CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12

Extension 2 Mathematics

Assessment Task 1 2014

Time allowed: 1 hour 40 minutes

Name: ______________________ **Teacher: Ms Strilakos**

Instructions:

- All questions should be attempted on your own paper.
- Show ALL necessary working in Section 2
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Please write on one side of each sheet of paper only.

	Multiple Choice	Q6	Q7	Total
E3				
Total	/5	/27	/25	/57

Section 1 – Multiple Choice

For each of the following questions please circle the correct answer:

1. If z is a non-zero complex number, then $\arg z + \arg \bar{z}$ is equal to:

   A. $\frac{\pi}{2}$ B. 1 C. x^2 D. 0

2. If $z = 4 + 3i$ and $w = 1 + i$ then $\frac{z}{w}$ is equal to:

   A. $2 + \frac{3i}{2}$ B. $\frac{7}{2} - \frac{1}{2}i$ C. $3+2i$ D. $2\sqrt{2} + \frac{i\,3\sqrt{2}}{2}$

3. The Cartesian equation of the locus described by the statement $|z| = |z - 2|$ is:

   A. $x^2 + y^2 = 4$ B. $x = 2$ C. $x = 1$ D. $y = 2x$

4. If $|w + z| = |w| + |z|$ for two non-zero complex numbers w and z then the expression for $\arg z$ in terms of $\arg w$ would be:

   A. $\arg(2w)$ B. $arg\ (w^2)$ C. $arg\ (w)$ D. $arg\ (0)$

5. The modulus-argument form of the complex number $z = 2i$ is:

   A. $2\ \text{cis}\frac{\pi}{2}$ B. $2cis\ \pi$ C. $\sqrt{2}\ cis\frac{\pi}{2}$ D. $\sqrt{2}\ cis\ \pi$

END OF SECTION 1

Section 2

Question 6

a) Find the Cartesian equation of the locus described by

i) $z.\bar{z} = 16$ 1

ii) $|z-2| = |z+3i|$ 2

b) Given z is a complex number such that $|z| = 1$ and $\arg z = \theta$, 3

where $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, show on an Argand diagram the set of points

representing i) z,

ii) z^2

iii) $1 - z^2$.

c) If x and y are real numbers such that $(x+iy)^2 = 4+3i$

i) Find real numbers a and b such that $(x-iy)^2 = a+bi$ 1

ii) Find the two square roots of $4+3i$ and hence the two square roots of $4-3i$. 3

iii) Solve the equation $(z^2-4)^2 + 9 = 0$ for z^2. 1

iv) Use your results from (i), (ii) and (iii) to solve $(z^2-4)^2 + 9 = 0$ for z. 1

d) Simplify, without the use of a calculator, showing all working: 2

$$\frac{\left(\cos\frac{\pi}{7} - i\,sin\frac{\pi}{7}\right)^3}{\left(\cos\frac{\pi}{7} + i\,sin\frac{\pi}{7}\right)^4}$$

e) If z and w are complex numbers with $Im\,z = 2$ and $Re\,w = -1$ and 3

$z + w = -zw$, find z and w.

f) i) If $z_1 = 6 + 8i$ and $|z_2| = 6$ 2

find the greatest and least values of $|z_1 + z_2|$

ii) If $|z_1 + z_2|$ takes its greatest value, then write z_2 in the form $x + iy$, 3

where x and y are real.

g) $OABC$ is a rhombus as shown in the diagram below, with $\overrightarrow{OA}$ representing z_1 and $\overrightarrow{OC}$ representing $z_2 = 4i$.

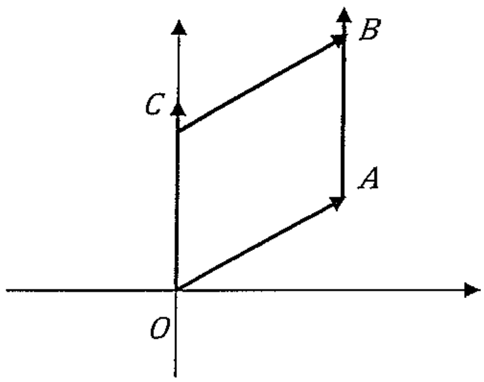
If $\arg z_1 = \frac{\pi}{6}$, find in $x + iy$ form the number representing:

i) $\overrightarrow{OA}$ 1

ii) $\overrightarrow{OB}$ 1

iii) $\overrightarrow{CA}$ 1

iv) z_1 after dilation by a factor of 3 followed by an anticlockwise rotation through an angle of $\frac{\pi}{6}$ radians. 2



Question 7

a) If $w^3 = 1,\ w \neq 1$ show that $1 + w^n + w^{2n}$ is equal to either 3 or zero. 3

Clearly state the conditions on n in both cases.

(you **do not** need to use Proof by Induction to show this clearly)

b) i) Find the three cubic roots of 8 2

ii) If $z_1^{\,3} = 8$ and $z_1 = \dfrac{z_2}{1-z_2}$, using your answer to part (i) 3

find the three roots of z_2.

c) Find the Cartesian equation of the locus of the point $P(x, y)$ where P 3

represents the complex number $z = x + iy$, given:

$$\arg(z-3) = \frac{2\pi}{3} + \arg z$$

d) Find all the complex numbers $z = a + ib$, where a and b are real, 3

such that $|z|^2 + 5\bar{z} + 10i = 0$

e) (i) Show that the Cartesian equation of the locus of a point P, 2

representing the complex number z, where $|z-1| = |z-3i|$

is $x - 3y + 4 = 0$

(ii) Sketch this locus on an Argand diagram and find z when $|z|$ 3

has its least value on this locus.

f) (i) Given that $z = cos\theta + i\sin\theta$, using De Moivre's Theorem, 3

show that $\cos 5\theta = cos\theta(5 - 20cos^2\theta + 16cos^4\theta)$

(ii) Hence find the general solutions of the equation 3

$$\cos 5\theta = 16cos^5\theta$$

You may find the following general expansion useful:

$$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \binom{n}{3}x^{n-3}a^3 \ldots \binom{n}{n}a^n$$

END OF PAPER

Section 1

Q.1. D

Q.2. $\frac{z}{w} = \frac{4+3i}{1+i} \times \frac{1-i}{1-i}$ B

$= \frac{4-4i+3i+3}{2}$

$= \frac{7}{2} - \frac{i}{2}$

Q.3 C.

Q4. C

Q.5. A

$2i$

Section 2

Question 6

a. i) $x^2+y^2=16$

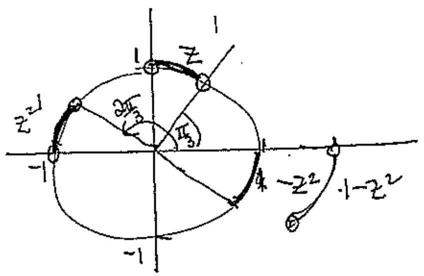
ii) $(x-2)^2+y^2 = x^2+(y+3)^2$

$-4x+4 = 6y+9.$

$4x+6y+5=0.$ $6y=-4x-5$ $y = -\frac{2}{3}x - \frac{5}{6}$

(3)

b.



(3)

SECTION 2

Q.6 (c) i) If $(x+iy)^2 = 4+3i$ x and y are real.

then $(x-iy)^2 = 4-3i$ (1) since $(x+iy)^2 = x^2-y^2+2xyi$

and $(x-iy)^2 = x^2-y^2-2xyi$.

So $a=4$ and $b=-3$

ii) $(x+iy)^2 = 4+3i$

$x^2-y^2+2xyi = 4+3i$

$x^2-y^2=4$ —(1) and $2xy=3$ $\therefore y = \frac{3}{2x}$. —(2)

Substitute (2) into (1) $x^2 - \frac{9}{4x^2} = 4$

ie $4x^4 - 16x^2 - 9 = 0$

$(2x^2+1)(2x^2-9)=0$ —(3)

Since x is real, the only solution for (3) are $x^2 = \frac{9}{2}$ $x = \pm\frac{3}{\sqrt{2}}$

ie $x = \pm\frac{3\sqrt{2}}{2}$

If $x = \frac{3\sqrt{2}}{2}$, $y = \frac{\cancel{3} \times \cancel{2}}{\cancel{2} \times \cancel{3}\sqrt{2}} = \frac{\sqrt{2}}{2}$

(3).

If $x = -\frac{3\sqrt{2}}{2}$, $y = -\frac{\sqrt{2}}{2}$

Thus the two square roots of $4+3i$ are $\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $-\frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

Since, for $4-3i$, $x^2-y^2=4$ and $y = -\frac{3}{2x}$,

the two square roots of $4-3i$ are $\frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ and $-\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

c) iii) $(z^2-4)^2+9=0.$

$(z^2-4)^2 = -9 = 9i^2$

$z^2-4 = \pm 3i.$ $\quad z^2 = 4 \pm 3i$ $\quad$ 1

v) To solve $(z^2-4)^2+9$ for z,

the solutions are: $\frac{3\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, $\frac{3\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$, $-\frac{3\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$, $-\frac{3\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ $\quad$ 1

$$\frac{\left(\cos\frac{\pi}{7}-i\sin\frac{\pi}{7}\right)^3}{\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)^4} = \frac{\left(\cos\left(-\frac{\pi}{7}\right)+i\sin\left(-\frac{\pi}{7}\right)\right)^3}{\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)^4} \quad 1$$

$$= \frac{\cos\left(-\frac{3\pi}{7}\right)+i\sin\left(-\frac{3\pi}{7}\right)}{\left(\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}\right)} \quad 1 \quad 2$$

$$= \cos(-\pi)+i\sin(-\pi) = -1. \quad 1$$

e) $\text{Im}\, z = 2 \qquad \text{Re}\, \omega = -1.$

Let $z = x+2i \qquad \omega = -1+yi$

$z+\omega = (x-1)+(2+y)i$

$-z.\omega = -(x+2i)(-1+yi)$ $\quad$ 1

$= -(-x+xyi-2i+2yi^2)$

$= x+2y+(2-xy)i$

Given $z+\omega = -z\omega$,

$(x-1)+(2+y)i = x+2y+(2-xy)i$

Equating real and imaginary parts: $x-1 = x+2y$ ie. $y = -\frac{1}{2}$

Q6 e) continued

$2+y = 2-xy$ ie. $y(1+x)=0$ $\quad$ Since $y \neq 0 \Rightarrow x=-1.$ $\quad$ 1

$\therefore z = -1+2i$, and $\omega = -1-\frac{1}{2}i$. $\quad$ 1 $\quad$ 3.

f) i) $z_1 = 6+8i \qquad |z_2| = 6$

$|z_1+z_2| = |z_1|+|z_2|$ $\quad$ for greatest value $\quad$ 1

$|z_1| = 10 \quad \therefore$ greatest value is 16. $\quad$ 2

$|z_1+z_2| = |z_1|-|z_2|$ $\quad$ for least value $\therefore$ least value is 4. $\quad$ 1

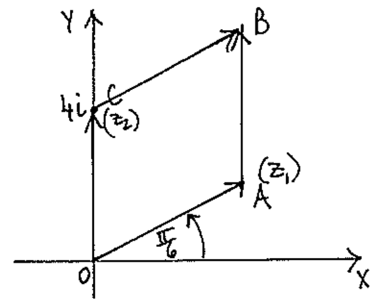
ii) If $|z_1+z_2| = 16$ $\quad$ this occurs when z_1 and z_2 are parallel. $\quad$ 1

$\therefore z_2 = kz_1 = k(6+8i) = 6k+8ki$ $\quad$ 1

$|z_2| = \sqrt{36k^2+64k^2} = 10k = 6 \quad \therefore k = \frac{3}{5}.$ $\quad$ 1 $\quad$ 3

$\therefore z_2 = \frac{3}{5}(6+8i) = \frac{18}{5}+\frac{24}{5}i$ $\quad$ 1

g)



i) $|\overrightarrow{OA}| = 4$ since its a rhombus.

$\overrightarrow{OA} = x+iy$ $\quad$ where $\quad x = 4\cos\frac{\pi}{6}$ $\quad$ and $\quad y = 4\sin\frac{\pi}{6}$

$x = 2\sqrt{3} \qquad y = 2$

$\therefore \overrightarrow{OA} = 2\sqrt{3}+2i$ $\quad$ 1

g) ii) $\vec{OB} = \vec{OA} + \vec{AB}$

$= \vec{OA} + \vec{OC}$ since $\vec{AB} = \vec{OC}$

$= 2\sqrt{3} + 2i + 4i$

$= 2\sqrt{3} + 6i$ (1)

iii) $\vec{CA} = \vec{CO} + \vec{OA}$

$= -4i + 2\sqrt{3} + 2i$

$= 2\sqrt{3} - 2i$ (1)

iv) $3z_1 \times (\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ 1

$= 3(2\sqrt{3} + 2i)(\frac{\sqrt{3}}{2} + \frac{i}{2})$ (2)

$= 3(3 + \sqrt{3}i + \sqrt{3}i - 1)$ 1

$= 6 + 6\sqrt{3}i$

uestion 7

i) $\omega^3 = 1$, $\omega \neq 1$.

$\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$

Since $\omega \neq 1$, we may conclude that $\omega^2 + \omega + 1 = 0$

$1 + \omega + \omega^2 = 0$ $\qquad \omega^0 + \omega^n + \omega^{2n}$

If $n = 0$ $\quad \omega^0 + \omega^0 + \omega^0 = 3$

$n = 1$ $\quad 1 + \omega + \omega^2 = 0$

$n = 2$ $\quad 1 + \omega^2 + \omega^4 = -\omega + \omega = 0$

$n = 3$ $\quad 1 + \omega^3 + \omega^6 = 1 + 1 + 1 = 3$

$n = 4$ $\quad 1 + \omega^4 + \omega^8 = 1 + \omega + \omega^2 = 0$

$n = 5$ $\quad 1 + \omega^5 + \omega^{10} = 1 + \omega^2 + \omega = 0$

$n = 6$ $\quad 1 + \omega^6 + \omega^{12} = 1 + 1 + 1 = 3.$

etc

$1 + \omega^n + \omega^{2n} = 3$

if n is a multiple of 3,

and equals zero otherwise. (3)

Q7(b) i) Let $z^3 = 8$

$z_A = 2$ $\qquad z_B = 2\,\text{cis}\,\frac{2\pi}{3}$ $\qquad z_C = 2\,\text{cis}\,-\frac{2\pi}{3}$

$= 2(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$ $\qquad = 2(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$

$= -1 + \sqrt{3}i$ $\qquad = -1 - \sqrt{3}i$ (2)

ii) $z_1 = \dfrac{z_2}{1 - z_2}$

$z_1 - z_1 z_2 = z_2$

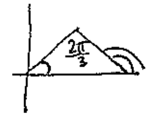
$z_2(1 + z_1) = z_1$ $\quad \therefore z_2 = \dfrac{z_1}{1 + z_1}$ 1 (3)

If $z_1 = 2$, $\quad z_2 = \frac{2}{3}$ 1

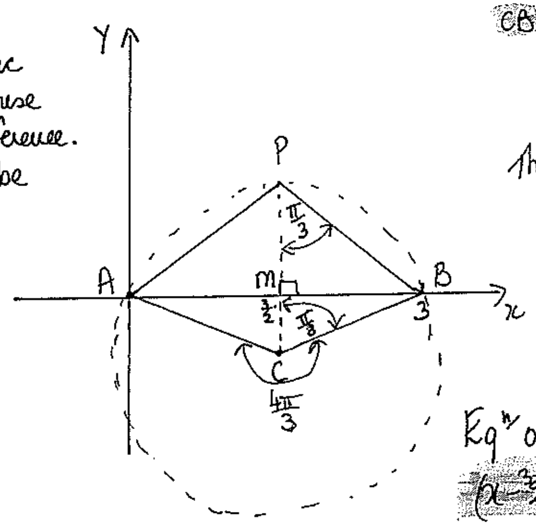
If $z_1 = -1 + \sqrt{3}i$, $\quad z_2 = \dfrac{-1 + \sqrt{3}i}{\sqrt{3}i} = \dfrac{-\sqrt{3}i + 3i^2}{3i^2} = \dfrac{-\sqrt{3}i - 3}{-3} = 1 + \dfrac{\sqrt{3}i}{3}$ 1

If $z_1 = -1 - \sqrt{3}i$, $\quad z_2 = \dfrac{-1 - \sqrt{3}i}{-\sqrt{3}i} = \dfrac{-\sqrt{3}i + 3}{3} = 1 - \dfrac{\sqrt{3}i}{3}$

c) $\arg(z - 3) = \frac{2\pi}{3} + \arg z$. $\qquad \therefore \arg(z-3) - \arg z = \frac{2\pi}{3}$



NOTE: Major arc subtends an obtuse angle at circumference. ∴ Centre must be below x-axis.



$CB = \frac{3}{2} \div \sin\frac{\pi}{3}$

$= \frac{3}{2} \times \frac{2}{\sqrt{3}} = \sqrt{3}$ 1

Thus radius is $\sqrt{3}$. (3)

$|MC| = CB \times \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. 1

$\therefore$ centre is at $(\frac{3}{2}, -\frac{\sqrt{3}}{2})$

Eq^n of locus is a circle

$(x - \frac{3}{2})^2 + (y + \frac{\sqrt{3}}{2})^2 = 3$ 1

7(d) $|z|^2 + 5\bar{z} + 10i = 0$.

Let $z = x + iy$

$|z| = \sqrt{x^2+y^2}$ $\qquad \bar{z} = x - iy$. $\qquad$ 3

$|z|^2 = x^2+y^2$

$|\bar{z}|^2 + 5\bar{z} + 10i = 0$

i.e. $x^2+y^2+5x-5yi+10i=0$ $\qquad$ 1

$\therefore$ $x^2+y^2+5x=0$ — ① $\quad$ and $\quad 10-5y=0$ — ② $\quad$ (Equating coeffs of $0+0i$)

$\Rightarrow y = 2$ $\qquad$ 1

Subst. $y=2$ into ① yields $\quad x^2+4+5x=0$
$(x+4)(x+1)=0 \quad \therefore x=-4,-1$.

$\therefore$ Solutions are $z=-4+2i$ and $z=-1+2i$. $\qquad$ 1

e) i) $|z-1| = |z-3i|$

Let $z = x+iy$

$|z-1| = \sqrt{(x-1)^2+y^2}$ $\quad$ and $\quad |z-3i| = \sqrt{x^2+(y-3)^2}$

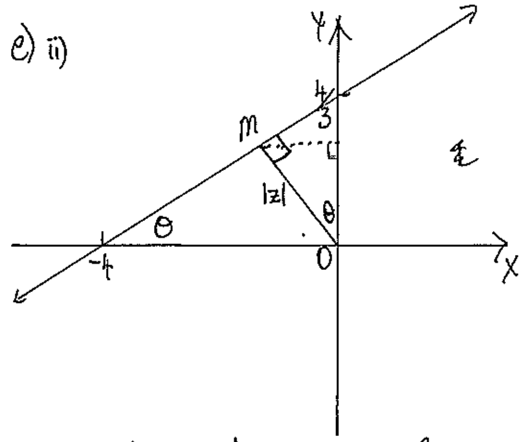
$\therefore$ If $|z-1| = |z-3i|$ then

$(x-1)^2+y^2 = x^2+(y-3)^2$ $\qquad$ 1 $\qquad$ 2

i.e. $\cancel{x^2}-2x+1+\cancel{y^2} = \cancel{x^2}+\cancel{y^2}-6y+9$ $\qquad$ 1

$-2x+6y-8=0$ $\quad$ i.e. $x-3y+4=0$ as required.
$3y = x+4$
$y = \frac{x}{3}+\frac{4}{3}$.

e) ii)



To find z: $|z| = \frac{2\sqrt{10}}{5}$

$\mathrm{Re}\,z = -\frac{2\sqrt{10}}{5}\sin\theta = -\frac{2\sqrt{10}}{5}\times\frac{1}{\sqrt{10}} = -\frac{2}{5}$. $\qquad$ 1

$\mathrm{Im}\,z = \frac{2\sqrt{10}}{5}\cos\theta = \frac{2\sqrt{10}}{5}\times\frac{3}{\sqrt{10}} = \frac{6}{5}$

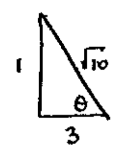
$z = -\frac{2}{5}+\frac{6i}{5}$ $\qquad$ 1

3.

$|z|$ takes least value when z lies on $\perp$ to the line from the origin.

i.e. $\sin\theta = \frac{|z|}{4}$ $\qquad |z| = 4\cdot\sin\theta$.

Now, $\tan\theta = \frac{4/3}{4} = \frac{1}{3}$ $\qquad \therefore |z| = 4/\sqrt{10} = \frac{2\sqrt{10}}{5}$



f) i) $z = \cos\theta + i\sin\theta$

$z^5 = (\cos\theta+i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

But $z^5 = \cos^5\theta + 5\cos^4\theta . i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta + 10\cos^2\theta\, i^3\sin^3\theta$
$+ 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$ (using Binomial Expansion.)

$= \cos^5\theta + 5\cos^4\theta . i\sin\theta - 10\cos^3\theta\sin^2\theta - 10\cos^2\theta\, i\sin^3\theta$
$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$.

$\therefore \cos 5\theta + i\sin 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$
$+ i\,(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating real and imaginary parts: $\qquad$ 1

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

i.e. $\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$ 1

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ 3

$= \cos\theta(5 - 20\cos^2\theta + 16\cos^4\theta)$ 1

ii) If $\cos 5\theta = 16\cos^5\theta$

then $5\cos\theta - 20\cos^3\theta = 0$.

i.e. $5\cos\theta(1 - 4\cos^2\theta) = 0$.

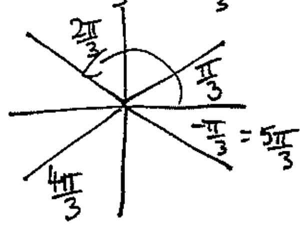
Either $\cos\theta = 0$ or $\cos^2\theta = \frac{1}{4}$ $\cos\theta = \pm\frac{1}{2}$. 1

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots$

$= \frac{n\pi}{2}$, $n = \pm 1, \pm 3, \dots$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \dots$

$-\frac{\pi}{3}, -\frac{2\pi}{3}, -\frac{4\pi}{3}, -\frac{5\pi}{3}, -\frac{7\pi}{3} \dots$



or $\pm\left(n\pi + \frac{\pi}{2}\right)$ 1 1

3