Lecture 7 (section 1.4)

Section 1.4 — algebraic properties of matrices; inverses

$\Longrightarrow$ Assuming that the operations below are defined, *i.e.,* we have compatible dimensions of A & B,

- commutativity of addition:

$$A + B = B + A$$

- associativity of addition:

$$(A + B) + C = A + (B + C)$$

- distributivity of multiplication:

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

- associativity of multiplication

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

- in general, non-commutativity of multiplication

$$A \cdot B \neq B \cdot A$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 13 & 2 \end{bmatrix}$$

$$\neq$$

$$B \cdot A = \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 3 & 6 \end{bmatrix}$$

$\Longrightarrow$ More algebraic properties in a book

$\Longrightarrow$ Recall, for any $\{a, b\} \in \mathbb{R}$ and $c \in \mathbb{R} \neq 0$

$$a \cdot c = b \cdot c \implies (a-b) \cdot c = 0 \implies a - b = 0 \implies a = b$$

This is known as a cancellation law for numbers.

$\Longrightarrow$ Cancellation law does not work for matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$$A \cdot C = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Note:

$$A \cdot B = A \cdot C \implies A \cdot (B - C) = 0$$

$$A \neq 0_{2\times 2} \qquad \textbf{but} \qquad B - C \neq 0_{2\times 2}$$

$\Longrightarrow$ Moral: it is easy to state whether A is $0_{m\times n}$; it is more subtle to define what it means for A to be non-zero with respect to cancellation law (we will revisit this issue later in the course).

Definition:

$n \times n$ matrix A is called an identity (a unit) matrix I_n if

$$I_n = [a_{ij}]\,, \qquad a_{ij} = \begin{cases} a_{ii} = 1\,, & i = 1, \cdots n \\ a_{ij} = 0\,, & i \neq j \end{cases}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow$ Note: for any $n \times n$ matrix A:

$$A \cdot I_n = A\,, \qquad I_n \cdot A = A$$

Notice that multiplication is commutative.

Claim:

If A is $n \times n$ matrix, its RREF is either:

- I_n (a unit matrix)
- has one or more zero rows

$\Longrightarrow$ Recall: for any $a \in \mathbb{R}$ and $a \neq 0$, $\exists$ a number a^{-1} (called the reciprocal or inverse), such that

$$a \cdot (a^{-1}) = 1 \qquad \text{and} \qquad (a^{-1}) \cdot a = 1$$

Definition:

Let A be $n \times n$ matrix. If $\exists$ a matrix A^{-1} (or dimension $n \times n$) such that $$A \cdot A^{-1} = I_n\,, \qquad \text{and} \qquad A^{-1} \cdot A = I_n$$ then A is an invertible matrix and A^{-1} is called the **inverse** of A.

$\Longrightarrow$ Note: not all nonzero square matrices are invertible

Definition:

A square non-invertible matrix is called **singular**.

$\Longrightarrow$ If A is an invertible matrix, then A^{-1} is unique.

Proof:

- Suppose there are 2 inverses of A,

$$A^{-1} = \begin{cases} B \\ C \end{cases}$$

then

$$A \cdot B = I_n\,, \qquad B \cdot A = I_n$$
$$A \cdot C = I_n\,, \qquad C \cdot A = I_n$$

- Consider:

$A \cdot B - A \cdot C = I_n - I_n = 0_{n\times n}$

$A \cdot (B - C) = 0_{n\times n}$

$\underbrace{B \cdot A}_{I_n} \cdot (B - C) = \underbrace{B \cdot 0_{n\times n}}_{0_{n\times n}}$

$I_n \cdot (B - C) = 0_{n\times n} \quad \Longrightarrow \quad B - C = 0_{n\times n} \quad \Longrightarrow \quad \boxed{B = C} \;\leftarrow$ must be the same

Theorem:

If A is 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

it is invertible only if the **determinant** of the matrix

$$\det(A) \stackrel{def}{\equiv} a \cdot d - b \cdot c \neq 0$$

The inverse matrix A^{-1} is given by

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof by direct computation: compute (at home) $A \cdot A^{-1}$

Example:

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \qquad \det(A) = ad - bc = 3 \cdot 2 - 5 \cdot 1 = 1 \neq 0$$

$$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$