CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Mathematics Extension 1

HSC ASSESSMENT Task 2 2013
Half-Yearly

Time allowed: $1\frac{1}{2}$ **hours**

Name: ________________________________ **Class:** _________

Teacher________ **Gong / Cheng / Strilakos**

Instructions:

- All questions should be attempted.
- Show ALL necessary working on your own paper.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Start each question on a new page and only write on one side of each sheet of paper.

	MC	Q1	Q2	Q3	Q4	Q5	TOTAL
HE1	/4				/10	/6	/20
HE2		/2				/4	/6
HE4		/8	/5	/10			/23
HE6			/5				/5
TOTAL	/4	/10	/10	/10	/10	/10	/54

SECTION 1

MULTIPLE CHOICE (4 marks)

Question 1

The gradient of the tangent to the curve described parametrically by the equations $x = 4t$ and $y = t^2$ at the point $t = 8$ is:

(A) 2 (B) 64 (C) 4 (D) $\frac{1}{4}$

Question 2

The value of $\int_{-\infty}^{0} \frac{e^x}{e^x+1}\, dx$ is:

(A) ∞ (B) 0 (C) $\ln 2$ (D) $-\infty$

Question 3

Which of the following is ***not true*** for the graph of $y = \frac{x^2}{x^2-1}$

The graph has an asymptote at:

(A) $x = -1$ (B) $y = 1$ (C) $x = 1$ (D) $y = 0$

Question 4

The parametric equations $x = t^2 + 2$ and $y = 3t$ represent which one of the following cartesian equations:

(A) $9x = y^2 + 2$ (B) $x = y^2 + 18$

(C) $9x + y^2 = 18$ (D) $9x - 18 = y^2$

END OF SECTION 1

SECTION 2

Question 1 (10 marks) (Start a new page)

(a) Complete the last steps of the following Proof by Mathematical Induction: 2

To Prove: $3^n > n(n+1)(n+2)$ for $n \geq 5$

Proof: Step1: Show it is true for $n = 5$.

If $n = 5$, $3^5 = 243$ and $n(n+1)(n+2) = 5(6)(7) = 210$

Since $243 > 210$, it is true for $n = 5$.

Step 2: Assume it is true for $n = k$ i.e. $3^k > k(k+1)(k+2)$.

Prove ...

(b) For the graph with equation $y = \frac{x^2-4}{(x-1)^2}$

(i) Find the coordinates of all points of intersection with the coordinate axes. 1

(ii) Find the equations of all asymptotes. 2

(iii) Find the coordinates of any points where the graph intersects its asymptotes. 1

(iv) Find the coordinates of, and nature of, any turning points. 2

(v) Sketch the graph, labelling all relevant points and clearly showing its asymptotic behaviour. 2

Question 2 (10 marks) (Start a new page)

(a) Given $f(x) = (x^2 + 2x + 2)e^{-x}$

(i) Find $f'(x)$ 1

(ii) Find a and b such that $a < b$ and $f''(a) = f''(b) = 0$. 2

(iii) Hence, or otherwise, evaluate $\int_a^b x^2 e^{-x}\, dx$ 2

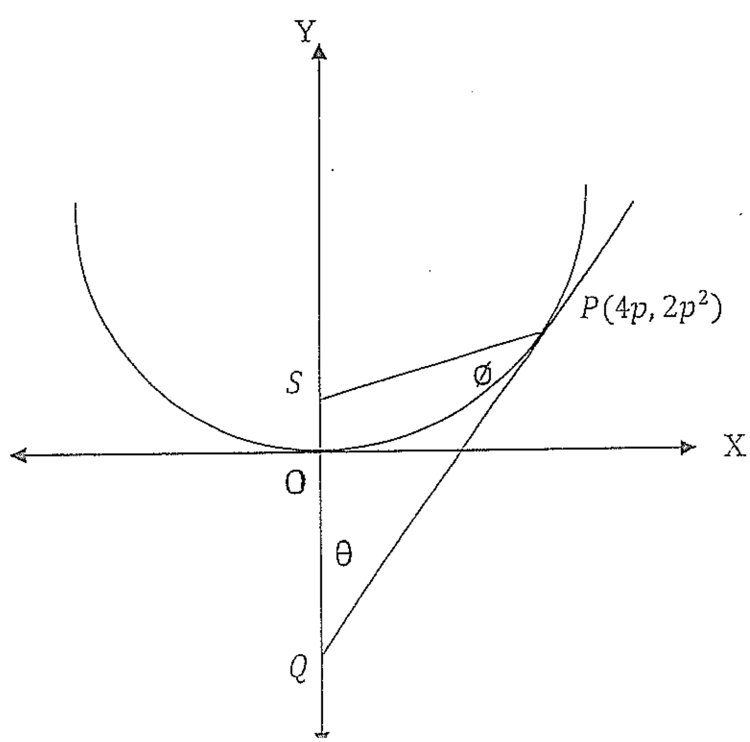
(b) The tangent to the parabola $x^2 = 8y$ at the point $P(4p, 2p^2)$ makes an angle θ with the axis of the parabola and it makes an angle ø with the focal chord which passes through the point P.

(i) State the coordinates of the focus S. 1

(ii) Find the equation of the tangent to the curve at the point P. 1

(iii) Find the co-ordinates of the point Q, where the tangent to P meets the axis of the parabola. 1

(iv) Hence, or otherwise, show that $\angle\theta = \angle ø$ 2



Question 3 (10 marks) (Start a new page)

(a) The tangent at $P(2ap, ap^2), p \geq 0$, to the parabola $4ay = x^2$ cuts the $x - axis$ at A. The normal to the parabola at P cuts the $y -$ axis at B.

(i) Find the equation of the tangent to the curve and the coordinates of the point A. 1

(ii) Find the equation of the normal to the curve and the coordinates of the point B. 1

(iii) Find the coordinates of the midpoint of AB. 1

(iv) Hence find the Cartesian equation of the locus of the midpoint of AB. 1

(b) (i) Show that the equation of the tangent to the curve $y = \log_e x$ at the point $P(e^2, 2)$ is $y = \dfrac{x}{e^2} + 1$. 1

(ii) Sketch the curve $y = \log_e x$ and its tangent at P on the same set of axes, labelling all relevant points.. 1

(iii) Express x in terms of y for $y = \log_e x$. 1

(iv) Calculate the volume of the solid generated when the area bounded by the $x - axis$, the $y - axis$, the curve $y = \log_e x$ and its tangent at $x = e^2$ is rotated about the $y - axis$. 3

Question 4 (10 marks) (Start a new page)

S is a set of points given by $x = t^4, y = t - 5$ for $0 \le t \le 5$.

(i) If R is the distance between the origin and a point P in S, show that $R^2 = t^8 + t^2 - 10t + 25$ 1

(ii) Hence find the minimum value of R. 3

(iii) Find the maximum value of R. 2

(iv) Where does S cut the x- and y- axes? 2

(v) The region bounded by S and the co-ordinate axes is rotated about the y-axis to form a solid of revolution, V. Find the volume of V. 2

Question 5 (10 marks) (Start a new page)

(a) Prove the following using Mathematical Induction: 4

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3} \quad \text{for } n \ge 1$$

Question 5 (cont.)

(b)

(i) Sketch the curve $y = e^{-x^2}$, showing any stationary points and points of inflexion. 3

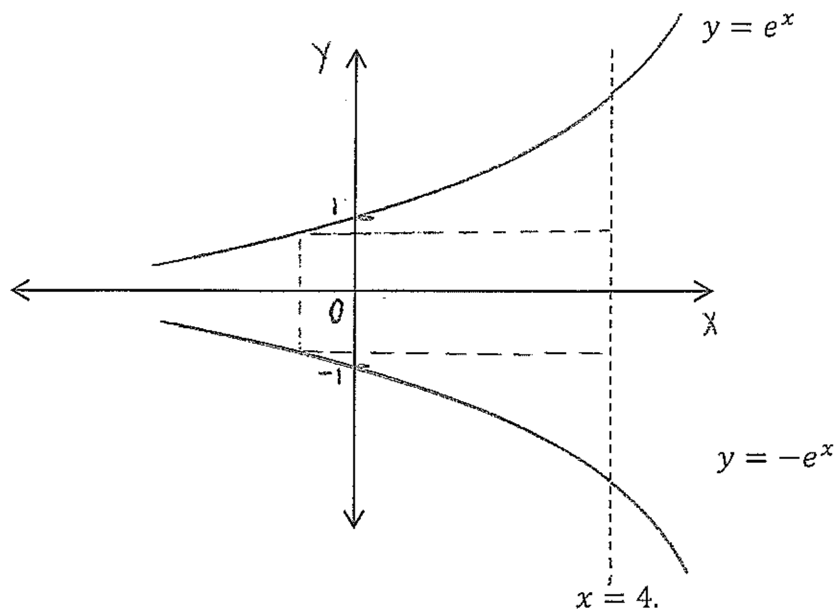
(ii) $ABCD$ is a rectangle inscribed in the curve, with AB lying on the x −axis. Find the area of the largest rectangle that can be inscribed under the curve. 3

END OF PAPER

Question 5 (cont.)

(b)

The graphs of $y = e^x$ and $y = -e^x$ are shown together on the graph below.



(i) Find an expression in terms of x for the area of a rectangle inscribed between these two curves and bounded by the line $x = 4$. [1]

(ii) Find the value of x for which this area is a maximum. [3]

(iii) Find this maximum area. [1]

(iv) Hence find the maximum area that would be enclosed by the curve $y = e^x$, the x-axis and the line $x = 4$. [1]

END OF PAPER

SECTION 1 MULTIPLE CHOICE

Q.1. C Q2. C Q3 B Q4 D

SECTION 2

QUESTION 1

(a) $3^k > k(k+1)(k+2)$ (Assumption)

Prove it is true for $n = k+1$ $n \geq 5$

ie Prove that $3^{k+1} > (k+1)(k+2)(k+3)$, $k \geq 5$

$LHS = 3^{k+1} = 3.3^k$

$> 3\left[k(k+1)(k+2)\right]$ from the assumption

Now, $3k(k+1)(k+2) > (k+1)(k+2)(k+3)$

since $3k > k+3$ for $k \geq 5$.

thus it is true for $n=k+1$ if it is true for $n=k$, and since it is true for $n=5$, it must be true for $n=6$, and $n=7$ and all $n \geq 5$.

(b) $y = \dfrac{x^2-4}{(x-1)^2}$

(i) Cuts x-axis at $y=0$ ie $x^2-4=0$, $x = \pm 2$ ∴ $(-2,0)$ $(2,0)$

cuts y-axis at $x=0$ ie $y = \frac{-4}{1} = -4$ ∴ $(0,-4)$

(ii) Vertical asymptote at $x=1$

For horizontal asymptote,

$y = 1 + \dfrac{2x-5}{(x-1)^2}$

$$\begin{array}{r|l} & 1 \\ x^2-2x+1 & x^2-4 \\ & x^2-2x+1 \\ \hline & 2x-5 \end{array}$$

Thus there is a horizontal asymptote at $y=1$.

(iii) Only horizontal asymptotes can be crossed.

$y=1$ when $2x-5=0$ ie $x = \frac{5}{2}$ ∴ $(\frac{5}{2}, 1)$

(iv) $\frac{dy}{dx} = 0$ for stat. pts.

$\dfrac{dy}{dx} = \dfrac{2x(x-1)^2 - 2(x-1)(x^2-4)}{(x-1)^4}$

$= \dfrac{(x-1)\left[2x(x-1) - 2(x^2-4)\right]}{(x-1)^4}$

$= \dfrac{(x-1)(-2x+8)}{(x-1)^4} = \dfrac{-2x+8}{(x-1)^3}$

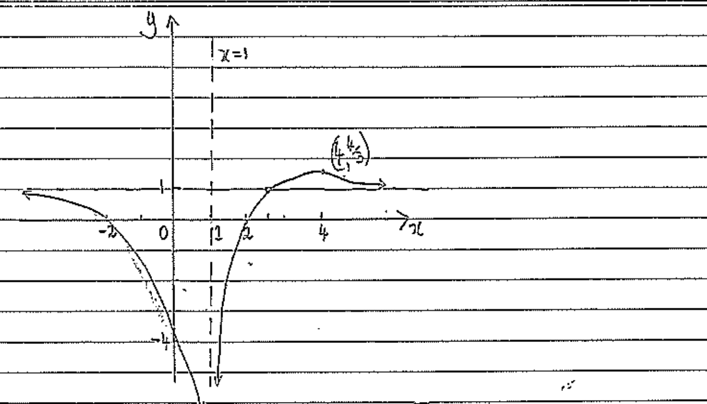
$= 0$ when $x = 4$ (Note: $x=1$ is vertical asymptote ∴ no t.p. there!)

TEST:

x	<4	4	>4
$\frac{dy}{dx}$	/	—	\

∴ maximum t.p. at $(4, \frac{4}{3})$

(v)



As $x \to +\infty$, $\dfrac{2x-5}{x^2-2x+1} = \dfrac{2-\frac{5}{x}}{x-2+\frac{1}{x}} \longrightarrow 0^+$

∴ $y = 1 + \dfrac{2x-5}{(x-1)^2} \longrightarrow 1^+$

As $x \to 1^+$, $(x-1)^2 \to 0$ ∴ $\dfrac{2x-5}{(x-1)^2} \to -\infty$ and $y \to -\infty$

As $x \to -\infty$, $\dfrac{2x-5}{x^2-2x+1} = \dfrac{2-\frac{5}{x}}{x-2+\frac{1}{x}} \longrightarrow 0^-$

∴ $y = 1 + \dfrac{2x-5}{(x-1)^2} \longrightarrow 1^-$

As $x \to 1^-$, $(x-1)^2 \longrightarrow 0$ ∴ $\dfrac{2x-5}{(x-1)^2} \to -\infty$ and $y \to -\infty$

QUESTION 2. $f(x) = (x^2+2x+2)e^{-x}$

(a) (i) $f'(x) = (2x+2)e^{-x} - e^{-x}(x^2+2x+2)$

$= e^{-x}(x^2)$

$= -x^2e^{-x}$

(ii) $f''(x) = (-2x)e^{-x} - e^{-x}(-x^2)$

$= -2xe^{-x} + x^2e^{-x}$

$= xe^{-x}(x-2)$

$= 0$ when $x=0$ or $x=2$

Since $a < b$, $a=0$ and $b=2$

(iii) $\displaystyle\int_a^b x^2e^{-x}\,dx = \int_0^2 x^2e^{-x}\,dx$

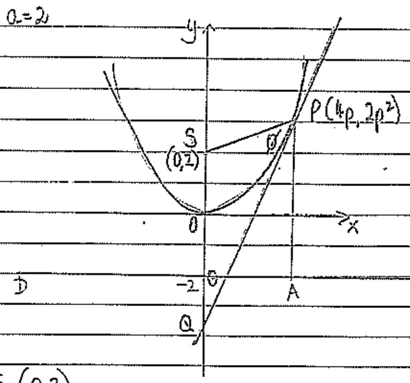
$= -\displaystyle\int_0^2 f'(x)\,dx$ {from (i)}

$= -\left[(x^2+2x+2)e^{-x}\right]_0^2$

$= -\left[(4+4+2)e^{-2} - 2e^0\right]$

(b) $x^2=8y$ $\quad P(4p,2p^2)$

$a=2$



(i) $S(0,2)$

(ii) $y=\dfrac{x^2}{8}$

$\dfrac{dy}{dx}=\dfrac{x}{4}$ $\quad$ At P, $\dfrac{dy}{dx}=p$.

$\therefore$ Eq^n of tangent at P is $\quad y-2p^2=p(x-4p)$

$y=px-2p^2$.

(iii) Tangent meets axis at $x=0$, $y=-2p^2$. $\quad (0,-2p^2)$

(iv) $d(QS)=2p^2+2$

$d(SP)=\sqrt{16p^2+(2p^2-2)^2}$

$=\sqrt{16p^2+4p^4-8p^2+4}$

$=\sqrt{4p^4+8p^2+4}$

$=\sqrt{(2p^2+2)^2}$

$=2p^2+2$

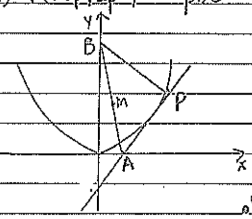
b(iv) cont. since $d(QS)=d(SP)$, $\quad \Delta PSQ$ is isosceles

$\therefore \angle\theta=\angle\phi$, equal angles opposite equal sides in an isosceles triangle.

Alternatively, $d(SP)=d(AP)$ by definition $\therefore d(SP)=2p^2-(-2)$
$=2p^2+2$

QUESTION 3

(a)(i) $P(2ap,ap^2)$ $\quad p\geq 0$ $\quad 4ay=x^2$.



$\dfrac{dy}{dx}=\dfrac{2x}{4a}=\dfrac{x}{2a}$

At P, $\dfrac{dy}{dx}=p$.

$\therefore$ Eq^n of tangent is:

$y-ap^2=p(x-2ap)$

i.e. $y=px-ap^2$.

Tangent cuts x-axis at $y=0$, $x=ap$ $\quad\therefore$ A is $(ap,0)$

(ii) Eq^n of normal $\quad y-ap^2=-\dfrac{1}{p}(x-2ap)$

$y=\dfrac{-x}{p}+2a+ap^2$

Cuts Y-axis at $x=0$, $y=2a+ap^2$

(iii) midpoint of AB is $\left(\dfrac{ap}{2},\ a+\dfrac{ap^2}{2}\right)$

3(a)(iv) $x=\dfrac{ap}{2}$ $\quad y=a+\dfrac{ap^2}{2}$ $\quad$ —(2)

$2x=ap$

$p=\dfrac{2x}{a}$ $\quad$ —(1)

Subst (1) into (2) $\quad y=a+a\left(\dfrac{4x^2}{2a^3}\right)=a+\dfrac{2x^2}{a}=\dfrac{a^2+2x^2}{a}$

OR $\quad ay=a^2+2x^2$

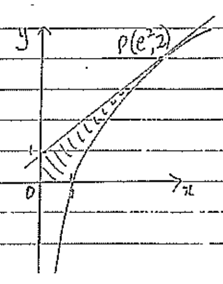
(b) $y=\log_e x$

(i) $\dfrac{dy}{dx}=\dfrac{1}{x}$ $\quad$ At $P(e^2,2)$ $\dfrac{dy}{dx}=\dfrac{1}{e^2}$

$\therefore$ Eq^n of tangent is $\quad y-2=\dfrac{1}{e^2}(x-e^2)$

$y=\dfrac{x}{e^2}+1$, as required.

(ii)



3(b)(iii) $y=\log_e x$

$\therefore x=e^y$.

(iv) $V=\pi\displaystyle\int_0^2 x_1^2\,dy-\pi\int_1^2 x_2^2\,dy$ $\qquad x_1=e^y$

$\dfrac{x_2}{e^2}=y-1$

$x_2=e^2(y$

$=\pi\displaystyle\int_0^2 (e^y)^2\,dy-\pi\int_1^2\left[e^2(y-1)\right]^2dy$

$=\pi\displaystyle\int_0^2 e^{2y}\,dy-\pi e^2\int_1^2(y^2-2y+1)\,dy$

$=\pi\left[\dfrac{1}{2}e^{2y}\right]_0^2-\pi e^2\left[\dfrac{y^3}{3}-y^2+y\right]_1^2$

$=\pi\left[\dfrac{1}{2}e^4-\dfrac{1}{2}\right]-\pi e^2\left[\left(\dfrac{8}{3}-4+2\right)-\left(\dfrac{1}{3}\cancel{-1}\cancel{+1}\right)\right]$

$=\pi\left[\dfrac{e^4}{2}-\dfrac{1}{2}-e^2\left(\dfrac{1}{3}\right)\right]$

$=\pi\left(\dfrac{e^4}{2}-\dfrac{e^2}{3}-\dfrac{1}{2}\right)$ cubic units

Question 4

$x=t^4 \quad y=t-5 \qquad 0\le t\le 5$

(i) $R^2 = x^2+y^2$

$= t^8 + (t-5)^2$

$= t^8 + t^2 - 10t + 25$

(ii) $\frac{dR^2}{dt} = 8t^7 + 2t - 10 = 0$ when $t=1$, by inspection.

t	<1	1	>1
$\frac{dR^2}{dt}$	\	—	/

$\therefore$ min^m when $t=1$.

When $t=1$,

$R^2 = 1+1-10+25$

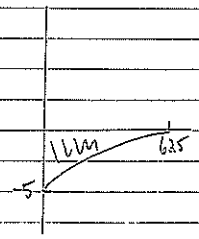
$= 17$

$\therefore R=\sqrt{17}$

(iv) S cuts x-axis at $y=0$, $t=5$ $\therefore x=625$

S cuts y-axis at $x=0$, $t=0$ $\therefore y=-5$

(v)



$V=\pi\int_{-5}^{0} x^2\,dy$

Q4 (v) $V=\pi\int_{-5}^{0} x^2\,dy \qquad x=t^4 \quad y=t-5$

$\therefore t=y+5$

$\therefore x=(y+5)^4$

$V=\pi\int_{-5}^{0}(y+5)^8\,dy$

$= \frac{\pi}{9}\left[(y+5)^9\right]_{-5}^{0}$

$= \frac{\pi}{9}(5^9)$

$= \frac{\pi}{9}.5^9$ cubic units.

Question 5

To Prove:

(a) $1^2+3^2+5^2+\dots+(2n-1)^2 = \frac{n(4n^2-1)}{3}, \quad n\ge 1$

Proof:

Show true for $n=1$

ie $1 = \frac{1(4-1)}{3} = 1$ $\therefore$ It is true.

Assume true for $n=k$

ie $1^2+3^2+5^2+\dots+(2k-1)^2 = \frac{k(4k^2-1)}{3}$

Prove true for $n=k+1$

ie $1^2+3^2+5^2+\dots+(2k-1)^2+[2(k+1)-1]^2 = \frac{(k+1)(4(k+1)^2-1)}{3}$

$= \frac{(k+1)(4k^2+8k+3)}{3}$

LHS $= \frac{k(4k^2-1)}{3} + (2k+1)^2$ from the assumption.

$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$

$= \frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$

$= \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$

$= \frac{(2k+1)(2k^2+5k+3)}{3}$

$= \frac{(2k+1)(2k+3)(k+1)}{3}$

$= \frac{(k+1)(4k^2+8k+3)}{3}$

= RHS as req. Thus true for $n=k+1$, if true for $n=k$. and since it is true for $n=1$, it must be true for $n=2, 3, \dots$ and all $n\ge 1$

5(b)

(i) $A = (4-x)(e^x - -e^x)$

$= (4-x).2e^x$

(ii) $\frac{dA}{dx} = -1.2e^x + 2e^x(4-x)$

$= 8e^x - 2e^x - 2xe^x$

$= 6e^x - 2xe^x$

$= 2e^x(3-x)$

$= 0$ when $x=3$

test:

x	<3	3	>3
$\frac{dA}{dx}$	/	—	\

$\therefore$ max^m at $x=3$

(iii) $A = (4-3)(2e^3)$

$= 2e^3$ sq units

(iv) $A_{max} = e^3$ sq units