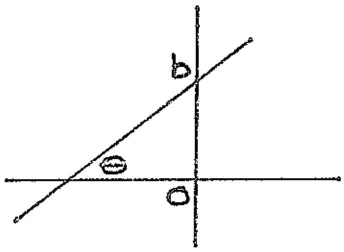
Curve Sketching

The concept of a function and its graph is of great importance in all branches of Mathematics. Graphs are a valuable aid to mathematical understanding and solving difficult problems.

Graphs of basic functions.

1. Linear : $ax + by + c = 0$
   $y = mx + b$



$m = \tan\theta$

2. Quadratic : $y = ax^2 + bx + c$

The graph of a quadratic function $y = ax^2 + bx + c$ is a parabola.
Its vertex lies on the axis of symmetry $x = -\frac{b}{2a}$.
If $a > 0$, y has a minimum at $x = -\frac{b}{2a}$.

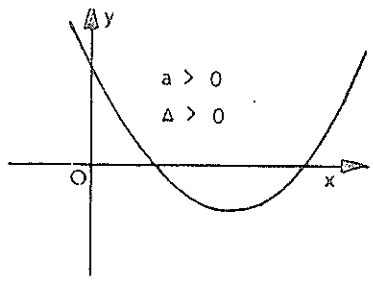
If $a < 0$, y has a maximum at $x = -\frac{b}{2a}$.

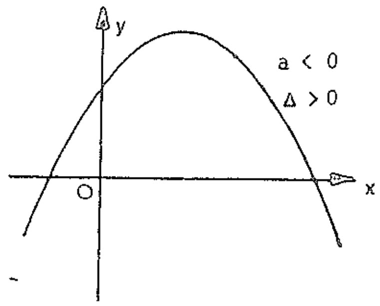
Discriminant $\Delta = b^2 - 4ac$.

If $\Delta < 0$, the curve does not intersect the x-axis.

If $\Delta = 0$, the curve touches the x-axis.

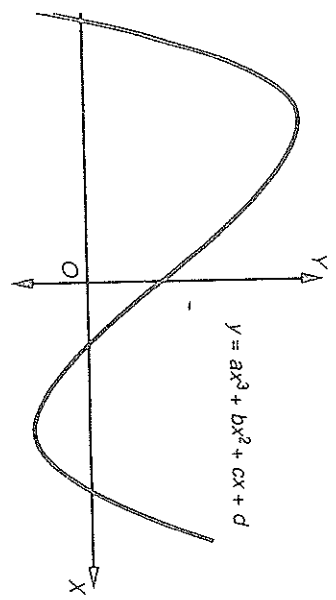
If $\Delta > 0$, the curve intersects the x-axis at $x = -\frac{b}{2a} \pm \frac{\Delta}{2a}$

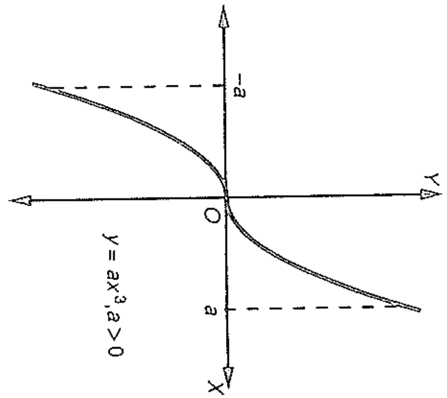




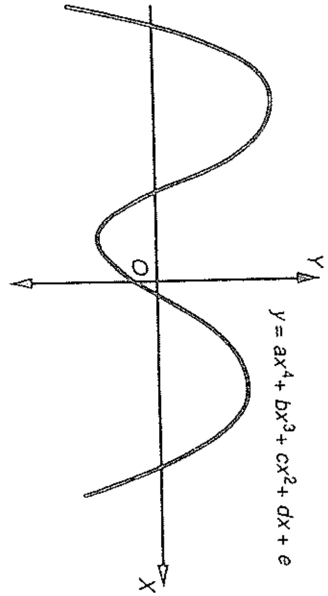
3. Polynomials : cubic $y = ax^3 + bx^2 + cx + d$
   quartic $y = ax^4 + bx^3 + cx^2 + dx + e$.
   others with form $y = bx^n$. $\Rightarrow$ special attention to $y = x^{1/2}$ $y = x^{1/3}$.

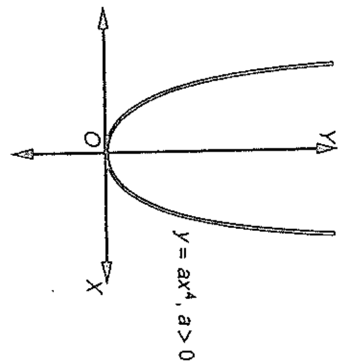
Graph of cubic function





Graph of quartic function





4. Graphs of $y = x^{1/2}$ and $y = x^{1/3}$

EXAMPLE: (1) Draw a sketch of $y = \sqrt{x}$ by analysing the behaviour of the function near $x = 0$.

SOLUTION:

The function $f(x) = \sqrt{x}$ exists and is continuous for $x \geqslant 0$.

Also $f(x) \geqslant 0$ for $x \geqslant 0$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$ for $x > 0$.

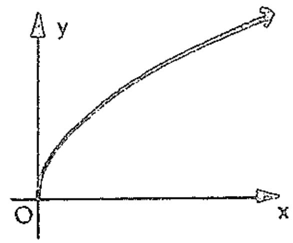
The point $x = 0$ is a critical point as $f'(0)$ does not exist. The curve $y = \sqrt{x}$ has a vertical tangent at $x = 0$.

Since $f'(x) > 0$ for $x > 0$, $f(x)$ is an increasing function and $f(0) = 0$ is the absolute minimum, but $f(x)$ has no absolute maximum.

It may be noted that since $f''(x) = -\frac{1}{4}x^{-3/2} < 0$ for $x > 0$, the curve is concave down.

This is sufficient information for drawing a reasonable graph of $y = \sqrt{x}$.

Note: $y = \sqrt{x}$ is the upper half of the parabola $y^2 = x$.



Graph of $y = \sqrt{x}$

EXAMPLE: (2)

Draw a sketch of $y = x^{1/3}$.

SOLUTION:

$$y = x^{1/3}, \quad \frac{dy}{dx} = \frac{1}{3}x^{-2/3}, \quad \frac{d^2y}{dx^2} = -\frac{2}{9}x^{-5/3}.$$

$f(x) = x^{1/3}$ exists for all real x

$f(-x) = (-x)^{1/3} = -x^{1/3} = -f(x)$, so the curve has point symmetry about the origin.

$f'(x)$ does not exist for $x = 0$, so the curve has a vertical tangent, at $x = 0$.

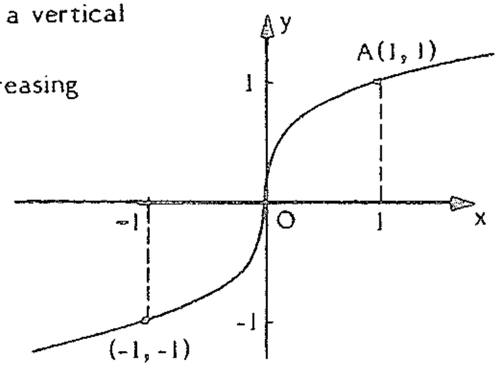
Also $f'(x) > 0$ for all x, $x \neq 0$, so it is an increasing curve. $x = 0$ is a critical point.

$f(x) \to \infty$ as $x \to \infty$

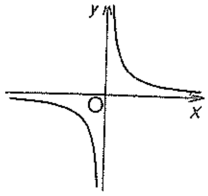
$f(x) \to -\infty$ as $x \to -\infty$

$f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$, so the origin is an inflection point.

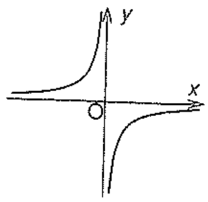
Also $f''(x) > 0$ for $x > 0$ tells us that the curve is concave down, while $f''(x) < 0$ tells us that the curve is concave up for $x < 0$. A few simple points are O, A(1,1), B(-1,-1).



4. Hyperbola – $y = \frac{k}{x}$ $x \neq 0$.



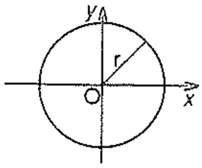
Hyperbola:
$y = \frac{k}{x}, k > 0$



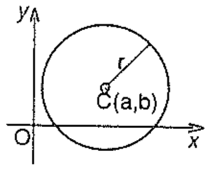
Hyperbola:
$y = \frac{-k}{x}, k > 0$

5. Circle – centre (h, k)
radius r.

– centre $(0, 0)$
radius r.

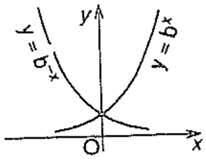


Circle:
$x^2 + y^2 = r^2$



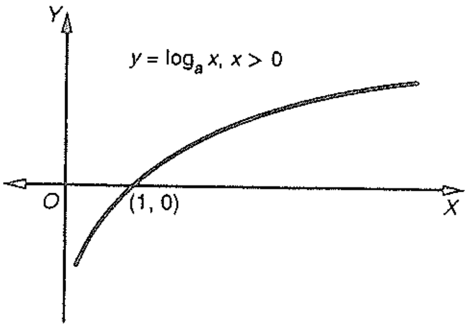
Circle:
$(x-a)^2 + (y-b)^2 = r^2$

6. Exponential: $y = a^x$ where a is any number (rational).
$y = e^x$ special case.
Both always pass thr' $(0, 1)$.



Exponential
curve: $y = b^x$

7. Logarithmic curve: $y = \log_a x$
$y = \log_e x$.



8. Trigonometric curves.

$y = a \sin bx$

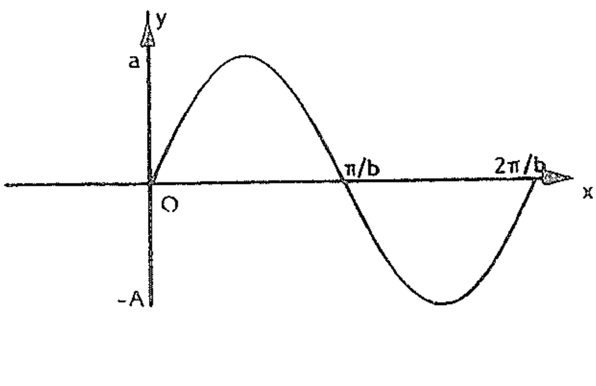
The amplitude $= a$

The period, $p = \frac{2\pi}{b}$

$f(x) = \sin x$

$f(-x) = -\sin x$

$\therefore$ $f(x) = \sin x$ is an odd function.

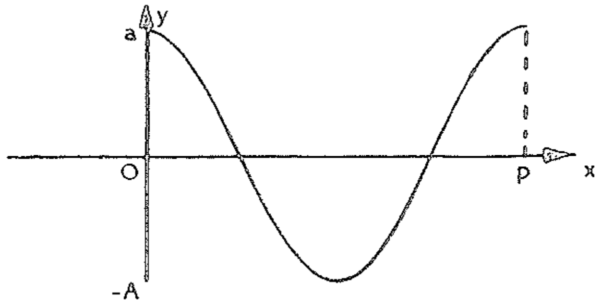


$y = A\cos bx$

Amplitude = A

Period $p = \frac{2\pi}{b}$

$f(x) = \cos x$ is an even function.



$y = \tan x$

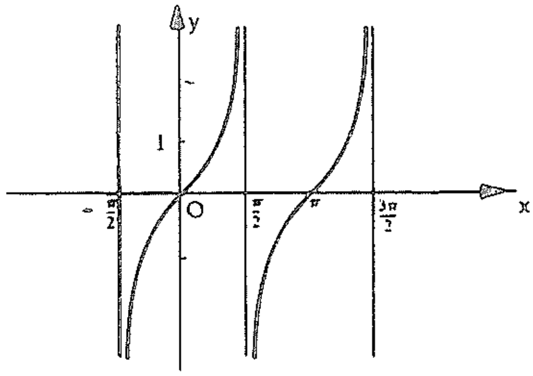
$y \to \infty$ as $x \to (2n-1)\frac{\pi}{2}$

The asymptotes are at

$x = (2n-1)\frac{\pi}{2}$,

$n = 0, \pm 1, \pm 2, \ldots$.

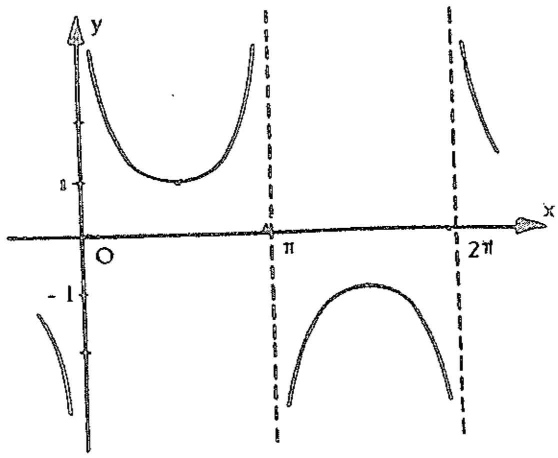
$f(x) = \tan x$ is an odd function.



(a) $y = \operatorname{cosec} x$, period 2π

cosec x is an odd function

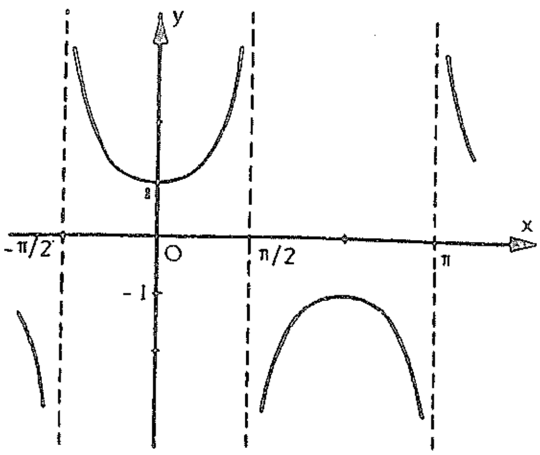
$\operatorname{cosec} x \to \infty$ at $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$



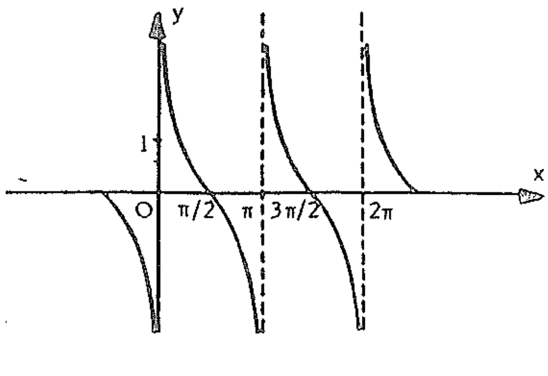
(b) $y = \sec x$, period 2π

sec x is an even function

$\sec x \to \infty$ at $x = (2n+1)\frac{\pi}{2}$, $(n = 0, \pm 1, \ldots)$

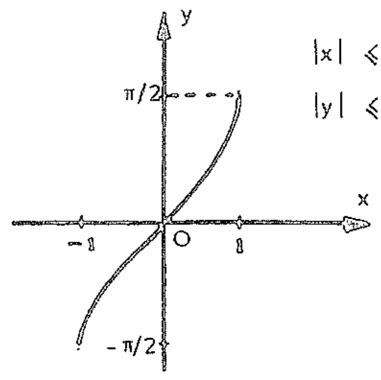


(c) $y = \cot x$ is an odd function, period π, with asymptotes at $x = n\pi$, $(n = 0, \pm 1, \ldots)$



9. Inverse trig.

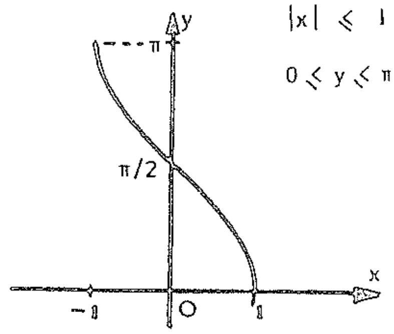
$y = \sin^{-1} x \qquad y = \cos^{-1} x \qquad y = \tan^{-1} x.$



$|x| \leqslant 1$

$|y| \leqslant \frac{\pi}{2}$

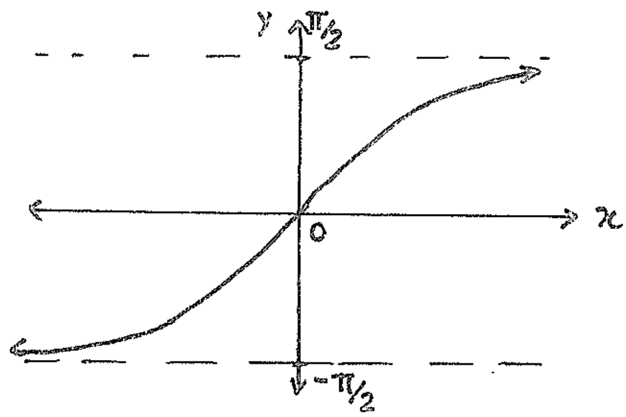
$y = \sin^{-1} x$



$|x| \leqslant 1$

$0 \leqslant y \leqslant \pi$

$y = \cos^{-1} x$



$y = \tan^{-1} x$

General rules applicable to every curve

1 Find the DOMAIN and RANGE if easy to find.

2 Find the x and y INTERCEPTS.

3 Decide if the curve has SYMMETRY:
  (a) LINE symmetry for EVEN functions
  (b) POINT symmetry for ODD functions

4 Decide on the values of x where y changes sign and decide where y is POSITIVE or NEGATIVE.

5 Find any ASYMPTOTES to the curve;
  that is, find y as $x \to \pm\infty$
  and find x as $y \to \pm\infty$

  Note: Any value of x that makes the denominator zero gives a vertical asymptote.

6 Find any STATIONARY points; that is, where $f'(x) = 0$, and find their nature.

7 Plot a few SPECIAL points to help with the general shape.

This is what we have usually done.
In this topic you will learn other techniques thus you don't need to follow these steps all the time!