Carlingford High School
Mathematics Extension 2
Year 12

HSC ASSESSMENT TASK 2
HALF YEARLY
Term 1 2016

Student Name:________________________________ Teacher: Mr GonG

- **Time allowed 2 Hours.**
- Start each question on a new page.
- Write on **ONE SIDE** of the paper only.
- Do not work in columns.
- Marks may be deducted for careless or badly arranged work.
- Only calculators approved by the Board of Studies may be used.
- All answers are to be completed in blue or black pen.except graphs and diagrams.
- There is to be NO LENDING OR BORROWING.

	MC	Q6	Q7	Q8	Q9	**Total**	
Complex Numbers	/2	/15				/17	
Graphs	/1		/15			/16	
Conics	/1			/16		/17	
Polynomials	/1				/14	/15	
Total	**/5**	**/15**	**/15**	**/16**	**/14**	**/65**	%

Section 1

Multiple Choice – **Start a new booklet** **(5 marks)**

1. The locus of a complex number z is the line $4x - 3y - 12 = 0$.
What is the minimum value of $|z|$?

A. 4 **B.** 3 **C.** $\dfrac{12}{5}$ **D.** 5

2. The fifth roots of $1+\sqrt{3}\,i$ are:

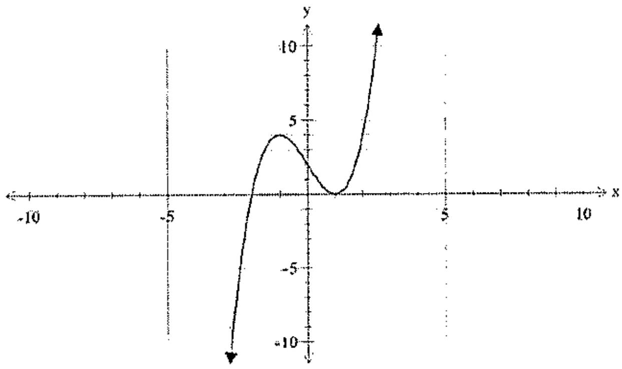
A. $2^5 cis\left(\dfrac{2k\pi}{5}+\dfrac{\pi}{30}\right),\ k=0,\ 1,\ 2,\ 3,\ 4$

B. $2^5 cis\left(\dfrac{2k\pi}{5}+\dfrac{\pi}{15}\right),\ k=0,\ 1,\ 2,\ 3,\ 4$

C. $2^{\frac{1}{5}} cis\left(\dfrac{2k\pi}{5}+\dfrac{\pi}{30}\right),\ k=0,\ 1,\ 2,\ 3,\ 4$

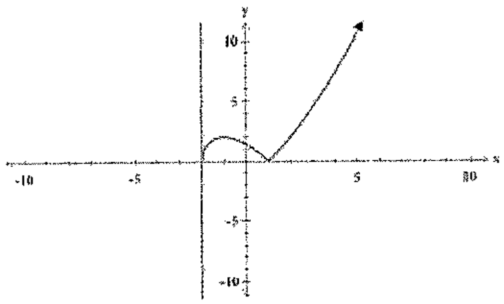
D. $2^{\frac{1}{5}} cis\left(\dfrac{2k\pi}{5}+\dfrac{\pi}{15}\right),\ k=0,\ 1,\ 2,\ 3,\ 4$

3. The diagram of $y = f(x)$ is drawn below.

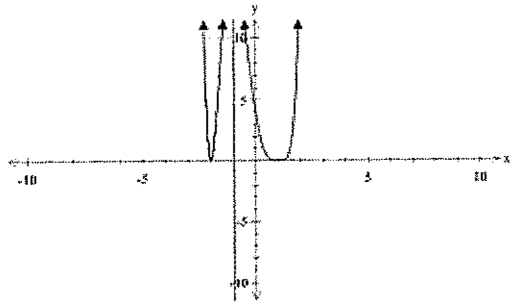


Which of the diagrams below best represents $y=\sqrt{f(x)}$?

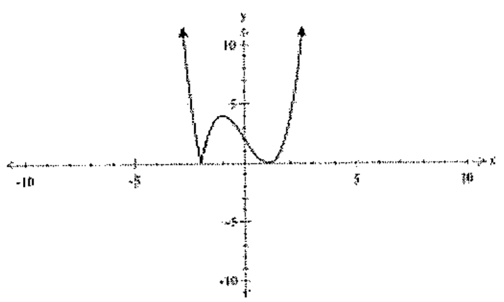
A.



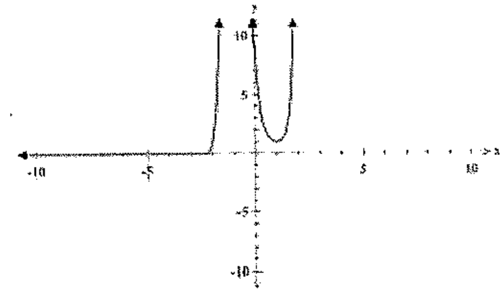
B.



C.



D.



4. The foci of the hyperbola $\frac{y^2}{8} - \frac{x^2}{12} = 1$ are:

A. $\left(\pm 2\sqrt{5},\ 0\right)$ **B.** $\left(\pm\sqrt{30},\ 0\right)$ **C.** $\left(0,\ \pm 2\sqrt{5}\right)$ **D.** $\left(0,\ \pm\sqrt{30}\right)$

5. What is the remainder when $P(x) = x^3 + x^2 - x + 1$ is divided by $(x - 1 - i)$?

A. $-3i - 2$ **B.** $3i - 2$ **C.** $3i + 2$ **D.** $2 - 3i$

Section 2

Question 6 **– Start a new booklet –** **(15 marks)** **Marks**

a) Let $A = 3 + 3\sqrt{3}i$ and $B = -5 - 12i$. Find:

i) $\overline{B}$ in $x + iy$ form **1**

ii) $\frac{A}{B}$ in $x + iy$ form **2**

iii) $\sqrt{B}$ in $x + iy$ form **2**

iv) The modulus and argument of A **2**

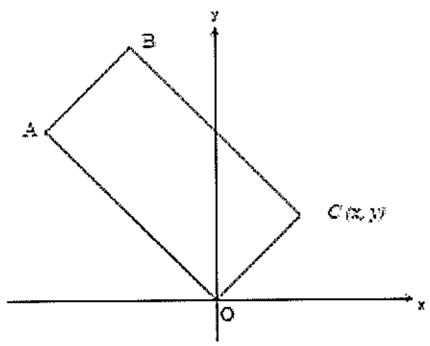
v) A^4 in $x + iy$ form **1**

b) If z represents the complex number $x + iy$, sketch the regions:

i) $|\arg z| < \frac{\pi}{4}$ **2**

ii) $Im\left(z^2\right) = 4$. **2**

c) On the Argand diagram shown OABC is a rectangle with the length OA being twice OC. OC represents the complex number $x + iy$.



Find the complex number represented by

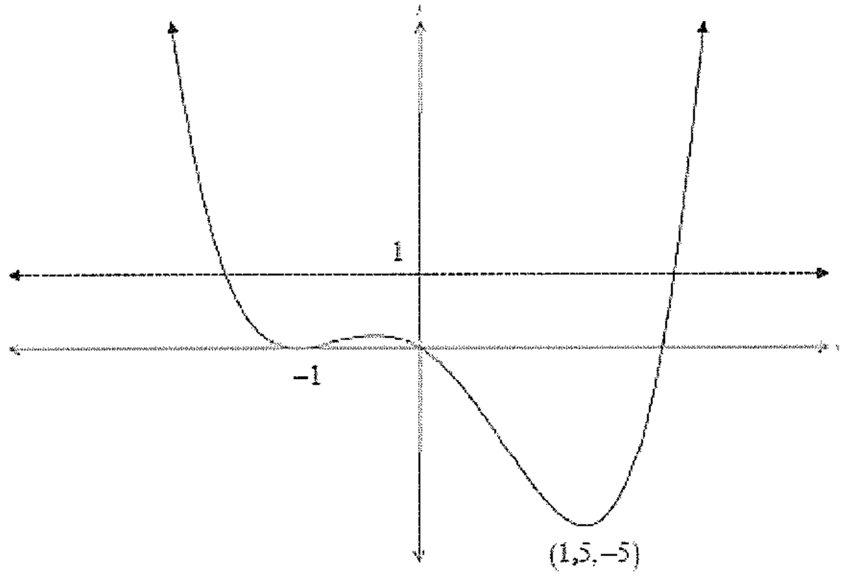
i) OA **1**

ii) OB **1**

iii) BC **1**

Question 7 – Start a new booklet – (15 marks) **Marks**

a) The graph of $y = f(x)$ is shown below.



Draw separate sketches for each of the following:

i) $y = |f(x)|$ **2**

ii) $y = \dfrac{1}{f(x)}$ **2**

iii) $y^2 = f(x)$ **2**

iv) $y = e^{f(x)}$ **2**

b) Without the use of calculus, sketch the graph of $y = \dfrac{x^3+1}{x}$, showing any asymptotes and intercepts with the coordinate axes. **3**

c) For the curve with equation $x^2 + 3xy - y^2 = 13$, determine the gradient of the tangent at the point (2, 3) on the curve. **4**

Question 8 - Start a new booklet - (16 marks) **Marks**

a) **i)** Show that the equation of the tangent at the point $P\left(ct,\ \frac{c}{t}\right)$ on the rectangular hyperbola $xy = c^2$ is $x + t^2y - 2ct = 0$. **2**

ii) Find the coordinates of A and B where this tangents cuts the x and y axes respectively. **2**

iii) Prove that the area of the triangle OAB is a constant, where O is the origin. **1**

b) Show that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(x_1,\ y_1)$ is given by the equation: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$. **3**

c) An ellipse has equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$. Show that this is the equation of the locus of a point $P(x, y)$ that moves such that the sum of its distances from $A(0, 3)$ and $B(0, -3)$ is 10 units. **4**

d) $A(a, 0)$, $P\ (a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\alpha,\ b\sin\alpha)$ are located on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ so that $\angle PAQ = 90°$. (See diagram below)

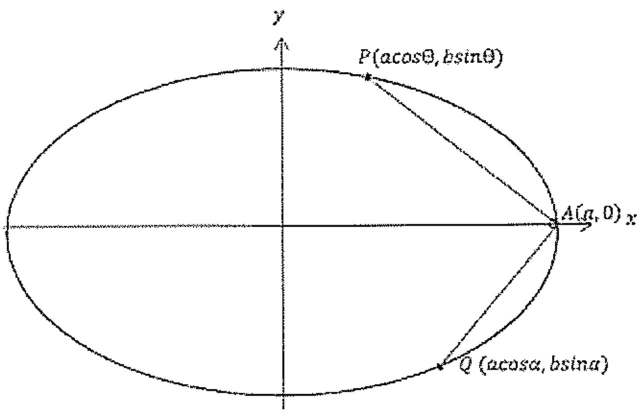


DIAGRAM NOT TO SCALE

Show that $\tan\frac{\alpha}{2} \cdot \tan\frac{\theta}{2} = -\frac{b^2}{a^2}$. **4**

Question 9 **- Start a new booklet –** **(14 marks)** **Marks**

a) The roots of the polynomial equation $2x^3 - 3x^2 + 4x - 5 = 0$ are α, β and γ.

Find the polynomial equation which has roots:

i) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. **2**

ii) 2α, 2β and 2γ. **2**

b) **i)** Find the values of A, B and C such that $\dfrac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+2}$ **2**

ii) Hence evaluate $\displaystyle\int \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x}\,dx$ **2**

c) Solve the equation $x^4 - 7x^3 + 17x^2 - x - 26 = 0$, given that $x = (3 - 2i)$ is a root of the equation. **3**

d) Given that $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0$, has a double root at $x = \alpha$, find the value of α. **3**

<u>END OF EXAM</u>

2016

Term 1 HSC Task 2 (HY) Examination

Ext 2 Mathematics

Solutions

HSC Task 2 (HY) Examination – Ext 2 Mathematics 2016

Section I **Multiple Choice Answer 1 Mark each**

	A	B	C	D
1.	○	○	●	○
2.	○	○	○	●
3.	●	○	○	○
4.	○	○	●	○
5.	○	●	○	○

Working Out

1	$\lvert z \rvert$ represents the length of the vector from the origin to z. Hence the minimum distance from the origin to z is the perpendicular distance from (0, 0) to $4x - 3y - 12 = 0$ $d = \left\lvert \frac{0+0-12}{\sqrt{4^2+(-3)^2}} \right\rvert = \left\lvert \frac{12}{5} \right\rvert = \frac{12}{5}$ → C	**4**	$\frac{y^2}{8} - \frac{x^2}{12} = 1$ $a = 2\sqrt{2}, b = 2\sqrt{3}$ $b^2 = a^2(e^2-1)$ $(2\sqrt{3})^2 = (2\sqrt{2})^2(e^2-1)$ $12 = 8(e^2-1)$ $\frac{12}{8} = e^2 - 1$ $e^2 = \frac{20}{8} = \frac{10}{4}$ $e = \frac{\sqrt{10}}{2}$	Foci = $(0, \pm ae) = \left(0, \pm 2\sqrt{2}\left(\frac{\sqrt{10}}{2}\right)\right)$ $= (0, \pm\sqrt{20})$ $= (0, \pm 2\sqrt{5})$ → C
2	$z^5 = 1 + \sqrt{3}i$ $R = \sqrt{1^2 + (\sqrt{3})^2} = 2$ Arg z: $tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ $\therefore z^5 = 2\,cis\,\frac{\pi}{3}$ $z = 2^{\frac{1}{5}} cis\left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$ → D	**5**	$P(x) = x^3 + x^2 - x + 1$ is divided by $(x - 1 - i)$ Let $x = 1 + i$ $x^2 = (1+i)^2 = 1 + 2i + i^2 = 2i$ $x^3 = 2i(1+i) = 2i + 2i^2 = 2i - 2$ Remainder $= P(1+i) = 2i - 2 + 2i - (1+i) + 1$ $= 4i - 1 - 1 - i$ $= 3i - 2$ → B	
3	Graph A			

Section II **Solutions**

Question 6

a) $A = 3 + 3\sqrt{3}i$ and $B = -5 - 12i$.

(i) $\bar{B} = \overline{-5 - 12i}$

$= -5 + 12i$

1 mark for answer

(ii) $\frac{A}{B} = \frac{3+3\sqrt{3}i}{-5-12i}$

$\frac{A}{B} = \frac{3+3\sqrt{3}i}{-5-12i} \times \frac{-5+12i}{-5+12i}$

1 mark for product

$= \frac{-15+36i-15\sqrt{3}i-36\sqrt{3}}{25-144i^2}$

$= \frac{(-15-36\sqrt{3})+(36-15\sqrt{3})i}{169}$

1 mark for answer

(iii) $\sqrt{B} = \sqrt{-5-12i}$

$Let\ (x+iy)^2 = -5-12i$

$\therefore\ x^2 + 2ixy - y^2 = -5-12i$

$\therefore\ x^2 - y^2 = -5$ ----------------(1)

$and\ \ 2xy = -12$

1 mark for correct equations

Sub $y = -\frac{6}{x}$ in [1]

$x^2 - \left(-\frac{6}{x}\right)^2 = -5$

$x^4 + 5x^2 - 36 = 0$

$(x^2+9)(x^2-4) = 0$

$x = \pm 2$ and $y = \mp 3$

$\therefore\ \sqrt{B} = 2-3i$ or $-2+3i$

1 mark for correct answer

(iv) Modulus $(r) = \sqrt{(3)^2 + (3\sqrt{3})^2} = \sqrt{36} = 6$

1 mark for correct mod

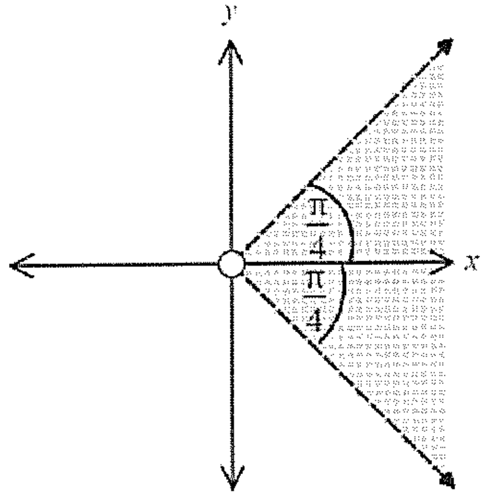
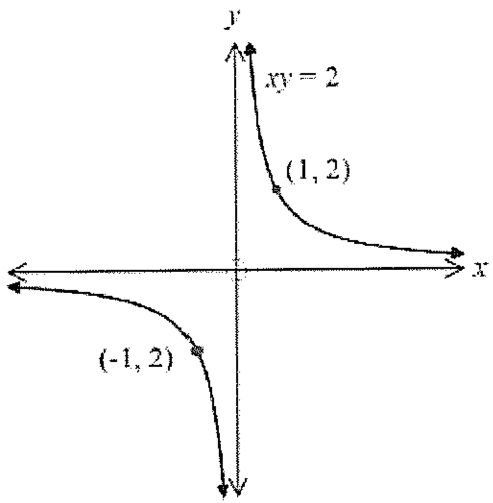
Argument: $tan\,\theta = \frac{3\sqrt{3}}{3} = \sqrt{3}$, $\theta = \frac{\pi}{3}$

1 mark for correct arg

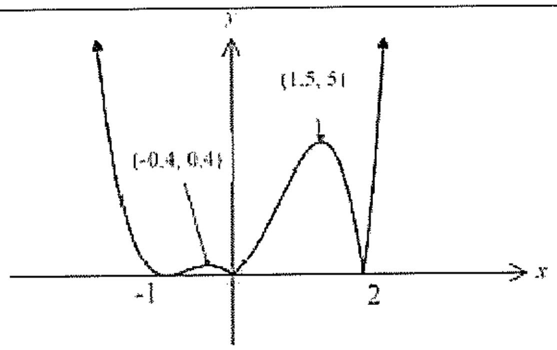
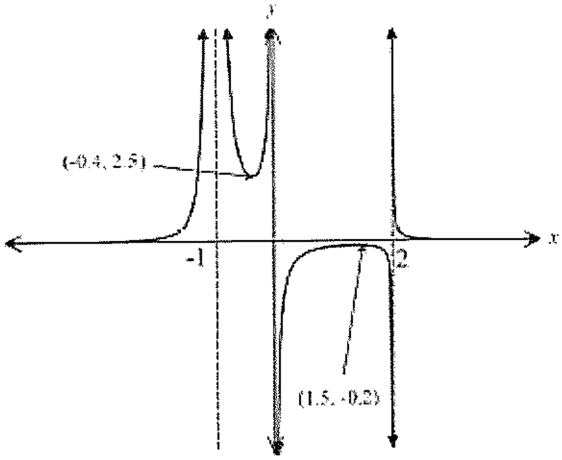
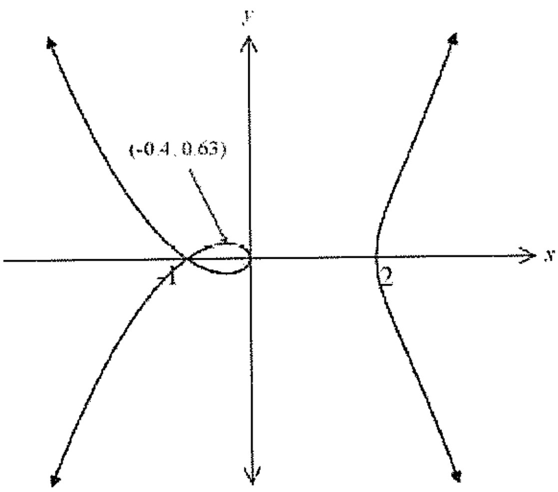
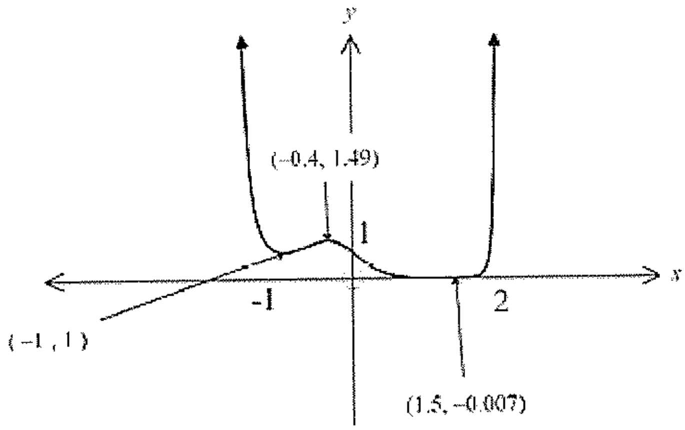
(v) $A^4 = \left(6\ cis\ \frac{\pi}{3}\right)^4 = 1296\ cis\ \frac{4\pi}{3} = 1296\ cis\ \frac{-2\pi}{3}$

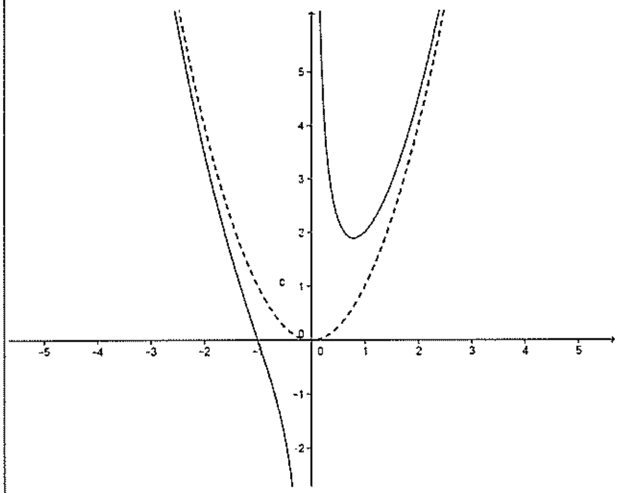
1 mark for correct form

$= -648 - 648\sqrt{3}i$

b)	i)	$\arg z = \theta$ where $\tan\theta = \frac{y}{x}$ If $\lvert\arg(z)\rvert < \frac{\pi}{4}$ then $-\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}$  	**1 mark for graph** **1 mark for feature**
	ii)	$z = x + iy$ $z^2 = (x+iy)^2 = x^2 + 2xyi - y^2$ $= x^2 - y^2 + 2xyi$ $Im(z^2) = 2xy$ Graph required is $Im(z^2) = 4$ $2xy = 4$ ie $xy = 2$ or $y = \frac{2}{x}$  	**1 mark for equation** **1 mark for graph**
c)	i)	$OA = 2(-y+ix)$	**1 mark**
	ii)	$OB = OA + AB$ $= -2y + 2ix + x + iy$ $= (x-2y) + (2x+y)i$	**1 mark**
	iii)	$BC = -OA$ $= 2y - 2xi$	**1 mark**

Question 7

a)	 i)	1 mark for correct *x-y* intercepts 1 mark for correct graph
	ii) 	1 mark for asymptotes & turning points 1 mark for correct graph
	iii) 	1 mark for critical point & *x*-intercepts 1 mark for correct graph
	iv) 	1 mark for correct *x-y* intercepts 1 mark for correct graph

b)	$y=\dfrac{x^3+1}{x}=x^2+\dfrac{1}{x}$ When $x=0$ then y is undefined When $y=0$ then $x=-1$ So vertical asymptote at $x=0$ As $x\to 0^+$, $y\to\infty$ As $x\to 0^-$, $y\to -\infty$ Also As $x\to\infty$, $y\to\left(x^2\right)^+$ and As $x\to-\infty$, $y\to\left(x^2\right)^-$		**1 mark for correct asymptotes** **1 mark for correct *x-y* intercepts** **1 mark for correct graph**
c)	$x^2+3xy-y^2=13$ $2x+3y+3x.\dfrac{dy}{dx}-2y.\dfrac{dy}{dx}=0$ $\dfrac{dy}{dx}(3x-2y)=-(2x+3y)$ $\dfrac{dy}{dx}=\dfrac{-(2x+3y)}{(3x-2y)}$ $\therefore at\ (2,3) \qquad \dfrac{dy}{dx}=-\dfrac{(4+9)}{6-6}$ $\therefore$ Gradient infinite $\therefore$ Tangent is vertical		**1 mark for line 1** **1 mark for correct *dy/dx* as a subject** **1 mark for correct substitution** **1 mark for correct conclusion**

Question 8

a)	**i)**	$xy = c^2$ $\qquad P\left(ct, \frac{c}{t}\right)$ By implicit differentiation $y + x\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{y}{x}$ At $P\left(ct, \frac{c}{t}\right)$ $\frac{dy}{dx} = -\frac{c}{t} \div ct$ $= -\frac{1}{t^2}$ $y - y_1 = m(x - x_1)$ $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ $t^2y - ct = -x + ct$ $x + t^2y - 2ct = 0$	**1 mark for correct gradient** **1 mark for correct working**
	ii)	When $y = 0$, $x + 0 - 2ct = 0$ $x = 2ct$ $\therefore A(2ct, 0)$ When $x = 0$, $0 + t^2y - 2ct = 0$ $y = \frac{2ct}{t^2} = \frac{2c}{t}$ $\therefore B\left(0, \frac{2c}{t}\right)$	**1 mark for correct coordinate *A*** **1 mark for correct coordinate *B***
	iii)	Now $OA = 2ct$ $OB = \frac{2c}{t}$ Area Triangle OAB $= \frac{1}{2}(2ct)\left(\frac{2c}{t}\right)$ $= 2c^2$ which is a constant as c is a constant.	**1 mark for correct working**
b)		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{-2b^2x}{2a^2y}$ At $P(x_1, y_1)$ $\frac{dy}{dx} = \frac{-b^2x_1}{a^2y_1}$ Normal $m = \frac{a^2y_1}{b^2x_1}$ $y - y_1 = m(x - x_1)$ $y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$ $b^2x_1y - b^2x_1y_1 = a^2y_1x - a^2y_1x_1$ $a^2y_1x - b^2x_1y = a^2y_1x_1 - b^2x_1y_1$ $(\div x_1y_1)$ $\therefore \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$	**1 mark for *dy/dx*** **1 mark for gradient of normal** **1 mark for substitute & manipulate.**

c)	Now $PA+PB=10$ i.e. $\sqrt{x^2+(y-3)^2}+\sqrt{x^2+(y+3)^2}=10$ ← $\sqrt{x^2+(y-3)^2}=10-\sqrt{x^2+(y+3)^2}$ ← $x^2+(y-3)^2=100-20\sqrt{x^2+(y+3)^2}+x^2+(y+3)^2$ $20\sqrt{x^2+(y+3)^2}=100+(y+3)^2-(y-3)^2$ $=100+12y$ $5\sqrt{x^2+(y+3)^2}=25+3y$ ← $25\left[x^2+(y+3)^2\right]=(25+3y)^2$ $25x^2+25y^2+150y+225=625+150y+9y^2$ $25x^2+16y^2=400$ ← $\frac{x^2}{16}+\frac{y^2}{25}=1$	**1 mark for line 1** **1 mark for line 2** **1 mark** **1 mark**
d)	Now $m_{PA}=-\frac{b\sin\theta}{a-a\cos\theta}$ and $m_{QA}=-\frac{b\sin\alpha}{a-a\cos\alpha}$ $=-\frac{b\sin\theta}{a(1-\cos\theta)}$ $=-\frac{b\sin\alpha}{a(1-\cos\alpha)}$ ← Now If $\angle PAQ=90^\circ$, then $m_{PA}\times m_{QA}=-1$ i.e. $-\frac{b\sin\theta}{a-a\cos\theta}\times-\frac{b\sin\alpha}{a-a\cos\alpha}=-1$ $\frac{b^2\sin\theta\sin\alpha}{a^2(1-\cos\theta)(1-\cos\alpha)}=-1$ $\therefore$ $-\frac{b^2}{a^2}=\frac{(1-\cos\theta)(1-\cos\alpha)}{\sin\theta\sin\alpha}$ ← Now let $t_1=\tan\frac{\theta}{2}$, $t_2=\tan\frac{\alpha}{2}$ then $-\frac{b^2}{a^2}=\frac{\left(1-\frac{1-t_1^2}{1+t_1^2}\right)\left(1-\frac{1-t_2^2}{1+t_2^2}\right)}{\left(\frac{2t_1}{1+t_1^2}\right)\times\left(\frac{2t_2}{1+t_2^2}\right)}$ ← $=\frac{1-\frac{1-t_2^2}{1+t_2^2}-\frac{1-t_1^2}{1+t_1^2}+\frac{(1-t_1^2)(1-t_2^2)}{(1+t_1^2)(1+t_2^2)}}{\frac{4t_1t_2}{(1+t_1^2)(1+t_2^2)}}$	**1 mark for the gradients** **1 mark** **1 mark for substitute the t formula**

	So multiply numerator & denominator by $(1+t_1^2)(1+t_2^2)$ We get $-\frac{b^2}{a^2}=\frac{(1+t_1^2)(1+t_2^2)-(1+t_1^2)(1-t_2^2)-(1-t_1^2)(1+t_2^2)+(1-t_1^2)(1-t_2^2)}{4t_1t_2}$ $=\frac{1+t_2^2+t_1^2+t_1^2t_2^2-(1-t_2^2+t_1^2-t_1^2t_2^2)-(1+t_2^2-t_1^2-t_1^2t_2^2)+1-t_2^2-t_1^2+t_1^2t_2^2}{4t_1t_2}$ $=\frac{4t_1^2t_2^2}{4t_1t_2}$ $-\frac{b^2}{a^2}=t_1t_2$ ⟵ Hence $-\frac{b^2}{a^2}=\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\alpha}{2}\right)$	**1 mark for this result**

Question 9			
a)	**i)**	$2x^3-3x^2+4x-5=0$ Let $X=\frac{1}{x}$, $\therefore x=\frac{1}{X}$ Therefore equation is $2\left(\frac{1}{X}\right)^3-3\left(\frac{1}{X}\right)^2+4\left(\frac{1}{X}\right)-5=0$ ⟵ i.e. $\frac{2}{X^3}-\frac{3}{X^2}+\frac{4}{X}-5=0$ Multiply by X^3 $2-3X+4X^2-5X^3=0$ ie $5x^3-4x^2+3x-2=0$ ⟵	**1 mark for correct substitution** **1 mark for correct equation**
	ii)	$2x^3-3x^2+4x-5=0$ Let $X=2x$ $\therefore x=\frac{X}{2}$ *Therefore equation is* $2\left(\frac{X}{2}\right)^3-3\left(\frac{X}{2}\right)^2+4\left(\frac{X}{2}\right)-5=0$ ⟵ $2\left(\frac{X^3}{8}\right)-3\left(\frac{X^2}{4}\right)+\frac{4X}{2}-5=0$ $\frac{X^3}{4}-\frac{3X^2}{4}+2X-5=0$ $X^3-3X^2+8X-20=0$ i.e. $x^3-3x^2+8x-20=0$ ⟵	**1 mark for correct substitution** **1 mark for correct equation**

b)	i) $\frac{4x^2-3x-4}{x^3+x^2-2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$ $\therefore\ 4x^2-3x-4 = A(x-1)(x+2) + B\,x\,(x+2) + C\,x\,(x-1)$ When $x = 0, \quad -4 = -2A \qquad \therefore A = 2$ $x = -2, \quad 18 = 6C \qquad \therefore C = 3$ $x = 1, \quad -3 = 3B \qquad \therefore B = -1$ $\therefore\ \frac{4x^2-3x-4}{x^3+x^2-2x} = \frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}$ ii) $\int \frac{4x^2-3x-4}{x^3+x^2-2x} = \int \left(\frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}\right) dx$ $= 2ln\,x - ln(x-1) + 3\,ln(x+2) + c$	**1 mark for correct working** **1 mark for correct values** **1 mark for correct integral** **1 mark for correct answer**
c)	$x^4 - 7x^3 + 17x^2 - x - 26 = 0$ $(3-2i)$ is a factor $\therefore\ (3+2i)$ is also a factor since coefficients are real $\therefore\ x^2 - 6x + 13$ is a factor. By division. $x^4 - 7x^3 + 17x^2 - x - 26 = (x^2 - 6x + 13)(x^2 - x - 2)$ $= (x^2 - 6x + 13)(x-2)(x+1)$ Therefore solution to $x^4 - 7x^3 + 17x^2 - x - 26 = 0$ is: $x = 3 \pm 2i, -1\ and\ 2$ OR USE SUMS AND PRODUCTS OF ROOTS $\alpha = 3-2i,\ \beta = 3+2i,\ \gamma = ?,\ \delta = ?$ $\sum \alpha = 6 + \gamma + \delta \rightarrow \gamma + \delta = 1$ $\prod \alpha = 13\gamma\delta = -26 \rightarrow \gamma\delta = -2$ $\delta = -\frac{2}{\gamma}$ so $\gamma - \frac{2}{\gamma} = 1$ $\gamma^2 - \gamma - 2 = 0$ $\gamma = 2, -1$ $\therefore$ roots are $3-2i, 3+2i, 2, -1$	**1 mark for using conjugate theorem** **1 mark for division** **1 mark for the correct answer**

d)	Let $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ $f'(x) = 4x^3 - 18x^2 + 18x + 4$ Double root when $f'(x) = f(x) = 0$ Test $x = \pm 1$ and $x = \pm 2$ (factors of 4) When $x = 2$, $f'(2) = 4(2^3) - 18(2^2) + 18(2) + 4$ $= 32 - 72 + 36 + 4 = 72 - 72 = 0$ $f(2) = (2^4) - 6(2^3) + 9(2^2) + 4(2) - 12$ $= 16 - 48 + 36 + 8 - 12 = 60 - 60 = 0$ $\therefore f'(2) = f(2) = 0$ $\therefore (x-2)$ is a repeated factor. $\therefore \alpha = 2$ is a double root.	**1 mark for using double root theorem & finding the derivative** **1 mark for testing for roots of $f'(x)$** **1 mark for testing $f(x)$ & stating the value of α**