Lecture 23 (sections 4.3,4.4)

Section 4.3 — continuation

$\Longrightarrow$ How does linear (in)dependence look for functions?

Consider

$$f_1(x)\,,\ f_2(x)\,,\cdots,\ f_n(x) \qquad \text{for} \qquad x\in(-\infty,\infty)$$

— a set of functions that are differentiable to order $(n-1)$:

$$f_i(x)\ \subset\ \mathcal{C}^{(n-1)}(-\infty,\infty)$$

We can construct a **Wronskian**, $W(x)$,

$$\underbrace{W(x)}_{\text{determinant of } n\times n \text{ matrix}} \quad \stackrel{def}{\equiv}\ \det\begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}$$

$\Longrightarrow$ Theorem:

If $W(x)$ is non-zero everywhere,

$$\{f_1(x)\,,\ f_2(x)\,,\cdots,\ f_n(x)\}$$

are linearly independent.

If $W(x)=0$ for all x, then

$$\{f_1(x)\,,\ f_2(x)\,,\cdots,\ f_n(x)\}$$

are linearly dependent.

■ Example:

$$f_1(x) = \sin(x)\,, \qquad f_2(x) = x\,, \qquad x \in (\frac{1}{2}, \frac{\pi}{2})$$

$$W = \det \begin{bmatrix} \sin(x) & x \\ \cos(x) & 1 \end{bmatrix} = \sin x - x\cos x$$

$\Longrightarrow$

$$W(\pi/4) = \sin\frac{\pi}{4} - \frac{\pi}{4}\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} \neq 0$$

It can be shown that $W(x) \neq 0$ for any $x \in (\frac{1}{2}, \frac{\pi}{2}) \Longrightarrow$ $\{f_1, f_2\}$ are linearly independent.

Section 4.4 — coordinates and basis

V is a vector space. A set of elements of V

$$\{\vec{v}_1\,, \vec{v}_2\,, \cdots, \vec{v}_n\}$$

is a **basis** of V if:

(i) $\vec{v}_1\,, \vec{v}_2\,, \cdots, \vec{v}_n$ are linearly independent
(ii) $\text{span}\{\vec{v}_1\,, \vec{v}_2\,, \cdots, \vec{v}_n\} = V$

If $\{\vec{v}_1\,, \vec{v}_2\,, \cdots, \vec{v}_n\}$ is a basis of $V \Longrightarrow$ any vector in V can be written in a unique was as a linear combination of the elements of the basis:

$$\vec{v} = c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \cdot + c_n \cdot \vec{v}_n$$

The coefficients

$$(c_1, c_2, \cdots, c_n)$$

are the coordinates of $\vec{v}$ in the given basis.

■ Example (1). Let $V = \mathbb{R}^3$ and

$$\hat{i} = (1,0,0)\,, \qquad \hat{j} = (0,1,0)\,, \qquad \hat{h} = (0,0,1)$$

We know by now that for any $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$

$$\vec{v} = v_1 \cdot \hat{i} + v_2 \cdot \hat{j} + v_3 \cdot \hat{h}$$

Thus

$$\{\hat{i}, \hat{j}, \hat{h}\}$$

form a basis (a standard basis) in $\mathbb{R}^3$

■ Example (2). What about

$$\vec{u}_1 = (1,1,0)\,, \qquad \vec{u}_2 = (1,0,1)\,, \qquad \vec{u}_3 = (0,1,1) \text{ ?}$$

Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a basis for $\mathbb{R}^3$?

- linear independence:

$$c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2 + c_3 \cdot \vec{u}_3 = \vec{0}$$

$\Longrightarrow$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot c_1 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot c_2 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Longleftrightarrow \qquad A \cdot \vec{c} = \vec{0}$$

Note:

$$\det(A) = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \underbrace{=}_{r_2 \to r_2 - r_1} \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = 1 \cdot (-1 - 1) = -2 \neq 0$$

$\implies$ Thus A is invertible:

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \cdot \vec{0} = 0$$

$\implies$ $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linear independent set.

- Do $\vec{u}_1, \vec{u}_2, \vec{u}_3$ span $\mathbb{R}^3$?
  i.e., $\exists \ (c_1, c_2, c_3)$ for any $\vec{b} \in \mathbb{R}^3$ such that

$$c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2 + c_3 \cdot \vec{u}_3 = \vec{b} \ ?$$

$\implies$

$$A \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\implies$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

exist and are unique since (as we already established) A is invertible.

Exercise: since $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ form a basis for $\mathbb{R}^3$, we should be able to find the coordinates of

$$\vec{b} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

this in basis.

Hint:

$$\vec{c} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\text{coordinates}} = A^{-1} \cdot \vec{b}$$

■ Example 3.

$$\mathcal{P}_n = \text{space of polynomials of degree } \leq n$$

$$\{1, x, x^2, \cdots x^n\}$$

form a basis for $\mathcal{P}_n$: for any $P \in \mathcal{P}_n$,

$$P = a_0 + a_1\ x + a_2\ x^2 + \cdots + a_n x^n$$

$\Longrightarrow$ the span condition is obvious; what about being linearly independent?

$$\mathcal{P}_3 : \qquad \{1, x, x^2, x^3\}$$

$$W(x) = \det \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 1 & 2x & 3x^2 \\ 0 & 2 & 6x \\ 0 & 0 & 6 \end{bmatrix} = 1 \cdot 1 \cdot \det \begin{bmatrix} 2 & 6x \\ 0 & 6 \end{bmatrix} = 1 \cdot 1 \cdot 2 \cdot 6 = 12 \neq 0$$

$\Longrightarrow$

$$\{1, x, x^2, x^3\}$$

are linearly independent