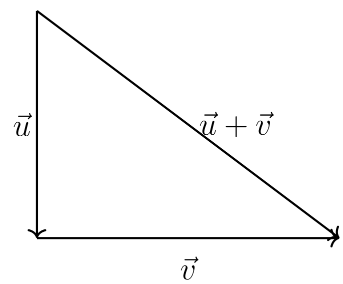
Lecture 32 (sections 6.2, 6.3)

Section 6.2 — continued

$\Longrightarrow$ Recall Pythagorean theorem:



$$\vec{u} \perp \vec{v} \qquad \Longrightarrow \qquad ||\vec{u}+\vec{v}||^2 = ||\vec{u}||^2 + |\vec{v}||^2$$

In general, if V is an inner product space, and u, v are two orthogonal vector, *i.e.*,

$$\langle u, v \rangle = 0$$

$\Longrightarrow$

$$||u+v||^2 = ||u||^2 + ||v||^2$$

Let

$$V \text{ is an inner product space}$$
$$W \text{ is a subspace of } V$$

then

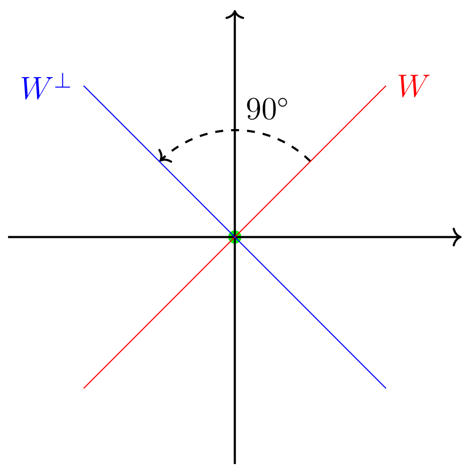
$$W^\perp \overset{def}{\equiv} \{u \in V \text{ such that } \langle u, w \rangle = 0 \text{ for any } w \in W\}$$

$W^\perp$ is called the orthogonal complement of W

Theorem:
$W^\perp$ is a subspace and

$$W \cap W^\perp = \{0\}$$

- Example 1. In $\mathbb{R}^2$:



$W \cap W^\perp = \{0\}$

■ Example 2.

$$\left(\mathbb{R}^2\right)^\perp = \{0\}$$

$$\{0\}^\perp = \mathbb{R}^2$$

Theorem:

Let W be a subspace of V (of finite dimension). Then:

$$\left(W^\perp\right)^\perp = W$$

Section 6.3 — Gram-Schmidt process

Consider $V = \mathbb{R}^3$ with

$$\vec{V}_1 = (0,1,0)\,, \qquad \vec{V}_2 = (1,0,1)\,, \qquad \vec{V}_3 = (1,0,-1)$$

■ Note:

$$\vec{V}_1 \cdot \vec{V}_2 = 0\,, \qquad \vec{V}_1 \cdot \vec{V}_3 = 0\,, \qquad \vec{V}_2 \cdot \vec{V}_3 = 0$$

Thus, $\{\vec{V}_1, \vec{V}_2, \vec{V}_3,\}$ is the orthogonal set of vectors

■ Vectors in $\{\vec{V}_1, \vec{V}_2, \vec{V}_3,\}$ are linearly independent:

$$c_1 \cdot \vec{V}_1 + c_2 \cdot \vec{V}_2 + c_3 \cdot \vec{V}_3 = 0 \qquad \Longrightarrow \qquad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{\text{1st column}} = (-1)\cdot\ \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = (-1)\cdot(-1\cdot 1 - 1\cdot 1) = 2 \neq 0$$

$\Longrightarrow \qquad \{\vec{V}_1, \vec{V}_2, \vec{V}_3,\}$ is a basis of $\mathbb{R}^3 \Longrightarrow \qquad$ actually it is an **orthogonal basis**.

An **orthonormal basis** is a set of mutually orthogonal basis vectors where each of them has the unit norm.

$\Longrightarrow \qquad$ we can go from $\{\vec{V}_1, \vec{V}_2, \vec{V}_3,\}$ to an orthonormal basis:

$$\hat{V}_1 = (0,1,0)\,, \qquad \hat{V}_2 = \frac{1}{\sqrt{2}} \cdot (1,0,1)\,, \qquad \hat{V}_3 = \frac{1}{\sqrt{2}} \cdot (1,0,-1)$$