Lecture 34 (section 6.3)

Section 6.3 — continued

- Example.

$V = \mathbb{R}^4$. Vectors

$$\vec{u}_1 = (0,2,1,0) \quad \vec{u}_2 = (1,-1,0,0)$$
$$\vec{u}_3 = (1,2,0,-1) \quad \vec{u}_4 = (1,0,0,1)$$

form a basis in V. Use G-S process to construct an orthogonal basis in $\mathbb{R}^4$

- $$\vec{V}_1 = \vec{u}_1 = (0,2,1,0)$$

- $$\vec{V}_2 = \vec{u}_2 - \text{proj}_{\{\vec{V}_1\}}\vec{u}_2$$

$\Longrightarrow$ guaranteed that $\vec{V}_2 \perp \vec{V}_1$ and $\{\vec{V}_1, \vec{V}_2\}$ span the same subspace as $\{\vec{u}_1, \vec{u}_2\}$

- Note:

$$\langle \vec{u}_2, \vec{V}_1 \rangle = (1,-1,0,0) \cdot (0,2,1,0) = -2$$

$$||\vec{V}_1||^2 = (0,2,1,0) \cdot (0,2,1,0) = 5$$

$$\vec{V}_2 = \underbrace{(1,-1,0,0)}_{\vec{u}_2} - \frac{\overbrace{(-2)}^{\langle \vec{u}_2, \vec{V}_1\rangle}}{\underbrace{5}_{||\vec{V}_1||^2}} \cdot \underbrace{(0,2,1,0)}_{\vec{V}_1} = (1,-1,0,0) + \left(0, \frac{4}{5}, \frac{2}{5}, 0\right) = \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

$\Longrightarrow$ Let's check that $\vec{V}_2 \perp \vec{V}_1$:

$$\underbrace{\left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)}_{\vec{V}_2} \cdot \underbrace{(0,2,1,0)}_{\vec{V}_1} = -\frac{2}{5} + \frac{2}{5} = 0$$

- $$\vec{V}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{V}_1 \rangle}{||\vec{V}_1||^2} \cdot \vec{V}_1 - \frac{\langle \vec{u}_3, \vec{V}_2 \rangle}{||\vec{V}_2||^2} \cdot \vec{V}_2 =$$

$$= (1,2,0,-1) - \frac{4}{5} \cdot (0,2,1,0) - \frac{1 \cdot 1 + 2 \cdot (-\frac{1}{5})}{1 + \frac{1}{25} + \frac{4}{25}} \cdot (1, -\frac{1}{5}, \frac{2}{5}, 0) =$$

$$= (1,2,0,-1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, -\frac{1}{10}, \frac{1}{5}, 0\right) =$$

$$= \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$\Longrightarrow$ check:

$$\vec{V}_3 \cdot \vec{V}_1 = \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \cdot (0,2,1,0) = 1 - 1 = 0$$

$$\vec{V}_3 \cdot \vec{V}_2 = \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \cdot (1, -\frac{1}{5}, \frac{2}{5}, 0) = \frac{1}{2} - \frac{1}{10} - \frac{2}{5} = 0$$

- do it yourself

$$\vec{V}_4 = \vec{u}_4 - \mathrm{proj}_{\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}} \vec{u}_4 = \cdots$$

- $\Longrightarrow$

$$\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4\}$$

an orthogonal basis on $\mathbb{R}^4$ (one of infinitely many)

$\Longrightarrow$ from here $\Longrightarrow$ orthonormal basis; normalize the vectors:

$$\underbrace{\left\{ \frac{\vec{V}_1}{||\vec{V}_1||} \quad \frac{\vec{V}_2}{||\vec{V}_2||} \quad \frac{\vec{V}_3}{||\vec{V}_3||} \quad \frac{\vec{V}_4}{||\vec{V}_4||} \right\}}_{\text{orthonormal basis on } \mathbb{R}^4}$$

<u>Review</u>

- Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

  - eigenvalues:

$$0 = \det(A - \lambda \cdot I_3) = \det \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \underbrace{=}_{\text{2nd column}} = (1-\lambda) \cdot \det \begin{bmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) \cdot (\lambda^2 - 5\lambda + 4 + 2) = (1-\lambda) \cdot (\lambda - 3) \cdot (\lambda - 2)$$

$\Longrightarrow$

$$\lambda_1 = 1\,, \qquad \lambda_2 = 2\,, \qquad \lambda_3 = 3$$

all eigenvalues are of multiplicity 1.

  - Eigenvector for $\lambda = \lambda_1 = 1$:

$$\begin{bmatrix} 4-\lambda_1 & 0 & 1 \\ -2 & 1-\lambda_1 & 0 \\ -2 & 0 & 1-\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ augmented matrix:

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{\substack{r_3 \to r_3 - r_2 \\ r_2 \to -\frac{1}{2} r_2 \\ r_1 \to r_1 - 3r_2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

  - continue with λ_2 and λ_3 $\cdots$