CARLINGFORD HIGH SCHOOL

DEPARTMENT OF MATHEMATICS

Year 12 Mathematics 2U

Term1 HALF YEARLY EXAM 2013

Time allowed: 2 hours

Name: ______________________________ **Class:** _________**Teacher**________

Gong/ Cheng/ Strilakos/ Nicolaou/ Lobejko/ Kellahan/ White

Instructions:

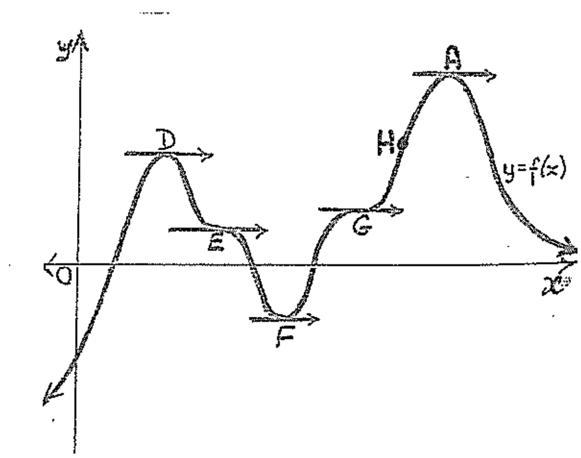
- All questions should be attempted.
- Show ALL necessary working on your own paper.
- Marks may not be awarded for careless or badly arranged work.
- Only board-approved calculators may be used.
- Start each question on a new page and only write on one side of each sheet of paper.

	MC	Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
H2							/10	/10
H5	/5	/11	/12	/11				/39
H6					/12			/12
H8						/11		/11
TOTAL	/5	/11	/12	/11	/12	/11	/10	/72

MULTIPLE CHOICE (5 marks) Circle the correct answer

Q1. Given the curve $y = f(x)$ for what section of the curve is $f'(x) > 0$ *and* $f''(x) > 0$

A. D to E B. E to F C. F to G D. G to H



Q2. What is the least value of the parabola $y = x^2 - 6x + 7$?

A. 2 B. 3 C. -3 D. -2

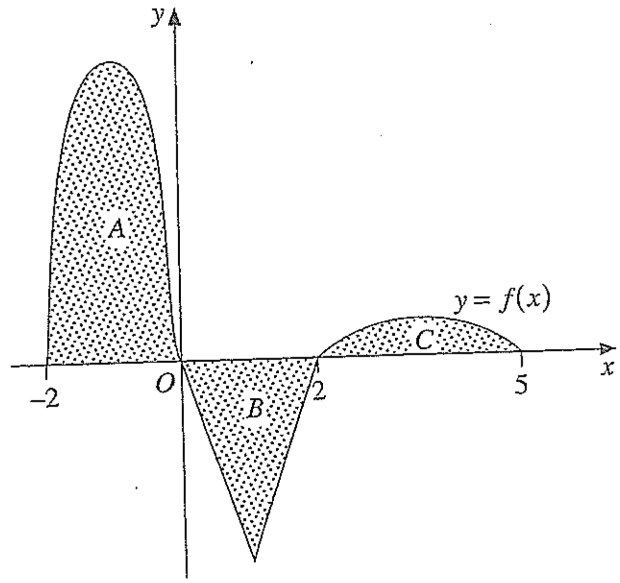
Q3. If the roots of the equation $2x^2 - 3x - 1 = 0$ are α *and* β what is the sum of the roots $\alpha + \beta$

A. $-\frac{1}{2}$ B. $\frac{3}{2}$ C. $-\frac{3}{2}$ D. -3

Q4. Allie invests \$500 at the beginning of each year in a superannuation scheme. If interest is paid at a rate of 6%p.a. on the investment (compounded yearly), what will the investment be worth after 10 years ?

A. $\$500\times 1.06^{10}$
B. $\$500\times (1.06^{10} + 1.06^9 + 1.06^8 + \ldots\ldots + 1.06)$
C. $\$500\times (1.06^9 + 1.06^8 + 1.06^7 + \ldots\ldots + 1.06)$
D. $\$500\times (1.06 \times 10)$

Q5. The graph of the function f is shown in the diagram below. The shaded areas are bounded by $y = f(x)$ and the x axis. The shaded area A is 8 square units, the shaded area B is 3 square units and the shaded area C is 1 square unit.



Evaluate $\int_{-2}^{5} f(x)dx$.

A. 12 B. 9 C. 6 D. 3

Question 1 (11 marks)

(a) (i) Write down the discriminant of $3x^2 + 2x + k = 0$ [1]

(ii) For what values of k does $3x^2 + 2x + k = 0$ have real roots? [2]

(b) Find the equation of the parabola with focus $(1,1)$ and directrix $y = -1$ [1]

(c) Find the primitive of $3 + \frac{1}{x^2}$ [1]

(d) Josh invests $\$1000$ in a term deposit that earns 3.5 % p.a. compounded annually. What is the value of the investment at the end of 20 years? [2]

(e) Differentiate $x^4 + 5x^{-1}$ with respect to x. [1]

(f) Evaluate $\sum_{n=2}^{4} n^2$ [1]

(g) Find the equation of the curve passing through (0,2) given that its gradient function is $2x + 8$ [2]

Question 2 (12 marks)

(a) The first term of a geometric series is 16 and the fourth term is $\frac{1}{4}$.

(i) Find the common ratio. [2]

(ii) Find the limiting sum of the series. [1]

(b) The first three terms of an arithmetic series are $-1 + 4 + 9 + \cdots$

(i) What is the 60th term? [2]

(ii) Hence, or otherwise, find the sum of the first 60 terms of the series. [2]

(c) A timber worker is stacking logs. The logs are stacked in layers, where each layer contains one log less than the layer below. There are five logs in the top layer, six logs in the next layer, and so on. There are n layers altogether.

(i) Write down the number of logs in the bottom layer. [2]

(ii) Show that there are $\frac{1}{2}n(n + 9)$ logs in the stack. [1]

(d) If $4, x, y$ form a geometric sequence and $x, y, 12$ form an arithmetic sequence, find the values of x and y. [2]

Question 3 (11 marks)

(a) Show that if $y = x(2x+3)^7$ then $\frac{dy}{dx} = (2x+3)^6\,(16x+3)$ [2]

(b) Find:

(i) $\int (3x+5)^4\,dx$ [2]

(ii) $\int \frac{2x^3-1}{x^2}\,dx$ [2]

(iii) $\int_1^4 \sqrt{x} + 2\ \,dx$ [2]

(c) Find the coordinates of the focus and equation of the directrix of a parabola with equation $y^2 - 4y + 8x - 12 = 0$ [3]

Question 4 (12 marks)

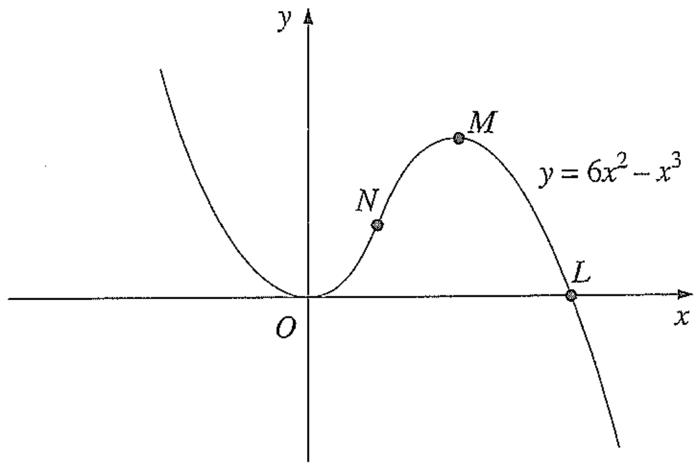
(a) Consider the curve given by $y = 7 + 4x^3 - 3x^4$

(i) Find the coordinates of the two stationary points. [2]

(ii) Determine the nature of the stationary points. [3]

(iii) Sketch the curve for the domain $-1 \leq x \leq 2$ [2]

(b) The diagram below shows a sketch of the curve $y = 6x^2 - x^3$. The curve cuts the x axis at L and has a local maximum at M and a point of inflexion at N.



(i) Find the coordinates of L [1]

(ii) Find the coordinates of M [2]

(iii) Find the coordinates of N [2]

Question 5 (11 marks)

(a) The following table lists the values of a function for five values of x.

x	1.0	2.0	3.0	4.0	5.0
$f(x)$	1.7	9.0	4.3	6.1	3.8

Use these five function values to estimate $\int_1^5 f(x)dx$ by: [4]

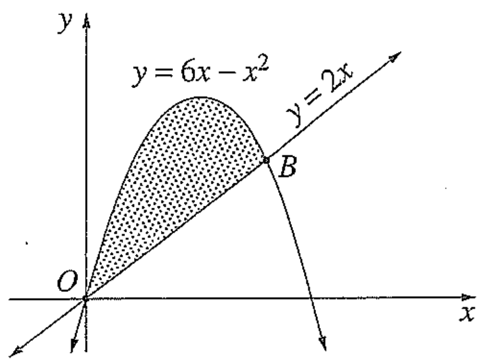
(i) Simpson's rule

(ii) Trapezoidal rule

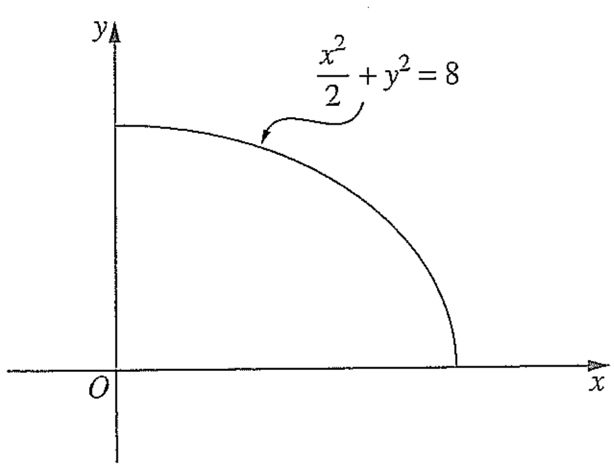
(b) The graphs of $y = 6x - x^2$ and $y = 2x$ intersect at the origin and point B. [4]

(i) Show that the coordinates of B are $(4,8)$

(ii) Find the shaded area bounded by $y = 6x - x^2$ and $y = 2x$



(c) The part of the curve $\frac{x^2}{2} + y^2 = 8$ that lies in the first quadrant is rotated about the x axis. Find the volume of the solid of revolution. [3]



Question 6 (10marks)

(a) Eddy invests \$30 000 in an account which earns 7%p.a. interest, compounded annually. He intends to withdraw \$M at the end of each year, immediately after the interest has been paid. He wishes to be able to do this for exactly 20 years, so that the account will then be empty.

(i) How much money does he have in the account immediately after he has made the first withdrawal? [1]

(ii) Write an expression in terms of M for the amount of money in the account immediately after his 20^{th} withdrawal. [2]

(iii) Calculate the value of M which leaves his account empty after the 20^{th} withdrawal. [2]

(b) A piece of wire $60cm$ long is cut into two parts. One part forms a rectangle, whose length is five times its width and the other part forms a square.

(i) Show that the sum of the two areas is: $A = 14x^2 - 90x + 225$, where x is the width of the rectangle. [2]

(ii) Show that the minimum sum of the two areas is $80\frac{5}{14}cm^2$ [3]

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

YEAR 12 Half Yearly 2unit Mathematics 2013 Solutions

MC Q1 D Q2 D Q3 B Q4 B Q5 C

Quest. 1

(a) ① (i) $\Delta = b^2 - 4ac$
$= (2)^2 - 4(3)(k)$
$= 4 - 12k$

(ii) Real roots when $\Delta \geq 0$
② $4 - 12k \geq 0$
$4 \geq 12k$
$k \leq \frac{1}{3}$

① (b) $(x-1)^2 = 4y$

① (c) $\int 3 + \frac{1}{x^2}\, dx = \int 3 + x^{-2}\, dx$
$= 3x + \frac{x^{-1}}{-1} + C$
$= 3x - \frac{1}{x} + C$

② (d) $A = P(1+r)^n$
$= 1000(1+0.035)^{20}$
$= \$1989.79$

① (e) $4x^3 - 5x^{-2}$

① (f) $(2)^2 + (3)^2 + (4)^2 = 29$

② (g) $\frac{dy}{dx} = 2x + 8$
$\therefore y = x^2 + 8x + C$
At $(0,2)$
$2 = C$
$\therefore y = x^2 + 8x + 2$ is eqn. of curve

Quest. 2

(a) ② (i) $T_4 = ar^3$
$\frac{1}{4} = 16r^3$
$r^3 = \frac{1}{64}$
$\therefore r = \frac{1}{4}$

① (ii) $S = \frac{a}{1-r}$
$= \frac{16}{1 - \frac{1}{4}}$
$= \frac{64}{3}$

(b) ② (i) $a = -1,\ n = 60,\ d = 5$
$T_{60} = -1 + (60-1)5$
$= -1 + 59(5)$
$= 294$

② (ii) $S_n = \frac{n}{2}(a + \ell)$
$S_{60} = \frac{60}{2}(-1 + 294)$
$= 8790$

(c) AP $5, 6, 7, 8, \ldots\ldots$
② (i) $T_n = a + (n-1)d$
$= 5 + (n-1).1$
$= 5 + (n-1)$
$= (n+4)$ logs

① (ii) $S_n = \frac{n}{2}(a + \ell)$
$= \frac{n}{2}(5 + n + 4)$
$= \frac{1}{2}n(n+9)$ logs

(d) ② Since $4, x, y$ form GP
$\frac{x}{4} = \frac{y}{x}$
$x^2 = 4y$
Since $x, y, 12$ form AP
$y - x = 12 - y$
$2y - x = 12$

Solving simult.
$2\left(\frac{x^2}{4}\right) - x = 12$
$\frac{x^2}{2} - x - 12 = 0$
$x^2 - 2x - 24 = 0$
$(x-6)(x+4) = 0$
$\therefore x = 6$ OR -4
When $x = 6$, $y = 9$

Quest 3

(a) $y = x(2x+3)^7$

2

$\frac{dy}{dx} = x \times 7(2x+3)^6 \times 2 + (2x+3)^7 \times 1$

$= 14x(2x+3)^6 + (2x+3)^7$

$= (2x+3)^6[14x + (2x+3)^1]$

$= (2x+3)^6(16x+3)$

(b) (i) 2 $\int (3x+5)^4 \, dx = \frac{(3x+5)^5}{5 \times 3} + C$

$= \frac{(3x+5)^5}{15} + C$

2 (ii) $\int \frac{2x^3 - 1}{x^2} \, dx = \int 2x - \frac{1}{x^2} \, dx$

$= \int 2x - x^{-2} \, dx$

$= \frac{2x^2}{2} - \frac{x^{-1}}{-1} + C$

$= x^2 + \frac{1}{x} + C$

2 (iii) $\int_1^4 \sqrt{x} + 2 \, dx = \int_1^4 x^{\frac{1}{2}} + 2 \, dx$

$= \left[\frac{2x^{\frac{3}{2}}}{3} + 2x\right]_1^4$

$= 2\left[\frac{\sqrt{x}^3}{3} + x\right]_1^4$

$= 2\left\{\left(\frac{8}{3} + 4\right) - \left(\frac{1}{3} + 1\right)\right\}$

$= \frac{32}{3}$

3 (c) $y^2 - 4y + 8x - 12 = 0$

$y^2 - 4y + 4 = 12 - 8x + 4$

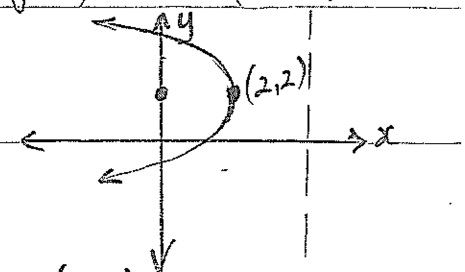
$(y-2)^2 = 16 - 8x$

$(y-2)^2 = -8(x-2)$

$4a = 8$

$\therefore a = 2$

Vertex $(2,2)$



Focus $(0,2)$

Directrix $x = 4$

3 (ii) $\frac{d^2y}{dx^2} = 24x - 36x^2$

$= 12x(2-3x)$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	$-$	0	$+$

change in concavity

$\therefore (0,7)$ is a horizonta point of inflexion

When $x = 1$, $\frac{d^2y}{dx^2} < 0$ ∩

$\therefore$ maximum TP at $(1,8)$

Quest. 4

(a) (i) $y = 7 + 4x^3 - 3x^4$

$\frac{dy}{dx} = 12x^2 - 12x^3$

$= 12x^2(1-x)$

Stat. pts. $\frac{dy}{dx} = 0$

$12x^2(1-x) = 0$

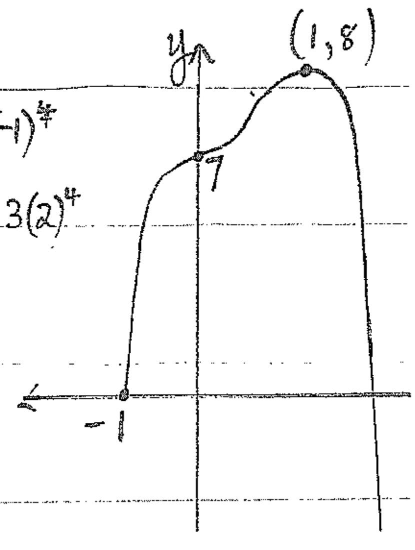
$x = 0$ OR 1

When $x = 0$, $y = 7$

2 (iii) End points

At $x = -1$, $y = 7 + 4(-1)^3 - 3(-1)^4$
$= 0$

At $x = 2$, $y = 7 + 4(2)^3 - 3(2)^4$
$= -9$



① (b) (i) $y = 6x^2 - x^3$

$= x^2(6-x)$

At L, $y=0$

$x^2(6-x) = 0$

$\therefore x = 0$ OR 6

$\therefore$ coordinates of L $(6,0)$

② (ii) $\frac{dy}{dx} = 12x - 3x^2$

At M, $\frac{dy}{dx} = 0$

$12x - 3x^2 = 0$

$3x(4-x) = 0$

$x = 0$ OR 4

When $x=4$,

$y = 6(4)^2 - (4)^3$

$= 32$

$\therefore$ Co-ordinates of M $(4,32)$

② (iii) $\frac{d^2y}{dx^2} = 12 - 6x$

At N, $\frac{d^2y}{dx^2} = 0$

$12 - 6x = 0$

$6x = 12$

$x = 2$

When $x = 2$, $y = 6(2)^2 - (2)^3$

$= 16$

$\therefore$ Co-ordinates of N are $(2,16)$

Quest.5

(a) (i) ② $A \doteq \frac{1}{3}\{1{\cdot}7 + (4\times 9{\cdot}0) + (2\times 4{\cdot}3) + (4\times 6{\cdot}1) + 3{\cdot}8\}$

$= 24{\cdot}8\dot{3}$ units^2

② (ii) $A \doteq \frac{1}{2}\{1{\cdot}7 + 2(9{\cdot}0 + 4{\cdot}3 + 6{\cdot}1) + 3{\cdot}8\}$

$= 22{\cdot}15$ units^2

(b) (i) Solving simult.

④ $6x - x^2 = 2x$

$4x - x^2 = 0$

$x(4-x) = 0$

$\therefore x = 0$ OR 4

At $x=4$, $y = 2(4)$

$= 8$

$\therefore$ Coordinates of B are $(4,8)$

(ii) Shaded Area $= \int_0^4 6x - x^2\,dx - \int_0^4 2x\,dx$

$= \int_0^4 4x - x^2\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \left(2(4)^2 - \frac{(4)^3}{3}\right) - (0)$

$= \frac{32}{3}$ units^2

$= 10\frac{2}{3}$

(c) Curve cuts x axis when $y=0$

③ $\therefore \frac{x^2}{2} + (0)^2 = 8$

$x^2 = 16$

$x = \pm 4$

In 1st quad. cut at $x = 4$

Now $\frac{x^2}{2} + y^2 = 8$

$y^2 = 8 - \frac{x^2}{2}$

$V = \pi \int_0^4 y^2\,dx$

$= \pi \int_0^4 \left(8 - \frac{x^2}{2}\right)dx$

$= \pi \left[8x - \frac{x^3}{6}\right]_0^4$

$= \pi \left(8(4) - \frac{(4)^3}{6}\right) - 0$

$= \pi \left(\frac{64}{3}\right)$

$(4\pi$ [illegible] 3

Quest. 6

(a) (i) (1)

$$A_1 = 30\,000(1+0{\cdot}07) - M$$
$$= 30\,000(1.07) - M$$

(ii) (2)

$$A_2 = A_1(1.07) - M$$
$$= [30000(1{\cdot}07) - M]\,1{\cdot}07 - M$$
$$= 30000(1{\cdot}07)^2 - 1{\cdot}07M - M$$

$$A_3 = A_2(1{\cdot}07) - M$$
$$= [30000(1{\cdot}07)^2 - 1{\cdot}07M - M]\,1{\cdot}07 - M$$
$$= 30\,000(1{\cdot}07)^3 - 1{\cdot}07^2M - 1{\cdot}07M - M$$

$$\vdots$$

$$A_{20} = 30000(1{\cdot}07)^{20} - 1{\cdot}07^{19}M - 1{\cdot}07^{18} - \ldots - M$$
$$= 30\,000(1{\cdot}07)^{20} - M(1{\cdot}07^{19} + 1{\cdot}07^{18} + \ldots + 1)$$

OR

$$30\,000(1{\cdot}07)^{20} - M\left[\frac{1{\cdot}07^{20}-1}{0{\cdot}07}\right]$$

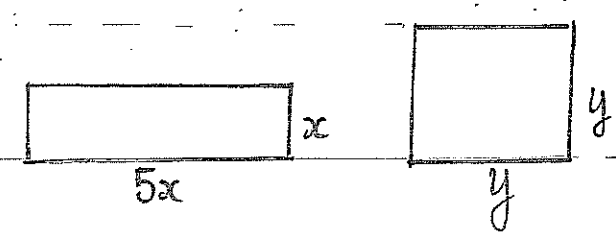
(2) (iii) A_{20} is zero after 20th withdraw

$$0 = 30000(1{\cdot}07)^{20} - M\left[\frac{1{\cdot}07^{20}-1}{0{\cdot}07}\right]$$

$$\therefore M = \frac{30\,000(1{\cdot}07)^{20}}{\left[\frac{1{\cdot}07^{20}-1}{0{\cdot}07}\right]}$$

$$= \$2831{\cdot}79$$

(b) (i) (2)



Sum of two areas is:

$$A = 5x^2 + y^2$$

But $5x + 5x + x + x + y + y + y + y = 60$

$$12x + 4y = 60$$
$$3x + y = 15$$
$$y = 15 - 3x$$

$$\therefore A = 5x^2 + (15-3x)^2$$
$$= 5x^2 + 225 - 90x + 9x^2$$
$$= 14x^2 - 90x + 225$$

(ii) (3) A is minimum when $\frac{dA}{dx} = 0$ & $\frac{d^2A}{dx^2} > 0$

$$\frac{dA}{dx} = 28x - 90$$
$$28x - 90 = 0$$
$$28x = 90$$
$$x = \frac{90}{28}$$
$$= 3\tfrac{3}{14}$$

$$A = 14 \times \left(3\tfrac{3}{14}\right)^2 - 90 \times \left(3\tfrac{3}{14}\right) + 225$$
$$= 80\tfrac{5}{14}\ \text{cm}^2$$

$$\frac{d^2A}{dx^2} = 28 > 0 \text{ (for all } x \text{ values)}$$

$\therefore$ A is a minimum