Lecture 12 (section 2.1)

Section 2.1 — determinants and cofactor expansion

$\Longrightarrow$ Recall,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a 2×2 matrix is invertible if

$$\det(A) = a\ d - b\ c \neq 0$$

and

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\Longrightarrow$ We want to generalize the concept of $\det(A)$ for any $n \times n$ matrix; in particular, we would like to have a simple singularity test:

$$\det(A) \neq 0 \qquad \Longleftrightarrow \qquad A \text{ is invertibel}$$

Definitions:

Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad \text{be} \qquad n \times n \qquad \text{matrix}$$

M_{ij} is the minor of entry a_{ij}, defined as a determinant of $(n-1) \times (n-1)$ matrix obtained by removing row i and column j from A

$$\underbrace{C_{ij}}_{\text{co-factor corrresponding to entry } a_{ij}} \qquad = \qquad (-1)^{i+j} M_{ij}$$

Theorems:

$$\underbrace{\det(A)}_{\text{independent on what row } i \text{ is used}} = a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \cdots + a_{in} \cdot C_{in}$$

$$\underbrace{\det(A)}_{\text{independent on what column } j \text{ is used}} = a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \cdots + a_{nj} \cdot C_{nj}$$

- 1×1 matrix:

$$A = \begin{bmatrix} a \end{bmatrix} \qquad \Longrightarrow \qquad \det(A) \overset{def}{=} a$$

- 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we know the answer:

$$\det(A) = a\ d - b\ c$$

But let's compute using different cofactor expansions:

- cofactor expansion corresponding to first row:

$$M_{11} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} d \end{bmatrix} = d$$

$$M_{12} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} c \end{bmatrix} = c$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = d\,, \qquad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -c$$

$$\det(A) = a_{11}\ C_{11} + a_{12}\ C_{12} = a\ d + b\ (-c) = \boxed{a\ d - b\ c}$$

- cofactor expansion corresponding to second column:

$$M_{12} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = c\,, \qquad M_{22} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a$$

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} = a_{12}\ (-1)^{1+2} M_{12} + a_{22}\ (-1)^{2+2} M_{22} = b\ (-1)c + d\ a = \boxed{a\ d - b\ c}$$

- 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 3 & 1 & -5 \end{bmatrix}$$

$\Longrightarrow$ chose cofactor expansion judiciously!
(most zeros in the expansion row or column)

- use column 1:

$$\det(A) = 1 \cdot C_{11} + 0 \cdot C_{21} + 3 \cdot C_{31} = M_{11} + 3M_{31}$$

$$M_{11} = \det \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 3 & 1 & -5 \end{bmatrix} = \det \begin{bmatrix} 2 & 4 \\ 1 & -5 \end{bmatrix} = 2 \cdot (-5) - 4 \cdot 1 = -14$$

$$M_{31} = \det \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 3 & 1 & -5 \end{bmatrix} = \det \begin{bmatrix} 2 & -3 \\ 2 & 4 \end{bmatrix} = 2 \cdot 4 - (-3) \cdot 2 = 14$$

$$\det(A) = (-14) + 3 \cdot (14) = 28$$

- use row 2:

$$\det(A) = 0 \cdot C_{21} + 2 \cdot C_{22} + 4 \cdot C_{23} = 2M_{22} - 4M_{23}$$

$$M_{22} = \det \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 3 & 1 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} = 1 \cdot (-5) - (-3) \cdot 3 = 4$$

$$M_{23} = \det \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 3 & 1 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = 1 \cdot 1 - 2 \cdot 3 = -5$$

$$\det(A) = 2 \cdot (4) - 4 \cdot (-5) = 28$$

- 4×4 matrix:

$$A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$$

$\implies$ best to use column 3:

$$\det(A) = 0 \cdot C_{13} + 0 \cdot C_{23} + (-3) \cdot C_{33} + 3 \cdot C_{43} = -3 \cdot M_{33} - 3 \cdot M_{43}$$

- $$M_{33} = \det \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix} = \det \begin{bmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{bmatrix} \underbrace{=}_{\text{use row 2}}$$

$$= 2 \cdot C_{21} + 2 \cdot C_{22} + (-2) \cdot C_{23} = -2 \cdot M_{21} + 2 \cdot M_{22} + 2 \cdot M_{23}$$

$$= -2 \ \det \begin{bmatrix} 3 & 5 \\ 10 & 2 \end{bmatrix} + 2 \ \det \begin{bmatrix} 3 & 5 \\ 2 & 2 \end{bmatrix} + 2 \ \det \begin{bmatrix} 3 & 3 \\ 2 & 10 \end{bmatrix}$$

$$= -2(6-50) + 2(6-10) + 2(30-6) = 128$$

- $$M_{43} = \det \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix} = \det \begin{bmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{bmatrix} \underbrace{=}_{\text{use row 3}} = 4 \cdot C_{31} + 1 \cdot C_{32}$$

$$= 4 \cdot M_{31} - M_{32} = 4 \ \det \begin{bmatrix} 3 & 5 \\ 2 & -2 \end{bmatrix} - \det \begin{bmatrix} 3 & 5 \\ 2 & -2 \end{bmatrix} = 3 \cdot (-6-10) = -48$$

$$\det(A) = -3 \cdot (128) - 3 \cdot (-48) = -240$$