Carlingford High School

Extension 1 Mathematics

Year 12

Half Yearly Exam

Term 4 2016

Time Allowed: 90 minutes

Name:______________________

Class: 12M1______

Teacher: Mr Gong Ms Kellahan Mrs Lobejko

- Answer each question in a new booklet.
- **Do not** work in columns.
- Marks may be deducted for careless or badly arranged work.
- Write in **blue or black** pen. Diagrams and graphs may be done in pencil.
- Only calculators approved by the Board of Studies may be used.
- There is to be NO LENDING OR BORROWING.
- **Reference sheets** and the **Multiple Choice Answer** sheet are attached at the back of the paper and may be removed.

	MC	**Q11**	**Q12**	**Q13**	**Mark**
Polynomials		/3			/3
Parametrics		/3			/3
Series & Applications		/3		/9	/12
Applications of Calculus		/6	/15	/4	/25
Total	/10	/15	/15	/13	/53

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Section I: Multiple choice
Answer on the multiple choice sheet.

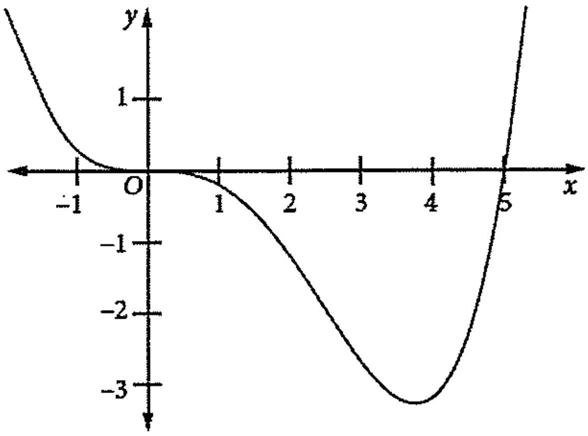
Mark

1. If $f(x) = e^{-x} - 3e^{-3x}$, then $f'(x)$ is: **1**

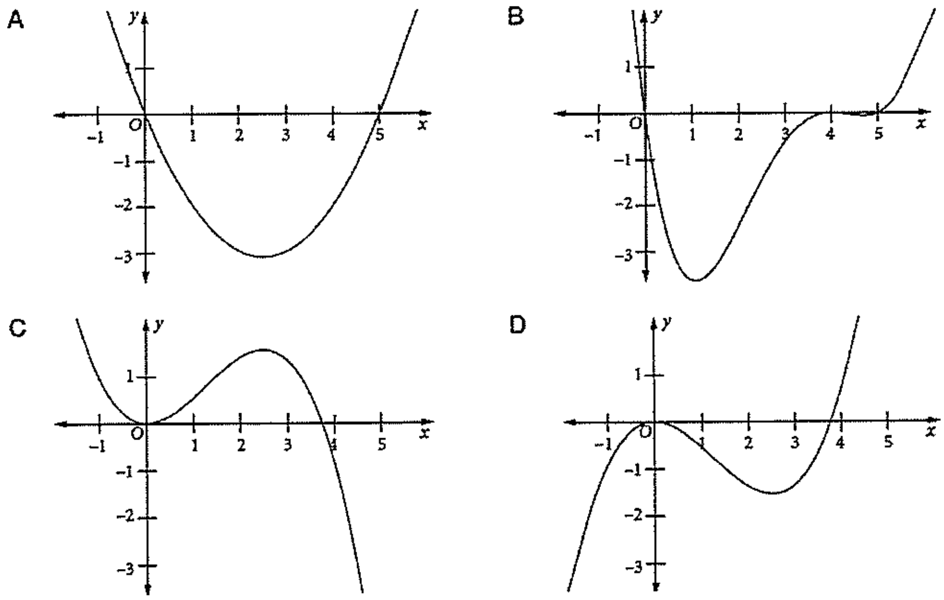
A. $-e^{-x} + 9e^{-3x}$ B. $-e^{-x} - 9e^{-3x}$

C. $e^{-x} + 9e^{-3x}$ D. $-e^{-x} + 3e^{-3x}$

2. The graph of $y = f(x)$ is shown. **1**



Which of the following could be the graph of $y = f'(x)$?



3. Solve $2\log_a x - \log_a 4 = 2\log_a 8$ **1**

A. ± 4 B. 16 C. 4 D. ± 16

4. Let $P(x) = 2x^3 - 3x - 7$.
Find the remainder when P(x) is divided $(x - 2)$

A. 3 B. –3 C. 17 D. –17

5. $\sum_{n=1}^{5} 3n^2 - 2n = ?$ **1**

A. 64 B. 65 C. 134 D. 135

6. The point $P(2t, 3t^2)$ lies on the parabola: **1**

A. $4x = 3y^2$ B. $3y = 4x^2$ C. $4y = 3x^2$ D. $3x = 4y^2$

7. Which of the following is an expression for the derivative of 5^x? **1**

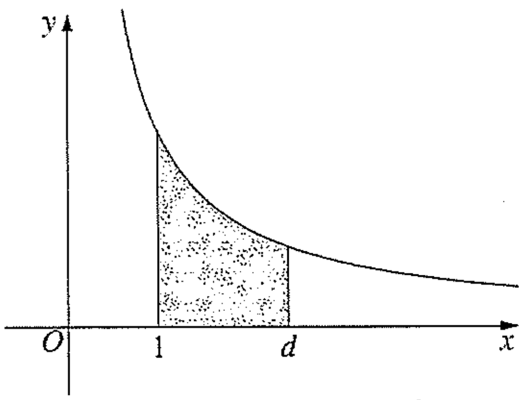
A. $x5^{x-1}$ B. 5^{x-1} C. 5^x D. $5^x \log_e 5$

8. The equation of the normal to the parabola $x^2 = 4ay$ at the variable point $P(2ap, ap^2)$ is given by $x + py = 2ap + ap^3$. **1**

How many different values of p are there such that the normal passes through the focus of the parabola?

A. 0 B. 1 C. 2 D. 3

9. The diagram shows the area under the curve $y = \frac{2}{x}$ from $x = 1$ to $x = d$. **1**



What value of d makes the shaded area equal to 2?

A. e B. e + 1 C. 2e D. e^2

10. The three roots of $-x^3 + 2x^2 - x = 0$ are α, β and γ. **1**

What is $\alpha^2 + \beta^2 + \gamma^2$?

A. 10 B. 6 C. 4 D. 2

End of Section I

Section II

Question 11. **Answer in a new booklet.** **15 Marks**

a. Differentiate $\ln\sqrt{\dfrac{1+x}{1-x}}$ **2**

b. If $\log_e \frac{b}{c} = 1.25$ what is the value of $\log_e \frac{c}{b}$? **2**

c. $\int \left(\sqrt{x}+2\right)^2 dx$ **2**

d. Use the principle of mathematical induction to prove that $3^{2n} + 7$ is divisible by 8, for positive integers $n \geq 1$ **3**

e. Solve $x^3 - 21x^2 + 126x - 216 = 0$, given that the roots are in a geometric progression. **3**

f. The point $P(2ap, ap^2)$ lies o the parabola $x^2 = 4ay$ whose focus is at S. The tangent at P given by $y - ap^2 = p(x - 2ap)$ meets the y –axis at Q.

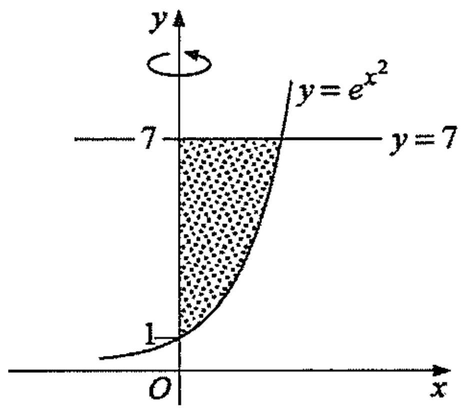
i) Find the coordinates at Q. **1**

ii) Show that $\angle SPQ = \angle SQP$ **2**

End of Question 11.

Question 12. Answer in a new booklet. 15 Marks

a. The shaded region bounded by $y = e^{x^2}$, y = 7 and the y –axis is rotated around the y-axis to form a solid.



i) Show that the volume of the solid is given by $V = \pi \int_1^7 \log_e y \, dy$. **2**

ii) Use the trapezoidal rule with 3 function values to approximate the volume of V, correct to 3 decimal places. **3**

b. Find the equation of the normal to the curve $y = e^{2x-1}$ at the point where x = 1, in terms of e. **3**

c. For the function $y = x^2 e^x$

i) State the domain and range. **2**

ii) Find any stationary points and determine their nature. **3**

iii) Sketch the curve showing all the main features. **2**

End of Question 12

Question 13. **Answer in a new booklet.** **13 Marks**

a. i) Sketch the curve $y = x^3 - 2x^2 - 3x$. **1**

ii) Hence find the area bounded by the curve $y = x^3 - 2x^2 - 3x$ and the x-axis. **3**

b. For the series $\ln 3 + \ln 6 + \ln 12 + \ldots$

i) Show that it is Arithmetic. **1**

ii) Find the 10^{th} term. **1**

iii) Evaluate $\sum_{n=1}^{10} \ln[3(2^{n+1})]$ **3**

c. Prove by mathematical induction that $n^2 > 3n + 11$ for $n > 5$. **4**

End of Question 13
End of Exam

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Year 12 Half Yearly HSC Examination

Mathematics Extension 1 2016

Section I – Multiple Choice Answer Sheet

Name ________________________.

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample:	$2 + 4 =$	(A) 2	(B) 6	(C) 8	(D) 9
		A ○	B ●	C ○	D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

Start here

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

t formulae

If $t = \tan\dfrac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at, at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1, y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\dfrac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0, y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos\left(nt + \alpha\right)$$

$$\ddot{x} = -n^2\left(x - b\right)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

Binomial theorem

$$\left(a+b\right)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180°$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

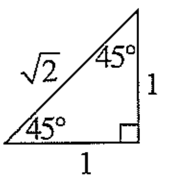
$$\sec\theta = \frac{1}{\cos\theta}$$

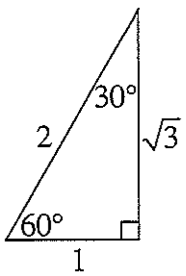
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios





Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Mathematics (continued)

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

Extension 1 Half Yearly Solutions

2015

Section I : Multiple Choice

1. A
2. D
3. ~~D~~ B
4. A
5. D
6. C
7. D
8. B
9. A
10. D.

Section II.

Question 11.

a)
$$\frac{d}{dx}\left(\ln\sqrt{\frac{1+x}{1-x}}\right)$$

$$= \frac{d}{dx}\left(\ln(1+x)^{1/2} - \ln(1-x)^{1/2}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)\right) \qquad 1$$

$$= \frac{1}{2}\times\frac{1}{1+x} - \frac{1}{2}\times\frac{-1}{1-x}$$

$$= \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right]$$

$$= \frac{1}{2}\left[\frac{1-x+1+x}{1-x^2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1-x^2}\right]$$

$$= \frac{1}{1-x^2} \qquad 1.$$

b) $\ln\frac{b}{c} = 1.25 = \frac{5}{4}$

$e^{5/4} = \frac{b}{c}$

$\therefore \frac{c}{b} = e^{-5/4}$. $\qquad$ 1

$\therefore \ln\frac{c}{b} = -\frac{5}{4}$ $\qquad$ 1

$= -1.25$.

Must have some working!

c) $\int(\sqrt{x}+2)^2\,dx$

$= \int(x + 4\sqrt{x} + 4)\,dx$

$= \frac{x^2}{2} + \frac{8}{3}x^{3/2} + 4x + c$. $\qquad$ 2

must have plus c.

d) Step 1: Prove true for $n=1$

$3^2+7=16$ which is divisible by 8.

$\therefore$ true for $n=1$. $\qquad$ 1

Step 2: Assume true for $n=k$.

$3^{2k}+7 = 8P$ (where P is an integer).

$3^{2k} = 8P-7$

Step 3: Prove true for $n=k+1$

$3^{2(k+1)} + 7 = 8Q$ (Q an integer)

$3^{2k+2} + 7 = 8Q$

$3^{2k} \times 3^2 + 7 = 8Q$ | 1

$(8P-7)\times 9 + 7 = 72P - 63 + 7$

$= 72P - 56$

$= 8(9P-7)$ | 1

$= 8Q$

$(Q = 9P - 7)$

$\therefore$ true for $n=k+1$.

Since true for $n=k+1$, when true for $n=k$ & true for $n=1$, therefore true for $n \geq 1$.

e) $x^3 - 21x^2 + 126x - 216 = 0$.

Let the root be $\frac{a}{r}, a, ar$.

$\alpha\beta\gamma = \frac{a}{r} \times a \times ar$

$= a^3$

$\therefore a^3 = -d/a$

$= 216$

$\therefore a = 6$. | 1

$\alpha + \beta + \gamma = \frac{6}{r} + 6 + 6r$

$= 21$.

$\therefore 6 + 6r + 6r^2 = 21r$

$6r^2 - 15r + 6 = 0$.

$2r^2 - 5r + 2 = 0$

$(2r-1)(r-2) = 0$

$\therefore r = \frac{1}{2}$ or $r = 2$. | 1

$\therefore$ roots are 6, 3 & 12. | 1.

f) i) At Q $x = 0$.

$\therefore y = p(0 - 2ap) + ap^2$

$= -2ap^2 + ap^2$

$= -ap^2$.

$\therefore Q(0, -ap^2)$ | 1

Must be written as a set of co-ordinates

ii) $SP^2 = (2ap)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2$

$= 2a^2p^2 + a^2p^4 + a^2$

$= a^2(2p^2 + p^4 + 1)$

$= a^2(p^2+1)^2$.

$\therefore SP = a(p^2+1)$

$SQ = SO + OQ$

$= a + ap^2$

$= a(1+p^2)$. | 1

$\therefore SP = SQ$

$\therefore SPQ$ is an isosceles triangle | 1

$\therefore \angle SPQ = \angle SQP$ ($\angle$'s opposite $=$ sides).

Question 12.

a) i) $y = e^{x^2}$

$\ln y = \ln e^{x^2}$

$\ln y = x^2 \ln e$

$\therefore x^2 = \ln y$ — 1

$\therefore V = \pi \int_1^7 x^2 \, dy$

$= \pi \int_1^7 \log_e y \, dy$ — 1.

ii)

y	1	4	7
$\ln y$	0	$\ln 4$	$\ln 7$

1

$V \doteq \pi \times \frac{3}{2}\left[0 + \ln 7 + 2\ln 4\right]$ — 1

$= 22 \cdot 235 \ u^3.$ — 1.

b) $y = e^{2x-1}$

$y' = 2e^{2x-1}$

At $x = 1$ $y' = 2e^{2-1} = 2e$

$y = e$

$\therefore$ m of normal is $-\frac{1}{2e}$ — 1.

$y - e = -\frac{1}{2e}(x - 1)$ — 1

$\therefore y = -\frac{1}{2e}x - 1 + e$

OR — 1

$x + 2ey - 2e^2 - 1 = 0.$

c)

i) D: all real x — 1

R: $y \geqslant 0$. — 1

ii) $y = x^2 e^x$.

$y' = x^2 e^x + 2xe^x$

$y'' = x^2 e^x + 2xe^x + 2xe^x + 2e^x$

$= x^2 e^x + 4xe^x + 2e^x.$

At $y' = 0$ stat. points.

$\therefore e^x(x^2 + 2x) = 0$

$e^x > 0$

$\therefore x^2 + 2x = 0$

$x(x+2) = 0$

$\therefore x = 0$ or $x = -2$ — 1

$y = 0$ or $y = 4e^{-2}$.

At $(0,0)$ $y'' = 2$

$y'' > 0$

$\therefore$ Minimum turning point. — 1

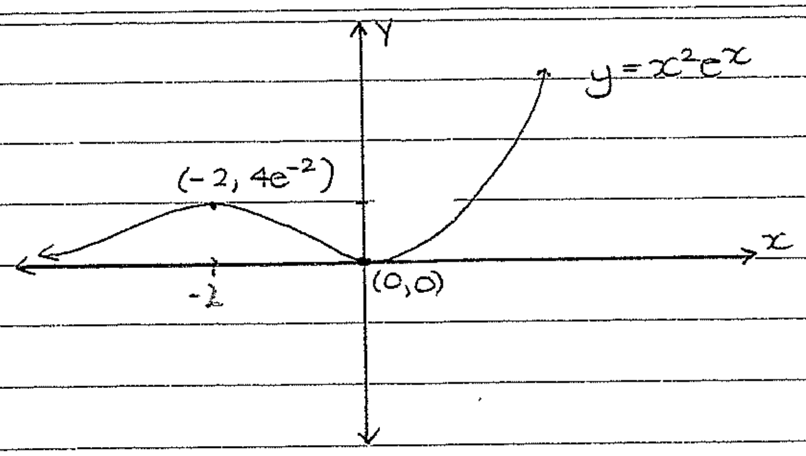
At $(-2, 4e^{-2})$ $y'' = e^{-2}(4 - 8 + 2)$

$= -2e^{-2}$

$y'' < 0$

$\therefore$ Maximum turning point. — 1.

iii)



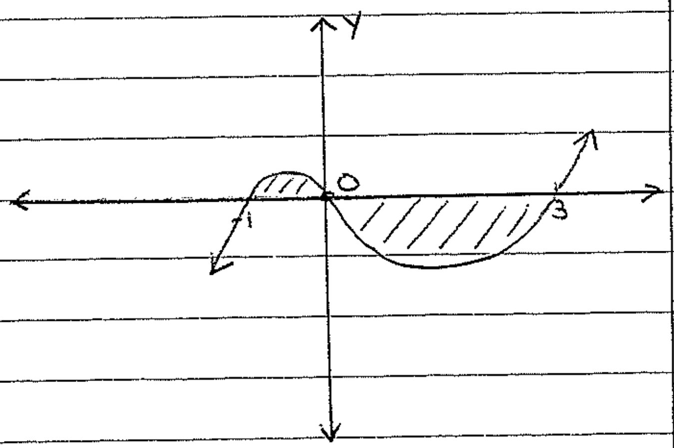
1 shape
1 main features.

Question 13.

a) $y = x^3 - 2x^2 - 3x$

$y = x(x^2 - 2x - 3)$

$y = x(x-3)(x+1)$



1

b) $A = \int_{-1}^{0} (x^3 - 2x^2 - 3x)\,dx$

$+ \left| \int_{0}^{3} (x^3 - 2x^2 - 3x)\,dx \right|$ 1

$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2} \right]_{-1}^{0}$

$+ \left| \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2} \right]_{0}^{3} \right|$ 1

$= \left[0 - \left(\frac{1}{4} + \frac{2}{3} - \frac{3}{2} \right) \right]$

$+ \left| \left[\left(\frac{3^4}{4} - \frac{2 \times 3^3}{3} - \frac{3 \times 3^2}{2} \right) - 0 \right] \right|$

$\therefore A = \frac{7}{12} + \frac{45}{4}$

$= \frac{71}{6} u^2.$ 1

b) i) $d_1 = \ln 6 - \ln 3$

$= \frac{\ln 6}{\ln 3}$

$= \ln 2.$

$d_2 = \ln 12 - \ln 6$

$= \frac{\ln 12}{\ln 6}$

$= \ln 2$

Since $d_1 = d_2$ it is an arithmetic series. 1

ii) $T_{10} = a + (n-1)d$

$= \ln 3 + 9 \times \ln 2$

$= \ln 3 + \ln 2^9$

$= \ln(3 \times 2^9)$ 1

$= \ln 1536.$

iii) next page.

iii)

$\ln 12 + (\ln 12 + \ln 2) + (\ln 12 + 2\ln 2) + \ldots$

$a = \ln 12 \quad d = \ln 2.$ 1

$S_{10} = \frac{n}{2}[2a + (n-1)d]$

$= \frac{10}{2}[2\ln 12 + 9 \times \ln 2]$ 1

$= 5[\ln 144 + \ln 2^9]$

$= 5\ln 73728.$ 1.

c). Step 1: Prove true for $n=6$.

$n = 6 \quad 36 > 3 \times 6 + 11$

$36 > 29.$

$\therefore$ true for $n=6$. 1.

Step 2: Assume true for $n=k$.

$S(k):\ k^2 > 3k + 11$

$\therefore k^2 - 3k - 11 > 0$

Step 3: Prove true for $n=k+1$.

$S(k+1)$:

$(k+1)^2 - 3(k+1) - 11 > 0.$

$k^2 + 2k + 1 - 3k - 3 - 11 > 0$

$(k^2 - 3k - 11) + 2k + 1 - 3 > 0$ 1

$S(k) + 2k - 2 > 0$

$\therefore S(k+1) > 0$ as $S(k) > 0$.

& $2k - 2 > 0$ as $k > 5$. 1

Since true for $n=k+1$ if true for $n=k$ & $n=6$. It is then true for $n>5$, n an integer. 1.