Carlingford High School

2014

TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

NAME:_______________ CLASS:______ TEACHER:_______________

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total Marks – 70

Section I Pages 2 – 4

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 5 – 10

60 marks

- Attempt Questions 11 – 14
- Allow about 1hour and 45 minutes for this section

	MC	Q11	Q12	Q13	Q14	TOTAL
MC	/10					/10
HE3			/5		/7	/12
HE4		/3	/4	/5		/12
HE5				/6		/6
HE6		/3				/3
HE7		/9	/6	/4	/8	/27
TOTAL	/10	/15	/15	/15	/15	/70

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet located on page 12 for Questions 1 – 10.

1. How many numbers greater than 5 000 can be formed with the digits 2, 4, 5, 6 and 9 if a digit cannot occur more than once in any number?

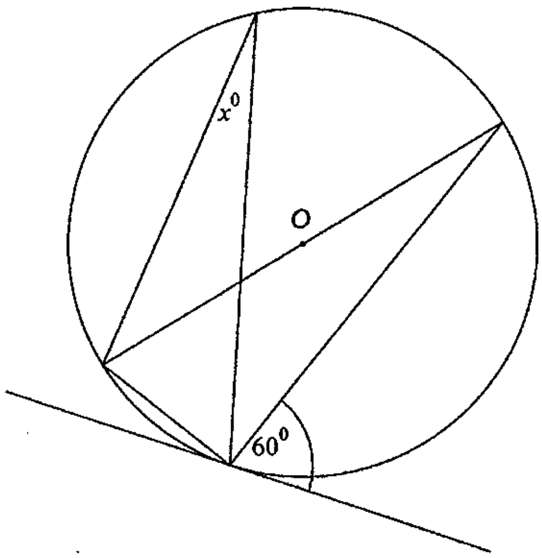
(A) 72

(B) 120

(C) 144

(D) 192

2. Find the value of x :



(A) $30°$

(B) $45°$

(C) $60°$

(D) $90°$

3. Find the derivative of $e^{\cos x}$

(A) $e^{\cos x}$

(B) $e^{\sin x}$

(C) $-\sin x\, e^{\cos x}$

(D) $\sin x\, e^{\cos x}$

4. The expansion needed to show $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ is:

(A) $\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

(B) $\sin(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

(C) $\sin(100^\circ - 25^\circ) = \sin 100^\circ \cos 25^\circ + \cos 100^\circ \sin 25^\circ$

(D) $\tan(45^\circ + 30^\circ) = \dfrac{\tan 45^\circ \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$

5. The Cartesian equation of the tangent to the parabola $x = t - 3$, $y = t^2 + 2$ at $t = -3$ is:

(A) $6x + y + 25 = 0$

(B) $6x + y + 36 = 0$

(C) $6x - y - 25 = 0$

(D) $6x + 2y - 25 = 0$

6. Using the substitution $u = 1 - x^3$, evaluate $\int_0^1 x^2\sqrt{1 - x^3}\, dx$.

(A) $-\frac{1}{9}$

(B) $\frac{1}{9}$

(C) $\frac{2}{9}$

(D) $\frac{1}{3}$

7. A particle moves in a straight line. Its position at any time t is given by

$x = 3\cos 2t + 4\sin 2t$.

The acceleration in terms of x is:

(A) $\ddot{x} = -3x$

(B) $\ddot{x} = -4x$

(C) $\ddot{x} = -16x^2$

(D) $\ddot{x} = -6\cos 2x + 8\sin 2x$

8. $\int \frac{dx}{\sqrt{1-3x^2}} =$

(A) $\left(\sin^{-1} 3x\right) + C$

(B) $\left(\tan^{-1} 3x\right) + C$

(C) $\frac{1}{\sqrt{3}}\left(\tan^{-1} \sqrt{3}\, x\right) + C$

(D) $\frac{1}{\sqrt{3}}\left(\sin^{-1} \sqrt{3}\, x\right) + C$

9. If α, β and γ are the roots of $x^3 - 7x^2 + 9x - 15 = 0$. Find the value of $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$.

(A) $-\frac{7}{15}$

(B) $\frac{7}{15}$

(C) $\frac{9}{15}$

(D) $\frac{15}{7}$

10. Using Newton's method once with a starting value of $x = 3$, find the approximate value of $\sqrt[3]{33}$.

(A) $2\frac{7}{9}$

(B) $3\frac{1}{5}$

(C) $3\frac{2}{9}$

(D) $3\frac{2}{3}$

Section II

60 marks

Attempt Questions 11 – 14.

Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A polynomial is given by $P(x) = x^3 + ax^2 + bx - 18$. Find the values of a and b if $(x + 2)$ is a factor of $P(x)$ and the remainder when $P(x)$ is divided by $(x - 1)$ is -24. **2**

(b) A and B are the points (1, 4) and (5, 2) respectively. Find the coordinates of the point M which divides the interval AB externally in the ratio 2:3. **1**

(c) Differentiate $y = \cos^{-1}(3x + 2)$ and state the values for which x is defined. **3**

(d) Find the volume of the solid of revolution formed when the curve $y = x^3 + 1$ is rotated about the y-axis from $y = 0$ to $y = a$. **3**

(e) Sketch $y = \frac{x-3}{x^2}$ showing all the main features including stationary points, inflexions and asymptotes. **3**

(f) Find the area bounded by the curve $y = \frac{1}{9 + x^2}$, the x axis and the lines $x = 0$ and $x = \sqrt{3}$ using the substitution $x = 3\tan\theta$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show $3\sin x\cos x = \frac{3}{2}\sin 2x$. **1**

(ii) Hence or otherwise, find the exact value of $\int_0^{\frac{\pi}{2}} 9\sin^2 x\cos^2 x\,dx$. **2**

(b) (i) Sketch the graph of $y = sin^{-1}(x^2)$ and state its domain and range. **3**

(ii) Solve the equation $sin^{-1}(x^2) = \frac{\pi}{6}$. **1**

(c) Greg designs a sky-diving simulator for a video game. He simulates the rate of change of the velocity of the skydivers as they fall by:

$\frac{dV}{dt} = -k(V-P)$, where k and P are constants.

The constant P represents the terminal velocity of the skydiver in the prone position which is 55 m/s.

(i) Show that $V = P + Ae^{-kt}$ is a solution of this differential equation. **1**

(ii) Initially the velocity of the skydiver is 0 m/s and the velocity after 10 seconds is 27 m/s. Find values for A and k. **2**

(iii) Find the velocity of the skydiver after 17 seconds. **1**

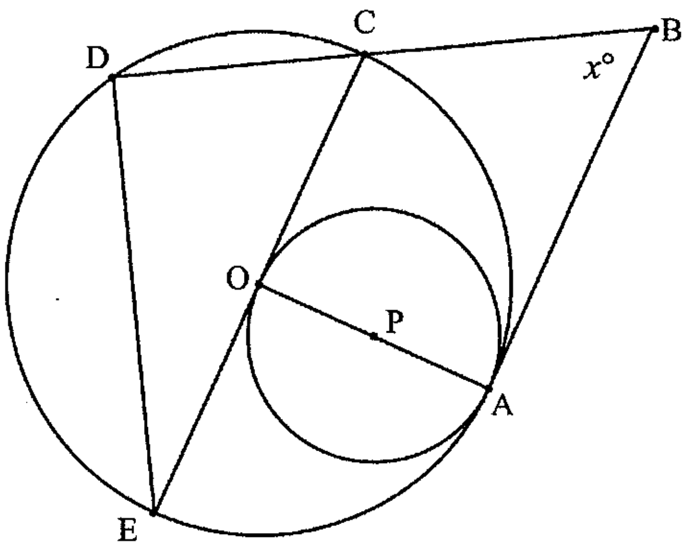
(iv) How long does it take the skydiver to reach a velocity of 50 m/s? **1**

Question 12 continues on page 7

Question 12 (continued)

(d) A circle, centre O, passes through the points A, C, D and E. **3**
Another circle, centre P, passes through the points A and O.
CE is a tangent to the circle centre P, with point of contact at O.
AB is a tangent to both circles with point of contact at A. OA is a diameter.
$\angle CBA = x°$.

Show that $\angle CED = (90 - x)°$



End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) By expressing $\cos x + \sin x$ in the form $r\sin(x+\alpha)$ solve the equation $\sin x + \cos x = 1$ for $0 \le x \le 2\pi$. **2**

(b) Consider the function $f(x) = \dfrac{e^x}{4+e^x}$.

(i) This function has no stationary points. Given that $f'(x) = \dfrac{4e^x}{\left(4+e^x\right)^2}$, find any points of inflexion. **2**

(ii) Explain why $f(x)$ has an inverse function. **1**

(iii) Find the inverse function $y = f^{-1}(x)$. **2**

(c) A particle moves on a line so that its distance from the origin at time t is x.

(i) Prove that $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ where v denotes velocity. **2**

(ii) If $\dfrac{d^2x}{dt^2} = -2x\left(x^2-20\right)$ and $v = 0$ *at* $x = 2$ find v^2 in terms of x. **3**

(iii) Is the motion simple harmonic? Why? **1**

(d) Prove by mathematical induction that **2**

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \frac{a\left(r^n - 1\right)}{r-1}$$

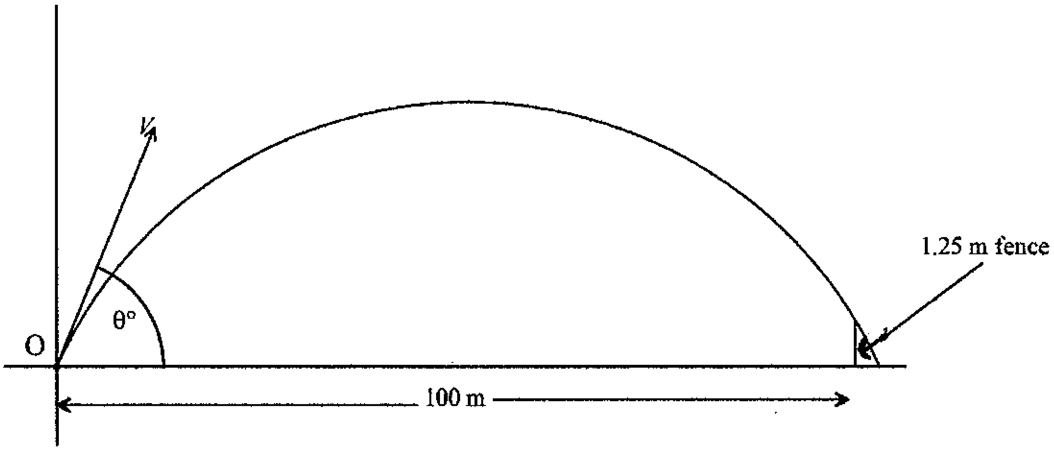
for all a and r, where n is a positive integer.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the general solutions of $sin^2x - cosx = 1$. **2**

(b) A ball is hit from the centre (O) of a cricket ground with a velocity of 34 m/s at an angle θ to the horizontal and towards a 1.25 metre high boundary fence which is 100 metres away.



(i) Derive the equations for horizontal and vertical displacement of the ball in flight. Air resistance may be neglected and acceleration can be taken as $-10 m/s^2$. **2**

(ii) Show the ball just clears the boundary fence when: **3**

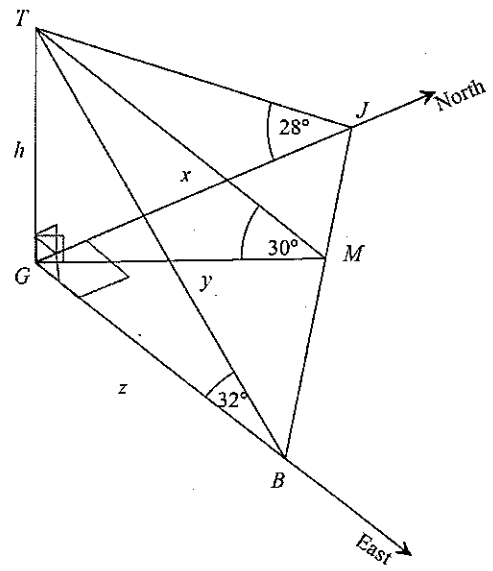
$$50000\tan^2\theta - 115600\tan\theta + 51445 = 0$$

(iii) Between what values does θ lie, for the ball to clear the boundary fence? **2**

Question 14 continues on page 10

Question 14 (continued)

(c) The Eiffel Tower (GT) is on flat ground in central Paris. Three friends Jordan, Maddy and Bella are observing the tower from a straight road on ground level. Jordan is due north of the tower, Bella is due east of the tower and Maddy is on the line of sight between Jordan and Bella. The angles of elevation to the summit of the tower from Jordan, Maddy and Bella are 28°, 30° and 32° respectively. The distances to the base of the tower from Jordan, Maddy and Bella are x, y and z respectively.



(i) Find expressions for $x, y, and\ z$. **1**

(ii) Find angle GJM to the nearest minute. **1**

(iii) Determine the bearing of Maddy from the base of the Eiffel Tower. **2**

(iv) Write an expression which is independent of h, for the ratio $\dfrac{MB^2}{JM^2}$. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Trial HSC Examination 2014

Mathematics Extension 1

Name ____________________ Teacher ________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✖ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ✖ B ✖ correct C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Ext 1 Trial 2014

Multiple Choice

1.	A ○	B ○	C ○	D ●
2.	A ●	B ○	C ○	D ○
3.	A ○	B ○	C ●	D ○
4.	A ●	B ○	C ○	D ○
5.	A ●	B ○	C ○	D ○
6.	A ○	B ○	C ●	D ○
7.	A ○	B ●	C ○	D ○
8.	A ○	B ○	C ○	D ●
9.	A ○	B ●	C ○	D ○
10.	A ○	B ○	C ●	D ○

Question 11		2014	
	Solution	Marks	Allocation of marks
(a)	$P(x)=x^3+ax^2+bx-18$ $P(-2)=0$ $P(1)=-24$ $P(-2)=0$ $0=(-2)^3+a(-2)^2+b(-2)-18$ $0=-8+4a-2b-18$ $0=4a-2b-26$............(1) $P(1)=-24$ $-24=1+a+b-18$ $0=a+b+7$...........(2) sub (2) into (1) $4(-7-b)-2b=26$ $-28-4b-2b=26$ $-28-6b=26$ $-6b=54$ $b=-9$ $a-9=-7$ $a=2$ $\therefore$ $a=2$ and $b=-9$	2	1 for finding the 2 equations 1 for solving the simultaneous equations
(b)	$x=\dfrac{mx_2+nx_1}{m+n}$ $=\dfrac{2\times 5+-3\times 1}{2-3}$ $=-7$ $y=\dfrac{my_2+ny_1}{m+n}$ $=\dfrac{2\times 2+-3\times 4}{2-3}$ $=8$ $\therefore$ pt $(-7, 8)$	1	
(c)	$y=\cos^{-1}(3x+2)$ Let $u=3x+2$ $\dfrac{du}{dx}=3$ $y=\cos^{-1}u$ $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$	3	
	$\dfrac{dy}{dx}=-\dfrac{1}{\sqrt{1-u^2}}\times 3$ $=-\dfrac{3}{\sqrt{1-(3x+2)^2}}$ $=-\dfrac{3}{\sqrt{-9x^2-12x-3}}$ $y=\cos^{-1}(3x+2)$is defined $-1\le 3x+2\le 1$ $-3\le 3x\le -1$ $-1\le x\le -\frac{1}{3}$ $\therefore \cos^{-1}(3x+2)$is defined for $-1\le x\le -\frac{1}{3}$		1 for using correct method to differentiate 1 correct differentiation 1 for stating defined values
(d)	$V=\pi\int_a^b x^2dy$ $y=x^3+1$ $y-1=x^3$ $x=\sqrt[3]{y-1}$ $x^2=(\sqrt[3]{y-1})^2$ $x^2=\left((y-1)^{\frac{1}{3}}\right)^2$ $x^2=(y-1)^{\frac{2}{3}}$ $V=\pi\int_0^a (y-1)^{\frac{2}{3}}dy$	3	1 for finding x^2

	$=\pi\left[\frac{3(y-1)^{\frac{5}{3}}}{5}\right]_0^a$ $=\pi\left[\frac{3(a-1)^{\frac{5}{3}}}{5}\right]-\left[-\frac{3}{5}\right]$ $=\frac{3\pi}{5}\left[\sqrt[3]{(a-1)^5}\ \ +1\right]$		1 for correct integration 1 for correct answer
(e)	$y=\frac{x-3}{x^2}$ $x^2\neq 0$ $\therefore\ x\neq 0$, vertical asymptote at $x=0$ $\lim\limits_{x\to\infty}\frac{x-3}{x^2}$ $=0$ $y'=\frac{x^2-2x(x-3)}{x^4}$ $=\frac{x^2-2x^2+6x}{x^4}$ $=\frac{-x^2+6x}{x^4}$ $=\frac{-x+6}{x^3}$ Stat pts $y'=0$ $0=-x+6$ $x=6$ When $x=6$ $y=\frac{1}{12}$ $\therefore$ Stat pt at $\left(6,\frac{1}{12}\right)$ $y''=\frac{-x-3x^2(-x+6)}{x^6}$ $=\frac{-x^3+3x^3-18x^2}{x^6}$ $=\frac{2x^3-18x^2}{x^6}$ $=\frac{2x-18}{x^4}$	3	1 for correct shape 1 for asymptote 1 for maximum Can use any valid method of sketching 1 or 2 can be awarded for finding some information

Inflexion $y''=0$

$0=2x-18$

$x=9$

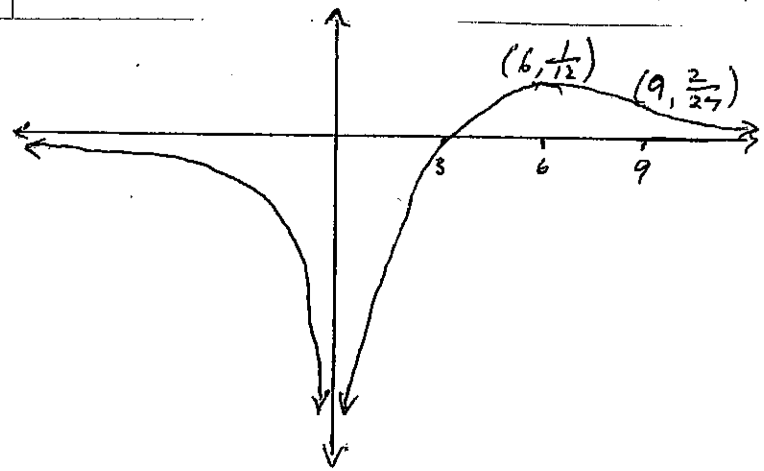
When $x=9$ $y=\frac{2}{27}$

Possible inflexion pt $\left(9,\frac{2}{27}\right)$ check concavity

Left $y''<0$ Right $y''>0$ change of concavity

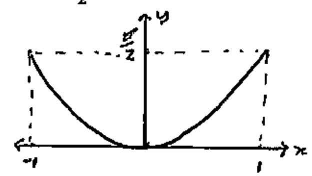
$\therefore$ inflexion pt $\left(9,\frac{2}{27}\right)$

Classify stat pt $\left(6,\frac{1}{12}\right)$ $y''<0$ concave down $\therefore$ Maximum pt



(f)	$y=\frac{1}{9+x^2}$ Let $x=3\tan\theta$ $\frac{dx}{d\theta}=3\sec^2\theta$ $y=\frac{1}{9+9\tan^2\theta}$ $=\frac{1}{9(1+\tan^2\theta)}$ $=\frac{1}{9\sec^2\theta}$	3	1 for correct substitution
	When $x=\sqrt{3}$ $\sqrt{3}=3\tan\theta$ $\tan\theta=\frac{\sqrt{3}}{3}$ $=\frac{1}{\sqrt{3}}$ $\theta=\frac{\pi}{6}$ When $x=0$ $0=3\tan\theta$ $\theta=0$ $\therefore\int_0^{\sqrt{3}}\frac{1}{9+x^2}\,dx$		1 for correct substitution and change of end points
	$=\int_0^{\frac{\pi}{6}}\frac{1}{9+x^2}\frac{dx}{d\theta}\times d\theta$ $=\int_0^{\frac{\pi}{6}}\frac{1}{9\sec^2\theta}\times 3\sec^2\theta\,d\theta$ $=\int_0^{\frac{\pi}{6}}\frac{1}{3}\,d\theta$ $=\left[\frac{1}{3}\theta\right]_0^{\frac{\pi}{6}}$ $=\left[\frac{\pi}{18}-0\right]$ $=\frac{\pi}{18}\text{ unit}^2$		1 for correct answer

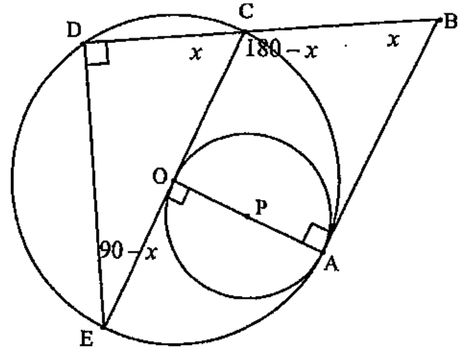
Question 12

	Solution	2014 Marks	Allocation of marks
(a)	(i) $3\sin x\cos x$ $=\sin x\cos x+2\sin x\cos x$ $=\frac{1}{2}\sin 2x+\sin 2x$ $=\sin 2x\left(\frac{1}{2}+1\right)$ $=\frac{3}{2}\sin 2x$	1	
	(ii) $\int_0^{\frac{\pi}{2}}9\sin^2 x\cos^2 x\,dx=\int_0^{\frac{\pi}{2}}(3\sin x\cos x)^2\,dx$ $=\int_0^{\frac{\pi}{2}}\frac{9}{4}\sin^2 2x\,dx$ $=\int_0^{\frac{\pi}{2}}\frac{9}{4}\times\left[\frac{1}{2}(1-\cos 4x)\right]dx$	2	1 for using part (i) or other method for changing form
	$=\int_0^{\frac{\pi}{2}}\frac{9}{8}(1-\cos 4x)dx$ $=\left[\frac{9}{8}x-\frac{9}{32}\sin 4x\right]_0^{\frac{\pi}{2}}$ $=\left[\frac{9\pi}{16}-\frac{9}{32}\sin 2\pi\right]-\left[0-\frac{9}{32}\sin 0\right]$ $=\frac{9\pi}{16}$		1 for correct integration
(b)	(i) D: $-1\le x\le 1$ R: $0\le y\le\frac{\pi}{2}$ 	1 1 1	
	(ii) $x^2=\sin\frac{\pi}{6}$ $=\frac{1}{2}$ $x=\pm\frac{1}{\sqrt{2}}$	1	

<table>
<tr><td>(c)</td><td>(i)
$V = P + Ae^{-kt}$
$\frac{dV}{dt} = -kAe^{-kt}$
but $Ae^{-kt} = V - P$
$\therefore \frac{dV}{dt} = -k(V-P)$
Or
$\frac{dV}{dt} = -kAe^{-kt}$
$= -k(P + Ae^{-kt} - P)$
$= -k(V-P)$</td><td>1</td><td>Either method is acceptable</td></tr>
<tr><td></td><td>(ii)
$V = 0 \quad P = 55 \quad t = 0$
$V = 55 + Ae^{-kt}$
Initially $t = 0 \quad v = 0$
$0 = 55 + Ae^{0}$
$A = -55$
$\therefore V = 55 - 55e^{-kt}$

When $t = 10 \quad V = 27$
$27 = 55 - 55e^{-10k}$
$55e^{-10k} = 28$
$e^{-10k} = \frac{28}{55}$
$-10k = \ln\left[\frac{28}{55}\right]$
$k = 0.067512867$</td><td>2</td><td>1 for A

1 for k</td></tr>
<tr><td></td><td>(iii)
When $t = 17$
$V = 55 - 55e^{-0.0675t}$
$V = 55 - 55e^{-0.0675 \times 17}$
$V = 37.5$m/s</td><td>1</td><td>Ignore any rounding either with k or the answer</td></tr>
</table>

<table>
<tr><td></td><td>(iv) $50 = 55 - 55e^{-0.0675 \times t}$
$e^{-0.0675 \times t} = \frac{5}{55}$
$t = \frac{\ln\left[\frac{5}{55}\right]}{-0.0675}$
$t = 35.5$ seconds
So it will take approximately 35.5 seconds.</td><td>1</td><td>1 for amount of time needed</td></tr>
<tr><td>(d)</td><td>

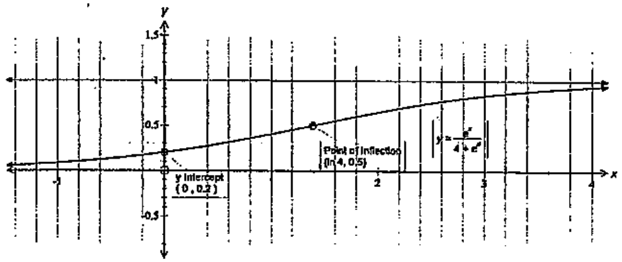
$\angle PAB = 90°$ (Angle between a tangent and radius)
$\angle POC = 90°$ (Angle between a tangent and radius)
$\therefore OC \parallel AB$ (cointerior angles are supplementary)
$\therefore \angle OCB = 180 - x°$ (Cointerior angles on $\parallel$ lines)
$\therefore \angle DCE = x°$ (angles on straight line)
$\angle EDC = 90°$ (angle in a semicircle is a right angle)
$\angle CED = 180° - 90° - x°$ (angle sum ΔCED)
$\therefore \angle CED = 90° - x°$</td><td>3</td><td>Alternative solutions possible, allocate marks as appropriate.

1 for showing parallel lines

1 for external angle $\angle DCE$.

1 for final answer</td></tr>
</table>

Question 13 — 2014

	Solution	Marks	Allocation of marks
a)	$\cos x + \sin x = r\sin(x+\alpha)$ $r = \sqrt{1^2+1^2}$ $r = \sqrt{2}$ $\tan\alpha = 1$ $\alpha = \frac{\pi}{4}$ $\cos x + \sin x = 1$ $\cos x + \sin x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right) = 1$ $\sin\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ Let $\theta = \left(x+\frac{\pi}{4}\right) \therefore \sin\theta = \frac{1}{\sqrt{2}}$ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$ $\therefore \quad \left(x+\frac{\pi}{4}\right) = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$ $x = 0, \frac{\pi}{2}, 2\pi$	2	1 for expressing in correct terms 1 for correct solutions
b)	(i) $f'(x) = \frac{4e^x}{(4+e^x)^2}$ $u = 4e^x \quad v = (4+e^x)^2$ $u' = 4e^x \quad v' = 2e^x(4+e^x)$ $f''(x) = \frac{4e^x(4+e^x)^2 - 8e^{2x}(4+e^x)}{((4+e^x)^2)^2}$ $f''(x) = \frac{(4+e^x)(4e^x(4+e^x) - 8e^{2x})}{(4+e^x)^4}$ $f''(x) = \frac{(4+e^x)(16e^x + 4e^{2x} - 8e^{2x})}{(4+e^x)^4}$ $f''(x) = \frac{4e^x(4-e^x)}{(4+e^x)^3}$	2	1 for finding the second derivative
	For inflexion points $f''(x) = 0$ $4 - e^x = 0$ $e^x = 4$ $x = \ln 4$ When $x = \ln 4$ $y = \frac{4}{4+4}$ $y = \frac{1}{2}$ $\therefore$ possible inflexion $= \left(\ln 4, \frac{1}{2}\right)$ Test $f''(1) > 0$ $f''(2) < 0$ $\therefore f''(x)$ changes sign $\therefore$ Inflexion at $= \left(\ln 4, \frac{1}{2}\right)$		1 for the inflexion with test
	(ii) $f(x)$ has an inverse because it is an increasing function. ie $f'(x) > 0$ for all x. If you check it graphically with a horizontal line test, it will only cut the function once. Therefore if you reflect the graph in the line $y=x$ it will pass the vertical line test. 	1	1 any valid explanation is acceptable, with or without a graph/sketch

	(iii) $f(x)=\dfrac{e^x}{4+e^x}$ ie $y=\dfrac{e^x}{4+e^x}$ Inverse $x=\dfrac{e^y}{4+e^y}$ $4x+xe^y=e^y$ $e^y(1-x)=4x$ $e^y=\dfrac{4x}{1-x}$ $y=\ln\left(\dfrac{4x}{1-x}\right)$	2	1 for interchanging x and y 1 for correct rearrangement for y
(c)	(i) $\dfrac{d^2x}{dt^2}=\dfrac{d}{dt}\left(\dfrac{dx}{dt}\right)=\dfrac{d}{dt}v=\dfrac{dv}{dt}$ $=\dfrac{dv}{dx}\times\dfrac{dx}{dt}$ $=\dfrac{dv}{dx}\times v$ $=v\dfrac{dv}{dx}$ $=\dfrac{d}{dv}\left(\dfrac{1}{2}v^2\right)\times\dfrac{dv}{dx}$ $=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$	2	Any acceptable proof may be used 1 1
	(ii) Since $\dfrac{d^2x}{dt^2}=-2x(x^2-20)$ $=40x-2x^3$ and $\dfrac{d^2x}{dt^2}=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ $\therefore \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)=40x-2x^3$ $\therefore \quad \dfrac{1}{2}v^2=\int 40x-2x^3\,dx$ $\dfrac{1}{2}v^2=20x^2-\dfrac{x^4}{2}+C$ When $v=0$ $x=2$ $\therefore \quad 0=20\times 2^2-\dfrac{2^4}{2}+C$ $\therefore \quad C=-72$	3	1 1 for correct integration
	$\therefore \dfrac{1}{2}v^2=20x^2-\dfrac{x^4}{2}-72$ $v^2=40x^2-x^4-144$ $=-(x^4-40x^2+144)$ $=-(x^2-36)(x^2-4)$		1 for expressing v^2 in terms of x

	(iii) The particle oscillates between the points x= 2 and x=6, this however is not simple harmonic motion as $\ddot{x}=-2x(x^2-20)$ Is not in the form $\ddot{x}=-n^2x$	1	1 for stating not in correct form for SHM
(d)	$a+ar+ar^2+ar^3+\ldots+ar^{n-1}=\dfrac{a(r^n-1)}{r-1}$ Step 1 Prove true for $n=1$ $LHS=a$ $RHS=\dfrac{a(r^1-1)}{r-1}$ $=a$ $\therefore LHS=RHS$	2	1 for step 1 and 2
	Step 2 Assume true for $n=k$ $a+ar+ar^2+ar^3+\ldots ar^{k-1}=\dfrac{a(r^k-1)}{r-1}$ Step 3 Prove true for $n=k+1$ ie Prove that $a+ar+ar^2+ar^3+\ldots ar^{k-1}+ar^{(k+1)-1}=\dfrac{a(r^{k+1}-1)}{r-1}$ $LHS=a+ar+ar^2+ar^3+\ldots ar^{k-1}+ar^{(k+1)-1}$ $=\dfrac{a(r^k-1)}{r-1}+ar^k$ $=\dfrac{(ar^k-a+ar^{k+1}-ar^k)}{r-1}$ $=\dfrac{ar^{k+1}-a}{r-1}$ $=\dfrac{a(r^{k+1}-1)}{r-1}$ $=RHS$ Therefore if true for $n=k$, also true for $n=k+1$. since true for $n=1$, by induction it is true for all positive integral values of n.		1 mark for Step 3

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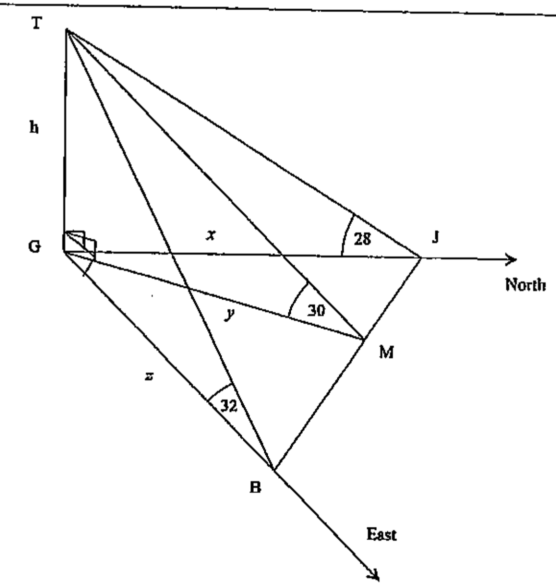
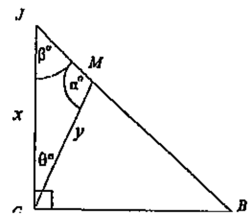
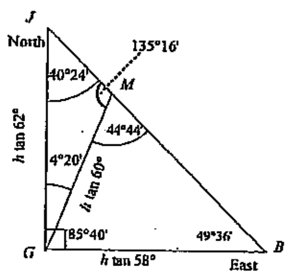
	Solution	Marks	Allocation of marks
a)	$1 - \cos^2 x - \cos x = 1$ $\cos^2 x + \cos x = 0$ $\cos x(\cos x + 1) = 0$ $\cos x = 0$ or $\cos x = -1$ $x = 2n\pi \pm \cos^{-1} 0$ or $2n\pi \pm \cos^{-1}(-1)$ $= 2n\pi \pm \frac{\pi}{2}$ or $2n\pi \pm \pi$	2	1 Mark
b)	(i) Horizontal Motion $\dot{x} = V\cos\theta$ $\dot{x} = 34\cos\theta$ $x = 34t\cos\theta + c_1$ $t = 0 \quad x = 0 \quad c_1 = 0$ $\therefore x = 34t\cos\theta$ $t = \frac{x}{34\cos\theta}$ Vertical Motion $\ddot{y} = -10$ $\dot{y} = -10t + C_2$ $t = 0 \quad \dot{y} = V\sin\theta$ $\therefore c_2 = 34\sin\theta$ $\therefore \dot{y} = -10t + 34\sin\theta$ $\therefore y = -5t^2 + 34\sin\theta + C_3$ $t = 0 \quad y = 0 \quad C_3 = 0$ $\therefore y = -5t^2 + 34t\sin\theta$	2	
	(ii) $y = -5t^2 + 34t\sin\theta$ $= -5\left[\frac{x}{34\cos\theta}\right]^2 + 34 \times \left[\frac{x}{34\cos\theta}\right] \times \sin\theta$ $= -\frac{5x^2}{1156}\sec^2\theta + x\tan\theta$ $= -\frac{5x^2}{1156}(1 + \tan^2\theta) + x\tan\theta$ $1156y = -5x^2 - 5x^2\tan^2\theta + 1156x\tan\theta$ When $x = 100$ $y = 1.25$ since the ball just clears the boundary fence. $1445 = -50000 - 50000\tan^2\theta + 1156000\tan\theta$	3	1 for subst in t correctly 1 for correct y 1 for correct values and
	(iii) For a six, when $x = 100$ $y > 1.25$ $50000\tan^2\theta - 115600\tan\theta + 51445 < 0$ $\tan\theta = = \frac{115600 \pm \sqrt{(115600)^2 - 4 \times 50000 \times 51445}}{2 \times 50000}$ $\tan\theta = 1.7105$ or 0.6015 $\theta = 59°41', 31°2'$ $\therefore \theta$ lies between $31°2'$ and $59°41'$	2	1 for using quadratic formula correctly 1 for the range of angles
c)	 (i) Let $GT = h$ $GJ = x$ $GM = y$ $GB = z$ $\angle TGJ = \angle TGM = \angle TGB = \angle JGB = 90°$ $\angle GTJ = 62°$ $\angle GTM = 60°$ $\angle GTB = 58°$ $\therefore \quad x = h\tan 62°$ $y = h\tan 60°$ $z = h\tan 58°$ 	1	1 for a correct diagram and expressions to find side lengths Accept $x = h\cot 28$ $y = h\cot 30$ $z = h\cot 32$

	(ii) Let angle GJM=β $\therefore \quad \tan\beta = \dfrac{h\tan 58°}{h\tan 62°}$ $\beta = 40° 24'$ (iii) $\dfrac{\sin\alpha}{h\tan 62°} = \dfrac{\sin 40°24'}{h\tan 60°}$ $\sin\alpha = \dfrac{\sin 40°24' \tan 62°}{\tan 60°}$ $= 0.70367$ $\alpha = \sin^{-1}(0.70367)$ $\alpha = 44°44'$ or $135° 16'$ Now if $\alpha = 44°44'$ and $\beta = 40°24'$, then $\theta = 180 - 44°44' - 40°24' = 94°52'$. But $\theta < 90°$, since it lies between north and east. $\therefore \alpha = 135°16'$ $\therefore \theta = 180° - 40°24' - 135°16'$ $= 4° 20'$ $= 4°$ $\therefore$ Maddy is on a bearing of $004°T$ from the Eiffel Tower	1 2	1 for finding the needed angles 1 for correct bearing 1 or 2 marks can be awarded if student was on the right track but has made a small error
	(iv) Using the angles calculated above we can obtain the diagram below:  Using the cos rule on ΔMBG. $MB^2 = (h\tan 60)^2 + (h\tan 58)^2 - 2 \times (h\tan 60)(h\tan 58)\cos 85°40'$ $= h^2\tan^2 60 + h^2\tan^2 58 - 2 \times h^2 . \tan 60. \tan 58. \cos 85°40'$ $= h^2(\tan^2 60 + \tan^2 58 - 2 \times \tan 60. \tan 58. \cos 85°40')$ Using the cos rule on ΔJMG. $JM^2 = (h\tan 60)^2 + (h\tan 62)^2 - 2 \times (h\tan 60)(h\tan 62)\cos 4°20'$ $= h^2\tan^2 60 + h^2\tan^2 62 - 2 \times h^2 . \tan 60. \tan 62. \cos 4°20'$ $= h^2(\tan^2 60 + \tan^2 62 - 2 \times \tan 60. \tan 62. \cos 4°20')$ $\dfrac{MB^2}{JM^2} = \dfrac{h^2(\tan^2 60 + \tan^2 58 - 2 \times \tan 60. \tan 58. \cos 85°40')}{h^2(\tan^2 60 + \tan^2 62 - 2 \times \tan 60. \tan 62. \cos 4°20')}$ $= \dfrac{(\tan^2 60 + \tan^2 58 - 2 \times \tan 60. \tan 58. \cos 85°40')}{(\tan^2 60 + \tan^2 62 - 2 \times \tan 60. \tan 62. \cos 4°20')}$	2	1 mark for obtaining an expression for at least one of MB^2 and/or JM^2. 1 mark for writing the ratio and simplifying out the terms in h. No need to evaluate the expression.