Carlingford High School
Mathematics Extension 2
Year 12

HSC ASSESSMENT TASK 3

Term 2 2016

Student Name: ______________________ Teacher: Mr GonG

- **Time allowed 55 minutes.**

- Start each question on a new page.
- Write on **ONE SIDE** of the paper only.
- Do not work in columns.
- Marks may be deducted for careless or badly arranged work.
- Only calculators approved by the Board of Studies may be used.
- All answers are to be completed in blue or black pen except graphs and diagrams.
- There is to be NO LENDING OR BORROWING.

	MC	Q6	Q7	Total	
Integration	/3	/16		**/19**	
Volumes	/2		/13	**/15**	
Total	**/5**	**/16**	**/13**	**/34**	

Section 1

Multiple Choice – **Start a new page** **(5 marks)**

1. Evaluate $\int \frac{dx}{x^2 - 4x + 13}$

A. $\frac{1}{3}\tan^{-1}\left(\frac{2x-4}{3}\right)+C$ **B.** $\frac{2}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$

C. $\frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$ **D.** $\frac{2}{3}\tan^{-1}\left(\frac{2x-4}{3}\right)+C$

2. The region bounded by the curves $y = x^2$ and $y = x^3$ in the first quadrant is rotated about the $y-$ axis. The volume of the solid of revolution formed can be found using:

A. $V = \pi\int_0^1 \left(y^{\frac{1}{3}} - y^{\frac{1}{2}}\right)dy$ **B.** $V = \pi\int_0^1 \left(y^{\frac{1}{2}} - y^{\frac{1}{3}}\right)dy$

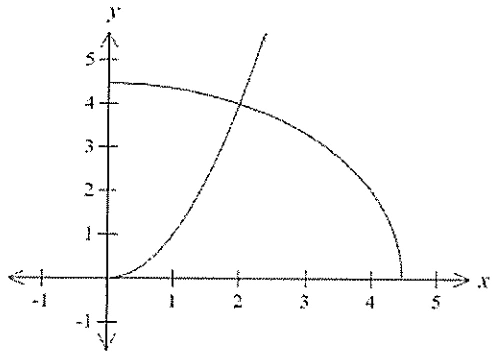
C. $V = \pi\int_0^1 \left(x^4 - x^6\right)dx$ **D.** $V = \pi\int_0^1 \left(y^{\frac{2}{3}} - y\right)dy$

3. Which of the following is an expression for $\int \frac{1}{1+\sin x+\cos x}dx$?

A. $\ln|t-1|+C$ **B.** $\ln|t+1|+C$

C. $\ln|t^2-1|+C$ **D.** $\ln|t^2+1|+C$

4. A solid is formed when the region bounded by the curves $y = x^2$, $y = \sqrt{20 - x^2}$ and the $y-$ axis is rotated about the $y-$ axis.

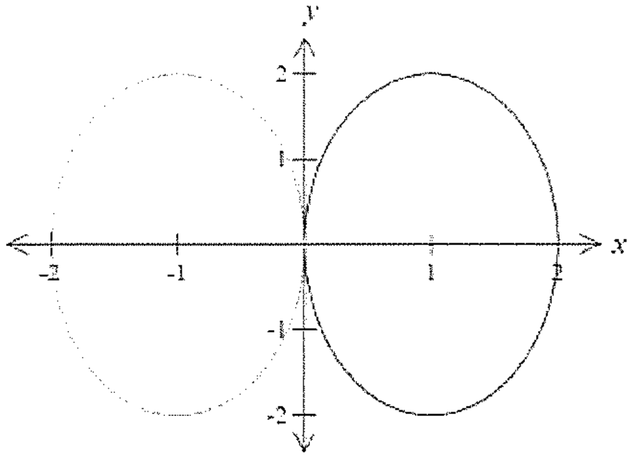


What is the correct expression for the volume of this solid using the method of cylindrical shells ?

A. $V = \int_0^2 2\pi\left(\sqrt{20-x^2} - x^2\right)dx$ **B.** $V = \int_0^2 2\pi\left(x^2 - \sqrt{20-x^2}\right)dx$

C. $V = \int_0^2 2\pi x\left(\sqrt{20-x^2} - x^2\right)dx$ **D.** $V = \int_0^2 2\pi x\left(x^2 - \sqrt{20-x^2}\right)dx$

5. The region enclosed by the ellipse $(x-1)^2+\frac{y^2}{4}=1$ is rotated about the y-axis to form a solid.



If the slices are taken perpendicular to the axis of rotation, what is the correct expression for the volume ?

A. $V=\int_{-2}^{2} 2\pi\sqrt{4-y^2}\,dy$

B. $V=\int_{-2}^{2} 2\pi\sqrt{1-y^2}\,dy$

C. $V=\int_{-2}^{2} \pi\sqrt{4-y^2}\,dy$

D. $V=\int_{-2}^{2} \pi\sqrt{1-y^2}\,dy$

Section 2

Question 6 – Start a new page – (16 marks) **Marks**

a) Find $\int \frac{dx}{\sqrt{9+16x-4x^2}}$. 3

b) Find $\int \frac{5x^2-3x+13}{(x-1)(x^2+4)}dx$. 3

c) Evaluate $\int_0^{\frac{\sqrt{\pi}}{2}} 3x\sin(x^2)dx$. 3

d) Evaluate $\int_1^e x^7 \ln x \, dx$. 3

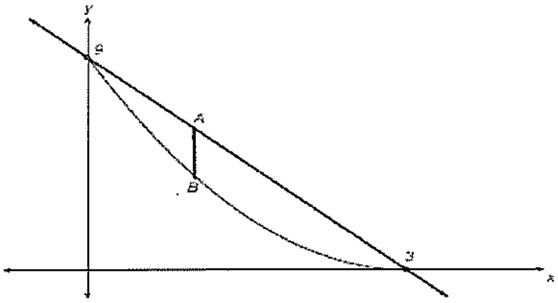
e) i) Derive the reduction formula: $\int x^n e^{-x^2}dx = -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}\int x^{n-2}e^{-x^2}dx$. 2

ii) Use this reduction formula to evaluate $\int_0^1 x^5 e^{-x^2}dx$. 2

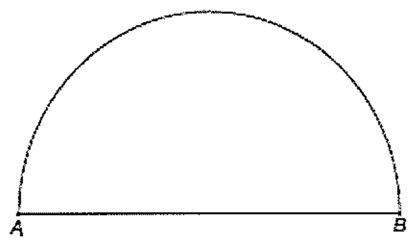
Question 7 - Start a new page – (13 marks) **Marks**

a) Use the method of cylindrical shells to find the volume of the solid generated by revolving the region enclosed by $y = 3x^2 - x^3$ and the x-axis around the y-axis. **4**

b)



Cross-section with base on AB



The diagram above shows the region enclosed by the parabola $y = (x-3)^2$ and the line $3x + y = 9$. The region forms the base of a solid.

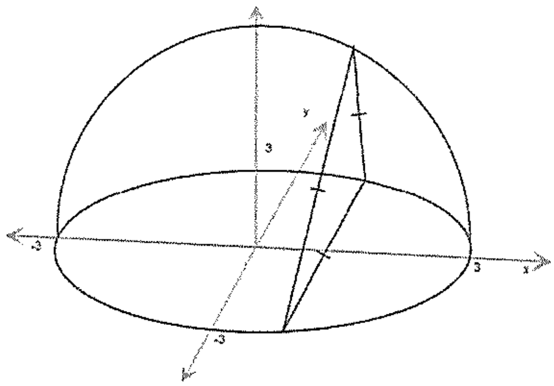
When the solid is sliced perpendicular to the x-axis, each cross-section is a semi-circle with diameter across the region. A typical cross-section is shown above.

i) If the solid is sliced along the line $x = a$, show that the area of the cross-section is $A = \frac{\pi}{8} a^2 (3-a)^2$, where $0 \le a \le 3$. **2**

ii) Find the volume of the solid. **3**

c) The diagram shows a solid which has the circle $x^2 + y^2 = 9$ as its base.

The cross-section perpendicular to the x-axis is an equilateral triangle.



i) Show that the area of a triangle is given by: $Area = \sqrt{3}(9 - x^2)$. **2**

ii) Hence or otherwise find the volume of the solid. **2**

<u>END OF EXAM</u>

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Section 1

Multiple Choice – Start a new page (5 marks)

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B. $\frac{2}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C$

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2. The region bounded by the curves $y=x^2$ and $y=x^3$ in the first quadrant is rotated about the $y-$axis. The volume of the solid of revolution formed can be found using:

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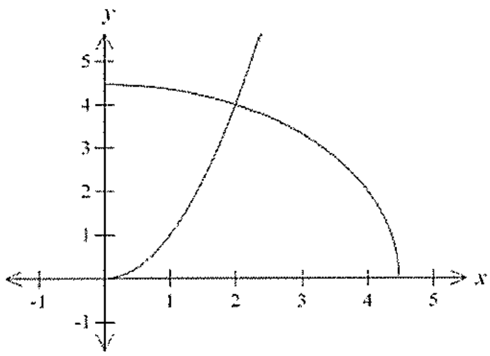

What is the correct expression for the volume of this solid using the method of cylindrical shells ?

A. $V=\int_0^2 2\pi\left(\sqrt{20-x^2}-x^2\right)dx$

B. $V=\int_0^2 2\pi\left(x^2-\sqrt{20-x^2}\right)dx$

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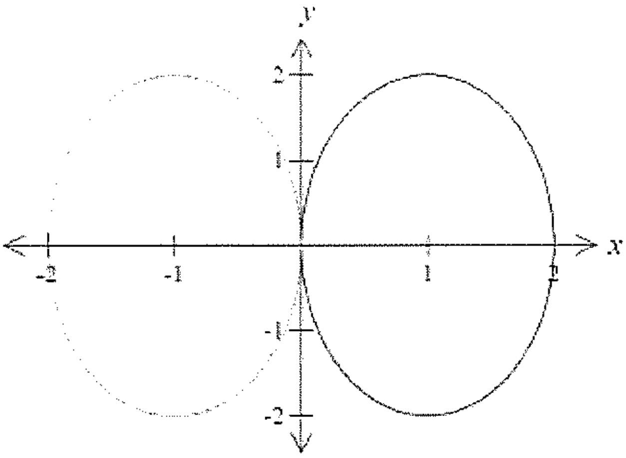
5. The region enclosed by the ellipse $(x-1)^2+\frac{y^2}{4}=1$ is rotated about the y-axis to form a solid.



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Section 2

Question 6 **– Start a new page –** **(16 marks)** **Marks**

a) Find $\int \frac{dx}{\sqrt{9+16x-4x^2}}$. **3**

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d) Evaluate $\int_1^e x^7 \ln x \, dx.$ **3**

e) **i)** Derive the reduction formula: $\int x^n e^{-x^2} dx = -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}\int x^{n-2}e^{-x^2} dx.$ **2**

ii) Use this reduction formula to evaluate $\int_0^1 x^5 e^{-x^2} dx.$ **2**

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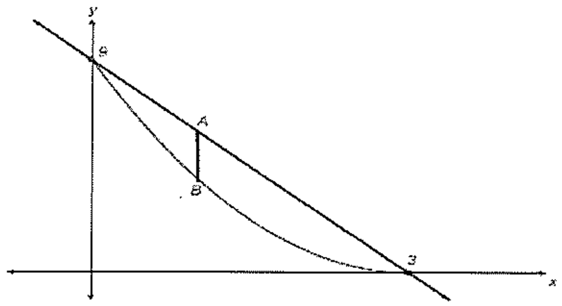
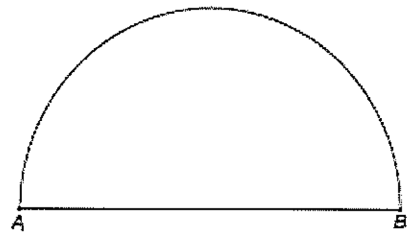


Cross-section with base on AB



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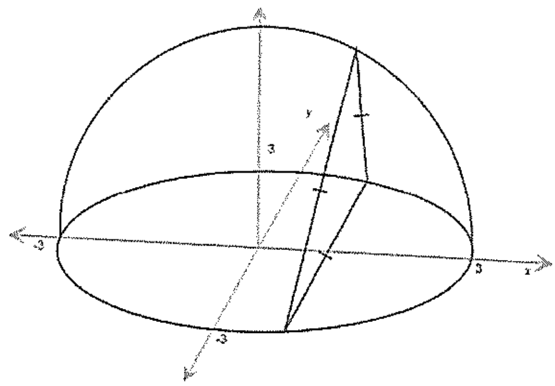


i) Show that the area of a triangle is given by: $Area = \sqrt{3}(9 - x^2)$. **2**

ii) Hence or otherwise find the volume of the solid. **2**

END OF EXAM

CARLINGFORD HS

ADVENTURE IN LEARNING

2016

Term 2 HSC Task 3 Examination

Ext 2 Mathematics

Solutions

HSC Task 3 Term 2 Examination – Ext 2 Mathematics 2016

Section I Multiple Choice Answer 1 Mark each

	A	B	C	D
1.	○	○	●	○
2.	○	○	○	●
3.	○	●	○	○
4.	○	○	●	○
5.	●	○	○	○

Multiple Choice Working Out

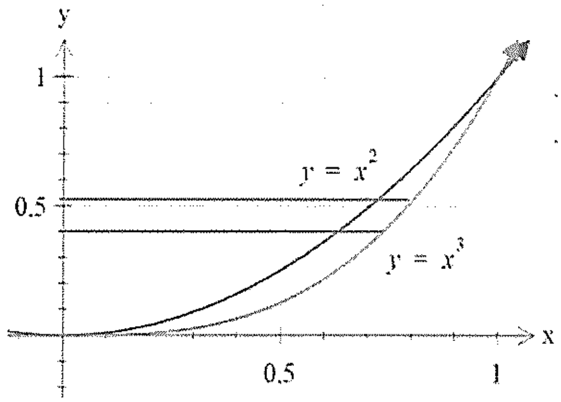
1

$$\text{Now } \int \frac{dx}{x^2-4x+13} = \int \frac{dx}{x^2-4x+4+9}$$
$$= \int \frac{dx}{(x-2)^2+9}$$
$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C \Rightarrow \text{C}$$

3

$$\int \frac{1}{1+\sin x+\cos x}\,dx = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}$$
$$= \int \frac{2}{1+t^2+2t+1-t^2}\,dt$$
$$= \int \frac{1}{1+t}\,dt$$
$$= \ln|t+1|+C \Rightarrow \text{B}$$

2

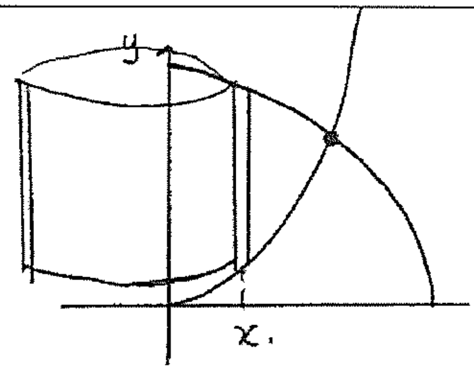


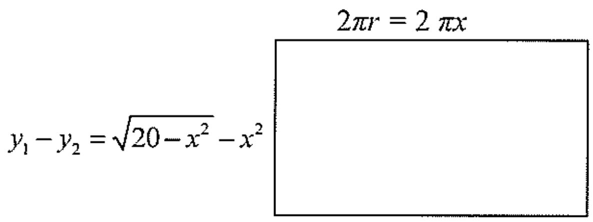
Now $y = x^3$ then $x = y^{\frac{1}{3}}$

and $y = x^2$ then $x = y^{\frac{1}{2}}$

$$\text{So } V = \pi\int_0^1\left[\left(y^{\frac{1}{3}}\right)^2-\left(y^{\frac{1}{2}}\right)^2\right]dy$$
$$= \pi\int_0^1\left(y^{\frac{2}{3}}-y\right)dy \Rightarrow \text{D}$$

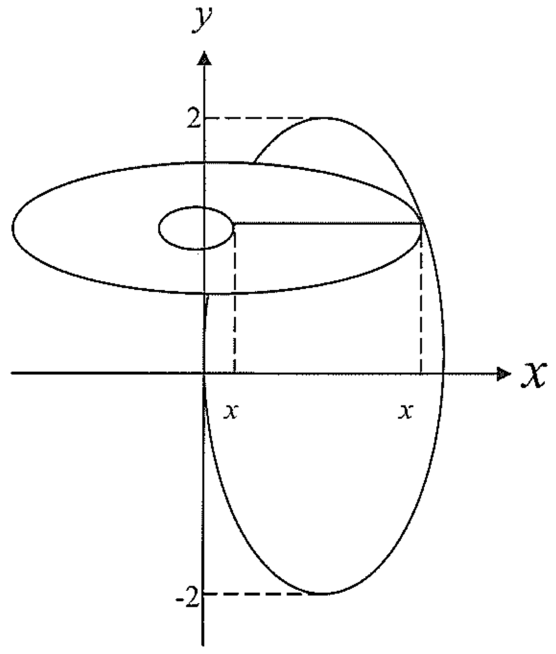
4





$$A = 2\pi x\left(\sqrt{20-x^2}-x^2\right)$$
$$V = \int_0^2 2\pi x\left(\sqrt{20-x^2}-x^2\right)dx \Rightarrow \text{C}$$

5



$\delta V = \pi\left(R+r\right)\left(R-r\right)\delta y$ where $R = x_2,\ r = x_1$

Taking $x_2,\ x_1$ as roots of equation $\left(x-1\right)^2 + \dfrac{y^2}{4} = 1$

at some fixed height y above $x-$axis:

$4(x^2-2x+1)+y^2-4=0$

$4x^2-8x+y^2=0$ has roots $x_1,\ x_2$

Sum of roots: $x_1 + x_2 = 2$

Product of roots: $x_1x_2 = \dfrac{y^2}{4}$

Now $\left(x_2-x_1\right)^2 = \left(x_1+x_2\right)^2 - 4x_1x_2$ where $x_2 - x_1 > 0$

So
$$\begin{aligned} x_2 - x_1 &= \sqrt{\left(x_1+x_2\right)^2 - 4x_1x_2} \\ &= \sqrt{4 - 4\cdot\frac{y^2}{4}} \\ &= \sqrt{4-y^2} \end{aligned}$$

$\therefore\ V = 2\pi\int_{-2}^{2}\sqrt{4-y^2}\,dy \quad \Rightarrow \text{A}$

Section II **Solutions**

Question 6

a)

$$\int \frac{dx}{\sqrt{9+16x-4x^2}}$$

$$9+16x-4x^2 = 9-4(x^2-4x)$$

$$= 9-4(x^2-4x+4)+16$$

$$= 25-4(x-2)^2$$

$$\int \frac{dx}{\sqrt{9+16x-4x^2}} = \int \frac{dx}{\sqrt{25-4(x-2)^2}}$$

$$= \frac{1}{5}\int \frac{dx}{\sqrt{1-\frac{4}{25}(x-2)^2}}$$

1 for correct manipulation

$$u = \frac{2(x-2)}{5} + c$$

$$du = \frac{2}{5}dx$$

$$dx = \frac{5}{2}\,du$$

$$= \frac{1}{5}\int \frac{\frac{5}{2}du}{\sqrt{1-u^2}}$$

1 for correct substitution

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2}\,sin^{-1}u$$

$$= \frac{1}{2}\,sin^{-1}\left(\frac{2(x-2)}{5}\right)$$

1 mark for correct answer

b)

Let $\dfrac{5x^2-3x+13}{(x-1)(x^2+4)} \equiv \dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+4}$

$$\equiv \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

$\therefore\ 5x^2-3x+13 = A(x^2+4)+(Bx+C)(x-1)$

1 mark for this line

When $x=1$ then $15=5A$

$A=3$

When $x=0$ then $13=12-C$

$C=-1$

When $x=-1$ then $21=15+(-B-1)(-2)$

$6=2B+2$

$B=2$

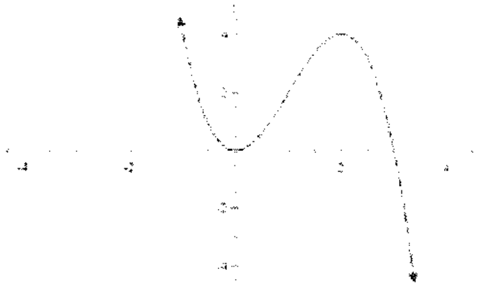
1 mark for the values of ***A, B & C***

	$\therefore\ I=\int\left(\frac{3}{x-1}+\frac{2x-1}{x^2+4}\right)dx$ $=3\ln\lvert x-1\rvert+\int\left(\frac{2x}{x^2+4}\right)dx-\int\left(\frac{1}{x^2+4}\right)dx$ $=3\ln\lvert x-1\rvert+\ln\lvert x^2+4\rvert-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$	**1 mark for correct answer**
c)	$\int_0^{\frac{\sqrt{\pi}}{2}} 3x\ \sin(x^2)\ dx$ Substitute $u = x^2$ $\frac{du}{dx}=2x$ $du = 2x\ dx$ $\frac{3}{2}du = 3x\ dx$ $x=0 \Rightarrow u=0^2=0$ $x=\frac{\sqrt{\pi}}{2} \Rightarrow u=\left(\frac{\sqrt{\pi}}{2}\right)^2=\frac{\pi}{4}$ $\int_0^{\frac{\sqrt{\pi}}{2}} 3x\ \sin(x^2)\ dx = \int_0^{\frac{\pi}{4}}\frac{3}{2}\sin u\ du$ $=\frac{3}{2}\int_0^{\frac{\pi}{4}}\sin u\ du$ $=\frac{3}{2}\Big[-\cos u\Big]_0^{\frac{\pi}{4}}$ $=-\frac{3}{2}\left(\cos\left(\frac{\pi}{4}\right)-\cos(0)\right)$ $=-\frac{3}{2}\left(\frac{1}{\sqrt{2}}-1\right)$ $=-\frac{3}{2}\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)$ $=\frac{3\sqrt{2}-3}{2\sqrt{2}}$ $=\frac{6-3\sqrt{2}}{4}$	**Using a Substitution** **1 for changing limits & variable** **1 for integral including correct limits** **1 for substitution & simplification to get answer** **OR**

		Working	Marks
		$\int_0^{\frac{\sqrt{\pi}}{2}} 3x\sin\left(x^2\right)dx$ $=-\frac{3}{2}\left[\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ $=-\frac{3}{2}\left[\cos\frac{\pi}{4}-\cos 0\right]$ $=-\frac{3}{2}\left(\frac{1}{\sqrt{2}-1}\right)$	**Without a Substitution** **1 mark for correct integration** **1 mark for correct working** **1 mark for correct answer**
d)		$I=\int_1^e x^7\ln x\,dx$ Let $u=\ln x$ then $u'=\frac{1}{x}$ & $v'=x^7$ then $v=\frac{x^8}{8}$ $=\left[\frac{x^8}{8}\ln x\right]_1^e-\frac{1}{8}\int_1^e x^7dx$ $=\frac{e^8}{8}-\frac{1}{8}\left[\frac{x^8}{8}\right]_1^e$ $=\frac{e^8}{8}-\frac{1}{8}\left(\frac{e^8}{8}-\frac{1}{8}\right)$ $=\frac{7e^8+1}{64}$	**1 mark for correct u' & v** **1 mark for correct substitution** **1 mark for correct answer**
e)	**i)**	$\int x^n e^{-x^2}dx$ Let $u = x^{n-1}$ $\quad v' = xe^{-x^2}$ $u' = (n-1)x^{n-2}$ $\quad v = -\frac{1}{2}e^{-x^2}$ $\int x^n e^{-x^2}dx = uv - \int vu'$ $= -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}\int x^{n-2}e^{-x^2}dx$	**1 mark for correct set up** **1 mark for correct answer**
	ii)	$\int_0^1 x^5e^{-x^2}dx = \left[-\frac{1}{2}x^4e^{-x^2}\right]_0^1 + \frac{4}{2}\int_0^1 x^3e^{-x^2}dx$ $= \frac{-1}{2e} + 2\int_0^1 x^3e^{-x^2}dx$ $= \frac{-1}{2e} + 2\left\{\left[-\frac{1}{2}x^2e^{-x^2}\right]_0^1 + 1\int_0^1 xe^{-x^2}dx\right\}$ $= \frac{-1}{2e} - 2\left(\frac{1}{2e}\right) + 2\left[-\frac{1}{2}x^0e^{-x^2}\right]_0^1 + \frac{1-1}{2}\int x^{1-2}e^{-x^2}dx$ $= \frac{-1}{2e} - \frac{1}{e} + \left[-e^{-x^2}\right]_0^1$ $= \frac{-1}{2e} - \frac{1}{e} - \frac{1}{e} + 1$ $= \frac{-1}{2e} - \frac{2}{2e} - \frac{2}{2e} + 1$ $= 1 - \frac{5}{2e}$	**1 mark for correct reduction** **1 mark for correct answer**

Question 7

a)



$$\left.\begin{aligned}\partial V &= 2\pi xy\partial x\\ &= 2\pi x\left(3x^2-x^3\right)\partial x\\ V &= \lim_{\partial x\to 0}\sum_{0}^{3} 2\pi x\left(3x^2-x^3\right)\partial x\end{aligned}\right\}$$

2 marks for correct set up

$$V=\int_a^b 2\pi xy\,dx$$

$$=\int_0^3 2\pi x\left(3x^2-x^3\right)dx$$

1 mark correct integral

$$=2\pi\int_0^3\left(3x^3-x^4\right)dx$$

$$=2\pi\left[\frac{3x^4}{4}-\frac{x^5}{5}\right]_0^3$$

$$=\frac{243\pi}{10}\text{ units}^3$$

1 mark for correct answer

b) i)

$$A=\frac{\pi}{2}r^2$$

$$=\frac{\pi}{2}\left[\frac{(9-3x)-(x-3)^2}{2}\right]^2$$

1 mark for correct r

$$=\frac{\pi}{2}\left[\frac{(9-3x)-(x-3)^2}{2}\right]^2$$

$$=\frac{\pi}{2}\frac{\left[3x-x^2\right]^2}{4}$$

$$=\frac{\pi}{8}\left[x(3-x)\right]^2$$

When $x=a$ then

$$A=\frac{\pi}{8}a^2(3-a)^2$$

1 mark for correct expression for A

ii)

$$V=\int_0^3\frac{\pi}{8}\left(9a^2-6a^3+a^4\right)da$$

1 mark for correct integral for V

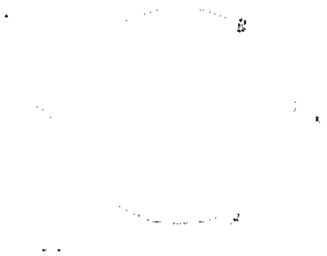
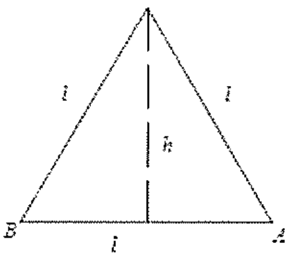
$$=\frac{\pi}{8}\left[3a^3-\frac{3a^4}{2}+\frac{a^5}{5}\right]_0^3$$

1 mark for correct integration

$$=\frac{\pi}{8}\left[\frac{81}{10}\right]$$

$$\therefore\ V=\frac{81\pi}{80}\text{units}^3$$

1 mark for correct answer

c)	i)	 	
		$x^2 + y^2 = 9$ $y = \sqrt{9 - x^2}$ $\therefore l = 2\sqrt{9 - x^2}$ $\sin 60 = \frac{h}{l}$ $h = l \sin 60$ $h = \frac{\sqrt{3}}{2} l$ $\therefore h = \sqrt{3}\sqrt{9 - x^2}$ $A(x) = \frac{1}{2} bh$ $= \frac{1}{2}\left(2\sqrt{9 - x^2}\right)\left(\sqrt{3}\sqrt{9 - x^2}\right)$ $= \sqrt{3}\left(9 - x^2\right)$ $OR \quad A = \; = \frac{1}{2} l^2 \sin 60°$ $= \frac{\sqrt{3}}{4} l^2$	**1 mark for correct h** **1 mark for correct area**
	ii)	$V = \int_{-3}^{3} \sqrt{3}\,(9 - x^2)\,dx$ $= \sqrt{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$ $= \sqrt{3}[(27 - 9) - (-27 + 9)]$ $= \sqrt{3}[18 + 18]$ $= 36\sqrt{3}$ **OR** $V = 2\int_0^3 \sqrt{3}\left(9 - x^2\right)dx$ $= 2\sqrt{3}\left[9x - \frac{x^3}{3}\right]_0^3$ $= 2\sqrt{3}\left[(27 - 9) - 0\right]$ $= 2\sqrt{3} \times 18$ $= 36\sqrt{3}$ units^3	**1 mark for correct integral** **1 mark for correct answer**

2016

Term 2 HSC Task 3 Examination

Ext 2 Mathematics

Solutions

HSC Task 3 Term 2 Examination – Ext 2 Mathematics 2016

Section I Multiple Choice Answer 1 Mark each

	A	B	C	D
1.	○	○	●	○
2.	○	○	○	●
3.	○	●	○	○
4.	○	○	●	○
5.	●	○	○	○

Multiple Choice Working Out

1

$$\text{Now } \int \frac{dx}{x^2-4x+13} = \int \frac{dx}{x^2-4x+4+9}$$

$$= \int \frac{dx}{(x-2)^2+9}$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+C \;\Rightarrow \text{C}$$

3

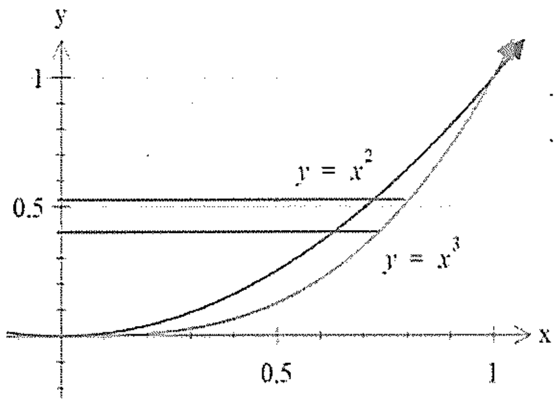
$$\int \frac{1}{1+\sin x+\cos x}\,dx = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}$$

$$= \int \frac{2}{1+t^2+2t+1-t^2}\,dt$$

$$= \int \frac{1}{1+t}\,dt$$

$$= \ln|t+1|+C \;\Rightarrow \text{B}$$

2



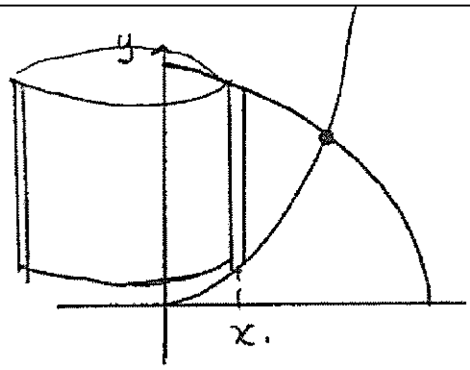
$$\text{Now } y = x^3 \text{ then } x = y^{\frac{1}{3}}$$

$$\text{and } y = x^2 \text{ then } x = y^{\frac{1}{2}}$$

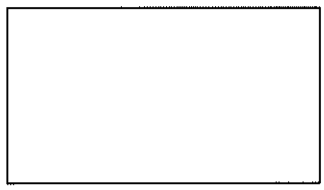
$$\text{So } V = \pi\int_0^1\left[\left(y^{\frac{1}{3}}\right)^2-\left(y^{\frac{1}{2}}\right)^2\right]dy$$

$$= \pi\int_0^1\left(y^{\frac{2}{3}}-y\right)dy \quad \Rightarrow \text{D}$$

4



$2\pi r = 2\,\pi x$

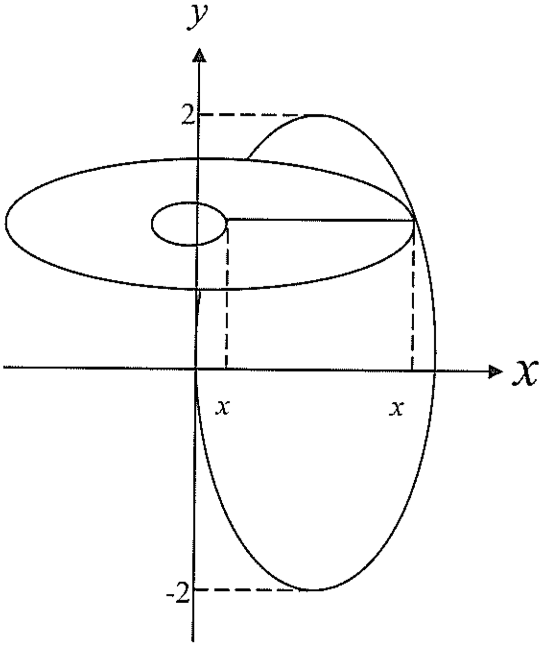


$$y_1-y_2=\sqrt{20-x^2}-x^2$$

$$A = 2\pi x\left(\sqrt{20-x^2}-x^2\right)$$

$$V = \int_0^2 2\pi x\left(\sqrt{20-x^2}-x^2\right)dx \quad \Rightarrow \text{C}$$

5



$\delta V = \pi\left(R+r\right)\left(R-r\right)\delta y$ where $R = x_2,\ r = x_1$

Taking $x_2,\ x_1$ as roots of equation $\left(x-1\right)^2 + \dfrac{y^2}{4} = 1$

at some fixed height y above $x-$axis:

$4(x^2 - 2x + 1) + y^2 - 4 = 0$

$4x^2 - 8x + y^2 = 0$ has roots $x_1,\ x_2$

Sum of roots: $x_1 + x_2 = 2$

Product of roots: $x_1 x_2 = \dfrac{y^2}{4}$

Now $\left(x_2 - x_1\right)^2 = \left(x_1 + x_2\right)^2 - 4x_1x_2$ where $x_2 - x_1 > 0$

So $\quad x_2 - x_1 = \sqrt{\left(x_1 + x_2\right)^2 - 4x_1x_2}$

$= \sqrt{4 - 4\cdot\dfrac{y^2}{4}}$

$= \sqrt{4 - y^2}$

$\therefore\ V = 2\pi\displaystyle\int_{-2}^{2}\sqrt{4-y^2}\,dy \quad \Rightarrow \text{A}$

Section II **Solutions**

Question 6

a)

$$\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$$

$$9 + 16x - 4x^2 = 9 - 4(x^2 - 4x)$$

$$= 9 - 4(x^2 - 4x + 4) + 16$$

$$= 25 - 4(x - 2)^2$$

$$\int \frac{dx}{\sqrt{9 + 16x - 4x^2}} = \int \frac{dx}{\sqrt{25-4(x-2)^2}}$$

$$= \frac{1}{5} \int \frac{dx}{\sqrt{1-\frac{4}{25}(x-2)^2}}$$

1 for correct manipulation

$$u = \frac{2(x-2)}{5} + c$$

$$du = \frac{2}{5} dx$$

$$dx = \frac{5}{2}\, du$$

$$= \frac{1}{5} \int \frac{\frac{5}{2} du}{\sqrt{1-u^2}}$$

1 for correct substitution

$$= \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2}\, sin^{-1} u$$

$$= \frac{1}{2}\, sin^{-1}\left(\frac{2(x-2)}{5}\right)$$

1 mark for correct answer

b)

Let $\dfrac{5x^2-3x+13}{(x-1)(x^2+4)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

$$\equiv \frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$$

$\therefore\ 5x^2 - 3x + 13 = A(x^2+4) + (Bx+C)(x-1)$ **1 mark for this line**

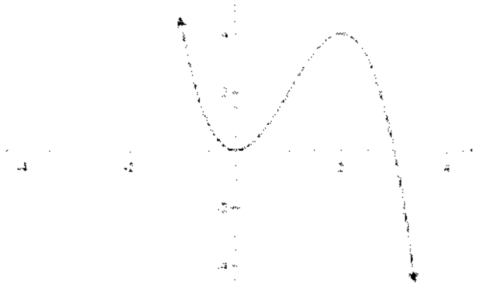
$$\left.\begin{array}{ll} \text{When } x = 1 \text{ then} & 15 = 5A \\ & A = 3 \\ \text{When } x = 0 \text{ then} & 13 = 12 - C \\ & C = -1 \\ \text{When } x = -1 \text{ then} & 21 = 15 + (-B-1)(-2) \\ & 6 = 2B + 2 \\ & B = 2 \end{array}\right\}$$

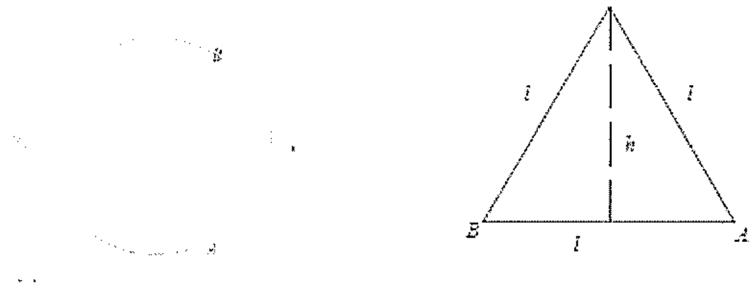
1 mark for the values of *A*, *B* & *C*

	$\therefore\ I=\int\left(\frac{3}{x-1}+\frac{2x-1}{x^2+4}\right)dx$ $=3\ln\lvert x-1\rvert+\int\left(\frac{2x}{x^2+4}\right)dx-\int\left(\frac{1}{x^2+4}\right)dx$ $=3\ln\lvert x-1\rvert+\ln\lvert x^2+4\rvert-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$	**1 mark for correct answer**
c)	$\int_0^{\frac{\sqrt{\pi}}{2}} 3x\ \sin(x^2)\,dx$ Substitute $u = x^2$ $\frac{du}{dx}=2x$ $du = 2x\,dx$ $\frac{3}{2}du = 3x\,dx$ $x=0 \Rightarrow u=0^2=0$ $x=\frac{\sqrt{\pi}}{2} \Rightarrow u=\left(\frac{\sqrt{\pi}}{2}\right)^2=\frac{\pi}{4}$ $\int_0^{\frac{\sqrt{\pi}}{2}} 3x\ \sin(x^2)\,dx = \int_0^{\frac{\pi}{4}}\frac{3}{2}\sin u\,du$ $=\frac{3}{2}\int_0^{\frac{\pi}{4}}\sin u\,du$ $=\frac{3}{2}\left[-\cos u\right]_0^{\frac{\pi}{4}}$ $=-\frac{3}{2}\left(\cos\left(\frac{\pi}{4}\right)-\cos(0)\right)$ $=-\frac{3}{2}\left(\frac{1}{\sqrt{2}}-1\right)$ $=-\frac{3}{2}\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)$ $=\frac{3\sqrt{2}-3}{2\sqrt{2}}$ $=\frac{6-3\sqrt{2}}{4}$	**Using a Substitution** **1 for changing limits & variable** **1 for integral including correct limits** **1 for substitution & simplification to get answer** **OR**

		$\int_0^{\frac{\sqrt{\pi}}{2}} 3x\sin\left(x^2\right)dx$ $=-\frac{3}{2}\left[\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ $=-\frac{3}{2}\left[\cos\frac{\pi}{4}-\cos 0\right]$ $=-\frac{3}{2}\left(\frac{1}{\sqrt{2}-1}\right)$	**Without a Substitution** **1 mark for correct integration** **1 mark for correct working** **1 mark for correct answer**
d)		$I=\int_1^e x^7\ln x dx$ Let $u=\ln x$ then $u'=\frac{1}{x}$ & $v'=x^7$ then $v=\frac{x^8}{8}$ $=\left[\frac{x^8}{8}\ln x\right]_1^e-\frac{1}{8}\int_1^e x^7 dx$ $=\frac{e^8}{8}-\frac{1}{8}\left[\frac{x^8}{8}\right]_1^e$ $=\frac{e^8}{8}-\frac{1}{8}\left(\frac{e^8}{8}-\frac{1}{8}\right)$ $=\frac{7e^8+1}{64}$	**1 mark for correct u' & v** **1 mark for correct substitution** **1 mark for correct answer**
e)	**i)**	$\int x^n e^{-x^2}dx$ Let $u = x^{n-1}$ $\quad v' = xe^{-x^2}$ $u' = (n-1)x^{n-2}$ $\quad v = -\frac{1}{2}e^{-x^2}$ $\int x^n e^{-x^2}dx = uv - \int vu'$ $= -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}\int x^{n-2}e^{-x^2}dx$	**1 mark for correct set up** **1 mark for correct answer**
	ii)	$\int_0^1 x^5 e^{-x^2}dx = \left[-\frac{1}{2}x^4 e^{-x^2}\right]_0^1 + \frac{4}{2}\int_0^1 x^3 e^{-x^2}dx$ $= \frac{-1}{2e} + 2\int_0^1 x^3 e^{-x^2}dx$ $= \frac{-1}{2e} + 2\left\{\left[-\frac{1}{2}x^2 e^{-x^2}\right]_0^1 + 1\int_0^1 xe^{-x^2}dx\right\}$ $= \frac{-1}{2e} - 2\left(\frac{1}{2e}\right) + 2\left[-\frac{1}{2}x^0 e^{-x^2}\right]_0^1 + \frac{1-1}{2}\int x^{1-2}e^{-x^2}dx$ $= \frac{-1}{2e} - \frac{1}{e} + \left[-e^{-x^2}\right]_0^1$ $= \frac{-1}{2e} - \frac{1}{e} - \frac{1}{e} + 1$ $= \frac{-1}{2e} - \frac{2}{2e} - \frac{2}{2e} + 1$ $= 1 - \frac{5}{2e}$	**1 mark for correct reduction** **1 mark for correct answer**

Question 7

a)	 $\partial V = 2\pi xy\partial x$ $= 2\pi x\left(3x^2 - x^3\right)\partial x$ $V = \lim_{\partial x \to 0}\sum_{0}^{3} 2\pi x\left(3x^2 - x^3\right)\partial x$		2 marks for correct set up
	$V = \int_a^b 2\pi xy dx$ $= \int_0^3 2\pi x\left(3x^2 - x^3\right)dx$		1 mark correct integral
	$= 2\pi\int_0^3\left(3x^3 - x^4\right)dx$ $= 2\pi\left[\frac{3x^4}{4} - \frac{x^5}{5}\right]_0^3$ $= \frac{243\pi}{10}$ units^3		1 mark for correct answer
b)	i)	$A = \frac{\pi}{2}r^2$ $= \frac{\pi}{2}\left[\frac{(9-3x)-(x-3)^2}{2}\right]^2$	1 mark for correct r
		$= \frac{\pi}{2}\left[\frac{(9-3x)-(x-3)^2}{2}\right]^2$ $= \frac{\pi}{2}\frac{\left[3x - x^2\right]^2}{4}$ $= \frac{\pi}{8}\left[x(3-x)\right]^2$ When $x = a$ then $A = \frac{\pi}{8}a^2(3-a)^2$	1 mark for correct expression for A
	ii)	$V = \int_0^3 \frac{\pi}{8}\left(9a^2 - 6a^3 + a^4\right)da$	1 mark for correct integral for V
		$= \frac{\pi}{8}\left[3a^3 - \frac{3a^4}{2} + \frac{a^5}{5}\right]_0^3$	1 mark for correct integration
		$= \frac{\pi}{8}\left[\frac{81}{10}\right]$ $\therefore\ V = \frac{81\pi}{80}$ units^3	1 mark for correct answer

c)	i)		
		$x^2 + y^2 = 9$ $y = \sqrt{9 - x^2}$ $\therefore l = 2\sqrt{9 - x^2}$ $\sin 60 = \frac{h}{l}$ $h = l \sin 60$ $h = \frac{\sqrt{3}}{2} l$ $\therefore h = \sqrt{3}\sqrt{9 - x^2}$ $A(x) = \frac{1}{2} bh$ $= \frac{1}{2}\left(2\sqrt{9 - x^2}\right)\left(\sqrt{3}\sqrt{9 - x^2}\right)$ $= \sqrt{3}\left(9 - x^2\right)$ OR $A = \frac{1}{2} l^2 \sin 60°$ $= \frac{\sqrt{3}}{4} l^2$	1 mark for correct h 1 mark for correct area
	ii)	$V = \int_{-3}^{3} \sqrt{3}\,(9 - x^2)\,dx$ $= \sqrt{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$ $= \sqrt{3}[(27 - 9) - (-27 + 9)]$ $= \sqrt{3}[18 + 18]$ $= 36\sqrt{3}$ **OR** $V = 2\int_0^3 \sqrt{3}\left(9 - x^2\right)dx$ $= 2\sqrt{3}\left[9x - \frac{x^3}{3}\right]_0^3$ $= 2\sqrt{3}\left[(27 - 9) - 0\right]$ $= 2\sqrt{3} \times 18$ $= 36\sqrt{3}$ units^3	**1 mark for correct integral** **1 mark for correct answer**