Lecture 35 (extra)

Section 3.5 cross-product

$\Longrightarrow$ Recall **dot product**:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$
$$(\vec{u}_1 + \vec{u}_2) \cdot \vec{v} = \vec{u}_1 \cdot \vec{v} + \vec{u}_2 \cdot \vec{v}$$
$$(\lambda \cdot \vec{u}) \cdot \vec{v} = \lambda \cdot (\vec{u} \cdot \vec{v})$$

"dot product" $\Longrightarrow$ a bilinear operation that takes 2 vectors in $\mathbb{R}^n$ and assigns a number in $\mathbb{R}$:

$$u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

$\Longrightarrow$ **Cross-product** is also a bilinear operations; but it not not defined in arbitrary $\mathbb{R}^n$. In $\mathbb{R}^3$ this operation takes 2 vectors and assigns to them a new vector in $\mathbb{R}^3$. This is an *antisymmetric/anti-commutative* operation:

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

The bilinear properties:

$$(\vec{u}_1 + \vec{u}_2) \times \vec{v} = \vec{u}_1 \times \vec{v} + \vec{u}_2 \times \vec{v}$$
$$(\lambda \cdot \vec{u}) \times \vec{v} = \lambda \cdot (\vec{u} \times \vec{v})$$

Computation of the cross-product

Let $\{\hat{i}, \hat{j}, \hat{h}\}$ be the standard orthonormal basis in $\mathbb{R}^3$:

$$\vec{u} = u_1 \cdot \hat{i} + u_2 \cdot \hat{j} + u_3 \cdot \hat{h}$$

$$\vec{v} = v_1 \cdot \hat{i} + v_2 \cdot \hat{j} + v_3 \cdot \hat{h}$$

$$\vec{u} \times \vec{v} \overset{def}{\equiv} \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{h} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \hat{i} \cdot \det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} - \hat{j} \cdot \det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} + \hat{h} \cdot \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$
$$= (u_2 v_3 - u_3 v_2) \cdot \hat{i} + (u_3 v_1 - u_1 v_3) \cdot \hat{j} + (u_1 v_2 - u_2 v_1) \cdot \hat{h}$$

■ Example. Given

$$\vec{u} = (1, 2, 3)\,, \qquad \vec{v} = (-1, 0, 5)$$

compute $\vec{u} \times \vec{v}$

$\Longrightarrow$

$$\vec{u} \times \vec{v} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{h} \\ 1 & 2 & 3 \\ -1 & 0 & 5 \end{bmatrix} \overbrace{=}^{\text{3rd row}} (-1) \cdot \det \begin{bmatrix} \hat{j} & \hat{h} \\ 2 & 3 \end{bmatrix} + 5 \cdot \det \begin{bmatrix} \hat{i} & \hat{j} \\ 1 & 2 \end{bmatrix} = -3\hat{j} + 2\hat{h} + 10\hat{i} - 5\hat{j}$$

$$= 10\hat{i} - 8\hat{j} + 2\hat{h} = \boxed{(10, -8, 2)}$$

$\Longrightarrow$ Cross-product properties

- $$\vec{u} \times \vec{u} = \vec{0}$$

- $$\vec{u} \times \vec{v} = \vec{0}$$

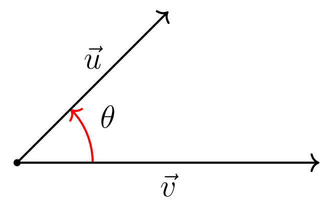
  if and only if two vectors are collinear, *i.e.*, parallel to each other: $\vec{u} || \vec{v}$

- $$\hat{i} \times \hat{j} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{h} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \overbrace{=}^{\text{1st row}} \hat{h} \det I_2 = \hat{h} \qquad \Longrightarrow \qquad \boxed{\hat{i} \times \hat{j} = \hat{h}}$$

  Similarly,

  $$\boxed{\hat{j} \times \hat{h} = \hat{i}}, \qquad \boxed{\hat{h} \times \hat{i} = \hat{j}}$$

$\Longrightarrow$ In general:



$$\vec{v} \times \vec{u} = ||\vec{v}|| \cdot ||\vec{u}|| \cdot \sin\theta \ \cdot \odot$$

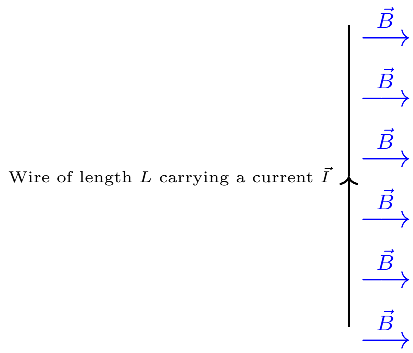
where $\odot$ is a vector directed **out of** the plane spanned by the vectors $\vec{v}$ and $\vec{u}$.

$\Longrightarrow$ Note:

$$\vec{u} \times \vec{v} = ||\vec{v}|| \cdot ||\vec{u}|| \cdot \sin\theta \ \cdot \otimes$$

where $\otimes = -\odot$ is a vector directed **into** the plane spanned by the vectors $\vec{v}$ and $\vec{u}$.

$\Longrightarrow$ Application:



$\vec{B}$ is the magnetic field.

$\Longrightarrow$ According to theory of electro-magnetism, there is a force $\vec{F}$ acting on the wire:

$$\vec{F} = L \cdot \vec{I} \times \vec{B} = L \cdot I \cdot B \cdot \sin 90^\circ \cdot \otimes = ILB \cdot \otimes$$

$\Longrightarrow$ Consequence of this force: