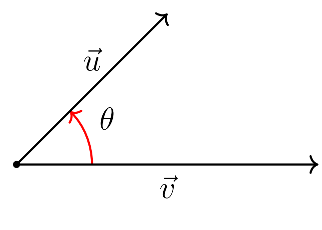
Lecture 17 (sections 3.2,3.3)

Section 3.2 continue — the dot product

In $\mathbb{R}^2$ or $\mathbb{R}^3$, consider two vectors $\vec{u}$ and $\vec{v}$:



$$\underbrace{\vec{u}\cdot\vec{v}}_{\text{dot product}} \quad = \quad ||\vec{u}||\cdot||\vec{v}||\cdot\cos\theta$$

If both $\vec{v} \neq 0$ and $\vec{u} \neq 0$ then

$$||\vec{u}|| \neq 0 \qquad \& \qquad ||\vec{v}|| \neq 0$$

thus

$$\vec{u}\cdot\vec{v} = 0 \qquad \Longrightarrow \qquad \cos\theta = 0 \qquad \Longrightarrow \qquad \theta = \{90^\circ \text{ or } 270^\circ\}$$

In this case we say that $\vec{u}$ and $\vec{v}$ are *mutually orthogonal*,

$$\vec{u} \ \perp \vec{v}$$

Compatible with above, in $\mathbb{R}^n$, for any two vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u}\cdot\vec{v} \ \stackrel{def}{\equiv} \ u_1\cdot v_1 + u_2\cdot v_2 + \cdots + u_n\cdot v_n$$

Properties:

- associativity:
$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$$
- commutativity:
$$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$$

$\Longrightarrow$ Problem. Find an angle between

$$\vec{u} = (3,0,1) \qquad \text{and} \qquad \vec{v} = (-1,2,1)$$

Solution:

$$||\vec{u}|| = \sqrt{3^2+0^2+1^2} = \sqrt{10}\,, \qquad ||\vec{v}|| = \sqrt{(-1)^2+2^2+1^2} = \sqrt{6}$$

$$\vec{u}\cdot\vec{v} = 3\cdot(-1)+0\cdot 2+1\cdot 1 = -2 = ||\vec{u}||\cdot||\vec{v}||\cdot\cos\theta = \sqrt{10}\cdot\sqrt{6}\cdot\cos\theta$$

$\Longrightarrow$

$$\cos\theta = \frac{-2}{\sqrt{10}\sqrt{6}} = -\frac{1}{\sqrt{15}} \qquad \Longrightarrow \qquad \theta = \arccos\left(-\frac{1}{\sqrt{15}}\right)$$

$\Longrightarrow$ In general, for two vectors in $\mathbb{R}^n$:

$$\vec{u}\cdot\vec{v} = \boxed{u_1\cdot v_1+\cdots u_n\cdot v_n} = ||\vec{u}||\cdot||\vec{v}||\cdot\cos\theta = \boxed{\sqrt{u_1^2+\cdots u_n^2}\cdot\sqrt{v_1^2+\cdots v_n^2}\cdot\cos\theta}$$

so

$$\cos\theta = \frac{u_1\cdot v_1+\cdots u_n\cdot v_n}{\sqrt{u_1^2+\cdots u_n^2}\cdot\sqrt{v_1^2+\cdots v_n^2}}$$

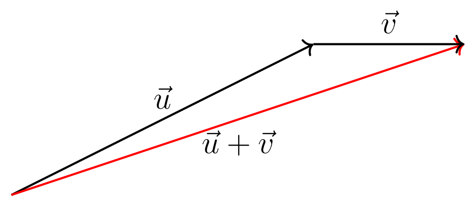
Theorem:

Triangular inequality

for any two vectors $\vec{u}$ and $\vec{v}$,

$$||\vec{u}+\vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$$

Indeed:



$\Longrightarrow$ Recall that the two vectors $\vec{u}$ and $\vec{v}$ are parallel if there exist a number a such that

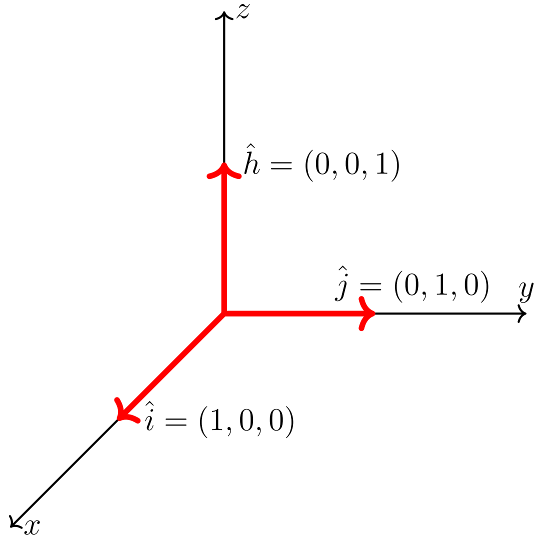
$$\vec{v} = a \cdot \vec{u}$$

Section 3.3 — orthogonality

If $\vec{u}, \vec{v} \neq 0$ and $\vec{u} \cdot \vec{v} = 0 \implies$

$$\vec{u} \perp \vec{v} \qquad \text{orthogonal}$$

Example:



$$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{h} = \hat{j} \cdot \hat{h} = 0 \qquad \implies$$

$$\{\hat{i}, \hat{j}, \hat{h}\} \qquad \text{are mutually orthogonal}$$

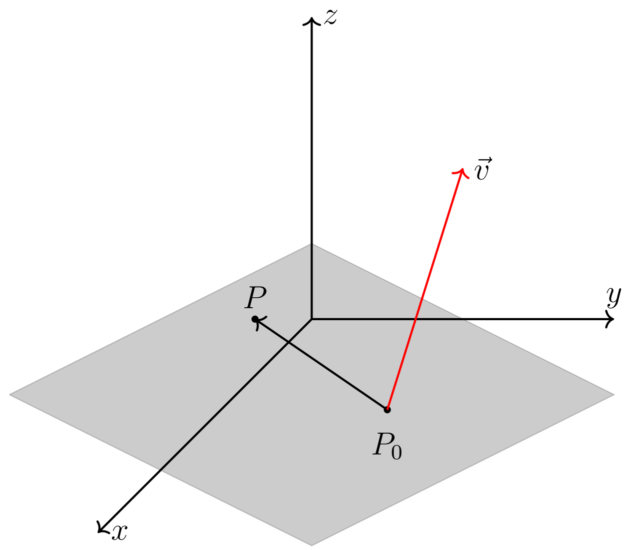
$\implies$ A standard basis in $\mathbb{R}^n$ is made of n mutually orthogonal vectors:

$$\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1\,, & i = j \\ 0\,, & \text{otherwise} \end{cases}$$

Why do we care about orthogonality?

$\implies$ Because it helps define complicated objects in space (like a *plane*) (other objects: *points*, *lines*)

$\Longrightarrow$ Equations of a plane in 3d:



$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \vec{v} = a \cdot \hat{i} + b \cdot \hat{j} + c \cdot \hat{h}$$

$P_0 = (x_0, y_0, z_0)$ ——— initial point of $\vec{v}$; a point on a plane

$P = (x, y, z)$ ——— any point on a plane

Definition:

Let

$$\vec{v} \perp \text{ plane} \qquad \Longleftrightarrow \qquad \vec{v} \perp \overrightarrow{P_0 P}, \qquad \text{for any } P \in \text{plane}$$

$\Longrightarrow$

$$\vec{v} \cdot \overrightarrow{P_0 P} = 0$$

$$a \cdot (x - x_0) + b \cdot (y - y_0) + c \cdot (z - z_0) = 0 \qquad \longleftarrow \qquad \text{equation for a plane}$$

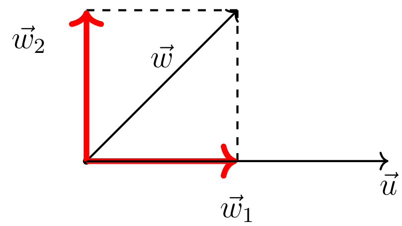
Alternatively:

$$\boxed{a \cdot x + b \cdot y + c \cdot z = d}, \qquad d \equiv a \cdot x_0 + b \cdot y_0 + c \cdot z_0$$

The projection theorem

$\Longrightarrow$ We are in $\mathbb{R}^n$; $\vec{u} \neq 0$ and $\vec{w}$ is any vector.

$\Longrightarrow$ We want:



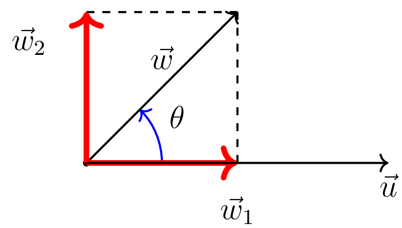
$$\vec{w} = \vec{w}_1 + \vec{w}_2 \qquad \text{(in a unique way)}$$

such that

$$\vec{w}_1 \; || \; \vec{u} \qquad \text{and} \qquad \vec{w}_2 \perp \vec{u}$$

$$\vec{w}_1 \overset{def}{\equiv} \underbrace{\mathrm{proj}_{\vec{u}}\vec{w}}_{\text{projection of } \vec{w} \text{ onto } \vec{u}}$$

Let θ be the angle between the vectors:



Note

$$\cos\theta = \frac{\vec{u} \cdot \vec{w}}{||\vec{u}|| \cdot ||\vec{w}||}$$

thus

$$||\vec{w}_1|| = ||\vec{w}|| \cdot \cos\theta = \frac{\vec{u} \cdot \vec{w}}{||\vec{u}||}$$

Because $\vec{w}_1 \; || \; \vec{u}$, the corresponding normalized vectors must be identical:

$$\hat{w}_1 = \hat{u} \qquad \Longleftrightarrow \qquad \frac{\vec{w}_1}{||\vec{w}_1||} = \frac{\vec{u}}{||\vec{u}||}$$

$\Longrightarrow$ uniquely,

$$\vec{w}_1 = \frac{||\vec{w}_1||}{||\vec{u}||} \cdot \vec{u} = \frac{\vec{u} \cdot \vec{w}}{||\vec{u}||} \cdot \frac{1}{||\vec{u}||} \cdot \vec{u} = \boxed{\frac{\vec{u} \cdot \vec{w}}{||\vec{u}||^2} \cdot \vec{u} = \mathrm{proj}_{\vec{u}}\vec{w}}$$

Note:

$$\vec{w} = \vec{w}_1 + \vec{w}_2 = \underbrace{\mathrm{proj}_{\vec{u}}\vec{w}}_{\vec{w}_1} + \vec{w}_2$$

Thus

$$\vec{w}_2 = \vec{w} - \mathrm{proj}_{\vec{u}}\vec{w} = \boxed{\vec{w} - \frac{\vec{u}\cdot\vec{w}}{||\vec{u}||^2}\cdot\vec{u}}$$

$\Longrightarrow$ Check at home that

$$\vec{w}_2 \cdot \vec{u} = 0$$

for any $\vec{w}$ and $\vec{u}$.