ACE
EXAM
PAPER

Student name: ________________

PAPER 2

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 2

General Instructions

- Working time - 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Given that $z = 3 + 4i$ and $w = 1 - i$, what is the value of zw?

(A) $-1 + i$

(B) $4 + 3i$

(C) $4 + 5i$

(D) $7 + i$

2. A body is in equilibrium under the action of three forces. If two of the forces are $F_1 = \underset{\sim}{i} - 2\underset{\sim}{j}$ and $F_2 = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ what is F_3 ?

(A) $-4\underset{\sim}{i} - \underset{\sim}{k}$

(B) $4\underset{\sim}{i} + \underset{\sim}{k}$

(C) $2\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$

(D) $3\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$

3. Which of the following is an expression for $\displaystyle\int \frac{1}{x^2 - 6x + 13}\,dx$?

(A) $\dfrac{1}{4}\tan^{-1}\dfrac{(x-3)}{2} + C$

(B) $\dfrac{1}{4}\tan^{-1}\dfrac{(x-3)}{4} + C$

(C) $\dfrac{1}{2}\tan^{-1}\dfrac{(x-3)}{2} + C$

(D) $\dfrac{1}{2}\tan^{-1}\dfrac{(x-3)}{4} + C$

4. The polynomial $P(z)$ has the equation $P(z) = z^4 - 4z^3 + Az + 20$ where A is real. Given that $3 + i$ is a zero of $P(z)$, which of the following expressions is $P(z)$ as a product of two real quadratic factors?

(A) $(z^2 - 2z + 2)(z^2 - 6z + 10)$

(B) $(z^2 + 2z + 2)(z^2 - 6z + 10)$

(C) $(z^2 - 2z + 2)(z^2 + 6z + 10)$

(D) $(z^2 + 2z + 2)(z^2 + 6z + 10)$

5. A force of magnitude 4 N acts in the north-east direction and another force of 3 N acts in the easterly direction. What is the resultant magnitude (in N) of these two forces?

(A) $\sqrt{25 - 12\sqrt{2}}$

(B) $\sqrt{25 + 12\sqrt{2}}$

(C) $25 + 12\sqrt{2}$

(D) $25 - 12\sqrt{2}$

6. Let $z = 2 - 3i$. What is the value of z^{-1} ?

(A) $-\frac{1}{5}(2 + 3i)$

(B) $\frac{1}{5}(2 - 3i)$

(C) $\frac{1}{13}(2 + 3i)$

(D) $\frac{1}{13}(2 - 3i)$

7. Which of the following is an expression for $\frac{\sin x \cos x}{5 + \sin x}dx$?

(A) $-5\ln|5 + \sin x| + C$

(B) $5\ln|5 + \sin x| + C$

(C) $-\sin x - 5\ln|5 + \sin x| + C$

(D) $\sin x - 5\ln|5 + \sin x| + C$

8. Suppose that n is a positive integer. For how many values of n is the number $9n^2 - 4$ a prime?

(A) 0

(B) 1

(C) 2

(D) 3

9. The velocity of a body moving in a straight line is given by $v = f(x)$ where x metres is the distance from origin and v is the velocity in metres per second. The acceleration of the body in ms^{-2} is given by:

(A) $f'(x)$

(B) $f'(v)$

(C) $xf'(x)$

(D) $f(x)f'(x)$

10. P, Q and R are three collinear points with position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively. Q lies between P and R. If $\left|\overrightarrow{QR}\right| = \frac{1}{2}\left|\overrightarrow{PQ}\right|$ then $\underset{\sim}{r}$ is equal to:

(A) $\dfrac{3}{2}\underset{\sim}{q} - \dfrac{1}{2}\underset{\sim}{p}$

(B) $\dfrac{3}{2}\underset{\sim}{p} - \dfrac{1}{2}\underset{\sim}{q}$

(C) $\dfrac{3}{2}\underset{\sim}{q} - \dfrac{3}{2}\underset{\sim}{p}$

(D) $\dfrac{1}{2}\underset{\sim}{p} - \dfrac{3}{2}\underset{\sim}{q}$

Section II

90 marks
Attempt questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) **Marks**

(a) Factorise the polynomial $x^4 + x^2 - 12$ completely over the field of:

(i) Rational numbers. **1**

(ii) Real numbers. **1**

(iii) Complex numbers. **1**

(b) Let $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$. Find z^6 **2**

(c) The points A and B have position vectors relative to the origin O given by:

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$.

(i) Find an expression for the vector $\overrightarrow{AB}$ in the form $x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$. **1**

(ii) Show that the cosine of the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ is $\frac{4}{9}$. **3**

(iii) Find the exact area of ΔOAB. **3**

(d) Find $\int xe^x \, dx$ **2**

Question 12 (15 marks) **Marks**

(a) Let $z = 3 - i$. Express the following in the form $a + ib$, where, a and b are real numbers:

(i) $i\bar{z}$ **1**

(ii) $\dfrac{1}{z}$ **1**

(b) Let $\underset{\sim}{u} = 5\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$.

(i) What is the scalar product of $\underset{\sim}{u}.\underset{\sim}{v}$? **1**

(ii) Find the unit vector of $\underset{\sim}{v}$. **2**

(c) The velocity of a particle moving in a straight line is given by $v = 10 - x$ where x metres is the distance from fixed point O and v is the velocity in metres per second. Initially the particle is at O.

(i) Let a be the acceleration in metres per second squared. Find an expression for a in terms of x. **1**

(ii) Show that $x = 10 - 10e^{-t}$ by integration. **2**

(iii) What is the limiting position of the particle? **1**

(d) Let $\overrightarrow{OA} = \left(2\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$ and $\overrightarrow{OB} = \left(5\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}\right)$ **2**

Find $\overrightarrow{OM}$ where M is the midpoint of AB.

(e) (i) Express each of $z_1 = \sqrt{3} + i$ and $z_2 = -\sqrt{2} + \sqrt{2}i$ in modulus-argument form. **2**

(ii) Find the exact value of $\arg\left(\dfrac{z_2}{z_1}\right) - \arg(z_1 + z_2)$ **2**

Question 13 (14 marks) **Marks**

(a) (i) Find real numbers A and B such that: **2**

$$\frac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)} = \frac{Ax + 1}{(x^2 + 1)} + \frac{B}{(x - 2)}$$

(ii) Hence find $\int \frac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)} dx$ **2**

(b) Prove by induction that $\frac{1}{n!} < \frac{1}{2^{n-1}}$ when $n \geq 3$ and n is an integer. **3**

(c) (i) Show that $z = 1 + i$ is a root of the following equation. **2**

$P(z) = z^2 - (3 - 2i)z + (5 - i) = 0$

(ii) Find the other root of the above equation. **1**

(d) A particle moves in a straight line under simple harmonic motion. Its displacement (x metres) from a fixed point O at any time (t seconds) is given by: $x = 4\cos^2 t - 1$.

(i) Find an expression for acceleration in terms of x. **2**

(ii) Sketch $x = 4\cos^2 t - 1$ for $0 \leq t \leq \pi$. **1**

Clearly show the times when the particle passes through O.

(iii) Find the time when the velocity of the particle is increasing most rapidly for $0 \leq t \leq \pi$. **1**

Question 14 (16 marks) **Marks**

(a) What is the exact value of $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$? **2**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ where n is apositive integer.

(i) Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x dx$ when $n \geq 2$. **2**

(ii) Prove that $I_n = \frac{(n-1)}{n} I_{n-2}$ when $n \geq 2$. **2**

(iii) What is the value of I_4 ? **1**

(c) A rock is projected vertically upwards from ground level. Assume air resistance is kv, where v is the velocity of the rock and k is a positive constant. The rock falls back to ground level under the influence of g, the acceleration due to gravity. Consider the rock's motion starting from maximum height. Let y be the displacement and t be the time elapsed after the rock has reached maximum height. Assume the rock has a unit mass.

(i) Explain why $\frac{dv}{dt} = kv - g$ while the rock is in motion. **1**

(ii) Show that $v = \frac{g}{k}\left(e^{kt} + 1\right)$ when $t \geq 0$. **3**

(iii) Show that $ky = v + \frac{g}{k}\ln(kgv - g^2)$ by using $\frac{dv}{dt} = v\frac{dv}{dy}$ **3**

(d) Find $\int \frac{\ln(\tan^{-1}x)}{1+x^2}\,dx$ **2**

Question 15 (17 marks) **Marks**

(a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2 + 2\sin x - \cos x}\,dx$ **4**

(b) (i) Prove that $(a + b + c)^2 \geq 3(ab + ac + bc)$. **2**
Where a, b and c are positive integers.

(ii) Hence or otherwise prove that **2**
$a^2b^2 + a^2c^2 + b^2c^2 \geq abc(a + b + c)$

(c) (i) On an Argand diagram shade the region where the inequalities **2**
$2 \leq |z| \leq 4$ and $0 \leq \arg z \leq \frac{\pi}{3}$ hold simultaneously.

(ii) Find the exact area of the shaded region. **1**

(d) There exists some real number x such that $x^2 = -1$ **2**
Show that the above statement is false.

(e) A particle moves in a straight line and its position at any time is given by:
$x = 1 + \sqrt{3}\cos 4t + \sin 4t$

(i) Prove the motion is simple harmonic. **1**

(ii) Find the amplitude of the motion. **1**

(iii) When does the particle first reach maximum speed after time t = 0? **2**

Question 16 (14 marks) **Marks**

(a) The point A, with coordinates $(0, a, b)$ lies on the line l_1, which has the equation:

$$\underset{\sim}{r} = 6\underset{\sim}{i} + 19\underset{\sim}{j} - \underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

(i) Find the values of a and b. **2**

(ii) The point P lies on l_1 and is such that OP is perpendicular to l_1, where O is the origin. Find the position vector of point P. **4**

(b) (i) Show that $a = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of the equation $z^5 + z - 1 = 0$. **2**

(ii) Find the monic cubic equation with real coefficients whose roots are also the roots of $z^5 + z - 1 = 0$ but do not include a? **2**

(c) Use mathematical induction to show that. $\tan\left[(2n-1)\frac{\pi}{4}\right] = (-1)^{n+1}$ for all positive integers $n \geq 1$. **4**

End of paper

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

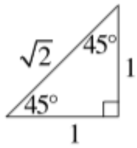
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

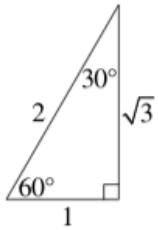


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

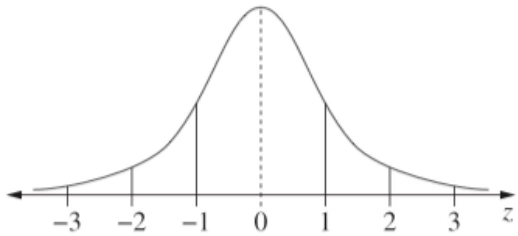
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$|\underset{\sim}{u}| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$