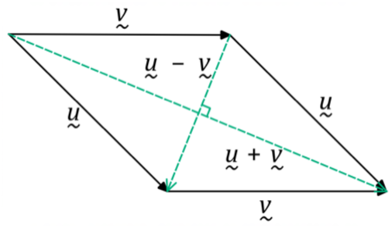
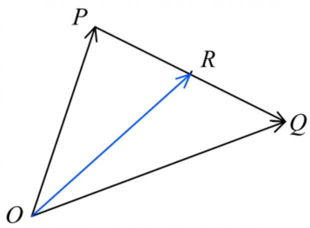
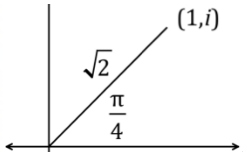
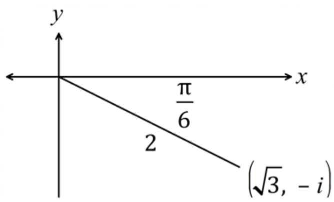
ACE Examination Paper 4
Year 12 Mathematics Extension 2 Yearly Examination
Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1	$z \div w = \dfrac{3-4i}{\sqrt{3}+i} \times \dfrac{\sqrt{3}-i}{\sqrt{3}-i} = \dfrac{3\sqrt{3}-4}{4} + \dfrac{\left(-4\sqrt{3}-3\right)i}{4}$	1 Mark: A
2	$\dfrac{d^2x}{dt^2} = 25 - 5x$ $= -5(x-5)$ Centre of motion at $x = 5$ (SHM $\dfrac{d^2x}{dt^2} = -n^2(x-b)$ with centre of motion at $x = b$)	1 Mark: B
3	The diagonals of a rhombus are perpendicular.  Two vectors are perpendicular if and only if $\underset{\sim}{u}.\underset{\sim}{v} = 0$ $(\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v}) = 0$	1 Mark: C
4	The contrapositive of the statement 'If A then B' is 'If not B then not A'. The contrapositive is true if and only if the statement itself is also true. (not B) $\Rightarrow$ (not A)	1 Mark: C
5	$v = 2\sqrt{1-x^2}$ $v^2 = 4(1-x^2)$ $\dfrac{1}{2}v^2 = 2(1-x^2)$ $a = \dfrac{d}{dx}(2(1-x^2))$ $a = -4x$	1 Mark: D
6	Let $x = \sin\theta$ with $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$ $dx = \cos\theta d\theta$ $(1-x^2)^{\frac{3}{2}} = (1-\sin^2\theta)^{\frac{3}{2}}$ $= (\cos^2\theta)^{\frac{3}{2}} = \cos^3\theta$ $\displaystyle\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}dx = \int \frac{\sin^2\theta}{\cos^3\theta}\cos\theta d\theta = \int \tan^2\theta\, d\theta$ $\displaystyle= \int (\sec^2\theta - 1)d\theta$ $= \tan\theta - \theta + C$ $= \dfrac{x}{(1-x^2)^{\frac{1}{2}}} - \sin^{-1}x + C$	1 Mark: C

7	Using the substitution $u = \ln x$ $du = \frac{1}{x}dx$ $\int \frac{\sec^2(\ln x)}{x}dx = \int \sec^2 u du$ $= \tan u + C$ $= \tan(\ln x) + C$	1 Mark: A
8	Let $(a+ib)^2 = 8+6i$ $a^2 + 2abi - b^2 = 8 + 6i$ $a^2 - b^2 = 8$ ① $2ab = 6$ ② Solving equations ① and ② simultaneously $a = 3$ and $b = 1$. $\therefore$ Root is $3 + i$ $a = -3$ and $b = -1$. $\therefore$ Root is $-3 - i$	1 Mark: A
9	Equations of projectile motion $x = Vt\cos\alpha$ $30 = 20t\cos\alpha$ $t = \frac{3}{2\cos\alpha}$ ① $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ $8.75 = -\frac{1}{2} \times 10 \times t^2 + 20t\sin\alpha$ $35 = -20t^2 + 80t\sin\alpha$ ② Substituting equation ① into equation ② $35 = -20 \times \left(\frac{3}{2\cos\alpha}\right)^2 + 80 \times \left(\frac{3}{2\cos\alpha}\right) \times \sin\alpha$ $35 = -45\sec^2\alpha + 120\tan\alpha$ $7 = -9(\tan^2\alpha + 1) + 24\tan\alpha$ $9\tan^2\alpha - 24\tan\alpha + 16 = 0$ $(3\tan\alpha - 4)^2 = 0$ $3\tan\alpha = 4$ $\tan\alpha = \frac{4}{3}$ $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$	1 Mark: D
10	$(2+2i)z^2 + 8iz - 4(1-i)$ $\Delta = (8i)^2 - 4 \times (2+2i) \times -4(1-i)$ $= -64 + 16(2 - 2i + 2i + 2)$ $= 0$	1 Mark: B

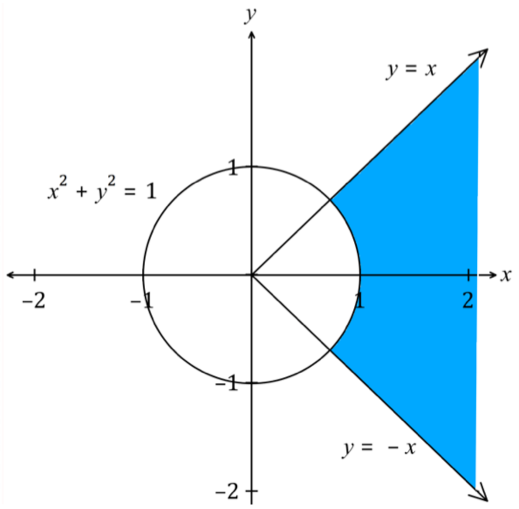
Section II

	Solution	**Criteria**
11(a) (i)	$z_1 + \overline{z_2} = 2i + (1 - 3i)$ $= 1 - i$	1 Mark: Correct answer.
11(a) (ii)	$z_1\, z_2 = 2i(1 + 3i)$ $= -6 + 2i$	1 Mark: Correct answer.
11(a) (iii)	$\dfrac{1}{z_2} = \dfrac{1 - 3i}{(1 + 3i)(1 - 3i)}$ $= \dfrac{1 - 3i}{1 + 9}$ $= \dfrac{1}{10} - \dfrac{3}{10}i$	1 Mark: Correct answer.
11(b)	$\displaystyle\int \frac{1}{\sqrt{5 + 4x - x^2}}dx = \int \frac{1}{\sqrt{9 - (x^2 - 4x + 4)}}dx$ $\displaystyle= \int \frac{1}{\sqrt{9 - (x - 2)^2}}dx$ $= \sin^{-1}\left(\dfrac{x - 2}{3}\right) + C$	2 Marks: Correct answer. 1 Mark: Completes the square.
11(c) (i)	$\overrightarrow{OR} = \overrightarrow{OP} + \dfrac{1}{2}\overrightarrow{PQ}$ $= p + \dfrac{1}{2}(q - p)$ $= \dfrac{1}{2}(p + q)$ 	2 Marks: Correct answer. 1 Mark: Shows some understanding.
11(c) (ii)	$\overrightarrow{PR} = -2\overrightarrow{PQ}$ $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$ $= \overrightarrow{OP} - 2\overrightarrow{PQ}$ $= p - 2(q - p)$ $= 3p - 2q$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
11(c) (iii)	$\overrightarrow{OR} = \overrightarrow{OP} + \dfrac{1}{3}\overrightarrow{PQ}$ $= p + \dfrac{1}{3}(q - p) = \dfrac{1}{3}(2p + q)$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
11(d)	Use the substitution $u = x^2 + 2x + 5$ $\dfrac{du}{dx} = 2x + 2$ $0.5du = (x + 1)dx$ When $x = 2$ then $u = 13$ and when $x = 3$ then $u = 20$ $\displaystyle\int_2^3 \frac{x + 1}{\sqrt{x^2 + 2x + 5}}dx = \int_{13}^{20} \frac{0.5du}{u^{\frac{1}{2}}}$ $= \left[u^{\frac{1}{2}}\right]_{13}^{20}$ $= \sqrt{20} - \sqrt{13}$	2 Marks: Correct answer. 1 Mark: Finds the primitive function or sets up the integration using substitution.

11(e)	$z^n = (\cos\theta + i\sin\theta)^n$ $\quad = \cos n\theta + i\sin n\theta$ $\dfrac{1}{z^n} = (\cos\theta + i\sin\theta)^{-n}$ $\quad = \cos n\theta - i\sin n\theta$ $\therefore z^n + \dfrac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ $\quad = 2\cos n\theta$	2 Marks: Correct answer. 1 Mark: Uses De Moivre's theorem.
12(a)	If $\|a\| < 1$ and $\|b\| < 1$ then $(1 - a^2)(1 - b^2) > 0$ $1 - a^2 - b^2 + a^2b^2 > 0$ $a^2 + b^2 + 2ab < 1 + a^2b^2 + 2ab$ $(a + b)^2 < (1 + ab)^2$ $\therefore \|a + b\| < \|1 + ab\|$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
12(b) (i)	$\dfrac{z_1}{z_2} = \dfrac{1 + i}{\sqrt{3} - i} \times \dfrac{\sqrt{3} + i}{\sqrt{3} + i}$ $\quad = \dfrac{\sqrt{3} - 1}{4} + i\dfrac{\sqrt{3} + 1}{4}$	1 Mark: Correct answer.
12(b) (ii)	$z_1 = \sqrt{2}(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4})$  $z_2 = 2\left\{\cos\left(-\dfrac{\pi}{6}\right) + i\sin\left(-\dfrac{\pi}{6}\right)\right\}$  	2 Marks: Correct answer. 1 Mark: Finds one of the points in modulus-argument form.
12(b) (iii)	$\dfrac{z_1}{z_2} = \dfrac{\sqrt{2}}{2}\left\{\cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right) + i\sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right)\right\}$ $\quad = \dfrac{\sqrt{3} - 1}{4} + i\dfrac{\sqrt{3} + 1}{4}$ Equating the real parts $\dfrac{1}{\sqrt{2}}\cos\left(\dfrac{5\pi}{12}\right) = \dfrac{\sqrt{3} - 1}{4}$ $\therefore \cos\left(\dfrac{5\pi}{12}\right) = \dfrac{\sqrt{6} - \sqrt{2}}{4}$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
12(c)	Integration by parts $\displaystyle\int \frac{\ln x}{x^2}\, dx = \int \ln x \times \frac{d}{dx}\left(-\frac{1}{x}\right) dx$ $\displaystyle\quad = -\frac{\ln x}{x} - \int \frac{d}{dx}\ln x \times -\frac{1}{x} dx$ $\displaystyle\quad = -\frac{\ln x}{x} - \frac{1}{x} + C$ $\displaystyle\quad = -\frac{\ln x + 1}{x} + C$	2 Marks: Correct answer. 1 Mark: Sets up integration by parts.

12(d)	Using the conjugate root theorem $1+i$ and $1+i$ are both roots of the equation $z^3+pz+q=0$ $(1+i)+(1-i)+\alpha=0$ (sum of the roots) $\alpha=-2$ $(1+i)\times(1-i)\times(-2)=-q$ (product of the roots) $(1+1)\times -2=-q$ $q=4$ $(1+i)(1-i)+(1-i)(-2)+(1+i)(-2)=p$ $p=-2$ $\therefore$ $p=-2$ and $q=4$.	2 Marks: Correct answer. 1 Mark: Recognises the conjugate root theorem.
12(e) (i)	$\dfrac{1}{(x+1)(x^2+2)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+2}$ $1=A(x^2+2)+(Bx+C)(x+1)$ Let $x=-1$ $1=3A$ or $A=\dfrac{1}{3}$ Equating the coefficients of x^2. $0=A+B$ $B=-\dfrac{1}{3}$ Equating the constants $1=2A+C$ $C=\dfrac{1}{3}$ $\therefore A=\frac{1}{3}, B=-\frac{1}{3}$ and $C=\frac{1}{3}$.	2 Marks: Correct answer. 1 Mark: Finds one of the pronumerals or shows some understanding.
12(e) (ii)	$\displaystyle\int\frac{1}{(x+1)(x^2+2)}dx=\int\frac{\frac{1}{3}}{x+1}+\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+2}dx$ $\displaystyle=\int\frac{1}{3}\times\frac{1}{x+1}-\frac{1}{3}\times\frac{x}{x^2+2}+\frac{1}{3}\times\frac{1}{x^2+2}dx$ $\displaystyle=\frac{1}{3}\ln\lvert x+1\rvert-\frac{1}{6}\ln(x^2+2)+\frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals
13(a)	Step 1: Prove the statement true for $n=1$ $7^1+15^1=22$ which is divisible by 11. True Step 2: Assume true for $n=k$ (where k is a positive odd integer) $7^k+15^k=11P$ (1) where P is an integer Step 3: Prove the result true for $n=k+2$ $7^{k+2}+15^{k+2}=11Q$ where Q is an integer	4 Marks: Correct answer. 3 Marks: Makes significant progress towards the solution. 2 Marks: Proves the result true for $n=1$ Attempts to use the result of $n=k$ to prove the result for $n=k+2$. 1 Mark: Proves the result true for $n=1$.

	LHS $= 7^{k+2} + 15^{k+2}$ $= 7^k \times 49 + 15^k \times 225$ $= 7^k \times 49 + (49 + 176) \times 15^k$ $= 49(7^k + 15^k) + 176 \times 15^k$ $= 49(11P) + 176 \times 15^k$ from 1 $= 11(49P + 16 \times 15^k)$ $= 11Q$ $=$ RHS Step 4: True by induction.	
13(b)	$\int \frac{e^{3x}+1}{e^x+1}dx = \int \frac{(e^x)^3+1^3}{e^x+1}dx$ $= \int \frac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1}dx$ $= \int (e^{2x}-e^x+1)dx$ $= \frac{1}{2}e^{2x} - e^x + x + C$	2 Marks: Correct answer. 1 Mark: Simplifies the integrand by factoring the sum of two cubes.
13(c) (i)	When $t = 1$ then $x = 2$ $2 = A\cos\left(\frac{\pi}{4} \times 1 + \alpha\right)$ $= A\left(\cos\frac{\pi}{4}\cos\alpha - \sin\frac{\pi}{4}\sin\alpha\right)$ $= A\left(\frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha\right)$ $2\sqrt{2} = A\cos\alpha - A\sin\alpha$ $\therefore A\sin\alpha - A\cos\alpha = -2\sqrt{2}$ When $t = 3$ then $x = -4$ $-4 = A\cos\left(\frac{\pi}{4} \times 3 + \alpha\right)$ $= A\left(\cos\frac{3\pi}{4}\cos\alpha - \sin\frac{3\pi}{4}\sin\alpha\right)$ $= A\left(-\frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha\right)$ $-4\sqrt{2} = -A\cos\alpha - A\sin\alpha$ $\therefore A\sin\alpha + A\cos\alpha = 4\sqrt{2}$	2 Marks: Correct answer. 1 Mark: Finds one of the equations or uses the compound angle formula with the given information.
13(c) (ii)	$A\sin\alpha - A\cos\alpha = -2\sqrt{2}$ 1 $A\sin\alpha + A\cos\alpha = 4\sqrt{2}$ 2 Adding equations 1 and 2 $2A\sin\alpha = 2\sqrt{2}$ Subtracting equation 1 from equation 2 $2A\cos\alpha = 6\sqrt{2}$	2 Marks: Correct answer. 1 Mark: Finds A or α. Alternatively shows some understanding of the problem.

	Now $(2A\sin\alpha)^2 + (2A\cos\alpha)^2 = \left(2\sqrt{2}\right)^2 + \left(6\sqrt{2}\right)^2$ $4A^2(\sin^2\alpha + \cos^2\alpha) = 8 + 72$ $A^2 = 20$ $\therefore A = 2\sqrt{5}$ Also $\dfrac{2A\sin\alpha}{2A\cos\alpha} = \dfrac{2\sqrt{2}}{6\sqrt{2}}$ $\tan\alpha = \dfrac{1}{3}$ $\therefore \alpha = \tan^{-1}\dfrac{1}{3}$	
13(c) (iii)	Particle passes through O when $x = 0$ $A\cos\left(\dfrac{\pi}{4}t + \alpha\right) = 0$ $\dfrac{\pi}{4}t + \alpha = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$ First passes through O $\dfrac{\pi}{4}t + \alpha = \dfrac{\pi}{2}$ $\dfrac{\pi}{4}t + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$ $\dfrac{\pi}{4}t = \dfrac{\pi}{2} - \tan^{-1}\dfrac{1}{3}$ $\dfrac{\pi}{4}t = \tan^{-1}3$ $\therefore t = \dfrac{4}{\pi}\tan^{-1}3 \approx 1.59$ seconds	2 Marks: Correct answer. 1 Mark: Finds $\dfrac{\pi}{4}t + \alpha = \dfrac{\pi}{2}$ or shows some understanding.
13(d)	$(a+b)^2 = (a-b)^2 + 4ab$ $\geq 4ab$ (since $(a-b)^2$ is a positive) $\dfrac{(a+b)}{ab} \geq \dfrac{4}{(a+b)}$ $\dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{4}{t}$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
14(a)		2 Marks: Correct answer. 1 Mark: Draws $\|z\| \geq 1$ or $-\dfrac{\pi}{4} \leq \arg z \leq \dfrac{\pi}{4}$.

14(b) (i)	Resolving forces $m\ddot{x} = -mk - mv^2$ $\ddot{x} = -k - v^2$ $\frac{1}{2}\frac{dv^2}{dx} = -k - v^2$ $\int \frac{dv^2}{k+v^2} = \int -2dx$ $\ln(k+v^2) = -2x + C$ Initial conditions $v = u, x = 0$ then $C = \ln(k+u^2)$ $\ln(k+v^2) = -2x + \ln(k+u^2)$ $x = -\frac{1}{2}\ln\frac{(k+v^2)}{(k+u^2)}$	2 Marks: Correct answer. 1 Mark: Resolves forces.
14(b) (ii)	$\ddot{x} = -k - v^2$ $\frac{dv}{dt} = -k - v^2$ $\frac{dv}{k+v^2} = -dt$ $\frac{1}{\sqrt{k}}\tan^{-1}\frac{v}{\sqrt{k}} = -t + C$ Initial conditions $\mathrm{t} = 0, v = u$ then $C = \frac{1}{\sqrt{k}}\tan^{-1}\frac{u}{\sqrt{k}}$ $t = \frac{1}{\sqrt{k}}\tan^{-1}\frac{u}{\sqrt{k}} - \frac{1}{\sqrt{k}}\tan^{-1}\frac{v}{\sqrt{k}}$ Particle is at rest when $v = 0$ $\therefore t = \frac{1}{\sqrt{k}}\tan^{-1}\frac{u}{\sqrt{k}}$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Recognises $\frac{dv}{dt} = -k - v^2$ or has some understanding of the problem.
14(c) (i)	$I_n = \int_0^1 \frac{1}{(1+x^2)^n}dx \qquad n = 1,2,3,\ldots$ $= [x(1+x^2)^{-n}]_0^1 - \int_0^1 x(-n)(1+x^2)^{-n-1}\,(2x)dx$ $= 2^{-n} + 2n\int_0^1 [(1+x^2)-1](1+x^2)^{-n-1}\,dx$ $= 2^{-n} + 2n\int_0^1 \left[(1+x^2)^{-n} - (1+x^2)^{-(n+1)}\right]dx$ $= 2^{-n} + 2nI_n - 2nI_{n+1}$ $2nI_{n+1} = (2n-1)I_n + 2^{-n}$ $I_{n+1} = \frac{2n-1}{2n}I_n + \frac{1}{n\times 2^{n+1}}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution 1 Mark: Correctly applies integration by parts.

14(c) (ii)	$I_3 = \frac{3}{4}I_2 + \frac{1}{16}$ $= \frac{3}{4}\left(\frac{1}{2}I_1 + \frac{1}{4}\right) + \frac{1}{16}$ $= \frac{3}{8}I_1 + \frac{1}{4}$ $I_1 = \int_0^1 \frac{1}{1+x^2}dx$ $= [\tan^{-1}x]_0^1$ $= \frac{\pi}{4}$ $\therefore \int_0^1 \frac{1}{(1+x^2)^3}dx = \frac{3\pi+8}{32}$	2 Marks: Correct answer. 1 Mark: Applies the recurrence relation to find an expression for I_3
14(d) (i)	$z = \frac{1+i\sqrt{3}}{2} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $= 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ $z^3 = 1(\cos\pi + i\sin\pi)$ $= -1$	2 Marks: Correct answer. 1 Mark: Expresses z in modulus argument form.
14(d) (ii)	$z^{10} = (z^3)^3 \times z$ $= (-1)^3 \times z$ $= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$	1 Mark: Correct answer.
15(a)	$\overrightarrow{OP} = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k},\ \overrightarrow{OQ} = 2\underset{\sim}{i} + \underset{\sim}{j},\ \overrightarrow{OR} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$ $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$ $= -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$ $= \underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$ $\lvert\overrightarrow{QP}\rvert = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$ $\lvert\overrightarrow{QR}\rvert = \sqrt{(1)^2 + (-3)^2 + (-2)^2} = \sqrt{14}$ $\overrightarrow{QP} \cdot \overrightarrow{QR} = (-1 \times 1) + (2 \times -3) + (-1 \times -2) = -5$ $\cos\angle PQR = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\lvert\overrightarrow{QP}\rvert\lvert\overrightarrow{QR}\rvert} = \frac{-5}{\sqrt{6}\sqrt{14}}$ $\angle PQR = 123.0618\ldots$ $\approx 123.1°$	4 Marks: Correct answer. 3 Marks: Uses the angle between two vectors. 2 Marks: Finds $\lvert\overrightarrow{QP}\rvert$ and $\lvert\overrightarrow{QR}\rvert$. 1 Mark: Finds $\overrightarrow{QP}$ and $\overrightarrow{QR}$.
15(b) (i)	$(1-3i)^2 = 1 - 6i + 9i^2$ $= 1 - 6i - 9$ $= -8 - 6i$	1 Mark: Correct answer.

15(b) (ii)	$2z^2 - 8z + (12 + 3i) = 0$ Using $\sqrt{-8-6i} = \pm(1-3i)$ $z = \dfrac{8 \pm \sqrt{64 - 4 \times 2 \times (12+3i)}}{4}$ $= \dfrac{8 \pm \sqrt{64 - 96 - 24i}}{4}$ $= \dfrac{8 \pm \sqrt{-32 - 24i}}{4} = \dfrac{8 \pm \sqrt{4(-8-6i)}}{4}$ $= \dfrac{4 \pm (1-3i)}{2}$ $z = \dfrac{5}{2} - \dfrac{3}{2}i$ or $z = \dfrac{3}{2} + \dfrac{3}{2}i$	2 Marks: Correct answer. 1 Mark: Uses the result from part (a) and shows some understanding.
15(c)	Step 1: To prove true for $n = 1$ $T_1 = (1+3)2^1 = 8$ Result is true for $n = 1$ To prove true for $n = 2$ $T_2 = (2+3)2^2 = 20$ Result is true for $n = 2$ Step 2: Assume true for $n = k$ $T_k = (k+3)2^k$ Step 3: To prove true for $n = k + 1$ $T_{k+1} = (k+4)2^{k+1}$ given $T_{k+1} = 4T_k - 4T_{k-1}$ LHS $= 4T_k - 4T_{k-1}$ $= 4(k+3)2^k - 4(k+2)2^{k-1}$ $= 4k2^k + 12 \times 2^k - 4k2^{k-1} - 8 \times 2^{k-1}$ $= 4k2^k + 12 \times 2^k - 2k2^k - 4 \times 2^k$ $= 2^{k+1}(2k + 6 - k - 2)$ $= (k+4)2^{k+1}$ $=$ RHS Step 4: True by induction	4 Marks: Correct answer. 3 Marks: Makes significant progress towards the solution. 2 Marks: Proves the result true for $n = 1$ and $n = 2$. Attempts to use the result of $n = k$ to prove the result for $n = k + 1$. 1 Mark: Proves the result true for $n = 1$ or $n = 2$.
15(d)	$\lvert \underset{\sim}{u} + \underset{\sim}{v} \rvert^2 = (\underset{\sim}{u} + \underset{\sim}{v}).(\underset{\sim}{u} + \underset{\sim}{v})$ $= \underset{\sim}{u} \cdot \underset{\sim}{u} + \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{v}$ $= \lvert \underset{\sim}{u} \rvert^2 + 2\underset{\sim}{u} \cdot \underset{\sim}{v} + \lvert \underset{\sim}{v} \rvert^2$ $= 6^2 + 2 \times -4 + 5^2 = 53$ $\lvert \underset{\sim}{u} + \underset{\sim}{v} \rvert = \sqrt{53}$	2 Marks: Correct answer. 1 Mark: Makes some progress.
15(e)	$t = \tan\dfrac{x}{2}$ $dt = \dfrac{1}{2}\sec^2\dfrac{x}{2}dx \quad dx = \dfrac{2}{1+t^2}dt$ When $x = 0$ then $t = 0$ and when $x = \frac{\pi}{2}$ then $t = 1$ $3 - \cos x - 2\sin x = \dfrac{3(1+t^2) - (1-t^2) - 4t}{1+t^2}$ $= \dfrac{4t^2 - 4t + 2}{1+t^2} = \dfrac{2(2t^2 - 2t + 1)}{1+t^2}$	4 Marks: Correct answer. 3 Marks: Makes significant progress. 2 Marks: Finds sin$x$ and cos$x$ in terms of t. 1 Mark: Shows some understanding.

	$\int_0^{\frac{\pi}{2}} \frac{1}{3-\cos x-2\sin x}dx = \int_0^1 \frac{1+t^2}{2(2t^2-2t+1)} \times \frac{2}{1+t^2}dt$ $= \int_0^1 \frac{1}{2t^2-2t+1}dt$ $= \int_0^1 \frac{1}{2\left(t^2-t+\frac{1}{2}\right)}dt$ $= \int_0^1 \frac{1}{2}\left[\frac{1}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}\right]dt$ $= \left[\tan^{-1}2\left(t-\frac{1}{2}\right)\right]_0^1$ $= \tan^{-1}1 - \tan^{-1}(-1) = \frac{\pi}{2}$	
16(a)	$\lvert\overrightarrow{OA}\rvert = \sqrt{3^2+2^2+(-4)^2}$ $= \sqrt{29}$ $\widehat{\overrightarrow{OA}} = \frac{\overrightarrow{OA}}{\lvert\overrightarrow{OA}\rvert}$ $= \frac{1}{\sqrt{29}}(3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k})$	2 Marks: Correct answer. 1 Mark: Finds the magnitude of $\overrightarrow{OA}$.
16(b)	Resolving forces (m =30 kg) $ma = -mg - kv^2$ $30a = -30g - kv^2$ $a = -g - \frac{1}{30}kv^2$ $\frac{dv}{dt} = -g - \frac{1}{30}kv^2$ $\frac{dt}{dv} = \frac{-30}{kv^2+30g}$ $t = \int \frac{-30}{kv^2+30g}dv$ The particle has an initial speed of u ms[-1] and reaches maximum height when $v = 0$. $t = \int_u^0 \frac{-30}{kv^2+30g}dv$ $= \frac{30}{\sqrt{k}}\int_0^u \frac{\sqrt{k}}{(\sqrt{30g})^2+(\sqrt{k}v)^2}dv$ $= \frac{30}{\sqrt{k}}\left[\frac{1}{\sqrt{30g}}\tan^{-1}\frac{\sqrt{k}v}{\sqrt{30g}}\right]_0^u$ $= \frac{30}{\sqrt{30gk}}\left[\tan^{-1}\frac{\sqrt{k}v}{\sqrt{30g}}\right]_0^u$ $= \sqrt{\frac{30}{gk}}\tan^{-1}\frac{\sqrt{k}u}{\sqrt{30g}}$	4 Marks: Correct answer. 3 Marks: Makes significant progress towards the solution. 2 Marks: Finds $t = \int \frac{-30}{kv^2+30g}dv$ 1 Mark: Resolves the forces

16(c)	Let $f(x) = x - \ln(1+x)$ and $f'(x) = 1 - \dfrac{1}{1+x}$ Minimum occurs when $f'(x) = 0$ $1 - \dfrac{1}{1+x} = 0$ $\dfrac{1+x}{1+x} - \dfrac{1}{1+x} = 0$ $1 + x - 1 = 0$ $x = 0 \ \ (x \neq -1)$ Test $f''(x) = \dfrac{1}{(1+x)^2}$ $f''(0) = 1 > 0$ Minima Therefore the least value of $f(x)$ is at $x = 0$ $f(0) = 0 - \ln(1+0) = 0$ hence $f(x) \geq 0$ $f(x) = x - \ln(1+x) \geq 0$ $\therefore x \geq \ln(1+x)$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up the function and correctly uses calculus.
16(d) (i)	$\lvert \underset{\sim}{u} \rvert = \sqrt{3^2 + m^2 + 1^2} = 10$ $\sqrt{10 + m^2} = 10$ $10 + m^2 = 100$ $m^2 = 90$ $m = \pm 3\sqrt{10}$	2 Marks: Correct answer. 1 Mark: Shows understanding of the magnitude of a vector.
16(d) (ii)	$\lvert \underset{\sim}{u} \rvert = \sqrt{3^2 + m^2 + 1^2}$ $= \sqrt{10 + m^2}$ $\underset{\sim}{u} = 3\underset{\sim}{\imath} + m\underset{\sim}{\jmath} + \underset{\sim}{k}$ y-axis: $\underset{\sim}{\jmath}$ (unit length of 1) $\underset{\sim}{u} \cdot \underset{\sim}{\jmath} = (3 \times 0) + (m \times 1) + (1 \times 0) = m$ $\cos\theta = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{\jmath}}{\lvert \underset{\sim}{u} \rvert \lvert \underset{\sim}{\jmath} \rvert} = \dfrac{m}{\sqrt{10 + m^2}} = \dfrac{1}{3}$ $\dfrac{m^2}{10 + m^2} = \dfrac{1}{9}$ $9m^2 = 10 + m^2$ $8m^2 = 10$ $m = \pm \dfrac{\sqrt{5}}{2}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Shows some understanding.