Lecture 3 (sections 1.1, 1.2)

Consider a system of linear equations of 3 equations with 3 unknows:

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + z = 5 \\ x + 3y + 0z = 1 \end{cases}$$

$\Longrightarrow$ Given a system of m equations with n unknows

$$\begin{cases} a_{11}\ x_1 + a_{12}\ x_2 + \cdots + a_{1n}\ x_n = b_1 \\ \dots \\ a_{m1}\ x_1 + a_{m2}\ x_2 + \cdots + a_{mn}\ x_n = b_m \end{cases}$$

we can assemble coefficients into a rectangular array of dimensions $m \times (n+1)$:

$$m - \text{rows} \left\{ \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}}_{(n+1)\text{-columns}} \right.$$

This array is called **augmented matrix** for the system.

$\Longrightarrow$ In our case:

$$m = 3 - \text{rows} \left\{ \quad \underbrace{\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 1 & 5 \\ 1 & 3 & 0 & 1 \end{bmatrix}}_{(n+1) = (3+1) = 4\text{-columns}} \right.$$

$\Longrightarrow$ In practice we care about <u>solutions</u>; different-looking systems are equivalent once they produce equivalent solutions:

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + z = 5 \\ x + 3y + 0z = 1 \end{cases} \underset{1-3\ switch}{\Longleftrightarrow} \begin{cases} x + 3y + 0z = 1 \\ 3x - y + z = 5 \\ x + 2y - 3z = 4 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 1 & 5 \\ 1 & 3 & 0 & 1 \end{bmatrix} \underset{1-3\ interchange}{\Longleftrightarrow} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 3 & -1 & 1 & 5 \\ 1 & 2 & -3 & 4 \end{bmatrix}$$

switch is an operation that produces an equivalent system of equations (*interchanging* two equations)

$$\boxed{\text{equivalent algebraic operations on a system} \longleftrightarrow \text{elementary operations on augmented matrix}}$$

$\Longrightarrow$ Why would we do it?

— to simplify system so that solution becomes obvious

$\Longrightarrow$ What are <u>all</u> the elementary row operations on the augmented matrix?

- (1) switch (interchange) of two rows
- (2) multiply a row by a nonzero constant; this corresponds to a multiplication of a corresponding equation by a nonzero constant

$$3x - y + z = 5 \qquad \Longrightarrow \qquad (-3)\,[3x - y + z] = (-3)5$$

$$-9x + 3y - 3z = -15$$

$$\begin{bmatrix} \cdots\cdots\cdots\cdots \\ 3 \quad -1 \quad 1 \quad 5 \\ \cdots\cdots\cdots\cdots \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} \cdots\cdots\cdots\cdots\cdots\cdots \\ (-3)\cdot[3 \quad -1 \quad 1 \quad 5] \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots\cdots\cdots\cdots\cdots \\ -9 \quad 3 \quad -3 \quad -15 \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix}$$

- (3) add to a row a constant multiple of another row:

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + z = 5 \end{cases} \Leftrightarrow \underset{\text{does not change the system of equations}}{(1) + 3\cdot(2)} \Leftrightarrow \begin{cases} 10x + -y + 0z = 19 \\ 3x - y + z = 5 \end{cases}$$

$\Longrightarrow$ Ultimate goal:

Use elementary row operations to reduce augmented matrix to **R**ow **E**chelon **F**orm (**REF**)

$\Longrightarrow$ Matrix is REF if:

- (1) If a row does not consist of zero's then the first nonzero number in the row is 1 (called **leading** 1)
- (2) If there are rows consisting entirely from 0's they are grouped at the bottom of the matrix
- (3) In two successive rows, not entirely zero, the leading 1 in the lower row occurs to the **right** of the leading 1 in the upper row

Example of REF matrix:

$$\begin{array}{ll} r_1: & \\ r_2: & \\ r_3: & \\ r_4: & \end{array} \begin{bmatrix} 0 & 1 & 2 & 6 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- leading 1's are circled (rows: $r_1 \cdots r_3$);
- row consisting entirely of 0's is at the bottom (row r_4);
- leading 1 in row r_2 is to the right of a leading 1 in row r_1; leading 1 in row r_3 is to the right of a leading 1 in row r_2;

$\Longrightarrow$ If (1)-(3) are true <u>and</u> each column that contains a leading 1 has zero everywhere else in that column, such a matrix is called **RREF** (**R**educed **R**ow **E**chelon **F**orm)

$$\begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array}$$
$$\begin{bmatrix} 0 & 1 & 2 & 6 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ REF, but not RREF: columns c_3 and c_5 violate the condition to be RREF

$\Longrightarrow$ It is straightforward to bring REF into RREF:

- starting point:

$$\begin{array}{c}r_1 : \\ r_2 : \\ r_3 : \\ r_4 :\end{array} \begin{bmatrix} 0 & 1 & 2 & 6 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $r_1 \to r_1 + (-2) \cdot r_2$

$$\begin{array}{c}r_1 : \\ r_2 : \\ r_3 : \\ r_4 :\end{array} \begin{bmatrix} 0 & 1 & 0 & 8 & -4 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $r_2 \to r_2 + (-2) \cdot r_3$ & $r_1 \to r_1 + 4 \cdot r_3$

$$\begin{array}{c}r_1 : \\ r_2 : \\ r_3 : \\ r_4 :\end{array} \begin{bmatrix} 0 & 1 & 0 & 8 & 0 & 5 \\ 0 & 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ RREF!

$\boxed{\textbf{Any matrix can be reduced} \quad \to \quad REF \quad \to \quad RREF}$

$\Longrightarrow$ Recall:

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + z = 5 \\ x + 3y + 0z = 1 \end{cases} \iff \begin{array}{cc} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 1 & 5 \\ 1 & 3 & 0 & 1 \end{bmatrix} & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

Sequence of elementary row operations:

- $r_2 \to r_2 + (-3) \cdot r_1$ & $r_3 \to r_3 + (-1) \cdot r_1$

$$\longrightarrow \begin{array}{cc} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 10 & -7 \\ 0 & 1 & 3 & -3 \end{bmatrix} & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

- $r_2 \leftrightarrow r_3$

$$\longrightarrow \begin{array}{cc} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 3 & -3 \\ 0 & -7 & 10 & -7 \end{bmatrix} & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

- $r_3 \leftrightarrow r_3 + 7 \cdot r_2$

$$\longrightarrow \begin{array}{cc} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 31 & -28 \end{bmatrix} & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

- $r_3 \leftrightarrow \frac{1}{31} \cdot r_3$

$$\longrightarrow \begin{array}{cc} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -\frac{28}{31} \end{bmatrix} & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

we are in REF (but not in RREF)

$\Longrightarrow$ corresponding system (which is guaranteed to have the **same** solution as the original one):

$$\begin{cases} x + 2y - 3z & = 4 \\ \quad y + 3z & = -3 \\ \qquad z & = -\frac{28}{31} \end{cases} \Longrightarrow \begin{aligned} x = 4 - 2y + 3z = & \ \frac{58}{31} \\ y = -3 - 3z = & -\frac{9}{31} \\ z = & -\frac{28}{31} \end{aligned}$$

as promised, have a trivial solution (when we go $\boxed{\textit{bottom} \ \to \ \textit{top}}$ equations)

$\Longrightarrow$ Solution is even simpler if augmented matrix is in RREF:

- start:

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -\frac{28}{31} \end{array}\right] & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

- $r_1 \to r_1 + (-2) \cdot r_2$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 0 & -9 & 10 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -\frac{28}{31} \end{array}\right] & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

- $r_1 \to r_1 + 9 \cdot r_3$ & $r_2 \to r_2 + (-3) \cdot r_3$

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{58}{31} \\ 0 & 1 & 0 & -\frac{9}{31} \\ 0 & 0 & 1 & -\frac{28}{31} \end{array}\right] & \begin{array}{l} : r_1 \\ : r_2 \\ : r_3 \end{array} \end{array}$$

we are in RREF $\Longrightarrow$

$$\begin{cases} x = \frac{58}{31} \\ y = -\frac{9}{31} \\ z = -\frac{28}{31} \end{cases}$$

there is nothing to solve!

More definitions:

- Every column in first n columns correspond to a variable:

$$\begin{cases} x+2y-3z=4 \\ 3x-y+z=5 \\ x+3y+0z=1 \end{cases} \iff \begin{array}{c} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 1 & 5 \\ 1 & 3 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \end{array}$$

$$\begin{array}{ll} c_1: & x - \text{column} \\ c_2: & y - \text{column} \\ c_3: & z - \text{column} \end{array}$$

$\Longrightarrow$ Variable in REF augmented matrix corresponding to columns with leading 1's are called **leading variables**

$\Longrightarrow$ Variable in REF augmented matrix corresponding to columns without leading 1's are called **free variables**

In our example:

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -\frac{28}{31} \end{bmatrix}$$

all variables are leading, there are no free variables

▪ If an augmented matrix in REF has a non-zero row with first n-elements being zero, the corresponding system is inconsistent

▪ If a consistent system has free variables, it has infinitely many solutions

Example I:

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$\Longrightarrow$

- 3×4 ($n = 3$) augmented matrix in RREF
- free variable: z (or x_3)
- corresponding system of linear equations is inconsistent (since the last row has no leading 1 in first $n = 3$ elements):

$$\begin{aligned} r_1 : \quad & x = 2 \\ r_2 : \quad & y = 7 \\ r_3 : \quad & 0 \cdot x + 0 \cdot y + 0 \cdot z = 1 \end{aligned}$$

Example II:

$$
\begin{array}{cccc}
x & y & z & \\
\end{array}
$$

$$
\left[\begin{array}{cccc}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 7 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

$\Longrightarrow$

- 3×4 $(n = 3)$ augmented matrix in RREF
- free variable: z
- corresponding system of linear equations has infinitely many solutions:

$$(x, y, z) = (2, 7, t)$$

where t is a parameter characterizing a solution (any value of t is allowed)

Example III:

$$\begin{array}{c c} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & \end{matrix} \\ \begin{matrix} r_1: \\ r_2: \\ r_3: \\ r_4: \end{matrix} & \begin{bmatrix} 1 & -6 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\Longrightarrow$

- 4×6 ($n = 5$) augmented matrix in RREF
-

$$\text{free variables}: \quad x_2\,,\, x_5$$

$$\text{leading variables}: \quad x_1\,,\, x_3\,,\, x_4$$

- corresponding system of linear equations is consistent
- general solution is represented expressing leading variables in terms of free variables. Steps:
  - assign different parameters to all free variables:

$$x_2 = t\,, \qquad x_5 = s$$

  - Use RREF of the augmented matrix to solve for leading variables (remember: solve bottom→top rows):

$$r_3: \qquad x_4 + 5x_5 = 8 \qquad \Longrightarrow \qquad x_4 = 8 - 5x_5 = 8 - 5s$$

$$r_2: \qquad x_3 + 4x_5 = 7 \qquad \Longrightarrow \qquad x_3 = 7 - 4x_5 = 7 - 4s$$

$$r_1: \qquad x_1 - 6x_2 + 3x_5 = 2 \qquad \Longrightarrow \qquad x_1 = 2 + 6x_2 - 3x_5 = 2 + 6t - 3s$$

  - general solution:

$$(x_1, x_2, x_3, x_4, x_5) = (2 + 6t - 3s, t, 7 - 4s, 8 - 5s, s)$$