Lecture 21 (sections 4.2,4.3)

Section 4.2 — continuation

A system of m linear equations with n unknows

$$A \cdot \overline{X} = \overline{0}$$

is called homogeneous.

$\Longrightarrow$ We can think of $\overline{X}$ as a vector in $\mathbb{R}^n$. Then, (theorem) the solution space of

$$A \cdot \overline{X} = \overline{0}_m$$

is a subspace of $\mathbb{R}^n$.

Proof:

- $\overline{X} = \overline{0}_n$ is a solution:

$$A \cdot \overline{0}_n = \overline{0}_m$$

- If $\overline{x}, \overline{y} \in \mathbb{R}^n$ are solutions, *i.e.*,

$$A \cdot \overline{x} = \overline{0} \qquad \text{and} \qquad A \cdot \overline{y} = \overline{0}$$

then $\overline{x} + \overline{y}$ is a solution as well:

$$A \cdot (\overline{x} + \overline{y}) = A \cdot \overline{x} + A \cdot \overline{y} = \overline{0} + \overline{0} = \overline{0}$$

- If $\overline{x} \in \mathbb{R}^n$ is a solutions, *i.e.*,

$$A \cdot \overline{x} = \overline{0}$$

and $k \in \mathbb{R}$, then $k \cdot \overline{x}$ is a solution as well:

$$A \cdot (k \cdot \overline{x}) = k \cdot A \cdot \overline{x} = k \cdot \overline{0} = \overline{0}$$

$\Longrightarrow$ Questions: what if system is inhomogeneous?

$$A \cdot \overline{X} = \overline{b}\,, \qquad \overline{b} \neq \overline{0}\ ?$$

$\Longrightarrow$ Not a subspace because $\overline{0}_n$ is not a solution

- Example: describe geometrically the solution space to

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

as a subspace of $\mathbb{R}^3$.

$\Longrightarrow$ Augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{matrix} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 3r_1 \end{matrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longleftarrow \text{RREF}$$

$\Longrightarrow$

free variables : y, z , leading variable : x

$\Longrightarrow$ solution space:

$$x + (-2) \cdot y + 3 \cdot z = 0$$

is a plane $\perp$ to $\vec{v} = (1, -2, 3)$ passing through the origin.

Let V be a vector space.

$$\{\vec{v}_1\,, \vec{v}_2\,, \cdot\,, \vec{v}_k\} \ \in \ V$$

is a linear independent set of vectors if

$$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \cdot + c_k \cdot \vec{v}_k = \vec{0}$$

has a unique solution:

$$c_1 = c_2 = \cdots = c_k = 0$$

$\implies$ Example: consider $\{\hat{\imath}, \hat{\jmath}, \hat{h}\} \in \mathbb{R}^3$. The standard basis is a linear independent set. Indeed:

$$c_1 \cdot \hat{\imath} + c_2 \cdot \hat{\jmath} + c_3 \cdot \hat{h} = \vec{0}$$

$$c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \implies \qquad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = I_3^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

a unique solution.

If a set of vectors is not linearly independent, it is called linearly dependent.

$\Longrightarrow$ Are vectors:

$$(1,2,1)\,, \qquad (-2,2,4)\,, \qquad (2,0,1)$$

linearly independent?

$$c_1 \cdot \begin{bmatrix}1\\2\\1\end{bmatrix} + c_2 \cdot \begin{bmatrix}-2\\2\\4\end{bmatrix} + c_3 \cdot \begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 2\\2 & 2 & 0\\1 & 4 & 1\end{bmatrix} \cdot \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \Longrightarrow \qquad A \cdot \vec{c} = \vec{0}\,, \qquad A = \begin{bmatrix}1 & -2 & 2\\2 & 2 & 0\\1 & 4 & 1\end{bmatrix}$$

Note,

$$\det(A) \underbrace{=}_{\text{3rd column expansion}} = 2 \cdot \det \begin{bmatrix}2 & 2\\1 & 4\end{bmatrix} + 1 \cdot \det \begin{bmatrix}1 & -2\\2 & 2\end{bmatrix} = 2 \cdot (8-2) + 1 \cdot (2+4) = 18 \neq 0$$

$\Longrightarrow$ A is invertible $\Longrightarrow$ there is a unique solution $\Longrightarrow$ the vectors are linearly independent