CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Extension 1 Mathematics

Term2 Assessment Task 2014

Time allowed: 55 minutes

Name: ______________________________ **Class:** ________**Teacher**________

Kellahan / White / Lobejko / Fardouly

Instructions:

- All questions should be attempted.
- Show ALL necessary working on your own paper.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Start each question on a new page and only write on one side of each sheet of paper.

Outcome	Q1 to Q8 & Q10	Q9	
HE4			/35
HE6			/5
	/35	/5	/40

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

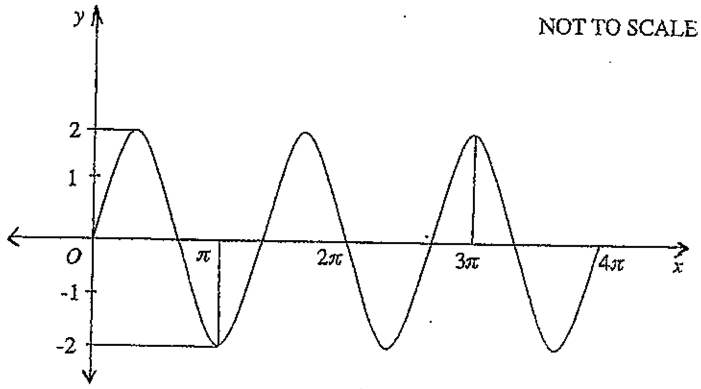
$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Year 12 HSC Term 2 Assessment Task (40 marks)

Mathematics Extension 1

1. Find the trigonometric equation for the graph below. [1]



2. Consider the function $f(x) = 3\sin^{-1}\frac{x}{2}$

   (a) Evaluate $f(2)$
   (b) State the domain and range of $y = f(x)$
   (c) Draw the graph of $y = f(x)$ [4]

3. Consider the function $f(x) = 1 + \frac{3}{(x-4)}$ for $x > 4$

   (a) Give the horizontal and vertical asymptotes for $y = f(x)$
   (b) Find the inverse function $f^{-1}(x)$
   (c) State the domain of $f^{-1}(x)$ [4]

4. Show that $tan\left(\sin^{-1}\frac{5}{13} - \cos^{-1}\frac{3}{5}\right)$ is equal to $-\frac{33}{56}$ [2]

5. (a) Solve the equation $2sin^2\theta = sin2\theta$ for $0 \leq \theta \leq 2\pi$ [2]

   (b) Find the general solutions of $2sin^3x - sinx - 2sin^2 x + 1 = 0$ [3]

6. (a) Differentiate $y = cos^2 3x$ [2]

   (b) Find $\frac{d}{dx}\cos^{-1}(3x^2)$ [1]

   (c) Show that the derivative of $y = \tan^{-1}\frac{x}{a}$ is $\frac{a}{(a^2+x^2)}$ [2]

7. (a) $\int \frac{dx}{x^2+7}$ [1]

(b) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-2x^2}} dx$ [3]

(c) $\int_0^{\frac{\pi}{2}} sin^2 3x \, dx$ [2]

8. Differentiate $x \sin^{-1} x + \sqrt{1-x^2}$ and hence evaluate $\int_0^1 \sin^{-1} x \, dx$ [4]

9. (a) Evaluate $\int_2^{10} \frac{x}{\sqrt{x-1}} dx$ using the substitution $x = t^2 + 1$ [3]

(b) Integrate $\frac{sec^2(\sin x)}{\sec x} dx$ using the substitution $u = \sin x$ [2]

10. Express $\sin 4x + \sqrt{3} \cos 4x$ in the form $R \sin(4x + \alpha)$, where α is in terms of π and hence find the general solution of the equation $\sin 4x + \sqrt{3} cos 4x = 0$, in exact form. [4]

YR 12 HSC Term 2 Ext 1 2014 Solutions

1. $y = 2\sin\dfrac{3x}{2}$

2. (a) $f(2) = 3\sin^{-1}1$

$= \dfrac{3\pi}{2}$

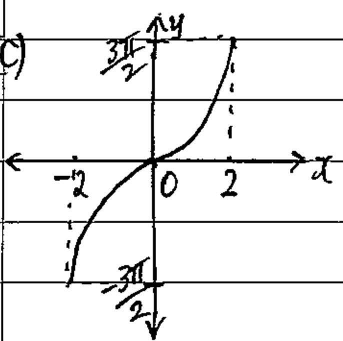
(b) ~~$D: -1 \le \dfrac{x}{2} \le 1$~~

$-2 \le x \le 2$

$R: -\dfrac{\pi}{2} \le \dfrac{y}{3} \le \dfrac{\pi}{2}$

$-\dfrac{3\pi}{2} \le y \le \dfrac{3\pi}{2}$

(c)



3. (a) Vert. asym: $x = 4$ ①

Horiz. asym: $y = 1$

(b) $x = 1 + \dfrac{3}{y-4}$

$x - 1 = \dfrac{3}{y-4}$ ②

$y - 4 = \dfrac{3}{x-1}$

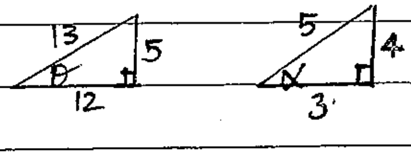
$y = \dfrac{3}{x-1} + 4$

$\therefore f^{-1}(x) = 4 + \dfrac{3}{x-1}$

(c) $D f^{-1}(x): \ x > 1$ ①

4. Let $\theta = \sin^{-1}\dfrac{5}{13}$ & $\alpha = \cos^{-1}\dfrac{3}{5}$

$\sin\theta = \dfrac{5}{13}$ $\quad \cos\alpha = \dfrac{3}{5}$



$\tan(\theta - \alpha) = \dfrac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$

$= \dfrac{\frac{5}{12} - \frac{4}{3}}{1 + \frac{20}{36}}$

$\therefore \tan\left(\sin^{-1}\dfrac{5}{12} - \cos^{-1}\dfrac{3}{5}\right) = -\dfrac{33}{56}$

5. (a) $2\sin^2\theta = \sin 2\theta \qquad 0 \le \theta \le 2\pi$

$2\sin^2\theta - \sin 2\theta = 0$

$2\sin^2\theta - 2\sin\theta\cos\theta = 0$

$2\sin\theta(\sin\theta - \cos\theta) = 0$

$\sin\theta = 0$ OR $\tan\theta = 0$

$\therefore \theta = 0, \pi, 2\pi, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

(b) $2\sin^3 x - \sin x - 2\sin^2 x + 1 = 0$

$\sin x(2\sin^2 x - 1) - 1(2\sin^2 x - 1) = 0$

$(2\sin^2 x - 1)(\sin x - 1) = 0$

$\sin^2 x = \dfrac{1}{2} \qquad \sin x = 1$

$\sin x = \pm\dfrac{1}{\sqrt{2}} \qquad \therefore x = n\pi + (-1)^n \sin^{-1} 1$

$\therefore x = n\pi \pm \dfrac{\pi}{4} \qquad x = n\pi + (-1)^n \dfrac{\pi}{2}$

6. (a) $y = \cos^2 3x$

$= (\cos 3x)^2$

$\dfrac{dy}{dx} = 2(\cos 3x) \times -\sin 3x \times 3$

$= -6\sin 3x\cos 3x$

$= -3\sin 6x$

(b) $\dfrac{d}{dx}\cos^{-1}(3x^2) = \dfrac{-6x}{\sqrt{1 - 9x^4}}$

(c) $y = \tan^{-1}\dfrac{x}{a}$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a}$$

$$= \frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a}$$

$$= \frac{1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}$$

$$= \frac{a^2}{a^2+x^2}\cdot\frac{1}{a}$$

$$= \frac{a}{(a^2+x^2)}$$

7 (a) $\displaystyle\int \frac{dx}{x^2+7} = \frac{1}{\sqrt7}\tan^{-1}\frac{x}{\sqrt7} + c$

(b) $$\int_{-\frac12}^{\frac12} \frac{1}{\sqrt{1-2x^2}}\,dx = \int_{-\frac12}^{\frac12}\frac{dx}{\sqrt2\sqrt{\frac12 - x^2}}$$

$$= \left[\frac{1}{\sqrt2}\sin^{-1}\sqrt2\,x\right]_{-\frac12}^{\frac12}$$

$\dfrac{\sqrt2}{2} = \dfrac{\sqrt2}{2}\times\dfrac{\sqrt2}{\sqrt2} = \dfrac{1}{\sqrt2}$

$$= \frac{1}{\sqrt2}\sin^{-1}\frac{\sqrt2}{2} - \frac{1}{\sqrt2}\sin^{-1}-\frac{\sqrt2}{2}$$

$$= \frac{1}{\sqrt2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{\sqrt2}\times\frac{\pi}{2}$$

$$= \frac{\pi}{2\sqrt2}$$

$$= \frac{\pi\sqrt2}{4}$$

(c) $$\int_0^{\frac{\pi}{2}} \sin^2 3x\,dx = \int_0^{\frac{\pi}{2}} \frac12(1-\cos 6x)\,dx$$

$$= \frac12\left[x - \frac{\sin 6x}{6}\right]_0^{\frac{\pi}{2}}$$

$$= \frac12\left\{\frac{\pi}{2} - \sin\frac{3\pi}{6} - (0)\right\}$$

$$= \frac12\times\frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

8 $$\frac{d}{dx}\, x\sin^{-1}x + \sqrt{1-x^2} = x\times\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x\times 1 + \frac{1}{\cancel{2}}(1-x^2)^{-\frac12}\times -\cancel{2}x$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x$$

8 cont.

$$\int_0^1 \sin^{-1}x = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$$

$$= \left(1\sin^{-1}1 + \sqrt{1-1^2}\right) - \left(0\sin^{-1}0 + \sqrt{1-0^2}\right)$$

$$= \frac{\pi}{2} - 1$$

9 (a) Let $x = t^2+1$

$$\frac{dx}{dt} = 2t$$

$$dx = 2t.dt$$

$$t^2 = x-1$$

$$\therefore t = \sqrt{x-1}$$

When $x=2$, $t=1$

When $x=10$, $t=3$

$$\int_2^{10}\frac{x}{\sqrt{x-1}}\,dx$$

$$= \int_1^3 \frac{t^2+1}{t}\,.\,2t\,.\,dt$$

$$= 2\int_1^3 t^2+1\ \ dt$$

$$= 2\left[\frac{t^3}{3}+t\right]_1^3$$

$$= 2\left\{\left(\frac{3^3}{3}+3\right) - \left(\frac{1^3}{3}+1\right)\right\}$$

$$= 2\times\frac{32}{3}$$

$$= \frac{64}{3}$$

$$= 21\tfrac13$$

9 (b) $\displaystyle\int \frac{\sec^2(\sin x)}{\sec x}\,dx = \int \frac{\sec^2(u)}{\cancel{\sec x}} \cdot \cancel{\sec x}\,.du$

let $u = \sin x$ $\qquad = \displaystyle\int \sec^2 u \; du$ 1

$\dfrac{du}{dx} = \cos x$ $\qquad = \tan u + C$

$dx = \dfrac{du}{\cos x}$ $\qquad = \tan(\sin x) + C$ 1

$\therefore dx = \sec x \,.du$

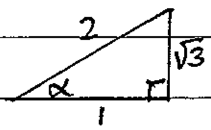
10 $\sin 4x + \sqrt{3}\cos 4x = R\sin(4x+\alpha)$

$\dfrac{1}{R}\sin 4x + \dfrac{\sqrt{3}}{R}\cos 4x = \sin(4x+\alpha)$

$= \sin 4x\cos\alpha + \cos 4x \sin\alpha$

$\cos\alpha = \dfrac{1}{R}$ $\qquad R = \sqrt{(\sqrt{3})^2 + 1^2}$ 1

$\sin\alpha = \dfrac{\sqrt{3}}{R}$ $\qquad = 2$



~~$\tan\alpha = \sqrt{3}$~~

$\therefore \alpha = \dfrac{\pi}{3}$ $\qquad \therefore \sin 4x + \sqrt{3}\cos 4x = 2\sin\left(4x + \dfrac{\pi}{3}\right)$ 1

$\sin 4t + \sqrt{3}\cos 4t = 0$

$\therefore 2\sin\left(4x + \dfrac{\pi}{3}\right) = 0$

$\sin\left(4x + \dfrac{\pi}{3}\right) = 0$

$4x + \dfrac{\pi}{3} = 0, \pm\pi, \pm 2\pi \ldots$ 1

$= n\pi$ where n is an integer

$\therefore 4x = n\pi - \dfrac{\pi}{3}$

$\therefore x = \dfrac{n\pi}{4} - \dfrac{\pi}{12}$ 1

OR $\dfrac{3n\pi - \pi}{12}$

OR $\dfrac{\pi(3n-1)}{12}$