Lecture 31 (sections 6.1,6.2)

Section 6.1 — inner products

In $\mathbb{R}^n$ we have the dot product: for two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

Properties:

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $(\vec{u} + \vec{w}) \cdot \vec{v} = \vec{u} \cdot \vec{v} + \vec{w} \cdot \vec{v}$
- $(k \cdot \vec{u}) \cdot \vec{v} = k \cdot (\vec{u} \cdot \vec{v})$
- $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

Let V be an abstract linear vector space. If $u, v \in V$, we define

$$(u, v) \quad \longrightarrow \quad \langle u, v \rangle \ \in \ \mathbb{R}$$

with the properties:

- 1
$$\langle u, v \rangle = \langle v, u \rangle$$
- 2
$$\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$$
- 3
$$\langle k \cdot u, v \rangle = k \cdot \langle u, v \rangle$$
- 4
$$\langle u, u \rangle \geq 0 \qquad \text{and} \qquad \langle u, u \rangle = 0 \iff u = 0$$

Then, $\langle , \rangle$ is called an **inner product** on V, and V itself is called an **inner product space**

- Example (1): $\mathbb{R}^2$ —

$$\langle u, v \rangle \overset{def}{\equiv} \vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2$$

- Example (2): $\mathbb{R}^n$ —

$$\underbrace{\langle u, v \rangle}_{\text{Euclidean inner product on } \mathbb{R}^n} \overset{def}{\equiv} \vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

- Example (3): $\mathbb{R}^2$ —

$$\underbrace{\langle u, v \rangle}_{\text{weighted inner product}} \overset{def}{\equiv} = 3u_1 \cdot v_1 + 7u_2 \cdot v_2$$

$\Longrightarrow$ Show that all the properties are OK

Norm of a vector:

$$||\vec{u}|| = \sqrt{\vec{u} \cdot \vec{u}}$$

From any inner product a norm can be defined:

$$||u|| = \sqrt{\langle u, u \rangle}$$

Distance between two points $\vec{u}$ and $\vec{v}$

$$d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||$$

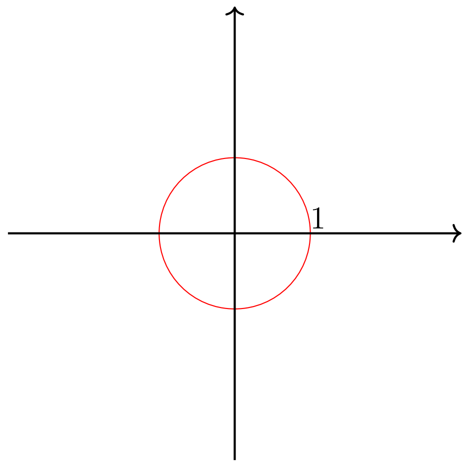
can also be generalized correspondingly for any pair of elements of the inner product space.

▪ Example 4. A circle of radius 1 centered at 0 is $\mathbb{R}^2$ is

$$||\vec{u}|| = 1$$

- if we use Euclidean inner product:

$$1 = ||\vec{u}|| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + u_2^2} \qquad \Longleftrightarrow \qquad \boxed{u_1^2 + u_2^2 = 1}$$

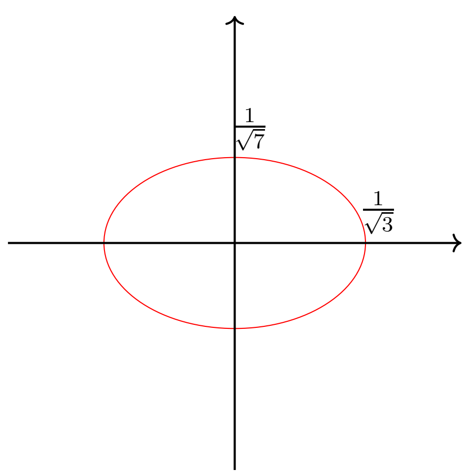


- let's use weighted inner product:

$$1 = d(\vec{u}, \vec{0}) = ||\vec{u} - \vec{0}|| = \sqrt{\langle \vec{u}, \vec{u}\rangle} = \sqrt{3u_1^2 + 7u_2^2}$$

$$\Updownarrow$$

$$\underbrace{\boxed{3u_1^2 + 7u_2^2 = 1}}_{\text{circle in weighted inner product space}} \qquad \Longrightarrow \qquad \frac{u_1^2}{\left(\frac{1}{\sqrt{3}}\right)^2} + \frac{u_2^2}{\left(\frac{1}{\sqrt{7}}\right)^2} = 1$$



- Example 5.

$$\mathcal{P}_n = \{\text{polynomials of degree } \leq n\}$$

We can turn $\mathcal{P}_n$ into an inner product space once we define inner product as follows:

$$p = a_0 + a_1 \cdot x + \cdots + a_n \cdot x^n$$
$$q = b_0 + b_1 \cdot x + \cdots + b_n \cdot x^n$$

$$\langle p, q \rangle \overset{def}{\equiv} a_0 \cdot b_0 + a_1 \cdot b_1 + \cdots + a_n \cdot b_n$$

Section 6.2 — angles and orthogonality

$\Longrightarrow$ Recall, if $\vec{u}, \vec{v}$ are nonzero vectors in $\mathbb{R}^n$,

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \cdot ||\vec{v}|| \cdot \cos\theta$$

$\Longrightarrow$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \cdot ||\vec{v}||}$$

If $\vec{u} \perp \vec{v} \Longrightarrow \quad \theta = \{\frac{\pi}{2}, \frac{3}{2}\pi\} \Longrightarrow$

$$\boxed{\vec{u} \cdot \vec{v} = 0 \qquad \Longleftrightarrow \qquad \vec{u} \perp \vec{v}}$$

We can do the same with the general inner vector space:

Given V and $\langle , \rangle$ defined on it, for any $u, v \neq 0$

$$\langle u, v \rangle \stackrel{def}{\equiv} ||u|| \cdot ||v|| \cdot \cos\theta$$

$$-1 \le \cos\theta \equiv \frac{\langle u, v \rangle}{||u|| \cdot ||v||} \le 1$$

$\Longrightarrow$

$$\cos\theta = 0 \qquad \Longleftrightarrow \qquad \langle u, v \rangle = 0 \qquad \Longleftrightarrow \qquad u \perp v$$