Carlingford High School
Mathematics Extension 2
Year 11

HSC ASSESSMENT TASK 1
Term 4 2018

Student Number : ______________________ **Teacher: Mr Cheng**

- **Time allowed 60 minutes.**
- Start each question on a new page.
- **Do not** work in columns or back to back.
- Marks may be deducted for careless or badly arranged work.
- Only calculators approved by the Board of Studies may be used.
- All answers are to be completed in blue or black pen except graphs and diagrams.
- There is to be NO LENDING OR BORROWING.

	MC	Q6	Q7	Total	
Complex Numbers	/5	/15	/15	/35	
Total	/5	/15	/15	/35	%

Section I

Write the correct answers on your answer sheet.

1 One rational root exists for $P(x) = 2x^3 - 3x^2 + 4x + 3$ such that $P(\frac{-1}{2}) = 0.$
When $P(x)$ is fully factorised over the complex field, what is the result?

(A) $(2x+1)(x^2 - 2x + 3)$

(B) $(2x+1)(x - 1 + i\sqrt{2})(x + 1 + i\sqrt{2})$

(C) $(2x+1)(x + 1 - i\sqrt{2})(x + 1 + i\sqrt{2})$

(D) $(2x+1)(x - 1 - i\sqrt{2})(x - 1 + i\sqrt{2})$

2 If $z = 1 - \sqrt{3}\,i = 2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$, then what is the value of z^{21} ?

(A) 2^{21}

(B) -2^{21}

(C) $(2^{21})i$

(D) $-(2^{21})i$

3 When the circle $|z - (3 + 4i)| = 5$ is sketched on the Argand Diagram the maximum value of $|z|$ occurs when z lies at the end of the diameter that passes through the centre and the origin.

What is the maximum value of $|z|$?

(A) $\sqrt{5}$

(B) 5

(C) 10

(D) $\sqrt{10}$

4 A square root of $8 + 6i$ is :

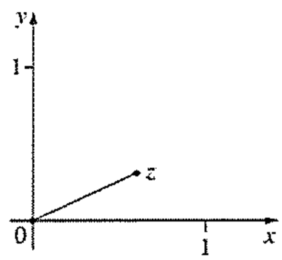
(A) $3 - i$

(B) $5 - 3i$

(C) $-3 - i$

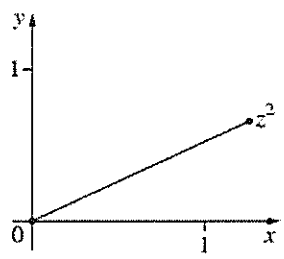
(D) $-3 + i$

5

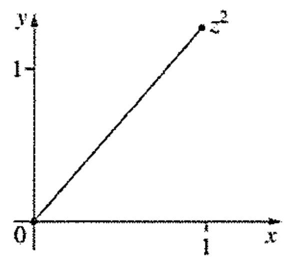


Which diagram best represents z^2?

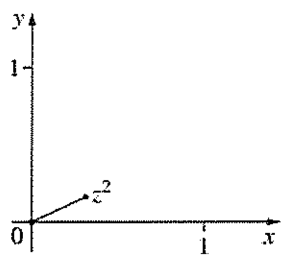
(A)



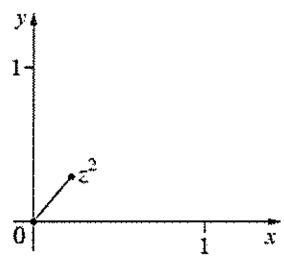
(B)



(C)



(D)



Question 6 (15 Marks) **Marks**

a) If $z = \dfrac{1 + \sqrt{3}\, i}{\sqrt{3}\ + i}$, find :

i) $|z|$ **1**

ii) $arg\,(z)$ **1**

iii) z^4 **1**

b) Sketch the region in the Argand plane consisting of those points *z* for which: **3**

$$|\arg(z+1)| < \frac{\pi}{6},\ z + \bar{z} \leq 6 \text{ and } |z+1| > 2.$$

c)

i) Use De Moivre's theorem to express $cos\,4\theta$ in terms of $cos\,\theta$. **2**

ii) Hence express $cot\,4\theta$ as a rational function of x where $x = cot\,\theta$. **2**

iii) By considering the roots of $cot\,4\theta = 0$, prove that **2**

$$cot\,\frac{\pi}{8}\,.\ cot\,\frac{3\pi}{8}\,.\ cot\,\frac{5\pi}{8}\,.\ cot\,\frac{7\pi}{8} = 1\,.$$

d) If z = cosΘ + isinΘ show that

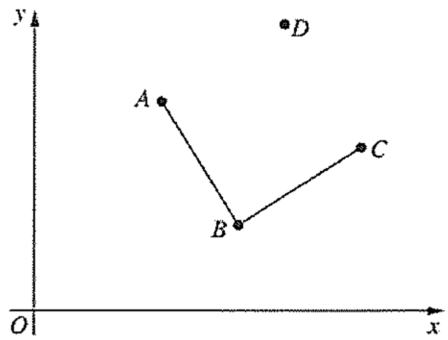
I) $z^n + \frac{1}{z^n} = 2cosn\Theta$ **1**

ii) Hence, express $cos^4\Theta$ in terms of cosnΘ **2**

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a)

(e)



In the diagram the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3, respectively. The triangle is isosceles and right-angled at B.

(i) Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$. 2 **2**

(ii) Suppose D is the point such that $ABCD$ is a square. Find the complex number, expressed in terms of z_1, z_2 and z_3, that represents D. 1 **2**

(b) (i) Solve the quadratic equation $z^2 + (2 + 3i)z + (1 + 3i) = 0$, giving your answers in the form $a + bi$, where a and b are real numbers. **2**

(ii) The polynomial $P(z)$ has the equation $P(z) = z^4 - 4z^3 + Az + 20$, where A is real. Given that $3+i$ is a zero of $P(z)$,

(α) Find A **1**

(β) Factorise completely over Complex Number. **2**

(iii) The locus of the complex number z, moving in the complex number plane such that

$\arg(z-2\sqrt{3}) - \arg(z-2i) = \frac{\pi}{3}$

(α) Sketch the locus in the Argand plane **1**

(β) Find the radius and the centre of the circle **2**

(iv) ϕ is a complex cube root of unity, i.e. $\phi^3 = 1$, $\phi \neq 1$.

(α) Find the value of $\phi + \phi^2$ **1**

(β) Prove that $(a-b)(a-\phi b)(a-\phi^2 b) = a^3 - b^3$ **2**

Ext 2 Solns

1 D

2 B

3 C

4 C

5 D

Q6a

i. $z = \frac{1+\sqrt{3}i}{\sqrt{3}+i}$

$|z| = \frac{|1+\sqrt{3}i|}{|\sqrt{3}+i|}$

$= \frac{\sqrt{1^2+(\sqrt{3})^2}}{\sqrt{(\sqrt{3})^2+1^2}}$

$= \frac{2}{2} = 1$ 1

or $z = \frac{1+\sqrt{3}i}{\sqrt{3}+i}$

$z = \frac{1+\sqrt{3}i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$z = \frac{\sqrt{3}-i+3i-\sqrt{3}i^2}{3-i^2}$

$z = \frac{2\sqrt{3}+2i}{4} = \frac{\sqrt{3}+i}{2}$

$|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1} = 1$

ii.

$Arg\, z = \frac{Arg\,(1+\sqrt{3}i)}{Arg\,(\sqrt{3}+i)}$

$= Arg\,(1+\sqrt{3}i) - Arg\,(\sqrt{3}+i)$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{6}$ 1

Or $Arg\left(\frac{\sqrt{3}+i}{2}\right) = \frac{\pi}{6}$

iii.

$z = cis\,\frac{\pi}{6}$

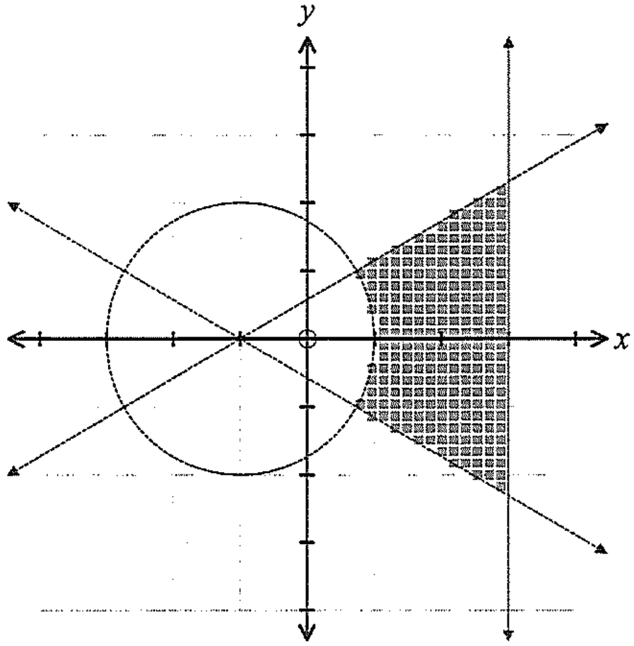
$z^4 = \left(cis\,\frac{\pi}{6}\right)^4$

$z^4 = cis\,\frac{2\pi}{3}$ or 1

$Z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

Question 12	Trial HSC Examination		2013
Part	**Solution**	**Marks**	**Comment**
(b)		3	One for each graph section

$x+iy + x-iy < 6$
$2x < 6$
$x < 3$

$|\arg(z+1)| < \frac{\pi}{\ }$, $z + \bar{z} < 6$ and $|z+1| > 2$.

(c)

i. $cis\, 4\theta = (cis\, \theta)^4 = cos^4\theta + 4cos^3\theta sin\theta i + 6cos^2\theta sin^2\theta i^2 + 4cos\theta sin^3 i^3 + sin^4\theta i^4$

$.cis\, 4\theta = (cis\, \theta)^4 = cos^4\theta + 4cos^3\theta sin\theta i - 6cos^2\theta sin^2\theta - 4icos\theta sin^3\theta + sin^4\theta$ — 1 — expansion

$cos\, 4\theta = cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta$

$= cos^4\theta - 6cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$

$= cos^4\theta - 6cos^2\theta + 6cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$

$cos\, 4\theta = 8cos^4\theta - 8cos^2\theta + 1$ — 1 — $cos\, 4\theta$

ii.

From above $\quad sin\, 4\theta = 4cos^3\theta\, sin\,\theta - 4\, cos\,\theta sin^3\theta$

$cos\, 4\theta = cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta$

$$cot\, 4\theta = \frac{cos\, 4\theta}{sin\, 4\theta} = \frac{cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta}{4cos^3\theta\, sin\,\theta - 4\, cos\,\theta sin^3\theta}$$ 1

Divide by $sin^4\theta$,

$$cot\, 4\theta = \frac{\frac{cos^4\theta}{sin^4\theta} - \frac{6cos^2\theta sin^2\theta}{sin^4\theta} + \frac{sin^4\theta}{sin^4\theta}}{\frac{4\sin\theta\, cos^3\theta}{sin^4\theta} - \frac{4\, cos\,\theta sin^3\theta}{sin^4\theta}}$$

$x = \cot\theta$

$$\cot 4\theta = \frac{x^4 - 6x^2 + 1}{4x^3 - 4x}$$ 1 — 1 — $\cot 4\theta$

iii.

If $cot\, 4\theta = 0$ $\qquad$ *Also* $\frac{x^4-6x^2+1}{4x^3-4x} = 0$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ $\qquad$ $x^4 - 6x^2 + 1 = 0$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

Since $x = \cot\theta$

Roots are $cot\frac{\pi}{8}, cot\frac{3\pi}{8}, cot\frac{5\pi}{8}, cot\,\frac{7\pi}{8}$ 1 — 1 — Roots in terms of θ.

From $x^4 - 6x^2 + 1 = 0$

product of the roots is 1

$\therefore\ cot\frac{\pi}{8}\ \ cot\frac{3\pi}{8}\ \ cot\frac{5\pi}{8}\ \ cot\,\frac{7\pi}{8} = 1$ 1 — 1

(d) $Z^n = \cos n\theta + i\sin n\theta$ ... (1)

$Z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ ... (2)

$= \cos n\theta - i\sin(n\theta)$ ... (2)

(1) + (2) $Z^n + Z^{-n} = 2\cos n\theta$

$\therefore Z^n + \frac{1}{Z^n} = 2\cos n\theta$

$(2\cos\theta)^4 = \left(Z + \frac{1}{Z}\right)^4$

$= Z^4 + 4Z^3\left(\frac{1}{Z}\right) + 6(Z^2)\frac{1}{Z^2} + 4Z\frac{1}{Z^3} + \frac{1}{Z^4}$

$= Z^4 + 4Z^2 + 6 + 4\cdot\frac{1}{Z^2} + \frac{1}{Z^4}$

$16\cos^4\theta = \left(Z^4 + \frac{1}{Z^4}\right) + 4\left(Z^2 + \frac{1}{Z^2}\right) + 6$

$= 2\cos 4\theta + 4(2\cos 2\theta + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore \cos^4\theta = \frac{1}{16}\left[2\cos 4\theta + 8\cos 2\theta + 6\right]$

$\cos^4\theta = \frac{1}{8}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$

Q7a

(a) (i) $\overrightarrow{BA} = z_1 - z_2$ and $\overrightarrow{BC} = z_3 - z_2$.

Rotating $\overrightarrow{BC}$ anticlockwise by 90° gives $\overrightarrow{BA}$. Hence, $z_1 - z_2 = i(z_3 - z_2)$.

Squaring both sides gives

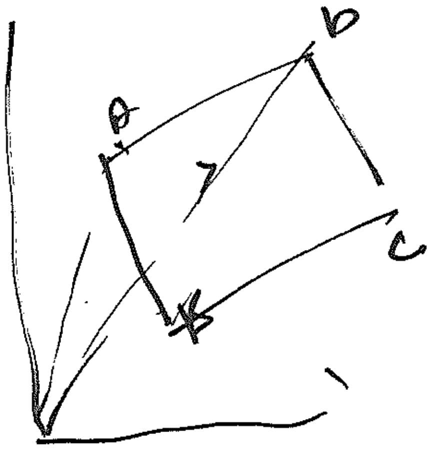
$$(z_1 - z_2)^2 = -(z_3 - z_2)^2.$$

(ii) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$. But $\overrightarrow{AD} = \overrightarrow{BC}$.

Therefore $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$

$= z_1 + (z_3 - z_2)$.

So D is represented by the complex number $z_1 - z_2 + z_3$.



7b(i)

$$z^2 + (2+3i)z + (1+3i) = 0$$

$a = 1$

$b = 2+3i$

$c = 1+3i$

$$z = \frac{-(2+3i) \pm \sqrt{(2+3i)^2 - 4(1)(1+3i)}}{2(1)}$$

$$= \frac{-(2+3i) \pm \sqrt{4+12i+9i^2-4-12i}}{2}$$

$$= \frac{-(2+3i) \pm \sqrt{9i^2}}{2}$$

$$= \frac{-(2+3i) \pm 3i}{2}$$

$z = \dfrac{-2-3i+3i}{2}$ and $z = \dfrac{-2-3i-3i}{2}$

$z = \dfrac{-2}{2}$ and $z = \dfrac{-2-6i}{2}$

$z = -1$ and $z = -1-3i$

$3+i$ is a zero $\therefore$ $3-i$ must be a zero

$(z-(3+i))(z-(3-i))$

$((z-3)-i)((z-3)+i)$

$\therefore (z-3)^2-(i)^2$

$= z^2-6z+9+1$

$= z^2-6z+10$

$z^2+2z+1+2-1$

$(z+1)^2+1$

$(z+1)^2-i^2$

$\therefore z^4-4z^3+Az+20 = (z^2-6z+10)(z^2+cz+2)$

$-4 = c-6 \qquad = (z^2-6z+10)(z^2+2z+2)$

$c = 2$

$\Rightarrow z^4+2z^3+2z^2-6z^3-12z^2-12z$
$+10z^2+20z+20$

$\Rightarrow z^4-4z^3+8z+20$

$\Rightarrow A=8$

(ii) $(z^2-6z+10)(z^2+2z+2)$

$= (z-(3+i))(z-(3-i))(z+1-i)(z+1+i)$

$= (z-3-i)(z-3+i)(z+1-i)(z+1+i)$

OR

$P(3+i) = 0$

$(3+i)^4 - 4(3+i)^3 + (3+i)A + 20 = 0$

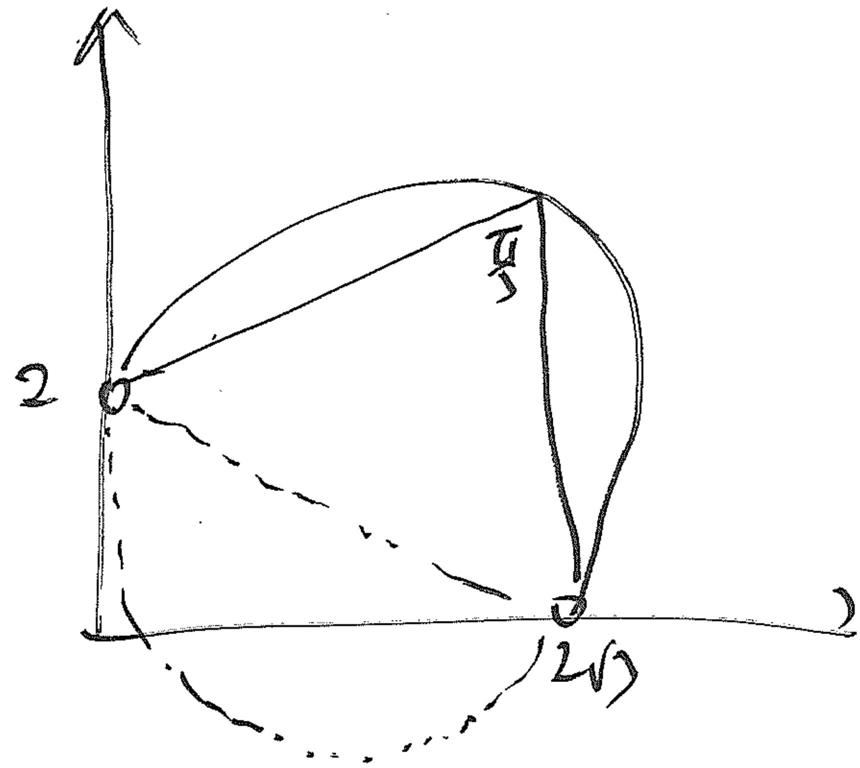
$-24 + 3A - 8i + Ai = 0$

$3A + Ai = 24 + 8i$

equate coeffs $\quad 3A = 24$

$A = 8$

(c)(ii)(a)



$$\arg\left(\frac{z-2\sqrt{3}}{z-\cancel{2\sqrt{3}}\;2i}\right) = \frac{\pi}{3}$$

$$\frac{z-2\sqrt{3}}{z-2i} = \frac{x+iy-2\sqrt{3}}{x+iy-2i}$$

$$= \frac{(x-2\sqrt{3})+iy}{x+(y-2)i} \times \frac{(x-(y-2)i)}{(x-(y-2)i)}$$

$$= \frac{x(x-2\sqrt{3})-(x-2\sqrt{3})(y-2)i+xyi+y(y-2)}{x^2+(y-2)^2}$$

$$= \frac{x^2-2\sqrt{3}x-\left[xy-2x-2\sqrt{3}y+4\sqrt{3}\right]i + xyi+y^2-2y\,]}{}$$

$$= \frac{x^2-2\sqrt{3}x+y^2-2y}{} + (2x+2\sqrt{3}y+4\sqrt{3})i$$

$$\arg\left(\frac{z-2\sqrt{3}}{z-2i}\right) = \tan^{-1}\left(\frac{2x+2\sqrt{3}y+4\sqrt{3}}{x^2-2\sqrt{3}x+y^2-2y}\right) = \frac{\pi}{3}$$

$$\frac{2x + 2\sqrt{3}y + 4\sqrt{3}}{x^2 - 2\sqrt{3}x + y^2 - 2y} = \tan\left(\frac{\pi}{3}\right)$$

$$\frac{2x + 2\sqrt{3}y - 4\sqrt{3}}{x^2 - 2\sqrt{3}x + y^2 - 2y} = \sqrt{3}$$

$$\sqrt{3}x^2 - 6x + \sqrt{3}y^2 - 2\sqrt{3}y = 2x + 2\sqrt{3}y - 4\sqrt{3}$$

$$\sqrt{3}x^2 \cancel{4} - 8x + \sqrt{3}y^2 - 4\sqrt{3}y = -4\sqrt{3}$$

$$x^2 - \frac{8}{\sqrt{3}}x + y^2 - 4y = -4$$

$$x^2 - \frac{8}{\sqrt{3}}x + \left(\frac{8}{2\sqrt{3}}\right)^2 + y^2 - 4y + (2)^2 = -4 + (2)^2 + \left(\frac{4}{\sqrt{3}}\right)^2$$

$$\left(x - \frac{4}{\sqrt{3}}\right)^2 + (y-2)^2 = \left(\frac{4}{\sqrt{3}}\right)^2$$

$r = \frac{4}{\sqrt{3}}$

Cen $\left(\frac{4}{\sqrt{3}}, 2\right)$

Question 4

(a) (i) $\phi^3 = 1$, $\phi \neq 1$

$\phi^3 - 1 = 0$

$(\phi - 1)(\phi^2 + \phi + 1) = 0$

$\therefore \phi^2 + \phi + 1 = 0 \quad (\phi \neq 1)$

$\therefore \underline{\underline{\phi^2 + \phi = -1}}$

(ii) $(a-b)(a-\phi b)(a-\phi^2 b)$

$= (a-b)(a^2 - \phi^2 ab - \phi ab + \phi^3 b^2)$

$= (a-b)(a^2 + b^2 - ab(\phi^2 + \phi))$

$= (a-b)(a^2 + b^2 + ab)$

$= \underline{\underline{a^3 - b^3}}$

(a) $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = z^{-1} = \cos\theta - i\sin\theta$

By De Moivre's theorem:

$\therefore \quad z^n = \cos n\theta + i\sin n\theta, \quad \frac{1}{z^n} = z^{-n} = \cos n\theta - i\sin n\theta$

Adding: $z^n + \frac{1}{z^n} = 2\cos n\theta$ ... (i)

Subtracting: $z^n - \frac{1}{z^n} = 2i\sin n\theta$ ... (ii)

(b) (i) $(2\cos\theta)^4 = (z + \frac{1}{z})^4 = (z^4 + \frac{1}{z^4}) + 4(z^2 + \frac{1}{z^2}) + 6$

$\therefore \quad 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ [using results (a)(i) n = 4, 2]

$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$