Lecture 26 (section 4.7)

Section 4.7 — continuation

- Problem 1

Find basis for the row space and the column space of the matrix:

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

$\Longrightarrow$ reduction to REF:

$$A \underbrace{\Longrightarrow}_{\substack{r_2 \to r_2 + 2r_1 \\ r_3 \to r_3 + r_1 \\ r_4 \to r_4 + 3r_1}} \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 6 & 1 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{\substack{r_3 \to r_3 - r_2 \\ r_4 \to r_4 - 2r_2}}$$

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_4 \to r_4 - r_3} \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ Basis of the row space of A^{REF}:

$$S_{row}^{REF} = \{(1, -2, 5, 0, 3)\,,\, (0, 1, 3, 0, 0)\,,\, (0, 0, 0, 1, 0)\}$$

The row space **does not change** under elementary row operations, so

$$S_{row}^{original} = S_{row}^{REF} = \{(1, -2, 5, 0, 3)\,,\, (0, 1, 3, 0, 0)\,,\, (0, 0, 0, 1, 0)\}$$

We can verify that any row of the original matrix can be expressed as a linear combination of the elements in S_{row}^{REF}. For example:

$$r_2^{orginal} = (-2, 5, -7, 0, -6) = (-2) \cdot (1, -2, 5, 0, 3) + 1 \cdot (0, 1, 3, 0, 0) + 0 \cdot (0, 0, 0, 1, 0)$$

$$r_3^{orginal} = (-1,3,-2,1,-3) = (-1)\cdot(1,-2,5,0,3) + 1\cdot(0,1,3,0,0) + 1\cdot(0,0,0,1,0)$$
$$r_4^{orginal} = (-3,8,-9,1,-9) = (-3)\cdot(1,-2,5,0,3) + 2\cdot(0,1,3,0,0) + 1\cdot(0,0,0,1,0)$$

$\Longrightarrow$ Basis for the column space of A^{REF}:

$$S_{column}^{REF} = \left\{ \underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{c_1^{REF}}, \underbrace{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}}_{c_2^{REF}}, \underbrace{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}_{c_4^{REF}} \right\}$$

The column space **<u>does change</u>** under elementary row operations, so

$$S_{column}^{original} \neq S_{column}^{REF}$$

However, we can identify basis vectors in $S_{column}^{original}$ by picking the columns of the original matrix A with the same index as in S_{column}^{REF}:

$$S_{column}^{original} = \left\{ \underbrace{\begin{bmatrix}1\\-2\\-1\\-3\end{bmatrix}}_{c_1^{original}}, \underbrace{\begin{bmatrix}-2\\5\\3\\8\end{bmatrix}}_{c_2^{orignal}}, \underbrace{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}_{c_4^{original}} \right\}$$

We can verify that any column of the original matrix can be expressed as a linear combination of the elements in $S_{column}^{original}$. For example:

$$c_3^{original} = \begin{bmatrix}5\\-7\\-2\\-9\end{bmatrix} = 11\cdot c_1^{original} + 3\cdot c_2^{original} + 0\cdot c_4^{original}$$

$$c_5^{original} = \begin{bmatrix}3\\-6\\-3\\-9\end{bmatrix} = 3\cdot c_1^{original} + 0\cdot c_2^{original} + 0\cdot c_4^{original}$$

Problem 2

Find 3×3 matrix whose null space is

- a point:

$$\begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\Longrightarrow}_{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow$ the only solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

- a line:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

leading : x_1, x_2

free : x_3

$\Longrightarrow$

$$\dim(\text{null space}) = \text{number of free variables} = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}, \qquad s \in \mathbb{R}$$

$\Longrightarrow$ z-axis

- a plane:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{leading} : x_1$$

$$\text{free} : x_2, x_3$$

$\Longrightarrow$

$$\dim(\text{null space}) = \text{number of free variables} \quad = \; 2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s - 2t \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}$$

$\Longrightarrow$ a plane perpendicular to $(1, -1, 2)$ vector:

$$x - y + 2z = 0 \qquad \Longleftrightarrow \qquad (x, y, z) \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0$$