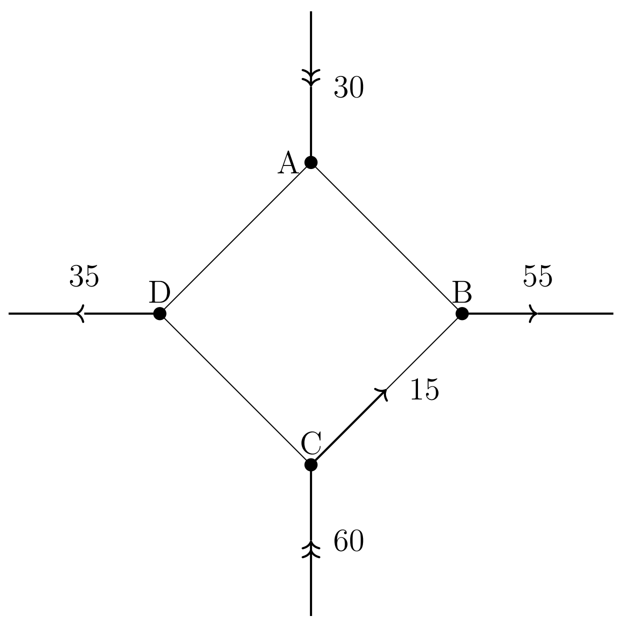
Lecture 1 (section 1.9)

Applications of linear systems

$\Longrightarrow$ Consider a network flow with 4 nodes: A,B,C,D (can represent the **flow** of information, objects — wifi-traffic, water,cars,$\cdots$)

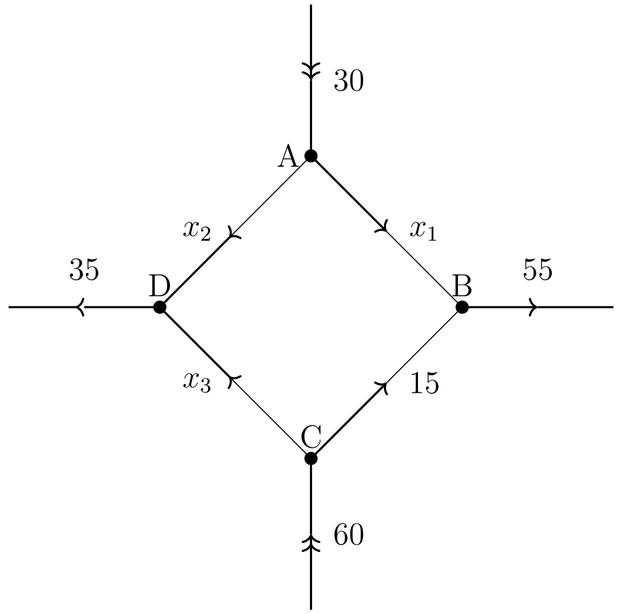


$\Longrightarrow$ A flow rates into nodes are given

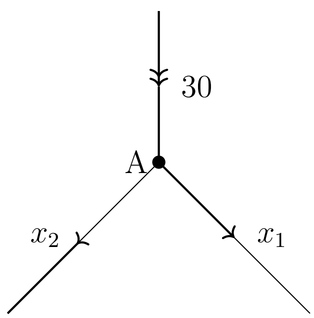
$\Longrightarrow$ Flow rate along CB edge is specified

Problem: find the remaining rates through each edge

Basic principle: flow conservation at each node



- Node A:



$$\underbrace{30}_{\text{“in”}} = \underbrace{x_1 + x_2}_{\text{“out”}}$$

- Node B:

$$x_1 + 15 = 55$$

- Node C:

$$60 = x_3 + 15$$

- Node D:

$$x_2 + x_3 = 35$$

Conservation laws $\Longrightarrow$ *system of linear equations*

$$\begin{cases} x_1 + x_2 = 30\,, & (1) \\ x_1 + 15 = 55\,, & (2) \\ x_3 + 15 = 60\,, & (3) \\ x_2 + x_3 = 35\,, & (4) \end{cases}$$

Solve problem $\Longrightarrow$ solve linear system

$$(2): \qquad x_1 = 55 - 15 = 40$$

$$(3): \qquad x_3 = 60 - 15 = 45$$

$$(1): \qquad x_2 = 30 - x_1 = 30 - 40 = -10 \qquad \text{(reversed direction)}$$

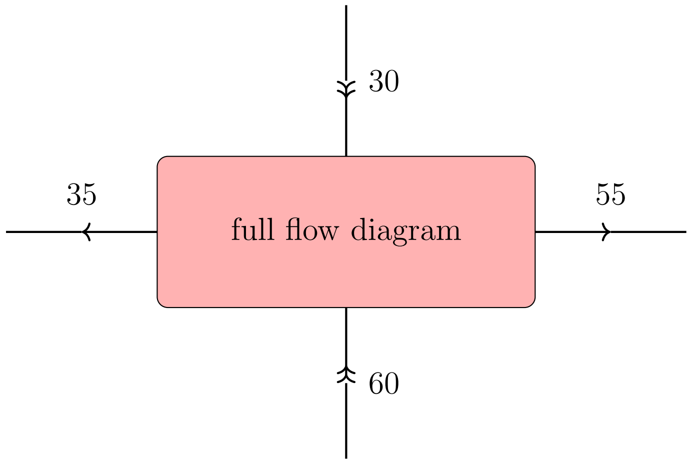
$$(4): \qquad \text{DOES NOT CARRY ANY NEW INFORMATION}$$

(4) is a *constraint*:

$$x_2 + x_3 = (-10) + 45 = 35 \qquad \stackrel{?}{=} \qquad 35 \;\; \text{(eq.4)}$$

$\Longrightarrow$ Problem is consistent, there is a unique solution

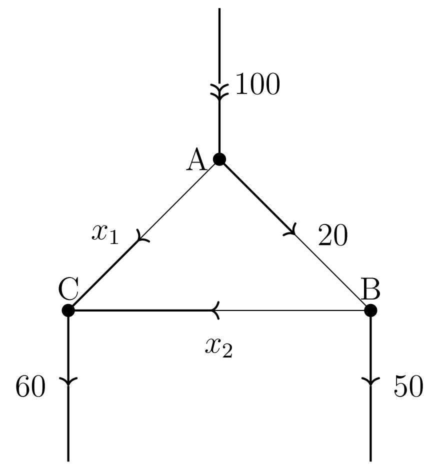
Why consistent?



$\Longrightarrow$ Because there is conservation at any node, there must be conservation for the whole network

$$\begin{array}{ccc} \text{"IN"} & & \text{"OUT"} \\ 30+60 & & 35+55 \\ || & & || \\ 90 & \overset{\checkmark}{=} & 90 \end{array}$$

Solution to linear system does not always exist:

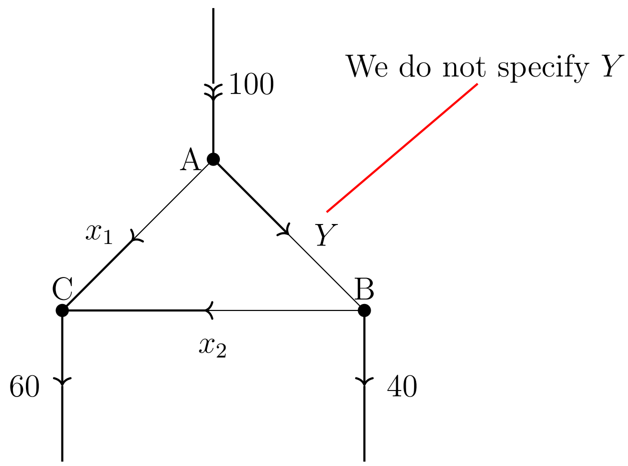


$$
\begin{array}{llcl}
\text{node A :} & 100 = x_1 + 20 & \longrightarrow & x_1 = 80 \\
\text{node B :} & 20 = x_2 + 50 & \longrightarrow & x_2 = -30 \\
\text{node C :} & x_1 + x_2 = 60 & \longrightarrow & 80 + (-30) = +50 \neq 60
\end{array}
$$

Reason:

$$
\begin{array}{ccc}
\text{"IN"} & & \text{"OUT"} \\
100 & \neq & 60 + 50 = 110
\end{array}
$$

Solution to linear system is not always unique:



$$\text{node A}: \quad 100 = x_1 + Y \quad \longrightarrow \quad x_1 = 100 - Y$$

$$\text{node B}: \quad Y = x_2 + 40 \quad \longrightarrow \quad x_2 = Y - 40$$

$$\text{node C}: \quad x_1 + x_2 = (100 - Y) + (Y - 40) = 60 \quad \longrightarrow \quad \overset{\checkmark}{=} \; 60$$

$\Longrightarrow$ True for any Y

Consistent?

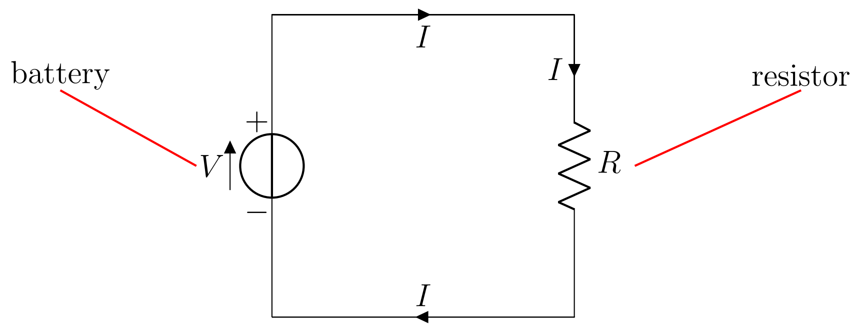
"IN"		"OUT"
100	$=$	$60 + 40 = 100$

$\Longrightarrow$ In the course we study when the system of linear equations is consistent, and when there is a unique solution

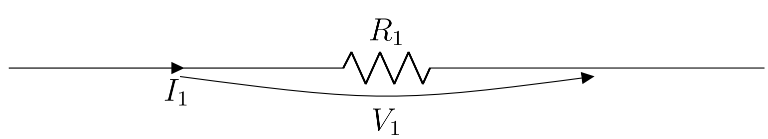
$\Longrightarrow$ In previous example setting $Y = 20$ gives a unique solution; without specifying Y system of equations is <u>underdetermined</u>

Electric circuits

$\Longrightarrow$ Elementary electric circuit:



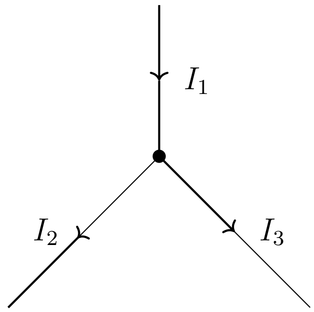
- **current** I flows from the positive pole of the battery to the negative pole
- battery is characterized by the **voltage** V (electric potential difference between the poles)
- current is characterized by a flow of positively charged particles (convention) — negatively charged particles (electrons) move in the direction opposite to that of the current
- **resistor** R reduces the *electric potential* according to **Ohm's law:**



V_1	$=$	R_1	$\cdot$	I_1
potential drop V_1		resistance of the resistor R_1		current I_1
accrose the resistor		property of the material		through the resistor
in **volts** (V)		in **Ohms** (Ω)		in **amperes** (A)

Kirchoff's laws of electric circuits:

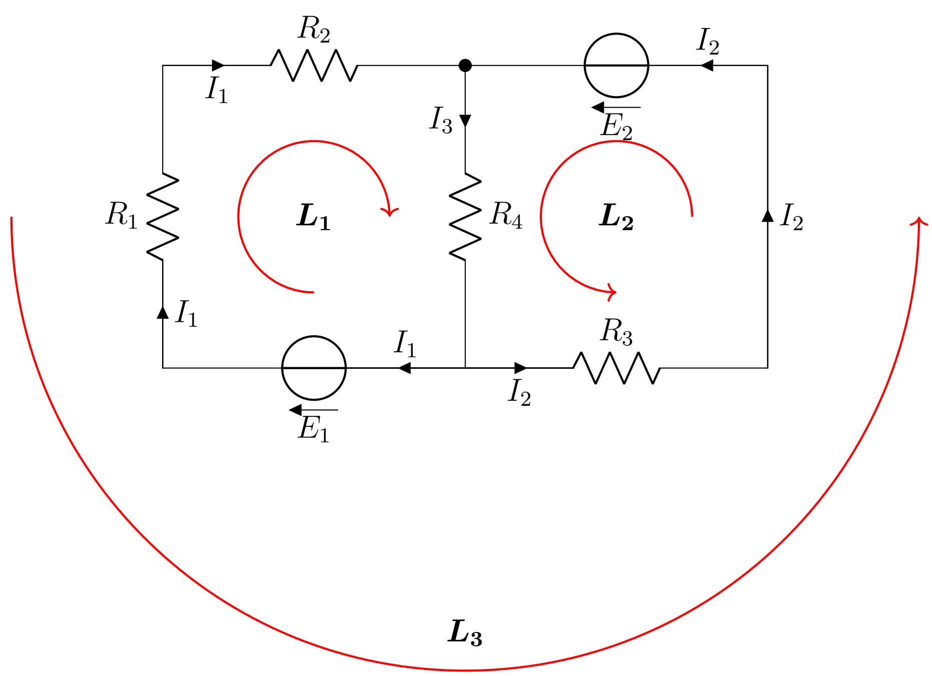
① Current conservation at each node



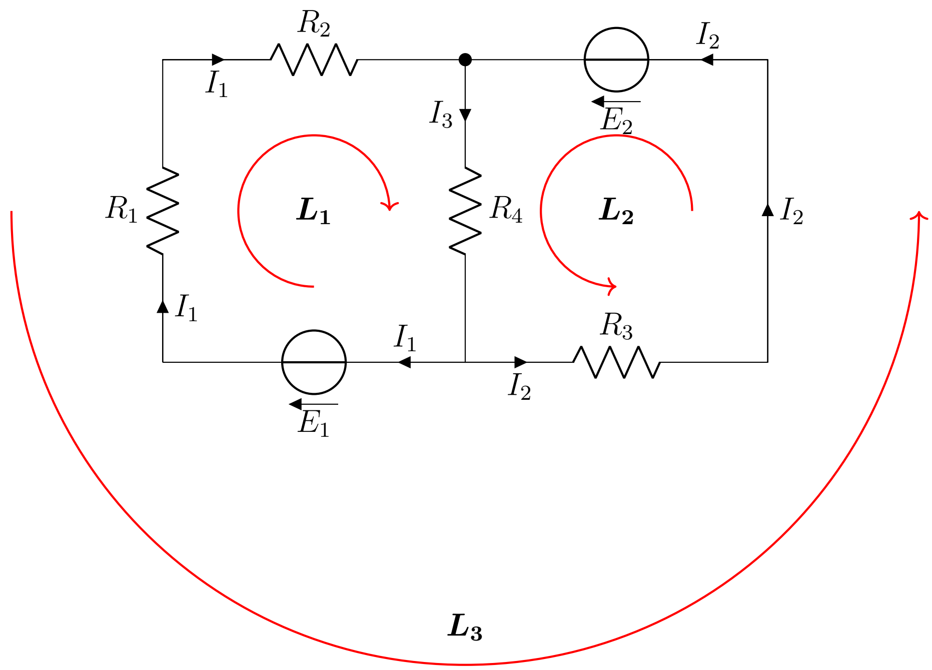
$$\underbrace{I_1}_{\text{“in”}} = \underbrace{I_2 + I_3}_{\text{“out”}}$$

② Sum of voltage rises = sum of voltage drops

$\Longrightarrow$ In normal language: electric potential change is zero for any closed loop of the circuit



$\Longrightarrow$ There are 3 closed loops: $\{\boldsymbol{L_1}, \boldsymbol{L_2}, \boldsymbol{L_3}\}$; orientation in red



$$\boldsymbol{L_1}: \qquad \underbrace{R_1 \cdot I_1 + R_2 \cdot I_1 + R_4 \cdot I_3}_{\text{“voltage drops”}} = \underbrace{E_1}_{\text{“voltage rises”}}$$

$$\boldsymbol{L_2}: \qquad \underbrace{R_4 \cdot I_3 + R_3 \cdot I_2}_{\text{“voltage drops”}} = \underbrace{E_2}_{\text{“voltage rises”}}$$

$$\boldsymbol{L_3}: \qquad \underbrace{R_3 \cdot I_2 + (-R_2 \cdot I_1) + (-R_1 \cdot I_1)}_{\text{“voltage drops”}} = \underbrace{(-E_1) + E_2}_{\text{“voltage rises”}}$$

$\Longrightarrow$ Note **minus** signs: they appear because we are going 'against' the current direction

$\Longrightarrow$ Note:

$$\boldsymbol{L_3 = L_2 - L_1}$$

that is, loop $\boldsymbol{L_3}$ is not *independent*