ACE
EXAM
PAPER

Student name: ____________________

PAPER 3

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 1

General Instructions

- Working time - 120 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt questions 11-14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Which of the following is the correct expression for $\int \frac{dx}{\sqrt{4-x^2}}$?

   (A) $\sin^{-1}2x + C$

   (B) $\cos^{-1}2x + C$

   (C) $\sin^{-1}\frac{x}{2} + C$

   (D) $\cos^{-1}\frac{x}{2} + C$

2. Which one of the following vectors is parallel to the vector $\overrightarrow{OT} = -8\underset{\sim}{i} - 12\underset{\sim}{j}$?

   (A) $\overrightarrow{OP} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$.

   (B) $\overrightarrow{OQ} = -6\underset{\sim}{i} + 9$.

   (C) $\overrightarrow{OR} = 4\underset{\sim}{i} + 6\underset{\sim}{j}$.

   (D) $\overrightarrow{OS} = 6\underset{\sim}{i} - 9\underset{\sim}{j}$.

3. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$

   Which are the values of m for which $y = e^{mx}$ satisfies the above differential equation?

   (A) $m = -2, m = 3$

   (B) $m = -1, m = 3$

   (C) $m = \pm 1$

   (D) $m = \pm 3$

4. Which expression is equal to $\cos x - \sin x$?

   (A) $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

   (B) $\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

   (C) $2\cos\left(x + \frac{\pi}{4}\right)$

   (D) $2\cos\left(x - \frac{\pi}{4}\right)$

5. Aaliyah of height 2 metres throws a ball to the top of a brick wall, at an angle $\theta°$ to the horizontal. The height of the brick wall is 16 metres. Aaliyah throws the ball at an initial velocity of 30 m/s when she is 18 metres from the base of the brick wall. What are the parametric equations of the path? Assume g = 10 ms^{-2}. (Take the origin at the ground level)

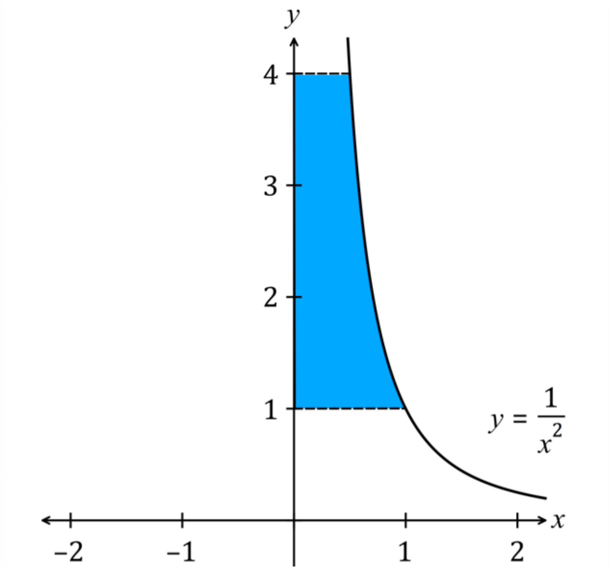
(A) $x = 30\sin\theta$ and $y = -5t^2 + 30t\sin\theta$

(B) $x = 30t\cos\theta$ and $y = -5t^2 + 30t\sin\theta$

(C) $x = 30t\sin\theta$ and $y = -5t^2 + 30t\sin\theta + 2$

(D) $x = 30t\cos\theta$ and $y = -5t^2 + 30t\sin\theta + 2$

6. The shaded region below shows the area bounded by the graph $y = \frac{1}{x^2}$ $(x > 0)$, the y-axis and the lines $y = 1$ and $y = 4$. What is the volume of the solid of revolution formed when the shaded region is rotated about the y-axis?



(A) π cubic units

(B) 4π cubic units

(C) πln3 cubic units

(D) πln4 cubic units

7. An examination consists of 36 multiple-choice questions, each question having three possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is $E(X)$?

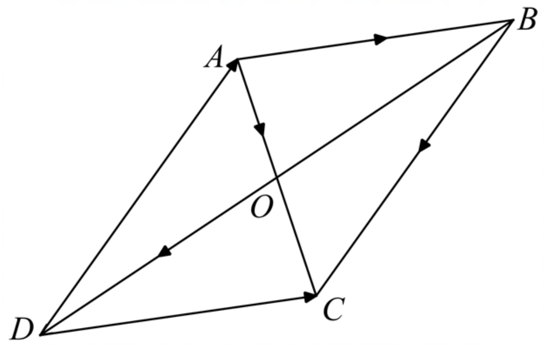
(A) 9

(B) 12

(C) 18

(D) 36

8. Parallelogram $ABCD$ has $\overrightarrow{AB} = \underset{\sim}{u}$, and $\overrightarrow{BC} = \underset{\sim}{v}$.
The point of intersection of the diagonals is O.



Which of the following is the vector $\overrightarrow{OD}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$?

(A) $\frac{1}{2}(\underset{\sim}{v} + \underset{\sim}{u})$

(B) $\frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$

(C) $\underset{\sim}{u} - \underset{\sim}{v}$

(D) $(\underset{\sim}{u} + \underset{\sim}{v})$

9. What are the solutions to the equation $\tan 2x + \tan x = 0$ for $0° < x < 180°$?

(A) $15°$ or $165°$

(B) $30°$ or $150°$

(C) $60°$ or $120°$

(D) $75°$ or 105

10. Mathematical induction is used to prove

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Which of the following is the correct expression for part of the induction proof?

(A) $\text{LHS} = (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right]$

(B) $\text{LHS} = (k+1)\left[\frac{1}{6}(2k+1) + (k+1)\right]$

(C) $\text{LHS} = k\left[\frac{1}{6}k(2k+1) + (k+1)\right]$

(D) $\text{LHS} = k\left[\frac{1}{6}(2k+1) + (k+1)\right]$

Section II

60 marks
Attempt questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (13 marks) **Marks**

(a) Show that $\tan^2 x - \tan^2 y = \dfrac{\cos^2 y - \cos^2 x}{\cos^2 x \cos^2 y}$ **2**

(b) $\overrightarrow{CD} = \underset{\sim}{i} - 2\underset{\sim}{j}$ and $\overrightarrow{DE} = 2\underset{\sim}{i} + 6\underset{\sim}{j}$. What is the magnitude of $\overrightarrow{CE}$? **2**

(c) What is the exact value of the definite integral $\int_0^{\frac{\pi}{3}} \sin^2 x dx$? **2**

(d) Find the value of $f'(x)$ if $f(x) = 3x^2 \cos^{-1} 3x$. **2**

(e) Using the substitution $u = x^2 - 9$ or otherwise, evaluate the indefinite integral: **2**

$$\int x\sqrt{x^2 - 9}\,dx$$

(f) Use mathematical induction to prove that $3^{2n} - 1$ is divisible by 8 when n is an integer greater than 0. **3**

Question 12 (16 marks) **Marks**

(a) Newton's law of cooling states that when an object at temperature T°C is placed in an environment at temperature T_0°C, the rate of the temperature loss is given by the equation:

$$\frac{dT}{dt} = -k(T - T_0)$$

where t is the time in seconds and k is a positive constant.

(i) Verify that $T = T_0 + Ae^{-kt}$ satisfies the above equation. **1**

(ii) A meal whose initial temperature is at 24°C is placed in a freezer in which the internal temperature is maintained at −40°C. After 5 seconds, the temperature of the meal is 19°C. How long will it take for the meal's temperature to reduce to 0°C. **4**

(b) Use the substitution $x = u^2$ $(u \geq 0)$ to find the value of $\int_1^3 \frac{dx}{(x+1)\sqrt{x}}$ **3**

Give your answer in simplest exact form.

(c) Show that $\frac{\cos A - \cos(A + 2\theta)}{2\sin\theta} = \sin(A + \theta)$ **2**

(d) Find $\int (\cos^2 3x)dx$ **2**

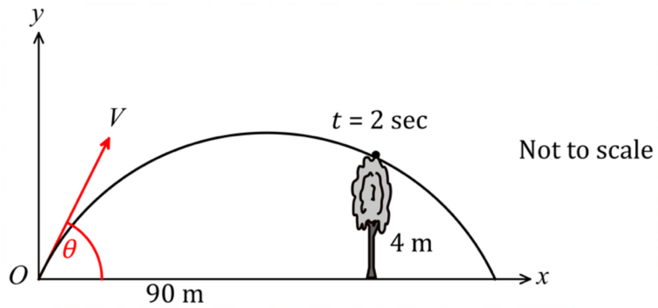
(e) (i) If $t = \tan\frac{\theta}{2}$ show that $4\sin\theta + 3\cos\theta + 5 = \frac{2(t+2)^2}{1+t^2}$ **2**

(ii) Hence solve the equation $4\sin\theta + 3\cos\theta + 5 = 0$ for $0 \leq \theta \leq 360°$. **2**

Question 13 (15 marks) **Marks**

(a) Six ordinary six-sided dice are thrown. What is the probability that exactly three of the dice land showing a four? **2**

(b) Carter hits a golf ball from a point O with speed V ms^{-1} at an angle $\theta°$ above the horizontal, where $0 < \theta < \frac{\pi}{2}$. The ball passes over a 4 m high tree after 2 seconds. The tree is 90 metres away from the point from which the ball was hit. Assume g = 10 ms^{-1}.

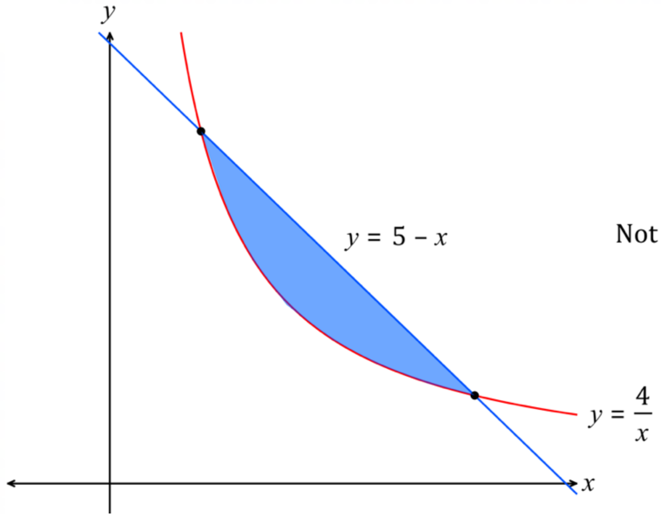


(i) Calculate the angle of projection of the golf ball to the nearest minute? Assume the horizontal and vertical displacements of the golf ball are given by $x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$. **3**

(ii) How far from where Carter hits the golf ball does it land? **2**

(c) (i) Where do the curves $y = 5 - x$ and $y = \dfrac{4}{x}$ intersect ? **1**

(ii) Find the area between the two curves. Answer correct to two decimal places. **2**



(d) What is the unit vector in the direction $\underset{\sim}{u} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$? **2**

(e) Prove by mathematical induction that, for $n \geq 1$ that: **3**

$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n-1)\,2^n$

Question 14 (16 marks) **Marks**

(a) The probability of winning a prize in a game of chance is 0.48. What is the least number of games that must be played to ensure that the probability of winning at least once is more than 0.95? **3**

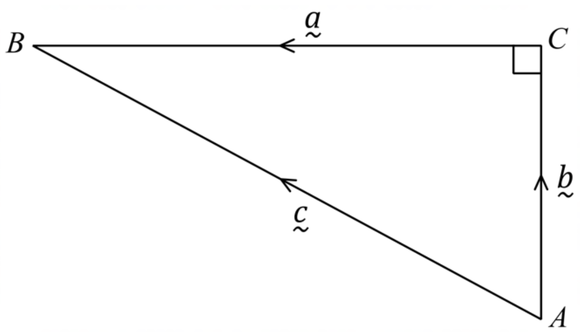
(b) A solid is formed when the region bounded by the x-axis and the graph $y = 3\sin 2x$ $0 \le x \le \frac{\pi}{2}$ is rotated around the x-axis. What is the volume of the solid? **3**

(c) Ryan on average can solve 70% of the problems in a mathematics paper. If a mathematics examination contains 7 problems, and a minimum of 5 problems is require for passing, find Ryan's chance of:

(i) solving exactly 5 problems. **2**

(ii) passing the examination. **2**

(d) ΔABC has a right angle at C.



(i) Show that $|\underset{\sim}{c}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a} + 2(\underset{\sim}{a} \cdot \underset{\sim}{b}) + \underset{\sim}{b} \cdot \underset{\sim}{b}$ **2**

(ii) Show that $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ **2**

(e) A rock drops into a pond, creating a circular ripple. The radius of the ripple increases from 0 cm, at a constant rate of 7 cms^{-1}. At what rate is the area enclosed within the ripple increasing when the radius is 16 cm? **2**

End of paper

NSW GOVERNMENT

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Financial Mathematics

$A = P(1 + r)^n$

Sequences and series

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, r \neq 1$

$S = \frac{a}{1 - r}, |r| < 1$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x - h)^2 + (y - k)^2 = r^2$

Logarithmic and Exponential Functions

$\log_a a^x = x = a^{\log_a x}$

$\log_a x = \frac{\log_b x}{\log_b a}$

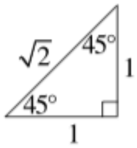
$a^x = e^{x \ln a}$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

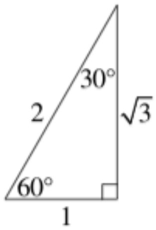


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$\cos A = \dfrac{1-t^2}{1+t^2}$

$\tan A = \dfrac{2t}{1-t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

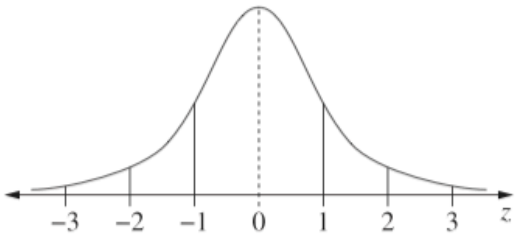
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \text{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\text{Var}(X) = np(1 - p)$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$