CARLINGFORD HIGH SCHOOL

YEAR 12 2016

HSC Assessment Term 2

MATHEMATICS (2 unit)

Time allowed: *55 minutes*

Instructions

1. Start each question on a NEW sheet of paper.
2. Write name, class and teacher on each page.
3. Board-approved calculators may be used in all parts of the test.
4. All necessary working should be shown in every question.
5. Marks may be deducted for careless or badly arranged work.

	Q1	Q2	Q3	Q4	Total
H6					
Total	/17	/18	/11	/11	/57

Question 1 (17 marks) **[START A NEW PAGE]**

(a) If $\log_x a = 3\cdot 6$ and $\log_x b = 2$ find:

(i) $\log_x ab$ **1**

(ii) $2\log_x a + \log_x b^3$ **2**

(b) Solve the following equations for x:

(i) $\log_x 64 = 3$ **2**

(ii) $\log_{27} x = -\frac{1}{3}$ **2**

(c) Express in simplest form $e^{2\ln x}$ **1**

(d) Show that $5\log_{32} x = \log_2 x$ **2**

(e) Find k if $3^{2k+1} \times 9^{k-1} = 1$ **2**

(f) Solve $2^x = 5$ correct to 1 decimal point 2

(g) Solve $log_2(x+1) - log_2(x-1) = 3$ 3

Question 2 (18 marks) **[START A NEW PAGE]**

(a) Differentiate with respect to x:

(i) $y = lnxe^x$ **2**

(ii) $y = \ell n\left(\dfrac{2x + 1}{2x - 1}\right)$ **2**

(b) (i) $\int e^{4x+1}dx$ **2**

(ii) $\int_1^e \dfrac{x + 1}{x}dx$ **2**

(iii) $\int \dfrac{x^2}{x^3 - 1}dx$ **2**

(c) Given that $y = e^{3x^2}$ find:

(i) $\dfrac{dy}{dx}$ **1**

(ii) Hence, find $\int_0^1 x e^{3x^2} dx$ **2**

(d) Find the equation of the curve that has $f''(x) = 12e^{2x}$

and a stationary point at $(0, -4)$. **3**

(e) Find the equation of the tangent to the curve $y = e^{\log_e x^2}$ at the point on the curve where $x = e$. **2**

Question 3 (11 marks) **[START A NEW PAGE]**

(a) Consider the curve $y = xe^x$

(i) Show that $\dfrac{dy}{dx} = e^x(x + 1)$ **2**

(ii) Show that the curve has one stationary point. **1**

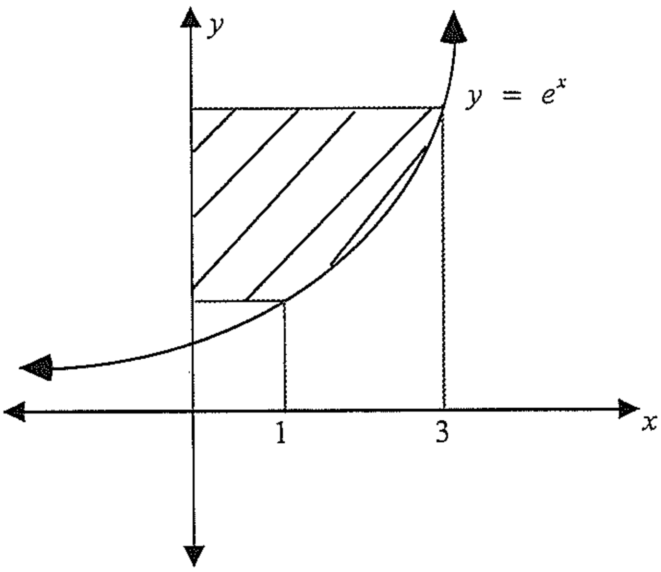
(iii) Given that $\dfrac{d^2y}{d^2x} = e^x(2 + x)$, determine the nature of the stationary point. **1**

(iv) Find any inflexion points. **2**

(v) Sketch the curve showing all the main features, noting that $\lim_{x \to \infty} xe^x = \infty$ and $\lim_{x \to -\infty} xe^x = 0$ **2**

(b) (i) Find the exact area bounded by the curve $y = e^x$, the x-axis and the lines $x = 1$ and $x = 3$ **1**

(ii) Hence, or otherwise, find the shaded area. **2**



Question 4 (11 marks) **[START A NEW PAGE]**

(a) Find the volume of the solid generated when the area bounded by the curve $y = e^{2x}$, the y-axis, the x-axis and the ordinate at $x = \log_e 2$ is rotated about the x-axis. **3**

(b) (i) Write down the domain of $y = \dfrac{\ell n x}{x}$ **1**

(ii) Find where the graph of this function cuts the x-axis. **1**

(iii) It is known that

$$\frac{dy}{dx} = \frac{1 - \ell n x}{x^2} \quad \text{(DO NOT PROVE THIS)}$$

Hence, show that $\dfrac{d^2y}{dx^2} = \dfrac{2\ell n x - 3}{x^3}$ **2**

(iv) Find the only stationary point and determine its nature. **2**

(v) Find the exact coordinates of the only point of inflection. **2**

END OF TEST

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

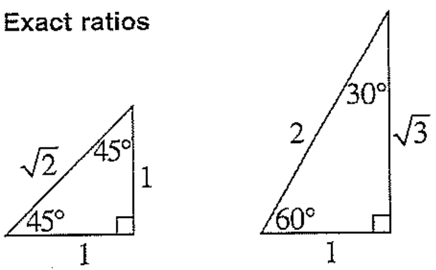
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\dfrac{dy}{dx} = F'(u)\dfrac{du}{dx}$

If $y = e^{f(x)}$, then $\dfrac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\dfrac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\dfrac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$\alpha + \beta = -\dfrac{b}{a}$ $\qquad$ $\alpha\beta = \dfrac{c}{a}$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

YEAR 12 MATHEMATICS ADVANCED 2016 TERM 2 SOLUTIONS

Q.1 Kon

QUESTION 1

(a) $\log_x a = 3.6 \qquad \log_x b = 2$

(i) $\log_x ab = \log_x a + \log_x b = 5.6$ 1

(ii) $2\log_x a + \log_x b^3 = 2\log_x a + 3\log_x b$ 1

$= 2 \times 3.6 + 3 \times 2 = 7.2 + 6 = 13.2$ 1

(b) (i) $\log_x 64 = 3$ 1

$x^3 = 64 \Rightarrow x = 4.$ 1

(ii) $\log_{27} x = -\frac{1}{3} = -3^{-1}.$

$27^{-1/3} = x$ i.e. 1 $x = \frac{1}{27^{1/3}} = \frac{1}{3}.$ 1

[redacted]

(c) $e^{2\ln x} = e^{\ln x^2} = x^2.$ 1

(d). $5\log_{32} x = \log_2 x.$ Ly [redacted]

Show that $\log_{32} x^5 = \log_2 x$

LHS $= \frac{\log_2 x^5}{\log_2 32}$ 1 $= \frac{\log_2 x^5}{\log_2 2^5} = \frac{1}{5}\log_2 x^5 = \log_2 x$ = RHS as required.

$\log_2 2 = 1$ 909 → 9.21

(e) $3^{2k+1} \times 9^{k-1} = 1$

$3^{2k} \times 3 \times 3^{2(k-1)} = 1$ 3^0 1

$\Rightarrow 3^{2k+1+2k-2} = 1$

$\Rightarrow 3^{4k-1} = 1$

$\therefore 4k-1 = 0 \quad k = \frac{1}{4}.$ 1

(f) $2^x = 5$ 1

$\log_2 5 = x \quad \ldots \quad x = 2.3$ 1

(g) $\log_2(x+1) - \log_2(x-1) = 3$

$\Rightarrow \log_2 \frac{x+1}{x-1} = 3$ 1

$2^3 = \frac{x+1}{x-1}$ 1 $\Rightarrow 8x - 8 = x + 1$

$7x = 9, \quad x = \frac{9}{7}$ 1

QUESTION 2

(a) (i) $y = \ln xe^x$ $\qquad y = \ln(xe^x)$

1st Interpretation: $\frac{dy}{dx} = \frac{1}{x} . e^x + e^x \ln x$ $\qquad$ 2nd Interpretation: $\frac{dy}{dx} = \frac{e^x + xe^x}{xe^x}$

$= \frac{1+x}{x}$

(ii) $y = \ln\left(\frac{2x+1}{2x-1}\right)$

$\frac{dy}{dx} = \frac{\frac{2(2x-1) - 2(2x+1)}{(2x-1)^2}}{\frac{2x+1}{2x-1}} = \frac{2(2x-1) - 2(2x+1)}{(2x-1)^{\cancel{2}}} \times \frac{\cancel{2x-1}}{2x+1}$

$= \frac{4x - 2 - 4x - 2}{(2x-1)(2x+1)} = \frac{-4}{(2x-1)(2x+1)}$

Q.2(b) (i) $\int e^{4x+1}\,dx$

$= \frac{1}{4}e^{4x+1} + c.$

(ii) $\int_1^e \frac{x+1}{x}\,dx = \int_1^e \left(1+\frac{1}{x}\right)dx$

$= \left[x + \ln x\right]_1^e$

$= e + \underbrace{\ln e}_{=1} - 1 - \cancel{\ln 1}^{\,0}$

$= e$

#

(iii) $\int \frac{x^2}{x^3-1}\,dx = \frac{1}{3}\ln(x^3-1) + c.$

(c) $y = e^{3x^2}$

(i) $\frac{dy}{dx} = 6x e^{3x^2}.$

(ii) $\int_0^1 x e^{3x^2}\,dx$

$= \frac{1}{6}\int_0^1 6x e^{3x^2}\,dx = \frac{1}{6}\left[e^{3x^2}\right]_0^1$

$= \frac{1}{6}\left[e^3 - e^0\right] = \frac{1}{6}e^3 - \frac{1}{6}.$

(d) $f''(x) = 12e^{2x}$

$f'(x) = 6e^{2x} + c$

At $(0,-4)$ $\frac{dy}{dx} = 0$ $\therefore$ $0 = 6e^{0} + c$ $\Rightarrow c = -6$. 1

$\therefore f'(x) = 6e^{2x} - 6$

$f(x) = 3e^{2x} - 6x + k$. 1

$(0,-4) \Rightarrow -4 = 3 - 0 + k \Rightarrow k = -7$

$\therefore$ $f(x) = 3e^{2x} - 6x - 7$ 1

(e) $y = e^{\log_e x^2} = x^2$.

$\frac{dy}{dx} = 2x$ At $x = e$, $\frac{dy}{dx} = 2e$

at $x = e$, $y = e^2$.

$y - e^2 = 2e(x-e) = 2ex - 2e^2$

$\Rightarrow y = 2ex - e^2$.

Question 3

(a) $y = xe^x$

(i) $\frac{dy}{dx} = e^x + xe^x$

(ii) $\frac{dy}{dx} = 0$ when $e^x(1+x) = 0$ ie $x = -1$

(iii) $\frac{d^2y}{dx^2} = e^x + e^x + xe^x = 2e^x + xe^x = e^x(x+2)$.

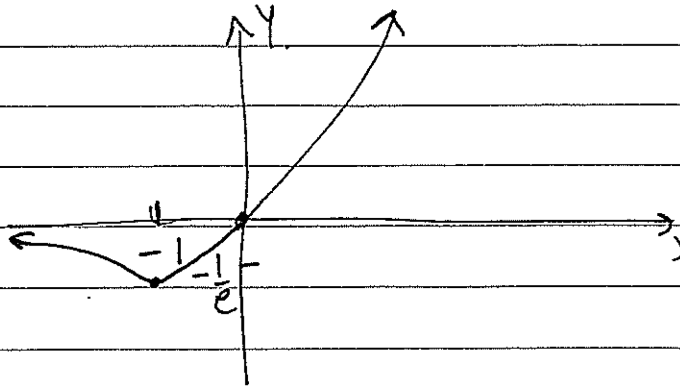
At $x = -1$, $\frac{d^2y}{dx^2} > 0$ $\therefore$ min^m stat. pt.

(iv) $\frac{d^2y}{dx^2} = 0$ for inflexion $\frac{d^2y}{dx^2} = 0$ at $x = -2$

If $x < -2$, $\frac{d^2y}{dx^2} < 0$, If $x > -2$, $\frac{d^2y}{dx^2} > 0$.

$\therefore$ Inflexion at $(-2, -2e^{-2})$.

(v)

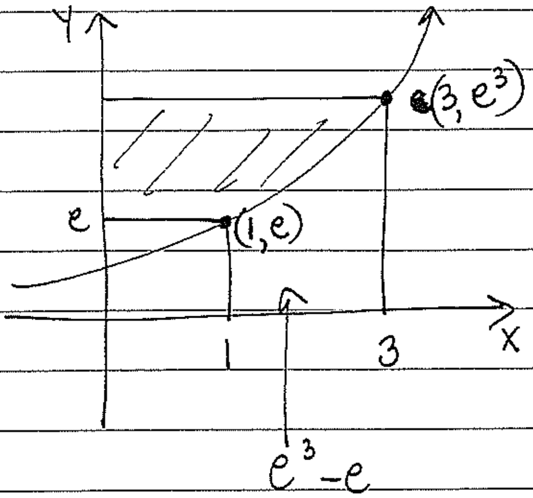


$y = xe^{x}$ at $x=0$, $y=0$.

Stat. pt. at $x=-1$, $y = \frac{-1}{e}$.

(b) (i) $A = \int_1^3 e^{x}\, dx = \left[e^{x}\right]_1^3 = e^3 - e^1$.

(ii)

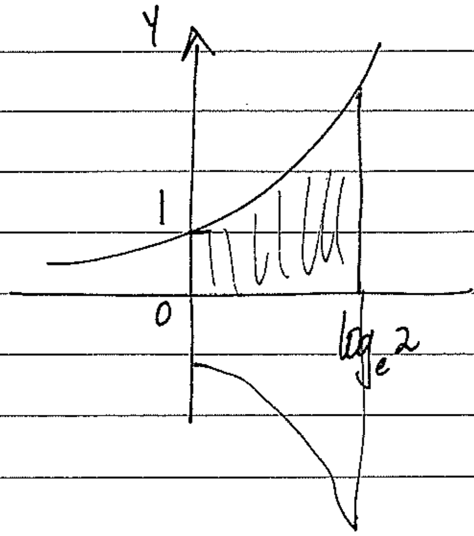


Shaded area $= 3e^3 - (e^3 - e) - e$

$= 2e^3$ square units.

Question 4

(a) $y = e^{2x}$



$$V = \pi \int_0^{\log_e 2} y^2 \, dx$$

$$= \pi \int_0^{\log_e 2} e^{4x} \, dx$$

$$= \frac{\pi}{4} \left[e^{4x} \right]_0^{\log_e 2}$$

$$= \frac{\pi}{4} \left[e^{4\log_e 2} - e^0 \right]$$

$$= \frac{\pi}{4} \left[2^4 - 1 \right] = \frac{\pi}{4} \times 15 \text{ cubic units.}$$

(b) (i) $y = \dfrac{\ln x}{x}$. domain $x > 0$.

(ii) cuts x-axis at $y = 0$. ie. $x = 1$.

(iii) $\dfrac{dy}{dx} = \dfrac{1 - \ln x}{x^2}$

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{x} \times x^2 - 2x(1 - \ln x)}{x^4}$$

$$= \frac{-x - 2x + 2x\ln x}{x^4}$$

$$= \frac{2\ln x - 3}{x^3}$$

Q4(b)(iv)

$\frac{dy}{dx} = 0$ when $\frac{1 - \ln x}{x^2} = 0$ ie $\ln x = 1$, $x = e$.

At $x = e$, $\frac{d^2y}{dx^2} = \frac{2-3}{e^3} = -\frac{1}{e^3}$. < 0

$\therefore$ max^{m} turning point $\left(e, \frac{1}{e}\right)$

(v) $\frac{d^2y}{dx^2} = 0$ for inflexion. When $\ln x = 3/2$

$x = e^{3/2}$.

At $x = e^{3/2}$ $y = \frac{\ln e^{3/2}}{e^{3/2}} = \frac{3}{2e^{3/2}}$. Thus inflexion at $\left(e^{3/2}, \frac{3}{2e^{3/2}}\right)$