Solutions to Extension 2 Maths Test Ass^T Task 1 2020

Q1 Consider the statement

For any integers a and b, $a+b \geq 15$ implies that $a \geq 8$ or $b \geq 8$,

(i) State the contrapositive of this statement

(ii) Hence prove this statement is true using contrapositive.

[1+2=3]

(i) For any integers a and b, $a<8$ and $b<8$ implies that $a+b<15$

(ii) Proof: Suppose that a and b are integers such that $a<8$ and $b<8$.

Since they are integers this implies that $a \leq 7$ and $b \leq 7$.

$\therefore a+b \leq 14 \Rightarrow a+b<15$ Which is true.

Q.2 (i) Let $x \in Z$. Prove by contradiction that if $5x - 7$ is odd, then x is even.

(ii) Hence prove directly that if $5x - 7$ is odd, then $9x + 2$ is even.

[2+~~3~~=~~5~~] 2 4 · -1

(i) To Prove by contradiction:

Assume that if $5x-7$ is odd, then x is odd.

Let $x = 2y+1$, $y \in \mathbb{Z}$

$\therefore$ $5x-7 = 5(2y+1) - 7 = 10y+5-7 = 10y-2 = 2(5y-1)$

Since $5y-1$ is an integer, $5x-7$ is even if x is odd.

thus by contradiction, $5x-7$ is odd if x is even.

(ii) If $5x-7$ is odd, then we have already proved that x is even.

$\therefore$ let $x = 2z$, $z \in \mathbb{Z}$.

Thus, $9x+2 = 9 \times 2z + 2 = 18z+2 = 2(9z+1)$

Since $9z+1$ is an integer, $9x+2$ must be even.

So, if $5x-7$ is odd, $9x+2$ is Even $\square$.

Q3. Let $x \in Z$. (i) Prove that if $3|x$, then $3|x^2$.

(ii) Prove that if $3 \nmid x$, then $3|(x^2-1)$, using cases.

[2+4=6] 3 5 -1.

(i) Proof: Assume that 3 divides x. Then $x = 3q$, $q \in \mathbb{Z}$ 1

Hence $x^2 = 9q^2 = 3(3q^2)$. 1

Since $3q^2 \in \mathbb{Z}$, it follows that $3|x^2$.

(ii) To Prove that if $3 \nmid x$, then $3|(x^2-1)$:

Proof: If $3 \nmid x$, then $x = 3q+1$ or $x = 3q+2$, $q \in \mathbb{Z}$ 1

<u>Case 1</u>: If $x = 3q+1$, $q \in \mathbb{Z}$ then

$$x^2-1 = (3q+1)^2-1 = 9q^2+6q+1-1 = 3(3q^2+2q)$$ 1

Since $3q^2+2q$ is an integer, $3|(x^2-1)$.

<u>Case 2</u>: If $x = 3q+2$, $q \in \mathbb{Z}$ then

$$x^2-1 = (3q+2)^2-1 = 9q^2+12q+4-1 = 3(3q^2+4q+1)$$ 1

Since $3q^2+4q+1$ is an integer, $3|(x^2-1)$

$\therefore$ It is true, if $3 \nmid x$, then $3|(x^2-1)$ $\square$

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Q.4 If $T(0) = 6$ and $T_n = 4T_{n-1} + 2^n$ for $n \geq 1$,

use Induction to prove that $T_n = 7 \cdot 4^n - 2^n$ [5]

Proof: Given $T(0) = 6$ and $T_n = 4T_{n-1} + 2^n$, $n \geq 1$.

Show true for $n=1$. $T_1 = 4T_0 + 2^1 = 4 \times 6 + 2 = 26$

$T_1 = 7.4^1 - 2^1 = 28 - 2 = 26$ thus true for $n=1$. 1

Assume true for $n=k$. i.e. $T_k = 7.4^k - 2^k$

Prove true for $n=k+1$ i.e. $T_{k+1} = 7.4^{k+1} - 2^{k+1}$ 1

Now, from the recursion formula,

$$\begin{aligned} T_{k+1} &= 4T_k + 2^{k+1} \\ &= 4[7.4^k - 2^k] + 2^{k+1} \quad 1 \\ &= 7.4^{k+1} - 4.2^k + 2^{k+1} \\ &= 7.4^{k+1} - 2^2.2^k + 2^{k+1} \quad 1 \\ &= 7.4^{k+1} - 2.2^{k+1} + 2^{k+1} \quad 1 \\ &= 7.4^{k+1} - 2^{k+1} \\ &= RHS \end{aligned}$$

$\therefore$ true using Induction.

more room

Q.5 Use a calculus method to prove that if $x \in R$, $x > 0$, then $x^4 + x^{-4} \geq 2$. [4]

Let $f(x) = x^4 + x^{-4}$

$f'(x) = 4x^3 - 4x^{-5} = 4x^{-5}(x^8 - 1) = 0$ for max/min / stat. points 1

Since $x > 0$, $4x^{-5} \neq 0$ $\therefore f'(x) = 0$ when $x^8 - 1 = 0$ i.e. $x = \pm 1$

But $x > 0$, $\therefore$ disregard $x = -1$. 1 For $x = 1$, $f(x) = 1 + 1 = 2$. 1

Now, as $x \to +\infty$, $f(x) = x^4 + x^{-4} = x^4 + \frac{1}{x^4} \to +\infty$ 1

4

and there are no other t.p.s.

$\therefore f(1) = 2$ is the minimum value of $f(x)$ and so

$x^4 + x^{-4} \geq 2 \quad \forall x$.

Q6 The diagram below shows two right angled triangles.

The left one has sides a, b and c where c is the length of the hypotenuse.

The triangle on the right has sides of length a+1, b+1 and c+1, where c+1 is the length of the hypotenuse. Show that a, b and c cannot all be integers.

[4]

<u>Proof</u>: For the 1st triangle, $a^2+b^2=c^2$

i.e. $a^2+b^2-c^2=0$ — ① 1

For the 2nd triangle,

$(b+1)^2+(a+1)^2=(c+1)^2$

i.e. $b^2+2b+1+a^2+2a+1=c^2+2c+1$

i.e. $\underbrace{a^2+b^2-c^2}_{=0 \text{ from ①}}+2(a+b)+1=2c$ 1

1

i.e. $2(a+b)+1=2c$

Now, since a, b, c are all positive, if a, b & c are all integers, then LHS is odd, and RHS is even, impossible! 1

∴ a, b, c cannot all be integers.

Q.7 Prove by contradiction, the proposition that:

For each real number x, if $0 < x < 1$, then

$$\frac{1}{x(1-x)} \geq 4$$

[4]

Proof: Using contradiction:

Assume there exists an x, $0<x<1$, such that

$$\frac{1}{x(1-x)} < 4 \quad -①$$

Now, since $0<x<1$, both x and $(1-x)$ are +ve, $\therefore x(1-x)>0$

$\times$ b.s. of ① by $x(1-x)$ to obtain:

$$1 < 4x(1-x)$$

i.e. $1 < 4x - 4x^2 \Rightarrow 4x^2 - 4x + 1 < 0$

$$(2x-1)^2 < 0$$

But since $(2x-1)$ is real, $(2x-1)^2 > 0$ which is a contradiction of the last statement.

Hence $\frac{1}{x(1-x)} \geq 4$ must be true.

4.

Q.8 (i) Show that $\frac{a}{b} + \frac{b}{a} \geq 2$ using the AM/GM inequality.

(ii) Hence show that, for a, b and c all positive reals, that

$$a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$$

[2+2=4]

(i) Let $\frac{a}{b} = x$ and $\frac{b}{a} = y$.

$\therefore \frac{x+y}{2} \geqslant \sqrt{xy}$ using Am/Gm Inequality. 1

$\therefore \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right) \geqslant \sqrt{\frac{a}{b}\frac{b}{a}}$ ie. $\frac{a}{b}+\frac{b}{a} \geqslant 2$. 1

(ii) $\because$ $a^3+a^3+b^3 \geqslant 3\sqrt[3]{a^3a^3b^3} = 3a^2b$

$b^3+b^3+c^3 \geqslant 3\sqrt[3]{b^3b^3c^3} = 3b^2c$ 1

$c^3+c^3+a^3 \geqslant 3\sqrt[3]{c^3c^3a^3} = 3c^2a$

Adding, we get $3(a^3+b^3+c^3) \geq 3(a^2b+b^2c+c^2a)$ 1

$\therefore a^3+b^3+c^3 \geqslant a^2b+b^2c+c^2a$

<u>or</u> see next sheet

Q 8(ii) Alternative Solution.

$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$

ie. Prove that:

$\Rightarrow \quad a^3+b^3+c^3 - (a^2b+b^2c+c^2a) \geq 0 \qquad - (3)$

Letting $A = a^2b$, $B = b^2c$, $C = c^2a$

$$\frac{A+B+C}{3} \geq \sqrt[3]{a^2b.b^2c.c^2a} = \sqrt[3]{a^3b^3c^3}$$

$\Rightarrow a^2b+b^2c+c^2a = A+B+C \geq 3\sqrt[3]{a^3b^3c^3} = 3abc \qquad - (1)$

Also, $\frac{a^3+b^3+c^3}{3} \geq \sqrt[3]{a^3b^3c^3}$

ie. $a^3+b^3+c^3 \geq 3\sqrt[3]{abc} = 3abc \qquad - (2)$

Subst. (1) & (2) into (3)

$3abc - 3abc \geq 0$ as required.

Q. 9 (i) Find the square roots of $-8-6i$.

(ii) Hence or otherwise, solve the equation $2x^2+(1+i)x+(1+i)=0$

[4]

(i) Let $z = x+iy$ and $z^2 = -8-6i$ $\quad x, y \in \mathbb{R}$.

$\therefore \quad z^2 = x^2-y^2+2xyi = -8-6i$

$\Rightarrow x^2-y^2=-8 \quad -(1) \qquad 2xy=-6 \Rightarrow y = -\frac{3}{x} \quad -(2)$ | 1

(2) into (1) yields $\quad x^2 - \frac{9}{x^2} = -8 \quad$ ie. $x^4+8x^2-9=0$

$(x^2-1)(x^2+9)=0$

Since x is real, $x = \pm 1 \quad$ If $x=1, y=-3$

If $x=-1, y=3$

1

Thus the square roots are: $z_1 = 1-3i$, $z_2 = -1+3i$

(ii) To solve, $\quad x = \frac{-(1+i) \pm \sqrt{(1+i)^2 - 4.2(1+i)}}{4} = \frac{-1-i \pm \sqrt{1+2i-1-8-8i}}{4}$

$= \frac{-1-i \pm \sqrt{-8-6i}}{4}$ | 1

ie. from (i) $\quad x = \frac{-1-i \pm (1-3i)}{4} = \frac{-1-i+1-3i}{4} = -i$

or

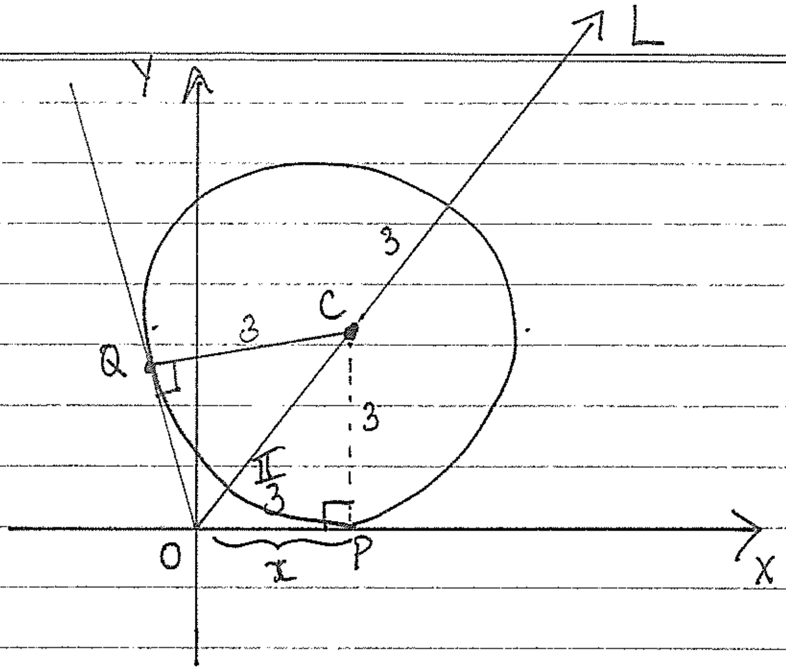
or $\frac{-1-i-1+3i}{4} = -\frac{1}{2} + \frac{i}{2}$

1

(4)

Q.10

(i)



①

(ii) For coords of centre, $\tan\frac{\pi}{3} = \frac{3}{x}$ $\therefore x = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$.

$y=3$ ②

$\therefore$ Circle has eqn $|z-\sqrt{3}-3i| = 3$

(iii) $\triangle OCP \equiv \triangle OCQ$ $\therefore \angle COQ = \frac{\pi}{3}$. ①

$\therefore$ max^{m} z_0 is $\frac{2\pi}{3}$

Q11. $\operatorname{Im}\{(2z - \bar{z}(1+i))\} = 0$ & $\operatorname{Re}(2z - \bar{z}(1+i)) < 4$

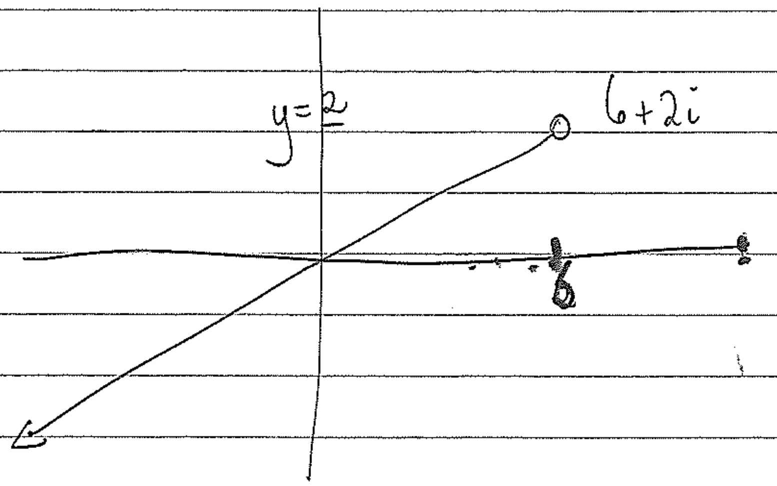
$$2z - \bar{z}(1+i) = 2(x+yi) - (x-yi)(1+i)$$
$$= 2x + 2yi - [x + xi - iy + y]$$
$$= x - y + 3yi - xi \quad 1$$
$$= (x-y) + i(3y - x)$$

(4)

$\operatorname{Im}(\quad) = 0$ when $3y = x$ } 1

$\operatorname{Re}(\quad) < 4$ when $x - y < 4$ ie $3y - y < 4$, $2y < 4$
$y < 2$ 1

So we are sketching $3y = x$ for $y < 2$. 1

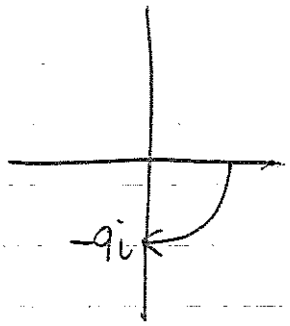


1 (5)

(1).

Q.12. $z^4 = -9i$

(i)



Let $z = r \operatorname{cis}\theta$

$z^4 = r^4 \operatorname{cis} 4\theta$

$= r^4(\cos 4\theta + i \sin 4\theta)$

$= -9i$

$\therefore \cos 4\theta = 0$ and $\sin 4\theta = -1$

$4\theta = \frac{3\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$

$\theta = \frac{3\pi}{8} + \frac{k\pi}{2} = \frac{3\pi + 4k\pi}{8}$

$= \frac{\pi}{8}(3+4k)$

$r^4 = 9$

$r = \sqrt{3}$ 2

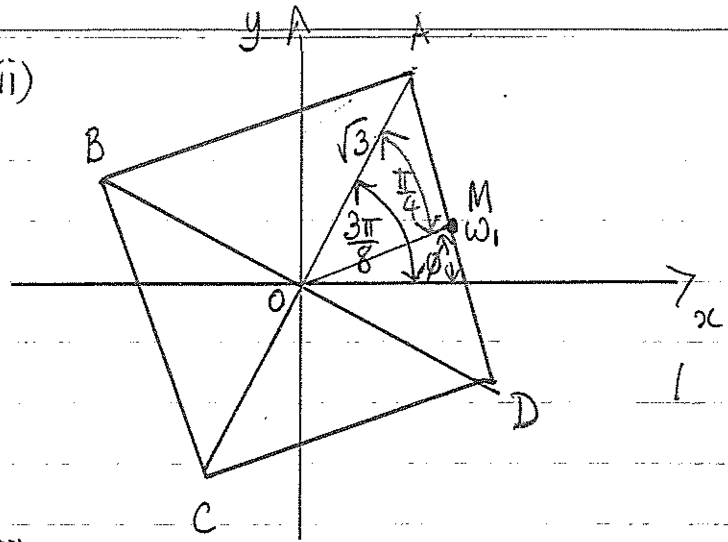
$\therefore z = \sqrt{3}\, e^{i\frac{\pi}{8}(3+4k)}$

$z_1 = \sqrt{3}\, e^{i\frac{3\pi}{8}}$ and $z_4 = \sqrt{3}\, e^{i\frac{15\pi}{8}}$

$z_2 = \sqrt{3}\, e^{i\frac{7\pi}{8}}$

$z_3 = \sqrt{3}\, e^{i\frac{11\pi}{8}}$

(ii)



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(iii)

$|OM| = \sqrt{3}\cos\frac{\pi}{4} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

$\omega_1 = |OM| \operatorname{cis} \phi$

where $\phi = \frac{3\pi}{8} - \frac{\pi}{4} = \frac{\pi}{8}$ 2

$\therefore \omega_1 = \frac{\sqrt{6}}{2} e^{i\frac{\pi}{8}}$

(iv) $\omega = \left(\frac{\sqrt{6}}{2} e^{i\frac{\pi}{8}}\right)^4$

$= \frac{36}{16} e^{i\frac{\pi}{2}} = \frac{9}{4} i$ 1

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Q.13 (i) $e^{i\theta} = \cos\theta + i\sin\theta$ — ①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$e^{-i\theta} = \cos\theta - i\sin\theta$ — ②

① + ② yields $2\cos\theta = e^{i\theta} + e^{-i\theta}$ $\therefore \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ 1

① − ② yields $2i\sin\theta = e^{i\theta} - e^{-i\theta}$ $\therefore \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ 1

Q.13.

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right) \quad -(1) \qquad \cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$$

(ii) $$\sin^3\theta\cos^2\theta = -\frac{1}{8i}\left(e^{i\theta}-e^{-i\theta}\right)^3 \cdot \frac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)^2$$

$$= -\frac{1}{8i}\left(e^{i3\theta}-3e^{i2\theta}.e^{-i\theta}+3e^{i\theta}.e^{-i2\theta}-e^{-i3\theta}\right)\cdot\frac{1}{4}\left(e^{i2\theta}+2e^{i\theta}.e^{-i\theta}+e^{-i2\theta}\right)$$

$$= -\frac{1}{32i}\left(e^{i3\theta}-e^{-i3\theta}-3e^{i\theta}+3e^{-i\theta}\right)\left(e^{i2\theta}+2+e^{-i2\theta}\right) \qquad 1$$

$$= -\frac{1}{32i}\left(e^{i5\theta}+2e^{i3\theta}+e^{i\theta}-e^{-i\theta}-2e^{-i3\theta}-e^{-i5\theta}-3e^{i3\theta}-6e^{i\theta}-3e^{-i\theta}\right.$$
$$\left.+3e^{i\theta}+6e^{-i\theta}+3e^{-i3\theta}\right)$$

This is a bit tedious

$$= -\frac{1}{32i}\left[\left(e^{i5\theta}-e^{-i5\theta}\right)-\left(e^{i3\theta}-e^{-i3\theta}\right)-\left(2e^{i\theta}-2e^{-i\theta}\right)\right]$$

$$= \frac{1}{32i}\left[2\left(e^{i\theta}-e^{-i\theta}\right)+\left(e^{i3\theta}-e^{-i3\theta}\right)-\left(e^{i5\theta}-e^{-i5\theta}\right)\right]$$

$$= \frac{1}{16}\left[\frac{2}{2i}\left(e^{i\theta}-e^{-i\theta}\right)+\frac{1}{2i}\left(e^{i3\theta}-e^{-i3\theta}\right)-\frac{1}{2i}\left(e^{i5\theta}-e^{-i5\theta}\right)\right]$$

$$= \frac{1}{16}\left[2\sin\theta+\sin3\theta-\sin5\theta\right] \qquad \text{from (1)} \qquad 1$$

(iii)

Now, to solve $\sin5\theta - \sin3\theta = 0$

$$\sin^3\theta\cos^2\theta = \frac{1}{16}\left[2\sin\theta+\sin3\theta-\sin5\theta\right] \qquad 1$$

$$\Rightarrow 16\sin^3\theta\cos^2\theta - 2\sin\theta = \sin3\theta - \sin5\theta$$

ie $$\sin5\theta - \sin3\theta = 2\sin\theta - 16\sin^3\theta\cos^2\theta$$

$$= 2\sin\theta\left(1-8\sin^2\theta\cos^2\theta\right)$$
$$= 0 \qquad 1$$

When either $\sin\theta = 0$ or

$1 = 8\sin^2\theta\cos^2\theta$

Using $\sin2\theta = 2\sin\theta\cos\theta$,

$1 = 2\left[\sin2\theta\right]^2$

$1 = 2\sin^2 2\theta$

$\therefore \frac{1}{2} = \sin^2 2\theta \qquad 1$

$\sin2\theta = \pm\frac{1}{\sqrt{2}} \qquad 1$

Q.13

Now, for $\sin\theta = 0$ and $0 \le \theta < \pi$

$$\theta = 0$$

and for $\sin 2\theta = \pm\frac{1}{\sqrt{2}}$, $2\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\therefore \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$ 1

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If using $\sin^3\theta\cos^2\theta = \sin^3\theta(1-\sin^2\theta)$
$= \sin^3\theta - \sin^5\theta$

$$\sin^5\theta = \left(\frac{1}{2i}\right)^5\left(e^{i\theta}-e^{-i\theta}\right)^5$$

$$= -\frac{1}{32i}\left(e^{i5\theta} - 5e^{i3\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-i3\theta} - e^{-i5\theta}\right)$$

$$= -\frac{1}{32i}\left(2i\sin 5\theta + 5\times 2i\sin 3\theta + 10\times 2i\sin\theta\right)$$

$$= -\frac{1}{32i}\left(2i\sin 5\theta + 10i\sin 3\theta + 20i\sin\theta\right)$$

$\therefore \sin^3\theta - \sin^5\theta = \ldots$

Q.14 $\quad z = \cos\theta + i\sin\theta$

(i) $\quad 1+z = 1+\cos\theta + i\sin\theta$ $\qquad$ using $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\qquad = 2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ $\qquad \Rightarrow 1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$\qquad = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$ $\qquad$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

(ii) $\quad |z_1| = |z_2| = 1 \qquad \arg z_1 = \alpha,\ \arg z_2 = \beta.$

$$\frac{z_1 + z_1 z_2}{z_1+1} = \frac{z_1(1+z_2)}{z_1+1}$$

Now, from part (i), $1+z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

Since $\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$ has modulus 1, $1+z$ has modulus $2\cos\frac{\theta}{2}$

$\therefore$ $1+z_2$ has modulus $2\cos\frac{\beta}{2}$ and $1+z_1$ has modulus $2\cos\frac{\alpha}{2}$

$$\therefore \left|\frac{z_1(1+z_2)}{z_1+1}\right| = \frac{1\times\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$$ since $|z_1| = 1$ also.

For $\arg\left(\frac{z_1(1+z_2)}{z_1+1}\right)$, $= \arg z_1 + \arg(1+z_2) - \arg(z_1+1)$

$$= \alpha + \frac{\beta}{2} - \frac{\alpha}{2} = \frac{\alpha}{2} + \frac{\beta}{2} = \frac{\alpha+\beta}{2}$$

Q-14(iii) If $\frac{z_1+z_1z_2}{z_1+1} = 2i$,

then the real part $=0$ and imaginary part $=2$

Now, $\frac{z_1+z_1z_2}{z_1+1} = \frac{z_1(1+z_2)}{z_1+1} = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}\left(\text{cis}\ \frac{\alpha+\beta}{2}\right)$ #

$$= \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}\left(\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right)$$

$Re\left(\frac{z_1+z_1z_2}{z_1+1}\right) = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \cdot \cos\left(\frac{\alpha+\beta}{2}\right) = 0$ — ① 1

and $Im\left(\frac{z_1+z_1z_2}{z_1+1}\right) = \frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \cdot \sin\left(\frac{\alpha+\beta}{2}\right) = 2.$ — ② 1

Now, assuming the number exists, $\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \neq 0$

$\therefore \cos\left(\frac{\alpha+\beta}{2}\right) = 0$ and $\sin\left(\frac{\alpha+\beta}{2}\right) > 0$ 1

$\Rightarrow \frac{\alpha+\beta}{2} = \frac{\pi}{2}$ —①' and $\sin\left(\frac{\alpha+\beta}{2}\right) = 1$ — ③

from here

1

$\therefore \frac{\alpha}{2} + \frac{\beta}{2} = \frac{\pi}{2}$ —① and $\sin\left(\frac{\alpha+\beta}{2}\right) = 1$ —② 1 —①''

$\Rightarrow \frac{\beta}{2} = \frac{\pi}{2} - \frac{\alpha}{2}$ $\therefore \cos\frac{\beta}{2} = \cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) \Rightarrow \cos\frac{\beta}{2} = \sin\frac{\alpha}{2}$ from ①

Q14 (continued)

Substituting ③ into ②.

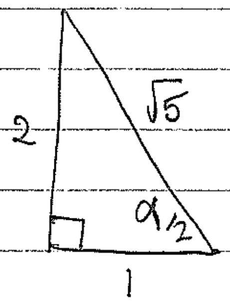
$$\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} \cdot 1 = 2$$

$$\therefore\ 2\cos\frac{\alpha}{2} = \cos\frac{\beta}{2}.$$

But $\cos\frac{\beta}{2} = \sin\frac{\alpha}{2}$ from ①".

$$\therefore\ 2\cos\frac{\alpha}{2} = \sin\frac{\alpha}{2} \Rightarrow \tan\frac{\alpha}{2} = 2$$

Now $\tan\frac{\alpha}{2} = 2$ $\quad\therefore\ \sin\frac{\alpha}{2} = \frac{2}{\sqrt5},\ \cos\frac{\alpha}{2} = \frac{1}{\sqrt5}$



$$\therefore\ \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{4}{5}.$$

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$

$$\therefore\ z_1 = -\frac{3}{5} + \frac{4}{5}i.$$

\#

Also, since $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\pi}{2}$, $\quad \alpha + \beta = \pi$ $\quad \therefore\ \cos\beta = \cos(\pi - \alpha)$

$$= -\cos\alpha$$

$$= \frac{3}{5}$$

$$\sin\beta = \sin(\pi - \alpha)$$

$$= \sin\alpha = \frac{4}{5}$$

$$\therefore\ z_2 = \frac{3}{5} + \frac{4}{5}i.$$

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