Senior Division

2014

1. The expression that has the same meaning as $9x^{-3}$ is

(A) $\dfrac{-9}{x^3}$ (B) $\dfrac{3}{x}$ (C) $\dfrac{1}{9x^3}$ (D) $\dfrac{3}{x^3}$ (E) $\dfrac{9}{x^3}$

▶ $9x^{-3} = 9 \times \dfrac{1}{x^3} = \dfrac{9}{x^3}$,

hence (E).

2. (Also J5)

The value of $\dfrac{1}{0.04}$ is

(A) 15 (B) 20 (C) 25 (D) 40 (E) 60

▶ $\dfrac{1}{0.04} = \dfrac{1 \times 100}{0.04 \times 100} = \dfrac{100}{4} = 25$,

hence (C).

3. (Also I3)

If $p = 9$ and $q = -3$ then $p^2 - q^2$ is equal to

(A) 64 (B) 72 (C) 84 (D) 90 (E) 96

▶ Now $p^2 = 81$ and $q^2 = 9$, so $p^2 - q^2 = 81 - 9 = 72$,

hence (B).

4. A circle has circumference π units. In square units, its area is

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{2}$ (C) π (D) 2π (E) 4π

▶ This circumference is $2\pi r = \pi$ so that $r = \dfrac{1}{2}$. Then $A = \pi r^2 = \dfrac{\pi}{4}$,

hence (A).

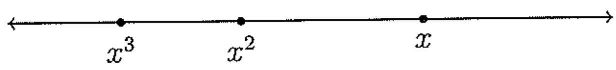
5. If $K = L + \dfrac{6}{R}$ and $L = 4$ and $K = 7$, then R equals

(A) -18 (B) 1 (C) 12 (D) 8 (E) 2

▶ We have $7 = 4 + \dfrac{6}{R}$ so that $\dfrac{6}{R} = 3$ and $R = 2$,

hence (E).

6. If x, x^2 and x^3 lie on a number line in the order shown below, which of the following could be the value of x?

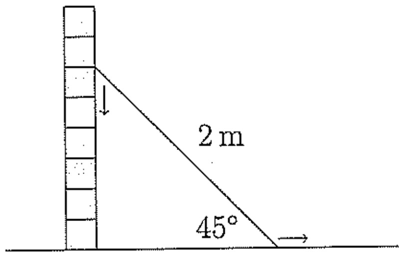


(A) -2 (B) $-\dfrac{1}{2}$ (C) $\dfrac{3}{4}$ (D) 1 (E) $\dfrac{3}{2}$

▸ We have $0 < x^2 < x$ so that x is positive and $x < 1$. The only possibility is $x = \frac{3}{4}$, and $x^3 = \frac{27}{64}$, $x^2 = \frac{9}{16} = \frac{36}{64}$ and $x = \frac{3}{4} = \frac{48}{64}$,

hence (C).

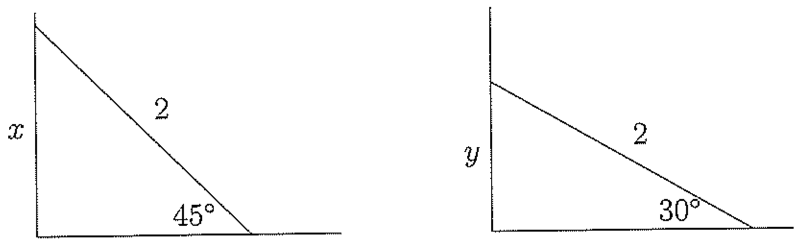
7. A 2 metre broom is leaning against a wall, with the bottom of the broom making an angle of 45° with the ground. The broom slowly slides down the wall until the bottom of the broom makes an angle of 30° with the ground.



How far, in metres, has the top of the broom slid down the wall?

(A) $\sqrt{2} - 1$ (B) $2 - \sqrt{3}$ (C) $\sqrt{3} - 1$ (D) $\sqrt{3} - \sqrt{2}$ (E) $2 - \sqrt{2}$

▸



Beforehand, $\frac{x}{2} = \sin 45° = \frac{1}{\sqrt{2}}$ so that $x = \sqrt{2}$. Afterwards, $\frac{y}{2} = \sin 30° = \frac{1}{2}$ so that $y = 1$. Then $x - y = \sqrt{2} - 1$,

hence (A).

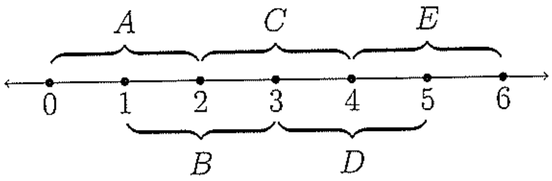
8. The base of a triangle is increased by 25% and its height is increased by 50%. Its area has increased by

(A) 25% (B) 50% (C) $66.\dot{6}$% (D) 75% (E) 87.5%

▸ Consider the formula $A = \frac{1}{2}bh$. As the triangle is enlarged, b is multiplied by $\frac{5}{4}$ and h is multiplied by $\frac{3}{2}$. In total, A is multiplied by $\frac{5}{4} \times \frac{3}{2} = \frac{15}{8} = 1.875$, which gives an increase of 87.5%,

hence (E).

9. On a section of the number line five intervals are marked as shown.



If a number x falls in the interval A and a number y falls in the interval D, then the number $\frac{1}{2}(x+y+1)$ must fall in which interval?

(A) A (B) B (C) C (D) D (E) E

▸ The number $x+y+1$ is larger than $0+3+1=4$ and less than $2+5+1=8$, so that $\frac{1}{2}(x+y+1)$ is between 2 and 4,

hence (C).

10. If $\dfrac{p}{p-2q}=3$ then $\dfrac{p}{q}$ equals

(A) -3 (B) 3 (C) $\dfrac{1}{3}$ (D) $\dfrac{2}{3}$ (E) 2

▸ We have $p=3(p-2q)$, so $6q=2p$ and $p=3q$. Then $\dfrac{p}{q}=3$,

hence (B).

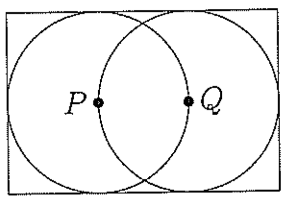
11. In a car park there are 3 Fords, 4 Holdens and 2 Hondas. If a parking inspector chooses 2 cars at random, the probability that both are Holdens is

(A) $\dfrac{1}{4}$ (B) $\dfrac{4}{27}$ (C) $\dfrac{1}{6}$ (D) $\dfrac{4}{9}$ (E) $\dfrac{16}{81}$

▸ The probability the first car is a Holden is $\frac{4}{9}$, and then the probability that the second is also a Holden is $\frac{3}{8}$. Thus the probability is $\frac{4}{9}\times\frac{3}{8}=\frac{1}{6}$,

hence (C).

12. In this figure, P and Q are the centres of two circles. Each circle has an area of $10\,\mathrm{m}^2$. The area, in square metres, of the rectangle is



(A) 20 (B) $20-\dfrac{10}{\pi}$ (C) $\dfrac{40}{\pi}$ (D) $\dfrac{50}{\pi}$ (E) $\dfrac{60}{\pi}$

▸ If r is the radius of both circles, then $\pi r^2=10$ and $r^2=\dfrac{10}{\pi}$. Then the area of the rectangle is $3r\times 2r=6r^2=\dfrac{60}{\pi}$,

hence (E).

13. The value of $\sqrt{1+2+3+4+\cdots+99+100}$ lies between

(A) 14 and 15 (B) 25 and 26 (C) 30 and 31 (D) 71 and 72 (E) 100 and 101

▶ Let the value be x, then $x^2 = 1 + 2 + 3 + \cdots + 99 + 100 = 50 \times (1 + 100) = 5050$ so that $x = \sqrt{5050}$. Since $\sqrt{4900} = 70$ and $\sqrt{6400} = 80$, this suggests that x is slightly more than 70. Checking, $71^2 = 5041$ and $72^2 = 5184$,

hence (D).

14. If $\dfrac{x-a}{x-b} = \dfrac{x-b}{x-a}$ and $a \neq b$, what is the value of x?

(A) $\dfrac{a-b}{2}$ (B) $\dfrac{a^2+b^2}{a+b}$ (C) $\dfrac{a^2+b^2}{2(a+b)}$ (D) $a+b$ (E) $\dfrac{a+b}{2}$

▶ *Alternative 1*

Multiplying the equation by both denominators yields the equivalent equation which we solve

$$\begin{aligned}(x-a)^2 &= (x-b)^2\\ -2ax + a^2 &= -2bx + b^2\\ x &= \frac{a^2-b^2}{2(a-b)} = \frac{a+b}{2}\end{aligned}$$

where the last cancellation is valid, since $a \neq b$,

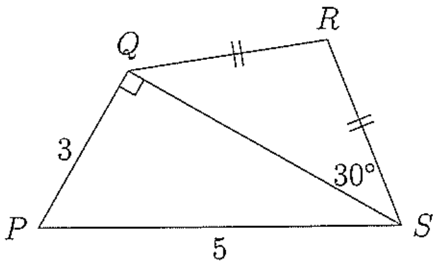
hence (E).

Alternative 2

With $u = \dfrac{x-a}{x-b}$ and $a \neq b$, we have $u \neq 1$ and $u = \dfrac{1}{u}$. Then $u^2 = 1$ and so $u = -1$. Consequently $x - a = -(x-b)$ and $2x = a + b$,

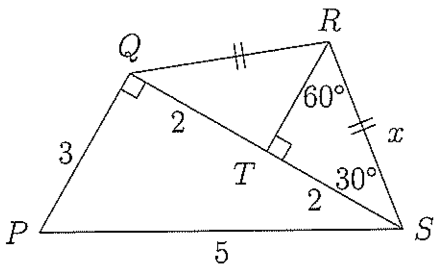
hence (E).

15. In the diagram, $PS = 5$, $PQ = 3$, $\triangle PQS$ is right-angled at Q, $\angle QSR = 30°$ and $QR = RS$. The length of RS is



(A) $\dfrac{\sqrt{3}}{2}$ (B) $\sqrt{3}$ (C) 2 (D) $\dfrac{4\sqrt{3}}{3}$ (E) 4

▶ Due to the right-angled triangle $\triangle PQS$, Pythagoras' theorem gives $QS = 4$. Then $\triangle QRS$ is isosceles, so its altitude RT bisects QS.



Now, $\triangle SRT$ is standard $30°, 60°, 90°$ triangle with $RT : RS : ST = 1 : 2 : \sqrt{3}$ so that $x = RS = \dfrac{2}{\sqrt{3}}ST = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$,

hence (D).

Comment
This problem can also be solved using trigonometry.

$$x = \frac{2}{\cos 30°} = \frac{4}{\sqrt{3}}$$

16. Billy, a seasonal worker in the town of Cowra, collected an even number of buckets of cherries on his first day. Each day after that he increased the number of buckets he picked by 2 buckets per day. In the first 50 days he collected 3250 buckets. The number of buckets Billy collected on the 50th day was

(A) 66 (B) 110 (C) 114 (D) 116 (E) 120

▶ *Alternative 1*
Let n be the number of buckets collected on the first day. Then he collected $n+2$ on the 2nd day, $n+4$ on the third day, and so on up to $n+98$ on the 50th day. The sum of this arithmetic progression is

$$\begin{aligned} 3250 &= \frac{50}{2}(n + (n+98)) \\ &= 50n + 2450 \end{aligned}$$

Then $n = (3250 - 2450) \div 50 = 16$ and $n + 98 = 114$,

hence (C).

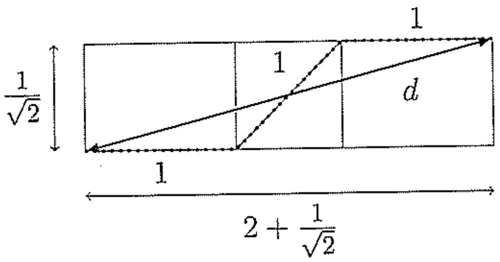
Alternative 2
On average, Billy picked $3250 \div 50 = 65$ buckets per day. His picking rate increased steadily from the 1st day to the 50th day with an increase of 98 buckets overall. So he must have started with $65 - 49 = 16$ buckets and finished with $65 + 49 = 114$ buckets,

hence (C).

17. A farmer walks 1 km east across his paddock, then 1 km north-east and then another 1 km east. Find the distance, in kilometres, between the farmer's initial position and his final position.

(A) $\sqrt{5+2\sqrt{2}}$ (B) $\sqrt{10}$ (C) $\sqrt{5}$ (D) $2+\sqrt{2}$ (E) $\sqrt{\frac{11}{2}+2\sqrt{10}}$

▶ Let d be the distance in kilometres.



$$\begin{aligned} d^2 &= \left(2+\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 \\ &= 4 + \tfrac{4}{\sqrt{2}} + \tfrac{1}{2} + \tfrac{1}{2} \\ &= 5 + 2\sqrt{2} \\ d &= \sqrt{5+2\sqrt{2}} \end{aligned}$$

hence (A).

18. Two machines move at constant speeds around a circle of circumference 600 cm, starting together from the same point. If they travel in the same direction then they next meet after 20 seconds, but if they travel in opposite directions then they next meet after 5 seconds. At what speed, in centimetres per second, is the faster one travelling?

(A) 60 (B) 65 (C) 70 (D) 75 (E) 85

▶ Let the speeds in centimetres per second be v (for the faster) and w. Then

$$5v + 5w = 600 \qquad \text{and} \qquad 20v - 20w = 600.$$

Then

$$v + w = 120 \qquad \text{and} \qquad v - w = 30,$$

so that $2v = 150$ and $v = 75$,

hence (D).

19. The equation $x^2 - kx + 374 = 0$ has two integer solutions. How many distinct values of k are possible?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

▶ If the solutions are α and β, then $\alpha\beta = 374$ and $\alpha + \beta = k$, and so α or β have the same sign. Since $374 = 2 \times 11 \times 17$ as prime factors and α and β are integers we have the following solutions.

α, β	k
$1, 374$	375
$2, 187$	189
$11, 34$	45
$17, 22$	39
$-1, -374$	-375
$-2, -187$	-189
$-11, -34$	-45
$-17, -22$	-39

hence (D).

20. Given that $f_1(x) = \dfrac{x}{x+1}$ and $f_{n+1}(x) = f_1(f_n(x))$, then $f_{2014}(x)$ equals

(A) $\dfrac{x}{2014x+1}$ (B) $\dfrac{2014x}{2014x+1}$ (C) $\dfrac{x}{x+2014}$ (D) $\dfrac{2014x}{x+1}$ (E) $\dfrac{x}{2014(x+1)}$

▶ *Alternative 1*

$$f_2(x) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{x}{x+x+1} = \frac{x}{2x+1}$$

$$f_3(x) = \frac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \frac{x}{x+2x+1} = \frac{x}{3x+1}$$

and in general, by induction

$$f_n(x) = \frac{x}{nx+1} \Longrightarrow f_{n+1}(x) = \frac{\frac{x}{nx+1}}{\frac{x}{nx+1}+1} = \frac{x}{x+nx+1} = \frac{x}{(n+1)x+1},$$

so $f_{2014}(x) = \dfrac{x}{2014x+1}$,

hence (A).

Alternative 2

Consider $\dfrac{1}{f_n(x)}$.

$$\frac{1}{f_1(x)} = 1 + \frac{1}{x} \implies \frac{1}{f_{n+1}(x)} = f_1(f_n(x)) = 1 + \frac{1}{f_n(x)}$$

$$\implies \frac{1}{f_{2014}(x)} = 1 + \frac{1}{f_{2013}(x)} = 2 + \frac{1}{f_{2012}(x)} = \cdots$$

$$\cdots = 2013 + \frac{1}{f_1(x)} = 2014 + \frac{1}{x} = \frac{2014x+1}{x}$$

Hence $f_{2014}(x) = \dfrac{x}{2014x+1}$,

hence (A).

21. (Also I23)

Starting with $\frac{2}{3}$ of a tank of fuel, I set out to drive the 550 km from Scone to Canberra. At Morisset, 165 km from Scone, I have $\frac{1}{2}$ of a tank remaining. If I continue with the same fuel consumption per kilometre and without refuelling, what happens?

(A) I will arrive in Canberra with $\frac{1}{9}$ of a tank to spare.

(B) I will arrive in Canberra with $\frac{1}{20}$ of a tank to spare.

(C) I will run out of fuel precisely when I reach Canberra.

(D) I will run out of fuel 110 km from Canberra.

(E) I will run out of fuel 220 km from Canberra.

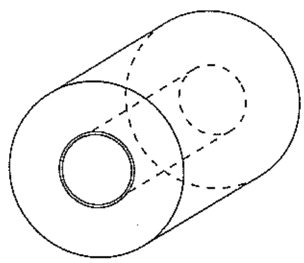
▶ Driving 165 km uses $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ of a tank, so on a full tank I can travel $6 \times 165 = 990$ km. Consequently the trip to Canberra uses $550 \div 990 = \frac{5}{9}$ of a tank. Since the car started with $\frac{2}{3} = \frac{6}{9}$ of a tank, I will make it to Canberra with $\frac{1}{9}$ of a tank remaining,

hence (A).

22. (Also I25)

Thanom has a roll of paper consisting of a very long sheet of thin paper tightly rolled around a cylindrical tube, forming the shape indicated in the diagram.

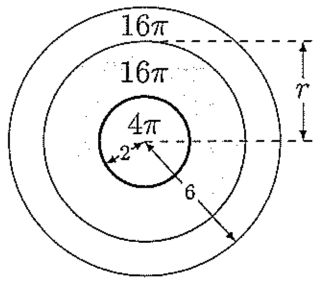
Initially, the diameter of the roll is 12 cm and the diameter of the tube is 4 cm. After Thanom uses half of the paper, the diameter of the remaining roll is closest to



(A) 6 cm (B) 8 cm (C) 8.5 cm (D) 9 cm (E) 9.5 cm

▶ Working in centimetres, let the half-roll's radius be r. From end on, the full roll had area $\pi(6^2 - 2^2) = 32\pi$, so half the roll has area 16π. Including the tube, the end of the half-roll has area $20\pi = \pi r^2$.

Then $r^2 = 20$, but $4.5^2 = 20.25$ and $4.4^2 = 19.36$, so that $4.4 < r < 4.5$, and the diameter is twice that,



hence (D).

23. For every 100 people living in the town of Berracan, 50 live in two-person households, 30 live in three-person households and 20 live in four-person households. What is the average number of people living in a household?

(A) 2.0 (B) 2.5 (C) 2.7 (D) 2.8 (E) 3.0

▶ *Alternative 1*

Let N be the population of Berracan, then

(i) $0.5N$ people live in 2-person households, in $\frac{0.5N}{2} = 0.25N$ houses

(ii) $0.3N$ people live in 3-person households, in $\frac{0.3N}{3} = 0.1N$ houses

(iii) $0.2N$ people live in 4-person households, in $\frac{0.2N}{4} = 0.05N$ houses.

Thus there are $0.4N$ households in Berracan and the average number of residents per household is $N \div 0.4N = 2.5$,

hence (B).

Alternative 2

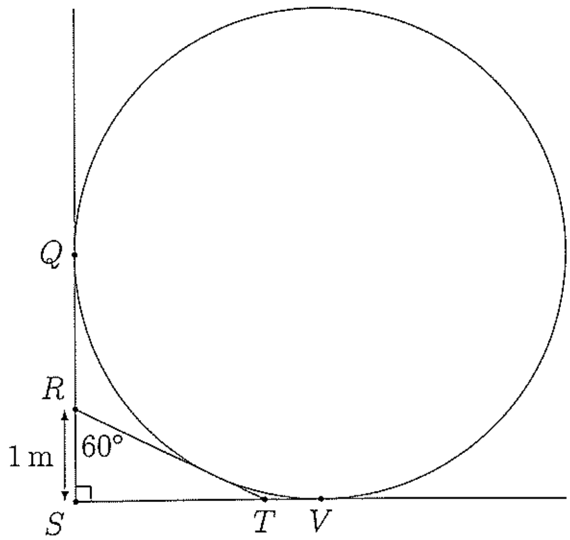
For every 100 people there are $\frac{50}{2} + \frac{30}{3} + \frac{20}{4} = 40$ households, so there are $100 \div 40 = 2.5$ people per household,

hence (B).

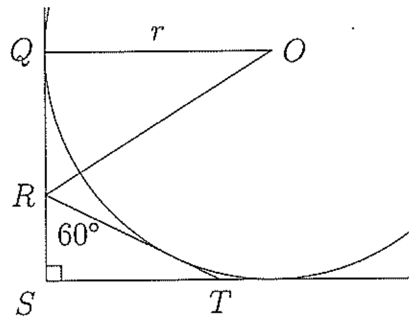
24. In the diagram, QS, RT and SV are tangents to the circle. The length of RS is 1 m and $\angle SRT = 60°$. What is the diameter of the circle, in metres?

(A) $3 + \sqrt{3}$ (B) 4 (C) $2\sqrt{3} + 2$

(D) $3\sqrt{3}$ (E) $\frac{9}{2}$



▸ *Alternative 1*
Let the radius be r metres, and label points thus:

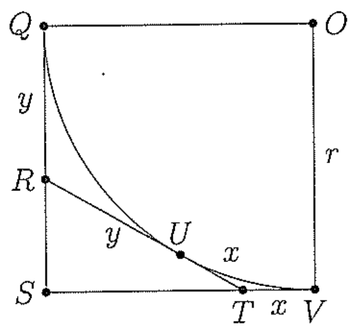


Then RO bisects the $120°$ angle $\angle QRT$, so that $\angle QRO = 60°$ and $r = QO = \sqrt{3}RQ$. Then

$$\begin{aligned} SQ &= SR + RQ \\ r &= 1 + \frac{1}{\sqrt{3}}r \\ (\sqrt{3}-1)r &= \sqrt{3} \\ 2r &= \sqrt{3}(\sqrt{3}+1) = 3+\sqrt{3}, \end{aligned}$$

hence (A).

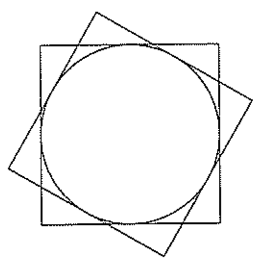
Alternative 2



In the diagram, note that the tangents from T to the circle are of equal length x and likewise the tangents from R are of length y. Then the perimeter of ΔRST is $3+\sqrt{3}$ but this is also $QS + SV = 2r$, the diameter of the circle,

hence (A).

Alternative 3
Draw two squares around the circle:



There are 8 congruent triangles with sides 1, $\sqrt{3}$ and 2. So the side of the square (and the diameter) is $1+\sqrt{3}+2$,

hence (A).

25. The sequence

$$2\ ,\ 2^2\ ,\ 2^{2^2}\ ,\ 2^{2^{2^2}}\ ,\ldots$$

is defined by $a_1 = 2$ and $a_{n+1} = 2^{a_n}$ for all $n \geq 1$. What is the first term in the sequence greater than 1000^{1000}?

(A) $a_4 = 2^{2^{2^2}}$ (B) $a_5 = 2^{2^{2^{2^2}}}$ (C) $a_6 = 2^{2^{2^{2^{2^2}}}}$ (D) $a_7 = 2^{2^{2^{2^{2^{2^2}}}}}$ (E) $a_8 = 2^{2^{2^{2^{2^{2^{2^2}}}}}}$

▶ We want $a_n > 1000^{1000} = 10^{3000}$. We know that $a_1 = 2$, $a_2 = 2^2 = 4$, $a_3 = 2^4 = 16$ and $a_4 = 2^{16} = 65536$, all less than 10^{3000}. Also $2^{10} = 1024 > 10^3$, so that we can estimate a_5,

$$a_5 = 2^{65536} = (2^{10})^{6553}\, 2^6 > (10^3)^{6553}\, 2^6 = 64 \times 10^{19659}$$

This is greater than 10^{3000},

hence (B).

26. (Also J30)
What is the largest three-digit number with the property that the number is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit?

▶ *Alternative 1*
Let the number be abc.
Then

$$\begin{aligned} 100a + 10b + c &= a + b^2 + c^3 \\ 99a + 10b - b^2 &= c(c^2 - 1) \\ 99a + b(10 - b) &= (c-1)c(c+1) \end{aligned}$$

Consider the possibilities:

$99a$	$b(10-b)$	$(c-1)c(c+1)$
$99 \times 1 = 99$	$1 \times 9 = 9$	$1 \times 2 \times 3 = 6$
$99 \times 2 = 198$	$2 \times 8 = 16$	$2 \times 3 \times 4 = 24$
$99 \times 3 = 297$	$3 \times 7 = 21$	$3 \times 4 \times 5 = 60$
$99 \times 4 = 396$	$4 \times 6 = 24$	$4 \times 5 \times 6 = 120$
$99 \times 5 = 495$	$5 \times 5 = 25$	$5 \times 6 \times 7 = 210$
$99 \times 6 = 594$	$6 \times 4 = 24$	$6 \times 7 \times 8 = 336$
$99 \times 7 = 693$	$7 \times 3 = 21$	$7 \times 8 \times 9 = 504$
$99 \times 8 = 792$	$8 \times 2 = 16$	$8 \times 9 \times 10 = 720$
$99 \times 9 = 891$	$9 \times 1 = 9$	

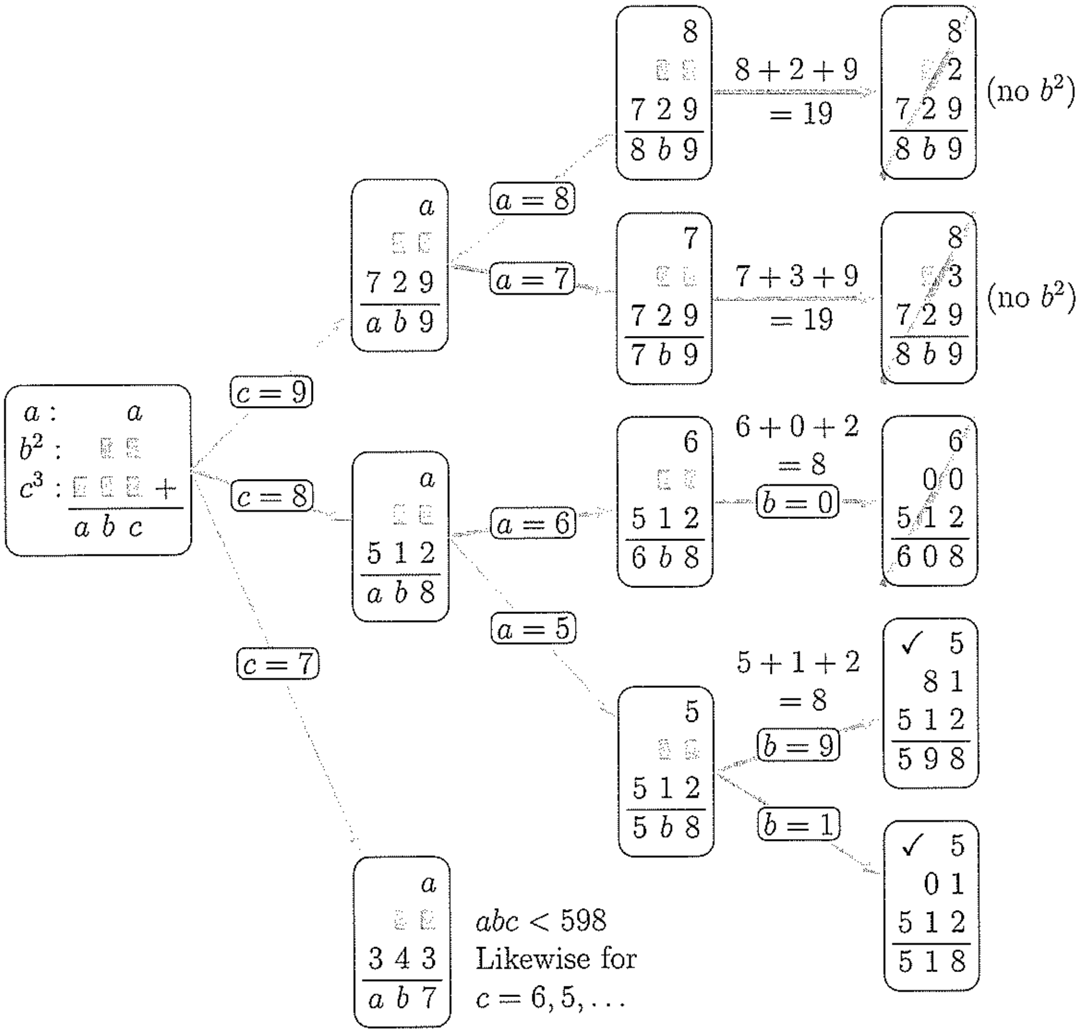
Looking at the possibilities for $99a + b(10-b) = (c-1)c(c+1)$, we have two:
$99 + 21 = 120 \Longrightarrow a = 1,\ b = 3 \text{ or } 7, c = 5 \Longrightarrow n = 135 \text{ or } n = 175.$
$495 + 9 = 504 \Longrightarrow a = 5,\ b = 1 \text{ or } 9, c = 8 \Longrightarrow n = 518 \text{ or } n = 598.$
So, there are four 3-digit numbers which satisfy the requirements and the largest of these four numbers is 598,

hence (598).

Alternative 2
The number abc is equal to $a + b^2 + c^3$, and these are the possible values of b^2 and c^3:

Digit	0	1	2	3	4	5	6	7	8	9
Square	0	1	4	9	16	25	36	49	64	81
Cube	0	1	8	27	64	125	216	343	512	729

We try these numbers in an addition grid, trying the large values of c first, then filling in possible values for a and b. This trial-and-error search is presented here as a tree.



The largest solution found is 598, and any solutions on branches $c = 7$, $c = 6$, $\ldots$, $c = 1$ must be less than this,

hence (598).

27. Igor wants to make a secret code of five-letter words. To make them easy to say, he follows these two rules:

(i) no more than two consonants or two vowels in succession

(ii) no word to start or end with two consonants

He rejects the letter 'Q' as too hard, so he has 20 consonants and 5 vowels to choose from. If N is the number of code words possible, what are the first three digits of N?

▶ *Alternative 1*
First find the various arrangements of consonants (C) and vowels(V):

Three consonants, two vowels: CVCCV, CVCVC, VCCVC.

Two consonants, three vowels: CVCVV, CVVCV, VCCVV, VCVCV, VCVVC, VVCCV, VVCVC.

One consonant, four vowels: VVCVV.

There are $20^3.5^2$ arrangements of each 3C2V type, $20^2.5^3$ arrangements of each 2C3V type and 20.5^4 arrangements of the 4C1V type. So the total number of possible words is

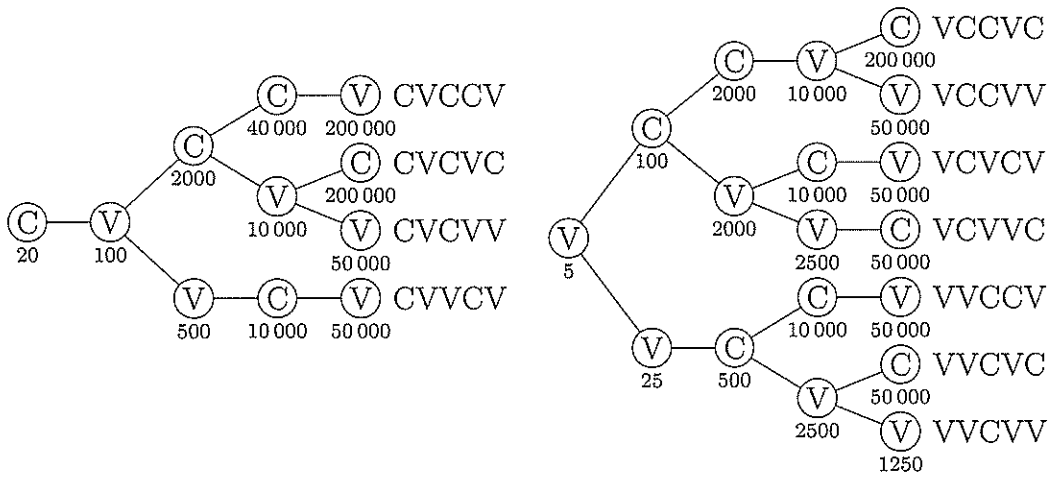
$$3.20^3.5^2 + 7.20^2.5^3 + 20.5^4 \;=\; 3.20.100^2 + 7.5.100^2 + 125.100 \;=\; 962\,500$$

The first three digits are 962,

hence (962).

Alternative 2

These tree diagrams show the number of choices available for each letter in the code word, depending on whether the letter chosen is a consonant (C) or a vowel (V).



Then the total number of possibilities is $3 \times 200\,000 + 7 \times 50\,000 + 12\,500 = 962\,500$,

hence (962).

28. (Also I30)

Consider the sequence $a_1, a_2, a_3, a_4, \ldots$ such that $a_1 = 2$ and for every positive integer n,

$$a_{n+1} = a_n + p_n, \qquad \text{where } p_n \text{ is the largest prime factor of } a_n.$$

The first few terms of the sequence are $2, 4, 6, 9, 12, 15, 20$. What is the largest value of n such that a_n is a four-digit number?

▶ *Alternative 1*

Let us write out some terms of the sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n	2	4	6	9	12	15	20	25	30	35	42	49
p_n	2	2	3	3	3	5	5	5	5	7	7	7

n	13	14	15	16	17	18	19	20	21	22	23	24
a_n	56	63	70	77	88	99	110	121	132	143	156	169
p_n	7	7	7	11	11	11	11	11	11	13	13	13

The crucial observation is that if p is a prime, then $a_{2p-2} = p^2$. If we assume that this is true, then we have $a_{192} = 97^2$. The next few terms can then be calculated as follows.

n	192	193	194	195	196	197	198	199
a_n	97^2	97×98	97×99	97×100	97×101	98×101	99×101	100×101
p_n	97	97	97	97	101	101	101	101

Since $a_{198} = 99 \times 101 = 9999$ and $a_{199} = 100 \times 101 = 10100$, the answer to the problem is 198.

In order to prove our crucial observation above, suppose that $a_{2p-2} = p^2$ for some prime p. Let q be the next largest prime after p. Then the next $q - p$ terms of the sequence after $a_{2p-2} = p^2$ are

$$p(p+1),\ p(p+2),\ p(p+3),\ \ldots,\ pq\,.$$

To see this, note that the difference between consecutive terms is p and that p divides each term. Furthermore, no prime larger than p can divide any term apart from the last. That is because each of those terms is of the form pk, where $p < k < q$. Since k lies between the consecutive primes p and q, it cannot be divisible by a prime larger than p.

The next $q - p$ terms of the sequence are

$$(p+1)q,\ (p+2)q,\ (p+3)q,\ \ldots,\ q^2\,.$$

To see this, note that the difference between consecutive terms is q and that q divides each term. Furthermore, no prime larger than q can divide any term. That is because each term is of the form kq, where $p < k \le q$. Since k is at most q, it cannot be divisible by a prime larger than q.

We have shown that if $a_{2p-2} = p^2$ for a prime p, then $2q - 2p$ terms further along in the sequence we find q^2. In other words, $a_{2q-2} = q^2$, where q is the next largest prime after p. By induction on the prime numbers, this shows that $a_{2p-2} = p^2$ for all primes p,

hence (198).

Alternative 2

Let $q_1, q_2, \ldots$ be the primes in ascending order. In the sequence described, the difference between two terms is always a prime. It can be seen from examining a few terms that the change in difference occurs at a product of two successive primes. So the length of each sequence of common differences q_m is $q_{m+1} - q_{m-1}$.

(Note that these terms with common differences q_m are the elements of a multiplication table for q_m between $q_{m-1}q_m$ and $q_m q_{m+1}$ and so can never divide by a higher prime.)

So adding the lengths of sequences with a common difference we find:

Difference	Number of terms	Cumulative count
q_1	$q_2 - 1$	1
q_2	$q_3 - q_1$	$q_3 - q_1 - 1$
q_3	$q_4 - q_2$	$q_4 + q_3 - q_1 - 1$
q_4	$q_5 - q_3$	$q_5 + q_4 - q_1 - 1$
q_5	$q_6 - q_4$	$q_6 + q_5 - q_1 - 1$

and, in general, the common difference changes to q_m at term $q_m + q_{m+1} - 2$, so that

$$a_{q_m+q_{m+1}-2} = q_m \times q_{m-1} \, .$$

Now, we find two consecutive primes with a product just less than 10000. $97 \times 101 = 9797$ and $101 \times 103 > 10000$ so $a_{97+101-2} = a_{196} = 9797$. From there, $a_{197} = 9898$, $a_{198} = 9999$,

hence (198).

29. A *lattice point* in the plane is a point whose coordinates are both integers. Consider a triangle whose vertices are lattice points $(0, 0)$, $(a, 0)$, and $(0, b)$, where $a \geq b > 0$. Suppose that the triangle contains exactly 74 lattice points in its interior, not including those lattice points on the sides of the triangle. Determine the sum of the areas of all such triangles.

▶ Inside the $a \times b$ rectangle, there are $(a-1)(b-1)$ lattice points, and with $g = \gcd(a, b)$, the diagonal passes through $g - 1$ of these. So the number of lattice points inside the triangle is

$$\frac{(a-1)(b-1) - g + 1}{2} = 74$$

and so we require

$$(a-1)(b-1) = 147 + g$$

Since we can write $a = cg$, $b = dg$ then

$$cdg^2 - (c+d)g = 146 + g$$

so that g is a divisor of $146 = 2 \times 73$, that is, $g \in \{1, 2, 73, 146\}$.
But $a \geq b \geq g$, so that

$$(g-1)^2 \leq (a-1)(b-1) = 147 + g$$

which fails for $g = 73$ and $g = 146$, so that $g = 1$ or $g = 2$.
Then $147 + g = 148$ or 149, which we need to factorise as $(a-1)(b-1)$. Since $148 = 2^2 \times 37$ and 149 is prime, there are only four cases to check.

g	$147+g$	$a-1$	$b-1$	a	b	$g = \gcd(a,b)$?	$A = \frac{1}{2}ab$
1	148	148	1	149	2	✓	149
1	148	74	2	75	3	×	
1	148	37	4	38	5	✓	95
2	149	149	1	150	2	✓	150
							394

So there are three such triangles, and the sum of their areas is 394,

hence (394).

30. A polynomial $p(x)$ is called *self-centered* if it has integer coefficients and $p(100) = 100$. If $p(x)$ is a self-centred polynomial, what is the maximum number of integer solutions k to the equation $p(k) = k^3$?

▶ Write $q(x) = p(x) - x^3$. Then we are looking for (distinct) zeros $a_1, \ldots, a_n$ of $q(x)$. By the factor theorem, each $(x - a_k)$ will be a factor of $q(x)$. Then using the long division algorithm to divide out each factor $(x - a_k)$ in turn, the quotient polynomial at each stage has integer coefficients. Consequently there is a factorisation $q(x) = (x - a_1) \ldots (x - a_n) q_0(x)$ where $q_0(x)$ is a polynomial with integer coefficients.
Then

$$q(100) = 100 - 100^3 = -999900 = -99 \cdot 100 \cdot 101 = -2^2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 101$$

but also we have a factorisation into integers

$$q(100) = (100 - a_1) \ldots (100 - a_n) q_0(100)$$

where the $(100 - a_k)$ are distinct factors of $q(100)$.
The largest number of distinct factors multiplying to give $q(100)$ is

$$q(100) = 1 \cdot (-1) \cdot 2 \cdot (-2) \cdot 3 \cdot (-3) \cdot 5 \cdot (-5) \cdot (-11) \cdot 101$$

and so $n \leq 10$. That is, every self-centred polynomial has 10 or fewer integer solutions to $p(x) = x^3$.
On the other hand, the above factorisation of -999900 into distinct integer factors allows us to write a polynomial

$$\begin{aligned} q(x) = (x - 99)(x - 101)(x - 98)(x - 102)(x - 97)(x - 103) \\ (x - 95)(x - 105)(x - 111)(x + 1) \end{aligned}$$

This has 10 integer zeros and $q(100) = -999900$. So if we write $p(x) = q(x) + x^3$ then $p(100) = 100$ and there are 10 integer solutions to the equation $p(x) = x^3$,
hence (10).