Lecture 6 (section 1.3)

Section 1.3

$\Longrightarrow$ Another matrix operation is matrix multiplication:

Consider two matricies A and B,

$$A: \quad m \times r \qquad (\text{m} - \text{rows}, \ \text{r} - \text{columns})$$
$$B: \quad r \times n \qquad (\text{r} - \text{rows}, \ \text{n} - \text{columns})$$

then $A \cdot B$ is a matrix of dimension $m \times n = (m \times r) \cdot (r \times n)$ defined so that

$$(A \cdot B)_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + \alpha_{ir} \cdot b_{rj}$$

for all $i = 1 \cdots m$ and $j = 1 \cdots n$

$$A = \begin{bmatrix} \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_{i1} \quad \cdots \quad a_{ir} \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix} \qquad \& \qquad B = \begin{bmatrix} \cdots & \cdots & b_{1j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & b_{rj} & \cdots & \cdots \end{bmatrix}$$

$\Longrightarrow$

$$A \cdot B = \begin{bmatrix} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots \quad \cdots \quad (A \cdot B)_{ij} \quad \cdots \quad \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix}$$

$$(A \cdot B)_{ij} \quad = \quad \begin{array}{c} a_{i1} \\ \times \\ b_{1j} \end{array} + \begin{array}{c} a_{i2} \\ \times \\ b_{2j} \end{array} + \cdots + \begin{array}{c} a_{ir} \\ \times \\ b_{rj} \end{array}$$

$\Longrightarrow$ Example:

$$A = \begin{bmatrix} 1 & -3 & -1 \\ 2 & 4 & 5 \end{bmatrix}, \qquad \dim(A): \; 2 \times 3$$

$$B = \begin{bmatrix} 1 & 0 & 2 & 5 \\ -1 & 1 & 3 & 1 \\ 2 & -2 & -1 & -3 \end{bmatrix}, \qquad \dim(B): \; 3 \times 4$$

$$C = A \cdot B\,, \qquad \dim(C) = (2 \times 3) \cdot (3 \times 4) = 2 \times 4$$

$$C = A \cdot B = \begin{bmatrix} \color{red}{1} & \color{red}{-3} & \color{red}{-1} \\ 2 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} \color{blue}{1} & 0 & 2 & 5 \\ \color{blue}{-1} & 1 & 3 & 1 \\ \color{blue}{2} & -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} \color{green}{2} & & & \\ & & & \end{bmatrix}$$

$$\color{green}{2} = \times \begin{array}{c} (\color{red}{1,\; -3,\; -1}) \\ \\ (\color{blue}{1,\; -1,\; 2}) \end{array} = 1 \cdot 1 + (-3) \cdot (-1) + (-1) \cdot 2 = 1 + 3 - 2$$

$\Longrightarrow$ Continue with the rest:

$$C = A \cdot B = \begin{bmatrix} 2 & -1 & -6 & 5 \\ 8 & -6 & 11 & -1 \end{bmatrix}$$

$\Longrightarrow$ Note: $B \cdot A$ is **not defined** since

$$(3 \times \underset{\uparrow\text{different}}{4)} \cdot \underset{\uparrow}{(2} \times 3)$$

$\Longrightarrow$ Comment: even if defined, in general,

$$A \cdot B \neq B \cdot A$$

Examples:

- $$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \neq \qquad B \cdot A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

- $$A = \begin{bmatrix} 1 & -1 & 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

$$\dim(A) = 1 \times 4, \qquad \dim(B) = 4 \times 1 \implies \dim(A \cdot B) = 1 \times 1, \qquad \dim(B \cdot A) = 4 \times 4$$

$$A \cdot B = \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 1 + 2 \cdot (-3) + 3 \cdot 4 = 7 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 & 6 \\ 1 & -1 & 2 & 3 \\ -3 & 3 & -6 & -9 \\ 4 & -4 & 8 & 12 \end{bmatrix}$$

$\Longrightarrow$ Why do we define matrix multiplication the way we did?

$\Longrightarrow$ ... many answers ... for us:

to be able to write system of linear equations in matrix form

Example:

$$\begin{cases} 3x - 2y + z = 7 \\ 4x - 6y + 2z = 3 \end{cases}$$

3 unknows and 2 equations;

- coefficient matrix of the system, A, $\dim(A) = 2 \times 3$:

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

- colum vector of unknows, $\overline{X}$, $\dim(\overline{X}) = 3 \times 1$

$$\overline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- colum vector of the system, $\overline{b}$, $\dim(\overline{b}) = 2 \times 1$

$$\overline{b} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$\Longrightarrow$

$$A \cdot \overline{X} = \overline{b} \qquad \Leftrightarrow \qquad \begin{cases} 3x - 2y + z = 7 \\ 4x - 6y + 2z = 3 \end{cases}$$

Note, in matrix multiplication $A \cdot \overline{X}$:

$$(2 \times 3) \cdot (3 \times 1) = 2 \times 1 \ = \ \dim(\overline{b})$$

$\Longrightarrow$ Definition:

Transpose of a matrix A, $\dim(A) = m \times n$, is a new matrix A^T of dimension $\dim(A^T) = n \times m$ such that

$$\left(A^T\right)_{ij} = A_{ji}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -5 & 6 \end{bmatrix} \qquad \Longrightarrow \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & -5 \\ -1 & 6 \end{bmatrix}$$

$\Longrightarrow$ Steps to construct a transpose matrix:

- take each column of A
- convert it into row (rotate counterclockwise 90°)
- stash at the bottom

$\Longrightarrow$ Note: for any matrix A,

$$\left(A^T\right)^T = A$$