Lecture 19 (sections 4.1,4.2)

Section 4.1 — continuation

$\Longrightarrow$ More vector spaces:

$$\mathcal{F} \equiv \Big\{ f : (-\infty, \infty) \to \mathbb{R} \Big\}$$

Is this a vector space?

■ Let $f, g \in \mathcal{F}$

$$\text{[addition]} \qquad (f+g)(x) \overset{def}{\equiv} f(x) + g(x)$$

$$\text{[multiplication]} \qquad (k \cdot f)(x) \overset{def}{\equiv} k \cdot f(x)$$

All axioms are satisfied;

"0" function:

$$0(x) \overset{def}{\equiv} 0 \qquad \text{for } x \in (-\infty, \infty)$$

$\mathcal{F}$ is a real (scalar multiplication by a real number) vector space

Section 4.2 — subspaces

$\Longrightarrow$ Let V be a real vector space. W is a subset of V.

Definition:

W is a subspace of V if W is a vector space itself with operations inherited from V.

$\Longrightarrow$ Simplified definition of a subspace:

- V is a vector space
- W is a subset of V: $W \in V$
- for any $u, v \in W$

$$u + v \in W$$

- for any $u \in W$ and $k \in \mathbb{R}$,

$$k \cdot u \in W$$

$\Longrightarrow$ W is closed under additions and multiplications

- Example: $V=\mathbb{R}^2$

Consider

$$W=\{(x,y)\in\mathbb{R}^2 \text{ such that } y=3x\}$$

i.e.,

$$W=\{(x,3x)\,, \qquad x\in\mathbb{R}\}$$

Let $u,v\in W \implies$

$$u=(x_1,3x_1)\,, \qquad v=(x_2,3x_2)$$

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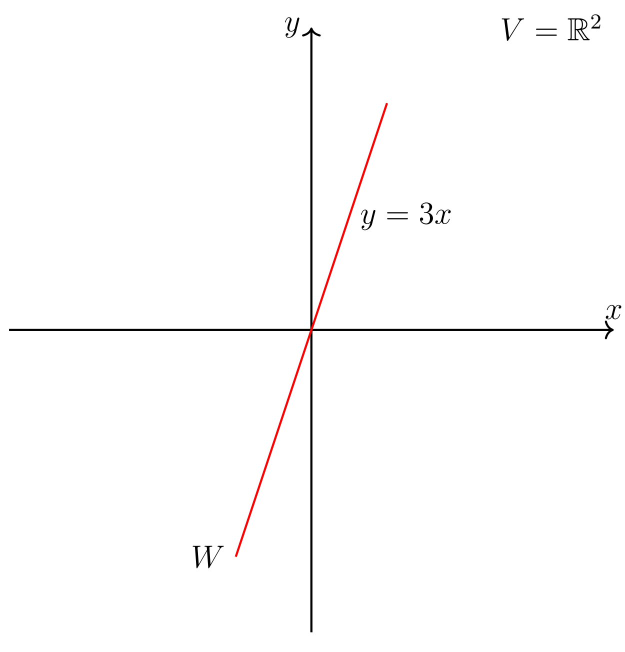
$$u+v=(x_1+x_2,3x_1+3x_2)=(x_1+x_2,3(x_1+x_2)) \ \in W$$

- $k\in\mathbb{R}$,

$$k\cdot u=(k\cdot x_1,k\cdot 3\cdot x_1)=(k\cdot x_1,3(k\cdot x_1)) \ \in W$$

$\implies \qquad W$ is a subspace of $\mathbb{R}^2$



W is a line passing through the origin

- Example: list all subspaces of $\mathbb{R}^2$:
  - (a) $\{0\}$
  - (b) any straight line passing through the origin
  - (c) $\mathbb{R}^2$ itself

- Example: let

$$V: \ M_{n\times n} = \{n \times n \ \text{ real matricies}\}$$

This is a vector space.

$$W_{n\times n} \overset{def}{\equiv} \{\underbrace{\text{symmetric}}_{A^T=A} \ n\times n \ \text{ real matricies}\}$$

Note:

$$W_{n\times n} \subset M_{n\times n}$$

Let $A, B \in W_{n\times n}$

$$(A+B)^T = A^T + B^T = A + B \qquad \Longrightarrow \qquad \in \ W_{n\times n}$$

$$(k\cdot A)^T = k\cdot A^T = k\cdot A \qquad \Longrightarrow \qquad \in \ W_{n\times n}$$

$\Longrightarrow \qquad W_{n\times n}$ is a subspace

- Example: consider a vector space of real functions

$$\mathcal{F} = \left\{ f: \ (-\infty, \infty) \to \mathbb{R} \right\}$$

Define $\mathcal{C} \subset \mathcal{F}$ as follows:

$$\mathcal{C} \overset{def}{\equiv} \left\{ f: \ (-\infty,\infty) \to \mathbb{R}\,, \text{ such that } f \text{ is a continuous function} \right\}$$

- Example: another subspace is a *space of polynomials of degree* $\leq n$ —

$$\mathcal{P}_n \overset{def}{\equiv} \left\{ a_0 + a_1 \cdot x + \cdot a_n \cdot x^n \right\}$$

$$P_1 \,\& P_2 \;\in \mathcal{P}_n \qquad \Longrightarrow \qquad P_1 + P_2 \in \mathcal{P}_n$$

$$P_1 \in \mathcal{P}_n\,,\; k \in \mathbb{R} \qquad \Longrightarrow \qquad k \cdot P_1 \in \mathcal{P}_n$$

$\Longrightarrow$ $\mathcal{P}_n$ is a subspace of $\mathcal{C}$, which is a subspace of $\mathcal{F}$:

$$\mathcal{P}_n \;\subset\; \mathcal{C} \;\subset\; \mathcal{F}$$

Intersection of two subspaces is a subspace

Definition:

Let V be a vector space. We say that u is a linear combination of

$$(v_1, v_2, \cdots v_n) \;\in\; V$$

if

$$u = c_1 \cdot v_1 + c_2 \cdot v_2 + \cdots c_n \cdot v_n$$

for some real numbers $c_1, c_2, \cdots c_n$.