Lecture 14 (sections 2.3,1.8)

Section 2.3 — continuation

Consider $n \times n$ matrix
$$A = \{a_{ij}\}$$

Define an *adjoint* matrix $\mathrm{Adj}(A)$:
$$\mathrm{Adj}(A) \overset{def}{=} \{C_{ij}\}^T$$
where C_{ij} are cofactors corresponding to a_{ij}.

Then:
$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{Adj}(A)$$

Example:
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Longrightarrow \qquad \underbrace{M}_{\text{matrix of minors}} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$\Longrightarrow \qquad \underbrace{C}_{\text{matrix of cofactors}} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \qquad \Longrightarrow \qquad \mathrm{Adj}(A) = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Longrightarrow \qquad A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \boxed{\frac{1}{a\,d - b\,c} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Cramer's rule for solving a system of linear equation:

Let

$$A \cdot \overline{X} = \overline{b}$$

be a system of n-linear equations in n-unknowns.

$$A: \qquad n \times n \qquad \text{matrix}$$
$$\overline{X} = \{x_i\}: \qquad \text{vector of variables}$$
$$\overline{b} = \{b_i\}: \qquad \text{RHS vector}$$

Assume $\det(A) \neq 0 \Longrightarrow$ there is a unique solution of the system

Introduce n-new matrices:

- first column of A is replaced with $\overline{b} \Longrightarrow$ A_1:

$$A_1 = \underbrace{}_{\overline{b}}\underbrace{\phantom{a_{12} \cdots a_{1n}}}_{\text{original columns}} \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- second column of A is replaced with $\overline{b} \Longrightarrow$ A_2:

$$A_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}$$

- i's column of A is replaced with $\overline{b} \Longrightarrow$ A_i: $\cdots$

Then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_i = \frac{\det(A_i)}{\det(A)}, \qquad x_n = \frac{\det(A_n)}{\det(A)}$$

$\Longrightarrow$ Example:

$$\begin{cases} 7x_1 - 2x_2 = 3 \\ 3x_1 + x_2 = 5 \end{cases}$$

$\Longrightarrow$ Follow Cramer's rule:

- matrix form

$$\begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \Longleftrightarrow \qquad A \cdot \overline{X} = \overline{b}$$

- construct A_i:

$$A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$$

- compute determinants:

$$\det(A) = 7{\cdot}1-(-2){\cdot}3 = 13\,, \qquad \det(A_1) = 3{\cdot}1-(-2){\cdot}5 = 13\,, \qquad \det(A_2) = 7{\cdot}5-3{\cdot}3 = 26$$

- compute x_i:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{13}{13} = 1\,, \qquad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{26}{13} = 2$$

Section 1.8 — matrix transformations

$\Longrightarrow$ Before we used matrices to solve linear equations. Matrices are indispensable in describing transformations between vector spaces

- Consider n-tuple (a vector)

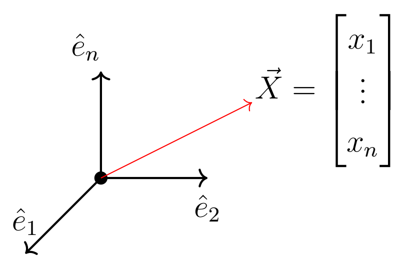
$$\vec{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where $x_1, \cdots x_n$ are real numbers.

- All such n-tuples form the vector space

$$\mathbb{R}^n$$

A useful way to think about components of $\vec{X}$ as coordinates of the point in $\mathbb{R}^n$:



- We can introduce standard basis vectors in $\mathbb{R}^n$:

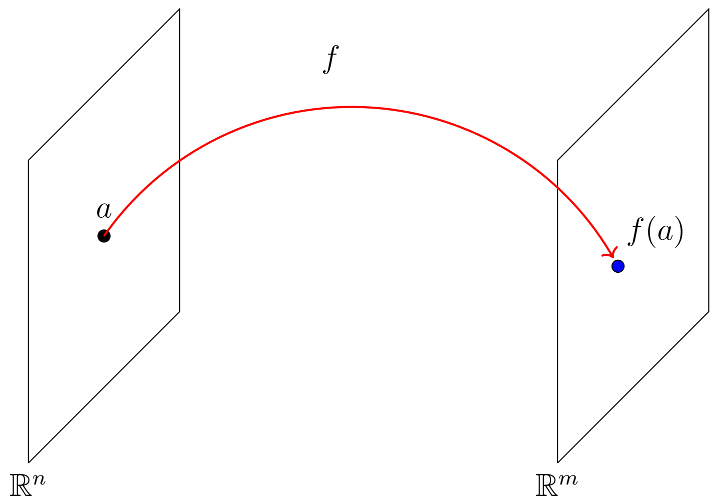
$$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots \qquad \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- Then, any vector

$$\vec{X} = \underbrace{x_1 \cdot \hat{e}_1 + x_2 \cdot \hat{e}_2 + \cdots + x_n \cdot \hat{e}_n}_{\text{decomposed in terms of the standard basis}}$$

$\Longrightarrow$ Recall, a function f is a rule that assigns a value in co-domain (the range) to a value in domain

Alternatively: a function f is a map between n-dimensional vector space $\mathbb{R}^n$ (domain) and m-dimensional vector space $\mathbb{R}^m$ (co-domain):



Alternatively: a function f is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

- In a special case $m = n$ the transformation is called an *operator* on $\mathbb{R}^n$
- we focus on maps arising from linear systems:

$$\underbrace{T_A}_{\text{transformation generated by } m\times n \text{ matrix } A} \quad : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

- $$T_A(\vec{X}) \stackrel{def}{\equiv} \underbrace{A \cdot \vec{X}}_{(m\times n)\cdot(n\times 1)} \longrightarrow \underbrace{(m \times 1) - \text{matrix}}_{\text{vector in } \mathbb{R}^m}$$

  note that the matrix multiplication in the definition of the linear transformation is well-defined.

$\Longrightarrow$ Examples:

- ①

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 4 & 3 \end{bmatrix} \qquad \begin{matrix} 3 & \times & 2 \\ \| & & \| \\ m & & n \end{matrix}$$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underbrace{2 \times 1}_{\text{vector in } \mathbb{R}^2\text{, a point on a plane}}$$

Thus,

$$T_A : \qquad \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

is the transformation that takes a point in with coordinates (x_1, x_2) on a plane and assigned to it a point with coordinates (w_1, w_2, w_3),

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = A \cdot \vec{X} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 3x_1 \\ 4x_1 + 3x_2 \end{bmatrix}$$

in $\mathbb{R}^3$.

A fancy way to say it is that T_A *embeds a plane into* 3*d space.*

- ② Note that if

$$A = 0_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad A \cdot \vec{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{X} \in \mathbb{R}^2$$

Such a transformation is called a zero transformation.

- ③ Consider an operator (a transformation $\mathbb{R}^n \to \mathbb{R}^n$) corresponding to

$$A = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly,

$$T_A(\vec{X}) = I_3 \cdot \vec{X} = \vec{X}\,, \qquad \vec{X} \in \mathbb{R}^3$$

Such an operator is called *identity operator* or *identity transformation*

Matrix transformations are linear:

- 1

$$T_A(k \cdot \vec{X}) \;=\; k \cdot T_A(\vec{X}) \qquad \underbrace{k \in \mathbb{R}}_{\text{a real number}}$$

- 2

$$T_A(\vec{X} + \vec{Y}) = T_A(\vec{X}) + T_A(\vec{Y})$$

(this follows from multiplication rules)