Marker - 1abcdf. AE
- 1e, 2d AF
- 2bc
- 3abc
- 3defg XC

Carlingford High School

2017

Mathematics
HSC Course

Year 12 Half Yearly Examination

Time allowed 2 hours

Name: ....

Teacher: *(Please Circle)*

Mr Fardouly
Mr Cheng
Mr Wilson
Mrs Wilson/Young
Mr Gong

General Instructions

- Start each question in a new booklet
- Do not write in columns
- Marks may be deducted for careless or badly arranged work
- Only calculators approved by the Board of Studies may be used
- All answers are to be completed in black pen except graphs and diagrams
- No lending or borrowing

Q1 Series & Applications	**Q2** Applications for Calculus	**Q3** Integration	**Total**
/19	/18	/26	/65

Attempt Questions 1-3

Answer all questions, starting each question on a ***new*** booklet with your name and question number at the top of the page.

Question 1 Use a SEPARATE writing booklet (19 marks)

a. The first two terms of a geometric sequence are 9 and 6.

i) What is the third term? 1

ii) What is the nth term? 2

b. Evaluate 2

$$\sum_{n=1}^{15}(5n+1)$$

c. The fifth term of an arithmetic series is 14 and the sum of the first 10 terms is 165. Find the first term of the series. 3

d. A plant is 50 cm high when first observed. In the first week of observation it grows 10 cm, and in each succeeding week the growth in height is 80% of the previous week's growth. If this pattern of growth continues, what will be its ultimate height? 2

e. Karen borrows \$450 000 to buy a house. The loan is charged 9% p.a. interest, compounded monthly over 25 years. Karen makes monthly repayments of \$$M$.

i) Show that the amount owing after 2 months (A_2) is 2

$$A_2 = 450000(1.0075)^2 - M(1.0075) - M$$

ii) Show that the amount of each repayment is \$3776.38 2

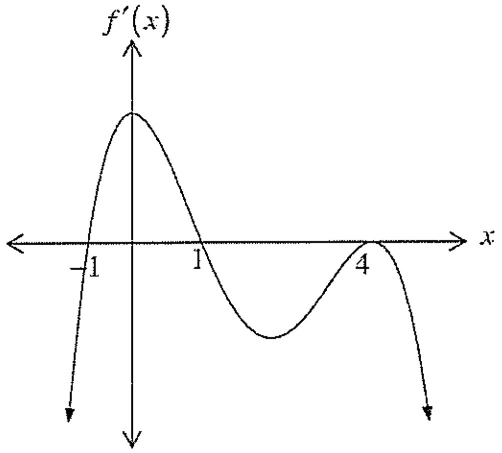
After 10 years (i.e. 120 repayments) the interest rate is lowered to 6% p.a.

iii) Calculate the amount that Karen still owes after 10 years. 2

f. A worker invests a certain amount at the beginning of each year in a superannuation fund for 35 years. Compound interest is paid at 9% p.a., compounded annually. Find the amount invested at the beginning of each year, if the value of the superannuation at the end of the 35^{th} year was \$658349. 3

Question 2 Use a SEPARATE writing booklet (18 marks)

a.



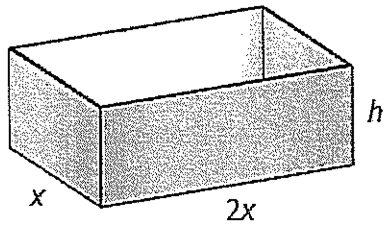
Not to Scale

The graph of $y = f'(x)$ is sketched above.

i) Write down the values of x where stationary points occur in the graph of $y = f(x)$. 1

ii) Sketch $y = f(x)$ given that the graph passes through (0, 0) and (4, -2). Clearly show any turning points or points of inflection. 2

b. Haris is building a small *open topped* lunch box. The box is twice as long as it is wide. The box has a total external surface area of 3750 cm^2.
Note: the box does not have a lid.



Let $x =$ the width of the box as shown.

i) Show that the height of the lunch box is given by $h = \dfrac{625}{x} - \dfrac{x}{3}$ 2

ii) Find the dimensions of the box which give a maximum volume. 3

c. Given that $y = x^2 - x$, show that $\dfrac{dy}{dx} - \dfrac{d^2y}{dx^2} = \dfrac{2y - x}{x}$ 3

d. Consider the function $f(x) = x^4 - 4x^3$

i) Show that $f'(x) = 4x^2(x-3)$ 1

ii) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature. 3

iii) Find the values of x for which the graph of $y = f(x)$ is concave down. 2

iv) Sketch the graph of the curve $y = f(x)$, showing the stationary points. 1

Question 3 Use a SEPARATE writing booklet (26 marks)

a. Find the indefinite integral

i) $\int (2x^2 - 5x + 6)dx$ 2

ii) $\int \frac{x^2+1}{x^2} dx$ 2

b. Evaluate

i) $\int_1^4 (x + \frac{1}{x^2})dx$ 2

ii) $\int_{-1}^2 (x-1)^2 dx$ 2

c. Consider the function $y = \frac{1}{4+x^2}$

i) Copy and complete the following table. 2

x	0	0.5	1	1.5	2
y					

ii) Apply Simpsons Rule with 5 function values to find an approximation for

$$\int_0^2 \frac{1}{4+x^2} dx$$ 2

d. Using the trapezoidal rule and 5 sub-intervals find an approximation for the definite integral.

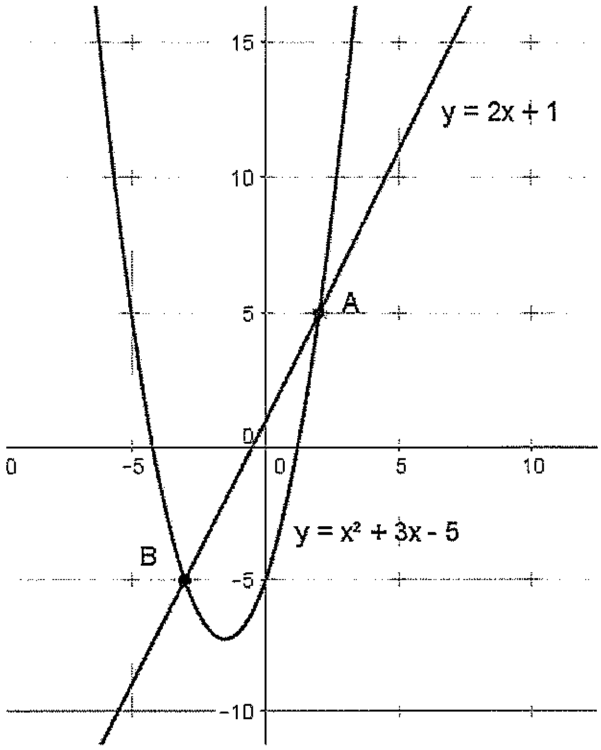
$$\int_0^2 (x^2 + x)\, dx$$ 3

e. Given $\frac{d^2x}{dt^2} = 6t - 6$ and that when $t = 0$, $\frac{dx}{dt} = 0$, $x = 5$, find x in terms of t. 3

f. The diagram shows the graphs of $y = x^2 + 3x - 5$ and $y = 2x + 1$.

i) Find the x values of the points of intersection, *A* and *B*. 1

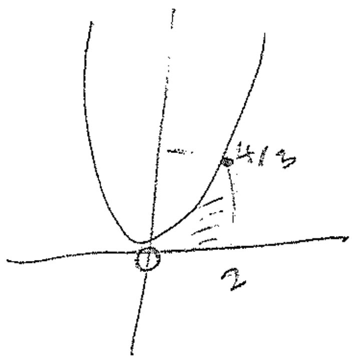
ii) Calculate the area of the enclosed region. 4



g. A bowl is formed by rotating the curve $y = \frac{x^2}{3}$ between $x = 0$ and $x = 2$ about the y-axis. 3

Find the volume of the solid formed.

End of Exam



Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

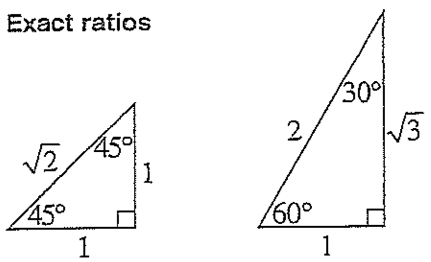
$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Mathematics (continued)

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

1) a) $T_5 = 14 \quad S_{10} = 165$

$a = ?$

$14 = a + 4d \ldots (1)$

$165 = \frac{10}{2}[2a + (10-1)d]$

$165 = 5[2a + 9d]$

$33 = 2a + 9d \ldots (2)$

$28 = 2a + 8d \ldots (1) \times 2$

$5 = d \ldots (3)$

Subst (3) into (1)

$14 = a + 4 \times 5$

$14 = a + 20$

$a = -6$

b) $a = 9 \quad T_2 = 6$

$\frac{6}{9} = r = \frac{2}{3}$

$T_3 = 6 \times \frac{2}{3}$

$= 4$

ii) $T_n = ar^{n-1}$

$= 9 \times (\frac{2}{3})^{n-1}$

1) c) $\sum_{n=1}^{15} (5n + 1)$

$6 + 11 + 16 + \ldots + 76$

$a = 6 \quad d = 5 \quad \therefore AP$

$S_{15} = \frac{15}{2}(6 + 76)$

$= 615$

d) $a = 10 \quad r = 0.8$

$S_\infty = \frac{10}{1 - 0.8}$

$= 50$

Plus the initial height

$\therefore 50 + 50 = 100cm$

Graham 1)e) 2) d

②

1)e) $r = \frac{0.09}{12} = 0.0075$

i)

$A_1 = 450000 \times 1.0075 - M$ ✓

$A_2 = (450000 \times 1.0075 - M) \times 1.0075 - M$ ✓

$= 450000 \times 1.0075^2 - M(1.0075) - M$

ii) $A_{300} = 450000 \times 1.0075^{300} - M(1 + 1.0075 + \ldots 1.0075^{299})$ ✓

$0 = 450000 \times 1.0075^{300} - \frac{M \times 1(1.0075^{300} - 1)}{1.0075 - 1}$

$\frac{M(1.0075^{300} - 1)}{0.0075} = 450000 \times 1.0075^{300}$

$m = \frac{450000 \times 1.0075^{300} \times 0.0075}{(1.0075^{300} - 1)}$ ✓

$= \$3776.38$

iii) $450000 \times 1.0075^{120} - \frac{3776.38(1.0075^{120} - 1)}{1.0075 - 1}$ ✓

$= \$372327.24$ ✓

iv) $0 = 372327.24 \times 1.00^{180} - \frac{m(1.005^{180} - 1)}{0.005}$

$m = \frac{372327.24 \times 1.005^{180} \times 0.005}{(1.005^{180} - 1)} = \$3141.91$

1) f)

$$658349 = P \times 1.09\frac{(1.09^{35}-1)}{1.09-1}$$

$$\frac{658349 \times (1.09-1)}{1.09(1.09^{35}-1)}$$

$= 2799.999.$

$\doteqdot 2800$

or

$A_1 = P \times 1.09^{35}$

$A_2 = P \times 1.09^{34}$

$A_{35} = P \times 1.09$

$A_{Total} = P(1.09 + \ldots 1.09^{35})$

$$658349 = P \times 1.09\frac{(1.09^{35}-1)}{1.09-1}$$

$$P = \frac{658349 \times (1.09-1)}{1.09(1.09^{35}-1)}$$

$= 2799.999$

$\doteqdot \$2800$

2) a) i) $x = -1, 1, 4$

ii) $x = -1$

x	-1.1	-1	-0.9
$f'(x)$	$-$	0	$+$

min at $x = -1$

$x = 1$

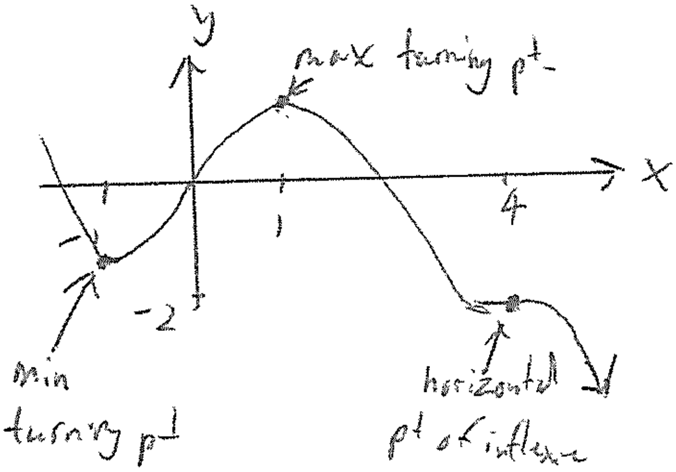
x	0.9	1	1.1
$f'(x)$	$+$	0	$-$

max at $x = 1$

$x = 4$

x	3.5	4	4.5
$f'(x)$	$-$	0	$-$

horizontal point of inflexion at $x = 4$



ii) b. $4xh + 2xh + 2x^2 = 3750$

i) $6xh + 2x^2 = 3750$

$3xh + x^2 = 1875$

$3xh = 1875 - x^2$

$h = \dfrac{1875 - x^2}{3x}$

$h = \dfrac{625}{x} - \dfrac{x}{3}$

ii) $V = 2x^2\left(\dfrac{625}{x} - \dfrac{x}{3}\right)$

$= 1250x - \dfrac{2}{3}x^3$

$V' = 1250 - 2x^2$

$1250 - 2x^2 = 0$

$x^2 = 625$

$x = 25$

$V'' = -4x$

when $x = 25$ $V'' = -100$

$\therefore$ max

$\therefore 25 \times 50 \times 16\frac{2}{3}$

c) $y = x^2 - x$

$y' = 2x - 1$

$y'' = 2$

$\dfrac{dy}{dx} - \dfrac{d^2y}{dx^2} = \dfrac{2y - x}{x}$

LHS RHS

$2x - 1 - 2$ $= \dfrac{2(x^2 - x) - x}{x}$

$2x - 3$ $= \dfrac{2x^2 - 2x - x}{x}$

$= \dfrac{2x^2 - 3x}{x}$

$= \dfrac{\cancel{x}(2x - 3)}{\cancel{x}}$

$= 2x - 3$

LHS = RHS.

Just (d) ie whole page.

2) d) $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$= 4x^2(x-3)$

ii) Stationary points when $f'(x) = 0$

$4x^2(x-3) = 0$

$\therefore x = 0$ or $x = 3$

y values

$(0,0)$ $(3,-27)$

$f''(x) = 12x^2 - 24x$

when $x = 3$ $f''(x) = 36$

$\therefore$ min

when $x = 0$ $f''(x) = 0$

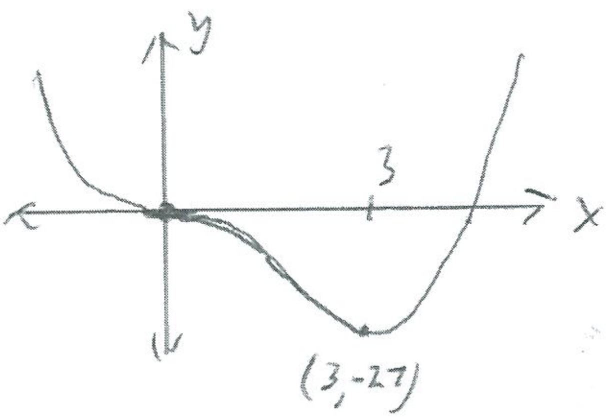
test for inflexion

x	-0.1	0	0.1
$f''(x)$	$+$	0	$-$

concavity changes

$\therefore (0,0)$ is horizontal point of inflexion

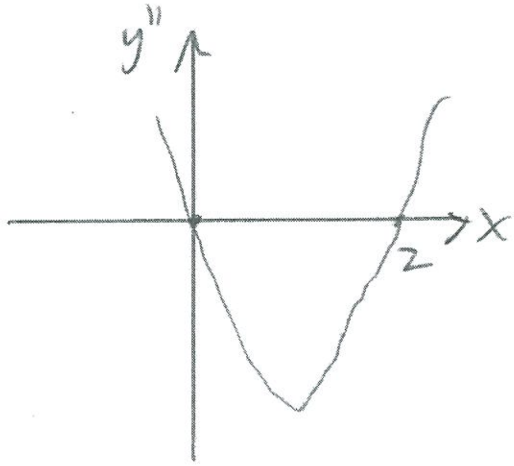
iii)



iv) Concave down when $f''(x) < 0$ (5)

$12x^2 - 24x < 0$

$12x(x-2) < 0$



$\therefore 0 < x < 2$

3) a) i)

$$\int (2x^2 - 5x + 6)\,dx$$

$$= \frac{2x^3}{3} - \frac{5x^2}{2} + 6x + c$$

ii) $$\int \frac{x^2+1}{x^2}\,dx$$

$$\int \left(\frac{x^2}{x^2} + \frac{1}{x^2}\right) dx$$

$$\int 1 + x^{-2}\,dx$$

$$= x - x^{-1} + c$$

$$= x - \frac{1}{x} + c$$

b) i) $$\int_1^4 \left(x + \frac{1}{x^2}\right) dx$$

$$= \int_1^4 (x + x^{-2})\,dx$$

$$\left[\frac{x^2}{2} - x^{-1}\right]_1^4$$

$$= 8 - \frac{1}{4} - \left(\frac{1}{2} - 1\right) = \frac{29}{4} \text{ or } 7\tfrac{1}{4}$$

ii) $$\int_{-1}^{2} (x-1)^2\,dx$$

$$\int_{-1}^{2} (x^2 - 2x + 1)\,dx$$

$$= \left[\frac{x^3}{3} - x^2 + x\right]_{-1}^{2}$$

$$\frac{8}{3} - 4 + 2 - \left(-\frac{1}{3} - 1 - 1\right)$$

$$= 3$$

c) $$y = \frac{1}{4+x^2}$$

i)

x	0	0.5	1	1.5	2
y	$1/4$	$4/17$	$1/5$	$4/25$	$1/8$

$\times$ 1 4 2 4 1

ii) $h = \frac{2-0}{4} = \frac{1}{2}$

$$\frac{\frac{1}{2}}{3}\left[\frac{1}{4} + \frac{16}{17} + \frac{2}{5} + \frac{16}{25} + \frac{1}{8}\right]$$

$$= 0.3927$$

3d) $\int_0^2 (x^2+x)\,dx$

$h = \frac{2-0}{5} = 0.4$

x	0	0.4	0.8	1.2	1.6	2
y	0	$\frac{14}{25}$	$\frac{36}{25}$	$\frac{66}{25}$	$\frac{104}{25}$	6

$\frac{0.4}{2}\left[0+2\left(\frac{14}{25}+\frac{36}{25}+\frac{66}{25}+\frac{104}{25}\right)+6\right]$

$= 4.72$

e) $y'' = 6t-6$

$y' = 3t^2-6t+c$

$0 = 3t^2-6t+c$

$c=0$

$\therefore \frac{dx}{dt} = 3t^2-6t$

$x = t^3-3t^2+c$

$5 = 0^3-0^2+c$

$c=5$

$\therefore x = t^3-3t^2+5$

f) i) $y = x^2+3x-5 \quad y = 2x+1$ 7

$x^2+3x-5 = 2x+1$

$x^2+x-6=0$

$(x+3)(x-2)=0$

$x = -3$ and 2

$B(-3,-5)\ A(2,5)$

ii) $\int_{-3}^{2}\left(2x+1-(x^2-3x-5)\right)dx$

$\int_{-3}^{2}(-x^2-x+6)\,dx$

$\left[-\frac{x^3}{3}-\frac{x^2}{2}+6x\right]_{-3}^{2}$

$= -\frac{8}{3}-2+12-\left(\frac{27}{3}-\frac{9}{2}-18\right)$

$= \cancel{\frac{2\ldots}{6}}\quad \frac{125}{6}$ or $20.8\dot{3}$

8

3) $y = \frac{x^2}{3}$ $x = 0, x = 2$

$3y = x^2$ $(0,0)$ $(2, \frac{4}{3})$

$\pi \int_0^{4/3} 3y \, dy$

$\pi \left[\frac{3}{2} y^2 \right]_0^{4/3}$

$\pi \times \frac{3}{2} \times \frac{16}{9} = \frac{8}{3} \pi u^3$