Lecture 8 (section 1.5)

Section 1.5 — elementary matrices; method for finding A^{-1}

$\Longrightarrow$ Elementary matrices:

E is an elementary matrix if E is obtained from I_n (the identity matrix) by performing a single elementary row operation.

Is elementary?

- $$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{yes} \qquad \Longrightarrow \qquad r_1 = 3 \cdot r_1^I$$

  row multiplication;

- $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{yes} \qquad \Longrightarrow \qquad r_1^I \leftrightarrow r_3^I$$

  row interchange;

- $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \qquad \text{yes} \qquad \Longrightarrow \qquad r_3 = r_3^I + (-4) \cdot r_1^I$$

  linear combination of 2 rows

$\Longrightarrow$ Note: every elementary matrix is invertible:

$$E \qquad E^{-1}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Theorem:

Let A be $n \times n$ (a square) matrix. The following statements are equivalent:

- A is invertible
- the homogeneous system

  $$A \cdot \overline{X} = \overline{0}$$

  has only zero solution
- The RREF of A is I_n
- A is a product of elementary matrices,

  $$A = E_1 \cdot E_2 \cdots E_k$$

$$\boxed{\begin{array}{l} \text{Algorithm for finding } A^{-1} \end{array}}$$

Algorithm for finding A^{-1}

- Form $n \times (2n)$ matrix:
$$\begin{bmatrix} A & | & I_n \end{bmatrix}$$
- perform elementary operations to bring the left half of this matrix to RREF
- repeat identical sequence of operations with the right half of the matrix
- if the left side is I_n, the right side is A^{-1}

$\Longrightarrow$ Example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Follow the algorithm:

$$\begin{bmatrix} A & | & I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \quad r_3 \to r_3 - r_1 \quad \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\to \quad r_3 \to r_3 - r_2 \quad \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \to \quad r_3 \to -\frac{1}{2} \cdot r_3 \quad \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \quad \begin{array}{c} r_1 \to r_1 - r_3 \\ r_2 \to r_2 - r_3 \end{array} \quad \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] = \begin{bmatrix} I_3 & | & A^{-1} \end{bmatrix}$$

$\Longrightarrow$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$\Longrightarrow$ Check that $A \cdot A^{-1} = A^{-1} \cdot A = I_3$