Lecture 22 (section 4.3)

Section 4.3 — continuation

$\Longrightarrow$ Any set of vectors that contains the zero vector is **linearly dependent**

Examples:

- 1 $V = \mathbb{R}^4$.

$$\vec{v_1} = (1, 2, 0, -2)\,, \qquad \vec{v_2} = (2, 0, -1, 3)\,, \qquad \vec{v_3} = (3, 2, -1, 1)$$

Are these vectors linearly independent?

$$c_1 \cdot \vec{v_1} + c_2 \cdot \vec{v_2} + c_3 \cdot \vec{v_3} = \vec{0}$$

$$c_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ Augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ -2 & 3 & 1 & 0 \end{bmatrix} \Longrightarrow \begin{matrix} r_2 \to r_2 + r_4 \\ r_4 \to r_4 + 2r_1 \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{matrix} r_2 \to \frac{1}{3} \cdot r_2 \\ r_3 \to r_3 + r_2 \\ r_4 \to r_4 - 7r_2 \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_1 \to r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longleftarrow \text{RREF}$$

$$\{c_1, c_2\} - \text{leading}\,, \qquad \{c_3\} - \text{free}$$

$\Longrightarrow$ Set $c_3 = t$,

$$(c_1, c_2, c_3) = (-t, -t, t)$$

$\Longrightarrow$ non-unique solution $\Longrightarrow$ set is $\boxed{\text{linearly dependent}}$

- ② $V = \mathcal{F}$

$$\mathcal{F} = \{\text{real functions defined on } (-\infty, \infty)\}$$

$$\mathcal{P}_n = \{\text{polynomials of degree } \leq n\}$$

Recall that $\mathcal{P}_n \subset \mathcal{F}$

Linearly independent set of polynomials:

$$\{1, x, x^2, \cdots, x^n\}$$

- ③ Note: $V = \mathbb{R}^n$ and

$$S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}, \qquad \text{and} \qquad r > n$$

$\Longrightarrow \qquad S$ is linearly dependent

- ④ When are two vectors linearly dependent?
  - — if one of the vectors is zero
  - — if both are nonzero, but:

$$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 = 0\,, \qquad\qquad c_1^2 + c_2^2 \underbrace{\neq}_{\text{at least one } c \text{ is nonzero}} 0$$

Let $c_2 \neq 0 \Longrightarrow$

$$\vec{v}_2 = -\frac{c_1}{c_2} \cdot \vec{v}_1$$

$\Longrightarrow \qquad \vec{v}_2$ is a multiple of $\vec{v}_1 \Longrightarrow$

$$\vec{v}_1 \underbrace{||}_{\text{parallel}} \vec{v}_2$$