Lecture 4 (section 1.2)

$\Longrightarrow$ Problem: find the most general solution using augmented matrix approach

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

- unknowns: $x_1, x_2, \cdots x_6$ (6 in total; $n = 6$)
- RHS:

$$b = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

- equations: 4 in total ($m = 4$)
- augmented matrix $m \times (n+1) \longrightarrow 4 \times 7$

$$\begin{array}{cc} & \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \equiv A \end{array}$$

$\Longrightarrow$ perform elementary row operations to bring A to REF

- $A \to A_{(1)}$:

$$\begin{aligned} r_2 &\to r_2 + (-2) \cdot r_1 \\ r_4 &\to r_4 + (-2) \cdot r_1 \\ r_3 &\to \frac{1}{5}\, r_3 \end{aligned}$$

$\Longrightarrow$

$$A_{(1)} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

- $A_{(1)} \to A_{(2)}$:

$$\begin{aligned} r_2 + r_3 = 0 &\to r_4 \\ r_2 &\to (-1) \cdot r_2 \\ r_4 &\to \frac{1}{2}\, r_4 \to r_3 \end{aligned}$$

$\Longrightarrow$

$$A_{(2)} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 4 & 0 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $A_{(2)} \to A_{(3)}$:

$$r_3 \to r_3 + (-2) \cdot r_2$$

$\Longrightarrow$

$$A_{(3)} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $A_{(3)} \to A_{(4)}$:

$$r_3 \ \to \ \frac{1}{3}\, r_3$$

$\Longrightarrow$

$$A_{(4)} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $A_{(4)}$ is REF:

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ A_{(4)} = & \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

  - leading variables: x_1, x_3, x_6
  - free variables: x_2, x_4, x_5

$\Longrightarrow$ system is consistent

$$\left.\begin{aligned} x_2 &= s \\ x_4 &= t \\ x_5 &= u \end{aligned}\right\} \quad \text{parametrization of free variables}$$

$\Longrightarrow$ solving for leading variables from bottom-to-top:

$r_3 :\quad x_6 = \dfrac{1}{3}$

$r_2 :\quad x_3 + 2x_4 + 3x_6 = 1 \quad \Longrightarrow \quad x_3 + 2t + 3 \cdot \dfrac{1}{3} = 1 \quad \Longrightarrow \quad x_3 = 1 - 1 - 2t = -2t$

$r_1 :\quad x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \quad \Longrightarrow \quad x_1 + 3s - 2(-2t) + 2u = 0 \quad \Longrightarrow$

$x_1 = -2u - 4t - 3s$

$\Longrightarrow$ Answer:

$$(x_1, x_2, x_3, x_4, x_5, x_6) = \left(-3s - 4t - 2u, s, -2t, t, u, \frac{1}{3}\right)$$

Definition:

It all elements on the right hand side of equations are zero, the system is called **homogeneous**:

$$b_1 = b_2 = \cdots = b_m = 0$$

$\Longrightarrow$ Example:

$$\begin{cases} 2x - y - 3z = 0 \\ -x + 2y - 3z = 0 \\ x + y + 4z = 0 \end{cases}$$

Note: homogeneous system has at least one solution — all variables zero.

In our case:

$$A = \begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \Longrightarrow \begin{matrix} r_3 \to r_1 \\ r_2 \to r_2 + r_3 \\ r_1 \to r_3 \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{matrix} r_2 \to \frac{1}{3} r_2 \\ r_3 \to r_3 + (-2) \cdot r_1 \end{matrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & -3 & -11 & 0 \end{bmatrix} \Longrightarrow r_3 \to r_3 + 3r_2$$

$$\Longrightarrow \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & -10 & 0 \end{bmatrix} \Longrightarrow r_3 \to \left(-\frac{1}{10}\right) \cdot r_3 \Longrightarrow \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Longrightarrow$ solving for leading variables (all in this case) from bottom-to-top:

$$\begin{aligned} r_3 :& \quad z = 0 \\ r_2 :& \quad y + \frac{1}{3} z = y + 0 = 0 \quad \Longrightarrow \quad y = 0 \\ r_1 :& \quad x + y + 4z = x + 0 + 0 = 0 \quad \Longrightarrow \quad x = 0 \end{aligned}$$

$\Longrightarrow$ Unique solution:

$$(x, y, z) = (0, 0, 0)$$