ACE
EXAM
PAPER

Student name: ____________________

PAPER 3

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 2

General Instructions

- Working time - 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Which of the following is an expression for $\int \frac{(x-1)^2}{x} dx$?

(A) $\frac{1}{2}x^2 + 2x - \ln|x| + C$

(B) $\frac{1}{2}x^2 - 2x + \ln|x| + C$

(C) $2x - 3 + \frac{2}{x} - \frac{1}{x^2} + C$

(D) $\frac{(x-1)^3}{3x} + C$

2. Let $\arg(z) = \frac{\pi}{5}$ for a certain complex number z. What is $\arg(z^7)$?

(A) $-\frac{7\pi}{5}$

(B) $-\frac{3\pi}{5}$

(C) $\frac{2\pi}{5}$

(D) $\frac{3\pi}{5}$

3. A particle is moving in simple harmonic motion in a straight line. At t seconds, it has a displacement of x metres from a fixed point O on the line, where x is given by $x = 4\sin^2 t - 1$. What is the centre of motion?

(A) $x = -1$

(B) $x = 0$

(C) $x = 1$

(D) $x = 2$

4. What is the value of $\int_1^3 x(x-2)^5\,dx$?

(A) $\frac{1}{7}$

(B) $\frac{2}{7}$

(C) $\frac{1}{3}$

(D) $\frac{2}{3}$

5. The points P and Q have position vectors relative to the origin O given by:

$\overrightarrow{OP} = \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

What is $\angle POQ$ to the nearest tenth of a degree?

(A) 67.6°

(B) 91.0°

(C) 112.4°

(D) 121.3°

6. Which of the following are the linear factors of $z^2 + 6z + 10$ over the complex field?

(A) $(z+3-i)(z+3+i)$

(B) $(z+3+i)^2$

(C) $(z-3-i)(z-3-i)$

(D) $(z+3+i)(z-3-i)$

7. A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $2m(v+v^2)$ Newtons when its speed is v ms^{-1}. At time t seconds, the particle has a displacement x metres from a fixed point O on the line and a velocity of v ms^{-1}. Which of the following is an expression for x in terms of v?

(A) $-\frac{1}{2}\int \frac{1}{1+v}\,dv$

(B) $-\frac{1}{2}\int \frac{1}{v(1+v)}\,dv$

(C) $\frac{1}{2}\int \frac{1}{1+v}\,dv$

(D) $\frac{1}{2}\int \frac{1}{v(1+v)}\,dv$

8. The converse of $A \Rightarrow B$ is:

(A) $(\text{not } A) \Leftrightarrow (\text{not } B)$

(B) $(\text{not } B) \Rightarrow (\text{not } A)$

(C) $B \Leftrightarrow B$

(D) $B \Rightarrow A$

9. A vector of magnitude 6 and with direction opposite to $\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$ is:

(A) $-6\underset{\sim}{i} + 12\underset{\sim}{j} - 2\underset{\sim}{k}$

(B) $-3\underset{\sim}{i} + 6\underset{\sim}{j} - 6\underset{\sim}{k}$

(C) $6\underset{\sim}{i} - 12\underset{\sim}{j} + 12\underset{\sim}{k}$

(D) $-2\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$

10 What is the value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1 - \sin^2 x}}\, dx$?

(A) $\ln\left(1 + \sqrt{2}\right)$

(B) $\ln\left(1 + \sqrt{3}\right)$

(C) $\dfrac{\pi}{2}$

(D) π

Section II

90 marks
Attempt questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (13 marks) **Marks**

(a) Three vectors are $\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j} + m\underset{\sim}{k}$ and $\underset{\sim}{w} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ where m is a real number.

(i) Find the value(s) of m for which $|\underset{\sim}{v}| = 2\sqrt{3}$. **2**

(ii) Find the value of m such that $\underset{\sim}{u}$ is perpendicular to $\underset{\sim}{v}$. **2**

(b) For $z = 1 - i, w = 3 - 2i$, find:

(i) $|z + w|$ **1**

(ii) $z^2 - w^2$ **2**

(c) The point A has position vector $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, point B has position vector $\overrightarrow{OB} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$ and point C has position vector $\overrightarrow{OC} = 4\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$ relative to an origin O. The point D is such that $ABCD$ is a parallelogram. Find the position vector of point D. **3**

(d) Use integration by parts to evaluate $\int e^x \sin x \, dx$. **3**

Question 12 (14 marks) **Marks**

(a) A particle is moving in a straight line in simple harmonic motion. If the amplitude of the motion is 3 cm and the period of the motion is 4 seconds, calculate the:

(i) maximum velocity of the particle. **2**

(ii) maximum acceleration of the particle. **2**

(iii) speed of the particle when it is 1 cm from the centre of the motion. **2**

(b) (i) Write $z = 1 + i$ in modulus-argument form. **1**

(ii) Find $|z^{10}|$ and $\arg(z^{10})$. **2**

(c) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that: **3**

$$\int_0^{\frac{\pi}{2}} \frac{1}{5 + 5\sin x - 3\cos x}\,dx = \int_0^1 \frac{1}{4t^2 + 5t + 1}\,dt$$

(ii) Hence evaluate in the simplest exact form. **2**

$$\int_0^{\frac{\pi}{2}} \frac{1}{5 + 5\sin x - 3\cos x}\,dx$$

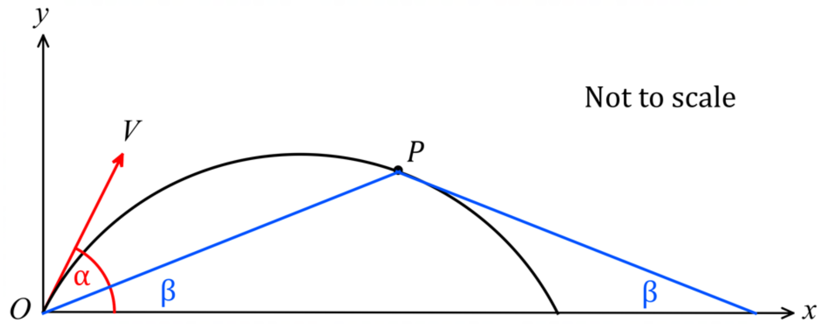
Question 13 (16 marks) **Marks**

(a) If $T_1 = 7$ and $T_n = 2T_{n-1} - 1$ for $n \geq 2$ show that: **3**

$T_n = 3.2^n + 1$ for $n \geq 1$

(b) Find the exact value of $\int_0^{\sqrt{2}} \sqrt{4 - x^2}\, dx$. **3**

(c) A particle is projected from a point O with speed V ms^{-1} at an angle of α to the horizontal. At a certain point P on its trajectory, the direction of motion of the particle and the line OP are inclined at equal angles β to the horizonal. **4**



Show that the time taken to reach P from O is $\dfrac{4V\sin\alpha}{3g}$.

(You can assume the six equations of motion)

(d) (i) Find the values of A, B, C and D such that: **2**

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

(ii) Hence evaluate $\displaystyle\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx$ **2**

(e) Show that $z.\bar{z} = |z|^2$ for any complex number. **2**

Question 14 (18 marks) **Marks**

(a) (i) For positive real numbers a, b show that $a^2 + b^2 \geq 2ab$. **1**

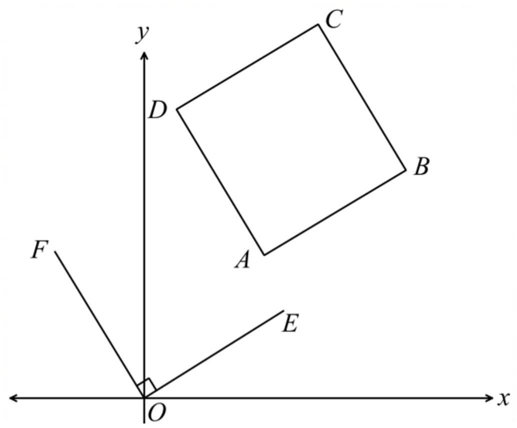
(ii) Hence show for positive real numbers a, b, c and d that: **2**

$3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$

(iii) Hence show for positive real numbers a, b, c and d and if **3**

$a + b + c + d = 1$ then $(ab + ac + ad + bc + bd + cd) \leq \frac{3}{8}$

(b) In the Argand diagram, $ABCD$ is a square, and OE and OF are parallel and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers z_1 and z_2 respectively.



(i) Explain why the point E corresponds to $z_2 - z_1$. **2**

(ii) What complex number corresponds to point F? **1**

(iii) What complex number corresponds to vertex D? **1**

(c) The point A, with coordinates $(0, a, b)$ lies on the line that has the equation: **2**

$\underset{\sim}{r} = 4\underset{\sim}{i} + 10\underset{\sim}{j} - \underset{\sim}{k} + \lambda(\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k})$

Find the values of a and b.

(d) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ where n is a non negative integer.

(i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\,dx \quad n \geq 2$. **2**

(ii) Deduce that $I_n = \dfrac{n-1}{n} I_{n-2}$. **2**

(iii) Evaluate I_4. **2**

Question 15 (14 marks) **Marks**

(a) Sketch the locus of z on the Argand diagram where the inequalities $|z-1| \leq 3$ and $\text{Im}(z) \geq 3$ hold simultaneously. **3**

(b) If a and b are two positive real numbers, prove that:

(i) $\dfrac{a+b}{2} \geq \sqrt{ab}$ **1**

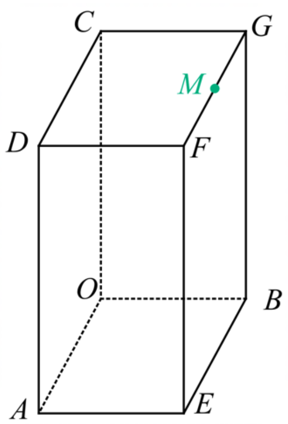
(ii) $(a+b)\left(\dfrac{1}{a}+\dfrac{1}{b}\right) \geq 4$ **2**

(c) (i) Find all pairs of integers x and y such that $(x+iy)^2 = -3-4i$. **2**

(ii) Hence or otherwise solve the quadratic equation: **2**

$z^2 - 3z + (3+i) = 0$

(d) A rectangular prism has $\overrightarrow{OA} = 3\underset{\sim}{i}$ $\overrightarrow{OB} = 5\underset{\sim}{j}$ and $\overrightarrow{OC} = 4\underset{\sim}{k}$



(i) Find $\overrightarrow{EB}$, $\overrightarrow{EF}$, $\overrightarrow{OE}$ and $\overrightarrow{OF}$. **2**

(ii) Find $\overrightarrow{OM}$ where M is the midpoint of FG. **2**

Question 16 (15 marks) **Marks**

(a) A speed boat of mass m is travelling at maximum power. At maximum power, its engine delivers a force F on the speed boat. The water exerts a resistive force proportional to the square of the speed boat's speed v.

(i) Show that $\ddot{x} = \frac{1}{m}(F - kv^2)$ where k is a positive constant. **2**

(ii) The speed boat increases its speed from v_1 to v_2. Show that the distance travelled during this period is: **3**

$$x = \frac{m}{2k}\ln\left(\frac{F - k{v_1}^2}{F - k{v_2}^2}\right)$$

(b) (i) Use the principle of mathematical induction to prove that: **4**

$$\sum_{i=1}^{n} i^2 = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$

(ii) Hence or otherwise evaluate $\lim\limits_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$. **1**

(c) Let $\overrightarrow{OA} = 3\underset{\sim}{i} + 4\underset{\sim}{k}$ and $\overrightarrow{OB} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$. Find $\overrightarrow{OC}$ where C is:

(i) the midpoint of the line segment AB. **2**

(ii) the point such that $\overrightarrow{AC} = \frac{4}{3}\overrightarrow{AB}$. **3**

End of paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

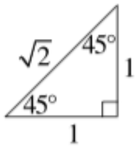
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

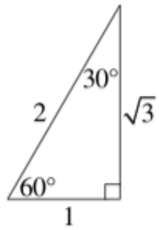
$A = \dfrac{1}{2}ab\sin C$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

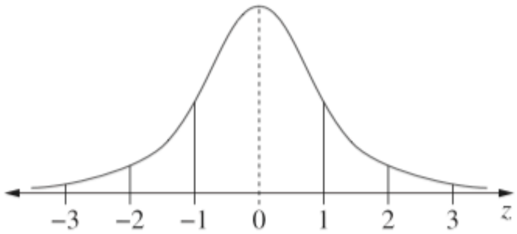
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$