Carlingford High School

Mathematics Extension 1

Year 12 Term 2 Examination

2015

Time allowed: 55 minutes

Name: ___________________________ Class: 12MA1___

12MA11 (Mrs Strilakos) 12MA12 (Mr Cheng)
12MA13 (Ms Wilson/Mr Gong)

Instructions

- Start each question on a new page
- Write on one side of the paper only, and not in columns
- Marks may be deducted for careless or badly arranged work
- Only calculators approved by the Board of Studies may be used
- All answers are to be completed in blue or black pen except graphs and diagrams
- No lending or borrowing

Outcome/Question	**1**	**2**	**3**	**Mark**
Logarithmic and Exponential Functions	/14			**/14**
Inverse Functions (including Inverse Trigonometric Functions)		/12		**/12**
Methods of Integration			/9	**/9**
Total	**/14**	**/12**	**/9**	**/35**

Question 1 **Marks**

(a) If $y = cx^n$, write an expression for $log_e x$, in terms of y. [2]

(b) Show that $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx = 2log_e(1+\sqrt{x}) + c$ [2]

(c) Given the function $y = xlog_e x$, for $x > 0$

(i) Show that there is a minimum turning point at $(\frac{1}{e}, -\frac{1}{e})$. [2]

(ii) Sketch the curve, for $0 < x \leq 5$, showing all turning points and intercepts. [3]

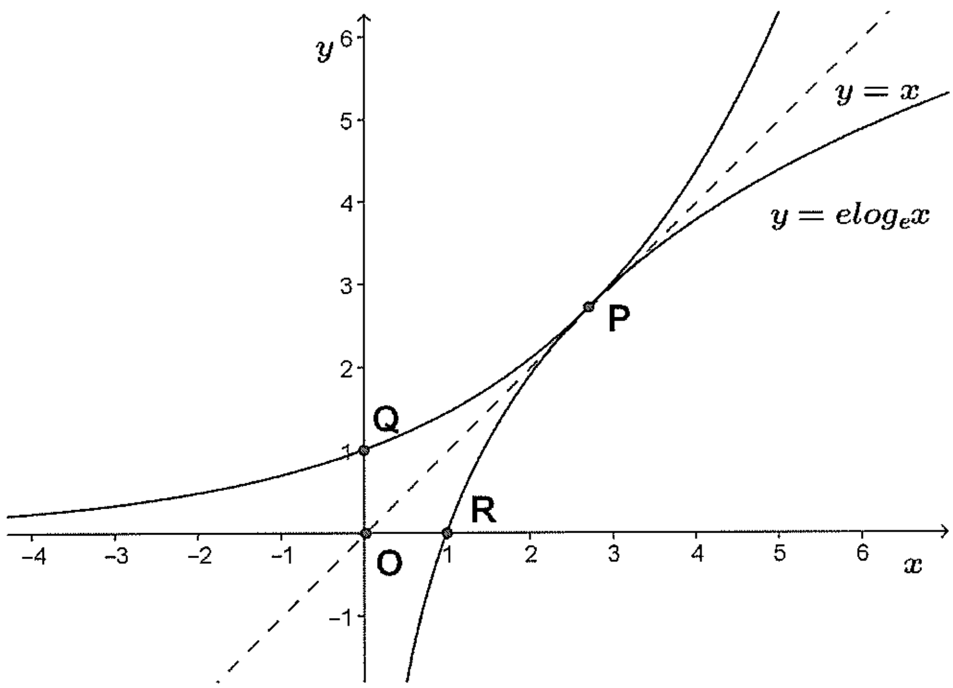
(d) The graphs of $y = elog_e x$ and its inverse are shown on the number plane below.

The two graphs intersect at the point *P* on the line $y = x$.

(i) Find the coordinates of *P*. [1]

(ii) Rewrite $y = elog_e x$ with x as the subject. [1]

(ii) Find the area of the region enclosed by the two graphs and the x and y axes. [3]



Question 2 (Please start a new page)

(a) Simplify $cos^{-1}a - cos^{-1}(-a)$ [1]

(b) (i) Differentiate $tan^{-1}(x+1)$ [1]

(ii) Hence, find the equation of the curve $y = f(x)$ that passes through the origin, given that $f'(x) = \frac{1}{x^2+2x+2}$ [2]

(c) Find $\int \frac{1}{\sqrt{25-4x^2}} dx$ [2]

(d) Let $f(x) = 1 + \sqrt{1-x}$

(i) Find the domain and range of $f(x)$ [2]

(ii) Write an expression for the inverse function $f^{-1}(x)$ [1]

(iii) On the same set of axes, sketch the graphs $y = f(x)$ and $y = f^{-1}(x)$, showing any axes intercepts and intersection points [3]

Question 3 (Please start a new page)

(a) Use the substitution $u = 1 + 2x$ to find $\int x\sqrt{1+2x}\, dx$ [2]

(b) (i) Write an expression for sin^2x in terms of $cos2x$ [1]

(ii) Hence, find the volume of the solid of revolution formed by revolving $y = sin3x$ about the x-axis, between $x = 0$ and $x = \frac{\pi}{6}$ [3]

(c) Use an appropriate substitution to evaluate $\int_0^{\frac{\pi}{2}} sinxcos^3x\, dx$ [3]

END OF TEST (now go back and check your work!)

TERM 2 EXAM SOLUTIONS - EXTENSION 1 2015

QUESTION 1

QN. 1 - STRILAKOS
2 - WILSON
3 - CHENG

(a) $\log y = \log cx^n$

$\log y = \log c + \log x^n$

$\log y = \log c + n\log x$ ✓

$n\log x = \log y - \log c$

$\left.\begin{aligned}\log x &= \frac{1}{n}(\log y - \log c)\\ &= \frac{1}{n}(\log \tfrac{y}{c})\end{aligned}\right\}$ ✓

(b) $\int \frac{x^{-1/2}}{1+x^{1/2}}\,dx$ ✓ $= 2\int \frac{\frac{1}{2}x^{-1/2}}{1+x^{1/2}}\,dx$

$= 2\log(1+\sqrt{x}) + c$ ✓

(c)(i) $y' = x \cdot \frac{1}{x} + \log x \cdot 1$

$= 1 + \log x$

$y'' = \frac{1}{x}$

when $y'=0$, $\log x = -1$

$x = e^{-1} = \frac{1}{e}$ ✓

when $x = \frac{1}{e}$, $y = \frac{1}{e}\log\frac{1}{e}$

$y = \frac{1}{e}\log e^{-1}$

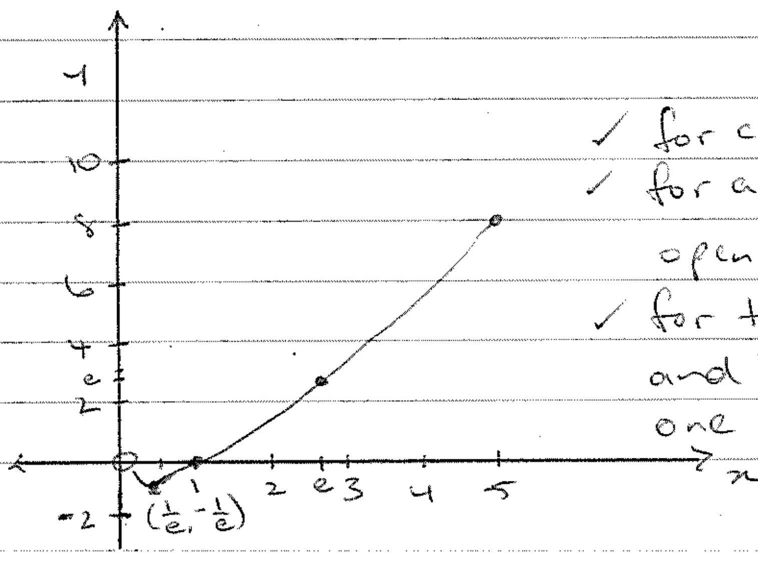
$y = -\frac{1}{e}\log e$

$y = -\frac{1}{e}$

when $x = \frac{1}{e}$, $y'' = \frac{1}{1/e} = e > 0$ ✓

$\therefore$ minimum turning point at $\left(\frac{1}{e}, -\frac{1}{e}\right)$

(ii)



✓ for correct curve
✓ for axes intercepts, including open circle at origin.
✓ for turning point plotted and labelled, and at least one other point on the curve.

(d)(i) $P(e, e)$ ✓ (no reason required)

(ii). $\frac{y}{e} = \log_e x$

$x = e^{y/e}$ ✓

(iii). Area between $y = e\log_e x$ and the y-axis, between $y = 0$ and $y = e$, is given by

$$A = \int_0^e e^{y/e}\,dy$$

$$= e\int_0^e \frac{1}{e}e^{y/e}\,dy$$

$$= e\left[e^{y/e}\right]_0^e$$

$$= e(e-1) \text{ units}^2 \checkmark$$

Area of PRO $= e(e-1) - \frac{1}{2}e^2$

$= e^2 - e - \frac{1}{2}e^2$

$= \frac{1}{2}e^2 - e$

$= e(\frac{1}{2}e - 1)$ units^2 ✓

Shaded area $= 2(\frac{1}{2}e^2 - e)$ ✓

$= e^2 - 2e$ units^2

QUESTION 2

(a) $\cos^{-1}a - (\pi - \cos^{-1}a) = 2\cos^{-1}a - \pi$ ✓

(b) (i) $\dfrac{1}{1+(1+x)^2}$ or $\dfrac{1}{x^2+2x+2}$ ✓

(ii) $f'(x) = \dfrac{1}{x^2+2x+2}$

$\therefore f(x) = \tan^{-1}(x+1) + c$

$f(0) = 0$

$\therefore 0 = \tan^{-1}1 + c$

$0 = \frac{\pi}{4} + c$

$c = -\frac{\pi}{4}$

$\therefore f(x) = \tan^{-1}(x+1) - \frac{\pi}{4}$ ✓

(c) $\displaystyle\int \frac{1}{\sqrt{4\left(\frac{25}{4}-x^2\right)}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - x^2}}\,dx$ ✓

$= \frac{1}{2}\sin^{-1}\frac{2x}{5} + c$ ✓

d) (i) Domain: $x \leq 1$ ✓
Range: $y \geq 1$ ✓

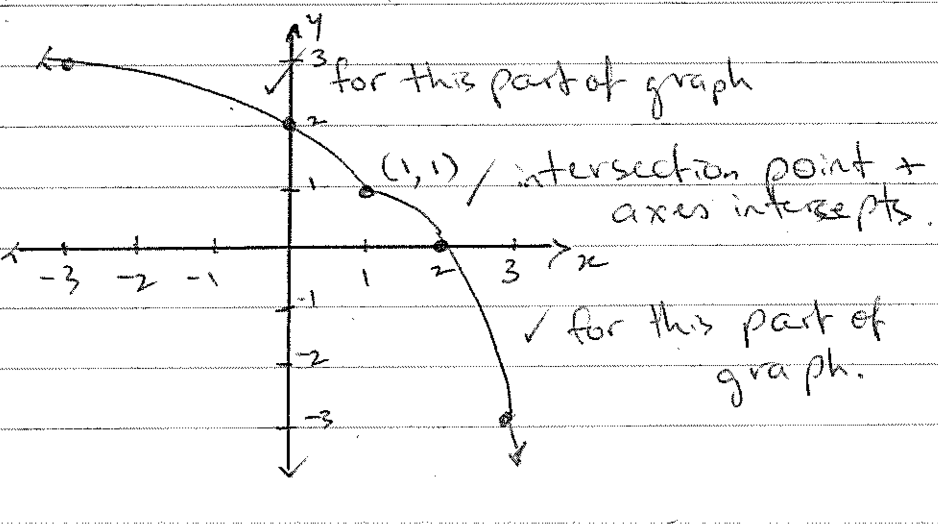
(ii) $x = 1 + \sqrt{1-y}$

$x - 1 = \sqrt{1-y}$

$(x-1)^2 = 1 - y$

$$\left.\begin{aligned} y &= 1-(x-1)^2 \\ &= 2x - x^2 \\ &= x(2-x) \end{aligned}\right\}$$ ✓ accept any of these.

(iii)



QUESTION 3

(a) $x = \frac{1}{2}(u-1)$

$du = 2\,dx \rightarrow dx = \frac{1}{2} \cdot du$

$\therefore \int x\sqrt{1+2x}\,dx = \frac{1}{2}\int (u-1)\,u^{1/2}\,du$ ✓

$= \frac{1}{2}\int u^{3/2} - u^{1/2}\,du$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + c$ ✓

(or equivalent – no marks lost for any attempt to simplify or change form)

(b) (i) $\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ ✓

(ii) $V = \pi \int_a^b y^2\,dx$

$= \pi \int_0^{\pi/6} \sin^2 3x\,dx$ ✓

$= \pi \int_0^{\pi/6} \frac{1}{2}(1 - \cos 6x)\,dx$ ✓

$= \frac{\pi}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\pi/6}$

$= \frac{\pi}{2}\left[\left(\frac{\pi}{6} - 0\right) - (0 - 0)\right]$

$= \frac{\pi^2}{12}$ units^3 ✓

(c) Let $u = \cos x$

$du = -\sin x \, dx$

when $x = 0$, $u = \cos 0 = 1$

$x = \frac{\pi}{2}$, $u = \cos \frac{\pi}{2} = 0$

$$\therefore \int_0^{\pi/2} \sin x \cos^3 x \, dx = -\int_1^0 u^3 \, du \quad \checkmark\checkmark$$

(1 mark only for $-\int_0^{\pi/2} u^3 \, du$)

$$= -\left[\frac{u^4}{4}\right]_1^0$$

$$= -\left(0 - \frac{1}{4}\right)$$

$$= \frac{1}{4} \quad \checkmark$$