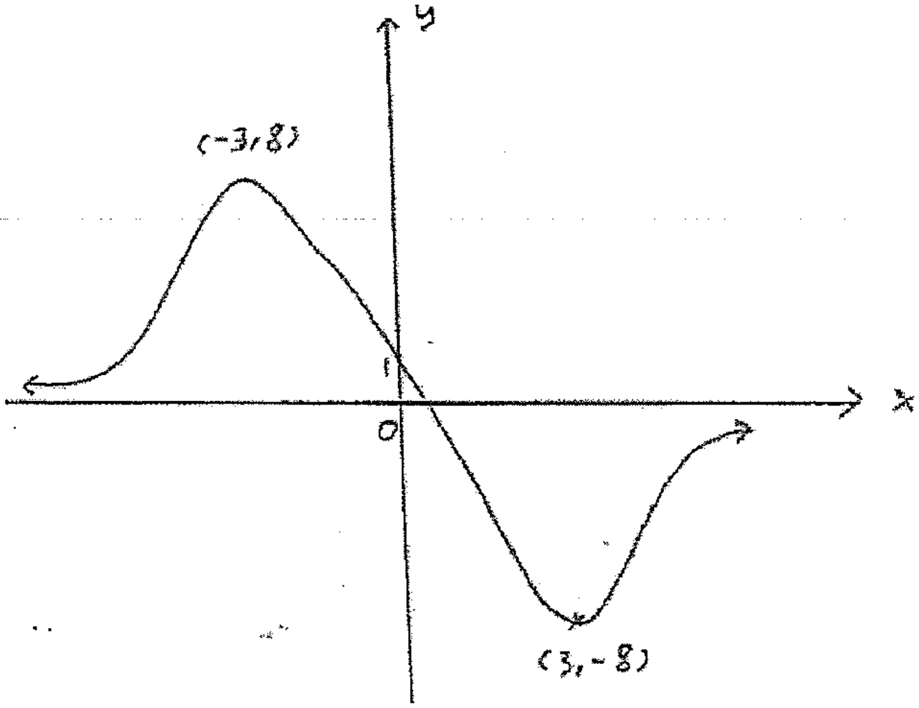
CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Extension 1 Mathematics

Half Yearly Examination 2018
Solutions

QUESTION 1 (18 marks)

a).	The graph of $y = f(x)$ should show these features.  	[3]
b).	For the given function $y = x^4 - 8x^2 + 7$. **i).** The y-intercept, when $x = 0$ i.e. $y = 0^4 - 8(0)^2 + 7$ $\therefore\ y = 7$	[1]
	ii). The x-intercepts, when $y = 0$ i.e. $x^4 - 8x^2 + 7 = 0$ $(x^2 - 7)(x^2 - 1) = 0$ $\therefore\ x = \pm\sqrt{7}$ or $x = \pm 1$	[2]
	iii). The turning points and their nature: $y' = 4x^3 - 16x$ and $y'' = 12x^2 - 16$ For stationary points, set $y' = 0$ i.e. $4x^3 - 16x = 0$ $4x(x^2 - 4) = 0$ $\therefore\ x = 0$ or $x = \pm 2$ and $y = 7$ or $y = -9$ Now test for nature: when $x = 0$ then $y'' = 12(0)^2 - 16$ $y'' < 0$, this implies maximum turning point So $(0, 7)$ is a maximum turning point. when $x = 2$ then $y'' = 12(2)^2 - 16$ $y'' > 0$, this implies minimum turning point So $(2, -9)$ is a minimum turning point. when $x = -2$ then $y'' = 12(-2)^2 - 16$ $y'' > 0$, this implies minimum turning point So $(-2, -9)$ is a minimum turning point.	[3]

iv). the points of inflexion, set $y'' = 0$ i.e. $12x^2 - 16 = 0$

$$\therefore \quad x = \pm\frac{2}{\sqrt{3}} \quad \text{and} \quad y = -\frac{17}{9}$$

So $\left(\pm\frac{2}{\sqrt{3}},\ -\frac{17}{9}\right)$ are possible inflexion points

[2]

Now test for concavity changes:

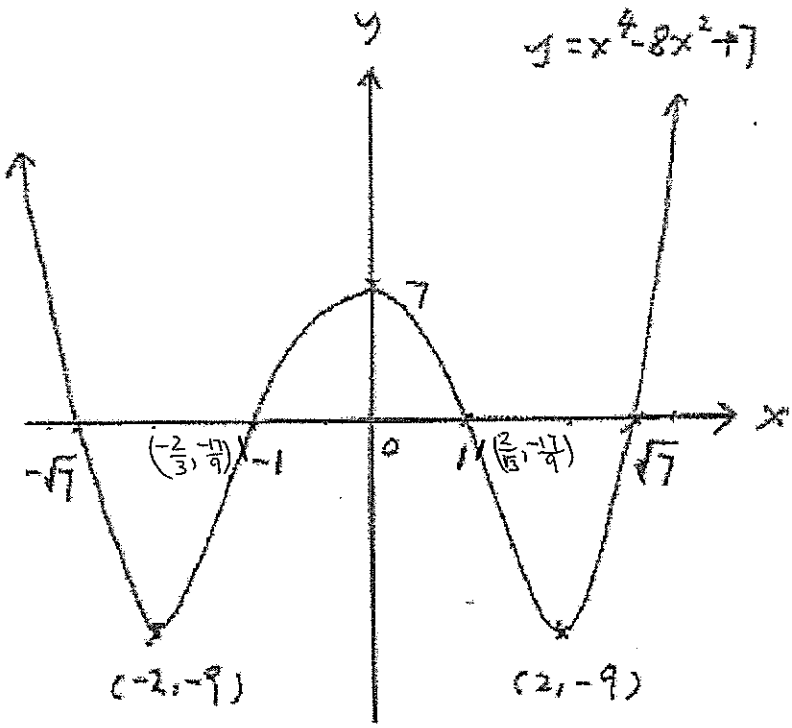
x	1.1	$\frac{2}{\sqrt{3}}$	1.2
y''	< 0	0	> 0

Since concavity changes then $\left(\frac{2}{\sqrt{3}},\ -\frac{17}{9}\right)$ is a point of inflexion.

x	-1.2	$-\frac{2}{\sqrt{3}}$	-1.1
y''	> 0	0	< 0

Since concavity changes then $\left(-\frac{2}{\sqrt{3}},\ -\frac{17}{9}\right)$ is a point of inflexion.

v). Now sketch the graph of the function.



[2]

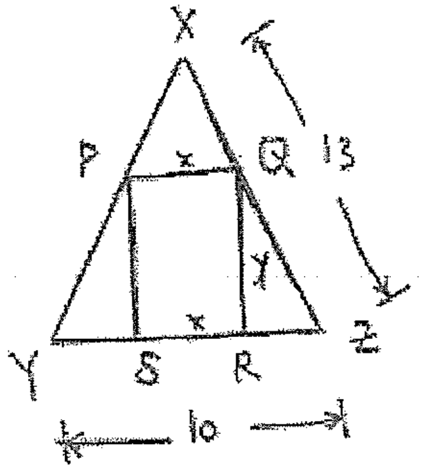
c).

i). $\dfrac{y}{12} = \dfrac{10-x}{10}$ (Matching sides of similar Δs are in the same proportion.)

$$10y = 12(10-x)$$

$$y = \frac{12}{10}(10-x)$$

$$\therefore\ y = 12 - \frac{6x}{5}$$



[2]

ii). $\because\ A = x \cdot y$

$$= x \cdot \left(12 - \frac{6x}{5}\right)$$

$$= 12x - \frac{6}{5}x^2$$

[1]

iii). Now $A = 12x - \dfrac{6}{5}x^2$ then

$$A' = 12 - \frac{12}{5}x \quad \text{and} \quad A'' = -\frac{12}{5}$$

For max / min put $A' = 0$

i.e. $12 - \dfrac{12}{5}x = 0$

$\therefore\ x = 5$

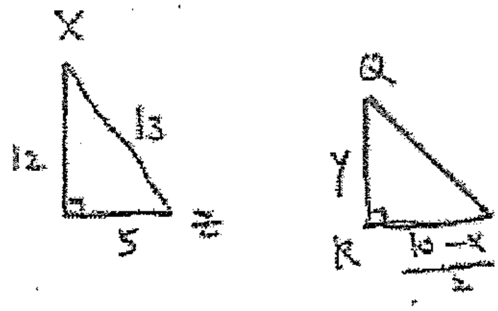
$\because\ A'' < 0$ this implies maximum

So $A_{max} = 12(5) - \dfrac{6}{5}(5)^2$

$= 30$

Thus greatest area of the rectangle is 30 cm^2.

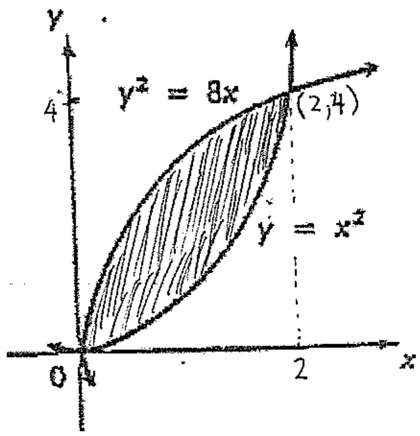
Similar triangles



[2]

QUESTION 2 (18 marks)

a).	**i)** $\int\left[5\left(\sqrt[3]{x^2}\right)+2\sqrt{x}-\frac{3}{x^2}\right]dx=\int\left(5x^{\frac{2}{3}}+2x^{\frac{1}{2}}-3x^{-2}\right)dx$ $=5\times\frac{3}{5}x^{\frac{5}{3}}+2\times\frac{2}{3}x^{\frac{3}{2}}-3\times\frac{1}{-1}x^{-1}+C$ $=3\left(\sqrt[3]{x^5}\right)+\frac{4}{3}\sqrt{x^3}+\frac{3}{x}+C$	**[2]**
	ii) $\int\frac{dx}{(2-5x)^3}=\int(2-5x)^{-3}$ $=\frac{(2-5x)^{-2}}{(-5\times-2)}+C$ $=\frac{1}{10(2-5x)^2}+C$	**[2]**
b).	**i)** $\int_1^2\frac{dx}{x}=[\ln x]_1^2=\ln 2-\ln 1=\ln 2$	**[2]**
	ii) $\int_1^2\frac{dx}{x}\approx\frac{2-1}{6}\left[\frac{1}{1}+4\times\frac{1}{1.5}+\frac{1}{2}\right]=\frac{25}{36}$	**[2]**
	iii). Now $\ln 2\approx\frac{25}{36}$ $2\approx e^{\frac{25}{36}}$ $2^{\frac{36}{25}}\approx e$ $\therefore\quad e\approx 2.713$	**[2]**

c).	i). Let $y^2 = 8x$ ……….. [1] and $y = x^2$ ……….. [2] substitute [2] into [1] get $(x^2)^2 = 8x$ $x^4 = 8x$ $x^4 - 8x = 0$ $x(x^3 - 8) = 0$ $\therefore\ x = 0$ or $x = 2$ The sketch of the region 		[2]
	ii). Now $A = \int_0^2 \left(\sqrt{8x} - x^2\right)dx$ $= \int_0^2 \left(\sqrt{8}x^{\frac{1}{2}} - x^2\right)dx$ $= \left[\sqrt{8} \times \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^2$ $= \frac{2\sqrt{8}}{3} \times 2^{\frac{3}{2}} - \frac{1}{3} \times 2^3 - 0$ $= \frac{2^4}{3} - \frac{2^3}{3}$ $= \frac{8}{3}$ units^2		[3]
	iii). Now $V = \pi \int_0^4 \left(y - \frac{y^4}{64}\right)dy$ $= \pi\left[\frac{1}{2}y^2 - \frac{y^5}{320}\right]_0^4$ $= \pi\left[\frac{1}{2} \times 4^2 - \frac{4^5}{320} - 0\right]$ $= \frac{24}{5}\pi$ units^3		[3]

QUESTION 3 (20 marks)

a).	$\int \frac{e^{2x}}{1-e^{2x}}dx = -\frac{1}{2}\int \frac{-2e^{2x}}{1-e^{2x}}dx$ $= -\frac{1}{2}\ln\left(1-e^{2x}\right)+C$	[2]
b).	i). $\frac{d}{dx}\left(3x^2\log_e x\right) = 3x^2 \times \frac{1}{x} + \log_e x\left(3\times 2x\right)$ $= 3x + 6x\log_e x$	[2]
	ii). $\frac{d}{dx}\left(\frac{e^{2x}}{2x+1}\right) = \frac{\left(2x+1\right)\times 2e^{2x} - e^{2x}\left(2\right)}{\left(2x+1\right)^2}$ $= \frac{2e^{2x}\left(2x+1-1\right)}{\left(2x+1\right)^2}$ $= \frac{4xe^{2x}}{\left(2x+1\right)^2}$	[2]
c).	$\frac{1}{2}\log_e x + \log_e y = \log_e z$ $\log_e x^{\frac{1}{2}} + \log_e y = \log_e z$ $\log_e\left(x^{\frac{1}{2}}y\right) = \log_e z$ $x^{\frac{1}{2}}y = z$ $x^{\frac{1}{2}} = \frac{z}{y}$ $\therefore\ x = \left(\frac{z}{y}\right)^2$	[2]

d).	i). $\log_{10}(x-2)+\log_{10}(2x-3)=1$. $\log_{10}\left[(x-2)(2x-3)\right]=\log_{10}10$ $(x-2)(2x-3)=10$ $2x^2-7x-4=0$ $(2x+1)(x-4)=0$ $\therefore\ x=-\frac{1}{2}$ (rejected) or $x=4$	[3]
	ii). $e^{x+1}-e^{x-2}=24$ $e\times e^x-\frac{1}{e^2}\times e^x=24$ $\left(e-\frac{1}{e^2}\right)e^x=24$ $e^x=24\div\left(e-\frac{1}{e^2}\right)$ $=24\div\left(\frac{e^3-1}{e^2}\right)$ $e^x=\frac{24e^2}{e^3-1}$ $x=\log_e\left(\frac{24e^2}{e^3-1}\right)$ $\therefore\ x\approx 2.229$	[2]
e).	Now $f'(x)=(1+e^x)(1-e^x)$ $f(x)=\int(1+e^x)(1-e^x)dx$ $=\int(1-e^{2x})dx$ $=x-\frac{1}{2}e^{2x}+C$ When $x=0, y=e$ then $e=0-\frac{1}{2}e^{2(0)}+C$ So $C=e+\frac{1}{2}$ Thus $f(x)=x-\frac{1}{2}e^{2x}+e+\frac{1}{2}$ $\therefore\ f(1)=1-\frac{1}{2}e^2+e+\frac{1}{2}$ $=\frac{3}{2}-\frac{1}{2}e^2+e$	[3]

f).	i). Now $\text{RHS} = \frac{1}{3(x-1)} + \frac{2}{3(x+5)}$ $= \frac{x+5+2(x-1)}{3(x-1)(x+5)}$ $= \frac{x+5+2x-2}{3(x-1)(x+5)}$ $= \frac{3x+3}{3(x-1)(x+5)}$ $= \frac{x+1}{(x-1)(x+5)}$ Thus LHS = RHS	[2]
	ii). Now $\int \frac{x+1}{(x-1)(x+5)}\,dx = \int\left(\frac{1}{3(x-1)} + \frac{2}{3(x+5)}\right)dx$ $= \frac{1}{3}\log_e\lvert x-1\rvert + \frac{2}{3}\log_e\lvert x+5\rvert + C$	[2]

QUESTION 4 (14 marks)

a).	Let the amount owed after its repayment be A_i i). Now $A_1 = \$35000(1 + 0.5\%) - M$ $= \$35000(1.005) - M$ So $A_2 = A_1 \times (1.005) - M$ $= [\$35000(1.005) - M] \times (1.005) - M$ $= \$35000(1.005)^2 - M(1.005) - M$	[2]
	ii). Now $A_3 = A_2 \times (1.005) - M$ $= [\$35000(1.005)^2 - M(1.005) - M] \times (1.005) - M$ $= \$35000(1.005)^3 - M(1.005)^2 - M(1.005) - M$ . . . $A_{78} = \$35000(1.005)^{78} - M(1.005)^{77} - \ldots - M(1.005) - M$ $= \$35000(1.005)^{78} - M\left[\frac{1.005^{78}-1}{0.005}\right]$ $= \$35000 \times 1.005^{78} - 200M(1.005^{78} - 1)$	[3]

	iii). Now $A_{78} = \$35000 \times 40\%$ i.e. $35000 \times 1.005^{78} - 200M(1.005^{78} - 1) = \$35000 \times 0.4$ $M = \dfrac{\$35000(1.005)^{78} - \$35000 \times 0.4}{200(1.005^{78} - 1)}$ $= \$395.80$	[3]
b).	Let $T_n = n(n+2)$ and $S_n = \dfrac{n}{6}(n+1)(2n+7)$ <u>Proof</u> <u>Step 1</u> If $n = 1$ then LHS $= T_1$ $= 1(1+2)$ $= 3$ RHS $= S_1$ $= \dfrac{1}{6}(1+1)(2\times1+7)$ $= \dfrac{1}{6}(2)(9)$ $= 3$ So LHS = RHS $\therefore$ The statement is true for $n = 1$. <u>Step 2</u> Assume statement is true for $n = k$, i.e. $S_k = \dfrac{k}{6}(k+1)(2k+7)$ Now prove the statement is true for $n = k + 1$, i.e. $S_{k+1} = \dfrac{k+1}{6}(k+1+1)\left[2(k+1)+7\right]$ $= \dfrac{k+1}{6}(k+2)(2k+9)$ Now $S_{k+1} = S_k + T_{k+1}$ $= \dfrac{k}{6}(k+1)(2k+7) + (k+1)(k+3)$ $= \dfrac{(k+1)}{6}\left[k(2k+7) + 6(k+3)\right]$ $= \dfrac{(k+1)}{6}\left[2k^2 + 13k + 18\right]$ $= \dfrac{(k+1)}{6}(k+2)(2k+9)$ $\therefore$ The statement is true for $n = k + 1$ if it is true for $n = k$. <u>Step 3</u> $\therefore$ If the statement is true for $n = k$, it is also true for $n = k + 1$. But it is true for $n = 1$. Thus it is true for all integers $n \geq 1$.	[3]

c).	Proof Let $S(n)$ be the statement that $7^{2n-1} + 5$ is divisible by 12. Step 1 For $n = 1$ then $7^{2\times 1-1} + 5 = 12$, which is divisible by 12. $\therefore$ $S(1)$ is true. Step 2 Assume $S(k)$ is true, i.e. $7^{2k-1} + 5 = 12M$ where M is an integer. So $7^{2k-1} = 12M - 5$ ....[1] Now required to prove $S(k+1)$ is true, i.e. $7^{2(k+1)-1} + 5 = 7^{2k+1} + 5$ is divisible by 12. $= 7^2 \times 7^{2k-1} + 5$ (sub [1] get) $= 49(12M - 5) + 5$ $= 12 \times 49M - 49 \times 5 + 5$ $= 12(49M - 20)$, which is divisible by 12 as M is an integer. $\therefore$ $S(k+1)$ is true when $S(k)$ is true. Step 3 Since the result is true when $n = 1$, hence it is true when $n = 2$, and so by mathematical induction the result is true for all $n \geq 1$.	[3]

END OF SOLUTIONS