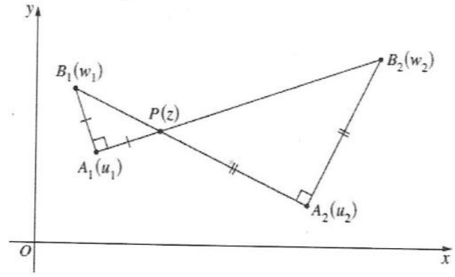
Complex Numbers Revision- Past HSC Questions involving Vectors

2012 Q12

(d) On the Argand diagram, the points A_1 and A_2 correspond to the distinct complex numbers u_1 and u_2 respectively. Let P be a point corresponding to a third complex number z.

Points B_1 and B_2 are positioned so that $\triangle A_1PB_1$ and $\triangle A_2B_2P$, labelled in an anti-clockwise direction, are right-angled and isosceles with right angles at A_1 and A_2, respectively. The complex numbers w_1 and w_2 correspond to B_1 and B_2, respectively.



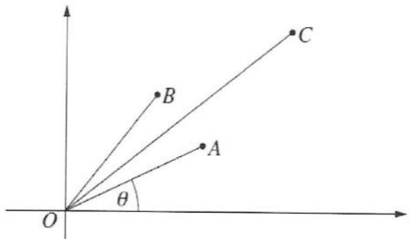
(i) Explain why $w_1 = u_1 + i(z - u_1)$. **1**

(ii) Find the locus of the midpoint of B_1B_2 as P varies. **2**

2010 Q2

(d) Let $z = \cos\theta + i\sin\theta$ where $0 < \theta < \frac{\pi}{2}$.

On the Argand diagram the point A represents z, the point B represents z^2 and the point C represents $z + z^2$.



Copy or trace the diagram into your writing booklet.

(i) Explain why the parallelogram $OACB$ is a rhombus. **1**

(ii) Show that $\arg(z + z^2) = \frac{3\theta}{2}$. **1**

(iii) Show that $|z + z^2| = 2\cos\frac{\theta}{2}$. **2**

(iv) By considering the real part of $z + z^2$, or otherwise, deduce that $\cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$. **1**

Solutions

(i) $\overrightarrow{A_1B_1} = \overrightarrow{A_1P}$ rotate 90^0 anticlockwise

$$w_1 - u_1 = (z-u_1)i$$

$$w_1 = u_1 + i(z-u_1) \text{ as required.}$$

(ii) $\overrightarrow{A_2P} = \overrightarrow{A_2B_2}$ rotate 90^0 anticlockwise

$$z - u_2 = (w_2 - u_2)i$$

$$z - u_2 + iu_2 = iw_2$$

$$-w_2 = iz - iu_2 - u_2$$

$$w_2 = u_2 + iu_2 - iz$$

$$w_2 = u_2 + i(u_2 - z)$$

The midpoint of B_1B_2 as P varies is:

$$\frac{w_1+w_2}{2} = \frac{u_1 + i(z-u_1) + u_2 + i(u_2-z)}{2}$$

$$= \frac{u_1+u_2}{2} + \frac{(u_2-u_1)}{2}i$$

which is a fixed point.

(d) (i)

$$OA = |\vec{OA}| = |z| = 1$$

$$OB = |\vec{OB}| = |z^2| = |z|^2 = 1$$

So OA=OB

OACB is a parallelogram with adjacent sides equal $\therefore$ it is a rhombus.

(ii) $Arg\left(z^2\right) = 2Arg(z)$

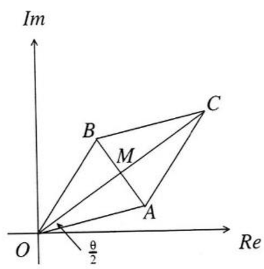
$$= 2\theta$$

$\therefore \angle BOA = \theta$

In a rhombus, the diagonals bisect the angles so $\angle COA = \frac{\theta}{2}$.

$$Arg(z+z^2) = \theta + \frac{\theta}{2} = \frac{3\theta}{2}.$$

(iii)



$|z+z^2| = OC$.

Let M be the midpoint of OC.

In a rhombus, the diagonals bisect each other at right angles.

In ΔOMA,

$$\cos\frac{\theta}{2} = \frac{OM}{OA}$$

$$= \frac{OM}{1}$$

$$= OM$$

$$OC = 2OM$$

So $OC = 2\cos\frac{\theta}{2}$

$$\therefore |z+z^2| = 2\cos\frac{\theta}{2}$$

(iv) $z+z^2 = 2\cos\frac{\theta}{2}\left(cis\frac{3\theta}{2}\right)$ from (ii) and (iii

$$= 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + i\times 2\cos\frac{\theta}{2}\sin\frac{3\theta}{2}$$

$$\text{Re}\left(z+z^2\right) = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Alternately,

$$z+z^2 = \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta$$

$$\text{Re}\left(z+z^2\right) = \cos\theta + \cos 2\theta$$

$$\therefore \cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$