CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Extension 2 Mathematics

Term 2

ASSESSMENT TASK 3

2017

Time allowed: **1½ hours**

Name: ________________________________ **Teacher: Ms Strilakos**

Instructions:

- All questions should be attempted on your own paper.
- Show ALL necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Please write on one side of each sheet of paper only, and do not use multiple columns on the page.

TOPIC	Polynomials	Integration	TOTAL
MARKS	/26	/30	/56

Question 1

(a) If α, β and γ are the roots of $2x^3 + 3x^2 - x - 1 = 0$, form the equation whose roots are $\alpha + 2,\ \beta + 2,\ \gamma + 2$.

[3 marks]

(b) A polynomial $P(x)$ is given by $P(x) = 4x^3 - 2x^2 + x + 5$.

(i) Find the remainder and the quotient when $P(x)$ is divided by $x^2 + 2x - 5$.

A second polynomial is given by $Q(x) = 4x^3 - 2x^2 + ax + b$.

(ii) Find the value of the constants a and b so that when $Q(x)$ is divided by $x^2 + 2x - 5$ there is no remainder.

[5 marks]

(c) Given $f(x) = x^3 + 3x^2 - 24x + 20, x \in R$

(i) Show that $(x - 1)$ is a factor of $f(x)$.

(ii) Hence factorise $f(x)$ as the product of a linear and a quadratic factor.

(iii) Find, in exact form, the solutions of $f(x) = 0$.

The line with equation $y = -8$ touches the graph of $f(x)$ at the point $Q(2, -8)$ and crosses the graph of $f(x)$ at another point P.

(iv) Determine the coordinates of P.

[5 marks]

(d) Resolve the following into partial fractions:

$$\frac{5}{(y^2+1)(y-2)}$$

[3 marks]

(e) One zero of the polynomial $P(x) = ax^3 + (a+1)x^2 + 10x + 15, \quad a \in R,$

is purely imaginary.

Find a and the zeros of the polynomial.

[5 marks]

(f) A real polynomial has the form $P(x) = 3x^4 - 12x^3 + cx^2 + dx + e.$

The graph of $y = P(x)$ has y-intercept 180. It cuts the x-axis at 2 and 6, and does not meet the x-axis anywhere else.

Given that the other two zeros are $m \pm ni,\ n > 0,$ find m and n.

[5 marks]

Question 2

(a) (i) Test to find if $\frac{x^3-2x}{\sqrt{x^4+1}}$ is an odd or an even function.

(ii) Evaluate the integral $\int_{-1}^{1} \frac{x^3-2x}{\sqrt{x^4+1}} dx$

[2 marks]

(b) Find the following integrals:

(i) $\int \frac{2x}{x^2+2x+1} dx$

(ii) $\int \frac{1}{1+sinx} dx$

(iii) $\int \frac{1}{(1+x^2)^2} dx$, using $x = \tan u$

[3+3+5=11 marks]

(c) (i) Find $\int \frac{1}{x^2+x} dx$

(ii) Hence find, by taking limits $\int_1^{\infty} \frac{1}{x^2+x} dx$

[5 marks]

(d) Find $\int \frac{1}{2+\cos x} dx$

[5 marks]

(e) Given $I_n = \int_0^1 x^n\sqrt{1-x^2}\, dx$, $n \in N$

(i) Show clearly that $(n+2)I_n = (n-1)I_{n-2}$, $n \geq 2$

(ii) Hence show that $\int_0^1 x^7\sqrt{1-x^2}\, dx = \frac{16}{315}$

[7 marks]

END OF TEST

BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180°$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

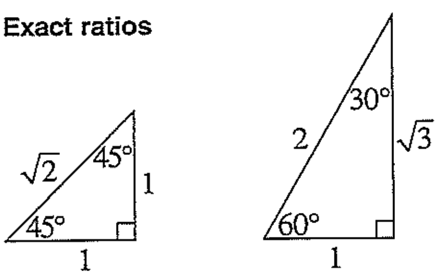
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivatives

If $y=x^n$, then $\dfrac{dy}{dx}=nx^{n-1}$

If $y=uv$, then $\dfrac{dy}{dx}=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$

If $y=\dfrac{u}{v}$, then $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

If $y=F(u)$, then $\dfrac{dy}{dx}=F'(u)\dfrac{du}{dx}$

If $y=e^{f(x)}$, then $\dfrac{dy}{dx}=f'(x)e^{f(x)}$

If $y=\log_e f(x)=\ln f(x)$, then $\dfrac{dy}{dx}=\dfrac{f'(x)}{f(x)}$

If $y=\sin f(x)$, then $\dfrac{dy}{dx}=f'(x)\cos f(x)$

If $y=\cos f(x)$, then $\dfrac{dy}{dx}=-f'(x)\sin f(x)$

If $y=\tan f(x)$, then $\dfrac{dy}{dx}=f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$\alpha+\beta=-\dfrac{b}{a}$ $\qquad$ $\alpha\beta=\dfrac{c}{a}$

Equation of a parabola

$$(x-h)^2=\pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+C$$

$$\int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+C$$

$$\int \frac{f'(x)}{f(x)}\,dx=\ln|f(x)|+C$$

$$\int \sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+C$$

$$\int \cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+C$$

$$\int \sec^2(ax+b)\,dx=\frac{1}{a}\tan(ax+b)+C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx\approx\frac{b-a}{2}\left[f(a)+f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx\approx\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Logarithms – change of base

$$\log_a x=\frac{\log_b x}{\log_b a}$$

Angle measure

$180°=\pi$ radians

Length of an arc

$l=r\theta$

Area of a sector

$$\text{Area}=\frac{1}{2}r^2\theta$$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$$

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

t **formulae**

If $t=\tan\frac{\theta}{2}$, then

$$\sin\theta=\frac{2t}{1+t^2}$$

$$\cos\theta=\frac{1-t^2}{1+t^2}$$

$$\tan\theta=\frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta=a,\quad \theta=n\pi+(-1)^n\sin^{-1}a$

$\cos\theta=a,\quad \theta=2n\pi\pm\cos^{-1}a$

$\tan\theta=a,\quad \theta=n\pi+\tan^{-1}a$

Division of an interval in a given ratio

$$\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2=4ay$,

$$x=2at,\quad y=at^2$$

At $\left(2at,\ at^2\right)$,

tangent: $y=tx-at^2$

normal: $x+ty=at^3+2at$

At $\left(x_1,\ y_1\right)$,

tangent: $xx_1=2a\left(y+y_1\right)$

normal: $y-y_1=-\frac{2a}{x_1}\left(x-x_1\right)$

Chord of contact from $\left(x_0,\ y_0\right)$: $xx_0=2a\left(y+y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x=b+a\cos(nt+\alpha)$$

$$\ddot{x}=-n^2(x-b)$$

Further integrals

$$\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\frac{x}{a}+C$$

$$\int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

Sum and product of roots of a cubic equation

$$\alpha+\beta+\gamma=-\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$$

$$\alpha\beta\gamma=-\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Q.1. $2x^3+3x^2-x-1=0.$

Let $y=x+2$ $\therefore$ $x=y-2$

$\Rightarrow$ $2(y-2)^3+3(y-2)^2-(y-2)-1=0$

$2\left[y^3-3.y^2.2+3y.2^2-8\right]+3\left[y^2-4y+4\right]-y+2-1=0$

$2y^3-12y^2+24y-16+3y^2-12y+12-y+1=0$

ie. $2y^3-9y^2+11y-3=0$

ie. $2x^3-9x^2+11x-3=0$ is the required equation

Q.2. $P(x)=4x^3-2x^2+x+5$

$$
\begin{array}{r|l}
 & 4x-10 \\
x^2+2x-5 & 4x^3-2x^2+x+5 \\
 & \underline{4x^3+8x^2-20x} \\
 & \quad -10x^2+21x+5 \\
 & \quad \underline{-10x^2-20x+50} \\
 & \qquad 41x-45
\end{array}
$$

$\therefore$ $P(x)=(x^2+2x-5)(4x-10)+(41x-45)$

$Q(x)=4x^3-2x^2+ax+b$ —— ①

Since there is no remainder, the factors of $Q(x)$ are:

$Q(x)=(x^2+2x-5)(4x+c)$

Q.2 (continued)

Now, $Q(x) = (x^2+2x-5)(4x+c)$ — ②

$Q(0) = b$ from ① and $Q(0) = -5c$ from ②

$\therefore\ b = -5c$ — ③

Also, $Q(1) = 4-2+a+b = (1+2-5)(4+c)$ from ① & ②

i.e. $2+a+b = -8-2c$

$\therefore\ a+b+2c = -10$ — ④

and $Q(2) = 32-8+2a+b = (4+4-5)(8+c)$

i.e. $24+2a+b = 24+3c$

$2a+b = 3c$ — ⑤

$b = 3c-2a$ — ⑤'

Solving ③, ④ and ⑤' :

③ = ⑤' i.e. $-5c = 3c-2a \Rightarrow -8c = -2a$

$\therefore\ a = 4c$ — ⑥

Subst. ③ & ⑥ into ④ to obtain:

$4c-5c+2c = -10 \Rightarrow c = -10$

$b = 50$

$a = -40$

Q.3. $f(x) = x^3+3x^2-24x+20$, $x \in \mathbb{R}$.

(i) $f(1) = 1+3-24+20 = 0$ $\therefore$ $(x-1)$ is a factor

(ii)

$$
\begin{array}{r|l}
 & x^2+4x-20 \\
\hline
x-1 & x^3+3x^2-24x+20 \\
 & \underline{x^3-x^2} \\
 & \quad 4x^2-24x+20 \\
 & \quad \underline{4x^2-4x} \\
 & \qquad -20x+20 \\
 & \qquad \underline{-20x+20} \\
 & \qquad\qquad 0.
\end{array}
$$

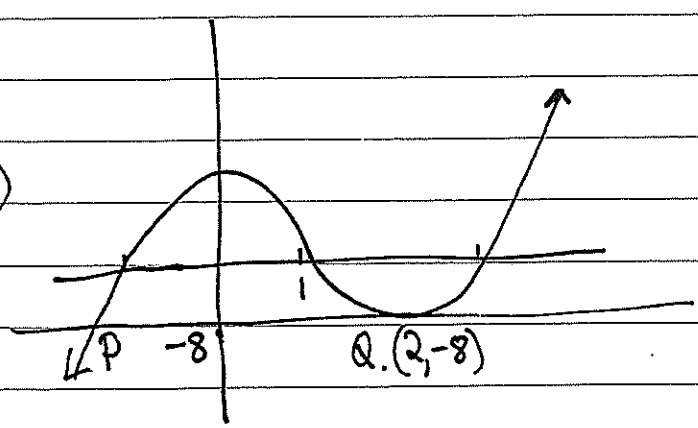
(ii) $\therefore$ $f(x) = (x-1)(x^2+4x-20)$

(iii) Solutions: $x=1$ and $x = \dfrac{-4\pm\sqrt{16+80}}{2}$

$= \dfrac{-4\pm\sqrt{96}}{2} = \dfrac{-4\pm 4\sqrt{6}}{2}$

$= -2\pm 2\sqrt{6}$

(iv)



$y=-8$
$\therefore$ cuts curve
at $x=-7$
$y=-8$

Line meets $f(x)$ when $f(x) = x^3+3x^2-24x+20 = -8$

and we want a double solution. $f(x) = x^3+3x^2-24x+28 = 0$

Q4

$$\frac{5}{(y^2+1)(y-2)} = \frac{a}{y-2} + \frac{by+c}{y^2+1}$$

$$\Rightarrow \quad 5 = a(y^2+1) + (by+c)(y-2)$$

If $y=2$, $5 = 5a \quad \therefore a=1$

If $y=0$, $5 = a - 2c \quad$ i.e. $c = -2$

If $y=1$, $5 = 2a + (b+c)(-1)$

$5 = 2a - b - c$

$= 2 - b + 2$

$\therefore b = -1$

$$\therefore \frac{5}{(y^2+1)(y-2)} = \frac{1}{y-2} - \frac{y+2}{y^2+1}$$

Q5. $P(x) = ax^3 + (a+1)x^2 + 10x + 15, \quad a \in \mathbb{R}$.

Let the purely imaginary zero be bi. Since $P(x)$ has real coeffs, $-bi$ is also a solution.

$\therefore (x-bi)(x+bi) = x^2+b^2$ is a factor.

Since constant term in $P(x)$ is 15

$$\therefore P(x) = (x^2+b^2)\left(ax + \frac{15}{b^2}\right)$$

product gives coeffs of x^3.

$$\therefore ax^3 + (a+1)x^2 + 10x + 15 = ax^3 + \frac{15}{b^2}x^2 + ab^2x + 15.$$

Q.5 (cont.)

Equating powers of x gives:

x^2: $a+1 = \frac{15}{b^2}$ — 1

x: $10 = ab^2 \Rightarrow a = \frac{10}{b^2}$ — 2

Substitute 2 into 1 to obtain:

$\frac{10}{b^2} + 1 = \frac{15}{b^2} \Rightarrow 1 = \frac{5}{b^2}$ $\therefore b^2 = 5$, $b = \pm\sqrt{5}$

$\therefore a = 2$

Substituting for a and b gives

$ax + \frac{15}{b^2} = (2x+3)$

$\therefore$ Solutions are: $\pm\sqrt{5}i$ and $-\frac{3}{2}$ ← from $(x^2+b^2)\left(ax+\frac{15}{b^2}\right)$

from (x^2+b^2)

Q6. $P(x) = 3x^4 - 12x^3 + cx^2 + dx + e$

Y-int. 180 $\therefore$ $e = 180$

X-int at $x = 2, 6$, $\therefore$ $x-2$ and $x-6$ are factors.

$(x-2)(x-6) = x^2 - 8x + 16$.

Other two roots are $m \pm ni$, $n > 0$.

$P(x) = 3x^4 - 12x^3 + cx^2 + dx + 180$

$= (x^2 - 8x + 16)(\qquad\qquad)$.

Now, Σ roots: $2 + 6 + (m+ni) + (m-ni) = \frac{12}{3} = 4$

ie. $8 + 2m = 4$, $2m = -4$, $m = -2$

Product of Roots: $2 \times 6 \times (m+ni)(m-ni) = \frac{180}{3} = 60$

ie. $12(m^2 + n^2) = 60$

$m^2 + n^2 = 5$, $n^2 = 1$, $n = 1$
Since $n > 0$.

$\therefore$ $m = -2$, $n = 1$.

Q.7 (i) Let $f(x) = \dfrac{x^3-2x}{\sqrt{x^4+1}}$

$$f(-x) = \frac{-x^3+2x}{\sqrt{x^4+1}} = -f(x) \quad \therefore \underline{\underline{\text{ODD}}}.$$

(ii) $\therefore \displaystyle\int_{-1}^{1} \frac{x^3-2x}{\sqrt{x^4+1}}\,dx = 0.$

Q.8. (i) $\displaystyle\int \frac{x-1}{x^2-2x+5}\,dx$

Let $u = x^2-2x+5 \qquad \dfrac{du}{dx} = 2x-2 = 2(x-1)$

$$\therefore\ I = \int \frac{1}{2u}\,\frac{du}{dx}\,.\,dx$$

$$= \frac{1}{2}\ln u + c = \frac{1}{2}\ln(x^2-2x+5) + c$$

(ii) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$ Let $\cancel{u = 1-x^2}$ $x = \sin\theta \quad \dfrac{dx}{d\theta} = \cos\theta$

$x^2 = \sin^2\theta$

$\cancel{x^2 = 1-u}$

$$I = \int \frac{\cancel{u^{1/2}}}{\cancel{1-u}}\ \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cdot\frac{dx}{d\theta}\,d\theta$$

$1-x^2 = 1-\sin^2\theta$

$= \cos^2\theta$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int r$$

Q.8. (i) $\displaystyle\int \frac{2x}{x^2+2x+1}\,dx$

$$= \int \frac{2x+2-2}{x^2+2x+1}\,dx$$

$$= \int \frac{2x+2}{x^2+2x+1}\,dx - \int \frac{2}{x^2+2x+1}\,dx$$

$$= \int \frac{1}{u}\,du - 2\int \frac{1}{v^2}\,dv \qquad v = x+1, \quad \frac{dv}{dx} = 1$$

$u = x^2+2x+1$

$\dfrac{du}{dx} = 2x+2$

$$= \ln u + \frac{2}{v} + C$$

$$= \ln(x^2+2x+1) + \frac{2}{x+1} + C.$$

(ii) $\displaystyle\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\,dx$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx$$

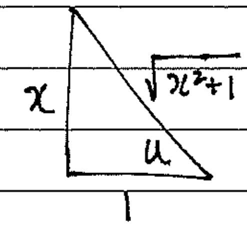
$$= \int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x\,dx - \int \tan x \sec x\,dx$$

(iii) $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$

Let $x = \tan u$ $\qquad \dfrac{dx}{du} = \sec^2 u$ $\qquad u = \tan^{-1} x$



$$I = \int \frac{1}{(1+\tan^2 u)^2} \cdot \frac{dx}{du} \cdot du$$

$$= \int \frac{1}{\sec^4 u} \cdot \sec^2 u \;\; du$$

$$= \int \cos^2 u \;\; du$$

$\sin 2u = 2 \sin u \cos u$

$$= 2 \cdot \frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{2x}{x^2+1}$$

$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$

$$I = \frac{1}{2}\int (1 + \cos 2u)\; du$$

$$= \frac{1}{2}\left[u + \frac{1}{2} \sin 2u \right] + c$$

$$= \frac{u}{2} + \frac{1}{4} \sin 2u + c$$

$$= \frac{1}{2} \tan^{-1} x + \frac{1}{4} \sin\left(2 \tan^{-1} u\right) + c.$$

$$= \frac{1}{2} \tan^{-1} x + \frac{1}{4} \cdot \frac{2x}{x^2+1} \quad + c$$

$$= \frac{1}{2} \tan^{-1} x + \frac{1}{2} \frac{x}{(x^2+1)} \quad + c$$

Q.10. (i) $\int \frac{1}{x^2+x}\,dx$

$= \int \frac{1}{x(x+1)}\,dx$

$$\frac{1}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1}$$

$$1 = a(x+1) + bx$$

$x=0$, $1 = a$

$x=-1$, $1 = -b$

$= \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \ln x - \ln(x+1) + c$

$\therefore \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

$= \ln \frac{x}{x+1} + c$

(ii) $\int_1^\infty \frac{1}{x^2+x}\,dx = \left[\ln\left(\frac{x}{x+1}\right)\right]_1^\infty$

Now, $x+1 \,\overline{)\, x}$ with quotient 1; $x+1$ subtracted, remainder -1

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$\therefore I = \left[\ln\left(1 - \frac{1}{x+1}\right)\right]_1^\infty$

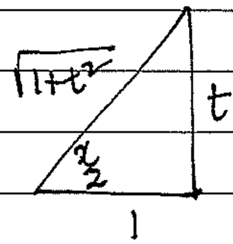
$\lim_{x\to\infty} \ln\left(1 - \frac{1}{x+1}\right) = 0$

$\therefore I = 0 - \ln\left(1 - \frac{1}{2}\right)$

$= -\ln\frac{1}{2} = \ln\left(\frac{1}{2}\right)^{-1}$

$= \ln 2$

Q.11 $\displaystyle\int \frac{1}{2+\cos x}\, dx$



Let $t = \tan\frac{x}{2}$

$$\cos\frac{x}{2} = \frac{1}{1+t^2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1+t^2}{2} \qquad \therefore \frac{dx}{dt} = \frac{2}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{dx}{dt} \cdot dt$$

$$= \int \frac{1}{\frac{2(1+t^2)+(1-t^2)}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1+t^2}{3+t^2} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{2}{3+t^2}\, dt$$

$$= \frac{2}{\sqrt{3}} \int \frac{\sqrt{3}}{3+t^2}\, dt$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\frac{t}{\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + c$$

2 (e)

$$I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx, \; n \in \mathbb{N}$$

To show, $(n+2) I_n = (n-1) I_{n-2}$, $n \geq 2$

$$I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$$

$$= \int_0^1 x^{n-1} \cdot x\sqrt{1-x^2}\, dx$$

Let $u = x^{n-1}$

$\frac{du}{dx} = (n-1)x^{n-2}$

and $\frac{dv}{dx} = x\sqrt{1-x^2}$

$v = -\frac{1}{3}(1-x^2)^{3/2}$

$$= \left[\frac{-1}{3}(1-x^2)^{3/2} \cdot x^{n-1}\right]_0^1 + \frac{1}{3}\int_0^1 (1-x^2)^{3/2}(n-1)x^{n-2}\, dx$$

$$= 0 + \frac{(n-1)}{3}\int_0^1 (1-x^2) \cdot (1-x^2)^{1/2} \cdot x^{n-2}\, dx$$

$$I_n = \frac{n-1}{3}\left[\underbrace{\int_0^1 (1-x^2)^{1/2} \cdot x^{n-2}\, dx}_{I_{n-2}} - \underbrace{\int_0^1 (1-x^2)^{1/2} \cdot x^n\, dx}_{I_n}\right]$$

$$\therefore I_n = \frac{(n-1)}{3} \times I_{n-2} - \frac{(n-1)}{3} \times I_n$$

$$\Rightarrow 3I_n = (n-1) \times I_{n-2} - (n-1) I_n$$

$$\Rightarrow (3+n-1) I_n = (n-1) I_{n-2}$$

ie. $(n+2) I_n = (n-1) I_{n-2}$

(ii) Show that $\displaystyle\int_0^1 x^7\sqrt{1-x^2}\,dx = \frac{16}{315}$

From (i) $I_n = \left(\dfrac{n-1}{n+2}\right) I_{n-2}$

$\therefore$ $I_7 = \displaystyle\int_0^1 x^7\sqrt{1-x^2}\,dx$

$= \dfrac{6}{9} I_5$

$I_5 = \dfrac{4}{7} I_3$, $I_3 = \dfrac{2}{5} I_1$

and $I_1 = \displaystyle\int_1^0 x\sqrt{1-x^2}\,dx$ Let $u = 1-x^2$ $x=1, u=0$

$\dfrac{du}{dx} = -2x$ $x=0, u=1$

$= \displaystyle\int_1^0 -\frac{1}{2} u^{1/2}\,du$

$= \left[-\dfrac{1}{\cancel{2}} \cdot \dfrac{\cancel{2}}{3} u^{3/2}\right]_1^0 = +\dfrac{1}{3}$

$\therefore$ $I_7 = \dfrac{6}{9}\left(\dfrac{4}{7}\left(\dfrac{2}{5} \times \dfrac{1}{3}\right)\right)$

$= \dfrac{\cancel{48}\,16}{\underset{21}{\cancel{63}} \times 15}$

$= \dfrac{16}{315}$ as required.