Lecture 33 (section 6.3)

Section 6.3 — continued

$\Longrightarrow$ Why care about orthogonal basis?

Let V be a vector space and

$$S = \{\vec{V}_1\,,\, \vec{V}_2\,,\, \cdots\,,\, \vec{V}_n\}$$

a basis of V. Then any vector $\vec{u} \in V$ can be represented uniquely as

$$\vec{u} = c_1 \cdot \vec{V}_1 + c_2 \cdot \vec{V}_2 + \cdots + c_n \cdot \vec{V}_n$$

$\{c_1, c_2, \cdots, c_n\}$ are the coordinates of $\vec{u}$ in the basis S.

If S is an orthogonal basis, then

$$\vec{u} = \underbrace{\frac{\langle \vec{u}, \vec{V}_1 \rangle}{||\vec{V}_1||^2}}_{=c_1} \cdot \vec{V}_1 + \underbrace{\frac{\langle \vec{u}, \vec{V}_2 \rangle}{||\vec{V}_2||^2}}_{=c_2} \cdot \vec{V}_2 + \cdots + \underbrace{\frac{\langle \vec{u}, \vec{V}_n \rangle}{||\vec{V}_n||^2}}_{=c_n} \cdot \vec{V}_n$$

If the basis is orthonormal $\Longrightarrow$

$$\vec{u} = \underbrace{\langle \vec{u}, \vec{V}_1 \rangle}_{c_1} \cdot \vec{V}_1 + \underbrace{\langle \vec{u}, \vec{V}_2 \rangle}_{c_2} \cdot \vec{V}_2 + \cdots + \underbrace{\langle \vec{u}, \vec{V}_n \rangle}_{c_n} \cdot \vec{V}_n$$

$\Longrightarrow$ Question: how do we produce an orthogonal basis starting from an arbitrary basis?

Theorem:

Let V be a vector space and W be a subspace of V.
Any vector $\vec{u} \in V$ can be decomposed (in a unique way)

$$\vec{u} = \vec{u}_1 + \vec{u}_2\,, \qquad \text{where} \qquad \vec{u}_1 \in W \qquad \text{and} \qquad \vec{u}_2 \in W^{\perp}$$

By definition,

$$\vec{u}_1 \;=\; \mathrm{proj}_W \vec{u}$$

is called a projection of $\vec{u}$ onto W.

$\Longrightarrow$ Example: if

$$\{\vec{V}_1\,,\,\vec{V}_2\,,\,\cdots\,,\,\vec{V}_k\}$$

is an orthogonal basis of W, then for $\vec{u} \in V$,

$$\mathrm{proj}_W \vec{u} = \frac{\langle \vec{u}, \vec{V}_1 \rangle}{||\vec{V}_1||^2} \cdot \vec{V}_1 + \frac{\langle \vec{u}, \vec{V}_2 \rangle}{||\vec{V}_2||^2} \cdot \vec{V}_2 + \cdots + \frac{\langle \vec{u}, \vec{V}_k \rangle}{||\vec{V}_k||^2} \cdot \vec{V}_k$$

$\Longrightarrow$

$$\vec{u}_2 = \vec{u} - \mathrm{proj}_W \vec{u} \;\in\; W^{\perp}$$

To show the last statement is true, choose $\vec{V}_i \in W$, then

$$\langle \vec{u}_2, \vec{V}_i \rangle = \langle \vec{u}, \vec{V}_i \rangle - \sum_{j=1}^{k} \frac{\langle \vec{u}, \vec{V}_j \rangle}{||\vec{V}_j||^2} \cdot \langle \vec{V}_j, \vec{V}_i \rangle \underbrace{=}_{\text{only } j=i \text{ is left}}$$

$$= \langle \vec{u}, \vec{V}_i \rangle - \frac{\langle \vec{u}, \vec{V}_i \rangle}{||\vec{V}_i||^2} \cdot \langle \vec{V}_i, \vec{V}_i \rangle = \langle \vec{u}, \vec{V}_i \rangle - \frac{\langle \vec{u}, \vec{V}_i \rangle}{||\vec{V}_i||^2} \cdot ||\vec{V}_i||^2 = \langle \vec{u}, \vec{V}_i \rangle - \langle \vec{u}, \vec{V}_i \rangle = 0$$

Let V be a vector space and

$$S_{original} = \{\vec{u}_1\,,\,\vec{u}_2\,,\,\cdots\,,\,\vec{u}_n\}$$

a basis of V. We want to convert it to an orthogonal basis

$$S = \{\vec{V}_1\,,\,\vec{V}_2\,,\,\cdots\,,\,\vec{V}_n\}$$

$\Longrightarrow$ Process:

- $$\vec{V}_1 = \vec{u}_1$$

- $$\vec{V}_2 = \vec{u}_2 - \text{proj}_{\vec{V}_1}\ \vec{u}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{V}_1\rangle}{||\vec{V}_1||^2}\cdot \vec{V}_1$$

  Note

  $$\langle \vec{V}_2, \vec{V}_1\rangle = 0 \qquad \text{because} \qquad \vec{V}_2 \in \{\vec{V}_1\}^{\perp}$$

- $$\vec{V}_3 = \vec{u}_3 - \text{proj}_{\{\vec{V}_1,\vec{V}_2\}}\ \vec{u}_3 = \vec{u}_3 - \left[\frac{\langle \vec{u}_3, \vec{V}_1\rangle}{||\vec{V}_1||^2}\cdot \vec{V}_1 + \frac{\langle \vec{u}_3, \vec{V}_2\rangle}{||\vec{V}_2||^2}\cdot \vec{V}_2\right]$$

  Note

  $$\langle \vec{V}_3, \vec{V}_1\rangle = \langle \vec{V}_3, \vec{V}_2\rangle = 0 \qquad \text{because} \qquad \vec{V}_2 \in \{\vec{V}_1, \vec{V}_2\}^{\perp}$$

- $$\vec{V}_4 = \vec{u}_4 - \text{proj}_{\{\vec{V}_1,\vec{V}_2,\vec{V}_3\}}\ \vec{u}_4$$

- $$\cdots$$

- $$\vec{V}_n = \vec{u}_n - \text{proj}_{\{\vec{V}_1,\vec{V}_2,\vec{V}_3,\cdots\vec{V}_{n-1}\}}\ \vec{u}_n$$