Q3

Carlingford High School

Year 11 Mathematics Extension 1

HSC Assessment Task 1

Term 4 2020

Time allowed: 50 minutes

Total marks: 33

Student number: Solutions

Instructions:

- Write your student number at the start of **every** question
- Use black pen. Pencil may be used for diagrams **only**
- Board approved calculators may be used
- Answer each question in the space provided
- Show all necessary working
- Marks may be deducted for illegible or badly set out work
- No lending or borrowing
- A formula sheet is provided

Student number: Solutions

HSC E1
AT1
Term 4 2020

Question 1 (13 marks)

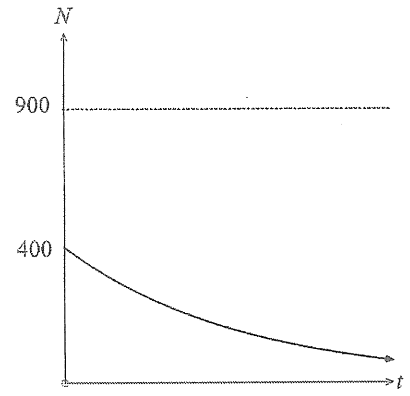
Answer this question in the space provided.

a) Circle the best answer.

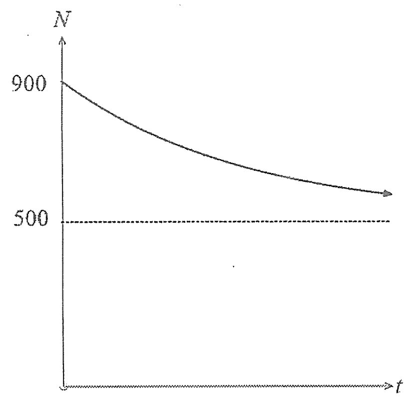
i) A quantity N has an initial value of 900 and the rate of change of N is given by the equation $\frac{dN}{dt} = 0.35(N - 500)$. **1**

Which graph shows the relationship between N and t ?

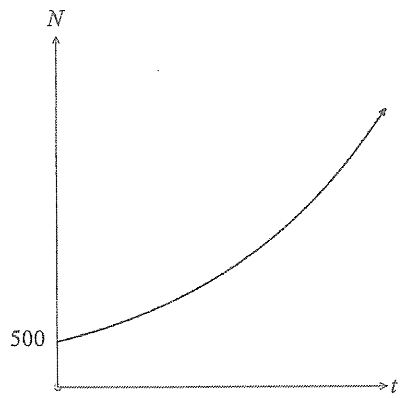
(A)



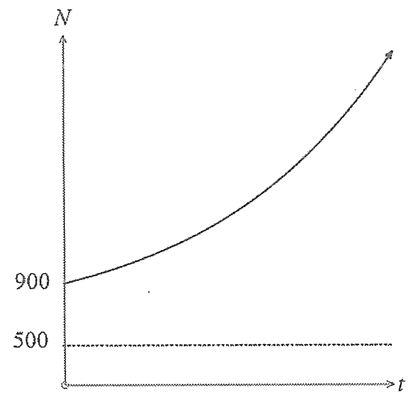
(B)



(C)



(D)



ii) Given that $\frac{dM}{dr} = 4$ and $\frac{dM}{dt} = 4t^3$, find the value of $\frac{dr}{dt}$ when $t = 2$. **1**

(A) 8

(B) 12

(C) 200

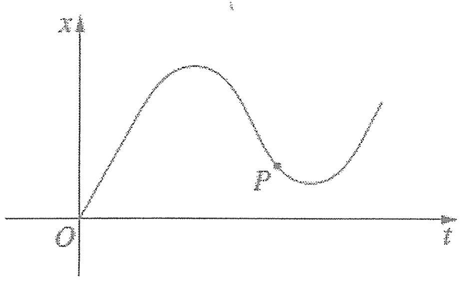
(D) 300

Q1 KC

Q2 LW

Q3 MB

iii) The graph shows the displacement x of a particle moving along a straight line as a function of time t. **1**



Which statement describes the motion of the particle at point P?

(A) The velocity is negative and the acceleration is negative.

(B) The velocity is negative and the acceleration is positive.

(C) The velocity is positive and the acceleration is negative.

(D) The velocity is positive and the acceleration is positive.

iv) Mrs Wilson wishes to prove by induction that statement S is true for all positive even integers. As a first step she proves $S(2)$. What should her second step be? **1**

(A) Prove $S(3)$

(B) Show that if $S(k)$ is true then $S(k+1)$ is true

(C) Show that if $S(k)$ is true then $S(k+2)$ is true

(D) She should have started with $S(1)$

b) The mass M of a radioactive substance at a time t satisfies the equation

$$M = M_0 e^{-kt}$$

where M_0 is the initial mass and k is constant.

i) If the half life of the substance is 8 hours, show that $k = \frac{1}{8}\ln 2$. **1**

$$e^{-8k} = \frac{1}{2}$$

$$\ln\left(\frac{1}{2}\right) = -8k$$

$$\ln 2 = 8k$$

$$k = \frac{1}{8}\ln 2$$

ii) If the instantaneous rate of change of the mass after 3 hours is -5.2 grams per hour, find M_0 correct to the nearest gram. **2**

$$\frac{dM}{dt} = -k M_0 e^{-kt}$$

$$-5.2 = -\frac{1}{8}\ln 2 . M_0 e^{-3/8 \ln 2}$$ ✓

$$M_0 = \frac{5.2}{1/8 \ln 2 \, e^{-3/8 \ln 2}}$$

$$= 77.83127\ldots$$

$$\approx 78g$$ ✓

c) The displacement of a particle moving along the x-axis is given by

$$x = 2t - \frac{1}{1+t},$$

where x is the displacement from the origin in metres, t is the time in seconds and $t \geq 0$.

i) What is the initial position of the particle? **1**

at $t=0$, $x = -1$

ii) Show that the acceleration of the particle is always negative. **2**

$$v = \frac{dx}{dt} = 2 + \frac{1}{(1+t)^2}$$

$$a = \frac{d^2x}{dt^2} = \frac{-2}{(1+t)^3} \quad \checkmark$$

$\therefore$ a is always negative since $1+t$ is always positive $\checkmark$

iii) What value does the velocity of the particle approach as t increases indefinitely? **1**

$$v = 2 + \frac{1}{(1+t)^2}$$

as $t \to \infty$, $\frac{1}{(1+t)^2} \to 0$

$\therefore v \to 2$

Student Number:______________________________

Question 2 (12 marks)

Answer this question in the space provided.

a) Let T be the temperature inside a room at time t and let A be the constant outside air temperature. According to Newton's law of cooling, $\frac{dT}{dt}$ is proportional to $(T - A)$,

i) Show that $T = A + Ce^{kt}$ (where C and k are constants), satisfies Newton's law of cooling. **1**

$$T = A + Ce^{kt}$$

$$\frac{dT}{dt} = kCe^{kt}$$

$$= kT - kA$$

$$= k(T - A), \text{ k constant}$$

$\therefore T = A + Ce^{kt}$ satisfies Newton's law of cooling

ii) The outside temperature is 5°C and a heating system breakdown causes the inside room temperature to fall from 20°C to 17°C in half an hour. After how many hours is the inside room temperature equal to 10°C? **3**

2 dps

at $t = 0$, $T = 20$ $A = 5$ $\therefore C = 15$

$T = 5 + 15e^{kt}$ ✓

at $t = \frac{1}{2}$ $T = 17$ $\therefore$ $17 = 5 + 15e^{k/2}$

$$12 = 15e^{k/2}$$

$$4/5 = e^{k/2}$$

$$\frac{k}{2} = \ln(0.8)$$

$$k = 2\ln(0.8) \approx -0.446$$ ✓

$$10 = 5 + 15e^{2\ln(0.8)t}$$

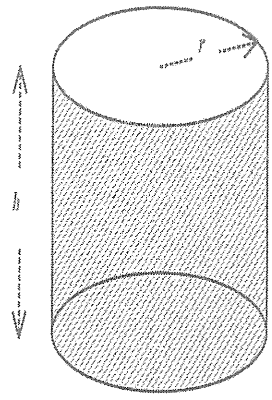
$$e^{2\ln(0.8)t} = 1/3$$

$$2\ln(0.8)t = -\ln(3)$$

$$t = -\frac{\ln(3)}{2\ln(0.8)}$$

2.46h OR $\approx$ 2h 28 minutes ✓

b) The solid shown below is a cylinder which has its height equal to twice its diameter.



A virtual 3-D model is created which maintains the ratio of height and diameter.

In an animation, the model is being scaled up so that its volume is increasing at a steady rate of 1200 cm^3 per second.

i) Show that the volume V is given by the equation $V = 4\pi r^3$ **1**

$V = \pi r^2 h$

$h = 2d = 4r$

$\therefore V = 4\pi r^3$

ii) Find an expression for $\frac{dr}{dt}$ in terms of r. **2**

$\frac{dV}{dr} = 12\pi r^2$

$\frac{dV}{dt} = 1200$ ✓

$\frac{dr}{dt} = \frac{1200}{12\pi r^2}$

$= \frac{100}{\pi r^2}$ ✓

iii) The animation reverses when $\frac{dr}{dt} = \frac{1}{4\pi}$ cm per second. What is the radius at this point? **1**

$\frac{100}{\pi r^2} = \frac{1}{4\pi}$

$\frac{\pi r^2}{100} = 4\pi$

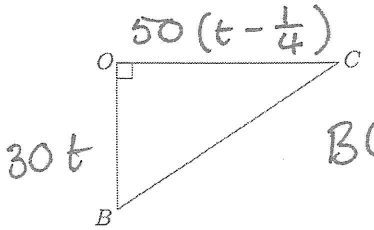
$r^2 = 400$

$r = 20$ cm

c) Two long straight roads meet at an angle of 90°. Bicycle B starts from this intersection and travels along one road at 30 km/h. Fifteen minutes later, Car C passes through the intersection and continues down the other road at 50 km/h.

i) Show that the distance between the car and the bicycle t hours after the bicycle leaves the intersection is given by **2**

$$BC = \sqrt{3400t^2 - 1250t + 156.25}$$



$$BC^2 = (30t)^2 + 50^2\left(t - \frac{1}{4}\right)^2$$
$$= 900t^2 + 2500\left(t^2 - \frac{1}{2}t + \frac{1}{16}\right)$$
$$= 3400t^2 - 1250t + 156.25$$

$$\therefore BC = \sqrt{3400t^2 - 1250t + 156.25}$$

ii) At what rate is the distance between the car and the bicycle increasing 1 hour after bicycle B left the intersection? (Answer to one decimal place.) **2**

$$\frac{d(BC)}{dt} = \frac{1}{2}(6800t - 1250)(3400t^2 - 1250 + 156.25)^{-1/2}$$

at $t = 1$

$$= \frac{\frac{1}{2} \times 5550}{\sqrt{2306.25}}$$
$$\approx 57.8 \text{ km/h}$$

Student Number:______________________

Question 3 (10 marks)

Answer this question in the space provided.

a) Prove by induction that $13 \times 6^n + 2$ is divisible by 5 for all integers $n \geq 0$. **4**

$n = 0 \quad 13 \times 6^0 + 2 = 15$
$= 5 \times 3$

$\therefore$ the statement is true for $n = 0$ ✓

Suppose that $13 \times 6^k + 2$ ~~= 15~~ is divisible by 5 for some integer $k \geq 0$.

Then: $13 \times 6^k + 2 = 5m$ for some integer $m > 0$. ✓

$$13 \times 6^{k+1} + 2 = 6[13 \times 6^k + 2] - 10$$
$$= 6 \times 5m - 10$$
$$= 5(6m - 2)$$

$\therefore 13 \times 6^{k+1} + 2$ is divisible by 5 ✓

We have shown that $13 \times 6^n + 2$ ~~is true~~ is divisible by 5 for $n = 0$ and that if it is divisible by 5 for $n = k$, it is divisible by 5 for $n = k+1$. Therefore, by mathematical induction, ~~the~~ $13 \times 6^n + 2$ is divisible by 5 for all integers $n \geq 0$.

✓

b) Prove by mathematical induction that $1 + 10 + 10^2 + \cdots + 10^{n-1} = \frac{1}{9}(10^n - 1)$ for all integers $n \geq 1$. 4

$n = 0$ ~~$1 = \frac{1}{9}(10-1)$~~ LHS $= 1$

RHS $= \frac{1}{9}(10-1) = 1$ ✓

$\therefore$ RHS $=$ LHS

Suppose the statement is true for $n = k$ for some integer $k \geq 0$ so that

$1 + 10 + 10^2 + \ldots + 10^{k-1} = \frac{1}{9}(10^k - 1)$ ✓

~~When $n = k+1$~~

then $1 + 10 + \ldots + 10^{k-1} + 10^k$

$$= \frac{1}{9}(10^k - 1) + 10^k$$
$$= \frac{1}{9}(10^k + 9 \times 10^k - 1)$$
$$= \frac{1}{9}(10 . 10^k - 1)$$
$$= \frac{1}{9}(10^{k+1} - 1)$$

$\therefore$ the statement holds for $n = k+1$ ✓

Since we have shown that the statement holds for $n = 1$ and for $n = k+1$ whenever it is true for $n = k$, by mathematical induction $1 + 10 + \ldots + 10^{k-1} = \frac{1}{9}(10^k - 1)$.

✓

c) Mr Fardouly notices that $2^3 - 1$, $2^5 - 1$ and $2^7 - 1$ are all prime numbers.

i) Prove that $2^n - 1$ is **not** always a prime number when n is odd. **1**

$2^1 - 1 = 1$ not prime

OR $2^9 - 1 = 511$

$= 7 \times 73$ not prime

OR $2^{11} - 1 = 2047$

ii) Explain why he should not try to prove by induction that $2^n - 1$ is prime whenever $n > 0$ is prime. **1**

Because we have no formula for the next prime number

[OR Because there is a counter example - if one is shown in part i]

End of Exam – Please check your work.