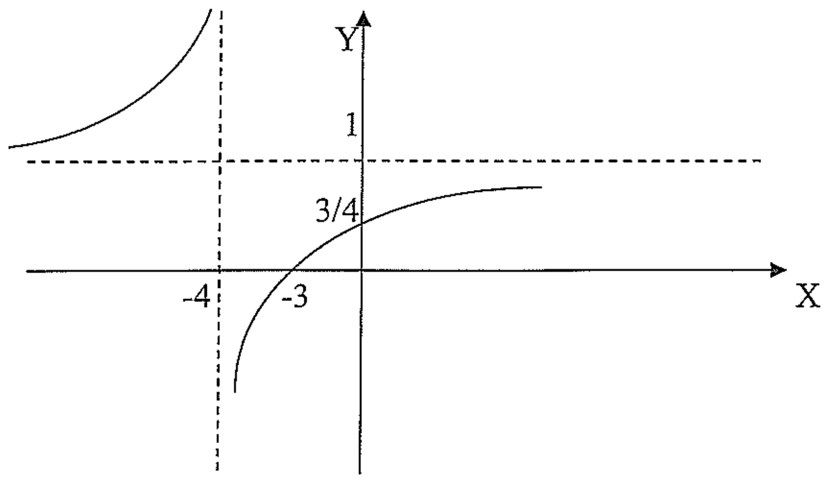
(d) (i) The point $P(a\, sec\emptyset, b\, tan\emptyset)$ is a point on the hyperbola $x = a\, sec\theta, y = b\, tan\theta$. **8**

If the line $y = mx + c$ is a tangent to the hyperbola at P, show that $m^2a^2 - b^2 = c^2$.

(ii) Find the equations of the tangents from the point $(-1, 3)$ to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1$

Question 7 (23 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the graph of $y = \dfrac{x+3}{x+4}$



Use the graph to:

(i) Find the largest possible domain of the function $y = \sqrt{\dfrac{x+3}{x+4}}$ **2**

(ii) Find the values of x for which the function $y = x - \log_e(x+4)$ is increasing. **2**

(iii) Copy the above diagram into your answer book, and on the same set of axes sketch the graph of $y = \left(\dfrac{x+3}{x+4}\right)^2$ clearly indicating which is each graph and labelling any axes intercepts. **2**

(iv) State the nature of the point (-3, 0) in part (iii). **1**

(v) On a new set of axes sketch the graph of $y^2 = \dfrac{x+3}{x+4}$ clearly showing all asymptotes and labelling axes intercepts. **2**

(vi) State the nature of the point (-3, 0) in part (v) **1**

(b) (i) Show that the locus specified by $|z-2| = 2(Rez - \frac{1}{2})$ is a branch of the hyperbola $\frac{x^2}{1} - \frac{y^2}{3} = 1$. **2**

(ii) Sketch the locus, and find the set of possible values of each of $|z|$ and $\arg z$ for a point on the locus. **4**

(c) z_1 and z_2 are two complex numbers such that $\frac{z_1+z_2}{z_1-z_2} = 2i$

(i) On an Argand diagram, show the vectors representing $z_1,\ z_2,\ z_1+z_2$, and z_1-z_2. **1**

(ii) Show that $|z_1| = |z_2|$ **2**

(iii) If α is the magnitude of the angle between the vectors representing z_1 and z_2, show that $\tan\frac{\alpha}{2} = \frac{1}{2}$ **2**

(iv) Show that $z_2 = \frac{1}{5}(3+4i)z_1$ **2**

Question 8 (20 marks) Use a SEPARATE writing booklet.

(a) Given $\arg(z-2) = 2\arg z$ **5**

(i) Show on an Argand diagram vectors representing z and $z-2$.

(ii) If the point P represents z, O is the origin, and Q has coordinates $(2,0)$ in this Argand diagram, what is the nature of ΔOPQ for non-real z?

(iii) Deduce that if z is non-real, then P lies on a circle and state its centre and radius.

(iv) On a new diagram, sketch the locus in the Argand diagram of a point representing z satisfying $\arg(z-2) = 2\,arg z$, for both real and non-real z.

(b) (i) Sketch the graph of $y = x^2 + \frac{2}{x}$, showing any intercepts on the coordinate axes, asymptotes, stationary points or points of inflexion. **6**

(ii) If the equation $x^2 + \frac{2}{x} - k = 0$ has exactly two different real solutions, find the value of k and the real solutions of the equation.

(c) (i) Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $T(2t, \frac{2}{t})$ has equation $x + t^2y = 4t$. **9**

(ii) This tangent cuts the x axis at the point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2x - y = 4t^3$.

(iii) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates $M\,(2t, -2t^3)$.

(iv) Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply.

END OF PAPER

MULTIPLE CHOICE

Q.1. B Q.2. C Q.3. D Q4. D Q5. B

Q6 (a) (i) $2|z| = 3(z+\bar{z})$

Let $z = x+iy$ $\quad |z| = \sqrt{x^2+y^2}$ $\quad \bar{z} = x-iy$

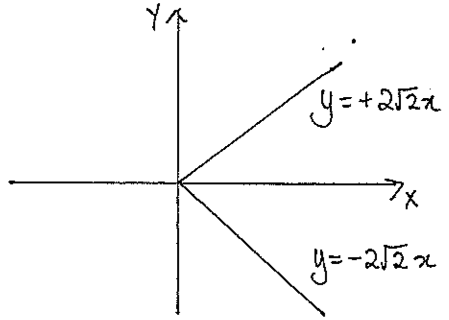
$\therefore$ $2|z| = 3(z+\bar{z})$ becomes

$2\sqrt{x^2+y^2} = 3(x+iy+x-iy) = 6x$ 1

Sq. b.s. $4(x^2+y^2) = 36x^2$ $\therefore$ $4y^2 = 32x^2$ i.e. $y^2 = 8x^2$. 1

$y = \pm 2\sqrt{2}\,x$, $x \geq 0$.

(ii)



1 (3)

(b) (i) Subst. $z = 1+i$ in $z^2 - (3-2i)z + (5-i) = 0$. to obtain

$(1+i)^2 - (3-2i)(1+i) + (5-i) = \cancel{1} + 2i - \cancel{1} - [3+i+2] + 5 - i$

$= 2i - i - 5 + 5 - i = 0$ 1

$\therefore$ $z = 1+i$ is a root.

(ii) Sum of roots $\alpha+\beta = \dfrac{-b}{a} = 3-2i$. 1 (3)

Let the second root be $a+ib$: $\quad 1+i+a+ib = 3-2i$ 1

Equating real and imag. parts: $\quad 1+a = 3$ $\quad$ and $(1+b) = -2$ $\therefore$ $b = -3$.

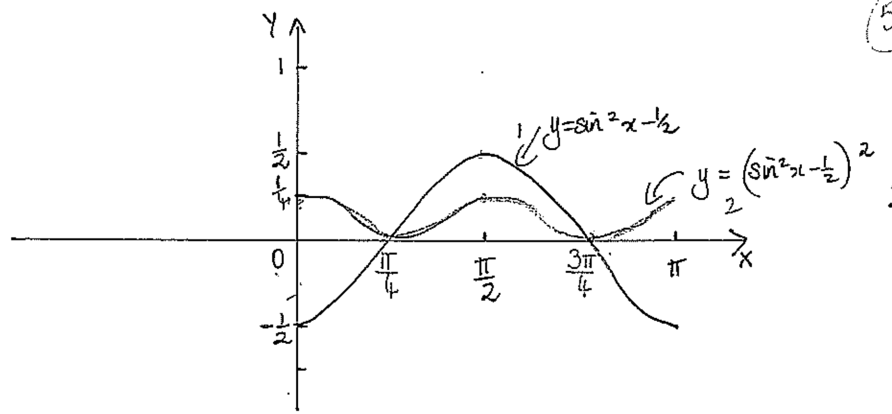
$\therefore a = 2$

Thus the other root is $2-3i$.

(c) (i) $y = (f(x))^2$.

$\dfrac{dy}{dx} = 2f(x).f'(x) = 0$ for stat. points when $f(x) = 0$ and when $f'(x) = 0$. 2

(ii) $f(x) = \sin^2 x - \frac{1}{2}$ $\quad 0 \leq x \leq \pi$. (5)



$y=mx+c$ is tangent to the hyperbola.

Now, $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ $\qquad x = a\sec\theta$, $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$y = b\tan\theta$, $\frac{dy}{d\theta} = b\sec^2\theta$.

$\therefore \frac{dy}{dx} = b\sec^2\theta \cdot \frac{1}{a\sec\theta\tan\theta}$

i.e. $m = \frac{b\sec\theta}{a\tan\theta} = \frac{b\sec\phi}{a\tan\phi}$ at P. $\quad$ —(1)

Also, since $y=mx+c$ at the point P,

$b\tan\phi = ma\sec\phi + c$. $\quad$ —(2)

$y \;=\; m\;x\;+c$.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\qquad$ and $y=mx+c$
$\Rightarrow y^2 = m^2x^2+c^2+2mcx$
i.e. $y^2 = (mx+c)^2$.

$\frac{x^2}{a^2} - \frac{m^2x^2+c^2+2mcx}{b^2} = 1$

$\times a^2b^2$: $\quad b^2x^2 - a^2m^2x^2 - a^2c^2 - 2a^2mcx = a^2b^2$ $\qquad$ (4)

$\Rightarrow x^2(b^2-a^2m^2) - 2a^2mcx - (a^2c^2+a^2b^2) = 0$.

If $y=mx+c$ is a tangent, then $\Delta = 0$ i.e. $4a^4m^2c^2 + 4(b^2-a^2m^2)(a^2c^2+a^2b^2) = 0$.

i.e. $\cancel{4a^4m^2c^2} + 4(a^2b^2c^2 + a^2b^4 - \cancel{a^4m^2c^2} - a^4b^2m^2) = 0$.

i.e. $a^2b^2(c^2+b^2-a^2m^2) = 0$.

$\therefore a^2m^2 - b^2 = c^2$ as required, since $a,b \neq 0$.

From (1) $\quad ma\tan\phi - b\sec\phi = 0$ $\quad$ —(1)'

$b\tan\phi - ma\sec\phi = c$ $\quad$ —(2)'.

$((1)')^2 \quad m^2a^2\tan^2\phi - b^2\sec^2\phi - 2mab\tan\phi\sec\phi = 0$.

$((2)')^2 \quad b^2\tan^2\phi - 2mab\tan\phi\sec\phi + m^2a^2\sec^2\phi = c^2$. $\qquad$ (4)

$((1)')^2 - ((2)')^2$ gives

$m^2a^2(\tan^2\phi - \sec^2\phi) + b^2(\sec^2\phi - \tan^2\phi) = -c^2$. $\quad$ —(3)

Now, $\tan^2\phi - \sec^2\phi = -1$. and $\sec^2\phi - \tan^2\phi = 1$

(3) becomes $\quad -m^2a^2 + b^2 = -c^2$.

i.e. $m^2a^2 - b^2 = c^2$ as required.

(ii) For equations of tangents from the point $(-1,3)$

to $\frac{x^2}{4} - \frac{y^2}{15} = 1$, $\quad a^2=4,\ b^2=15$.

Tangents have equations of the form $y=mx+c$.

$\therefore 3 = -m+c$ i.e. $c = m+3$ $\qquad$ Since $(-1,3)$ lies on both tangents.

Also, we know that $m^2a^2 - b^2 = c^2$

$\therefore 4m^2 - 15 = c^2$. $\qquad c^2 = (m+3)^2$

$\therefore 4m^2 - 15 = (m+3)^2$

i.e. $4m^2 - 15 = m^2 + 6m + 9$.

$3m^2 - 6m - 24 = 0$.

$(3m + 6)(m - 4) = 0$. (4)

$\therefore m = -2$ or $m = 4$.

If $m = -2$, $c = 1$. If $m = 4$, $c = 7$.

$\therefore$ Eqns of tangents are: $y = -2x + 1$ and $y = 4x + 7$.

Q.8. (a) (i) $\frac{x+3}{x+4} \geq 0$ $\therefore$ Domain is $x \geq -3$ or $x < -4$ (2)

i.e. $x+3 \geq 0$ and $x+4 > 0$ or $x+3 \leq 0$ and $x+4 < 0$.
$x \geq -3$ & $x \geq -4$ i.e. $x \geq -3$ or $x \leq -3$ & $x < -4$ i.e. $x < -4$

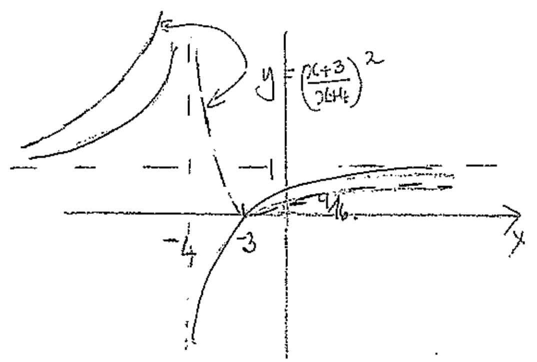
(ii) $\frac{dy}{dx} > 0$ for increasing values of the function. $y = x - \log_e(x+4)$

Now, $\frac{dy}{dx} = 1 - \frac{1}{x+4} = \frac{x+3}{x+4}$. (2)

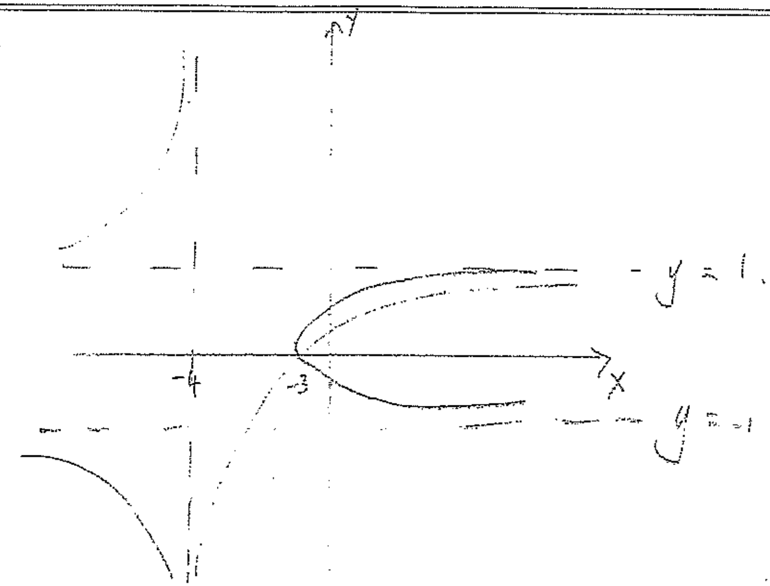
So wherever this graph is positive we have $\frac{dy}{dx} > 0$.

i.e. $\underline{x > -3}$ since $\log_e(x+4)$ is not defined for $x \leq -4$.

(iii) (2)



(iv) (2)



(v) in (iii) $(-3, 0)$ is a minimum t.p. (1).

in (iv) $(-3, 0)$ is a critical point where $\frac{dy}{dx}$ is not defined. (1)

Q8(b) (i) $|z-2| = 2(\text{Re } z - \frac{1}{2})$

Let $z = x+iy$

$\sqrt{(x-2)^2+y^2} = 2(x-\frac{1}{2})$ $\quad x \geq \frac{1}{2}$.

Sq. b.s. $(x-2)^2+y^2 = 4(x-\frac{1}{2})^2$

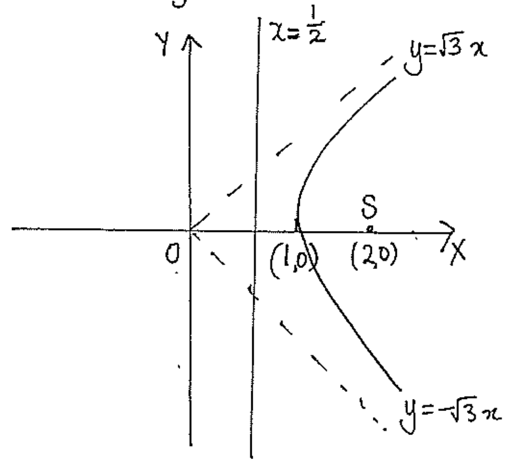
$x^2-4x+4+y^2 = 4x^2-4x+1$

$3x^2-y^2 = 3$

$x^2 - \frac{y^2}{3} = 1$

2.

(ii)



$a=1, \quad b=\sqrt{3}$

asymptotes: $y = \pm\frac{b}{a}x$

i.e. $y = \pm\sqrt{3}\,x$.

focus: $(ae, 0)$

$b^2=a^2(e^2-1)$

$3 = e^2-1$, $e=2$

$\therefore$ focus is at $(2,0)$

vertex $a=1$.

Directrix: $x = \frac{a}{e} = \frac{1}{2}$

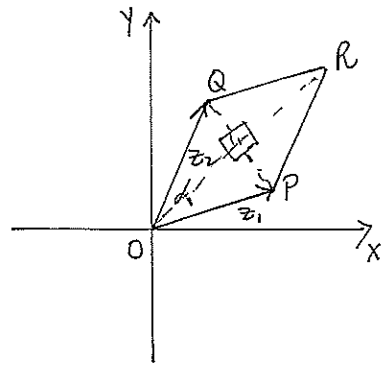
Argument is between $\tan\theta = \sqrt{3}$ and $\tan\theta = -\sqrt{3}$

i.e. $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$.

$|z| \geq 1$, since closest pt. to origin is $(1,0)$.

(c) $\frac{z_1+z_2}{z_1-z_2} = 2i$

(i)



Let $\vec{OP}$ be z_1 and $\vec{OQ}$ be z_2

$\therefore$ $\vec{OR}$ is z_1+z_2 and $\vec{QP}$ is z_1-z_2.

(ii) To show $|z_1|=|z_2|$

$\frac{z_1+z_2}{z_1-z_2} = 2i$ $\quad\therefore z_1+z_2 = 2i(z_1-z_2)$.

$\therefore \vec{OR} = 2i\,\vec{QP}$ $\quad$ i.e. $\vec{OR} \perp \vec{QP}$.

$\therefore$ OPRQ is a rhombus, since $\vec{OQ} \parallel \vec{PR}$ $\quad\therefore |\vec{OP}| = |\vec{OQ}|$

i.e. $|z_1| = |z_2|$.

(iii) Since the diagonals of a rhombus bisect the angles at the vertices,

$\angle ROP = \frac{\alpha}{2}$.

Note: $\tan\frac{\alpha}{2} = \frac{\text{opp}}{\text{adj}}$.

The diagonals also bisect at right angles.

$\therefore \tan\frac{\alpha}{2} = \frac{\frac{1}{2}|\vec{QP}|}{\frac{1}{2}|\vec{OR}|}$ $\quad$ But $\vec{OR} = 2i\,\vec{QP}$ $\quad\therefore |\vec{OR}| = 2|\vec{QP}|$

$\therefore \tan\frac{\alpha}{2} = \frac{\frac{1}{2}|\overrightarrow{QP}|}{\frac{1}{2}\times 2|\overrightarrow{QP}|} = \frac{1}{2}$ as required.

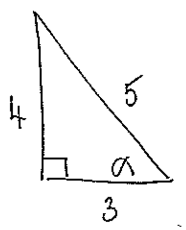
(iv) To show that $z_2 = \frac{1}{5}(3+4i)z_1$,

since $|z_2| = |z_1|$,

$z_2 = (\cos\alpha + i\sin\alpha)z_1$ i.e. z_1 is rotated anticlockwise through α.

Now, $\tan\frac{\alpha}{2} = \frac{1}{2}$ and $\tan\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$

$= \dfrac{2\times\frac{1}{2}}{1-(\frac{1}{2})^2} = \dfrac{1}{3/4} = \dfrac{4}{3}$



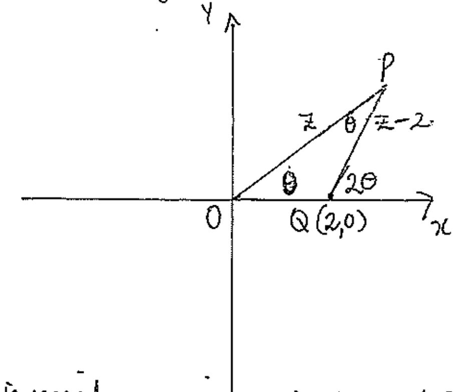
$\therefore \cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$.

$\therefore z_2 = \left(\frac{3}{5} + i\frac{4}{5}\right)z_1$

$z_2 = \frac{1}{5}(3+4i)z_1$ as required.

Q9. (a) $\arg(z-2) = 2\arg z$.

(i)



1

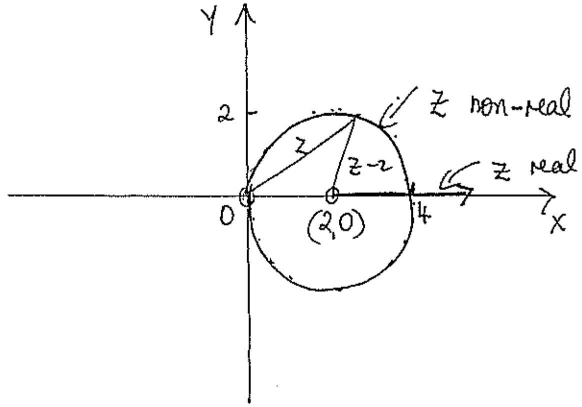
(ii) ΔOPQ is isosceles, since $\angle PQX = \angle POQ + \angle OPQ$
i.e. $2\theta = \theta + \angle OPQ$ $\therefore \angle OPQ = \theta$. 1

Thus P does not lie on OQ, given z is not real. (OQ lies on the real axis).

(iii) Since ΔOPQ is isosceles, $|z-2| = |\overrightarrow{OQ}| = 2$ 1

$\therefore$ P lies on a circle with its centre at (2,0) and radius 2 units for z non-real.

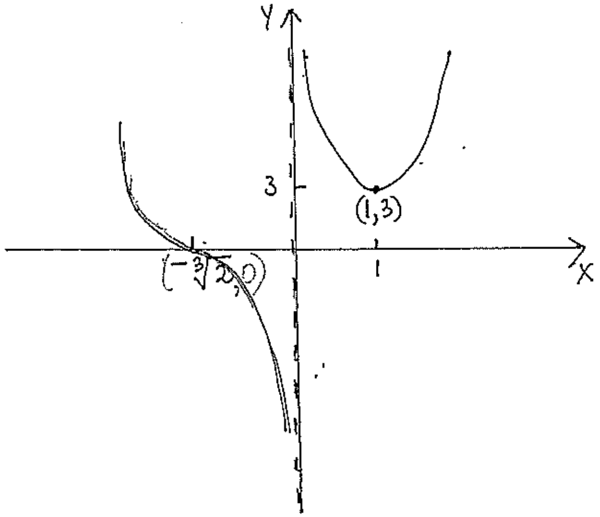
(iv)



z non-real $(x-2)^2+y^2=4$ $x\neq 0$.

z real, $x>2$ $y=0$. 2

Q9(b) (i)



For Stat. Pts.

$y = x^2 + \frac{2}{x} = \frac{x^3+2}{x}$

$\frac{dy}{dx} = 2x - \frac{2}{x^2} = 0$

When $2x^3 - 2 = 0$

$x^3 - 1 = 0$ 1

$x = 1.$

for $x<1$, $\frac{dy}{dx} < 0$
for $x>1$, $\frac{dy}{dx} > 0$. $\Big\}$ min^m t.p at $(1, 3)$

For Axes Intercepts.

$x \neq 0$, $y = 0$ $x = \sqrt[3]{-2} = -\sqrt[3]{2}$ 1

Asymptotic Behav.

As $x \to -\infty$, $y \to (x^2)^-$

As $x \to +\infty$, $y \to (x^2)^+$

As $x \to 0^+$, $y \to +\infty$

As $x \to 0^-$, $y \to -\infty$

(4).

Inflexion

$\frac{d^2y}{dx^2} = 2 + \frac{4}{x^3} = 0$

When $\frac{2x^3+4}{x^3} = 0$

$x^3 + 2 = 0$

$x = -\sqrt[3]{2}$

$x < -\sqrt[3]{2}$, $\frac{d^2y}{dx^2} > 0$
$x > -\sqrt[3]{2}$, $\frac{d^2y}{dx^2} < 0$ $\Big\}$ ∴ Inflexion at $(-\sqrt[3]{2}, 0)$ 1

Q9(b)(ii)

$x^2 + \frac{2}{x} - k = 0$

$x^2 + \frac{2}{x} = 0$ has one real sol^n.

To have exactly two, the parabolic part of curve needs to <u>touch</u> x-axis at x=1

∴ $k = +3$.

One real sol^n is ∴ $x = 1$. 1

To find the other, $x^2 + \frac{2}{x} - 3 = 0$

$x^3 + 2 - 3x = 0$.

Double root $x=1$, for touching at $x=1$, $y=0$. (2)

The roots of the cubic are 1, 1 and α.

Sum of roots is $-\frac{b}{a} = 1 + 1 + \alpha = 0$ ∴ $\alpha = -2$.

Hence the real solutions are $x=1$ and $x=-2$. 1

Q9 (c)(i) $xy=4$ $T\left(2t, \frac{2}{t}\right)$

$$y + x\frac{dy}{dx} = 0$$

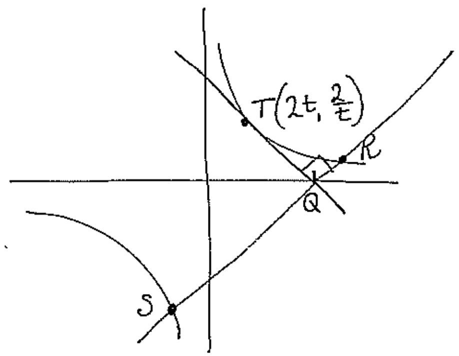
$$\frac{dy}{dx} = -\frac{y}{x} \qquad \text{At } T,\ \frac{dy}{dx} = -\frac{2}{2t^2} = -\frac{1}{t^2}. \qquad 1$$

$\therefore$ eqn of tangent at T is $y - \frac{2}{t} = -\frac{1}{t^2}(x-2t)$

$\times t^2$ $t^2y - 2t = -x + 2t$

i.e. $x + t^2y = 4t$ — ① as required. 1

(ii) Tangent cuts x-axis at Q.



m of $\perp$ to tangent is t^2.

Q occurs at $y=0$ for ①

$\therefore$ $x = 4t$, $y = 0$. at Q 1

$\therefore$ eqn of $\perp$ at Q is:

$$y = t^2(x-4t) \quad \text{i.e. } y = t^2x - 4t^3. \quad \text{— ②} \qquad 1$$

(iii) $\perp$ at Q cuts $xy=4$ at R & S.

For coords of R & S: solve $xy=4$ and ②

i.e. $y = \frac{4}{x}$ & $y = t^2x - 4t^3$

$$\frac{4}{x} = t^2x - 4t^3 \qquad 1$$

$\times x$ $4 = t^2x^2 - 4t^3x$

Q9(c)(iii) (cont) (long way)

$$x^2t^2 - 4t^3x - 4 = 0$$

$$x = \frac{4t^3 \pm \sqrt{16t^6 + 16t^2}}{2t^2}$$

$$= \frac{4t^3 \pm \sqrt{16t^2(t^4+1)}}{2t^2}$$

$$= \frac{4t^3 \pm 4t\sqrt{t^4+1}}{2t^2}$$

$$= \frac{2t^2 \pm 2\sqrt{t^4+1}}{t} \qquad 1$$

We see the midpt. of these two x-values is $x=2t$.

At $x=2t$, from ② $y = t^2 \times 2t - 4t^3 = -2t^3$.

$\therefore$ M has coords $(2t, -2t^3)$. 1

(iv) $x = 2t$, $t = \frac{x}{2}$. $y = -2t^3 = -2\frac{x^3}{8} = \frac{-x^3}{4}$ 1

This excludes the case where $t=0$, i.e. $(0,0)$. 1