Lecture 20 (section 4.2)

Section 4.2 — continuation

Let V be a vector space.

$$S = \{w_1, w_2, \cdots, w_n\}$$

a set of vectors in V.

$$W \overset{def}{\equiv} \text{ set of all possible linear combinations of elements in } S$$

$$\{c_1 \cdot w_1 + c_2 \cdot w_2 + \cdots + c_n \cdot w_n\}, \qquad c_i \in \mathbb{R}$$

$\Longrightarrow$ Clearly, W is a subspace of V:

$$W \overset{def}{\equiv} \text{span}(S)$$

S is called a spanning set for W

Examples:

- 1

$$\mathcal{P}_n = \{\text{polynomials of degree } \leq n\}$$

$$S_n \equiv \{1, x, x^2, \cdots, x^n\}$$

$\mathcal{P}_n$ is a subspace of $\mathcal{F}$; S_n is a spanning set for $\mathcal{P}_n$:

$$\mathcal{P}_n = \text{span}(S_n)$$

■ 2 Let

$$\vec{u} = (1, 2, -1)\,, \qquad \vec{v} = (6, 4, 2)\,, \qquad \vec{w} = (9, 2, 7) \ \in \mathbb{R}^3$$

Is $\vec{w}$ a linear combination of $\vec{u}$ and $\vec{v}$?

$\implies$ If true, $\exists \ \{c_1, c_2\} \in \mathbb{R}$ such that

$$\vec{w} = c_1 \cdot \vec{u} + c_2 \cdot \vec{v}$$

$$\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \quad \implies \quad \begin{cases} c_1 + 6c_2 = 9 \\ 2c_1 + 4c_2 = 2 \\ -c_1 + 2c_2 = 7 \end{cases}$$

$\implies$ Augmented matrix

$$\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \implies \begin{matrix} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 + r_1 \end{matrix} \implies \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{bmatrix} \implies r_3 \to r_3 + r_2$$

$$\implies \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \implies r_2 \to -\tfrac{1}{8} \cdot r_2 \implies \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Longleftarrow \text{REF}$$

$$\implies r_1 \to r_1 - 6 \cdot r_2 \implies \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Longleftarrow \text{RREF}$$

$\implies$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Yes, $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$

$$\vec{w} = -3 \cdot \vec{u} + 2 \cdot \vec{v}$$

■ 3 Do

$$S = \{(1,1,2),\ (1,0,1),\ (2,1,3)\}$$

span $\mathbb{R}^3$?

In other words, is it true that

$$\text{span}(S) = \mathbb{R}^3$$

If true, then for **any** vector

$$\vec{b} = (b_1, b_2, b_3)\ \in \mathbb{R}^3$$

there must be coefficients (c_1, c_2, c_3), such that

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Solution of

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

must exist for any $\vec{b}$.

$\Longrightarrow$ Augmented matrix

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \Longrightarrow \begin{matrix} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \Longrightarrow r_3 \to r_3 - r_2$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} \Longrightarrow r_2 \to -r_2 \Longrightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} \Longleftarrow \text{REF}$$

Above system is consistent only if

$$b_3 - b_1 - b_2 = 0$$

$\Longrightarrow$

$$\mathrm{span}(S) \neq \mathbb{R}^3$$

Explicit counterexample:

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$$

can not be represented as a linear combination of vectors in S.