Lecture 25 (sections 4.5,4.7)

Section 4.5 — continuation

$\Longrightarrow$ Problem. Consider the subspaces of $\mathbb{R}^4$:

- all vectors of the form: $W : \{(a, b, c, 0)\}$

$$\text{basis :} \qquad S = \{(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0)\}$$

$$\dim(W) = 3$$

- all vectors of the form (a, b, c, d) where $d = a + b\,,\ c = a - b$

$$W \subset \mathbb{R}^4\,, \qquad W:\ \{a, b, a-b, a+b\}\,, \qquad a, b \in \mathbb{R}$$

$$\text{basis :} \qquad S = \{(1,0,1,1),\ (0,1,-1,1)\}$$

$$\dim(W) = 2$$

- all vectors of the form (a, b, c, d) where $a = b = c = d$

$$W \subset \mathbb{R}^4\,, \qquad W:\ \{a, a, a, a\}\,, \qquad a \in \mathbb{R}$$

$$\text{basis :} \qquad S = \{(1,1,1,1)\}$$

$$\dim(W) = 1$$

Consider $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

■ Definition 1

The row space of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A. Note that the rows of A are vectors in $\mathbb{R}^n$:

$$\begin{aligned} \vec{r}_1 &= (a_{11}, a_{12}, \cdots, a_{1n}) \\ \vec{r}_2 &= (a_{21}, a_{22}, \cdots, a_{2n}) \\ &\cdots \\ \vec{r}_m &= (a_{m1}, a_{m2}, \cdots, a_{mn}) \end{aligned}$$

■ Definition 2

The column space of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. Note that the columns of A are vectors in $\mathbb{R}^m$:

$$\vec{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \qquad \vec{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \qquad \cdots, \qquad \vec{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$\Longrightarrow$ Example. Consider 2×3 matrix:

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 7 \end{bmatrix}$$

- rows of A:

$$\begin{array}{l} \vec{r}_1 = (1,4,3) \\ \vec{r}_2 = (0,2,7) \end{array} \qquad \in \mathbb{R}^3$$

Row space of A is

$$\operatorname{span}\{(1,4,3)\,,\,(0,2,7)\}$$

$\vec{r}_1$ and $\vec{r}_2$ are linearly independent (check)

$$\dim(\text{row space}) = 2$$

- columns of A:

$$\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{c}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \vec{c}_3 = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \qquad \in \mathbb{R}^2$$

Column space of A is

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 4 \\ 2 \end{bmatrix} , \cancel{\begin{bmatrix} 3 \\ 7 \end{bmatrix}} \right\}$$

$\vec{c}_1$ and $\vec{c}_2$ are linearly independent (check), however,

$$\vec{c}_3 = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{7}{2} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} - 11 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dim(\text{column space}) = 2 \qquad (\#\text{ of linearly independent vectors})$$

Definition:

The **null space** of A is the space of the solutions of the homogeneous system $$A \cdot \vec{X} = \vec{0}$$ It is a subspace of $\mathbb{R}^n$

Theorem 1

The null space of a matrix is not changed by elementary row operations.

Theorem 2

The row space of a matrix is not changed by elementary row operations.

$\Longrightarrow$ Not true for the column space!

- Example: consider the following matrix in REF,

$$\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - the rows that contain the leading 1's form the basis for the row space

$$S_{row} = \text{span}\{(1,3,0,7)\,,\ (0,1,4,0)\,,\ (0,0,0,1)\}$$

  - the columns that contain the leading 1's will be the basis for the column space

$$S_{column} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\Longrightarrow$ Note that $\vec{c}_3$ is not linearly independent:

$$\vec{c}_3 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = 4 \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} - 12 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$