Lecture 11 (sections 1.6,1.7)

$\cdots$ Continue with

$$A \cdot \overline{X} = \overline{b}$$

$\Longrightarrow$ what if either:

- A is not a square matrix?
- A is a square matrix but is not invertible?

$\Longrightarrow$ then linear system is consistent only for some choice of $\overline{b}$.

Example:

$$\begin{cases} x_1 + x_2 + 2x_3 = b_1 \\ x_1 + x_3 = b_2 \\ 2x_1 + x_2 + 3x_3 = b_3 \end{cases}$$

$\Longrightarrow$ we determine consistency conditions on $\overline{b}$ using augmented matrix approach:

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \rightarrow \quad \begin{matrix} r_2 \rightarrow r_2 - r_1 \\ r_3 \rightarrow r_3 - 2r_1 \end{matrix} \quad \rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} r_3 \rightarrow r_3 - r_2 \\ r_2 \rightarrow -r_2 \end{matrix} \quad \rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} \quad \Longleftarrow \quad \text{REF} - \text{like}$$

"REF-like" because the last row (r_3) is not entirely zero in general.

$\Longrightarrow$ focus on r_3: system is consistent only if

$$b_3 - b_1 - b_2 = 0$$

Assuming

$$b_3 - b_1 - b_2 = 0$$

$\Longrightarrow$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \qquad r_1 \to r_1 - r_2 \qquad \to \left[\begin{array}{cccc} 1 & 0 & 1 & b_2 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longleftarrow \text{RREF}$$

- leading variables: x_1, x_2
- free variable: x_3

$\Longrightarrow$ There are infinitely many solutions for any

$$\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 = b_1 + b_2 \end{bmatrix}$$

parameterized by $x_3 = s$:

$$\overline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_2 - s \\ b_1 - b_2 - s \\ s \end{bmatrix}$$

However, if

$$b_3 - b_1 - b_2 \neq 0$$

$\Longrightarrow$ the system of linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 = b_1 \\ x_1 + x_3 = b_2 \\ 2x_1 + x_2 + 3x_3 = b_3 \end{cases}$$

is inconsistent (there is no solution)

Section 1.7 — diagonal, triangular and symmetric matrices

A square matrix A is diagonal if the only non-zero elements are on the main diagonal:

$$\underbrace{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{diagonal matricies}}$$

$\Longrightarrow$ Note:

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \qquad B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In general, if

$$A = \underbrace{\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}}_{n\times n \text{ diagonal matrix}} \quad \stackrel{def}{\equiv} \quad \operatorname{diag}(a_1, \cdots a_n)$$

then

$$A^k = \underbrace{A \cdots A}_{k-\text{times}} = \underbrace{\begin{bmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{bmatrix}}_{n\times n \text{ diagonal matrix}}$$

Lemma:

Diagonal matrix is invertible if all diagonal matrix elements are non-zero:

$$A = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} \qquad \equiv \qquad \mathrm{diag}(a_1, \cdots a_n)$$

If

$$a_1 \cdot a_2 \cdots a_n \neq 0$$

then

$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_n} \end{bmatrix}$$

A square matrix is upper triangular (UTM) if all elements below the main diagonal are zero

$$A^{UTM} = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

A square matrix is lower triangular (LTM) if all elements above the main diagonal are zero

$$A^{LTM} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 7 & -3 & 1 & 3 \end{bmatrix}$$

$\Longrightarrow$ Properties:

- Transpose of a LTM is an UTM:

$$\left(A^{LTM}\right)^T = B^{UTM}$$

- Transpose of UTM is LTM:

$$\left(A^{UTM}\right)^T = B^{LTM}$$

- A triangular matrix (LTM or UTM)

$$A = \{a_{ij}\}\,, \qquad \begin{array}{c} i = 1 \cdots n \\ j = 1 \cdots n \end{array}$$

  is invertible if all of its diagonal elements are non-zero:

$$a_{11} \cdot a_{22} \cdots a_{nn} \neq 0$$

- Products of LTMs is an LTM; products of UTMs is an UTM

- Inverses of LTMs or UTMs (if exist) are triangular matrices of the same type

A is a symmetric matrix if

$$A^T = A$$

$\Longrightarrow$ Properties:

- if A and B are symmetric $n \times n$ matrices and $k \in \mathbb{R}$, then the following matrices:

$$A + B\,, \qquad A - B\,, \qquad k \cdot A$$

  are symmetric.

- Note: if A and B are symmetric, $A \cdot B$ is not necessarily symmetric. Indeed:

$$(A \cdot B)^T = B^T \cdot A^T = B \cdot A \qquad \underbrace{=}_{\text{only if the matricies commute}} \qquad A \cdot B$$

  Example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}}_{A \cdot B} \Longleftarrow \qquad (A \cdot B)^T \neq A \cdot B$$

$$\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}}_{B \cdot A} \Longleftarrow \qquad B \cdot A \neq A \cdot B$$

- If A is a symmetric invertible matrix, A^{-1} is symmetric

- Note that for any $n \times n$ matrix A, matrices

$$A \cdot A^T\,, \qquad \text{and} \qquad A^T \cdot A$$

  are symmetric.

  Indeed:

$$\left(A \cdot A^T\right)^T = \left(A^T\right)^T \cdot A^T = A \cdot A^T$$
$$\left(A^T \cdot A\right)^T = A^T \cdot \left(A^T\right)^T = A^T \cdot A$$