Lecture 13 (sections 2.2,2.3)

Section 2.2 — determinants by row reduction

Properties of determinants under elementary row operations:

- interchange 2 rows $\Longrightarrow$ determinant will change sign
- multiply a row by a constant $\Longrightarrow$ determinant is multiplied by the same constant
- perform on any row i:

$$r_i \to r_i + \text{const} \cdot r_j\,, \qquad j \neq i\,,$$

$\Longrightarrow$ the determinant is unchanged

$\Longrightarrow$ Examples: compute $\det(A)$

- $$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$\Longrightarrow$ transform A via elementary row operations:

$$\det(A) \underbrace{=}_{r_2 \to r_2 - r_3} \det \begin{bmatrix} 0 & 1 & 5 \\ 1 & -12 & 8 \\ 2 & 6 & 1 \end{bmatrix} \underbrace{=}_{r_3 \to r_3 - 2 \cdot r_2} \det \begin{bmatrix} 0 & 1 & 5 \\ 1 & -12 & 8 \\ 0 & 30 & -15 \end{bmatrix}$$

$$\underbrace{=}_{r_3 \leftarrow 15 r_3} 15 \det \begin{bmatrix} 0 & 1 & 5 \\ 1 & -12 & 8 \\ 0 & 2 & -1 \end{bmatrix} \underbrace{=}_{r_1 \leftrightarrow r_2} -15 \det \begin{bmatrix} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 0 & 2 & -1 \end{bmatrix} \underbrace{=}_{\text{cofactors wrt column 1}} -15 \cdot 1 \cdot C_{11}$$

$$= -15 \cdot M_{11} = -15 \cdot \det \begin{bmatrix} 1 & 5 \\ 2 & -1 \end{bmatrix} = -15 \cdot (-1 - 10) = 165$$

- $$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$\Longrightarrow$ transform A via elementary row operations:

$$\det(A) \underbrace{=}_{r_1 \to r_1 - 2 \cdot r_2} \det \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \underbrace{=}_{r_1 \leftrightarrow r_2} -\det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\underbrace{=}_{r_3 \to r_3 - 2 \cdot r_2 \ \& \ r_4 \to r_4 - r_2} -\det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \underbrace{=}_{r_4 \to r_4 + r_3} (-1) \cdot \det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} \qquad \Longleftarrow \text{UTM}$$

$\Longrightarrow$

$$\det(A) = (-1) \cdot \left(\underbrace{1 \cdot 1 \cdot (-1) \cdot 6}_{\text{det of UTM}} \right) = 6$$

Let A and B be $n \times n$ matrices.

- Then

$$\det(A \cdot B) \;=\; \det(A) \cdot \det(B)$$

$$\det(A^T) = \det(A)$$

- But

$$\det(A + B) \quad \underbrace{\neq}_{\text{in general}} \quad \det(A) + \det(B)$$

$\Longrightarrow$ Theorem:

Let A be $n \times n$ matrix. Then A is invertible if and only if

$$\det(A) \;\neq\; 0$$

$\Longrightarrow$

$$A \text{ is singular} \qquad \Longleftrightarrow \qquad \det(A) = 0$$