CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

HSC Trial Examination 2020

Mathematics Extension 2

General Instructions **Student Number:**____________________________

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Section 1

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10

Q.1 Consider the following statement for $n \in Z$:

If $n^2 + 4n + 1$ is even, then n is odd.

Which of the following statements is the contrapositive of this statement for $n \in Z$?

(A) If n is even, then $n^2 + 4n + 1$ is odd.

(B) If $n^2 + 4n + 1$ is odd, then n is even.

(C) If n is odd, then $n^2 + 4n + 1$ is even.

(D) If $n^2 + 4n + 1$ is even, then n is even.

Q.2 Which of the following expressions is equal to $\int x^2 e^{-x} dx$?

(A) $-x^2 e^{-x} + \int 2xe^{-x} dx$

(B) $-2xe^{-x} - \int 2xe^{-x} dx$

(C) $-x^2 e^{-x} - \int 2xe^{-x} dx$

(D) $-2xe^{-x} + \int 2xe^{-x} dx$

Q.3 Which of the following expressions is the partial fraction form of the algebraic fraction

$\frac{x-4}{(x-3)^2(x^2+2)}$?

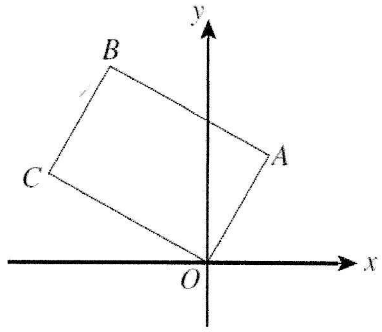
(A) $\frac{A}{(x-3)^2} + \frac{B}{x^2+2}$

(B) $\frac{A}{(x-3)^2} + \frac{Bx+C}{x^2+2}$

(C) $\frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{x^2+2}$

(D) $\frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+2}$

Q.4 The Argand diagram shows the rectangle OABC where OC = 2OA. Vertex A corresponds to the complex number w.



Which of the following complex numbers corresponds to vertex *C*?

(A) $-2iw$

(B) $2iw$

(C) $-2w$

(D) $2w$

Q.5 Which of the following is the complex number $4\sqrt{3} - 4i$?

(A) $4e^{-\frac{i\pi}{6}}$

(B) $4e^{\frac{5\pi}{6}}$

(C) $8e^{-\frac{i\pi}{6}}$

(D) $8e^{\frac{5\pi}{6}}$

Q.6 A particle moves in simple harmonic motion along the x-axis about the origin. Initially, the particle is at its extreme positive position. The amplitude of the motion is 12 metres and the particle returns to its initial position every 3 seconds.

What is the equation for the position of the particle at time t seconds?

(A) $x = 12\cos\frac{2\pi t}{3}$

(B) $x = 24\cos\frac{2\pi t}{3}$

(C) $x = 12\cos 3t$

(D) $x = 24\cos 3t$

Q.7 What is the Cartesian equation of a sphere with centre $\underset{\sim}{c} = -3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ that passes through $\underset{\sim}{a} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$?

(A) $(x-3)^2 + (y+1)^2 + (z-2)^2 = 7$

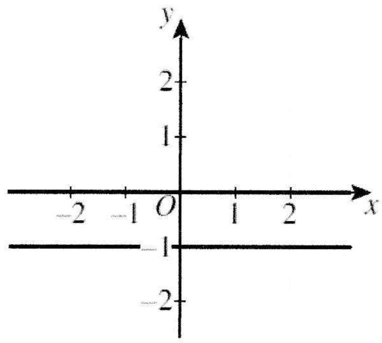
(B) $(x+3)^2 + (y-1)^2 + (z+2)^2 = 7$

(C) $(x-3)^2 + (y+1)^2 + (z-2)^2 = 49$

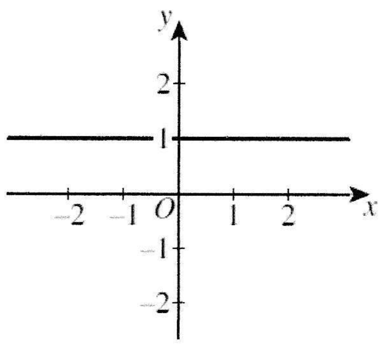
(D) $(x+3)^2 + (y-1)^2 + (z+2)^2 = 49$

Q.8 Which of the following diagrams shows the subset of the complex plane satisfied by the relation $i\bar{z} - iz = 2$?

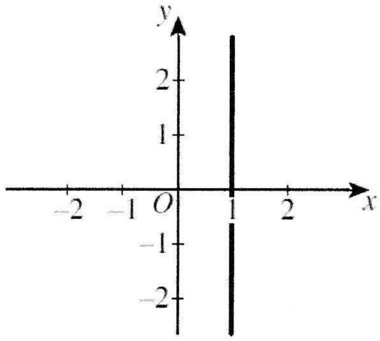
(A)



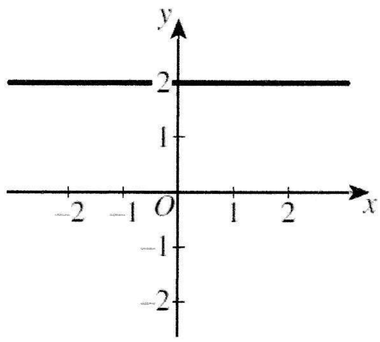
(B)



(C)



(D)



Q.9 A particle is moving along a straight line. At time t, its velocity is v and its displacement from a fixed origin is x.

If $\dfrac{dv}{dx} = \dfrac{1}{2v}$, which of the following best describes the particle's acceleration and velocity?

(A) constant acceleration and constant velocity

(B) constant acceleration and decreasing velocity

(C) constant acceleration and increasing velocity

(D) increasing acceleration and increasing velocity

Q.10 Which of the methods below would be suitable to use when proving the following statement?

"For any infinite sequence of integers, there will always be two numbers in that sequence that differ by a multiple of 5243."

(A) Proof by Contradiction

(B) Proof by Induction

(C) Proof using the Pigeonhole Principle

(D) Proof using Probability Theory

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the value of b given that $-i$ is a root of the equation $z^2 + bz + (1 - i) = 0$. **2**

(b) Consider the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = m\underset{\sim}{i} + n\underset{\sim}{j}$.

(i) Find the values of m and n such that $(\underset{\sim}{a} + \underset{\sim}{c})$ is parallel to $\underset{\sim}{b}$. **2**

(ii) Find all values of m and n such that $\underset{\sim}{c}$ is a unit vector perpendicular to $\underset{\sim}{a}$. **2**

(c) Using the substitution $u = 1 - \sin 2x$, evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 - \sin 2x}\,(1 - 2\cos^2 x)\,dx$. **4**

(d) Let $z = \sqrt{3} + i$.

(i) Express z in modulus-argument form. **2**

(ii) Find the smallest positive integer n such that $z^n - \overline{z}^n = 0$. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider two complex numbers, u and v, such that $\text{Im}(u) = 2$ and $\text{Re}(v) = -1$. **2**

Given that $u + v = -uv$, find the values of u and v.

(b) Solve the equation $\left|e^{2i\theta} + e^{-2i\theta}\right| = 1$ where $-\pi < \theta \leq \pi$. **3**

(c) Given that $y = \dfrac{1}{1+x}$, prove by mathematical induction that $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^n n!}{(1+x)^{n+1}}$ for all positive integers n. **3**

(d) A subset of the complex plane is described by the relation $\text{Arg}(z - 2i) = \dfrac{\pi}{6}$.

(i) Show that the Cartesian equation of this relation is $y = \dfrac{1}{\sqrt{3}}x + 2, x > 0$. **2**

(ii) Draw a sketch of this relation. **1**

(iii) Given that z is a complex number that satisfies the relation $\text{Arg}(z - 2i) = \dfrac{\pi}{6}$, find the least possible exact value of $|z - 3 + i|$. **4**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Prove that $\log_2 5$ is an irrational number. **3**

(b) Relative to a fixed origin, O, a particle is moving in a straight line with simple harmonic motion of period $\frac{2\pi}{n}$ seconds and amplitude a metres. Initially, the particle is $\frac{a}{2}$ metres from O and is moving away from O.

(i) Find an expression for the particle's displacement, x, at time t. Give your answer in the form $x = a\sin(nt + \alpha)$. **1**

(ii) Find the time when the particle will first reach an extreme position. **1**

(iii) The particle has speed V m s^{-1} when it is $\frac{a}{3}$ metres from an extreme position. **2**

Find, in terms of V, the particle's maximum speed.

(c) Consider two lines, l_1 and l_2, with vector equations $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ respectively.

(i) Find $\underset{\sim}{r}_1$, the vector equation of l_1, in the direction of $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and passing through the point $(-1, 2, -3)$. **1**

The line l_2 has the vector equation $\underset{\sim}{r}_2 = (-t + 1)\underset{\sim}{i} + (2t - 2)\underset{\sim}{j} + (3t + 6)\underset{\sim}{k}$ where $t \in R$.

(ii) Find a vector parallel to l_2. **1**

(iii) Find the point of intersection of l_1 and l_2. **3**

(iv) Find the acute angle between l_1 and l_2. Give your answer in degrees correct to one decimal place. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) If $z = x + iy$ and $e^{iz} = 3i$

(i) Find expressions for $|e^{iz}|$ and $\arg(e^{iz})$ in terms of x and y. **1**

(ii) Hence find all z. **3**

(b) For d, an integer where $d > 1$,

(i) Show that $\frac{1}{d^2} < \frac{1}{d(d-1)}$

1

(ii) Noting that $\frac{1}{d^2 - d} = \frac{1}{d-1} - \frac{1}{d}$ show that, for a positive integer n : **3**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\dots\dots + \frac{1}{n^2} < 2.$$

(c) Consider the equation $z^5 = (z+1)^5$ where $z \in C$.

(i) Explain why this equation does NOT have five roots. **1**

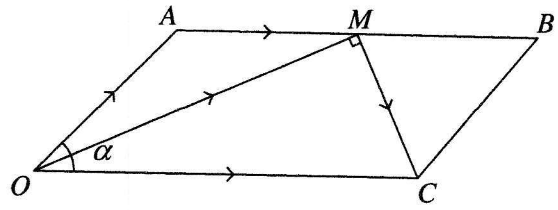
(ii) Solve $z^5 = (z+1)^5$, giving your answer in the form $a + bi\cot\theta$. **4**

(iii) Describe the geometrical relationship between the roots of the equation $iz^5 = (iz+1)^5$ and the roots of the equation $z^5 = (z+1)^5$. **2**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows parallelogram $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{b}$ and $\left|\overrightarrow{OC}\right| = 2\left|\overrightarrow{OA}\right|$. The angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is α.

M is a point on AB such that $\overrightarrow{AM} = k\overrightarrow{AB}$, $0 \le k \le 1$ and $\overrightarrow{OM} \cdot \overrightarrow{MC} = 0$.



(i) Use a vector method to show that $|\underset{\sim}{a}|^2(1-2k)(2\cos\alpha - (1-2k)) = 0$. **4**

(ii) Find the set of values for α such that there are two possible positions for M. **2**

(b) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{2}{3+5\cos x}\,dx$.

(i) Using the substitution $t = \tan\frac{x}{2}$, show that $I = \displaystyle\int_0^1 \frac{2}{4-t^2}\,dt$. **2**

(ii) Hence find the value of I. Give your answer in the form $\ln\sqrt{k}$ where k is a positive integer. **2**

Question 15 continues on next page

Question 15 (continued)

(c) A particle is projected from a point O above horizontal ground. At time t seconds, the particle's position vector is $\underset{\sim}{r} = gt\cos\theta\underset{\sim}{i} + \left(\frac{g}{4} + gt\sin\theta - \frac{g}{2}t^2\right)\underset{\sim}{j}$ where θ is the angle of projection and g is the acceleration due to gravity.

(i) The particle's time of flight is T seconds. **3**

Show that $T = \frac{1}{\sqrt{2}}(\sqrt{1-\cos 2\theta} + \sqrt{2-\cos 2\theta})$.

(ii) The particle's range is R metres. **1**

Show that $R = \frac{g}{2}(\sqrt{1-\cos^2 2\theta} + \sqrt{2+\cos 2\theta - \cos^2 2\theta})$.

(iii) The particle's maximum range occurs when $\cos 2\theta = \frac{1}{5}$. (Do NOT prove this.) **1**

Find the extra distance attained by projecting the particle at this angle rather than at an angle of 45°. Give your answer in the form $\frac{g}{2}(\sqrt{a} - \sqrt{b} - c)$ where a, b and c are positive integers.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int \frac{dx}{(x^m+1)^n}$, where m and n are positive integers.

(i) Using Integration by Parts, show that $I_{n+1} = \frac{x}{mn(x^m+1)^n} + \frac{mn-1}{mn} I_n$. **4**

(ii) Hence, using your result to part (i), find $\int \frac{dx}{(x^2+1)^3}$ **3**

(b) Consider two positive real numbers a_1 and a_2.

(i) Prove that $\frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2}$ **1**

Let $a_1, a_2, \dots, a_n$ be n positive real numbers.

If $a_1, a_2, \dots, a_n = 1$ then $a_1 + a_2 + \dots + a_n \geq n$.(Do NOT prove this.)

(ii) Prove that $\frac{a_1 + a_2 + \dots + a_n}{n} \geq (a_1 a_2 \dots a_n)^{\frac{1}{n}}$ **3**

(iii) Hence prove that $2^n - 1 > n\sqrt{2^{n-1}}$ for integers $n \geq 1$. **4**

End of paper