ACE
EXAM
PAPER

Student name: ____________________

PAPER 2

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 1

General Instructions

- Working time - 120 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt questions 11-14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. If $A = \binom{2}{5}$ and $B = \binom{11}{1}$, what is vector $\overrightarrow{AB}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$?

   (A) $2\underset{\sim}{i} + 5\underset{\sim}{j}$

   (B) $9\underset{\sim}{i} - 4\underset{\sim}{j}$

   (C) $9\underset{\sim}{i} + 4\underset{\sim}{j}$

   (D) $13\underset{\sim}{i} + 6\underset{\sim}{j}$

2. What is $\dfrac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$?

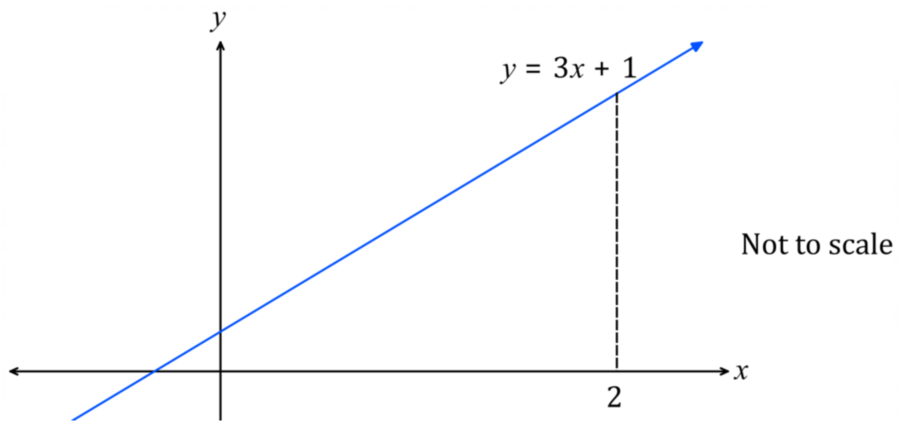
   (A) $\dfrac{-2}{\sqrt{1-x^2}}$

   (B) $\dfrac{-1}{\sqrt{1-x^2}}$

   (C) $\cos^{-1}x$

   (D) $\sin^{-1}x$

3. A region in the first quadrant is bounded by the line $y = 3x + 1$, the x-axis, the y-axis, and the line $x = 2$.



   What is the volume of the solid of revolution formed when this region is rotated about the x-axis?

   (A) 8 units^3

   (B) 38 units^3

   (C) 8π units^3

   (D) 38π units^3

4. Which of the following is an expression for $\int \frac{x}{(2-x^2)^3}\,dx$? Use the substitution $u = 2 - x^2$

(A) $\frac{1}{2(2-x^2)^2} + C$

(B) $\frac{1}{4(2-x^2)^2} + C$

(C) $\frac{1}{4(2-x^2)^4} + C$

(D) $\frac{1}{8(2-x^2)^4} + C$

5. Which one of the following differential equations is $y = 2xe^{2x}$ a solution?

(A) $\frac{d^2y}{dx^2} - 4y = 8e^{2x}$

(B) $\frac{d^2y}{dx^2} - 4y = e^{2x}$

(C) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

(D) $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 0$

6. A trigonometric expression is shown below.

$$\frac{\sin 2x}{\cos 2x - 1}$$

Which of the following is equal to the above expression?

(A) $-\cot x$

(B) $\cot(2x) - 1$

(C) $\sin(2x) + \sec(2x)$

(D) $\sin(2x) - \tan(2x)$

7. If $\frac{dy}{dx} = \frac{1}{5}(y-1)^2$ and $y = 0$ when $x = 0$ then:

(A) $y = \frac{x}{x+5}$

(B) $y = \frac{5}{1-x} - 5$

(C) $y = \frac{5}{x+5} - 1$

(D) $y = \frac{5}{x+5} + 1$

8. At a checkpoint 6% of the vehicles are trucks. A random sample of 30 vehicles is photographed passing through the checkpoint. What is the probability that three of the 30 vehicles will be trucks?

(A) 0.100

(B) 0.165

(C) 0.195

(D) 0.216

9. A ball is projected with a velocity of 25 ms^{-1} at an angle of 30° to the horizontal. What is the maximum height reached by the ball? Let g be 9.8 ms^{-2} ?

(A) 0.32 metres

(B) 7.97 metres

(C) 15.94 metres

(D) 31.13 metres

10. Aurora made an error proving that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. Part of the proof is shown below.

Step 1: To prove $2^n + (-1)^{n+1}$ is divisible by 3

(m is an integer)

To prove true for $n = 1$

$2^1 + (-1)^{1+1} = 2 + 1 = 3 \times 1$ Line 1

Result is true for $n = 1$

Step 2: Assume true for $n = k$

$2^k + (-1)^{k+1} = 3m$ where m is an integer Line 2

Step 3: To prove true for $n = k + 1$

$2^{k+1} + (-1)^{k+1+1} = 2(2^k) + (-1)^{k+2} = 3p$ where p is an integer Line 3

LHS $= 2[3m + (-1)^{k+1}] + (-1)^{k+2}$ Line 4

$= 2 \times 3m + 2 \times (-1)^{k+2} + (-1)^{k+2}$

$= 3[2m + (-1)^{k+2}]$

$= 3p$

$=$ RHS

Step 4: True by induction

In which line did Aurora make an error?

(A) Line 1

(B) Line 2

(C) Line 3

(D) Line 4

Section II

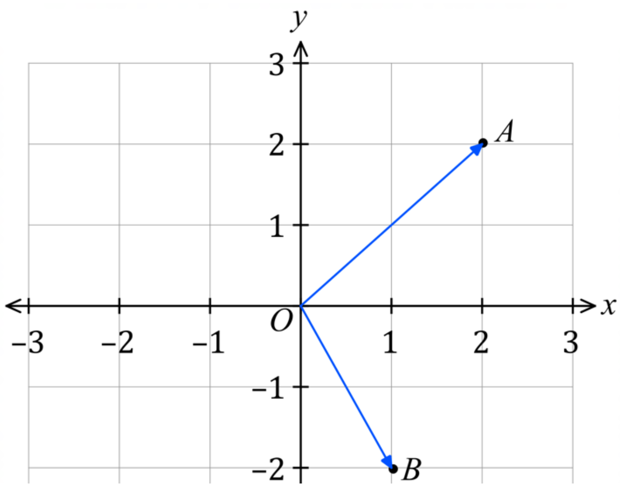
60 marks
Attempt questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) **Marks**

(a) Find the exact value of $\int_{-1}^{1} \sqrt{4-x^2}dx$, using the substitution $x = 2\sin\theta$. **3**

(b) The vectors $\overrightarrow{OA}$, and $\overrightarrow{OB}$ are shown below. **2**



Find the size of $\angle AOB$ to the nearest degree.

(c) (i) Prove that $\cot\theta - 2\cot 2\theta = \tan\theta$. **2**

(ii) Hence or otherwise show that: **2**

$\tan\theta + 2\tan 2\theta + 4\tan 4\theta = \cot\theta - 8\cot 8$

(d) Use the substitution $x = u^2 + 1$ to evaluate $\int_{2}^{10} \frac{x}{\sqrt{x-1}}dx$. **2**

(e) Prove by mathematical induction that, for $n \geq 1$ that: **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

(f) The area enclosed by the curve $y = x^4$, the y-axis and the line $y = 4$ is rotated about the y-axis. Find the volume of the solid of revolution. **2**

Question 12 (16 marks) **Marks**

(a) (i) Express $2\sin x - \cos x$ in the form $A\sin(x - \alpha)$ for $A > 0$ and $0 \leq \alpha \leq 90°$. **2**

(ii) Hence solve the equation $2\sin x - \cos x = 1$ for $0 \leq x \leq 360°$. **2**

(b) $\overrightarrow{PQ} = -\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\overrightarrow{QR} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$. What is the magnitude of $|\overrightarrow{PR}|$? **2**

(c) After time t years the number N of animals in a national park decreases according to the equation:

$$\frac{dN}{dt} = -0.09(N - 100)$$

The initial number of animals in the national park is 500.

(i) Verify that $N = 100 + Ae^{-0\cdot 09t}$ is a solution of the above equation, where A is a constant. **1**

(ii) After one year the number of animals in the national park is 400. **2**
Find the time taken for the number of animals to reach 200.
Answer correct to three significant figures.

(d) Use the substitution $x = \frac{1}{4}\tan\theta$ to evaluate $\int \frac{1}{1 + 16x^2}\,dx$. **3**

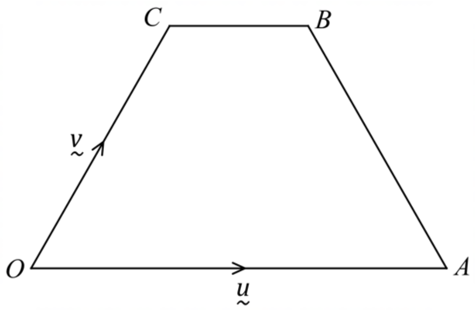
(e) A clay shooter hits the target 95% of the time. In a competition he will have forty shots at the target.

(i) What is the probability he hits 36 targets? Answer correct to 4 decimal places. **2**

(ii) What is the probability he misses at most two times? Answer correct to 4 decimal places. **2**

Question 13 (14 marks) **Marks**

(a) $OABC$ is a trapezium, $\overrightarrow{OA} = \underset{\sim}{u}$, $\overrightarrow{OC} = \underset{\sim}{v}$ and $\overrightarrow{OA} = 3\overrightarrow{CB}$. **2**



Express $\overrightarrow{AB}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

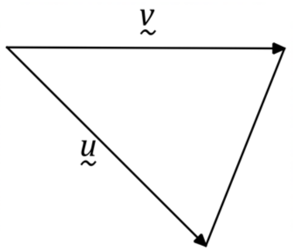
(b) A rock is projected horizontally from the top of a 25 metre high cliff.
The rock is thrown with an initial velocity of 40 ms^{-1}. Assume g = 10 ms^{-2}.

(i) Determine the parametric equations of the path. **2**
(Take the origin at the base of the cliff.)

(ii) How far from the base of the cliff does the rock hit the sea? **2**

(c) Prove by mathematical induction that $5^n + 12n - 1$ is divisible by 16 for all positive integers n $(n \geq 1)$ **3**

(d) An equilateral triangle of side 5 units is shown below. **2**



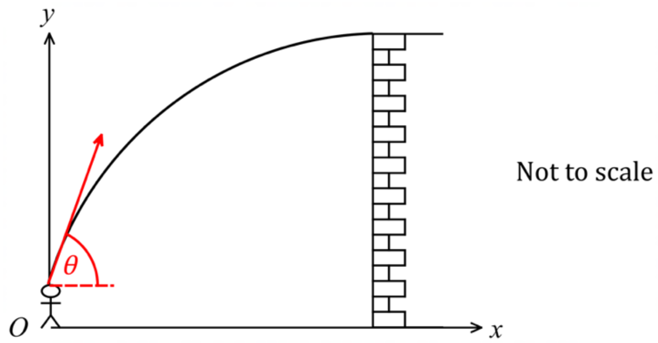
Vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are represented in the diagram. What is the value of $\underset{\sim}{u} \cdot \underset{\sim}{v}$?

(e) Find the exact value of $\displaystyle\int_0^{\pi} \sin^4 x dx$. **3**

Question 14 (14 marks) **Marks**

(a) What is the exact value of $\sin(-105°)$? **2**

(b) George of height 2 metres throws a ball to the top of a brick wall. The height of the brick wall is 15 metres. George throws the ball at an initial velocity of 25 m/s when he is 20 metres from the base of the brick wall.



You may assume that: $x = 25t\cos\theta,\ \ y = -5t^2 + 25t\sin\theta + 2$

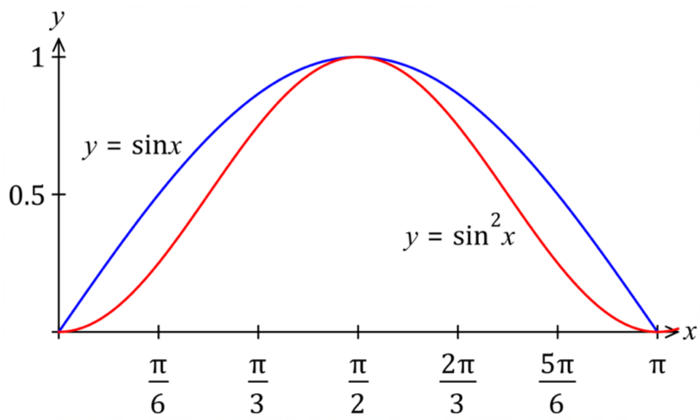
where x and y are the horizontal and vertical displacements of the ball in metres from O at time t seconds after being thrown.

(i) Show that the cartesian equation of the balls path is: **2**

$$y = -\frac{x^2}{125}(1 + \tan^2\theta) + x\tan\theta + 2$$

(ii) What are the two angles of projection he must throw the ball between to ensure that the ball lands at the top of the brick wall? **2**

(c) The curves $y = \sin x$ and $y = \sin^2 x$ between $0 \le x \le \pi$ are shown below. **2**



Find the area bounded between these curves in the domain.

(d) Two standard dice are rolled together, and the sum of the numbers rolled is noted. The result is recorded as either a 5 or not a 5.

(i) If p is the probability of a sum of 5, find the value of p. **1**

(ii) Find $E(X)$ and $\mathrm{Var}(X)$. **2**

(e) Solve the following differential equation giving y as a function of x. **3**

$$\frac{dy}{dx} = 1 - 2y \text{ where } y = -1 \text{ when } x = 0$$

End of paper

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

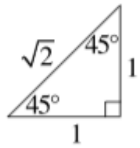
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

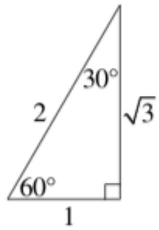
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

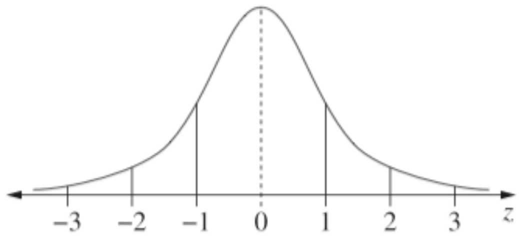
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$|\underset{\sim}{u}| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}|\,|\underset{\sim}{v}|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$