Carlingford High School

Mathematics Extension 2

Year 11

HSC ASSESSMENT TASK 1

Term 4 2013

Time Allowed: 60 minutes

Name:______________________ Teacher: Ms Kellahan

- Start each question on a new page.
- Write on ONE SIDE of the paper only.
- Marks may be deducted for careless or badly arranged work.
- Only calculators approved by the Board of Studies may be used.
- All answers are to be completed in blue or black pen except graphs and diagrams.
- There is to be NO LENDING OR BORROWING.

	MC	Q6	Q7	Mark	
E3					%
Total	5	15	15	35	

E3: Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

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Section 1 – Multiple Choice

Circle your answer.

1. If $z = 2 + 3i$ and $w = -5 - 2i$, what is the value of zw? 1

   A. -4 B. $-3 + i$ C. $-4 - 19i$ D. $-16 - 19i$

2. The argument of $(1 + i)^3$ is: 1

   A. $\frac{\pi}{2}$ B. $\frac{\pi}{4}$ C. π D. $\frac{3\pi}{4}$

3. If z and w are two complex numbers then: 1

   A. $\overline{z+w} = \overline{z} - \overline{w}$ B. $\overline{z+w} = \overline{z} \times \overline{w}$

   C. $\overline{z+w} = \overline{z} \div \overline{w}$ D. $\overline{z+w} = \overline{z} + \overline{w}$

4. The reciprocal of $3i$ is: 1

   A. $\frac{1}{3i}$ B. $\frac{-i}{3}$ C. $-3i$ D. $3i$

5. When the circle $|z-(3+4i)| = 5$ is sketched on the Argand Diagram the maximum value of $|z|$ occurs when z lies at the end of the diameter that passes through the centre and the origin. 1

   What is the maximum value of $|z|$?

   A. $\sqrt{5}$ B. 5 C. 10 D. $\sqrt{10}$

END OF SECTION 1

Section 2

Question 6 - Start a new page (15 Marks)

a. Let $z = 3 + i$ and $w = 1 - i$.
Find in rectangular form:

i) $2z + iw$ 1

ii) $\bar{z}w$ 1

b. If $z = \dfrac{1+\sqrt{3}\,i}{\sqrt{3}+i}$, find :

i) $|z|$ 1

ii) $Arg\,(z)$ 1

iii) z^4 1

iv) the five fifth roots of z. 2

c. i) Find all pairs of integers a and b such that $(a + ib)^2 = 8 + 6i$ 1

ii) Hence solve $z^2 + 2z(1+2i) - (11+2i) = 0$ 2

d. Sketch the region in the Argand plane consisting of those points z for which: 3

$$|\arg(z+1)| < \frac{\pi}{6},\ z + \bar{z} \le 6 \text{ and } |z+1| > 2.$$

e. Show that for the complex number, $z = \dfrac{1-t^2+2it}{1+t^2}$, $|z| = 1$ for all values of t 2

Question 7– Start a new page (15 marks)

a. Describe the following transformation $z \rightarrow \alpha z$ where $\alpha = 6+6i$. Illustrate on an Argand diagram for $z = 2i$. 2

b. i) Use De Moivre's theorem to express $\cos 4\theta$ in terms of $\cos\theta$. 2

ii) Hence express $\cot 4\theta$ as a rational function of x where $x = \cot\theta$. 1

iii) By considering the roots of $\cot 4\theta = 0$, prove that: 2

$$\cot\frac{\pi}{8}\cdot\cot\frac{3\pi}{8}\cdot\cot\frac{5\pi}{8}\cdot\cot\frac{7\pi}{8} = 1$$

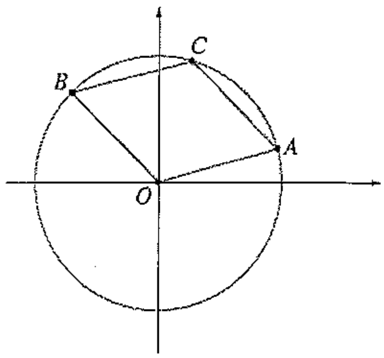
c. In an Argand diagram the points P, Q and R represent the complex numbers z_1, z_2 and $z_2 + i(z_2 - z_1)$.

i) Show that PQR is a right angled triangle 1

ii) Find in terms of z_1 and z_2 the complex number represented by the point S such that PQRS is a rectangle. 2

d. The diagram shows two distinct points A and B that represent the complex numbers z and w respectively. The points A and B lie on the circle of radius r centred at O. The point C representing the complex number z + w also lies on this circle.

Copy the diagram into your paper.



i) Using the fact that C lies on the circle, show geometrically that $\angle AOB = \frac{2\pi}{3}$. 2

ii) Hence show that $z^3 = w^3$ 2

iii) Show that $z^2 + w^2 + zw = 0$ 1

END OF EXAM

EXTENSION 2 TERM 4 TASK 1

Solutions.

1. $zw = (2+3i)(-5-2i)$

$= -10 - 4i - 15i + 6$

$= -4 - 19i$

(C)

2. $(1+i) = 1^4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \cos\pi + i\sin\pi$

(D).

3. (D) $\overline{z+w} = \overline{z} + \overline{w}$

4. $\frac{1}{3i} \times \frac{-3i}{-3i} = \frac{-3i}{9}$

$= -\frac{i}{3}$.

(B)

5. (C).

Q6.

a) $z = 3+i \qquad w = 1-i$

i) $2z + iw = 2(3+i) + i(1-i)$

$= 6 + 2i + i + 1$

$= 7 + 3i$ (1)

ii) $\overline{z}w = (3-i)(1-i)$

$= 3 - 3i + i - 1$

$= 2 - 4i$ (1)

b) $z = \frac{1+\sqrt{3}i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \frac{\sqrt{3} - i + 3i + \sqrt{3}}{3+1}$

$= \frac{2\sqrt{3} + 2i}{4}$

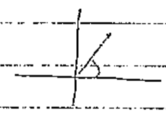
$= \frac{\sqrt{3}+i}{2}$

i) $|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \sqrt{\frac{3}{4} + \frac{1}{4}}$

$= 1$ (1)

ii) $\text{Arg}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$



$\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$\therefore \text{Arg } \frac{\pi}{6}$ (1)

iii) $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$z^4 = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^4$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ (1)

IV: $z^5 = \text{cis}\frac{\pi}{6}$.

$z = \dfrac{\text{cis}\frac{\pi}{6} + 2k\pi}{5}$ $\quad k = 0, 1, 2, 3, 4$

$= \text{cis}\left(\frac{\pi}{30} + \frac{2k\pi}{5}\right)$

$z_0 = \text{cis}\frac{\pi}{30}$

$z_1 = \text{cis}\left(\frac{\pi}{30} + \frac{2\pi}{5}\right)$

$= \text{cis}\left(\frac{13\pi}{30}\right)$

$z_2 = \text{cis}\left(\frac{\pi}{30} + \frac{4\pi}{5}\right)$

$= \text{cis}\frac{5\pi}{6}$

$z_3 = \text{cis}\left(\frac{\pi}{30} + \frac{6\pi}{30}\right)$

$= \text{cis}\left(\frac{37\pi}{30}\right)$

$z_4 = \text{cis}\left(\frac{\pi}{30} + \frac{8\pi}{30}\right)$

$= \text{cis}\left(\frac{49\pi}{30}\right)$

c) $(a+ib)^2 = 8+6i$

$a^2 + 2aib + b^2 = 8+6i$

$a^2 - b^2 = 8$ (1) $\quad 2ab = 6$

$a^2 - \left(\frac{3}{a}\right)^2 = 8$ $\quad ab = 3$

$a^2 - \frac{9}{a^2} = 8$ $\quad b = \frac{3}{a}$

$a^4 - 9 = 8a^2$

$a^4 - 8a^2 - 9 = 0$

$(a^2-9)(a^2+1) = 0$

a real

$\therefore a^2 = 9$

$a = \pm 3$

$\therefore b = \frac{3}{\pm 3}$

$b = \pm 1$

$\therefore \sqrt{8+6i} = \pm(3+i)$ 1

ii) $z^2 + 2z(1+2i) - (11+2i) = 0$

$z = \dfrac{-2(1+2i) \pm \sqrt{4(1+2i)^2 + 4(11+2i)}}{2}$

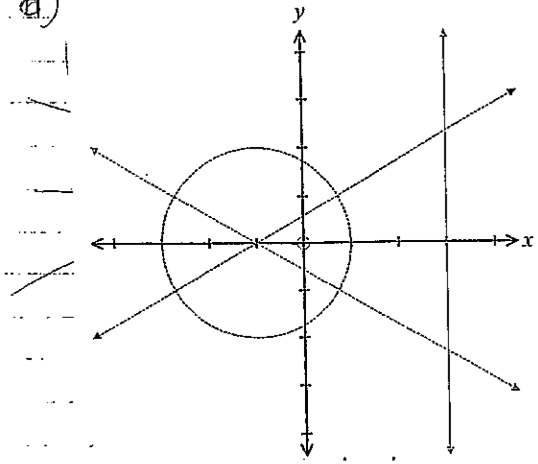
$= \dfrac{-2(1+2i) \pm \sqrt{32+24i}}{2}$

$= \dfrac{-2+4i \pm 2\sqrt{8+6i}}{2}$ 1

$= \dfrac{-2-4i \pm 2(3+i)}{2}$

$= 2-i$ or $-4-3i$ 1

d)



1 per graph.

$|\arg(z+1)| < \frac{\pi}{6}$

$z + \bar{z} \leq 6$

$|z+1| > 2$

e) $|z| = \left|\frac{1 - t^2 + 2it}{1+t^2}\right|$

$= \sqrt{\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2}$

$= \sqrt{\frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2}}$

$= \sqrt{\frac{1 + 2t^2 + t^4}{(1+t^2)^2}}$

$= \sqrt{\frac{(1+t^2)^2}{(1+t^2)^2}}$

$= 1.$ which is indep. of t.

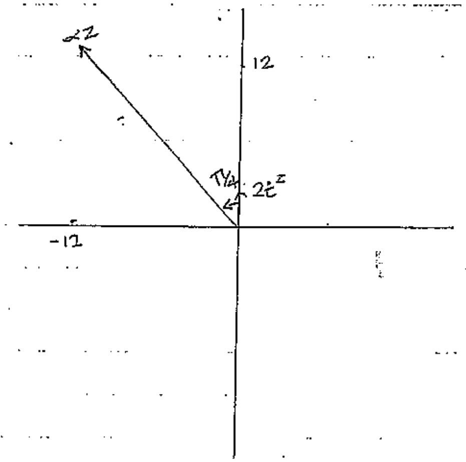
Q7. a) $z \to \alpha z$

$|\alpha| = \sqrt{6^2 + 6^2}$
$= 6\sqrt{2}$

$\arg\alpha = \frac{\pi}{4}$

$(6i + 6) = 6i + 6$
$= 6\sqrt{2}\left(\frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$
$= 6\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$\alpha z = 2i(6i + 6)$
$= -12 + 12i$



Transformation is
$\frac{\pi}{4}$ anticlockwise and an enlargement of $6\sqrt{2}$.

b) i) $\therefore \text{cis}\,4\theta = (\text{cis}\,\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\sin^3\theta\, i^3 + \sin^4\theta\, i^4$

$= \underline{\cos^4\theta} + 4\cos^3\theta\, i\sin\theta \underline{- 6\cos^2\theta\sin^2\theta} - 4\cos\theta\, i\sin^3\theta + \underline{\sin^4\theta}$,

$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 2\cos^4\theta - 8\cos^2\theta + 6\cos^4\theta + 1$.

$= 8\cos^4\theta - 8\cos^2\theta + 1$

ii) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$\cot 4\theta = \dfrac{\cos 4\theta}{\sin 4\theta}$

$= \dfrac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$

$= \dfrac{\dfrac{\cos^4\theta}{\sin^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\sin^4\theta} + \dfrac{\sin^4\theta}{\sin^4\theta}}{\dfrac{4\cos^3\theta\sin\theta}{\sin^4\theta} - \dfrac{4\cos\theta\sin^3\theta}{\sin^4\theta}}$

$= \dfrac{\cot^4\theta - \dfrac{6\cos^2\theta}{\sin^2\theta} + 1}{\dfrac{4\cos^3\theta}{\sin^3\theta} - \dfrac{4\cos\theta}{\sin\theta}}$

$= \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$

$= \dfrac{x^4 - 6x + 1}{4x^3 - 4x}$.

iii) $\cot 4\theta = 0$.

$\therefore \dfrac{x^4 - 6x^2 + 1}{4x^3 - 4x} = 0$

$\therefore x^4 - 6x^2 + 1 = 0$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

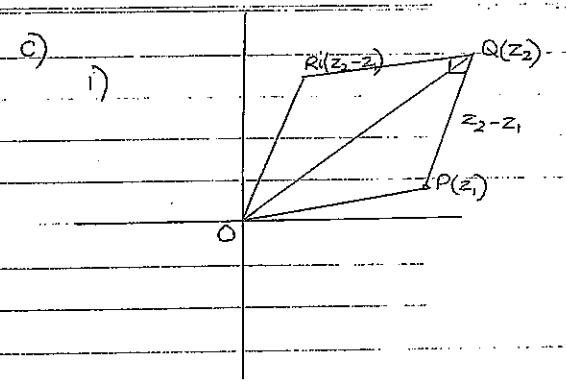
$\therefore$ roots are $\cot\frac{\pi}{8}, \cot\frac{3\pi}{8}, \cot\frac{5\pi}{8}, \cot\frac{7\pi}{8}$.

From $x^4 - 6x^2 + 1 = 0$

Product of roots is 1

$\therefore \cot\frac{\pi}{8} \times \cot\frac{3\pi}{8} \times \cot\frac{5\pi}{8} \times \cot\frac{7\pi}{8}$.

c) i)



$\vec{PQ} = z_2 - z_1$

$\vec{PQ}$ rotated anticlockwise by $\frac{\pi}{2}$ gives $i(z_2 - z_1)$

$\vec{OR} = z_2 + i(z_2 - z_1)$

$\angle PQR = \frac{\pi}{2}$

$\therefore \Delta PQR$ is right angle

ii) If $PQRS$ is a rectangle $\vec{PS} = \vec{QR}$

& $\vec{PS} \parallel \vec{QR}$.

Hence $\vec{PS}$ represents $i(z_2 - z_1)$

$\vec{OS} = \vec{OP} + \vec{PS}$

$\therefore \vec{OS} = z_1 + i(z_2 - z_1)$.

d i)

A B C z, w & $z+w$.

$\vec{OA} = z \quad \vec{OB} = w \quad \vec{OC} = z+w$.

In $\triangle OAC$.

$\vec{OC} = \vec{OA} + \vec{AC}$

$z+w = z + \vec{AC}$

$\therefore \vec{AC} = w$.

That is $AC = OB$ & $AC \parallel OB$.

$\therefore$ OACB is a parallelogram.

$\therefore AC =$ radius of circle OB.

$= r$

$\therefore \triangle OAC \Rightarrow OA = OC = AC = r$.

$\therefore \triangle OAC$ is an equi. $\triangle$

$\therefore \angle OAC = \frac{\pi}{3}$

Also $BC = OA = r$ (opp sides of para.)

In $\triangle OBC$ $BC = OC = OB = r$

$\therefore \triangle OBC$ equi $\triangle$.

$\therefore \angle COB = \frac{\pi}{3}$

$\angle AOB = \angle AOC + \angle COB$

$= \frac{\pi}{3} + \frac{\pi}{3}$

$= \frac{2\pi}{3}$.

ii)

$\angle xOA = \alpha$

$\angle xOB = \alpha + \frac{2\pi}{3}$

$z = OA(\cos\angle xOA + i\sin\angle xOB)$

$= r(\cos\alpha + i\sin\alpha)$

$w = OB\left(\cos\left(\alpha + \frac{2\pi}{3}\right) + i\sin\left(\alpha + \frac{2\pi}{3}\right)\right)$

$= r\left[\cos\left(\alpha + \frac{2\pi}{3}\right) + i\sin\left(\alpha + \frac{2\pi}{3}\right)\right]$

$z^3 = r^3(\cos 3\alpha + i\sin 3\alpha)$.

$w^3 = r^3\left(\cos 3\left(\alpha + \frac{2\pi}{3}\right) + i\sin 3\left(\alpha + \frac{2\pi}{3}\right)\right)$

$= r^3(\cos(3\alpha + 2\pi) + i\sin(3\alpha + 2\pi))$

$= r^3(\cos 3\alpha + i\sin 3\alpha)$

$= z^3$

iii)

$z^3 - w^3 = (z-w)(z^2 + zw + w^2)$

but $z^3 = w^3$ part ii).

$z^3 - w^3 = 0$

$(z-w)(z^2 + zw + w^2) = 0$

Noting $z \neq w$ then $z^2 + w^2 + zw = 0$.