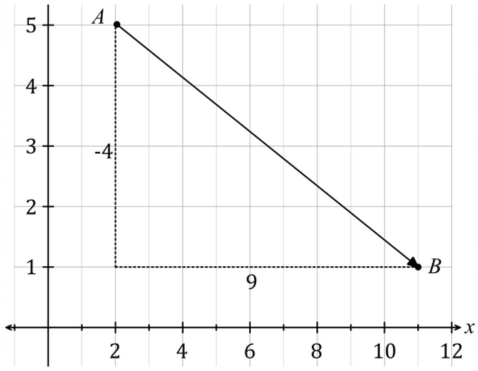
ACE Examination Paper 2
Year 12 Mathematics Extension 1 Yearly Examination
Worked solutions and marking guidelines

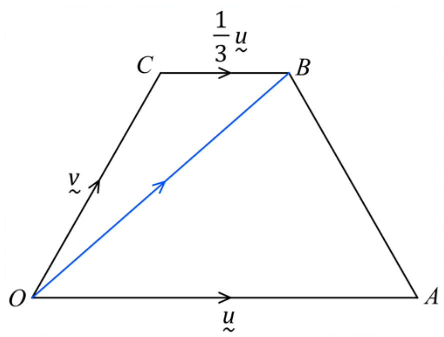
Section I		
	Solution	**Criteria**
1.	$\overrightarrow{AB} = 9\underset{\sim}{i} - 4\underset{\sim}{j}$ 	1 Mark: B
2.	$\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$ $= x \times \frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x - \frac{1}{2\sqrt{1-x^2}} \times -2x$ $= \cos^{-1}x$	1 Mark: C
3.	$V = \pi\int_a^b y^2\,dx = \pi\int_0^2 (3x+1)^2\,dx$ $= \pi\int_0^2 (9x^2+6x+1)dx$ $= \pi[3x^3+3x^2+x]_0^2$ $= \pi[(3\times 2^3 + 3\times 2^2 + 2) - 0]$ $= 38\pi$ cubic units	1 Mark: D
4.	$u = 2 - x^2$ $\frac{du}{dx} = -2x$ or $-\frac{1}{2}du = xdx$ $\int \frac{x}{(2-x^2)^3}dx = -\frac{1}{2}\int \frac{1}{u^3}du$ $= -\frac{1}{2} \times -\frac{1}{2}u^{-2} + C$ $= \frac{1}{4(2-x^2)^2} + C$	1 Mark: B
5.	$y = 2xe^{2x}$ $\frac{dy}{dx} = 2x \times 2e^{2x} + e^{2x} \times 2 = 4xe^{2x} + 2e^{2x}$ $\frac{d^2y}{dx^2} = 4x \times 2e^{2x} + e^{2x} \times 4 + 4e^{2x} = 8xe^{2x} + 8e^{2x}$ (A) $\frac{d^2y}{dx^2} - 4y = 8e^{2x}$ LHS $= 8xe^{2x} + 8e^{2x} - 4(2xe^{2x})$ $= 8e^{2x} =$ RHS	1 Mark: A

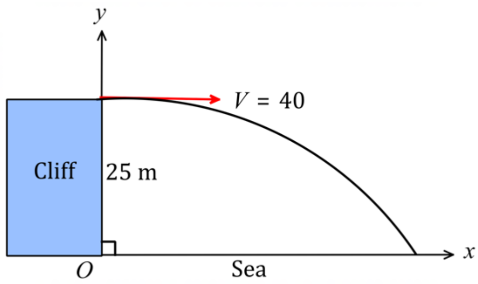
	Solution	**Criteria**
6.	$\dfrac{\sin 2x}{\cos 2x - 1} = \dfrac{2\sin x\cos x}{\cos^2 x - \sin^2 x - 1}$ $= \dfrac{2\sin x\cos x}{-2\sin^2 x}$ $= -\cot x$	1 Mark: A
7.	$\dfrac{dy}{dx} = \dfrac{1}{5}(y-1)^2$ $\dfrac{dx}{dy} = \dfrac{5}{(y-1)^2}$ $x = \displaystyle\int \dfrac{5}{(y-1)^2}\,dy = -5\,(y-1)^{-1} + C$ Now $y = 0,\ x = 0 \Rightarrow C = -5$ $x = \dfrac{-5}{(y-1)} - 5$ $\dfrac{x+5}{5} = \dfrac{-1}{(y-1)}$ $(x+5)(y-1) = -5$ $y - 1 = \dfrac{-5}{(x+5)}$ $y = \dfrac{-5}{(x+5)} + 1$ $= \dfrac{-5+x+5}{(x+5)} = \dfrac{x}{(x+5)}$	1 Mark: A
8.	Let p be the probability of a truck. $p = 0.06, n = 30$ $P(X = x) = {}^{30}C_x\,(0.06)^x(0.94)^{30-x}$ $P(X = 3) = {}^{30}C_3\,(0.06)^3(0.94)^{27}$ $= 0.1649\ldots$ ≈ 0.165	1 Mark: C
9.	$h = \dfrac{V^2\sin^2\theta}{2g}$ $= \dfrac{(25)^2 \times \sin^2 30}{2 \times 9.8}$ $= 7.9719\ldots$ ≈ 7.97 metres	1 Mark: B
10.	$\text{LHS} = 2[3m - (-1)^{k+1}] + (-1)^{k+2}$ $= 2 \times 3m + 2 \times (-1)^{k+2} + 1 \times (-1)^{k+2}$ $= 3[2m + (-1)^{k+2}]$ $= 3p$ $= \text{RHS}$ $\therefore$ Error in line 4.	1 Mark: D

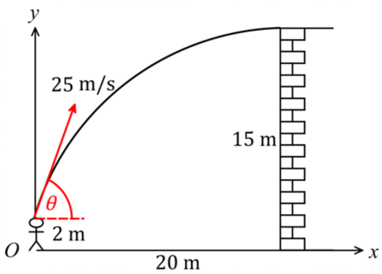
Section II		
11(a)	$x = 2\sin\theta$ $\frac{dx}{d\theta} = 2\cos\theta$ $dx = 2\cos\theta d\theta$ When $x = 1$, $1 = 2\sin\theta$, $\theta = \frac{\pi}{6}$ When $x = -1$, $-1 = 2\sin\theta$, $\theta = -\frac{\pi}{6}$ $\int_{-1}^{1} \sqrt{4 - x^2}\, dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{4 - 4\sin^2\theta} \times 2\cos\theta d\theta$ $= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2\theta d\theta = 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(1 + \cos 2\theta) d\theta$ $= 4\left[\frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$ $= 2 \times \left[\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - \left(-\frac{\pi}{6} + \frac{1}{2}\sin -\frac{\pi}{3}\right)\right]$ $= \frac{2\pi}{3} + \sqrt{3}$	3 Marks: Correct answer. 2 Marks: Uses the substitution and simplifies the integral. 1 Mark: Adjusts the limits and finds *dx*.
11(b)	$\cos\angle AOB = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert}$ $= \frac{2 \times 1 + 2 \times (-2)}{\sqrt{2^2 + 2^2} \times \sqrt{1^2 + (-2)^2}}$ $= \frac{-2}{\sqrt{40}}$ $\angle AOB = 108.4349 \ldots$ $\approx 108°$	2 Marks: Correct answer. 1 Mark: Uses the formula for the angle between two vectors.
11(c) (i)	$\text{LHS} = \cot\theta - 2\cot 2\theta$ $= \frac{1}{\tan\theta} - 2 \times \frac{1}{\tan 2\theta}$ $= \frac{1}{\tan\theta} - 2 \times \frac{1 - \tan^2\theta}{2\tan\theta}$ $= \frac{1}{\tan\theta} - \frac{1}{\tan\theta} + \tan\theta$ $= \tan\theta$ $= \text{RHS}$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
11(c) (ii)	$\tan\theta = \cot\theta - 2\cot 2\theta$ from part (i) $\tan 2\theta = \cot 2\theta - 2\cot 4\theta$ $\tan 4\theta = \cot 4\theta - 2\cot 8\theta$ Substitute these results into the identity. $\text{LHS} = \tan\theta + 2\tan 2\theta + 4\tan 4\theta$ $= (\cot\theta - 2\cot 2\theta) + 2(\cot 2\theta - 2\cot 4\theta) + 4(\cot 4\theta - 2\cot 8\theta)$ $= \cot\theta - 8\cot 8\theta$ $= \text{RHS}$	2 Marks: Correct answer. 1 Mark: Uses the result in part (i) and makes some progress.

11(d)	$x = u^2 + 1$ $dx = 2udu$ when x = 10, u = 3 and x = 2, u = 1 $\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx = \int_1^3 \frac{u^2+1}{u}(2udu)$ $= \int_1^3 2u^2 + 2du$ $= 2\left[\frac{u^3}{3} + u\right]_1^3$ $= 2[(9+3) - \left(\frac{1}{3} + 1\right)]$ $= 21\frac{1}{3}$	2 Marks: Correct answer. 1 Mark: Sets up the integration using the substitution.
11(e)	Step 1: To prove true for n = 1 $\text{LHS} = \frac{1}{2!} = \frac{1}{2}$ $\text{RHS} = 1 - \frac{1}{(1+1)!} = \frac{1}{2}$ Result is true for n = 1 Step 2: Assume true for n = k $S_k = 1 - \frac{1}{(k+1)!}$ Step 3: To prove true for n = k + 1 $S_{k+1} = 1 - \frac{1}{(k+2)!}$ $S_k + T_{k+1} = S_{k+1}$ $\text{LHS} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$ $= 1 - \frac{(k+2)-(k+1)}{(k+2)!}$ $= 1 - \frac{1}{(k+2)!}$ $= \text{RHS}$ Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for n = 1 and attempts to use the result of n = k to prove the result for n = k + 1 1 mark: Proves the result true for n = 1.
11(f)	$V = \pi\int_0^4 x^2\,dy$ $= \pi\int_0^4 y^{\frac{1}{2}}\,dy$ $= \pi\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_{10}^4$ $= \frac{2\pi}{3}\left(4^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$ $= \frac{16\pi}{3}$ cubic units	2 Marks: Correct answer. 1 Mark: Uses volume formula with at least one correct value.

12(a) (i)	$2\sin x - \cos x = A\sin(x - \alpha)$ $= A\sin x\cos\alpha - A\cos x\sin\alpha$ $A\cos\alpha = 2$ ① $A\sin\alpha = 1$ ② Equation ② divided by equation ① $\tan\alpha = \frac{1}{2}$ $\alpha = 26°34'$ Squaring and adding the equations $A^2(\sin^2\alpha + \cos^2\alpha) = 1^2 + 2^2$ $A^2 = 5$ $A = \sqrt{5}\ (A > 0)$ $\therefore 2\sin x - \cos x = \sqrt{5}\sin(x - 26°34')$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
12(a) (ii)	$\sqrt{5}\sin(x - 26°34') = 1$ $\sin(x - 26°34') = \frac{1}{\sqrt{5}}$ $x - 26°34' = 26°34'$ or $153°26'$ $x = 53°8'$ or $180°$	2 Marks: Correct answer. 1 Mark: Finds one solution or makes some progress using part (a).
12(b)	$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$ $= (-\underset{\sim}{i} + 4\underset{\sim}{j}) + (4\underset{\sim}{i} - 3\underset{\sim}{j})$ $= 3\underset{\sim}{i} + \underset{\sim}{j}$ $\lvert\overrightarrow{PR}\rvert = \sqrt{3^2 + 1^2}$ $= \sqrt{10}$	2 Marks: Correct answer. 1 Mark: Finds $\overrightarrow{PR}$.
12(c) (i)	$N = 100 + Ae^{-0\cdot 09t}$ $\frac{dN}{dt} = -0.09 \times Ae^{-0\cdot 09t}$ $= -0.09(N - 100)$	1 Mark: Correct answer.
12(c) (ii)	When $t = 1$ then $N = 400$ $400 = 100 + Ae^{-0\cdot 09\times 1}$ $Ae^{-0\cdot 09} = 300$ $A = \frac{300}{e^{-0\cdot 09}} = 328.2522 \ldots$ We need to find t when $N = 200$ $200 = 100 + \frac{300}{e^{-0\cdot 09}}e^{-0\cdot 09t}$ $e^{-0\cdot 09t} = 100 \div \frac{300}{e^{-0\cdot 09}} = 0.3046 \ldots$ $t = \frac{\ln(0.3046 \ldots)}{-0.09}$ $= 13.2068 \ldots$ ≈ 13.2 years $\therefore$ It will 13.2 years for the number of animals to reach 200.	2 Marks: Correct answer. 1 Mark: Finds the value of A or shows similar understanding of the problem.

12(d)	$x = \frac{1}{4}\tan\theta$ $\frac{dx}{d\theta} = \frac{1}{4}\sec^2\theta d\theta$ $1 + 16x^2 = 1 + 16\left(\frac{1}{4}\tan\theta\right)^2 = 1 + \tan^2\theta = \sec^2\theta$ $\int \frac{1}{1 + 16x^2} dx = \int \frac{\frac{1}{4}\sec^2\theta}{\sec^2\theta} d\theta$ $= \frac{1}{4}\int 1\, d\theta$ $= \frac{1}{4}\theta + \mathrm{C}$ Now $x = \frac{1}{4}\tan\theta$ $4x = \tan\theta$ $\theta = \tan^{-1}4x$ $\therefore \int \frac{1}{1 + 16x^2} dx = \frac{1}{4}\tan^{-1}4x + C$	3 Marks: Correct answer. 2 Marks: Finds the solution in terms of θ. 1 Mark: Sets up the integration using the substitution.
12(e) (i)	Let p be the probability of hitting the target. $p = 0.95, n = 40$ $P(X = x) = {}^{40}C_x(0.95)^x(0.05)^{10-x}$ $P(X = 36) = {}^{40}C_{36}(0.95)^{36}(0.05)^{40-36}$ $= 0.09012\ldots$ ≈ 0.0901	2 Marks: Correct answer. 1 Mark: Makes some progress.
12(e) (ii)	Misses at most 2 targets then X= 38, 39 and 40 $P(X = x) = {}^{40}C_x(0.95)^x(0.05)^{10-x}$ Expression is: $P(X \geq 38) = P(X = 38) + P(X = 39) + P(X = 40)$ $= {}^{40}C_{38}(0.95)^{38}(0.05)^2 + {}^{40}C_{39}(0.95)^{39}(0.05)^1 + {}^{40}C_{40}(0.95)^{40}$ $= 0.676735\ldots$ ≈ 0.6767	2 Marks: Correct answer. 1 Mark: Uses the complementary event or shows some understanding.
13(a)	$\overrightarrow{CB} = \frac{1}{3}\overrightarrow{OA} = \frac{1}{3}\underset{\sim}{u}$ $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$ $= \underset{\sim}{v} + \frac{1}{3}\underset{\sim}{u}$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $= \underset{\sim}{v} + \frac{1}{3}\underset{\sim}{u} - \underset{\sim}{u}$ $= \underset{\sim}{v} - \frac{2}{3}\underset{\sim}{u}$ 	2 Marks: Correct answer. 1 Mark: Finds $\overrightarrow{OB}$.

13(b) (i)	 **Horizontally** $a_x = \ddot{x} = 0$ $v_x = \dot{x} = c_1$ At $t = 0, v_x = 40 \Rightarrow c_1 = 40$ $v_x = \dot{x} = 40$ $x = 40 + c_2$ When $t = 0, x = 0 \Rightarrow c_2 = 0$ $x = 40t$ **Vertically** $a_y = \ddot{y} = -10$ $v_y = \dot{y} = -10t + c_3$ At $t = 0, v_y = 0 \Rightarrow c_3 = 0$ $v_y = \dot{y} = -10t$ $y = -5t^2 + c_4$ When $t = 0$ $y = 25 \Rightarrow c_4 = 25$ $y = -5t^2 + 25$	2 Marks: Correct answer. 1 Mark: Finds horizontal or vertical parametric equations or shows some understanding of the problem.
13(b) (ii)	Particle reaches the sea when $y = 0$ $0 = -5t^2 + 25$ $t = \sqrt{5}\ (t > 0)$ Horizontal distance from the base of the cliff. $x = 40t$ $\quad = 40 \times \sqrt{5}$ $\quad = 40\sqrt{5}$ m $\therefore$ The rock hits the sea $40\sqrt{5}$ metres from the base of the cliff.	2 Marks: Correct answer. 1 Mark: Finds the time taken to reach the sea.
13(c)	Step 1: To prove true for $n = 1$ $5^1 + 12 \times 1 - 1 = 16$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $5^k + 12k - 1 = 16m$ where m is an integer Step 3: To prove true for $n = k + 1$ $5^{k+1} + 12(k+1) - 1 = 16p$ where p is an integer LHS $= 5^{k+1} + 12(k+1) - 1$ $\quad = 5^{k+1} + 12k + 11$ $\quad = 5(5^k + 12k - 1) - 48k + 16$ $\quad = 5(5^k + 12k - 1) + 16(1 - 3k)$ $\quad = 5(16m) + 16(1 - 3k)$ $\quad = 16(5m + 1 - 3k)$ $\quad = 16p$ $\quad =$ RHS Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$. 1 mark: Proves the result true for $n = 1$.
13(d)	$\underset{\sim}{u} . \underset{\sim}{v} = \lvert \underset{\sim}{u} \rvert \lvert \underset{\sim}{v} \rvert \cos\theta$ $\quad = 5 \times 5 \times \cos 60°$ $\quad = 12.5$	2 Marks: Correct answer. 1 Mark: Makes some progress.

13(e)	$\int_0^{\pi} \sin^4 x dx = \int_0^{\pi} (\sin^2 x)^2 dx$ $= \int_0^{\pi} \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$ $= \int_0^{\pi} \frac{1}{4}(\cos^2 2x - 2\cos 2x + 1) dx$ $= \frac{1}{4}\int_0^{\pi} \frac{1}{2}(1+\cos 4x) - 2\cos 2x + 1\, dx$ $= \frac{1}{4}\left[\frac{1}{2}x + \frac{1}{8}\sin 4x - \sin 2x + x\right]_0^{\pi}$ $= \frac{1}{4}\left[\left(\frac{\pi}{2}+\pi\right) - 0\right]$ $= \frac{3\pi}{8}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: : Applies double angle trig identity.
14(a)	$\sin(-105°) = \sin 255°$ $= -\sin 75° = -\sin(30° + 45°)$ $= -(\sin 30° \cos 45° + \cos 30° \sin 45°)$ $= -\left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)$ $= \frac{1+\sqrt{3}}{2\sqrt{2}}$	2 Marks: Correct answer. 1 Mark: uses the compound angle formula with at least one correct value.
14(b) (i)	$x = 25t\cos\theta$ ① $y = -5t^2 + 25t\sin\theta + 2$ ② Eqn ① $t = \frac{x}{25\cos\theta}$ into eqn ② $y = -5\frac{x^2}{25^2\cos^2\theta}$ $+25\frac{x}{25\cos\theta}\sin\theta + 2$ $y = -\frac{x^2}{125}\sec^2 40 + x\tan\theta + 2$ $y = -\frac{x^2}{125}(1+\tan^2\theta) + x\tan\theta + 2$ 	2 Marks: Correct answer. 1 Mark: makes some progress towards the solution.
14(b) (ii)	When $x = 20$ then $y = 15$ Now $(20, 15)$ satisfies the equation in part (i) $15 = -\frac{20^2}{125}(1+\tan^2\theta) + 20\tan\theta + 2$ $75 = -16(1+\tan^2\theta) + 100\tan\theta + 10$ $16\tan^2\theta - 100\tan\theta + 81 = 0$ $\tan\theta = \frac{100 \pm \sqrt{100^2 - 4 \times 16 \times 81}}{2 \times 16}$ $= \frac{100 \pm \sqrt{4816}}{32}$b $= 0.9563\ldots$ or $0.52936\ldots$ $\theta \approx 44°$ or $79°$ $\therefore$ The ball must be thrown between 44° and 79°.	2 Marks: Correct answer. 1 Mark: Determines the quadratic equation.

14(c)	$A = \int_0^{\pi} \sin x\, dx - \int_0^{\pi} \sin^2 x\, dx$ $= \int_0^{\pi} \sin x\, dx - \int_0^{\pi} \frac{1}{2}(1 - \cos 2\mathrm{x})\, dx$ $= \left[-\cos x - \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\pi}$ $= \left[-\cos\pi - \frac{1}{2}\left(\pi - \frac{1}{2}\sin 2\pi\right)\right] - \left[-\cos 0 - \frac{1}{2}\left(0 - \frac{1}{2}\sin 0\right)\right]$ $= 2 - \frac{\pi}{2}$ square units	2 Marks: Correct answer. 1 Mark: Sets up the integrand correctly.
14(d) (i)	A Bernoulli trial. There are only two outcomes: either 5 or not 5. Total outcomes: $6 \times 6 = 36$ Successful outcomes: (1,4) (2,3) (3,2) (4,1) $p = \frac{4}{36} = \frac{1}{9}$	1 Mark: Correct answer.
14(d) (ii)	$E(X) = p = \frac{1}{9}$ $Var(X) = p(1-p)$ $= \frac{1}{9} \times \left(1 - \frac{1}{9}\right)$ $= \frac{8}{81}$	2 Marks: Correct answer. 1 Mark: Finds $E(X)$ or $\mathrm{Var}(X)$.
14(e)	$\frac{dy}{dx} = 1 - 2\mathrm{y}$ $\frac{dx}{dy} = \frac{1}{1 - 2\mathrm{y}}$ $x = \int \frac{1}{1-2y}\, dy$ $= -\frac{1}{2}\ln(1-2y) + C$ Now $x = 0$ $y = -1 \Rightarrow C = \frac{1}{2}\ln 3$ $x = -\frac{1}{2}\ln(1-2y) + \frac{1}{2}\ln 3$ $x = \frac{1}{2}\ln\left(\frac{3}{1-2y}\right)$ $2x = \ln\left(\frac{3}{1-2y}\right)$ $e^{2x} = \frac{3}{1-2y}$ $(1-2y)e^{2x} = 3$ $-2ye^{2x} = 3 - e^{2x}$ $y = \frac{3 - e^{2x}}{-2e^{2x}}$ $\therefore y = \frac{1}{2}(1 - 3e^{-2x})$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Separates the variables and attempts to integrate.