Lecture 10 (section 1.6)

Section 1.6 — more on linear systems and invertible matrices

$\Longrightarrow$ Theorem:

A linear system may have either:

- no solutions $\longrightarrow$ inconsistent
- unique solution $\longrightarrow$ consistent
- infinite number of solutions $\longrightarrow$ consistent

$\Longrightarrow$ Consider a system with an equal number of equation and unknowns:

$$\underbrace{A}_{n\times n \text{ matrix}} \cdot \underbrace{\overline{X}}_{n-\text{vector}} = \underbrace{\overline{b}}_{n-\text{vector}}$$

If A is invertible $\Longrightarrow$ there is a unique solution:

$$A \cdot \overline{X} = \overline{b}$$
$$A^{-1} \cdot \left(A \cdot \overline{X}\right) = A^{-1} \cdot \overline{b}$$
$$\underbrace{\left(A^{-1} \cdot A\right)}_{I_n} \cdot \overline{X} = A^{-1} \cdot \overline{b}$$
$$\overline{X} = A^{-1} \cdot \overline{b}$$

Example:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ x_1 + 8x_3 = 17 \end{cases}$$

$\Longrightarrow$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \qquad \overline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \overline{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

- Finding A^{-1}:

$$\begin{bmatrix} A & | & I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \to \quad \begin{array}{c} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - r_1 \end{array} \quad \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\to \quad \begin{array}{c} r_1 \to r_1 - 2r_2 \\ r_3 \to r_3 + 2r_2 \end{array} \quad \to \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \to \quad \begin{array}{c} r_1 \to r_1 + 9r_3 \\ r_2 \to r_2 - 3r_3 \\ r_3 \to -r_3 \end{array} \quad \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] = \begin{bmatrix} I_3 & | & A^{-1} \end{bmatrix} \qquad \Longrightarrow \qquad A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

- $$\overline{X} = \underbrace{\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}}_{A^{-1}} \cdot \underbrace{\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}}_{\overline{b}} = \begin{bmatrix} -40 \cdot 5 + 16 \cdot 3 + 9 \cdot 17 \\ 13 \cdot 5 + (-5) \cdot 3 + (-3) \cdot 17 \\ 5 \cdot 5 + (-2) \cdot 3 + (-1) \cdot 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\Longrightarrow$ unique solution:

$$\overline{X} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

The following statements are equivalent (A is $n \times n$ matrix):

- A is invertible
- an equation
$$A \cdot \overline{X} = \overline{0}$$
only has a zero solution
- The RREF of A is I_n
- A is a product of elementary matrices
- an equation
$$A \cdot \overline{X} = \overline{b}$$
is always consistent and
- $$A \cdot \overline{X} = \overline{b}$$
always has a unique solution.

always means for any vector $\overline{b}$