Lecture 28 (sections 4.9,5.1)

Section 4.9 — basic matrix transformations in $\mathbb{R}^2$ ($\mathbb{R}^3$)

$$\mathbb{R}^2 : \qquad \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or} \qquad (x, y)$$

2-component vector

$\Longrightarrow$ Matrix transformations/operations: A is 2×2 matrix,

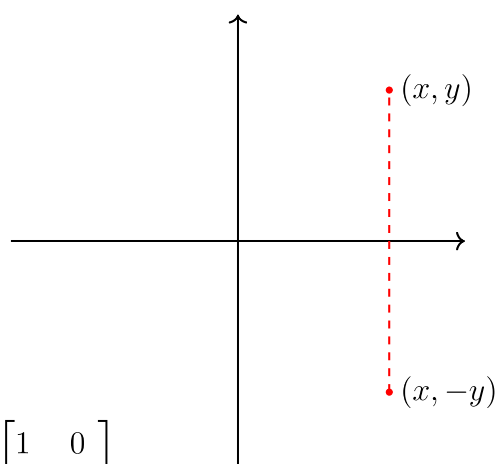
$$T_A : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$\Longrightarrow$ Specific operations:

- reflection about x-axis

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{T_{R_x}} \begin{bmatrix} x \\ -y \end{bmatrix}$$



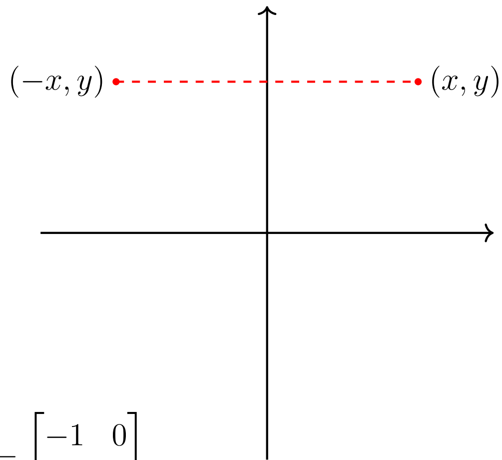
$$T_{R_x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

$T_{R_x} \Longrightarrow$ matrix of reflections about x-axis; $\det(T_{R_x}) = -1$

- reflection about y-axis

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{T_{R_y}} \begin{bmatrix} -x \\ y \end{bmatrix}$$



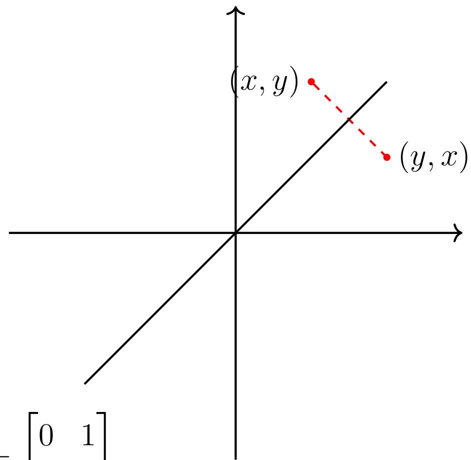
$$T_{R_y} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

$T_{R_y} \Longrightarrow$ matrix of reflections about y-axis; $\det(T_{R_y}) = -1$

- reflection about $x = y$ line

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{T_{R_{x=y}}} \begin{bmatrix} y \\ x \end{bmatrix}$$

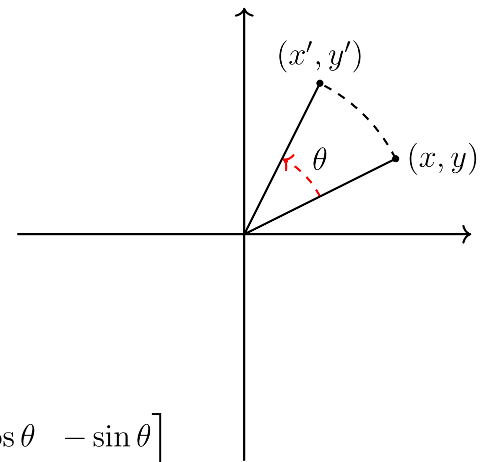


$$T_{R_{x=y}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

$T_{R_{x=y}} \Longrightarrow$ matrix of reflections about $x = y$ line; $\det(T_{R_{x=y}}) = -1$

- rotation operators:



$$T_{R_\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$T_{R_\theta} \Longrightarrow$ matrix of rotation about the origin by a degree θ counterclockwise for $\theta > 0$

$\Longrightarrow$ Note: for $\theta = 90°$,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{T_{R_{90°}}} \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

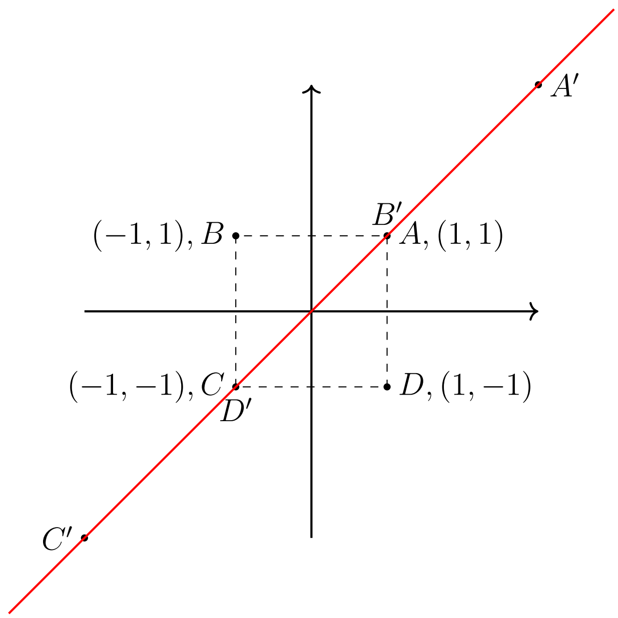
$\Longrightarrow$ you can check other points

- Example ①

$\Longrightarrow$ What if A is singular, *i.e.*, $\det(A) = 0$?

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x + 2y \end{bmatrix}$$



$$\begin{array}{rcll} A & \to & A' & (3,3) \\ B & \to & B' & (1,1) \\ C & \to & C' & (-3,-3) \\ D & \to & D' & (-1,-1) \end{array}$$

$\Longrightarrow$ by a singular transformation,

$$\text{a square} \quad \Longrightarrow \quad \text{a line}$$

- Example 2

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}$$

$\Longrightarrow$ This will transform any point or shape into a single point: the origin $(0, 0)$

Let A be $n \times n$ matrix and $\vec{V} \in \mathbb{R}^n$.
$\vec{V} \neq \vec{0}$ is an eigenvector of A with an associated eigenvalue λ if

$$A \cdot \vec{V} = \lambda \cdot \vec{V}$$

$\Longrightarrow$ When is $\vec{V}$ an eigenvector?

$$A \cdot \vec{V} = \lambda \cdot \vec{V} \qquad \Longleftrightarrow \qquad A \cdot \vec{V} = \lambda \cdot I_n \cdot \vec{V} \qquad \Longrightarrow$$

$$\boxed{(A - \lambda \cdot I_n) \cdot \vec{V} = \vec{0}}$$

For $\vec{V} \neq \vec{0} \Longrightarrow$ this system should have a nonzero solution $\Longrightarrow$ $(A - \lambda \cdot I_n)$ must be a singular matrix $\Longrightarrow$

$$\boxed{\det(A - \lambda \cdot I_n) = 0} \quad \Longrightarrow \quad \text{characteristic equation}$$

Characteristic equation is used to find the eigenvalues of A

■ Example.

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$\Longrightarrow$ characteristic equation:

$$0 = \det(A - \lambda \cdot I_2) = \det\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

$$0 = (1-\lambda) \cdot (3-\lambda) - 4 \cdot 2 = 3 - 4\lambda + \lambda^2 - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5) \cdot (\lambda + 1)$$

$\Longrightarrow$ eigenvalues

$$\{\lambda_1 = -1\,, \qquad \lambda_2 = 5\}$$

next: find eigenvectors