Mechanics

A particle moves in a st line away from a fixed pt O

st $\frac{1}{v} = A + Bt$ +ve constants A and B

a) Show that ~~retardation~~ slowing down of particle is proportional to the square of the speed

b) If retardation is $1\ m/s^2$ and velocity $80\ m/s$ when $t=0$ find the values of A and B.

Express x in terms of t and v in terms of x

$$v = \frac{1}{A+Bt}$$

$$\frac{dv}{dt} = -\frac{1}{B}\frac{1}{(A+Bt)^2}$$

$$= -\cancel{B} \cdot v^2$$

b $t=0$ $v = \frac{1}{A} = 80$ $A = \frac{1}{80}$

$$\frac{dv}{dt} = \cancel{-\frac{1}{B} \cdot \frac{1}{A^2}} = -B \cdot v^2$$

$$B = 6400$$

$$\frac{dx}{dt} = \frac{1}{A+Bt}$$

$$x = \frac{1}{B}\ln(A+Bt) + C$$

$$= 6400\ln\left(\frac{1}{80} + \frac{t}{6400}\right) + c - \ln\left(\frac{1}{80}\right)$$

$$= 6400\ln(1 + t/80)$$

$$x = \frac{1}{B}\ln\left(\frac{1}{v}\right) + \ln(80)$$

$$x = \frac{1}{B}\ln\left(\frac{80}{v}\right)$$

$$e^{Bx} = \frac{80}{v}$$

$$\frac{v}{80} = e^{-Bx}$$

$$v = 80e^{-\frac{1}{6400}x}.$$

The force of attraction experienced by a mass m at a distance $x > r$ from the centre of the earth is $\frac{mgr^2}{x^2}$, r radius of earth

A particle of mass m starts from the surface with speed u away from earth

a) find speed when particle distance x from earth.

b) deduce that particle will escape if $u^2 > 2gr$

a) $F = ma$

$= \frac{mgr^2}{x^2}$ towards centre of earth

$\therefore a = -\frac{gr^2}{x^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$v^2 = \int -\frac{gr^2}{x^2}\,dx$

$= \frac{2gr^2}{\cancel{2}x} + c$

when $x = r$ $v = u$

$u^2 = \frac{2gr}{\cancel{2}} + c$ $\quad c = u^2 - \frac{gr}{\cancel{2}}$

$\frac{r}{x} - 1 = \frac{1-x}{x}$

~~$v^2 = u^2 - \frac{gr}{2}\left(\frac{x-1}{x}\right)$~~ $\quad v^2 = u^2 - \frac{2gr}{\cancel{2}}\left(1 - \frac{r}{x}\right)$

speed $= v =$ ~~$\sqrt{u^2 - \frac{2gr}{2}\left(\frac{x-r}{x}\right)}$~~ $\sqrt{u^2 - 2gr\left(1 - \frac{r}{x}\right)}$

$v = $

$\frac{r}{x}$ always < 1

~~$v^2 = a^2$~~

~~As x gets large $v^2 \to$~~

SHM

At ground level where $g = 9.81\ m/s$ a simple pendulum beats exact seconds (each ½ oscillation takes 1 second)
If it is taken up a mountain to a place where it loses 30 seconds per day, find the value of g at the new location

$$\text{Period} = 2s \qquad n = \frac{2\pi}{\pi} \qquad 2s = 2\pi\sqrt{\frac{l}{g}}$$

$$\ddot{x} = -n^2 x$$

New location

$$\text{new period} = 2 \times \frac{24 \times 60 \times 60}{24 \times 60 \times 60 - 30} = 2 \cdot \frac{86400}{86370}$$

$$= 2.000694686$$

$$g_{new} = \left(\frac{2\pi}{T_{new}}\right)^2 \times$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{T_{new}}{T_{old}} = \frac{2\pi\sqrt{l\ g_{old}}}{2\pi\sqrt{l\ g_{new}}}$$

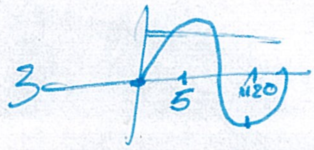
$$g_{new} = \left(\frac{T_{old}}{T_{new}}\right)^2 \times g_{old}$$

$$= \left(\frac{86370}{86400}\right)^2 \times 9.80$$

$$0.9993056\ldots$$

$$= 9.803188683\ldots$$

$$\approx 9.8$$



Assume motion of tides is SHM

mid point at 1.50

high tide 5am 9m ← mid point at 8:10

low tide 11.20 am 3m

latest time before noon

a) Find when a ship which needs ~~12.5~~ 7.5 m of water can enter the harbour

~~b) rate at which the water level is changing when the depth of water is 13 m~~

Period = 6h 20 min × 2 = 12h 40 min

$= 760$ min

$T = 2\pi/n \quad \therefore \quad n = \frac{2\pi}{T} = \pi/380$

Amplitude = 3m = $\frac{1}{2}(9-3)$

$x = 6 + 3\sin\left(\frac{\pi}{380}t\right)$ where t is mins after 1:50 am

or $\cancel{x = 6 + 3\sin\left(\frac{\pi}{380}(t + 1{:}50)\right)}$

$x = 7.5 \therefore \sin\left(\frac{\pi}{380}t\right) = \frac{1}{2}$

$\frac{\pi}{380}t = \frac{\pi}{6}, \frac{5\pi}{6}$

$t = \frac{380}{6}, \frac{5 \times 380}{6}$

$63\frac{1}{3}, \quad 316.66$

1h 3' 20'', 5h 16' 40''

\+ 1 50

1:53:20 7:06' 40

✓

Resisted motion in a straight line

HSC 87

$$\overset{R}{\longleftarrow}\bullet\longrightarrow V$$

$R = v + v^3$ $\qquad\qquad$ $m\ddot{x} = -\cancel{m}(v + v^3)$ $\qquad$ $m = 1$

$$\ddot{x} = \cancel{\frac{dv}{dx}}\, v\frac{dv}{dx} \qquad v\frac{dv}{dx} = -(v + v^3)$$

$$\frac{dv}{dx} = -(1 + v^2)$$

$$\frac{dx}{dv} = -\frac{1}{1+v^2}$$

$$x = -\tan^{-1}(v) + c$$

when $x = 0$ $v = Q$

$$0 = -\tan^{-1}Q + c \quad \therefore c = \tan^{-1}Q$$

$$x = \overset{a}{\tan^{-1}Q} - \underset{b}{\tan^{-1}(v)}$$

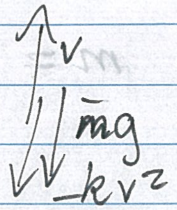
$$\tan x = \tan(a - b)$$

$$= \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$$\tan x = \frac{Q - v}{1 + Qv}$$

$$x = \tan^{-1}\left(\frac{Q - v}{1 + Qv}\right).$$

Motion upward - a problem



$$\ddot{x} = -g - kv^2$$

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$x = \int \frac{-v}{g+kv^2}\,dv$$

$$= -\frac{1}{2k}\ln(g+kv^2) + c \quad \text{where } c = \ln a$$

$$x = -\frac{1}{2k}\ln\left(a(g+kv^2)\right)$$

When $x = 0$ $v = U$

$$a = \frac{1}{g+kU^2}$$

$$x = -\frac{1}{2k}\ln\left(\frac{g+kv^2}{g+kU^2}\right) = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right)$$

Greatest height when $v = 0$

$$\cancel{x} = -\frac{1}{2k}\ln\left(\frac{g}{g+kU^2}\right) = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g}\right)$$

$$= \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)$$

Corresponding time

~~vdx~~

Now

$$a = \frac{dv}{dt}$$

$$= -g - kv^2$$

$$\therefore \frac{dt}{dv} = -\frac{1}{g+kv^2}$$

$$t = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(\sqrt{\frac{k}{g}}\,v\right) + c$$

$$c = \frac{1}{}$$

$$x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right)$$

$$e^{2kx} = \frac{g+kU^2}{g+kv^2}$$

$$g + kv^2 = (g+kU^2)e^{-2kx} - g$$

$$\left(\frac{dx}{dt}\right)^2$$

$$\frac{dt}{dx} = \left[\frac{1}{k}\left[(g+kU^2)e^{2kx} - g\right]\right]^{-\frac{1}{2}}$$

$$g = f(x)^{-1/2}$$

$$g' = f'(x)\cdot\left(-\tfrac{1}{2}\right)f(x)^{-3/2}$$

$$f' = -\frac{2k}{k}$$

time when $x = h$

<u>Remember</u> $a = \frac{dv}{dt} \therefore \frac{dv}{dt} = -g - kv^2$

$$\frac{dt}{dv} = \frac{-1}{g + kv^2}$$

$$t = -\frac{1}{g} \int \frac{1}{1 + \frac{k}{g}v^2} dv$$

$$= -\frac{1}{g} \cdot \sqrt{\frac{g}{k}} \tan^{-1}\left(\sqrt{\frac{k}{g}}\, v\right) + c$$

when $t = 0$ $v = u$ $\therefore$ $c = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}\, u\right)$

$$t = \frac{1}{\sqrt{gk}}\left(-\tan^{-1}\left(\sqrt{\frac{k}{g}}\, v\right) + \tan^{-1}\sqrt{\frac{k}{g}}\, u\right)$$

when $v = 0$ $t = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}\, u\right)$.

Motion downwards

$\cancel{m}a = \cancel{m}g - \cancel{m}kv$

$\cancel{v \frac{dv}{dx} = g - kv}$

$\frac{dv}{dt} = g - kv$

$\frac{dt}{dv} = \frac{1}{g - kv}$

$t = -\frac{1}{k} \ln(g - kv) + C$

$v = 0$ when $t = 0$

$C = \frac{1}{k} \ln g$

$\cancel{-kt = \ln(g - kv)}$

$\cancel{e^{-kt} = g - kv}$

$\cancel{kv = g - e^{kt}}$

$\cancel{v = \frac{1}{k}(g - e^{kt})}$

$t = \frac{1}{k} \ln g - \frac{1}{k} \ln(g - kv)$

$t = \frac{1}{k} \ln\left(\frac{g}{g - kv}\right)$

$e^{-kt} = \frac{g - kv}{g}$

$ge^{-kt} = g - kv$

$v = \frac{g}{k}(1 - e^{-kt})$.

Terminal velocity ; ~~when $g = kv$~~

as $t \to \infty$

$v \to g/k$

$\cancel{\text{i.e. } g = g - e^{kt}}$ ~~[doesn't help~~

OR $\frac{dv}{dt} = 0$

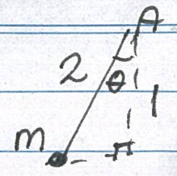
$g = kv$

$v = g/k$.

$r = 2\sin\theta$

Example 2

$T\sin\theta = mr\omega^2$



$\cos\theta = \frac{1}{2}$

$\theta = 60^\circ$

$h = 1 = g/\omega^2$

$\therefore \omega^2 = g$

$\omega = \sqrt{g}$

$T\cos\theta = mg$

$\frac{T}{2} = 6g$

$T = 12g$

Fitzpatrick exc 7.5 Q5

Banked Tracks

~~Lateral force = $-r\omega^2\cos\theta$~~ ~~$v =$~~ $\boxed{v = r\omega}$

a) $\boxed{a = \frac{v^2}{r}}$ $a = r\omega^2$ ~~α~~ $=$ ~~v~~

need v ~~of~~ in m/s.

$F = ma = \frac{50.0\cancel{0}\cancel{0} \times (50\,000)^2 \div 60^{\cancel{2}}}{40\cancel{0}}$

$\doteq 24113N$

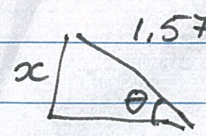
b) ~~$\tan\theta =$~~ $\tan\theta = \frac{v_0^2}{rg} \approx 0.492$

no slip

$\tan\theta = \frac{\frac{mv_0^2}{r}}{mg} = \frac{v_0^2}{rg}$

$\theta = \tan^{-1}\left(\frac{(50000/60^2)^2}{400 \times 9.8}\right)$

$2^\circ 49'$

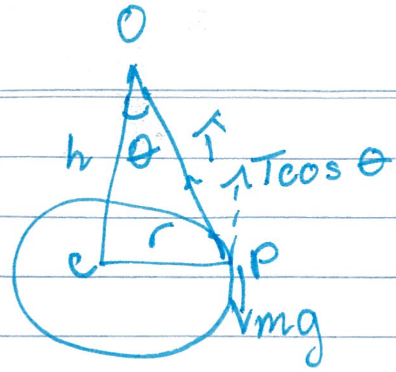


$\theta = 2^\circ 49'$

$x = 1.575\sin 2^\circ 49'$

$= 0.077\,m$ (~~7.7~~ cm) 7.7 ~~Error~~

$\approx 8\,cm.$



Let $\theta = \angle COP$
and T the tension in the string

a) Vertically $T\cos\theta = mg$
Horizontally $T\sin\theta = mr\omega^2$

$$\therefore \tan\theta = \frac{mr\omega^2}{mg}$$
$$= \frac{r}{h}$$
$$\therefore \frac{r\omega^2}{g} = \frac{r}{h}$$
$$\omega^2 = \frac{g}{h}$$
$$\omega = \sqrt{\frac{g}{h}}$$

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$$

b) From a) $\omega^2 = \frac{g}{h}$ i.e.
$$\therefore h = \frac{g}{\omega^2}$$

2 revs/second $\quad \omega = 2\times 2\pi$ radian/sec
$$= 4\pi$$
$$h = g/16\pi^2$$

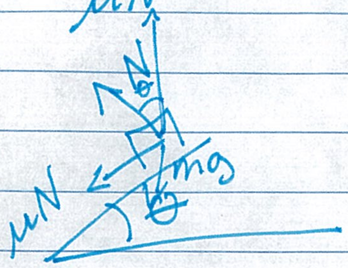
1 rev/s $\quad \omega = 2\pi$
$$h = g/4\pi^2$$

Difference in height $= \frac{g}{16\pi^2} - \frac{g}{4\pi^2}$
$$= -\frac{3g}{16\pi^2} \text{ cm}$$

The level of the circle is lowered by $\frac{3g}{16\pi^2}$ cm.

2015 Q14 c)

speed v tendency to slide up the track opposed by force μN



vertically $\cancel{mg} + N\cos\theta \overset{?}{-} \mu N\sin\theta = \cancel{0}\ mg$ (1)

Horizontally $N\sin\theta + \mu N\cos\theta = \frac{mv^2}{r}$ (2)

(2) ÷ (1) $\quad \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$

$$v^2 = rg\left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right).$$

$V^2 = rg$ still a tendency to slide up

show $\mu < 1$

$\frac{\tan\theta + \mu}{\cancel{1 - \mu\tan}} = 1 - \mu\tan\theta$

$0 < \theta < \frac{\pi}{2}$

$\Rightarrow \tan\theta > 0$

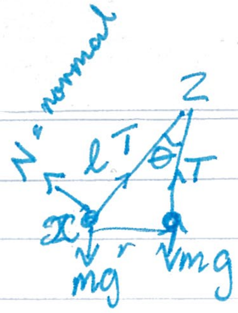
$\mu(1+\tan\theta) = 1 - \tan\theta$

$\mu = \frac{1-\tan\theta}{1+\tan\theta}$ $\begin{matrix} <1 \\ >1 \end{matrix}$

$1 - \tan\theta < 1$ and $1 + \tan\theta > 1$

$\therefore \mu < 1$

11.



at x $T\cos\theta + N\sin\theta = mg$ ①

$N\cos\theta + mr\omega^2 = T\sin\theta$ ②

on y $T = mg$

$\therefore\ mg\cos\theta + N\sin\theta = mg$

$$N = \frac{mg(1-\cos\theta)}{\sin\theta}$$

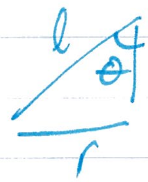
sub into ② $\cancel{m}g(1-\cos\theta)\cdot\frac{\cos\theta}{\sin\theta} + \cancel{m}r\omega^2 = \cancel{m}g\sin\theta$

$g(1-\cos\theta)\cdot\cos\theta + r\omega^2\sin\theta = g\sin^2\theta$

$g\cos\theta - g = -r\omega^2\sin\theta$

$$\omega^2 = \frac{g(\cos\theta - 1)}{-r\sin\theta}$$

$$g\cdot\frac{1-\cos\theta}{r\sin\theta}\ \frac{1+\cos\theta}{1+\cos\theta}$$



$$g\ \frac{\sin^2\theta}{r\sin\theta(1+\cos\theta)}$$

$$= \frac{g\sin\theta}{r(1+\cos\theta)} \qquad l = \frac{r}{\sin\theta}$$

$$= \frac{g}{l(1+\cos\theta)}$$

iii) Deduce that $\frac{g}{2\omega^2} < l < \frac{g}{\omega^2}$

since $0 \le \theta \le \frac{\pi}{2}$

now $0 < \cos\theta < 1$

$1 < 1+\cos\theta < 2$