Lecture 16 (sections 3.1,3.2)

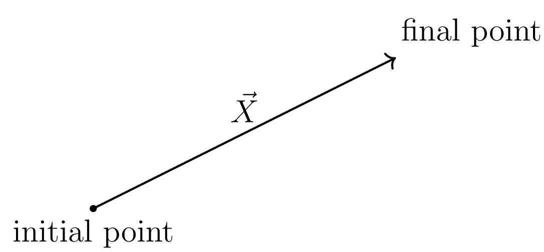
Section 3.1 — Euclidean vector spaces $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$

$$\mathbb{R}^2 : \qquad \vec{X} = (x_1, x_2)\,, \qquad \text{where } x_1, x_2 \in \mathbb{R} \implies \text{components of } \vec{X}$$

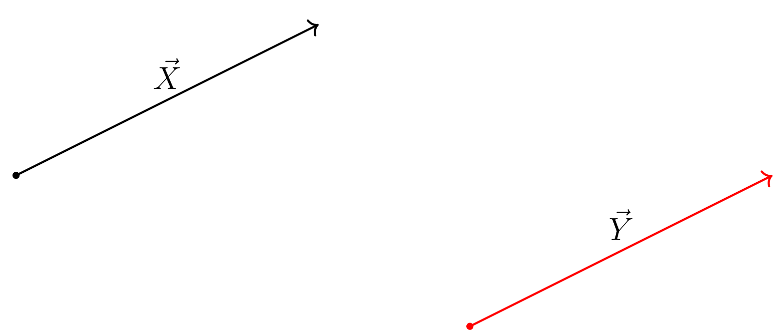
$\vec{X}$ is a vector:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\implies$ A vector is characterized by a *magnitude* (the length) and the *direction*



$\implies$ All vectors of the same magnitude and the same direction are equivalent:



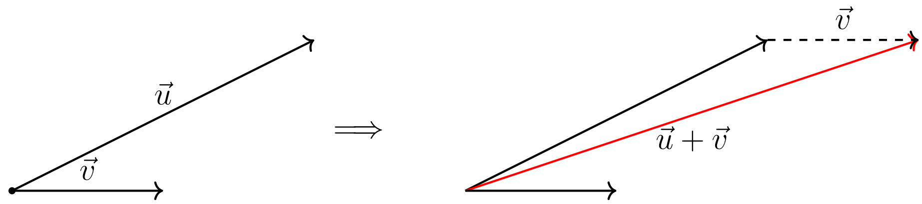
The two vectors $\vec{X}$ and $\vec{Y}$ are equivalent,

$$\vec{X} = \vec{Y}$$

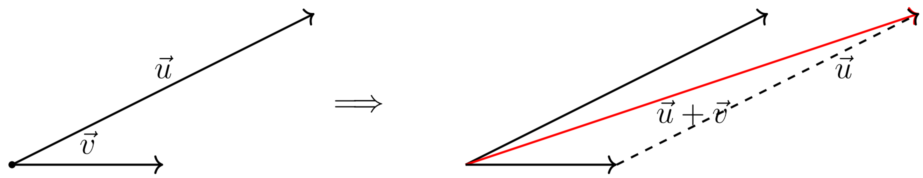
$\implies$ Note: initial points of equivalent vectors don't have to be the same!

Algebraic operations on vectors:

- addition:



Alternatively:



$\Longrightarrow$ Properties of vector addition:

- commutative:

$$\vec{u}+\vec{v}=\vec{v}+\vec{u}$$

- associative:

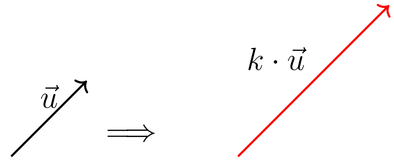
$$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$$

$\Longrightarrow$ If $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$ then

$$\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2)$$

(properties of vector operations follow from algebraic properties of operations on vector components)

- multiplication by a number (a scalar) $k\in\mathbb{R}$:



Magnitude is changed by a factor $|k|$; direction is the same if $k>0$, opposite if $k<0$. In components:

$$k\cdot\vec{u}=(k\cdot u_1,k\cdot u_2)$$

$\Longrightarrow$ Note:

$$0 \cdot \vec{X} = \vec{0}\,, \qquad \vec{X} + \vec{0} = \vec{X}$$

for any $\vec{X}$, where $\vec{0} = (0, 0)$ (in components).

$\Longrightarrow$ Above can be generalized to vectors in n-dim:

$$\mathbb{R}^n : \qquad \vec{X} \in \mathbb{R}^n\,, \qquad \vec{X} = (x_1, \cdots x_n)\,, \qquad \text{or} \qquad \vec{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$\Longrightarrow$ same operations, same properties

Consider a set (a collection) of vectors

$$\vec{X}_1\,,\ \vec{X}_2\,,\ \cdots\,,\ \vec{X}_m\ \in \mathbb{R}^n$$

$$\vec{X}_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix}, \qquad \cdots \qquad , \qquad \vec{X}_m = \begin{bmatrix} x_{m1} \\ x_{12} \\ \vdots \\ x_{mn} \end{bmatrix}$$

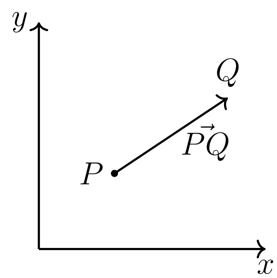
and a set of m numbers $c_1, c_2, \cdots c_m \in \mathbb{R}$.

Then we can form a *linear* combination of vectors in the set as

$$c_1 \cdot \vec{X}_1 + c_2 \cdot \vec{X}_2 + \cdots + c_m \cdot \vec{X}_m \ = \ \underbrace{\sum_{i=1}^{m}}_{\text{summation symbol}} \ c_i \cdot \vec{X}_i$$

$\Longrightarrow$ Last thing: (in 2d)

Let $P = (x_1, y_1)$ be the initial point of the vector; $Q = (x_2, y_2)$ be the terminal point of the vector



then

$$\vec{PQ} \overset{def}{\equiv} (x_2 - x_1, y_2 - y_1)$$

Section 3.2 — norm, dot product, distance

$\Longrightarrow$ *Distance* is a measure of separation between points in space

"Distance" depends on what space is it?

Examples:

- in $\mathbb{R}^3$, the distance between the North Pole and the South Pole of a sphere of radius R is

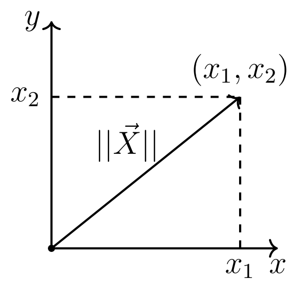
$$\text{distance }_{\mathbb{R}^3}[N - S] = 2R$$

- on S^2 (a 2-dim sphere, a surface of a ball),

$$\text{distance }_{S^2}[N - S] = \pi R \neq \text{distance }_{\mathbb{R}^3}[N - S]$$

A *norm* of a vector is its magnitude; it is a distance between the initial and the terminal point of a vector:

$$\vec{X} = (x_1, x_2) \in \mathbb{R}^2$$



$$\underbrace{||\vec{X}||}_{\text{notation for the norm}} \stackrel{def}{\equiv} \sqrt{x_1^2 + x_2^2}$$

$\Longrightarrow$ Consider two points

$$P = (x_1, y_1) \qquad \text{and} \qquad Q = (x_2, y_2)$$

then

$$\text{distance}(P, Q) \stackrel{def}{\equiv} ||\vec{PQ}|| \quad = \quad ||(x_2 - x_1, y_2 - y_1)|| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$\Longrightarrow$ Properties:

- If
$$||\vec{v}|| = 0 \qquad \Longrightarrow \qquad \vec{v} = \vec{0}$$

-
$$||k \cdot \vec{v}|| = |k| \cdot ||\vec{v}||$$

- in $\mathbb{R}^n$: for any $\vec{X} = (x_1, \cdots, x_n)$,
$$||\vec{X}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Recall a standard basis in $\mathbb{R}^n$:

$$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \qquad , \ \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Any vector $\vec{X} \in \mathbb{R}^n$,

$$\vec{X} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$$

has a unique decomposition in basis vectors:

$$\vec{X} = x_1 \cdot \hat{e}_1 + x_2 \cdot \hat{e}_2 + \cdots + x_n \cdot \hat{e}_n$$

Example: in $\mathbb{R}^3$ basis vectors typically denoted $\{\hat{i}, \hat{j}, \hat{h}\}$. Then,

$$\vec{u} = (1, -3, 4) \ = \ 1 \cdot \hat{i} + (-3) \cdot \hat{j} + 4 \cdot \hat{h}$$

$\vec{u}$ is a unit vector if

$$||\vec{u}|| = 1$$

Example of unit vectors in $\mathbb{R}^3$:

$$||\hat{i}|| = ||\hat{j}|| = ||\hat{h}|| = 1$$

$\Longrightarrow$ Note: any vector can be *normalized* (to make its norm=1)

Indeed, let

$$\vec{u} = u_1 \cdot \hat{i} + u_2 \cdot \hat{j} + u_3 \cdot \hat{h}$$

then

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Lets denote the corresponding normalized vector as

$$\hat{u} = k \cdot \vec{u}\,, \qquad k \in \mathbb{R}_+$$

This vector has the same direction as $\vec{u}$ since $k > 0$. Then

$$1 = ||\hat{u}|| = k\ ||\vec{u}|| \qquad \Longrightarrow \qquad k = \frac{1}{||\vec{u}||}$$

Thus:

$$\hat{u} = \frac{1}{||\vec{u}||} \cdot \vec{u}$$