# Lecture 29 (sections 5.1,5.2)

Section 5.1 — continued

- Example 1.

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$\Longrightarrow$ characteristic equation:

$$0 = \det(A - \lambda \cdot I_2) = \det\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

$$0 = (1-\lambda)\cdot(3-\lambda) - 4\cdot 2 = 3 - 4\lambda + \lambda^2 - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)\cdot(\lambda + 1)$$

$\Longrightarrow$ eigenvalues

$$\{\lambda_1 = -1\,, \qquad \lambda_2 = 5\}$$

$\Longrightarrow$ If eigenvector $\vec{V}$ corresponds to an eigenvalue $\lambda$, then

$$A \cdot \vec{V} = \lambda \cdot \vec{V}$$

- $\lambda = \lambda_1 = -1$

$$(A - \lambda_1 \cdot I_2)\cdot \vec{V} = \vec{0} \qquad \Longleftrightarrow \qquad \begin{bmatrix} 1-\lambda_1 & 4 \\ 2 & 3-\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ sub $\lambda_1 = -1 \Longrightarrow$

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ augmented matrix

$$\begin{bmatrix} 2 & 4 & 0 \\ 2 & 4 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_2 \to r_2 - r_1} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_1 \to \frac{1}{2} r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ eigenvector $\vec{V_1} = (-2, 1)$ corresponds to eigenvalue $\lambda_1 = -1$

- $\lambda = \lambda_2 = 5$

$$(A - \lambda_2 \cdot I_2) \cdot \vec{V} = \vec{0} \qquad \Longleftrightarrow \qquad \begin{bmatrix} 1-\lambda_2 & 4 \\ 2 & 3-\lambda_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ sub $\lambda_2 = 5 \Longrightarrow$

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ augmented matrix

$$\begin{bmatrix} -4 & 4 & 0 \\ 2 & -2 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_2 \to r_2 + \frac{1}{2} r_1} \begin{bmatrix} -4 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_1 \to -r_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ eigenvector $\vec{V_2} = (1, 1)$ corresponds to eigenvalue $\lambda_2 = 5$

■ Example 2.

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$\Longrightarrow$ characteristic equation:

$$0 = \det(A - \lambda \cdot I_3) = \det \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} \Longrightarrow \text{second column expansion for } \det \Longrightarrow$$

$$0 = (2-\lambda) \cdot \det \begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda) \cdot (-\lambda \cdot (3-\lambda) - (-2) \cdot 1)$$

$$0 = (2-\lambda) \cdot (\lambda^2 - 3\lambda + 2) = (2-\lambda) \cdot (\lambda - 2) \cdot (\lambda - 1) = (1-\lambda) \cdot (2-\lambda)^2$$

$$\boxed{2 \; - \; \text{multiplicity of eigenvalue}}$$

$\Longrightarrow$ eigenvalues

$$\{\lambda_1 = 1\,, (m_1 = 1); \qquad \lambda_2 = 2\,, (m_2 = 2)\}$$

where $m$ denotes the multiplicity of the corresponding eigenvalue.

- $\lambda = \lambda_1 = 1\,,\ (m_1 = 1)$

$$(A - \lambda_1 \cdot I_3) \cdot \vec{V} = \vec{0} \qquad \Longleftrightarrow \qquad \begin{bmatrix} -\lambda_1 & 0 & -2 \\ 1 & 2-\lambda_1 & 1 \\ 1 & 0 & 3-\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ sub $\lambda_1 = 1 \Longrightarrow$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ augmented matrix

$$\begin{bmatrix} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{\substack{r_3 \to r_3 + r_1 \\ r_1 \to -r_1}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_2 \to r_2 - r_1} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{leading variables} : v_1, v_2$$
$$\text{free variable} : v_3$$

$$v_3 = s\,, \qquad v_2 = v_3 = s\,, \qquad v_1 = -2v_3 = -2s$$

$\Longrightarrow$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ eigenvector $\vec{V_1} = (-2, 1, 1)$ corresponds to eigenvalue $\lambda_1 = 1$ of multiplicity $m_1 = 1$

- $\lambda = \lambda_2 = 2\,,\ (m_2 = 2)$

$$(A - \lambda_2 \cdot I_3) \cdot \vec{V} = \vec{0} \qquad \Longleftrightarrow \qquad \begin{bmatrix} -\lambda_2 & 0 & -2 \\ 1 & 2-\lambda_2 & 1 \\ 1 & 0 & 3-\lambda_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ sub $\lambda_2 = 2 \Longrightarrow$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ augmented matrix

$$\begin{bmatrix} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{\substack{r_3 \to r_3 + \frac{1}{2}r_1 \\ r_2 \to r_2 + \frac{1}{2}r_1}} \begin{bmatrix} -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underbrace{\Longrightarrow}_{r_1 \to -\frac{1}{2}r_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{leading variable} : v_1$$
$$\text{free variables} : v_2, v_3$$
$$\{v_2, v_3\} = \{s, t\}\,, \qquad v_1 = -v_3 = -t$$

$\Longrightarrow$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\Longrightarrow$ eigenvectors

$$\{\vec{V}_{2,1}, \vec{V}_{2,2}\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

form a basis for the eigenspace corresponding to eigenvalue $\lambda_2 = 2$ of multiplicity $m_2 = 2$

Theorem:

- If $\lambda = 0$ is an eigenvalue of $A$, then $A$ is singular
- If $\lambda \neq 0$ is not an eigenvalue of $A$, then $A$ is nonsingular

Section 5.2 — diagonalization

If $A$ is $n \times n$ matrix, and $P$ is an invertible $n \times n$ matrix, a transformation

$$A \longrightarrow P^{-1} \cdot A \cdot P$$

is called a **similarity transformation**

- If by similarity transformation we obtain a diagonal matrix, $A$ is called **diagonalizable**

Theorem:

- If $A$ has $n$ linearly independent eigenvectors, it is diagonalizable