Lecture 18 (section 4.1)

Section 4.1 — real vector spaces

$\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ as vector spaces

- addition
- multiplication by a scalar (with certain properties)

Consider a collection of objects (a set) V on which two operations are defined:

- 1 addition —
  If $u, w \in V$, then there is a rule that defines their sum

$$u + w \in V$$

which has the properties

(i) u+w=w+u

(ii) u+(v+w)=(u+v)+w

(iii) there is a zero element $0 \in V$ such that

$$u + 0 = u$$

(iv) for each $u \in V$ there is an element $(-u)$ such that

$$u + (-u) = 0$$

- (2) multiplication by a scalar (a real number) —

$$v \in V\,, \quad \lambda \in \mathbb{R} \qquad\qquad \Longrightarrow \qquad\qquad \lambda \cdot v \in V$$

properties

(i)

$$(\lambda_1 + \lambda_2) \cdot v = \lambda_1 \cdot v + \lambda_2 \cdot v$$

(ii)

$$(\lambda_1 \cdot \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v)$$

(iii)

$$1 \cdot v = v$$

(iv)

$$\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v\,, \qquad u, v \in V$$

If all above is satisfied, V is called a **real vector space**

Examples:

- (a) The real vectors spaces we know:

$$\mathbb{R}^2\,, \qquad \mathbb{R}^3\,, \qquad \mathbb{R}^n$$

- (b) Zero vector space: it has a single element, the zero vector — $\vec{0}$

$$\vec{0} + \vec{0} = \vec{0}$$

$$k \cdot \vec{0} = \vec{0}\,, \qquad 1 \cdot \vec{0} = \vec{0}$$

Note: all axioms of vector space are satisfied

- (c) space of matrices:

$$V: \qquad \{M_{m\times n}\} = \{\text{set of all } m \times n \text{ real matricies}\}$$

addition: if A, B are $m \times n$ matrices, $A + B$ is $m \times n$ matrix;
$0_{m\times n}$ is zero element, $A + 0_{m\times n} = A$; $\lambda \cdot A$ is defined

- d space of functions of a single variable

$$f \text{ is a function } f(x)$$
$$g \text{ is a function } g(x)$$

All axioms and properties can be verified:

$$f + g \overset{def}{\equiv} f(x) + g(x) \qquad \Longrightarrow \qquad \text{defined}$$

$$0 = \text{zero function}$$

$$1 \cdot f \overset{def}{\equiv} 1 \cdot f(x)$$