Junior Division

2014 AMC

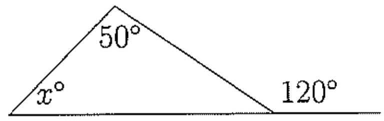
1. What is the value of $17 + 16 + 14 + 13$?

(A) 60 (B) 61 (C) 63 (D) 68 (E) 70

▸ As $17 + 13 = 30$ ~~and~~ $16 + 14 = 30$, so $17 + 16 + 14 + 13 = 60$,

hence (A).

2. (Also I2)
In the diagram the value of x is



(A) 80 (B) 70 (C) 60 (D) 50 (E) 40

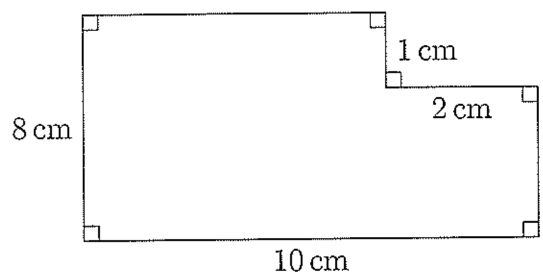
▸ *Alternative 1*
The three interior angles of the triangle are $x°$, $50°$ and $60°$. These add to $180°$, so $x + 110 = 180$ and $x = 70$,

hence (B).

Alternative 2
Using the exterior angle formula, $x + 50 = 120$, so that $x = 70$,

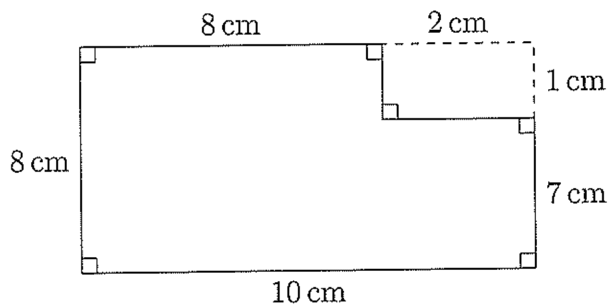
hence (B).

3. What is the perimeter of the figure below in centimetres?



(A) 21 (B) 30 (C) 36 (D) 39 (E) 78

▸ The combined height of the two right-hand boundaries is 8 cm and the combined length of the two top boundaries is 10 cm. So the total perimeter is $2 \times 10 + 2 \times 8 = 36$ cm,



hence (C).

4. (Also UP5)
This week at my lemonade stand I sold \$29 worth of lemonade, but I had spent \$34 on lemons and \$14 on sugar. My total loss for the week was

(A) \$1 (B) \$9 (C) \$19 (D) \$21 (E) \$29

▶ In dollars, the total amount spent was $34 + 14 = 48$, so the loss was $48 - 29 = \$19$,
hence (C).

5. (Also S2)
The value of $\dfrac{1}{0.04}$ is

(A) 15 (B) 20 (C) 25 (D) 40 (E) 60

▶ $\dfrac{1}{0.04} = \dfrac{1 \times 100}{0.04 \times 100} = \dfrac{100}{4} = 25$,

hence (C).

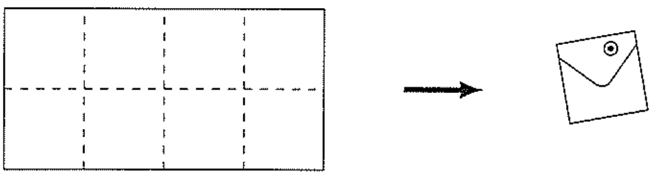
6. (Also I5)
If $\dfrac{5}{6}$ of a number is 30, what is $\dfrac{3}{4}$ of the number?

(A) 22.5 (B) 24 (C) 25 (D) 27 (E) 40

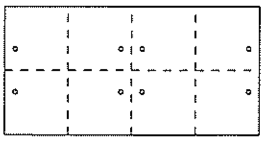
▶ One-sixth of the number is 6, so the number is 36. Then $\frac{1}{4}$ of the number is 9 and $\frac{3}{4}$ of the number is 27,

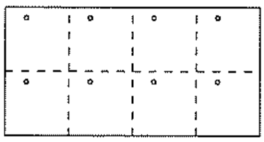
hence (D).

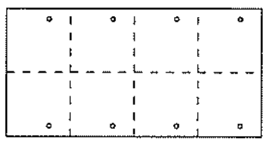
7. (Also UP14)
A map, 40 cm wide and 20 cm high, is folded along the dashed lines indicated to form a 10 cm × 10 cm square so that it just fits in its envelope. It is then pinned to a notice board.

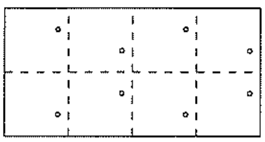


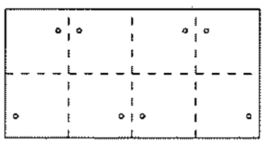
Which one of the following could be the pattern of pinholes on the map?

(A) 

(B) 

(C) 

(D) 

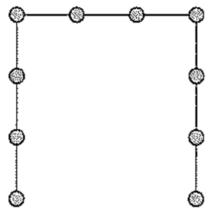
(E) 

- Wherever there is a fold, the two 10 cm × 10 cm squares either side will be mirror images of each other, with the line of symmetry being the fold,

hence (A).

8. (Also I6)

This diagram is called an *open square* of order 4, since the three sides are all the same length and each side has four posts spaced evenly along it. The total number of posts which would be evenly spaced along an open square of order 10 would be



(A) 26 (B) 27 (C) 28 (D) 30 (E) 32

- There will be 10 posts on the left and on the right, and ignoring the corner posts, 8 posts along the top, giving a total of 28 posts,

hence (C).

9. A train is scheduled to leave the station at 10:14 am and it takes 2 hours and 47 minutes to arrive at its destination. If the train leaves 8 minutes late, when does it arrive?

(A) 7:28 am (B) 7:35 am (C) 12:09 pm (D) 1:01 pm (E) 1:09 pm

- The train leaves at 10:22 am and arrives 2 hours and 47 minutes later. Since $22+47 = 69$, this is at 13:09, or 1:09 pm,

hence (E).

Comment

The train will arrive 2 hours and 55 minutes after 10:14 am, or almost 3 hours. So counting back from 1:14 pm is a natural way of arriving at the answer.

10. Consecutive numbers are written on five separate cards, one on each card. If the sum of the smallest three numbers is 60, what is the sum of the largest three numbers?

(A) 62 (B) 63 (C) 64 (D) 65 (E) 66

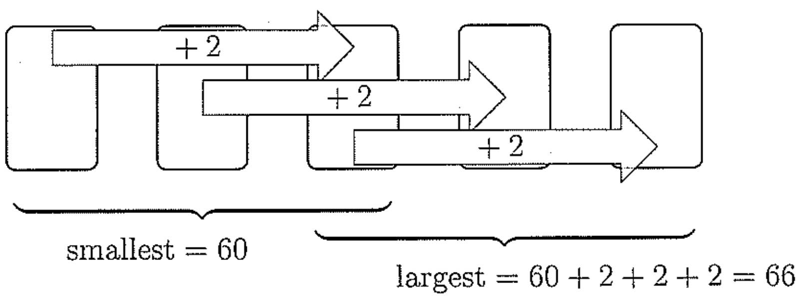
- *Alternative 1*

The average of the three smallest is 20, so they will be 19, 20 and 21. Then the largest three numbers are 21, 22 and 23, which add to 66,

hence (E).

Alternative 2

Comparing the largest three to the smallest three:

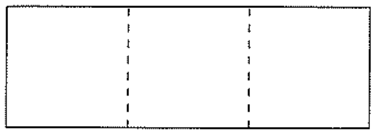


hence (E).

11. The width of a rectangle is one-third of its length. If its area is $108\,\text{cm}^2$ then its perimeter in centimetres is

(A) 54 (B) 48 (C) 42 (D) 36 (E) 24

▶ *Alternative 1*
The rectangle can be divided into three squares.



Then each square has area $108 \div 3 = 36\,\text{cm}^2$. So each square is $6\,\text{cm} \times 6\,\text{cm}$, the rectangle is $6\,\text{cm} \times 18\,\text{cm}$ and its perimeter is $6 + 18 + 6 + 18 = 48\,\text{cm}$,

hence (B).

Alternative 2
Suppose the rectangle has width w cm, then it has length $3w$ cm and area $3w \times w = 3w^2\,\text{cm}^2$. Then

$$3w^2 = 108$$
$$w^2 = 36$$
$$w = 6$$

So the rectangle has width 6 cm, length 18 cm and perimeter $2 \times (6 + 18) = 48\,\text{cm}$,

hence (B).

12. Six people are standing in a line. The height of the first person is 150 cm and the height of the sixth person is 180 cm. The height of each other person is the average of the heights of the person directly in front and the person directly behind. What is the height of the fourth person in the line?

(A) 165 cm (B) 168 cm (C) 170 cm (D) 172 cm (E) 174 cm

▶ Let the people be A, …, F. Then B's height is halfway between A's and C's, C's height is halfway between B's and D's, and so on. Consequently, the six heights are equally spaced from 150 cm to 180 cm. Since there are five jumps in height, they are 6 cm apart and so the heights of A, …, F must be 150, 156, 162, 168, 174 and 180 centimetres. The fourth person is 168 cm tall,

hence (B).

13. An unusual tower is built with cubes starting with one in the bottom layer, then 4 in the second layer, 9 in the third, then 16, and so on. Altogether 91 cubes are used to build the tower. How many layers does the tower have?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

▶ The layers are square numbers, and as we progressively add square numbers we find that

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$$

The tower has 6 layers,

hence (B).

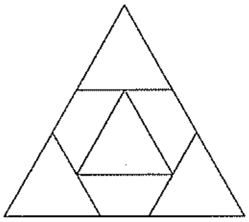
14. At my school, there are 76 students who are placed as evenly as possible in six classes, so that no two classes differ in size by more than one student. How many classes at the school have exactly 12 students?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

▶ The classes cannot all be the same size, since $76 \div 6 = 12\,\mathrm{r}4$. The difference between the smallest and largest classes must be equal to 1, so if we put 12 students in each class, the remaining 4 students must go into 4 different classes. The classes have sizes 12, 12, 13, 13, 13 and 13,

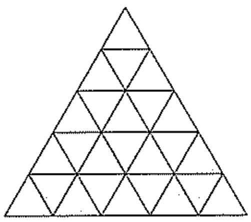
hence (A).

15. Four equilateral triangles of the same size are arranged with horizontal bases inside a larger equilateral triangle, as shown. What fraction of the area of the larger triangle is covered by the smaller triangles?



(A) $\frac{2}{3}$ (B) $\frac{1}{2}$ (C) $\frac{4}{9}$ (D) $\frac{4}{7}$ (E) $\frac{16}{25}$

▶ The shortest length in the figure is half the side of one of the shaded triangles, so we use this length as the basis of a triangular grid:



Then 16 out of the 25 equal triangles are shaded,

hence (E).

16. After 9 weeks Mikayla has an average mark of 5 out of 10 in the weekly spelling tests. What is the minimum number of extra weeks now required to raise her average to 7?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

▶ *Alternative 1*

Total marks gained after 9 weeks is $9 \times 5 = 45$. Suppose she can get full marks for the remaining n weeks, then she scores $45 + 10n$ marks in $9 + n$ weeks.

$$\begin{aligned} \frac{45+10n}{9+n} &= 7 \\ 45 + 10n &= 63 + 7n \\ 3n &= 18 \\ n &= 6 \end{aligned}$$

Therefore 6 more weeks are needed,

hence (C).

Alternative 2
Mikayla's target is 7 marks per test. After the 9 weeks, she is 18 marks behind her target. However, the best she can do after this is to score 10, which is 3 marks per test over target. Consequently the earliest she can make up the 18 marks is in 6 weeks. Then in all 15 weeks, she obtains $9 \times 5 + 6 \times 10 = 105$ marks, for an average of $105 \div 15 = 7$ marks,

hence (C).

17. Anne has four cards, each with a different number written on it. She makes a list of all the different totals that can be obtained by choosing two or more cards and adding the numbers on them. What is the maximum number of different totals that she could have in her list?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

▶ Let the numbers on the cards be a, b, c and d.
The possible sums if two numbers are chosen are

$$a+b, \quad b+c, \quad c+d, \quad a+c, \quad b+d, \quad a+d\,.$$

The possible sums if three numbers are chosen are

$$a+b+c, \quad a+b+d, \quad a+c+d, \quad b+c+d\,.$$

There is only one way to choose all four numbers, giving the sum

$$a+b+c+d$$

This gives 11 different ways of adding the numbers. These will give 11 different totals for most choices of a, b, c, d. For instance, using the coin denominations $a = 5$, $b = 10$, $c = 20$, $d = 50$ will give 11 different totals,

hence (E).

18. In the months of March, April and May, my lawn grows 0.7 cm every day. On the day that it reaches a height of 20 cm, I always mow it back to a height of 2.5 cm. If I mow my lawn on the first day of March, how many times in total do I need to mow the lawn during these three months?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

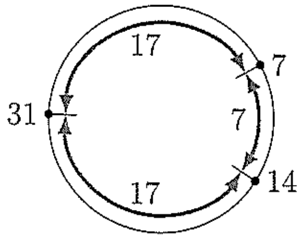
▶ Between mowings, the grass grows from 25 mm to 200 mm, giving 175 mm of growth, at 7 mm per day. The number of days between mowings is $175 \div 7 = 25$. The months in question have a total of 92 days, so the lawn is mowed on the 1st, 26th, 51st and 76th days but not the 101st,

hence (C).

19. There are n people sitting equally spaced around a circle. The people are numbered in order around the circle from 1 up to n. Person 31 notices that person 7 and person 14 are the same distance from him. How many people are sitting around the circle?

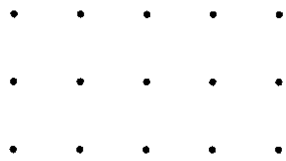
(A) 42 (B) 41 (C) 40 (D) 39 (E) 38

▶ There are 7 places from 7 to 14, and since there are 17 places from 14 to 31, there are also 17 places from 31 to 7. So in all there are $7 + 17 + 17 = 41$ places around the table,



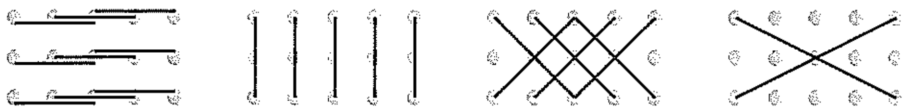
hence (B).

20. (Also UP26, I16)
A 3 by 5 grid of dots is set out as shown. How many straight line segments can be drawn that join two of these dots and pass through exactly one other dot?



(A) 14 (B) 20 (C) 22 (D) 24 (E) 30

▶ *Alternative 1*
We draw all such line segments—horizontal, vertical, at 45°, and others:



$9 + 5 + 6 + 2 = 22$,

hence (C).

Alternative 2
The line segments can be classified by the midpoint dot, since then each line segment is counted only once. Also the number of line segments through each dot form a symmetric pattern of numbers:

	1	1	1	
1	4	6	4	1
	1	1	1	

There are 22 line segments,

hence (C).

21. What is the sum of ten consecutive two-digit whole numbers where the first and last numbers are perfect squares?

(A) 205 (B) 210 (C) 215 (D) 225 (E) 230

▶ The difference between the first and last numbers is 9. The two-digit square numbers are 16, 25, 36, 49, 64 and 81 and the differences between these are increasing odd numbers 9, 11, 13, 15 and 17. Consequently the only two-digit square numbers that differ by 9 are 16 and 25.

So the numbers are $16, 17, \ldots, 25$ and $16 + 17 + \cdots + 24 + 25 = \frac{10}{2}(16 + 25) = 205$,

hence (A).

22. (Also I17)

A hotel has rooms that can accommodate up to two people. Couples can share a room, but otherwise men will share only with men and women only with women. How many rooms are needed to guarantee that any group of 100 people can be accommodated?

(A) 50 (B) 51 (C) 67 (D) 98 (E) 99

▶ At worst, 51 rooms will be needed. There are an even number of people coming as couples. So if there are an even number of single men there must be an even number of single women too, so that everyone can be paired up and 50 rooms will do. If, on the other hand, there are an odd number of single men, there will be an odd number of single women. So everyone can be paired except for one man and one woman, with 98 pairs in 49 rooms, plus one man and one woman who need a room each,

hence (B).

23. A three-digit number, written abc, is called *fuzzy* if abc is divisible by 7, the two-digit number bc is divisible by 6, the digit c is divisible by 5 and the three digits a, b and c are all different. How many fuzzy numbers are there?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

▶ If c is divisible by 5 then it must be 0 or 5.

If bc is divisible by 6, then c must be even, so that $c = 0$ and $b \neq 0$. Thus bc must be 30, 60 or 90.

If $abc = ab0$ is divisible by 7 then so is ab, where $b = 3, 6$ or 9. The multiples of 7 ending in $3, 6$ or 9 are 49, 56 and 63, so abc must be one of $490, 560$ or 630. Therefore there are 3 fuzzy numbers,

hence (D).

24. If a is the number $1111\ldots1111$, with 100 digits all 1, and b is the number $999\ldots999$ with 50 digits all 9, how many digits are 1 in the number $a - b$?

(A) 49 (B) 50 (C) 97 (D) 98 (E) 99

▶ *Alternative 1*

We can see from the subtraction

$$\begin{array}{ccccccccccccc}
1 & 1 & 1 & \cdots & 1 & {}^{0}\cancel{1} & {}^{10}\cancel{1} & {}^{10}\cancel{1} & \cdots & {}^{10}\cancel{1} & {}^{10}\cancel{1} & {}^{1}1 & \\
 & & & & & & 9 & 9 & \cdots & 9 & 9 & 9 & - \\
\hline
1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 2 &
\end{array}$$

that the units digit in the result is 2, then there are 49 digits 1, one digit of 0 and the remaining 49 digits are 1, so there are 98 digits equal to 1,

hence (D).

Alternative 2

Adding 1 to both a and b,

$$\underbrace{111111\ldots111111}_{100 \text{ digits}} - \underbrace{999\ldots999}_{50 \text{ digits}} = \underbrace{111111\ldots111112}_{100 \text{ digits}} - \underbrace{1000\ldots000}_{51 \text{ digits}}$$

$$= \underbrace{11111\ldots1111011111\ldots11112}_{100 \text{ digits}}$$

So 98 of the digits in $a-b$ are 1,

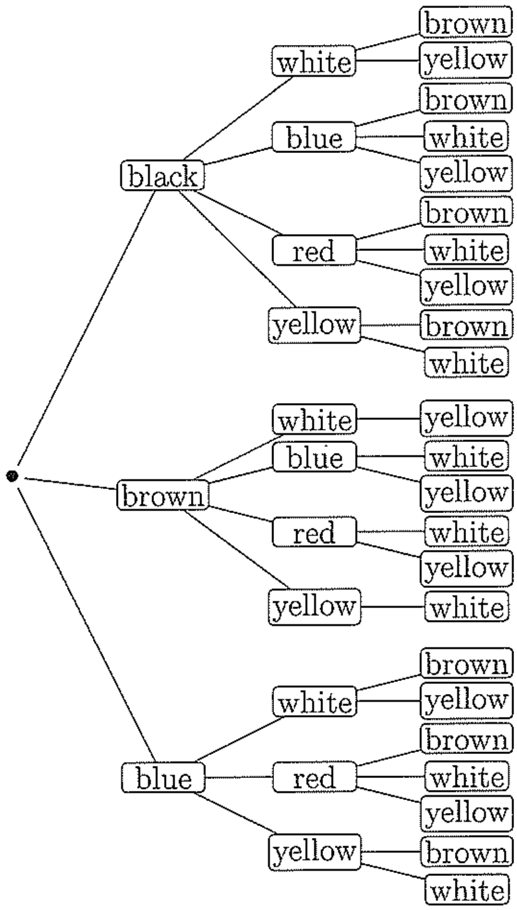
hence (D).

25. Zac has three jackets, one black, one brown and one blue. He has four shirts, one white, one blue, one red and one yellow. He has three pairs of trousers, one brown, one white and one yellow. How many combinations of jacket, shirt and trousers are possible if no two items are of the same colour?

(A) 23 (B) 25 (C) 26 (D) 27 (E) 29

▶ *Alternative 1*

This tree shows all the possibilities of choosing jacket, then shirt, then trousers, each of a different colour:



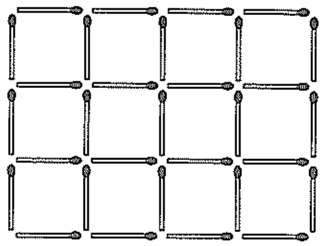
hence (A).

Alternative 2

If colour does not matter, there are $3 \times 4 \times 3 = 36$ combinations. Of these 4 have two browns, 3 have two blues, 3 have two whites and 3 have two yellows, and none have all three the same. So there are $36 - 13 = 23$ combinations,

hence (A).

26. The diagram shows a grid 3 units high and 4 units wide that uses 31 matches. How many matches would you need to create a grid of squares that is 13 units high and 33 units wide?

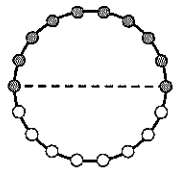


▶ The required grid contains 14 rows of horizontal matches and each row contains 33 matches. So there are $14 \times 33 = 462$ horizontal matches. The grid contains 34 columns of vertical matches and each column contains 13 matches. So there are $34 \times 13 = 442$ vertical matches. Therefore, the total number of matches needed is $462 + 442 = 904$,

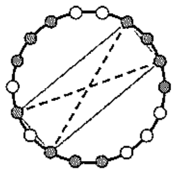
hence (904).

27. Eighteen points are equally spaced on a circle, from which you will choose a certain number at random. How many do you need to choose to guarantee that you will have the four corners of at least one rectangle?

▶ The four corners of an inscribed rectangle appear as the ends of two diameters. It is possible to choose 10 points without having two complete diameters, as for example, the 10 consecutive points shaded below:



However, once 11 or more points are chosen, then at most 7 diameters are incomplete. So at least 2 diameters are complete, forming a rectangle.

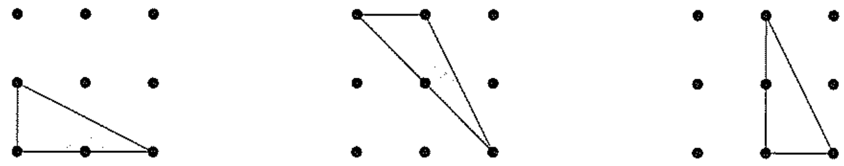


Consequently 11 points are needed to guarantee one rectangle,

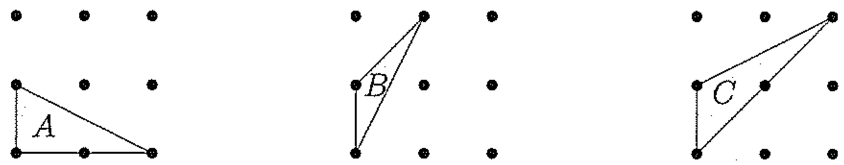
hence (11).

28. (Also I26)

In a 3×3 grid of points, many triangles can be formed using 3 of the points as vertices. Three such triangles are shown below. Of all these possible triangles, how many have all three sides of different lengths?



▶ Here are three non-congruent scalene triangles in the grid.



To see that there are no others, consider the longest side of such a triangle.

(i) If the longest side has length $\sqrt{2^2 + 2^2} = 2\sqrt{2}$, then it must be a diagonal of the grid. The third vertex cannot be a corner dot, so it must be an edge dot, giving triangle C.

(ii) If the longest side has length $\sqrt{2^2 + 1^2} = \sqrt{5}$ then the other sides can only be 1, $\sqrt{2}$ or 2. With 1 and $\sqrt{2}$ we get triangle B and with 1 and 2 we get triangle A. But with 2 and $\sqrt{2}$ we don't get a grid point.

(iii) If the longest side has length 2 then the other sides must be 1 and $\sqrt{2}$, but there is no triangle with sides 1, $\sqrt{2}$ and 2 on the grid.

Triangle A can be in 8 different orientations—4 rotations and the reflections of these. Since it only occupies 2×1 cells on the grid, each of these orientations can be in 2 positions, giving 16 triangles. Triangle B is similar, also giving 16 triangles.

Triangle C can be in 8 different orientations, but it occupies 2×2 cells on the grid, and so each orientation can only be in one position, giving 8 triangles. So in total there are $16 + 16 + 8 = 40$ triangles,

hence (40).

29. (Also UP30)

How many three-digit numbers are there in which one of the digits is the sum of the other two?

▶ The possibilities can be counted by considering three cases.

(i) The smallest digit is 0, so the digits are 0, a and a, for $a = 1, \ldots, 9$. For example, 404. These digits can form two numbers like 404 and 440. So this case has 18 numbers.

(ii) The two smallest digits are equal, so the digits are a, a and $2a$, for $a = 1, 2, 3, 4$. For example, 633. These digits can form three numbers like 336, 363 and 633. So this case has 12 numbers.

(iii) All three digits are different, so there is a smallest digit a, a second smallest digit b and a largest digit $a+b$. For example, 385. These digits can form six numbers like 358, 385, 538, 583, 835 and 853.

To fully count case (iii) we need to know how many pairs of digits a and b are possible. This table shows possible choices of a and b, and the three digits obtained.

a \ b	2	3	4	5	6	7	8
1	123	134	145	156	167	178	189
2		235	246	257	268	279	
3			347	358	369		
4				459			

For each of these $7+5+3+1=16$ selections of digits, six numbers can be formed, and so there are $16 \times 6 = 96$ numbers in case (iii).
There are $18+12+96=126$ numbers in total,

hence (126).

30. (Also S26)
What is the largest three-digit number with the property that the number is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit?

▶ *Alternative 1*
Let the number be abc.
Then

$$\begin{aligned} 100a+10b+c &= a+b^2+c^3 \\ 99a+10b-b^2 &= c(c^2-1) \\ 99a+b(10-b) &= (c-1)c(c+1) \end{aligned}$$

Consider the possibilities:

$99a$	$b(10-b)$	$(c-1)c(c+1)$
$99 \times 1 = 99$	$1 \times 9 = 9$	$1 \times 2 \times 3 = 6$
$99 \times 2 = 198$	$2 \times 8 = 16$	$2 \times 3 \times 4 = 24$
$99 \times 3 = 297$	$3 \times 7 = 21$	$3 \times 4 \times 5 = 60$
$99 \times 4 = 396$	$4 \times 6 = 24$	$4 \times 5 \times 6 = 120$
$99 \times 5 = 495$	$5 \times 5 = 25$	$5 \times 6 \times 7 = 210$
$99 \times 6 = 594$	$6 \times 4 = 24$	$6 \times 7 \times 8 = 336$
$99 \times 7 = 693$	$7 \times 3 = 21$	$7 \times 8 \times 9 = 504$
$99 \times 8 = 792$	$8 \times 2 = 16$	$8 \times 9 \times 10 = 720$
$99 \times 9 = 891$	$9 \times 1 = 9$	

Looking at the possibilities for $99a+b(10-b)=(c-1)c(c+1)$, we have two:
$99+21=120 \Longrightarrow a=1,\ b=3$ or $7, c=5 \Longrightarrow n=135$ or $n=175$.
$495+9=504 \Longrightarrow a=5,\ b=1$ or $9, c=8 \Longrightarrow n=518$ or $n=598$.
So, there are four 3-digit numbers which satisfy the requirements and the largest of these four numbers is 598,

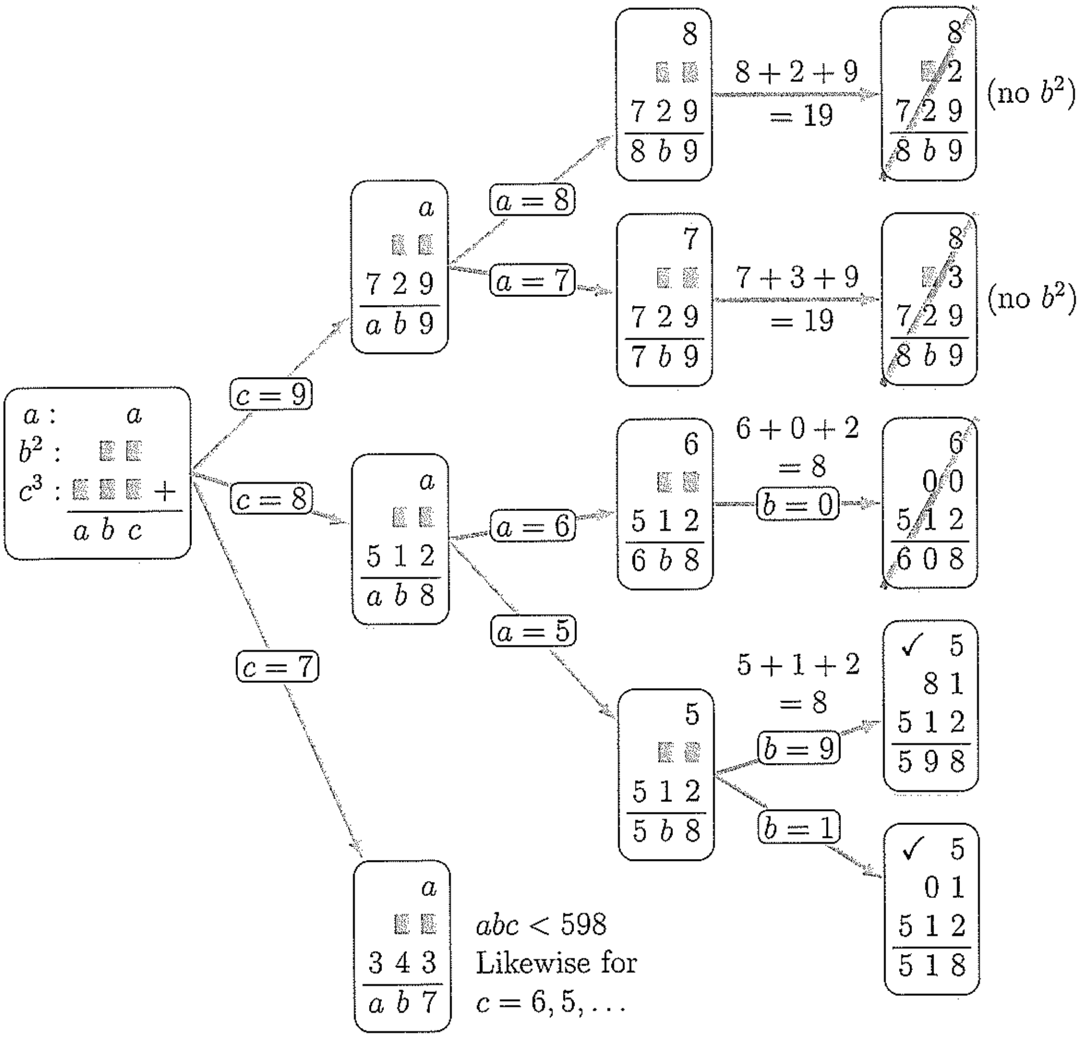
hence (598).

Alternative 2

The number abc is equal to $a + b^2 + c^3$, and these are the possible values of b^2 and c^3:

Digit	0	1	2	3	4	5	6	7	8	9
Square	0	1	4	9	16	25	36	49	64	81
Cube	0	1	8	27	64	125	216	343	512	729

We try these numbers in an addition grid, trying the large values of c first, then filling in possible values for a and b. This trial-and-error search is presented here as a tree.



The largest solution found is 598, and any solutions on branches $c = 7$, $c = 6$, ..., $c = 1$ must be less than this,

hence (598).