Carlingford High School Mathematics Extension 2 Trial Exam 2018

Solutions

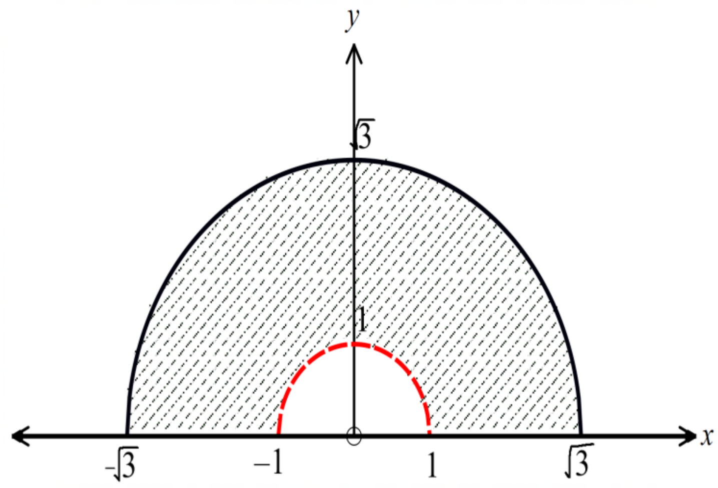
Multiple Choice

1	D	**6**	D
2	C	**7**	C
3	A	**8**	D
4	B	**9**	B
5	A	**10**	A

Question 11

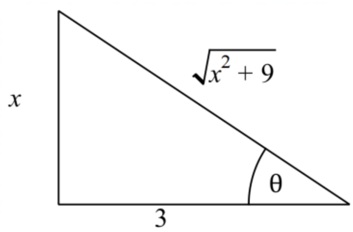
	Solution	Marks	Allocation of marks
(a)	$\lvert z\rvert = 2 \; and \arg z = \frac{\pi}{3}$ (i) $z = 2cis\,\frac{\pi}{3}$ $z^5 = \left(2cis\,\frac{\pi}{3}\right)^5$ $= 2^5\left(cis\,\frac{5\pi}{3}\right)$ $= 32cis\,\frac{5\pi}{3}$ $= 32cis\frac{-\pi}{3}$	**1**	1 mark for correct answer
	(ii) $z = 2cis\,\frac{\pi}{3}$ $= 2\left(cos\frac{\pi}{3} + isin\,\frac{\pi}{3}\right)$ $= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $= 1 + \sqrt{3}i$	**1**	1 mark for correct answer
	(iii) $\frac{1}{z} = \frac{1}{1+\sqrt{3}i}$ $= \frac{1}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$ $= \frac{1-\sqrt{3}i}{1+3}$ $= \frac{1-\sqrt{3}i}{4}$	**2**	2 marks for correct solution with real denominator 1 mark if correct method of realising the denominator is used with an error in calculation

	Solution	Marks	Allocation of marks
	(iv) $\omega^2 z = (2-3i)^2(1+\sqrt{3}i)$ $= (4-12i-9)(1+\sqrt{3}i)$ $= (-5-12i)(1+\sqrt{3}i)$ $= -5-5\sqrt{3}i-12i+12\sqrt{3}$ $= (-5+12\sqrt{3})-(5\sqrt{3}+12)i$	**1**	1 mark for correct answer
(b)	(i) $\frac{x+1}{(x+3)(x+2)^2} = \frac{a}{x+3}+\frac{b}{x+2}+\frac{c}{(x+2)^2}$ $x+1 = a(x+2)^2+b(x+2)(x+3)+c(x+3)$ When $x=-2$ $\quad -1 = 0+0+c \quad \therefore\ c = -1$ $x=-3 \quad -2 = a+0+0 \quad \therefore\ a = -2$ Coefficient of x^2: $\ 0 = -2+b \quad \therefore b = 2$	**2**	2 marks for correct values of *a, b* and *c* 1 mark if the correct method is used with an error in calculation
	(ii) $\int \frac{x+1}{(x+3)(x+2)^2}\,dx = \int \frac{-2dx}{x+3} + \int \frac{2dx}{x+2} - \int \frac{dx}{(x+2)^2}$ $= -2\,ln(x+3)+2ln(x+2)-\int (x+2)^{-2}dx$ $= 2[ln(x+2)-ln(x+3)] - \frac{(x+2)^{-1}}{-1}+c$ $= 2\ln\left(\frac{x+2}{x+3}\right)+\frac{1}{x+2}+c$	**2**	2 marks for correct solution 1 mark if correct method is used with an error in calculation or algebra
(c)	(i) $cos\,x + cos\,3x = 4cos^3x - 2\,cos\,x$. LHS = $cos\,x + cos\,3x$ $= cos\,x + cos(2x+x)$ $= cos\,x + cos\,2x\,cos\,x - sin\,2x\,sin\,x$ $= cos\,x + (2cos^2x-1)\,cos\,x - (2\,sin\,x\,cos\,x)\,sin\,x$ $= cos\,x + 2cos^3x - cos\,x\ - 2\,sin^2x cos\,x$ $= \ 2cos^3x - 2(1-\ cos^2x)\,cos\,x$ $= \ 2cos^3x - 2\,cos\,x + 2cos^3x$ $= 4cos^3x - 2\,cos\,x$ = RHS	**2**	2 marks for correct reasoning to achieve required result 1 mark if some correct working is provided but which does not achieve correct result, or is incomplete or has an error in calculation, logic or algebra

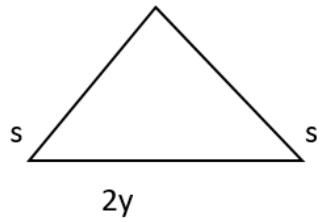
	Solution	Marks	Allocation of marks
	(ii) $(\cos x + \cos 3x) = 0$ $4\cos^3 x - 2\cos x = 0$ $2\cos x\,(2\cos^2 x - 1) = 0$ $2\cos x = 0$ or $2\cos^2 x = 1$ $\cos x = 0$ or $\cos x = \pm\frac{1}{\sqrt{2}}$ $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$	**2**	2 marks for correct solution with all values provided 1 mark if some correct working is shown with an error in calculation, logic or algebra or if not all values are given
(d)	Region in the Argand diagram where $1 < z\bar{z} \leq 3$ and $Im\,(z) \geq 0$ $z\bar{z} = (x+iy)(x-iy) = x^2 + y^2$ So $1 < x^2 + y^2 \leq 3$ and $y \geq 0$. 	**2**	2 marks where the correct graphs are shown with the required region shaded with broken and unbroken lines in correct positions. 1 mark if a graph is incorrect or the required region is not shaded or if broken and unbroken lines are not in correct positions.

Question 12

	Solution	Marks	Allocation of marks
(a)	(i) $\int sec^4 x \tan x \; dx$ Let $u = sec\,x$ $du = sec\,x\,tan\,x$ $\int sec^4 x \tan x \; dx = \int sec^3 x\, sec\,x\,tan\,x\,dx$ $= \int u^3\,du$ $= \frac{u^4}{4} + c$ $= \frac{1}{4}\,sec^4 x + c$	**2**	2 marks for correct solution 1 mark if correct method is used with an error in calculation or algebra

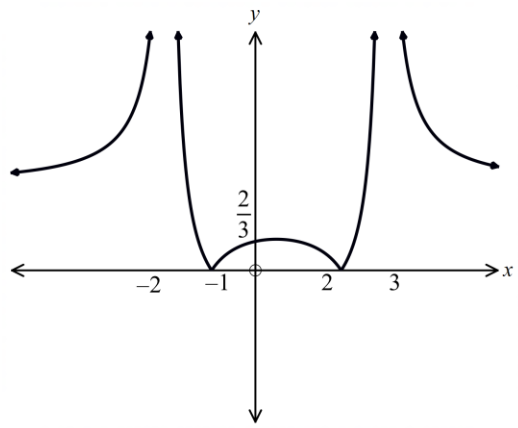
	Solution	Marks	Allocation of marks
	ALTERNATIVE SOLUTION: Different but equivalent answer $\int sec^4x \tan x\, dx = \int sec^2x \sec^2 x \tan x\, dx$ $= \int sec^2x(\tan^2 x + 1) \tan x\, dx$ $= \int sec^2x \tan^3 x + \sec^2 x \tan x\, dx$ $= \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + c$		
	(ii) $\int \frac{dx}{\sqrt{7+4x-x^2}} = \int \frac{dx}{\sqrt{7-(x^2-4x)}}$ $= \int \frac{dx}{\sqrt{7-(x^2-4x+4)+4}}$ $= \int \frac{dx}{\sqrt{11-(x-2)^2}}$ $= sin^{-1}\left(\frac{x-2}{\sqrt{11}}\right) + c$	**2**	2 marks for correct solution 1 mark if correct method is used with an error in calculation or algebra
	(iii) $$\int \frac{dx}{x^2\sqrt{9+x^2}}$$ $x = 3\tan\theta$ $dx = 3sec^2\theta\, d\theta$ $$\int \frac{dx}{x^2\sqrt{9+x^2}} = \int \frac{3sec^2\theta\, d\theta}{(3\tan\theta)^2\sqrt{9+(3\tan\theta)^2}}$$ $= \int \frac{3sec^2\theta\, d\theta}{9tan^2\theta\sqrt{9(1+tan^2\theta)}}$ $= \int \frac{3sec^2\theta\, d\theta}{9tan^2\theta\sqrt{9sec^2\theta}}$ $= \int \frac{3sec^2\theta\, d\theta}{9tan^2\theta.3\sec\theta}$ $= \frac{1}{9}\int \frac{sec\,\theta\, d\theta}{tan^2\theta}$ Since $\frac{sec\,\theta}{tan^2\theta} = \frac{1}{cos\,\theta} \div \frac{sin^2\theta}{cos^2\theta} = \frac{1}{cos\,\theta} \times \frac{cos^2\theta}{sin^2\theta} = \frac{cos\,\theta}{sin^2\theta}$ $= \frac{1}{9}\int \frac{\cos\theta}{sin^2\theta} d\theta$ $= -\frac{1}{9} cosec\,\theta + C$ $= -\frac{1}{9}\frac{\sqrt{x^2+9}}{x} + C$	**3**	3 marks for correct solution 2 marks for working that includes correct substitution and some correct manipulation toward the required integral 1 mark for working that includes correct substitution OR some correct manipulation toward the required integral 

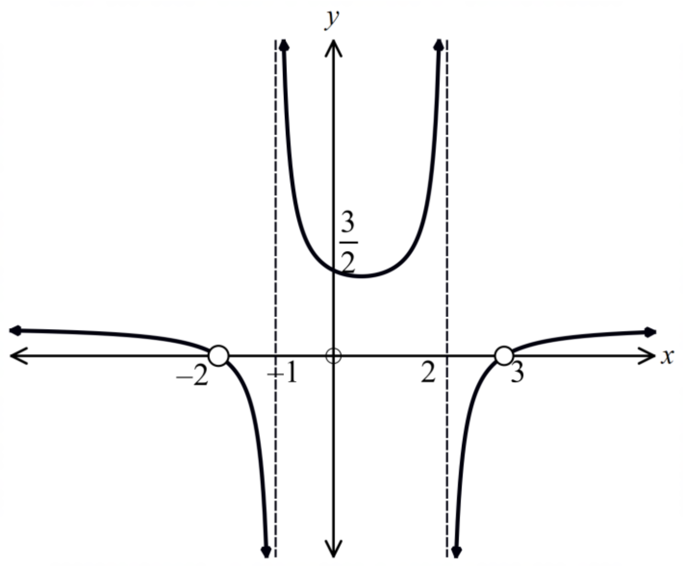
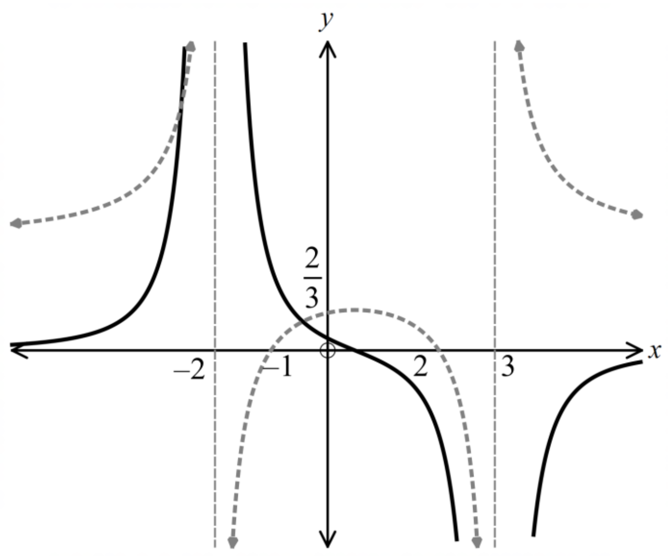
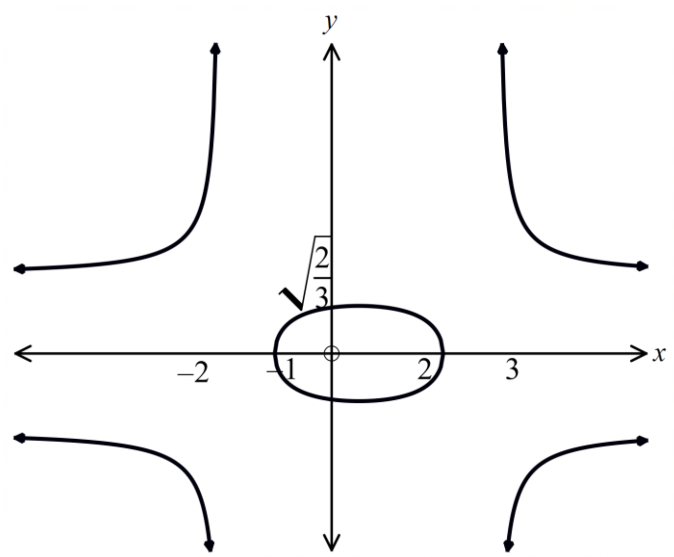
	Solution	Marks	Allocation of marks
(b)	Use the substitution $u = x^2 + 2x + 5$ $$\frac{du}{dx} = 2x + 2$$ $$0.5du = (x+1)dx$$ When $x = 2$ then $u = 13$ and when $x = 3$ then $u = 20$ $$\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}dx = \int_{13}^{20} \frac{0.5du}{u^{\frac{1}{2}}}$$ $$= \left[u^{\frac{1}{2}}\right]_{13}^{20}$$ $$= \sqrt{20} - \sqrt{13}$$	**2**	2 marks: Correct answer. 1 mark: Finds the primitive function or sets up the integration using substitution.
(c)	$$x^2 + xy + y^2 = 7$$ Normal at (1, 2) $$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$ $$(x+2y)\frac{dy}{dx} = -2x - y$$ $$\frac{dy}{dx} = \frac{-2x-y}{x+2y}$$ At (1, 2) $m = \frac{dy}{dx} = \frac{-2(1)-2}{1+2(2)} = \frac{-4}{5}$ $m(normal) = \frac{5}{4}$ $$y - y_1 = m(x - x_1)$$ $$y - 2 = \frac{5}{4}(x-1)$$ $$4y - 8 = 5x - 5$$ $$5x - 4y + 3 = 0$$	**2**	2 marks for correct equation of normal 1 mark for working that includes correct differentiation or gradient found correctly and some correct and relevant algebraic manipulation

	Solution	Marks	Allocation of marks
(d)	$x^2 + y^2 = 16$ $y = \sqrt{16 - x^2}$ $\therefore 2y = 2\sqrt{16 - x^2}$  By Pythagoras $s^2 + s^2 = \left(2\sqrt{16 - x^2}\right)^2$ $2s^2 = 4\,(16 - x^2)$ $s^2 = 2(16 - x^2)$ $Area = \frac{1}{2}s^2 = \frac{1}{2}[2(16 - x^2)]$ $= 16 - x^2$ $V = \int_{-4}^{4}(16 - x^2)\,dx \qquad OR = 2\int_0^4(16 - x^2)$ $= \left[16x - \frac{x^3}{3}\right]_{-4}^{4}$ $= \left[\left(64 - \frac{64}{3}\right) - \left(-64 - \frac{-64}{3}\right)\right]$ $= \frac{256}{3}$ u^3	**4**	4 marks for correct answer 3 marks for attempting to use correct method for volume with minor error in area or integration 2 marks for finding area or equivalent merit 1 mark for some correct working relevant to the solution

Question 13

	Solution	Marks	Allocation of marks
(a)	(i) $m(PQ) = \dfrac{\frac{3}{q} - \frac{3}{p}}{3q - 3p} = \dfrac{3(p-q)}{pq} \times \dfrac{1}{3(q-p)} = -\dfrac{1}{pq}$ $y - y_1 = m(x - x_1)$ $y - \frac{3}{p} = -\frac{1}{pq}(x - 3p)$ $pqy - 3q = -x + 3p$ $x + pqy = 3(p + q)$	**2**	2 marks for correct equation 1 mark for working that includes correct gradient and some correct and relevant algebraic manipulation
	(ii) Tangent at $P\left(3p, \frac{3}{p}\right)$ $xy = 9$ $y + x\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{y}{x} = -\frac{3}{p} \div 3p = -\frac{3}{p} \times \frac{1}{3p} = -\frac{1}{p^2}$ $y - y_1 = m(x - x_1)$ $y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$ $p^2y - 3p = -x + 3p$ $x + p^2y = 6p$	**2**	2 marks for correct equation 1 mark for working that includes correct derivative and some correct and relevant algebraic manipulation

	Solution	Marks	Allocation of marks
	(iii) Tangent at P is $x + p^2y = 6p$ - - - (1) Tangent at Q is $x + q^2y = 6q$ - - - (2) (1) − (2) $(p^2 - q^2)y = 6(p - q)$ $y = \frac{6}{p+q}$ $\therefore x + \frac{6p^2}{p+q} = 6p$ $x = 6p - \frac{6p^2}{p+q} = \frac{6p(p+q) - 6p^2}{p+q} = \frac{6p^2 + 6pq - 6p^2}{p+q} = \frac{6pq}{p+q}$ $\therefore T\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$	**2**	2 marks for correct values of *x* and *y* 1 mark for working that includes correct value of *x* or *y*
	(iv) $T\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$ $m = -\frac{1}{pq}$ $y - y_1 = m(x - x_1)$ $y - \frac{6}{p+q} = -\frac{1}{pq}\left(x - \frac{6}{p+q}\right)$ $pqy - \frac{6pq}{p+q} = -x + \frac{6pq}{p+q}$ $x + pqy = \frac{12pq}{p+q}$ Passes through (0, 6) $\therefore 0 + 6pq = \frac{12pq}{p+q}$ $6pq(p+q) = 12pq$ $i.e. p + q = 2$	**2**	2 marks for correct reasoning to achieve required result 1 mark if some correct working is provided but which does not achieve correct result, or is incomplete or has an error in calculation, logic or algebra
(b)	(i)  	**1**	Correct graph

	Solution	Marks	Allocation of marks
	(ii) 	2	2 marks for correct graph, 1 mark for graph with some correct features but some errors Points of discontinuity at -2 and 3 because they existed there in the original graph
	(iii) 	2	2 marks for correct graph, 1 mark for graph with some correct features but some errors
	(iv) 	2	2 marks for correct graph, 1 mark for graph with some correct features but some errors

Question 14

	Solution	Marks	Allocation of marks
(a)	(i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}$ (*) Now α^2, β^2 and γ^2 are zeroes of $x^{3/2}-3x+4x^{\frac{1}{2}}-6=0$, hence satisfy $(x^{3/2}+4x^{\frac{1}{2}})^2=(3x+6)^2$ $x^3+8x^2+16x=9x^2+36x+36$ $x^3-x^2-20x-36=0$ (**) $\therefore\ \beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2=-20$ and $\alpha^2\beta^2\gamma^2=36$ Substituting into (*), $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=-\frac{5}{9}$. Note that we have also found $\alpha^2+\beta^2+\gamma^2=1$ which can be used in (ii).	**3**	3 marks for correct value 2 marks for significant working toward correct result with minor errors in calculation or algebra. 1 mark for working that includes a correct sum/product and some correct and relevant algebraic manipulation
	(ii) $x^3-3x^2+4x-6=0 \rightarrow x^3=3x^2-4x+6$ $\therefore\ \alpha^3=3\alpha^2-4\alpha+6$ $\beta^3=3\beta^2-4\beta+6$ $\gamma^3=3\gamma^2-4\gamma+6$ $\alpha^3+\beta^3+\gamma^3=3(\alpha^2+\beta^2+\gamma^2)-4(\alpha+\beta+\gamma)+18$ $=3(1)-4(3)+18$ $=9$	2	2 marks for correct value 1 mark for working that includes a correct sum/product and some correct and relevant algebraic manipulation
(b)	(i) $p(x)=ax^3+bx^2+cx+d$ $p'(x)=3ax^2+2bx+c$ $\Delta=4b^2-12ac$ $=4(b^2-3ac)$ <0 if $b^2-3ac<0$ as given. $\therefore$ $p'(x)$=0 has no real solutions hence $p(x)$ has no stationary points and is always either increasing or decreasing. Therefore $p(x)$ cuts the x-axis only once.	**2**	2 marks for correct solution. 1 mark for finding $p'(x)$ or equivalent.

	Solution	Marks	Allocation of marks
	(ii) $p\left(-\frac{b}{3a}\right)=0$ $p'\left(-\frac{b}{3a}\right)=3a\left(-\frac{b}{3a}\right)^2+2b\left(-\frac{b}{3a}\right)+c$ $=\frac{b^2}{3a}-\frac{2b^2}{3a}+c$ $=-\frac{1}{3a}(b^2-3ac)$ $=0$ Therefore at least a double root. $p''(x)=6ax+2b$ $p''\left(-\frac{b}{3a}\right)=-2b+2b$ $=0$ $x=-\frac{b}{3a}$ is a triple root of $p(x)$.	**2**	2 marks for correct solution. 1 mark for showing $p'\left(-\frac{b}{3a}\right)=0$ or equivalent.
(c)	(i) $z^n=\cos n\theta+i\sin n\theta$ $z^{-n}=\frac{1}{z^n}=\cos n\theta-i\sin n\theta$ $z^n+\frac{1}{z^n}=(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)$ $=2\cos n\theta$	**1**	1 mark for working to achieve required result
	(ii) $(2\cos\theta)^5=\left(z+\frac{1}{z}\right)^5=z^5+5z^4\left(\frac{1}{z}\right)+10z^3\left(\frac{1}{z^2}\right)+10z^2\left(\frac{1}{z^3}\right)+5z\left(\frac{1}{z^4}\right)+\frac{1}{z^5}$ $32\cos^5\theta=z^5+5z^3+10z+10\left(\frac{1}{z}\right)+5\left(\frac{1}{z^3}\right)+\frac{1}{z^5}$ $=\left(z^5+\frac{1}{z^5}\right)+5\left(z^3+\frac{1}{z^3}\right)+10\left(z+\frac{1}{z}\right)$ $=2\cos 5\theta+5(2\cos 3\theta)+10(2\cos\theta)$ $\cos^5\theta=\frac{2}{32}\cos 5\theta+\frac{10}{32}\cos 3\theta+\frac{20}{32}\cos\theta$ $\cos^5\theta=\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta$ $\therefore A=\frac{1}{16}, B=\frac{5}{16}$ and $C=\frac{5}{8}$	**2**	2 marks for correct values 1 mark for working that includes a correct expansion and some correct and relevant algebraic manipulation
	(iii) $\int\cos^5\theta\,d\theta=\int\left(\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta\right)d\theta$ $=\frac{1}{80}\sin 5\theta+\frac{5}{48}\sin 3\theta+\frac{5}{8}\sin\theta+c$	**1**	1 mark for correct integral

	Solution	Marks	Allocation of marks
(d)	$2-3i$ is a root of the equation $x^2-(6-2i)x+k=0$. (i) Let $\alpha = 2-3i$ and other root be β. $$\alpha+\beta = -\frac{b}{a} = -\frac{-(6-2i)}{1}$$ $$\therefore 2-3i+\beta = 6-2i$$ $$\therefore \beta = 4+i$$	**1**	1 mark for finding the other complex root
	(ii) $\alpha\beta = \frac{c}{a} = \frac{k}{1} = k$ $\therefore \alpha\beta = (2-3i)(4+i) = k$ $\therefore k = 8+2i-12i-3i^2$ $k = 11-10i$	**1**	1 mark for correct value for k.

Question 15

	Solution	Marks	Allocation of marks
(a)	(i) Area of annulus $A = \pi(R^2-r^2)$ where $r = 2-x = 2-y^2$ and $R = 2-(-1) = 3$ $$A = \pi(3^2-(2-y^2)^2)$$ $$= \pi(9-(4-4y^2+y^4))$$ $$= \pi(5+4y^2-y^4)$$	**1**	1 mark for correct solution
	(ii) $V = \lim_{\delta y\to 0}\sum_{y=-1}^{1}\pi(5+4y^2-y^4)$ $= \pi\int_{-1}^{1} 5+4y^2-y^4 dy$ $= 2\pi\left[5y+\frac{4}{3}y^3-\frac{1}{5}y^5\right]_0^1$ by symmetry $= \frac{184\pi}{15}$ units^3	**2**	
(b)	$$x^4-5x^3+17x^2+37x-50 = 0$$ If $3-4i$ is a factor then $3+4i$ is also a factor. $$\therefore divisible\ by\ x^2-(\alpha+\beta)x+\alpha\beta = 0$$ $\alpha+\beta = 6$ $\alpha\beta = 9+16 = 25$ $$i.e. divisible\ by\ x^2-6x+25$$ By division, $$x^4-5x^3+17x^2+37x-50 = (x^2-6x+25)(x^2+x-2)$$ $$x^4-5x^3+17x^2+37x-50 = (x^2-6x+25)(x+2)(x-1)$$ $$If\ x^4-5x^3+17x^2+37x-50 = 0$$ Then $x = 3\pm 4i, -2\ and\ 1$.	**3**	3 marks for correct factors 2 marks for significant working toward correct result with minor errors in calculation or algebra. 1 mark for working that includes some correct and relevant algebraic manipulation
(c)	(i) $$I_n = \int sec^n x\ dx$$ $$= \int sec^{n-2}x\, sec^2 x\, dx$$ $Let\ u = sec^{n-2}x \qquad v' = sec^2 x$ $u' = (n-2)(sec\,x\,tan\,x)sec^{n-3}x \qquad v = tan\,x$	**2**	2 marks for correct use of integration by parts to show required result 1 mark for working that includes use of integration

	Solution	Marks	Allocation of marks
	$$I_n = uv - \int vu'$$ $I_n = sec^{n-2}x \ tan\,x - (n-2)\int sec^{n-2}x tan^2 x\,dx$ $= sec^{n-2}x \ tan\,x - (n-2)\int sec^{n-2}x\,(sec^2 x - 1)dx$ $= sec^{n-2}x \ tan\,x - (n-2)\int sec^n x - \ sec^{n-2}x\,dx$ $\therefore I_n + (n-2)I_n = \ sec^{n-2}x \ tan\,x + (n-2)I_{n-2}$ $(n-1)I_n = \ sec^{n-2}x \ tan\,x + (n-2)I_{n-2}$ $\therefore I_n = \frac{1}{n-1}\ sec^{n-2}x tan\,x + \frac{n-2}{n-1}\ I_{n-2}$		by parts and some correct and relevant algebraic manipulation but does not achieve result or is incomplete
	(ii) $\int_0^{\frac{\pi}{4}} sec^n x\,dx = \left[\frac{1}{3}sec^2 x tan\,x\right]_0^{\frac{\pi}{4}} + \frac{2}{3}\ I_2$ $= \left[\frac{1}{3}sec^2 x tan\,x + \frac{2}{3}tan\,x\right]_0^{\frac{\pi}{4}}$ $= \left[\frac{1}{3}sec^2\frac{\pi}{4}tan\,\frac{\pi}{4} + \frac{2}{3}tan\frac{\pi}{4}\right] - \left[\frac{1}{3}sec^2(0)tan\,(0) + \frac{2}{3}tan(0)\right]$ $= \left[\frac{1}{3}\left(\sqrt{2}\right)^2(1) + \frac{2}{3}\ (1)\right] - 0$ $= \frac{2}{3} + \frac{2}{3}$ $= 1\frac{1}{3}$	**1**	1 mark for correct answer
(d)	(i) $9x^2 - 16y^2 = 144$ $$\frac{x^2}{16} - \frac{y^2}{9} = 1$$ $a^2 = 16 \qquad b^2 = 9$ $a = 4 \qquad b = 3$ Eccentricity: $b^2 = \ a^2(e^2 - 1)$ $9 = 16\ (e^2 - 1)$ $\frac{9}{16} + 1 = \ e^2$ $e^2 = \frac{25}{16}$ $e = \frac{5}{4}$ Foci $(\pm ae, 0)$ $ae = 4 \times \frac{5}{4} = 5$ $\therefore S(5, 0)$ $S' = (-5, 0)$	**2**	2 marks for correct eccentricity and foci 1 mark for working that has either eccentricity or focus correct
(d)	(ii) $P(x_1, y_1)$ is an arbitrary point on $9x^2 - 16y^2 = 144$ $\frac{x^2}{16} - \frac{y^2}{9} = 1$ $\frac{2x}{16} - \frac{2y}{9}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{2x}{16} \div \frac{2y}{9}$ $= \frac{2x}{16} \times \frac{9}{2y} = \frac{9x}{16y}$ At $P(x_1, y_1)$ $\frac{dy}{dx} = \frac{9x_1}{16y_1}$ $y - \ y_1 = m(x - \ x_1)$ $y - \ y_1 = \frac{9x_1}{16y_1}(x - \ x_1)$ $16yy_1 - 16{y_1}^2 = 9xx_1 - 9{x_1}^2$ $9{x_1}^2 - \ 16{y_1}^2 = \ 9xx_1 - \ 16yy_1$	**1**	1 mark for correct working to achieve required equation of tangent

<table>
<tr><th></th><th>Solution</th><th>Marks</th><th>Allocation of marks</th></tr>
<tr><td></td><td>$\therefore 9xx_1 - 16yy_1 = 144$</td><td></td><td></td></tr>
<tr><td></td><td>(iii) Since $9x^2 - 16y^2 = 144$
The tangent cuts the x – axis when $y = 0$
$\therefore 9xx_1 = 144$
$x = \frac{144}{9x_1} = \frac{16}{x_1}$
$G\left(\frac{16}{x_1}, 0\right)$</td><td>1</td><td>1 mark for coordinates of G</td></tr>
<tr><td></td><td>(iv)
$P(x_1, y_1)$ $S(5,0)$ $S'(-5,0)$ $G\left(\frac{16}{x_1}, 0\right)$

<u>NOTE:</u>
$9x^2 - 16y^2 = 144$
$y^2 = \frac{9x^2 - 144}{16}$

$SG = 5 - \frac{16}{x_1} = \frac{5x_1 - 16}{x_1}$ $S'G = 5 + \frac{16}{x_1} = \frac{5x_1 + 16}{x_1}$

$SP^2 = (x_1 - 5)^2 + {y_1}^2$
$= {x_1}^2 - 10x_1 + 25 + \frac{9x^2 - 144}{16}$
$= \frac{16{x_1}^2 - 160x_1 + 400 + 9{x_1}^2 - 144}{16}$
$= \frac{(5x_1 - 16)^2}{16}$
$\therefore SP = \frac{5x_1 - 16}{4}$

Similarly, $S'P = \frac{5x_1 + 16}{4}$
Now, $\frac{SP}{S'P} = \frac{5x_1 - 16}{4} \div \frac{5x_1 + 16}{4} = \frac{5x_1 - 16}{5x_1 + 16}$
$\frac{SG}{S'G} = \frac{5x_1 - 16}{x_1} \div \frac{5x_1 + 16}{x_1} = \frac{5x_1 - 16}{5x_1 + 16}$
Therefore, $\frac{SP}{S'P} = \frac{SG}{S'G}$</td><td>2</td><td>2 marks for working that correctly derives the required result

1 mark for working that includes a correct use of the results proved previously or other hyperbola properties</td></tr>
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Question 16

	Solution	Marks	Allocation of marks
(a)	(i) $(a-b)^2 \geq 0$ $\therefore\ a^2 - 2ab + b^2 \geq 0$ $a^2 + b^2 \geq 2ab$ Add $2ab$ to both sides $a^2 + 2ab + b^2 \geq 4ab$ $(a+b)^2 \geq 4ab$ $a + b \geq 2\sqrt{ab}$ $\therefore a + b - 2\sqrt{ab} \geq 0$	**1**	1 mark for correct working to achieve required result
	(ii) $a + b \geq 2\sqrt{ab}$ $b + c \geq 2\sqrt{bc}$ $c + a \geq 2\sqrt{ac}$ $\therefore\ (a+b)(b+c)(c+a) \geq 8\sqrt{abbcca}$ $\therefore\ (a+b)(b+c)(c+a) \geq 8\sqrt{a^2b^2c^2}$ $\therefore\ (a+b)(b+c)(c+a) \geq 8abc$	**2**	2 marks for correct working to achieve required result 1 mark for working that includes relevant working toward result showing some knowledge of properties of inequalities
(b)	(i) Let $n = 1$. Then $x_1 = 1$ and $2\left(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\right) = 2\left(\frac{2/3}{4/3}\right) = 1.$ For n an integer ≥ 2 we need to show that $\frac{4 + x_{n-1}}{1 + x_{n-1}} = 2\left(\frac{1+\alpha^n}{1-\alpha^n}\right), \qquad \alpha = -\frac{1}{3} \quad (*)$ Suppose that (*) holds for some integer $k \geq 1$ and let $n = k + 1$. Then $LHS = \frac{4 + 2\left(\frac{1+\alpha^k}{1-\alpha^k}\right)}{1 + 2\left(\frac{1+\alpha^k}{1-\alpha^k}\right)}$ $= \frac{4(1-\alpha^k) + 2(1+\alpha^k)}{(1-\alpha^k) + 2(1+\alpha^k)}$ $= \frac{6 - 2\alpha^k}{3 + \alpha^k}$ $= \frac{2 + 2\alpha^{k+1}}{1 - \alpha^{k+1}}$ (multiplying top and bottom by $-\alpha = 1/3$) $= RHS.$ Therefore since the statement is true for $n = 1$ and $n = 2$, and if it is true for a positive integer $n = k$ then it is also true for $n = k + 1$, by induction it is true for all integers $n \geq 1$.	**4**	1 mark to check n=1 1 mark 1 mark 1 mark
	(ii) As $n \to \infty$, $\alpha^n \to 0$ since $\alpha < 1$. Therefore the limiting value is 2.	**1**	

<table>
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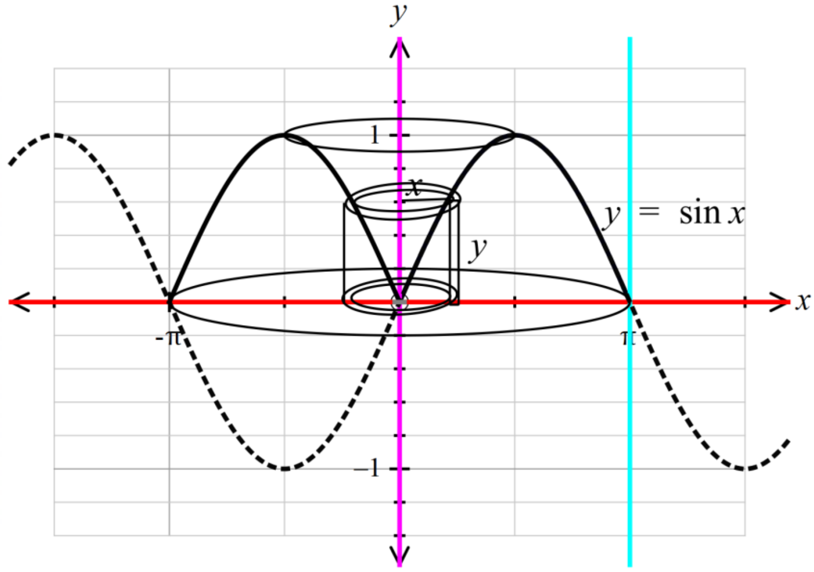
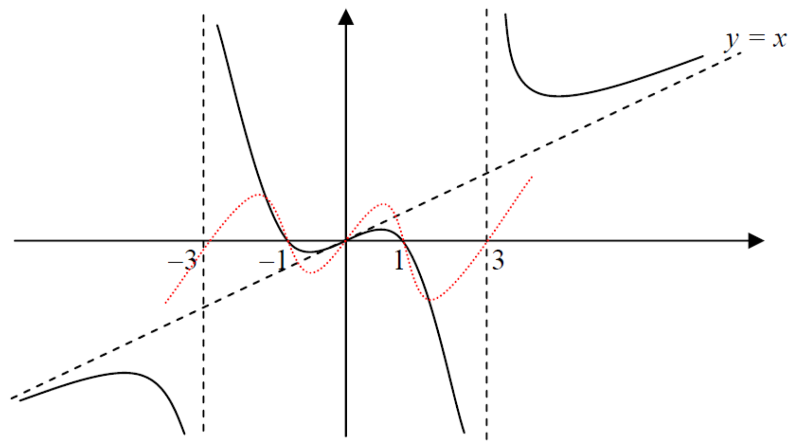
<tr><td>(c)</td><td>

Using the shell method
$$\delta V = \pi((x+\delta x)^2 - x^2)y$$
$$= 2\pi x \delta x \cdot y, \text{ ignoring terms in } (\partial x)^2$$
$$V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} 2\pi x y \delta x$$
$$= 2\pi \int_0^{\pi} xy\,dx$$
$$= 2\pi \int_0^{\pi} x \sin x\,dx$$
$Let\ u = x \quad v' = \sin x$
$\quad u' = 1 \quad v = -\cos x$
$$= 2\pi\left\{[-x\cos x]_0^{\pi} - \int_0^{\pi} -\cos x\,dx\right\}$$
$$= 2\pi\left\{[-x\cos x]_0^{\pi} - [-\sin x]_0^{\pi}\right\}$$
$$= 2\pi[-x\cos x + \sin x]_0^{\pi}$$
$$= 2\pi\left\{[-\pi\cos\pi + \sin\pi] - 0\right\}$$
$$= 2\pi(\pi)$$
$$= 2\pi^2$$</td><td>3</td><td>3 marks for correct answer

2 marks for significant working toward correct result with minor errors in logic, integration, calculation or algebra.

1 mark for working that includes a correct integral and some correct and relevant algebraic manipulation</td></tr>
<tr><td>(d)</td><td>$$y = x + \frac{8x}{x^2-9} = \frac{x^3-x}{x^2-9} = \frac{x(x+1)(x-1)}{(x+3)(x-3)}$$
$\therefore$ Asymptotes: $y = x$, and $x = \pm 3$
$\therefore x$-intercepts: $(0,0)$, $(\pm 1, 0)$.
(The dotted curve is the “guidegraph” of equation $y = (x+3)(x+1)x(x-1)(x-3)$)

</td><td>4</td><td>4 marks correct answer

3 marks most features of graph correct

2 marks asymptotes and x-intercepts identified or equivalent

1 mark asymptotes identified correctly or equivalent</td></tr>
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