Lecture 9 (section 1.6)

Solving a system of LE using the inverse matrix

$\Longrightarrow$ Consider:

$$\begin{cases} 3x - 4y = 3 \\ 4x - 5y = 5 \end{cases}$$

$\Longrightarrow$ representing above in a matrix form:

$$A \cdot \overline{X} = \overline{b}$$

$$A = \begin{bmatrix} 3 & -4 \\ 4 & -5 \end{bmatrix}, \qquad \overline{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \overline{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

- Assume that A is invertible; then $\exists\ A^{-1}$ such that $A^{-1} \cdot A = I_2 \Longrightarrow$

$$\begin{aligned} A^{-1} \cdot \left(A \cdot \overline{X}\right) &= A^{-1} \cdot \overline{b} \\ \left(A^{-1} \cdot A\right) \cdot \overline{X} &= A^{-1} \cdot \overline{b} \\ I_2 \cdot \overline{X} &= A^{-1} \cdot \overline{b} \\ \overline{X} &= A^{-1} \cdot \overline{b} \end{aligned}$$

- In our case:

$$\det(A) = 3 \cdot (-5) - (-4) \cdot 4 = -15 + 16 = 1 \neq 0$$

A is invertible:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1} \cdot \begin{bmatrix} -5 & 4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -4 & 3 \end{bmatrix}$$

- Unique (by construction and because inverse is unique) solution:

$$\overline{X} = A^{-1} \cdot \overline{b} = \begin{bmatrix} -5 & 4 \\ -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \Longrightarrow \qquad \begin{aligned} x &= 5 \\ y &= 3 \end{aligned}$$

Properties of invertible matrices

- If A & B are $n \times n$ invertible matrices, then $A \cdot B$ is also invertible and

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

Proof:

$$(A \cdot B)^{-1} \cdot (A \cdot B) \stackrel{?}{=} B^{-1} \cdot \underbrace{A^{-1} \cdot A}_{I_n} \cdot B = B^{-1} \cdot I_n \cdot B = B^{-1} \cdot B = I_n$$

Similarly,

$$(A \cdot B) \cdot (A \cdot B)^{-1} \stackrel{?}{=} A \cdot \underbrace{B \cdot B^{-1}}_{I_n} \cdot A^{-1} = A \cdot I_n \cdot A^{-1} = A \cdot A^{-1} = I_n$$

Note, if $B = A$, $\quad A^2 \stackrel{def}{\equiv} A \cdot A \quad$ is invertible; similarly:

$$A^k \stackrel{def}{\equiv} \underbrace{A \cdot A \cdots A}_{k- \text{ times}}, \qquad \left(A^k\right)^{-1} = \left(A^{-1}\right)^k$$

- If A is invertible, A^T is invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Recall,

$$(A \cdot B)^T = B^T \cdot A^T, \qquad \text{so}$$

$$\left(A^T\right)^{-1} \cdot A^T \stackrel{?}{=} \left(A^{-1}\right)^T \cdot A^T = \left(A \cdot A^{-1}\right)^T = I_n^T = I_n$$

Similarly,

$$A^T \cdot \left(A^T\right)^{-1} \stackrel{?}{=} A^T \cdot \left(A^{-1}\right)^T = \left(A^{-1} \cdot A\right)^T = I_n^T = I_n$$

$\Longrightarrow$ Problem ①

Find the inverse of:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$$

Follow the algorithm:

$$\left[A \;\mid\; I_4\right] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right] \to \quad \begin{array}{c} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_1 \\ r_4 \to r_4 - r_1 \end{array} \quad \to$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right] \to \quad \begin{array}{c} r_3 \to r_3 - r_2 \\ r_4 \to r_4 - r_2 \\ r_2 \to \frac{1}{3} \cdot r_2 \end{array} \quad \to$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right] \to \quad \begin{array}{c} r_4 \to r_4 - r_3 \\ r_3 \to \frac{1}{5} \cdot r_3 \end{array} \quad \to$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right] \to \quad r_4 \to \frac{1}{7} \cdot r_4 \quad \to$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{array}\right] = \left[I_4 \;\mid\; A^{-1}\right] \quad \Longrightarrow \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

$\Longrightarrow$ Problem 2

Given

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

find an elementary operation E such that $E \cdot A = B$

- Long way: assuming A^{-1} exists,

$$\begin{aligned} (E \cdot A) \cdot A^{-1} &= B \cdot A^{-1} \\ E \cdot \underbrace{\left(A \cdot A^{-1}\right)}_{I_3} &= B \cdot A^{-1} \\ E \cdot I_3 &= B \cdot A^{-1} \\ E &= B \cdot A^{-1} \end{aligned}$$

- Cheat: note that B is obtained from A by a single elementary operation $r_1 \leftrightarrow r_3$; elementary matrix $E_{1\leftrightarrow 3}$ responsible for such an interchange is:

$$\left[\underbrace{\begin{matrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3}\right] \rightarrow \qquad r_1 \leftrightarrow r_3 \qquad \rightarrow \left[\underbrace{\begin{matrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{matrix}}_{B} \;\middle|\; \underbrace{\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}}_{E_{1\leftrightarrow 3}}\right]$$

$\Longrightarrow$

$$E_{1\leftrightarrow 3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ Check at home:

$$E_{1\leftrightarrow 3} \cdot A = B$$