Lecture 24 (sections 4.4,4.5)

Section 4.4 — continuation

$\Longrightarrow$ In any vector space, a set that contains the zero vector is always linearly dependent.

Why?

Let:

$$\{\vec{0}, \vec{v}_1, \cdots \vec{v}_k\}$$

Note:

$$\underbrace{7}_{7 \neq 0} \cdot \vec{0} + 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_k = \vec{0}$$

Section 4.5 — dimension

Any vector space V will have a basis. In fact, V can have many different basis. All such basis of V have the same number of vectors.

$$\text{this number} \quad \overset{def}{\equiv} \quad \text{the dimension of } V$$

Examples:

- 1

$$\dim(\mathbb{R}^2) = 2\,, \qquad \dim(\mathbb{R}^3) = 3\,, \qquad \dim(\mathbb{R}^n) = n$$

- 2

$$\mathcal{P}_3 : \qquad \{1, x, x^2, x^3\}$$

$\Longrightarrow$

$$\dim(\mathcal{P}_3) = 4$$

$\Longrightarrow$

$$\dim(\mathcal{P}_n) = n + 1$$

- 3

$$M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \{a, b, c, d\} \in \mathbb{R} \right\}$$

$\Longrightarrow$

$$\dim\left(M_{2\times 2}\right) = 4$$

$$\dim\left(M_{m\times n}\right) = m \cdot n$$

- 4

$$\mathcal{F} = \{f : \ (-\infty, \infty) \to \mathbb{R}\}$$

a vector space of functions of a single variable $\Longrightarrow$

$$\dim(\mathcal{F}) = \infty$$

Why?

$\Longrightarrow$ $\mathcal{F}$ includes polynomials of arbitrary high degree, *i.e.*, $\mathcal{P}_n \subset \mathcal{F}$; and (as we note a bit later) because of this inclusion,

$$\dim(\mathcal{P}_n) \leq \dim(\mathcal{F}) \qquad \Longrightarrow$$

$$\lim_{n\to\infty} \dim(\mathcal{P}_n) = \lim_{n\to\infty}(n+1) = \infty \qquad \leq \qquad \dim(\mathcal{F})$$

Dimension of V equals # of coordinates

Theorem:

If $W \subset V$ is a subspace of V, then

$$\dim W \ \leq \ \dim V$$

If

$$\dim W \ = \ \dim V \qquad \Longrightarrow \qquad W = V$$

Definition:

If $W \neq \{\vec{0}\} \subset V$ is a subspace of V, and $$\dim W \; < \; \dim V$$ then W is called a proper subspace of V

$\Longrightarrow$ Recall: the set of all solutions of a homogeneous system of m equations (with n unknowns)

$$A_{m\times n} \cdot \vec{X} = \vec{0}$$

is a subspace of $\mathbb{R}^n$

- Find the dimension of the solution space to:

$$\begin{cases} x_1 + 3x_2 - x_3 = 0 \\ 2x_1 + 6x_2 - 2x_3 = 0 \end{cases}$$

$\Longrightarrow$ $m \times n = 2 \times 3$; augmented matrix:

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 6 & -2 & 0 \end{bmatrix} \quad \underbrace{\Longrightarrow}_{r_2 \to r_2 - 2r_1} \quad \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longleftarrow \text{RREF}$$

$$\text{leading variable :} \quad x_1$$
$$\text{free variables :} \quad x_2, x_3$$

$\Longrightarrow$ Set

$$x_2 = s\,, \qquad x_3 = t \qquad \Longrightarrow \qquad x_1 = -3x_2 + x_3 = -3s + t$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{solution space}} = \begin{bmatrix} -3s+t \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ $\{s, t\}$ are coordinates on the solution space;

$$\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are basis vectors of the solution space

$$\dim(\text{solution space}) \;=\; 2 \qquad \text{(same as the number of free variables)}$$