CARLINGFORD HIGH SCHOOL

Year 12 Mathematics Extension 1

HSC Assessment Task 3 2019

Time allowed: 55 minutes

Student Number: ________________________________

Instructions:

- All questions should be attempted
- Show all necessary working
- Marks may not be awarded for careless or badly arranged work
- Only board-approved calculators may be used
- Start each question on a new sheet

	Question 1	Question 2	Question 3	Mark
Integration Techniques	/3	/3		/6
Exponential and Logarithmic Functions	/9			/9
Trigonometric Functions		/9		/9
Inverse Functions			/12	/12
	/12	/12	/12	/36

Question 1

(a) Use the substitution $u = 1 + 2x$ to find the exact value of $\int_1^2 \frac{x}{1+2x}\, dx$. **[3]**

(b) \$500 is deposited into an account that pays $1.25\%p.a.$ interest, compounded annually. **[3]**

How many years and months will it take for this deposit to grow to \$1000?

(c) (i) Differentiate $2xe^{-x}$. **[1]**

(ii) Hence find $\int 2xe^{-x} dx$. **[2]**

(d) Find the volume of the solid of revolution formed when $y = \frac{2}{\sqrt{2-x}}$ is revolved around the x-axis between $x = 0$ and $x = 1$. (Leave your answer in exact form) **[3]**

Question 2

(a) Find $\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\, dx$ by making the substitution $x = 2sin\, u$. **[3]**

(b) Given $cosA = \frac{2}{5}$ and $tanB = \frac{1}{2}$, where $\pi \leq A \leq 2\pi$ and $\pi \leq B \leq 2\pi$, find the exact value of $\sin(2A + B)$. **[3]**

(c) (i) Write $\sqrt{3}cos\, \theta - sin\, \theta$ in the form $Rcos(\theta + \alpha)$. **[1]**

(ii) Hence or otherwise, solve $sec\, \theta + tan\, \theta = \sqrt{3}$, for $0 \leq \theta \leq \frac{\pi}{2}$. **[2]**

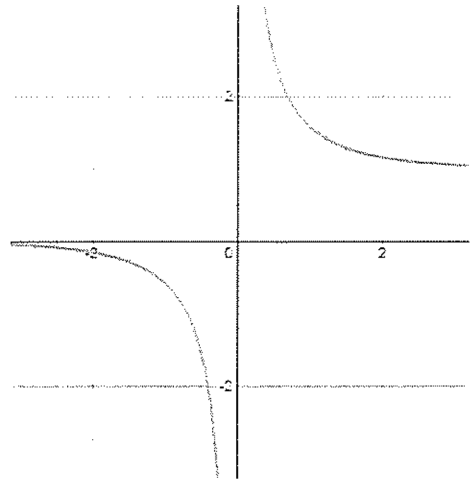
(d) Evaluate $\lim_{x \to 0} \frac{sin\frac{x}{2}}{x}$. **[1]**

(e) Evaluate $\int_0^{\frac{\pi}{3}} cos^2 3x\, dx$. **[2]**

Question 3

(a) The graph of $f(x) = \frac{e^x}{e^x-1}$ is shown below. **[3]**

Find $f^{-1}(x)$, and state its domain.



(b) Explain why $sin^{-1}(cos^{-1}(-1))$ cannot be evaluated. **[2]**

(c) Evaluate $\int_0^1 \frac{-1}{\sqrt{4-x^2}} dx$. **[2]**

(d) Sketch $y = sin^{-1}(\frac{x}{2}) + cos^{-1}(\frac{x}{2})$, for the correct domain. **[2]**

(e) (i) Find the domain and range for the function $y = 2tan^{-1}(\frac{x}{3} + 1)$. **[1]**

(ii) Hence or otherwise, sketch $= 2tan^{-1}(\frac{x}{3} + 1)$, labelling any axes intercepts. **[2]**

END OF TEST

MATHEMATICS EXTENSION 1 - HSC AT3 2019 - SOLUTIONS

QUESTION 1 (GF)

(a) $$\int_1^2 \frac{x}{1+2x}\,dx = \int_3^5 \frac{\frac{u-1}{2}}{u}\cdot\frac{1}{2}\,du \quad ✓$$

$u = 1+2x \rightarrow x = \frac{u-1}{2}$

$du = 2\,dx \rightarrow dx = \frac{1}{2}du$

$x=1,\ u=3$

$x=2,\ u=5$

$$= \frac{1}{4}\int_3^5 \frac{u-1}{u}\,du$$

$$= \frac{1}{4}\int_3^5 1-\frac{1}{u}\,du$$

$$= \frac{1}{4}\Big[u - \ln u\Big]_3^5 \quad ✓$$

$$= \frac{1}{4}\Big[5-\ln 5-(3-\ln 3)\Big]$$

$$= \frac{1}{4}(2-\ln 5+\ln 3)$$

or

$$\frac{1}{4}\left(2+\ln\frac{3}{5}\right) \quad ✓$$

(b) $A = P(1+r)^n$

$1000 = 500(1+1.25\%)^n$ ✓

$2 = 1.0125^n$

$\cancel{\frac{\log 1.0125}{\log 2}}$

$\log_{1.0125} 2 = \log_{1.0125}(1.0125)^n$

$$\therefore n = \log_{1.0125} 2 = \frac{\log 2}{\log 1.0125} \quad ✓$$

$= 55.7976\ldots$

✓ $= 55$ years 10 months

(or 55 years 9.6 months)

QUESTION 1 (cont.) (GF)

(c) (i) $y = 2xe^{-x}$

$$\left.\begin{aligned} y' &= 2e^{-x} + (-2x)e^{-x} \\ &= 2e^{-x} - 2xe^{-x} \\ &= 2e^{-x}(1-x) \end{aligned}\right\} \checkmark$$

(ii) $\int 2e^{-x} - 2xe^{-x}\,dx = 2xe^{-x} + c$ from (i).

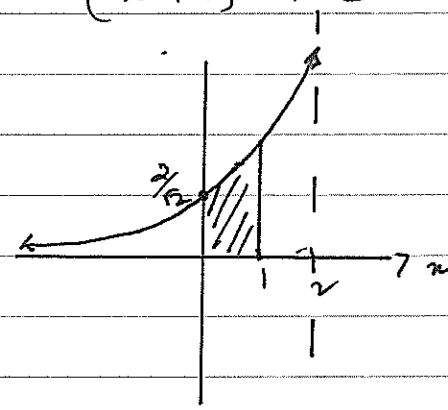
$$\int 2e^{-x}\,dx - \int 2xe^{-x}\,dx = 2xe^{-x} + c.$$

$$-\int 2xe^{-x}\,dx = 2xe^{-x} - \int 2e^{-x}\,dx + c \quad \checkmark$$

$$-\int 2xe^{-x}\,dx = 2xe^{-x} + 2e^{-x} + c.$$

$$\left.\begin{aligned} \therefore \int 2xe^{-x}\,dx &= -2xe^{-x} - 2e^{-x} + c \\ &= -2e^{-x}(x+1) + c \end{aligned}\right\} \checkmark$$

(d) $V = \pi \int_a^b y^2\,dx$



$= \pi \int_0^1 \left(\frac{2}{\sqrt{2-x}}\right)^2 dx$ ✓

$= \pi \int_0^1 \frac{4}{2-x}\,dx$

$= -4\pi \int_0^1 \frac{-1}{2-x}\,dx$ (diagram not required)

$= -4\pi \left[\ln(2-x)\right]_0^1$ ✓

$= -4\pi\left[\ln 1 - \ln 2\right]$

$= -4\pi\left[-\ln 2\right]$

$= 4\pi \ln 2$ units^3 ✓

QUESTION 2 (55)

(a) $$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \frac{4\sin^2 u}{(4-4\sin^2 u)^{3/2}} \cdot 2\cos u \, du$$

$x = 2\sin u$

$dx = 2\cos u \, du$

$$= \int \frac{4(1-\cos^2 u) \cdot 2\cos u \, du}{4^{3/2}(1-\sin^2 u)^{3/2}}$$

$$= \int \frac{(1-\cos^2 u)(\cos u)}{\cos^3 u}\,du$$

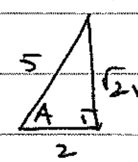
$$= \int \frac{1-\cos^2 u}{\cos^2 u}\,du$$

$$= \int \frac{1}{\cos^2 u} - 1 \, du$$

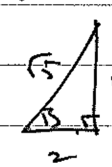
$$= \int \sec^2 u - 1 \, du$$

$$= \tan u - u + c$$
$$= \tan\left(\sin^{-1}\frac{x}{2}\right) - \sin^{-1}\frac{x}{2} + c$$

(b) $\cos A = \frac{2}{5}$



$\tan B = \frac{1}{2}$



$$\sin(2A+B) = \sin 2A\cos B + \cos 2A \sin B$$
$$= 2\sin A\cos A\cos B + (\cos^2 A - \sin^2 A)\sin B$$

A in 4th quadrant
B in 3rd quadrant

or $2\cos^2 A - 1$
or $1 - 2\sin^2 A$.

$$= 2\left(\frac{-\sqrt{21}}{5}\right)\left(\frac{2}{5}\right)\left(\frac{-2}{\sqrt{5}}\right) + \left[\left(\frac{2}{5}\right)^2 - \left(\frac{-\sqrt{21}}{5}\right)^2\right]\left(\frac{-1}{\sqrt{5}}\right)$$

$$= \frac{8\sqrt{21}}{25\sqrt{5}} + \left(\frac{4}{25} - \frac{21}{25}\right)\left(\frac{-1}{\sqrt{5}}\right)$$

$$= \frac{8\sqrt{21}}{25\sqrt{5}} + \frac{17}{25\sqrt{5}}$$

QUESTION 2 (cont.) (SS)

(c) (i) $R = \sqrt{3^2+1^2}$ $\quad \alpha = \tan^{-1}\frac{1}{\sqrt{3}}$

$= 2 \qquad\qquad = \pi/6$

$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{6}\right)$ ✓

(ii) $\sec\theta + \tan\theta = \sqrt{3}$

$\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \sqrt{3}$

$\frac{1+\sin\theta}{\cos\theta} = \sqrt{3}$

$1 + \sin\theta = \sqrt{3}\cos\theta$

$\therefore \sqrt{3}\cos\theta - \sin\theta = 1$ ✓

$2\cos\left(\theta + \frac{\pi}{6}\right) = 1$ from (i).

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

$\theta + \frac{\pi}{6} = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$ ✓

(d) $\lim_{x\to 0} \frac{\sin\frac{x}{2}}{x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$

$= \frac{1}{2}\times 1 \quad \left(\text{as } \lim_{x\to 0}\frac{\sin\theta}{\theta} = 1\right)$.

$= \frac{1}{2}$ ✓

QUESTION 2 (cont.). (ss)

(e). $\cos 2x = 2\cos^2 x - 1$

$\therefore \cos 6x = 2\cos^2 3x - 1$

$2\cos^2 3x = \cos 6x + 1.$

$\cos^2 3x = \frac{1}{2}(\cos 6x + 1)$

$$\therefore \int_0^{\pi/3} \cos^2 3x \, dx = \frac{1}{2}\int_0^{\pi/3} (\cos 6x + 1) \, dx$$

$$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + x\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{1}{6}\sin 2\pi + \frac{\pi}{3} - \frac{1}{6}\sin 0 - 0\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{6}$$

QUESTION 3 (KC)

(a) $y = \frac{e^x}{e^x - 1}$

f^{-1}: $x = \frac{e^y}{e^y - 1}$

$xe^y - x = e^y$ ✓

$xe^y - e^y = x$

$e^y(x-1) = x$

$e^y = \frac{x}{x-1}$

$y = \ln\left(\frac{x}{x-1}\right)$ ✓

Range of $y = f(x)$: $y < 0$, $y > 1$ (from diagram)

$\therefore$ Domain of $y = f^{-1}(x)$: $x < 0$, $x > 1$ ✓

(We can check this by solving $\frac{x}{x-1} > 0$).

(b). $\cos^{-1}(-1) = \pi$ ✓

but $\sin^{-1}(\pi)$ has no solution as π lies outside the domain $-1 \leq x \leq 1$ ✓

(c). $\int_0^1 \frac{-1}{\sqrt{4-x^2}} \, dx = \left[\cos^{-1}\frac{x}{2}\right]_0^1$ ✓

$= \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(0)$

$= \frac{\pi}{3} - \frac{\pi}{2}$ ✓

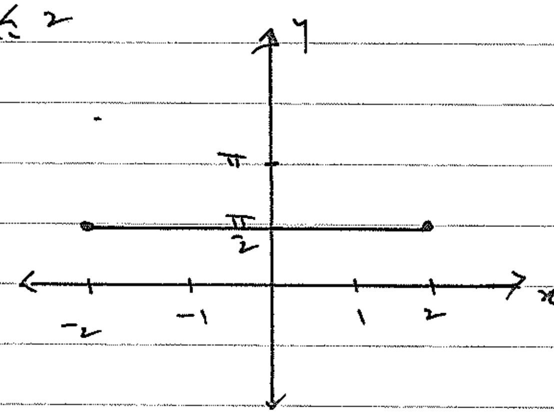
$= -\frac{\pi}{6}$ ✓

QUESTION 3 (cont.) (KC)

(d) $y = \sin^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{2}$ $\left(\text{as } \sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}\right)$.

$-1 \leq \frac{x}{2} \leq 1$ (the natural domain for $\sin^{-1}\frac{x}{2}$ and $\cos^{-1}\frac{x}{2}$)

$\therefore -2 \leq x \leq 2$



✓ for $y = \frac{\pi}{2}$ graph

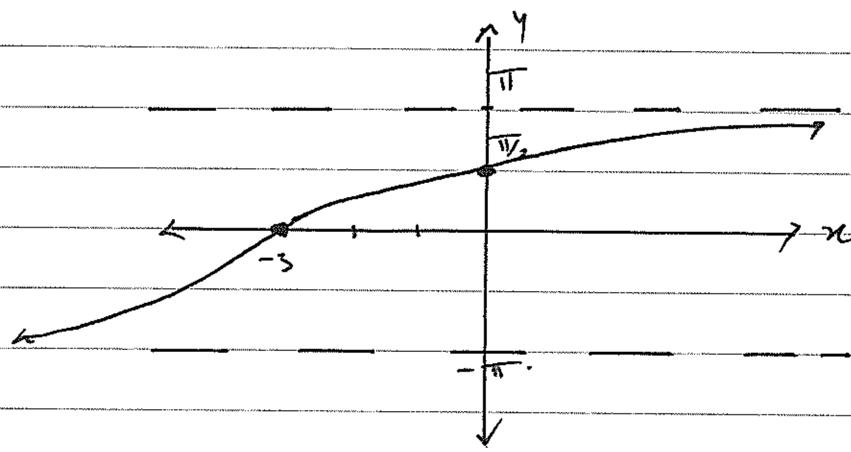
✓ for limiting the domain to $-2 \leq x \leq 2$

(e). (i) Domain : all real x ✓

Range : $-\pi < y < \pi$ $\left(\text{as } \frac{-\pi}{2} < \tan^{-1}\theta < \frac{\pi}{2} \quad \therefore -\pi < 2\tan^{-1}\theta < \pi\right)$.

(ii). $x = 0$: $y = 2\tan^{-1} 1$
$= 2\,\frac{\pi}{4}$
$= \frac{\pi}{2}$

$y = 0$: $2\tan^{-1}\left(\frac{x}{3}+1\right) = 0$
$\tan^{-1}\left(\frac{x}{3}+1\right) = 0$
$\frac{x}{3} + 1 = 0$
$\frac{x}{3} = -1$
$x = -3$.



✓ correct intercepts.

✓ correct graph (showing asymptotes)