Lecture 30 (section 5.2)

Section 5.2 — continued

- From example 2 earlier:

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

is a 3×3 matrix. Is A diagonalizable?

$\Longrightarrow$ characteristic equation:

$$0 = \det(A - \lambda \cdot I_3) = \det \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} \Longrightarrow \text{second column expansion for } \det \Longrightarrow$$

$$0 = (2-\lambda) \cdot \det \begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda) \cdot (-\lambda \cdot (3-\lambda) - (-2) \cdot 1)$$

$$0 = (2-\lambda) \cdot (\lambda^2 - 3\lambda + 2) = (2-\lambda) \cdot (\lambda - 2) \cdot (\lambda - 1) = (1-\lambda) \cdot (2-\lambda)^2$$

$\Longrightarrow$ eigenvalues:

$$\{\lambda_1 = 1\,, (m_1 = 1); \qquad \lambda_2 = 2\,, (m_2 = 2)\}$$

where m denotes the multiplicity of the corresponding eigenvalue.

$\Longrightarrow$ eigenvectors:

$$\underbrace{\vec{V}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\lambda=\lambda_1} \qquad \underbrace{\{\vec{V}_{2,1}, \vec{V}_{2,2}\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\lambda=\lambda_2}$$

$\Longrightarrow$ Question: are vectors

$$\vec{V}_1\,,\ \vec{V}_{2,1}\,, \vec{V}_{2,2}$$

linearly independent?

$\Longrightarrow$

$$c_1 \cdot \vec{V}_1 + c_2 \cdot \vec{V}_{2,1} + c_3 \cdot \vec{V}_{2,2} = 0$$

$$\underbrace{\begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{\text{call it matrix } M} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ use det test to check the singularity of M:

$$\underbrace{\det(M)}_{\text{2nd column expansion}} = 1 \cdot \det \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = 1 \cdot (-2 \cdot 1 - (-1) \cdot 1) = -1 \neq 0$$

$\Longrightarrow$

$$\{\vec{V}_1\,,\ \vec{V}_{2,1}\,, \vec{V}_{2,2}\}$$

are linearly independent.

How do we find the P matrix?

$\Longrightarrow$

$$P = \begin{bmatrix} \vec{V}_1 & \vec{V}_{2,1} & \vec{V}_{2,2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow$ Next step: compute P^{-1}:

$$\left[P \mid I_3\right] = \left[\begin{array}{ccc|ccc} -2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \begin{array}{c} r_3 \leftrightarrow r_1 \\ r_2 \to r_2 - r_1 \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -2 & 0 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\to \quad r_3 \to r_3 + 2r_1 \quad \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right] \to \begin{array}{c} r_1 \to r_1 - r_3 \\ r_2 \to r_2 + r_3 \end{array} \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right] = \left[I_3 \mid P^{-1}\right]$$

$\Longrightarrow$

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$\Longrightarrow$ We can now compute matrix transform of A due to P:

$$A \to P^{-1} \cdot A \cdot P = \underbrace{\begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}}_{\text{compute first}} \cdot \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 2 \\ 2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \equiv \underbrace{D}_{\text{diagonal form of } A}$$

$\{1, 2, 2\} \quad \Longrightarrow \quad$ eigenvalues of A with multiplicities

$\Longrightarrow$ D is not unique. It depends on the order of the eigenvectors used to generate the matrix P

- we used:

$$P = \begin{bmatrix} \vec{V}_1 & \vec{V}_{2,1} & \vec{V}_{2,2} \end{bmatrix} \qquad \Longrightarrow \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

corresponding to a sequence of eigenvalues:

$$\lambda_1, \lambda_2, \lambda_2$$

- if instead we used

$$\tilde{P} = \begin{bmatrix} \vec{V}_{2,1} & \vec{V}_1 & \vec{V}_{2,2} \end{bmatrix} \qquad \Longrightarrow \qquad \tilde{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Problem:

Given

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

calculate A^{2018}

Too long and painful to do directly. So let's be sneaky, assume that A is diagonalizable:

$$A \to P^{-1} \cdot A \cdot P = D \qquad \Longrightarrow \qquad A = P \cdot D \cdot P^{-1}$$

$\Longrightarrow$

$$A^2 = A \cdot A = P \cdot D \cdot \underbrace{P^{-1} \cdot P}_{=I_n} \cdot D \cdot P^{-1} = P \cdot D^2 \cdot P^{-1}$$

$\Longrightarrow$

$$A^{2018} = P \cdot D^{2018} \cdot P^{-1}$$

and it is much easier to compute D^{2018}!

- Find eigenvalues:

$$0 = \det(A - \lambda I_2) = \det \begin{bmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix} = (-\lambda \cdot (-1-\lambda) - 3 \cdot 2) = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

$\Longrightarrow$

$$\lambda_1 = -3\,, \qquad \lambda_2 = 2$$

- eigenvector $\vec{V}_1$, corresponding to $\lambda = \lambda_1$:

$$\begin{bmatrix} -\lambda_1 & 3 \\ 2 & -1-\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ augmented matrix approach

$$\begin{bmatrix} 3 & 3 & 0 \\ 2 & 2 & 0 \end{bmatrix} \rightarrow \begin{array}{c} r_1 \rightarrow \frac{1}{3} r_1 \\ r_2 \rightarrow \frac{1}{2} r_2 \\ r_2 \rightarrow r_2 - r_1 \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$

$$\vec{V}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- eigenvector $\vec{V}_2$, corresponding to $\lambda = \lambda_2$:

$$\begin{bmatrix} -\lambda_2 & 3 \\ 2 & -1-\lambda_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ augmented matrix approach

$$\begin{bmatrix} -2 & 3 & 0 \\ 2 & -3 & 0 \end{bmatrix} \rightarrow \begin{array}{c} r_2 \rightarrow r_2 + r_1 \\ r_1 \rightarrow -\frac{1}{2} r_1 \end{array} \rightarrow \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$

$$\vec{V}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- compute P, P^{-1}, D:

$$P = \begin{bmatrix} \vec{V_1} & \vec{V_2} \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{\det P} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-2-3} \begin{bmatrix} 2 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$D = P^{-1} \cdot A \cdot P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

- compute D^{2018}:

$$D^{2018} = \begin{bmatrix} (-3)^{2018} & 0 \\ 0 & 2^{2018} \end{bmatrix} = \begin{bmatrix} 3^{2018} & 0 \\ 0 & 2^{2018} \end{bmatrix}$$

- compute A^{2018}:

$$A^{2018} = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3^{2018} & 0 \\ 0 & 2^{2018} \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{3}{5} \cdot 2^{2018} + \frac{2}{5} \cdot 3^{2018} & \frac{3}{5} \cdot 2^{2018} - \frac{3}{5} \cdot 3^{2018} \\ \frac{2}{5} \cdot 2^{2018} - \frac{2}{5} \cdot 3^{2018} & \frac{2}{5} \cdot 2^{2018} + \frac{3}{5} \cdot 3^{2018} \end{bmatrix}$$