Lecture 15 (section 1.8, review)

Linear transformation — continued

$$\mathbb{R}^n \quad \xrightarrow{T} \quad \mathbb{R}^m$$

T is a linear transformation if

- $$T(k\vec{X}) = k\ T(\vec{X})$$
- $$T(\vec{X} + \vec{Y}) = T(\vec{X}) + T(\vec{Y})$$

Theorem:

Any linear transformation is a matrix transformation.

$\Longrightarrow$ In other words, if T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then there exists a matrix A, of size $m \times n$, such that

$$T(\vec{X}) = A \cdot \vec{X}$$

$\Longrightarrow$ How can we find A, given T?

- Let $\hat{e}_1 \cdots, \hat{e}_n$ be the standard basis of $\mathbb{R}^n$, then

$$A = \begin{bmatrix} T(\hat{e}_1) & T(\hat{e}_2) & \cdots & T(\hat{e}_n) \end{bmatrix}$$

where $T(\hat{e}_i)$ is m-component vector representing coordinates of $T(\hat{e}_i)$ in a standard basis in $\mathbb{R}^m$

$\Longrightarrow$ Problems:

- 1 Find A corresponding to a transformation:

$$\mathbb{R}^3 \xrightarrow{T} \mathbb{R}^4 : \qquad \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{n=3} \xrightarrow{T} \underbrace{\begin{bmatrix} x_1 - 2x_3 \\ -40x_1 + x_2 \\ x_1 - \pi\ x_2 + 10x_3 \\ 4x_2 + 100x_3 \end{bmatrix}}_{m=4}$$

Evaluate $T(\hat{e}_i)$:

$$T(\hat{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -40 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\hat{e}_2) = \begin{bmatrix} 0 \\ 1 \\ -\pi \\ 4 \end{bmatrix}, \qquad T(\hat{e}_3) = \begin{bmatrix} -2 \\ 0 \\ 10 \\ 100 \end{bmatrix}$$

$\Longrightarrow$ $m \times n = 4 \times 3$ matrix A realizing the transformation is

$$A = \begin{bmatrix} T(\hat{e}_1) & T(\hat{e}_2) & T(\hat{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ -40 & 1 & 0 \\ 1 & -\pi & 10 \\ 0 & 4 & 100 \end{bmatrix}$$

- (2) A given by

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 0 \\ 1 & 2 & -4 \end{bmatrix}$$

Is A invertible?

$\Longrightarrow$ Note: we are not asked to find the inverse of A! Recall:

$$\det(A) = \begin{cases} 0\,, \text{ singular, } i.e., \text{ not invertible} \\ \neq 0\,, \text{ non} - \text{singular, } i.e., \text{ invertible} \end{cases}$$

$$\det(A) \underset{r_3 \to r_3 - r_1}{=} \det \begin{bmatrix} 1 & 2 & -4 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \qquad \Longrightarrow \qquad \text{not invertible}$$

(you can imagine doing co-factor expansion using the 3rd row).

$\Longrightarrow$ Moral: if a matrix (a square one) has identical rows (or columns — can u prove this?) then this matrix is not invertible.

- (3) A given by

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 3 \\ 6 & 2 & -4 \end{bmatrix}$$

Is A invertible?

$\Longrightarrow$ No: $r_3 = 2r_1 \Longrightarrow \quad \det(A) = 0$

- ④ Compute the determinant of

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$

$$\det(A) = \begin{matrix} r_1 \leftrightarrow r_5 \\ r_2 \leftrightarrow r_4 \end{matrix} = (-1)^{1+1} \det \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

$$= (-1)^2 \cdot \underbrace{5 \cdot 2 \cdot (-1) \cdot (-4) \cdot (-3)}_{\text{diagonal matrix}} = -120$$

- ⑤ Given that A is 3×3 matrix such that

$$\det(A) = -7$$

compute $\det(3 \cdot A)$

$\Longrightarrow$ Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \Longrightarrow \qquad 3 \cdot A = \begin{bmatrix} 3a & 3b & 3c \\ 3d & 3e & 3f \\ 3g & 3h & 3i \end{bmatrix}$$

$$\det(3 \cdot A) = \begin{matrix} r_1 \to 3 \cdot r_1 \\ r_2 \to 3 \cdot r_2 \\ r_3 \to 3 \cdot r_3 \end{matrix} = 3^{1+1+1} \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 27 \cdot (-7) = -189$$