1

Alex

All teachers

one - 20 → 23.
own multiple choice

Section 1

Use the multiple-choice answer sheet for Questions 1 – 8

1. Max has a drawer which contains forty unmatched socks of identical style, with equal numbers of white, blue, grey and brown socks. Max reaches into the drawer in the dark to retrieve a pair of socks. What is the minimum number of socks that Max must take from the drawer, to ensure that his selection includes a pair of matching socks of any colour?
   A. 3
   B. 4
   C. 5
   D. 6

2. A particle moves along the x-axis such that its displacement (x metres) from the origin at a time t seconds is given by the equation:

$$x = 5t^2 - \frac{4}{t^5}.$$

   What is the acceleration (in m/s^{-2}) of the particle when $t = 1$?
   A. -120
   B. -110
   C. 0
   D. 30

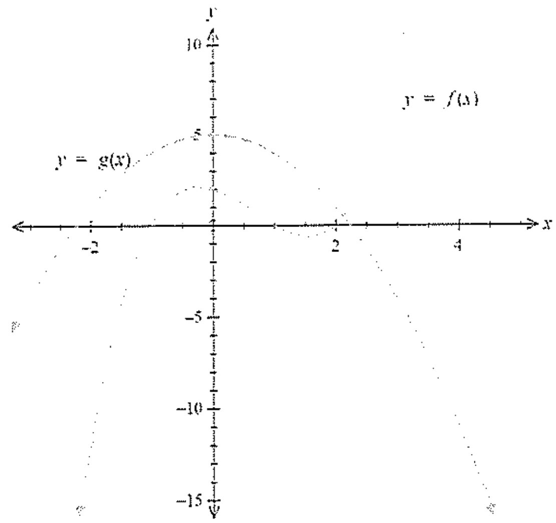
3. What is the expression for $\frac{dr}{dt}$ if $V = \frac{4}{3}\pi r^3$ and $\frac{dV}{dt} = 2$?

A.	$\frac{1}{2\pi r^2}$	C.	$8\pi r^2$
B.	$\frac{32\pi}{3}$	D.	16π

4. A population grows according to the equation $N(t) = 120e^{0.025t}$, where t is measured in months.
   By how many individuals, does the population grow over the first 3 years?

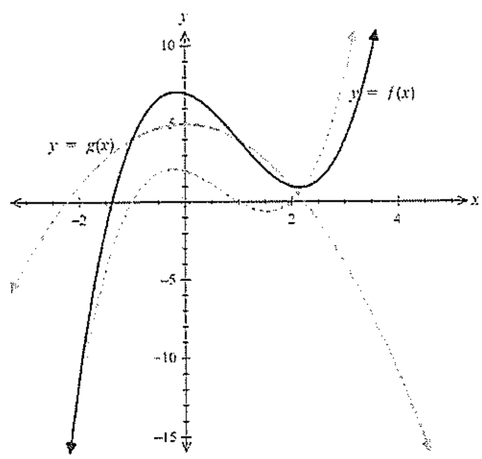
   A. 120
   B. 125
   C. 175
   D. 295

5. The graphs of $y = f(x)$ and $y = g(x)$ are shown on the grid below.

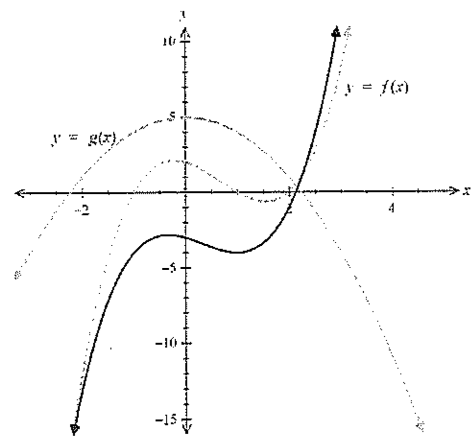


In which diagram has the graph of $y = f(x) + g(x)$ been drawn on the same grid?

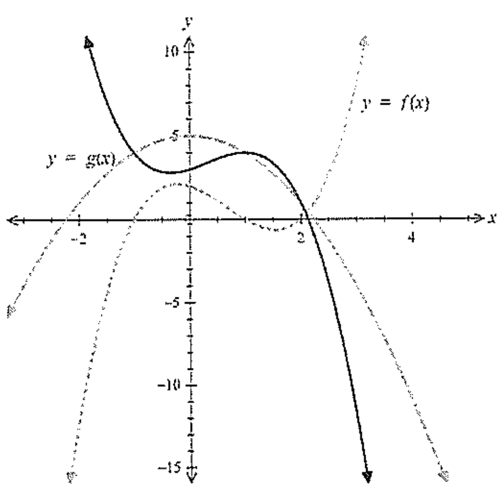
A.



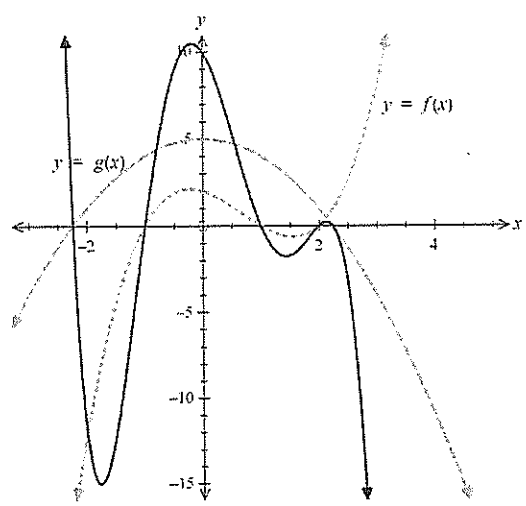
B.



C



D.



6. Given that $\cos^{-1}\left(\sqrt{1-t^2}\right) = \alpha$, which expression represents $\sin \alpha$.

A. $\sqrt{1-t^2}$
B. $\sqrt{1+t^2}$
C. 1
D. t

7. The expansion of $\cos(2x + 60°)$ equals:

A. $\frac{\cos 2x}{2} + \frac{\sqrt{3}\sin 2x}{2}$
B. $\frac{\cos 2x}{2} - \frac{\sqrt{3}\sin 2x}{2}$
C. $\frac{\sqrt{3}\cos 2x}{2} + \frac{\sin 2x}{2}$
D. $\frac{\sqrt{3}\cos 2x}{2} - \frac{\sin 2x}{2}$

8. What is the multiplicity of the root $x = 1$ if $P(x) = 3x^5 - 5x^4 + 5x - 3$?

A. 1
B. 2
C. 3
D. 4

Section 2

Question 9 **(5 marks)**

There are 8 girls and 7 boys in a class.

(a) The class elects a captain and vice-captain. In how many ways is this possible? **1**

~~8 girls available, 2 to select~~

15 students available, 2 students to select (order.

$${}^{15}P_2 = \frac{15!}{(15-2)!} = 14 \times 15$$

$$= 210$$

(b) The class elects four representatives for the student council. How many different groups of representatives are possible? **2**

15 students available, 4 students to select

$${}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1}$$

∴ There are 1365 ways to select.

(c) How many groups of representatives for the student council are possible, if the class decide that they want two girls and two boys in the group? **2**

2 girls from 8 girls

2 boys from 7 boys

$${}^{8}C_2 \times {}^{7}C_2 = 588$$

Question 10 (3 marks) Marks

The equation $x^3 + 2x^2 + 3x + 6 = 0$ has roots α, β and γ. Find the value of 3

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

$x^3 + 2x^2 + 3x + 6 = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{3}{1} = 3$ ✓ 1

$\gamma\beta\gamma = -\frac{d}{a} = -\frac{6}{1} = -6$ ✓ 1

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓ 1

$= \frac{-3}{6} = -\frac{1}{2}$

Question 11 (3 marks)

Consider the polynomial $P(x) = 2x^3 - x^2 + ax + 6$ where a is a real number.

Let $(x - 1)$ be a factor of $P(x)$.

(a) What is the value of a? 2

$P(1) = 2 - 1 + a + 6$ ✓ 1

$0 = 1 + 6 + a$

$-7 = a$ ✓ 1

(b) Find the remainder when $P(x)$ is divided by $(2x + 1)$. 1

$x = \frac{1}{2} = 0.5$ Remainder theorm

$P(0.5) = 2(0.5)^3 - (0.5)^2 + a(0.5) + 6$

$= 9$

Question 12 (3 marks) 3

Solve: $\frac{2x-4}{3-x} \geq 2$.

$\frac{2x-4}{3-x} \times (3-x)^2 \geq 2(3-x)^2$

$(3-x) \times (2x-4) \geq 2(9-6x+x^2)$

$6x-12-2x^2+4x \geq 18-12x+2x^2$

$22x-30-4x^2 \geq 0$

$2x^2-11x+15 \leq 0$

$2x^2-6x-5x+15 \leq 0$

$2x(x-3)-5(x-3) \leq 0$

$(2x-5)(x-3) \leq 0$

$2\frac{1}{2} \leq x < 3$

Question 13 (3 marks) 3

Find the number of ways in which 3 males and 3 females can be arranged in a line so that the two end positions are occupied by males and no two males are next to each other.

Possible arrangements

MFMFFM or MFFMFM

$2 \times 3! \times 3! = 72$

Question 14 (5 marks)

For the function $f(x) = 4\sin^{-1}\left(\frac{x-5}{2}\right)$.

(a) Give the domain and range of the function. **2**

Domain

$-1 \leq \frac{x-5}{2} \leq 1$

$-2 \leq x-5 \leq 2$

$3 \leq x \leq 7$

Domain $[3, 7)$

Range

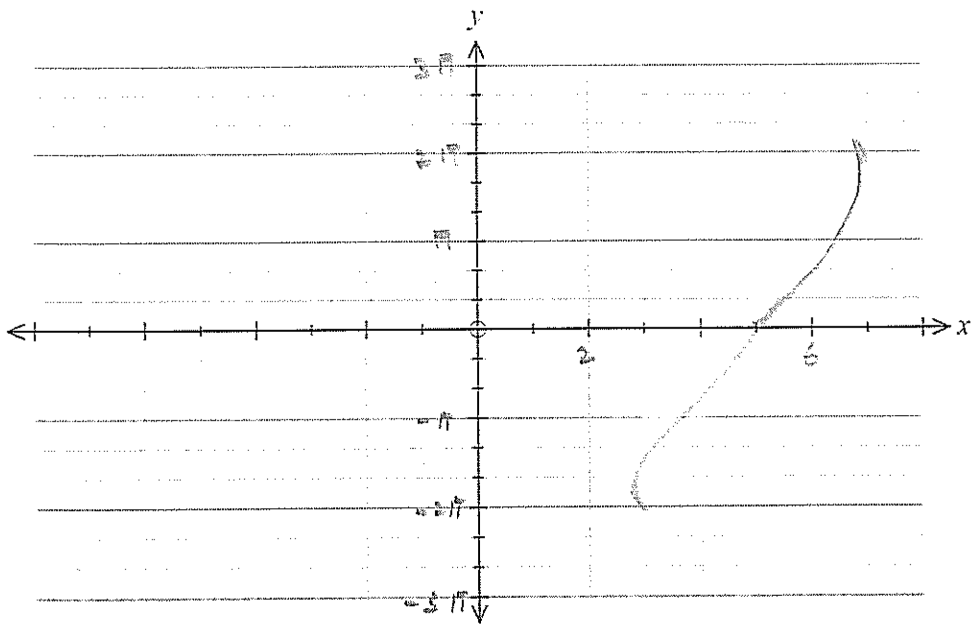
$-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x-5}{2}\right) \leq \frac{\pi}{2}$

$4\left(-\frac{\pi}{2}\right) \leq 4\sin^{-1}\left(\frac{x-5}{2}\right) \leq 4\left(\frac{\pi}{2}\right)$

$-2\pi \leq y \leq 2\pi$

Range $[-2\pi, 2\pi]$

(b) Draw a sketch of the function. **3**



Question 15 (6 marks)

The number of bacteria N a person has after being infected with a virus after t hours is given by:

$$N = 10\,000e^{0.05t}$$

(a) Find the number of bacteria after 10 hours. **1**

$N = 10{,}000\,e^{0.05 \times 10}$

$= 16487.2127$

$= 16487$

(b) Find the time required for the number of bacteria to reach 100 000. Answer to the nearest hour. **3**

$N = 10{,}000\,e^{0.05 \times t}$

$100.000 = 10.000\,e^{0.05t}$

$e^{0.05t} = 10$

$0.05t = \log_e 10$

$t = \dfrac{\log_e 10}{0.05}$

$= 46.0517$ **2**

$= 46$ hrs

(c) What is the rate at which bacteria is increasing after one day?

$\dfrac{dN}{dt} = 10000 \times 0.05 \times e^{0.05t}$

$= 500\,e^{0.05t}$

Rate after 24 hrs

$\dfrac{dN}{dt} = 500\,e^{0.05 \times 24}$

$= 1660.0584$

$= 1660$ bacteria per hour.

Question 16 (3 marks) 3

Show that $\frac{cos3\beta}{sin\beta} + \frac{sin3\beta}{cos\beta} = 2cot2\beta$

$$\frac{\cos 3\beta \cos\beta + \sin 3\beta \sin\beta}{\sin\beta \cos\beta}$$

$$\frac{\cos(3\beta - \beta)}{\frac{1}{2}\sin 2\beta}$$

$$2\,\frac{\cos 2\beta}{\sin 2\beta}$$

$$= 2\cot 2\beta.$$

Question 17 (3 marks)

Find the Cartesian equation that is represented by the pair of parametric equations below:

$$x = 2p + 3 \quad \ldots (1)$$
$$y = p^2 - 6p + 9 \quad \ldots (2)$$

$x = 2p + 3$

$y = p^2 - 6p + 9$

Factorise (2)

$y = (p-3)^2 \rightarrow (3)$

$x - 3 = 2p$

$\frac{x-3}{2} = p$

$\frac{x-3}{2} - 3 = p - 3$

$p - 3 = \frac{x-9}{2}$

substitute in (3)

$y = \left(\frac{x-9}{2}\right)^2$

or $y = \frac{(x-9)^2}{4}$

$4y = (x-9)^2$

$4y = x^2 - 18x + 81$

Question 18 (6 marks)

(a) Use the substitution $t = \tan\frac{\theta}{2}$ to show that the equation $2\sin\theta - 3\cos\theta = 1$ is equivalent to the equation $t^2 + 2t - 2 = 0$. 3

$2\sin\theta - 3\cos\theta = 2 \times \frac{2t}{1+t^2} - 3 \times \frac{1-t^2}{1+t^2}$

$= \frac{3t^2 + 4t - 3}{1+t^2}$

as $2\sin\theta - 3\cos\theta = 1$

$\frac{3t^2 + 4t - 3}{1+t^2} = 1$

$3t^2 + 4t - 3 = 1 + t^2$

$2t^2 + 4t - 4 = 0$

$t^2 + 2t - 2 = 0$

(b) Hence, solve the equation $2\sin\theta - 3\cos\theta = 1$ for $0 \le \theta \le 360°$.
Answer correct to the nearest degree. 3

$t^2 + 2t - 2 = 0$

$(t+1)^2 = 3$

$t = -1 \pm \sqrt{3}$

$t = \tan\frac{\theta}{2}$ for $0 < \theta < 90^\circ$

$\tan\frac{\theta}{2} = -1 \pm \sqrt{3}$

$\frac{\theta}{2} = 36.2060$ or $\frac{\theta}{2} = 110.1039$

$\theta = 72.4120$ or $\theta = 220.20$

72° $\quad$ 220°

Question 19 (3 marks) 3

Show that: $2\sin(A+B)\cos(A-B) = \sin 2A + \sin 2B$.

L.H.S $2(\sin A\cos B + \cos A\sin B)(\cos A\cos B + \sin A\sin B)$

$2(\sin A\cos A\cos^2 B + \sin B\cos B\sin^2 A + \sin B\cos B\cos^2 A + \sin A\cos A\sin^2 B)$

$= 2\sin A\cos B(\sin^2 B + \cos^2 B) + 2\sin B\cos B(\sin^2 A + \cos^2 A)$

$\sin 2A + \sin 2B$

$=$ RHS

Question 20 (7 marks)

A particle moves along the x-axis such that its displacement (x metres) from the origin at a time t seconds is given by the equation:

$$x = 2t^3 - 15t^2 + 24t\ .$$

(a) The motion of the particle begins at the origin when $t = 0$. After how many seconds does the particle first return to the origin? (Answer to the nearest tenth of a second). **2**

$x = 2t^3 - 15t^2 + 24t$

$= t(2t^2 - 15t + 24)$

when $x = 0$, $t = 0$

$2t^2 - 15t + 24 = 0$ ✓

$t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{+15 \pm \sqrt{15^2 - 4 \times 2 \times 24}}{2 \times 2}$

$= \dfrac{15 \pm \sqrt{33}}{4} \Rightarrow 2.3$ or 5.2

Ans is first time 2.3 ✓

(b) Find the times at which the particle is at rest. **3**

$x = 2t^3 - 15t^2 + 24t$

$\dfrac{dx}{dt} = 6t^2 - 30t + 24$ ✓

$\dfrac{dx}{dt} = 0$

$6t^2 - 30t + 24 = 0$ ✓

$6(t^2 - 5t + 4) = 0$

$6(t-4)(t-1) = 0$ ✓

$t = 1,\ t = 4$

(c) Find the initial acceleration of the particle. **2**

$\dfrac{dx}{dt} = 6t^2 - 30t + 24$

$\dfrac{d^2x}{dt^2} = 12t - 30$ ✓

when $t = 0$

$\dfrac{d^2x}{dt^2} = -30\ ms^{-2}$ ✓

7

Question 21 (3 marks)

The displacement of a particle at time t (in seconds) is given by: 3

$$x = 3e^{-2t} + 4e^{-t} + 2t$$

Find the exact time at which the particle comes to rest.

$x = 3e^{-2t} + 4e^{-t} + 2t$

$\frac{dx}{dt} = -6e^{-2t} - 4e^{-t} + 2 = 0$ ✓

particle comes to rest when $\frac{dx}{dt} = 0$

$-6e^{-2t} - 4e^{-t} + 2 = 0$

$-2(3e^{-2t} + 2e^{-t} - 1) = 0$

$(3e^{-t} - 1)(e^{-t} + 1) = 0$ ✓

but $e^{-t} + 1 \neq 0$

hence $3e^{-t} - 1 = 0$

$3e^{-t} = 1$

$e^{-t} = \frac{1}{3}$

$t = -\log_e \frac{1}{3}$

$= \log_e 3$ ✓

$= 1.09$ seconds

(2)

Question 22 (8 marks)

Consider the function $f(x) = (x + 2)^2 - 5$.

(a) What is the largest domain of $f(x)$, including x = 0 that has an inverse? **1**

Vertex form is $(x-h)^2 + k \Rightarrow$ Vertex $= (-2, -5)$

thus $x \geq -2$ is domain

that includes $x = 0$ and has

an inverse

(b) Find the equation of the inverse function $f^{-1}(x)$. **2**

$x = (y+2)^2 - 5$

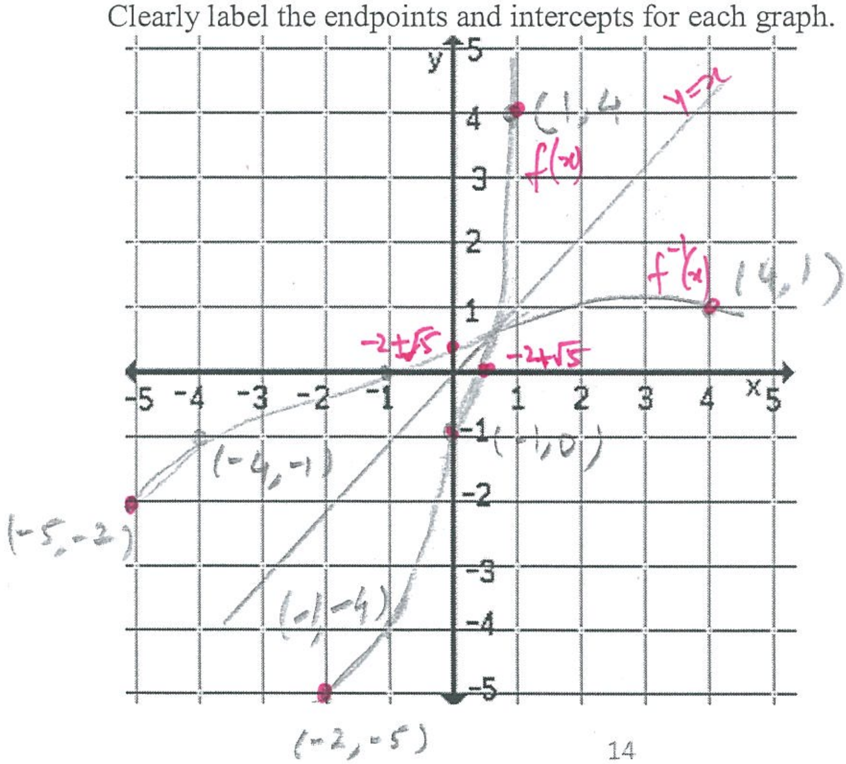
$x + 5 = (y+2)^2$

$y + 2 = \pm\sqrt{x+5}$ only positive case as $x \geq -2$

$y = -2 + \sqrt{x+5}$

(c) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same number plane. **3**

Clearly label the endpoints and intercepts for each graph.



$(x+2)^2 - 5 = 0$

$x + 2 = \pm\sqrt{5}$

$x = -2 \pm \sqrt{5}$

1 mk for shape.

1 mk for x-y intercepts

1 mk for end-pts.

6

(d) What is the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$? Answer in simplest exact form. **2**

graph intersect on the line $y = x$

$x = (x+2)^2 - 5$

$x = x^2 + 4 + 4x - 5$

$x^2 + 4x - x + 4 - 5 = 0$

$x^2 + 3x - 1 = 0$

$x = \dfrac{-(+3) \pm \sqrt{(+3)^2 - 4 \times (1) \cdot (-1)}}{2}$

$x = \dfrac{-3 \pm \sqrt{9+4}}{2} = \dfrac{-3 \pm \sqrt{13}}{2}$

as $x > 0$ $\quad x = \dfrac{-3 + \sqrt{13}}{2}$

Question 23 (6 marks)

(a) Show that $(1-x)^5\left(1+\frac{1}{x}\right)^5 = \left(\frac{1}{x} - x\right)^5$.

LHS $= \left((1-x)\left(1+\frac{1}{x}\right)\right)^5$

$= \left(1 + \frac{1}{x} - x - 1\right)^5$

$= \left(\frac{1}{x} - x\right)^5 =$ RHS.

(b) Write the expansion of $\left(\frac{1}{x} - x\right)^5$, leaving coefficients in the form $\binom{n}{r}$. **2**

$\left(\frac{1}{x} - x\right)^5 = \binom{5}{0}\left(\frac{1}{x^5}\right) - \binom{5}{1}\left(\frac{1}{x^4} \cdot x\right) + \binom{5}{2}\left(\frac{1}{x^3} \cdot x^2\right)$

$- \binom{5}{3}\left(\frac{1}{x^2} \cdot x^3\right) + \binom{5}{4}\left(\frac{1}{x} \cdot x^4\right) - \binom{5}{5} \cdot x^5$

$\left(\frac{1}{x} - x\right)^5 = \binom{5}{0}\left(\frac{1}{x^5}\right) - \binom{5}{1}\left(\frac{1}{x^3}\right) + \binom{5}{2}\left(\frac{1}{x}\right) - \binom{5}{3}(x) + \binom{5}{4}x^3$

$- \binom{5}{5} \cdot x^5$

(c) Determine the coefficient of x^3 in the expansion of $(1-x)^5\left(1+\frac{1}{x}\right)^5$ and hence show that $\binom{5}{4}\binom{5}{1}-\binom{5}{5}\binom{5}{2}-\binom{5}{3}\binom{5}{0}=\binom{5}{4}$. **2**

$= \left(\binom{5}{0} - \binom{5}{1}x + \binom{5}{2}x^2 - \binom{5}{3}x^3 + \binom{5}{4}x^4 - \binom{5}{5}x^5\right) \times$

$\left(\binom{5}{0} + \binom{5}{1}\frac{1}{x} + \binom{5}{2}\frac{1}{x^2} + \binom{5}{3}\frac{1}{x^3} + \binom{5}{4}\frac{1}{x^4} + \binom{5}{5}\frac{1}{x^5}\right)$

coefficients of x^3 come from $x^5 \times \frac{1}{x^2}$, $x^4 \times \frac{1}{x}$ and $x^3 \times 1$

Coefficients of $x^3 = -\binom{5}{5}\binom{5}{2} + \binom{5}{4}\binom{5}{1} - \binom{5}{3}\binom{5}{0}$

Equating with coefficients of x^3 from other expansion

End of Paper

$-\binom{5}{5}\binom{5}{2} + \binom{5}{4}\binom{5}{1} - \binom{5}{3}\binom{5}{0} = \binom{5}{4}$

$\Rightarrow \binom{5}{4}\binom{5}{1} - \binom{5}{5}\binom{5}{2} - \binom{5}{3}\binom{5}{0} = \binom{5}{4}$

2