Solutions – Senior Division

2015

1. $21 \times 2015 = 42\,315$,

hence (C).

2. The first equation is $85 = 4 + R^2$, which simplifies to $R^2 = 81$ and has positive solution $R = \sqrt{81} = 9$,

hence (C).

3. Ngoc's share is $\$1000 \times \frac{3}{8} = \$375$,

hence (D).

4. The angles in the regular polygons are 60°, 90° and 108°, a total of 258°. The remaining angle is $x = 360° - 258° = 102°$,

hence (E).

5. $3^{-2} - 2^{-3} = \frac{1}{9} - \frac{1}{8} = \frac{8}{72} - \frac{9}{72} = -\frac{1}{72}$,

hence (C).

6. (Also J17, I11)

Alternative 1

Jenna must leave out a longer side and Dylan a shorter side, where the longer side is 8 cm longer than the shorter side. So the sides are x cm and $(x + 8)$ cm. Then Jenna's measurement is $2x + x + 8 = 80$, so that $3x = 72$ and $x = 24$. The rectangle is 24 cm by 32 cm. The perimeter is then $2 \times 24 + 2 \times 32 = 112$ cm,

hence (A).

Alternative 2

Suppose the rectangle has width w and height h. Dylan's and Jenna's measurements are $2w + h$ and $2h + w$. Adding these, $3w + 3h = 80 + 88 = 168$ and so $w + h = 168 \div 3 = 56$. Then the perimeter is $2(w + h) = 112$ cm,

hence (A).

7. (Also J22, I14)

The score is a multiple of 6 if one of the dice is 6 or if one of the dice is even (2, 4 or 6) and the other is 3. We tabulate these possibilities amongst the 36 equally likely rolls.

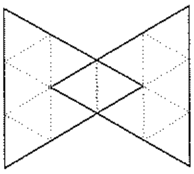
Second dice

First dice	1	2	3	4	5	6
1						✓
2			✓			✓
3		✓		✓		✓
4			✓			✓
5						✓
6	✓	✓	✓	✓	✓	✓

Then the probability that the score is a multiple of 6 is $\frac{15}{36} = \frac{5}{12}$,

hence (B).

8. *Alternative 1*

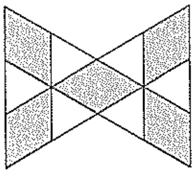


In this grid of equilateral triangles 2 triangles out of 16 are shaded,

hence (B).

Alternative 2

For each triangle, rotate the rhombus about the triangle's centre by $\pm 120°$.



The remaining 6 triangles are equilateral, equal in area to 3 rhombuses. Consequently the total area of the hexagon is equal to 8 times the area of the shaded rhombus,

hence (B).

9. *Alternative 1*

Let the two numbers be x and y, with x the larger.

From the first statement, $x = 20 + y$. From the second statement, $x + 4 = 3(y + 4)$ so that $x = 8 + 3y$.

Then $20 + y = 8 + 3y$, which solves to $y = 6$. Then $x = 20 + y = 26$,

hence (A).

Alternative 2

The difference doesn't change when 4 is added. Then the two 'new' numbers have ratio 3 and difference 20, so they must be 10 and 30. So the original numbers were 6 and 26,

hence (A).

10. (Also I12)

Eight days is $8 \times 24 = 192$ hours. Hence the speed in kilometres per hour is $\frac{11500}{192} \approx \frac{12000}{200} = 60\,\text{km/h}$,

hence (E).

11. The 9 possible products give $0, 0, 0, 3, 6, x, 6, 12, 2x$, which is the set

$$C = \{0, 3, 6, 12, x, 2x\}.$$

For C to have exactly 5 elements, either (i) $x \in \{0, 3, 6, 12\}$ and $2x \notin \{0, 3, 6, 12\}$ or (ii) $2x \in \{0, 3, 6, 12\}$ and $x \notin \{0, 3, 6, 12\}$.

Case (ii) is not possible, since $2x \in \{0, 3, 6, 12\}$ implies that $x = 0, 3$ or 6.

For case (i), the only $x \in \{0, 3, 6, 12\}$ with $2x \notin \{0, 3, 6, 12\}$ is $x = 12$, $2x = 24$,

hence (A).

12.

$$\begin{aligned}\frac{1}{\frac{1}{\frac{1}{2}+\frac{1}{3}}+\frac{1}{\frac{1}{4}+\frac{1}{5}}} &= \frac{1}{\frac{1}{\frac{5}{6}}+\frac{1}{\frac{9}{20}}}\\ &= \frac{1}{\frac{6}{5}+\frac{20}{9}}\\ &= \frac{1}{\frac{154}{45}}\\ &= \frac{45}{154}\end{aligned}$$

hence (D).

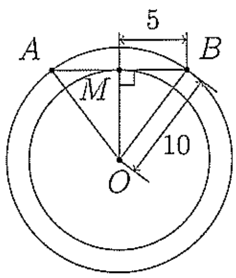
13. There are $200 \div 4 = 50$ blocks. Each block is 3 cm further to the right than the previous block. There are 49 'steps' between blocks, so the top block is $3 \times 49 = 147$ cm further right than the bottom block. Then $L = 147 + 8 = 155$,

hence (B).

14. Of the 15 boys, 5 are fifteen and 10 are sixteen. So of 14 sixteen-year-old students, 4 are girls. This is $\frac{4}{25}$ of the class,

hence (C).

15.



In the circular cross-section, let the cut line be AB, its midpoint be M and let O be the centre of the circle.

Then $MB = 5$ cm and $OB = 10$ cm. Let $r = OM$, the internal radius, then using Pythagoras' theorem on $\triangle MBO$, $r^2 = 10^2 - 5^2 = 75$ so that $r = \sqrt{75} = 5\sqrt{3}$.

The thickness of the wall of the pipe is the difference in radii $10 - 5\sqrt{3} = 5(2 - \sqrt{3})$,

hence (E).

16. Suppose the ages of Euleraptor, Fermatops and Gaussasaurus are e, f and g, measured in millions of years, so that $e + f + g = 360$ and $e = 2g$.

Then g million years ago, Euleraptor was g million years old and so Fermatops was $0.5g$ years old. Hence $f = 1.5g$ and so we solve

$$\begin{aligned}e + f + g &= 360\\ 2g + 1.5g + g &= 360\\ g &= \frac{360}{4.5} = 80\end{aligned}$$

and then $f - g = 0.5g = 40$,

hence (B).

17. If the assembly was on the Friday, the timetable would look like this.

		Mon	Tue	Wed	Thu	Fri
	1	1	6	4	2	7
	2	2	7	5	3	1
Period	3	3	1	6	4	2
	4	4	2	7	5	3
	5	5	3	1	6	4

Mon	Tue	Wed	Thu	Fri
5	3	1	6	A
6	4	2	7	4
7	5	3	1	5
1	6	4	2	6
2	7	5	3	7

Subjects 5, 3, 1 and 6 (shading in period 1) can be first period in the second week. If the assembly moves forward to day X, then the last subject on day X (shading in period 5) becomes the first subject on the next day, as do any subjects in period 5 on following days up to Thursday. So subjects 2, 7, 5 and 3 can also be the first period in the second week.

The only subject that cannot be in the first period in week 2 is subject 4,

hence (C).

18. Factored into primes, $2015 = 5 \times 13 \times 31$, which suggests $2^y - 1 = 2^5 - 1 = 31$ and $2^x + 1 = 2^6 + 1 = 65$. Then $xy = 5 \times 6 = 30$.

This is the only possibility, since the factors of 2015 are $\{\underline{1}, \underline{5}, 13, \underline{31}, \underline{65}, 165, 403, 2015\}$ of which only the underlined factors are one more or one less than a power of 2,

hence (D).

19. If N is even, then $42 = N + \frac{N}{2} = \frac{3N}{2}$, and so $N = \frac{2}{3} \times 42 = 28$.

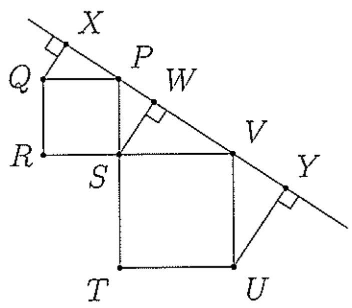
If N is odd, then its second-largest divisor is $k \leq \frac{1}{3}N$. Then $42 = N + k \leq \frac{4}{3}N$, so $N \geq \frac{3}{4} \times 42 = 31.5$. We check the odd values of N from 33 to 41.

N	33	35	37	39	41
k	11	7	1	13	1
$N + k$	44	42	38	52	42

The possible values of N are 28, 35 and 41,

hence (A).

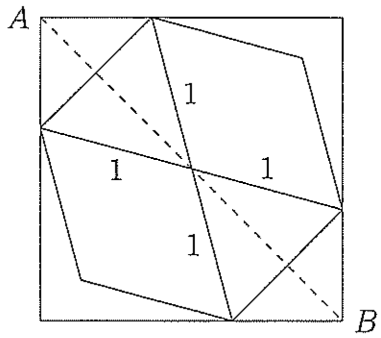
20. Construct SW perpendicular to PV.



Then $\triangle PXQ$ is congruent to $\triangle SWP$, since they have all pairs of corresponding angles equal and one pair of corresponding sides equal. Hence $QX = PW$. Similarly $YU = WV$. Therefore $QX + YU = PV$,

hence (E).

21. Let A be the top-left vertex of the square, and B the bottom-right vertex. Then AB is composed of two heights of an isosceles right triangle with hypotenuse 1, and two heights of an equilateral triangle with sides of length 1.



Thus we have $AB = 1 + \sqrt{3}$. So the side of the square has length $\frac{1}{\sqrt{2}}AB = \frac{\sqrt{2}}{2}\left(1+\sqrt{3}\right) = \frac{1}{2}\left(\sqrt{2}+\sqrt{6}\right)$,

hence (D).

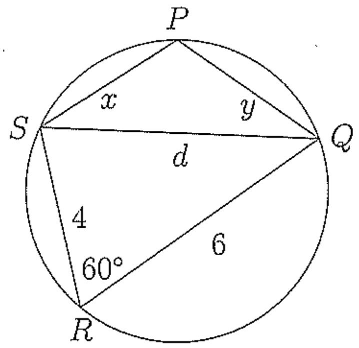
22. The sequence begins

$$3 \quad 5 \quad 15 \quad \boxed{\tfrac{1}{3}} \quad 5 \quad \boxed{\tfrac{1}{15}} \quad \tfrac{1}{3} \quad \boxed{\tfrac{1}{5}} \quad \tfrac{1}{15} \quad \boxed{3} \quad \tfrac{1}{5} \quad \boxed{15} \quad 3 \quad \boxed{5}$$

so it cycles every 12 terms. Since $2015 \div 12$ has a remainder of 11, the 2015th term is the same as the 11th, which is $\frac{1}{5}$,

hence (E).

23. *Alternative 1*
Join SQ and let $SQ = d$.



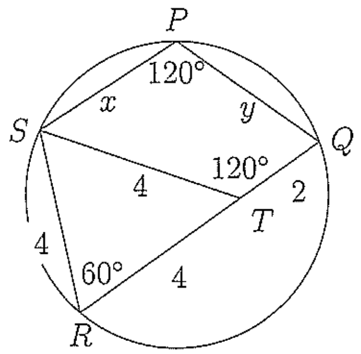
As $PQRS$ is a cyclic quadrilateral, $\angle SPQ = 120°$.
Then, from the cosine rule

$$\begin{aligned}
x^2 + y^2 - 2xy\cos 120° &= d^2 = 4^2 + 6^2 - 2\times 4\times 6\times \cos 60° \\
x^2 + y^2 + xy &= 16 + 36 - 24 = 28 \\
(x+y)^2 - xy &= 28 \qquad (1)
\end{aligned}$$

Try $x + y = 6$ then $36 - xy = 28$ and possible values for x and y are $x = 4$ and $y = 2$,
hence (A).

Note that none of the other values satisfy (1).

Alternative 2
Since $PQRS$ is cyclic, the angle at P is $120°$. Place an equilateral triangle $\triangle SRT$ on the diagram as shown.



The quadrilateral $PQTS$ has angles $\angle SPQ$ and $\angle STQ$ equal. An easy way to ensure that x and y are integers is to make $\triangle PSQ$ congruent to $\triangle TSQ$ either by making $PQRS$ a kite or a parallelogram. The kite solution has $x = 4$, $y = 2$,

hence (A).

24. *Alternative 1*
For $1 \le i \le 6$, let $b_i = \frac{a_{i+1}}{a_i}$ be the quotient of successive terms of the sequence. The conditions of the problem imply that b_i is an integer greater than or equal to 2 and that $a_7 = a_1b_1b_2b_3b_4b_5b_6 < 100$.

- If $a_1 \ge 2$, then we have $a_7 \ge 2^7 = 128$, which is a contradiction. Therefore, we must have $a_1 = 1$.
- If one of the b_i is greater than or equal to 4, then we have $a_7 \ge 1 \times 2^5 \times 4 = 128$, which is a contradiction. Therefore, we must have $b_i \le 3$ for $1 \le i \le 6$.
- If two of the b_i are equal to 3, then we have $a_7 \ge 1 \times 2^4 \times 3^2 = 144$, which is a contradiction.

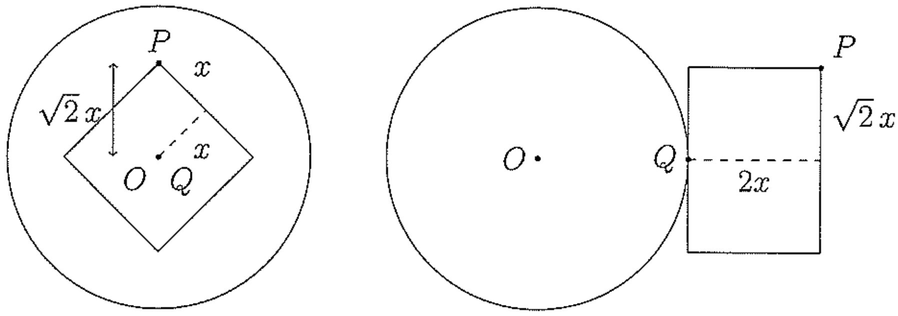
Therefore, we deduce that the integers $(b_1, b_2, b_3, b_4, b_5, b_6)$ must be $(2, 2, 2, 2, 2, 2)$ or $(2, 2, 2, 2, 2, 3)$ in some order. This yields seven possibilities in total. For each such sequence $b_1, b_2, b_3, b_4, b_5, b_6$, one can recover a unique sequence $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ that satisfies the conditions of the problem,

hence (B).

Alternative 2
The smallest a_7 could be is in the sequence $1, 2, 4, 8, 16, 32, 64$.
Replacing one of the six '$\times 2$' steps with '$\times 3$' will make $a_7 = 96$. So there are six possibilities with $a_7 = 96$.
Any ratio $a_{k+1} : a_k$ greater than 3 or pair of such ratios greater than 2 will make $a_7 > 100$.
In all, there are 7 possibilities,

hence (B).

25. Suppose the cube has edge $2x$ and it touches the outer sphere at P and the inner sphere at Q. It is clear that Q is the centre of a face and P is a vertex.
Here are two views of the inner sphere and cube, with P uppermost.



Then

$$(OP)^2 = (2+2x)^2 + (\sqrt{2}\,x)^2 \; = \; 4 + 8x + 6x^2$$

But $OP = 4$, so

$$\begin{aligned} 6x^2 + 8x + 4 &= 16 \\ 3x^2 + 4x - 6 &= 0 \\ x &= \frac{\sqrt{22}-2}{3} \end{aligned}$$

and the cube's edge is $2x = \frac{2}{3}(\sqrt{22}-2)$,

hence (D).

26. (Also I27)
Alternative 1
Suppose

$$\frac{1}{3} + \frac{1}{n} = \frac{n+3}{3n} = \frac{a}{b}$$

where $b < n$ and $\gcd\{a, b\} = 1$. Let $k = \gcd(n+3, 3n)$, the factor that is cancelled.
Then $k = \dfrac{3n}{b} > \dfrac{3b}{b} = 3$.
Since k divides $n+3$, it divides $3n+9$. However, k also divides $3n$, so k is a divisor of 9. Since $k > 3$, we have $k = 9$. Then $9a = n+3$ and $9b = 3n$. Then $n = 9a - 3$ and $b = 3a - 1 < n$.
So n is one of $6, 15, 24, 33, \ldots$. We can verify that all such n work:

$$\frac{1}{3} + \frac{1}{n} = \frac{1}{3} + \frac{1}{9a-3} = \frac{3a-1}{9a-3} + \frac{1}{9a-3} = \frac{3a}{9a-3} = \frac{a}{3a-1} = \frac{a}{b}.$$

To count these, solve $n < 2015$ where $n = 9a - 3$ and a is a whole number.

$$9a - 3 < 2015 \implies 9a < 2018 \implies a < 224\frac{2}{9} \implies a \le 224\,,$$

hence (224).

Alternative 2

If n is not a multiple of 3, then the common denominator is $3n$ which has no common factor with the numerator $n+3$. So we check where n is a multiple of 3.

n	3	6	9	12	15	18	21	24	27	$\cdots$
$\frac{1}{3}+\frac{1}{n}$	$\frac{2}{3}$	$\frac{3}{6}$	$\frac{4}{9}$	$\frac{5}{12}$	$\frac{6}{15}$	$\frac{7}{18}$	$\frac{8}{21}$	$\frac{9}{24}$	$\frac{10}{27}$	$\cdots$
Simplification		$\frac{1}{2}$			$\frac{2}{5}$			$\frac{3}{8}$		$\cdots$

The pattern for $n = 6, 15, 24, \ldots$ continues, since cancellation occurs only when the numerator $\frac{n}{3}+1$ is a multiple of 3. In particular, n is 3 less than a multiple of 9.

The last multiple of 3 before 2015 is 2013, which is 3 less than $2016 = 224 \times 9$. So from $n = 6 = 1 \times 9 - 3$ up to $n = 2013 = 224 \times 9 - 3$ there are 224 values of n where the simplification is possible,

hence (224).

27. In the plane $VUSP$ the small cubes form a 100×100 grid of $1 \times \sqrt{2}$ rectangles. The line VZ passes across a diagonal of a rectangle of 33×100 grid rectangles.

Since 33 and 100 have no common factor, VZ does not pass through any of the grid points, and so it separately passes through 32 grid lines in one direction and 99 grid lines in the other direction. That is, it passes from one cube to another 131 times. Consequently it passes through 132 cubes,

hence (132).

28. (Also I29)

In each fifteen-minute interval starting from noon, a train arrives after 0 minutes, 3 minutes, 5 minutes, 6 minutes, 9 minutes, 10 minutes, 12 minutes and 15 minutes. Between these arrivals, there are two 1-minute intervals, two 2-minute intervals, and three 3-minute intervals. Hence, we have the following facts.

- The probability of a train arriving during a 1-minute interval is $\frac{2 \times 1}{15}$ and the average wait is 30 seconds.
- The probability of a train arriving during a 2-minute interval is $\frac{2 \times 2}{15}$ and the average wait is 60 seconds.
- The probability of a train arriving during a 3-minute interval is $\frac{3 \times 3}{15}$ and the average wait is 90 seconds.

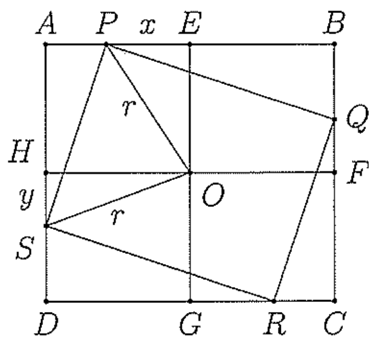
So on average, the number of seconds that I should expect to wait is

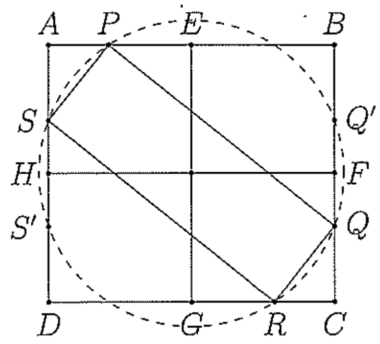
$$\frac{2}{15} \times 30 + \frac{4}{15} \times 60 + \frac{9}{15} \times 90 = 4 + 16 + 54 = 74,$$

hence (74).

29. *Alternative 1*

The two rectangles share a centre O and then $OP = OS$. Let E, F, G, H be the midpoints of the sides of $ABCD$ as pictured. Then $PQRS$ will be in one of these configurations, or a mirror image:





However, in the case on the right, the rectangle $PQ'RS'$ satisfies the conditions of the question. Furthermore, $\triangle PRS$ and $\triangle PRS'$ have the same base PR but $\triangle PRS'$ has greater altitude, hence greater area. Consequently $PQ'RS'$ has greater area so that $PQRS$ can't be the maximum.

Now in $\triangle EOP$, $x^2 + 16^2 = r^2$ and in $\triangle HOS$, $y^2 + 19^2 = r^2$. Consequently

$$x^2 - y^2 = 19^2 - 16^2 = 3 \times 35 = 105$$

$$(x + y)(x - y) = 3 \cdot 5 \cdot 7$$

where $x + y < 35$. Consequently either (i) $x + y = 21$ and $x - y = 5$ or (ii) $x + y = 15$ and $x - y = 7$.

In case (i) $x = 13$ and $y = 8$, then $\triangle APS$ and $\triangle CRQ$ each have area $\frac{1}{2} \cdot 6 \cdot 24 = 72$, and $\triangle BQP$ and $\triangle DSR$ each have area $\frac{1}{2} \cdot 8 \cdot 32 = 128$. Then $PQRS$ has area $38 \times 32 - 2 \times 72 - 2 \times 128 = 816$.

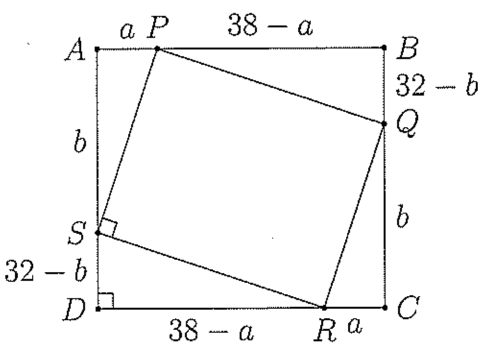
In case (ii) $x = 11$ and $y = 4$, then $\triangle APS$ and $\triangle CRQ$ each have area $\frac{1}{2} \cdot 8 \cdot 20 = 80$, and $\triangle BQP$ and $\triangle DSR$ each have area $\frac{1}{2} \cdot 12 \cdot 30 = 180$. Then $PQRS$ has area $38 \times 32 - 2 \times 80 - 2 \times 180 = 696$.

There are no other cases, so the first case is the largest,

hence (816).

Alternative 2

Label lengths and calculate the area, as shown.



$$\begin{aligned} \text{Area} &= 38 \times 32 - ab - (38 - a)(32 - b) \\ &= 38b + 32a - 2ab \end{aligned}$$

We can assume $1 \leq a \leq 19$. Since $\triangle APS$ and $\triangle BQP$ are similar, $\frac{a}{b} = \dfrac{32 - b}{38 - a}$, so $b^2 - 32b + a(38 - a) = 0$. This quadratic in b has solution

$$\begin{aligned} b &= \frac{32 \pm \sqrt{32^2 - 4a(38 - a)}}{2} \\ &= 16 \pm \sqrt{256 - a(38 - a)} \end{aligned}$$

Then we require $256 - a(38-a) = (a-19)^2 - 105$ to be a perfect square. We check $a = 1, \ldots, 19$:

a	1	2	3	4	5	6	7	8	9	10...19
$(a-19)^2 - 105$	219	184	151	120	91	64	39	16	−5	< 0
						✓		✓		

With $a = 6$, $b = 16 \pm 8$ and with $a = 8$, $b = 16 \pm 4$ so that the solutions (a, b) are $(6, 8), (6, 24), (8, 12), (8, 20)$. These give areas 400, 816, 520 and 696,

hence (816).

30. *Alternative 1*

To obtain the minimum possible average value, we must have $S = \{0, 1, 2, \ldots, k, 2015\}$ for some integer k satisfying $0 \le k \le 2014$. The average value of the numbers in S is

$$\frac{(0+1+2+\cdots+k)+2015}{k+2} = \frac{\frac{1}{2}k(k+1)+2015}{k+2} = \frac{1}{2}\left(k+2+\frac{4032}{k+2}\right) - \frac{3}{2}. \qquad (\star)$$

We can use the fact that squares are non-negative to deduce that

$$\left(\sqrt{k+2} - \sqrt{\frac{4032}{k+2}}\right)^2 \ge 0 \qquad \Rightarrow \qquad k+2+\frac{4032}{k+2} \ge 2\sqrt{4032}.$$

Equality occurs if and only if $k + 2 = \sqrt{4032} = 63.49\ldots$, which implies that $k = 61.49\ldots$. Since $k+2+\frac{4032}{k+2}$ is a convex function of k for $0 \le k \le 2014$, the minimum possible value of $(\star)$ for k a positive integer must occur when $k = 61$ or $k = 62$. When $k = 61$, the average value of the numbers in S is $\frac{1}{2}\left(63 + \frac{4032}{63}\right) - \frac{3}{2} = 62$, and when $k = 62$, the average value of the numbers in S is $\frac{1}{2}\left(64 + \frac{4032}{64}\right) - \frac{3}{2} = 62$. Hence, the minimum possible average value of the numbers in S is 62,

hence (62).

Alternative 2

For $2 \le n \le 2015$, consider $S_n = \{0, 1, 2, \ldots, n-2, 2015\}$, the n-element set with all elements as small as possible. Let m_n be the mean value of S_n.

$$m_n = \frac{\frac{1}{2}(n-1)(n-2)+2015}{n} = \frac{n^2-3n+4032}{2n}$$

Since each $S_{n+1} = S_{n-1} \cup \{n-1\}$, the sequence $m_2, m_3, m_4, \ldots$ will be decreasing $(m_n > m_{n+1})$ so long as $n - 1 < m_n$, constant when $n - 1 = m_n$, and increasing otherwise. Since $n > 0$ this inequality is equivalent to $2n(n-1) < 2nm_n$.

$$2n^2 - 2n < n^2 - 3n + 4032$$
$$n^2 + n - 4032 < 0$$

Rather than using the quadratic formula, which requires $\sqrt{16121}$, note that $4032 = 2^6 \times 3^2 \times 7 = 63 \times 64$. Then

$$(n-63)(n+64) < 0$$
$$-64 < n < 63$$

In particular, the sequence $m_2, m_3, m_4, \ldots$ decreases down to $m_{63} = m_{64} = 62$ and increases thereafter,

hence (62).