Lecture 27 (section 4.8)

Section 4.8 — rank and nullity of a matrix

Let A be $m \times n$ matrix.

$\Longrightarrow$ row space and column space of A always have the same dimension. This dimension is called a **rank** of A:

$$\text{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A)$$

- row space is a subspace of $\mathbb{R}^n$
- column space is a subspace of $\mathbb{R}^m$ $\Longrightarrow$

$$\text{rank}(A) \ \leq \ \min(m, n)$$

$$\text{nullity}(A) \ \stackrel{def}{\equiv} \ \text{dimension of the null space of } A$$

- Problem: find the rank and nullity of the matrix $m \times n = 4 \times 5$

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$$

$\Longrightarrow$

$$\text{rank}(A) \leq \min(4, 5) = 4$$

$\Longrightarrow$ let's bring A to REF:

$$A \quad \underbrace{\Longrightarrow}_{r_3 \to r_3 + 2r_1} \quad \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix} \quad \underbrace{\Longrightarrow}_{\substack{r_4 \to r_2 \\ r_4 \to r_4 - r_3}} \quad \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{\Longrightarrow}_{r_3 \to r_3 + r_2} \qquad \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Longleftarrow \text{ REF}$$

$$\text{leading variables} : x_1, x_2, x_4$$
$$\text{free variables} : x_3, x_5$$

$\Longrightarrow$

$$\text{rank}(A) = 3\,, \qquad \text{nullity}(A) = 2$$

Note:

$$\text{rank}(A) + \text{nullity}(A) = 3 + 2 = 5 \ = \ \#\ \text{number of columns of } A$$

Some theorems:

■ Let A be $m \times n$ matrix $\Longrightarrow$

$$\text{rank}(A) + \text{nullity}(A) = \text{ \# number of columns of } A$$

$$\text{rank}(A) + \text{nullity}(A) = n$$

■ Let A be $n \times n$ (a square) matrix. If

$$\text{rank}(A) = n$$

then:

- A is nonsingular (A is invertible)
- $\det(A) \neq 0$
- $A \cdot \overline{X} = \overline{b}$ always has a unique solution:

$$\overline{X} = A^{-1} \cdot \overline{b}$$

- $A \cdot \overline{X} = \overline{0}$ only has a zero solution
- $$\text{nullity}(A) = 0$$

■ Problem.

Let A be 7×6 matrix such that $A \cdot \overline{X} = \overline{0}$ only has the trivial solution (zero solution). Find the rank and the nullity of A

$\Longrightarrow$

$$\text{nullity}(A) = \text{\# free variables} = 0$$

$$\text{rank}(A) + \text{nullity}(A) = n = 6 \qquad \Longrightarrow \qquad \text{rank}(A) = 6$$

$\Longrightarrow$ Consider

$$A \cdot \overline{X} = \overline{b}$$

A is $m \times n$ matrix
$\overline{X}$ is $n \times 1$ matrix
$\overline{b}$ is $m \times 1$ matrix

$m = \#$ of equations; $n = \#$ of unknowns

- If $m > n$ system is called **overdetermined**; there is always $\overline{b}$ such that

$$A \cdot \overline{X} = \overline{b}$$

  is inconsistent (does not have a solution)

- If $m < n$ the system is called **underdetermined**; in this case for *any* $\overline{b}$ the system

$$A \cdot \overline{X} = \overline{b}$$

  either
  - (i) has an infinite number of solutions
  - (ii) is inconsistent

$\Longrightarrow$ Example for (ii):

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \end{cases}$$

here $m \times n = 2 \times 3 \Longrightarrow \quad m < n$