Lecture 7.5: **Centroids**

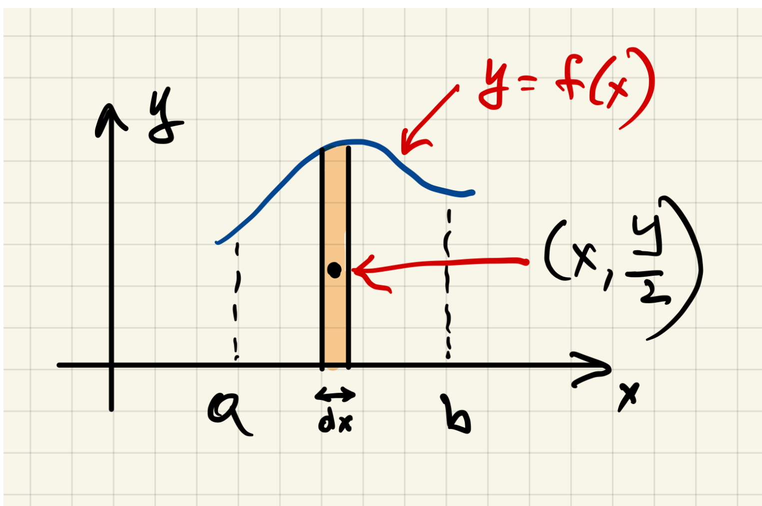
$\Longrightarrow$ A centroid is a center of mass for an object of a constant density. $\Longrightarrow$ To find the centroid, simply replace the density with 1 in all the CM formulas, *e.g.*, for 2-D object:

$$\bar{x} = \frac{M_{x=0}}{A}\,, \qquad \bar{y} = \frac{M_{y=0}}{A}$$

where $M_{x=0}$ and $M_{y=0}$ are moments relative to $x = 0$ and $y = 0$ of an object of area A and surface density $\sigma = 1$.

$\Longrightarrow$ Consider a region $\mathcal{R}$,

$$\mathcal{R}: \qquad a \le x \le b\,, \qquad 0 \le y \le f(x)$$



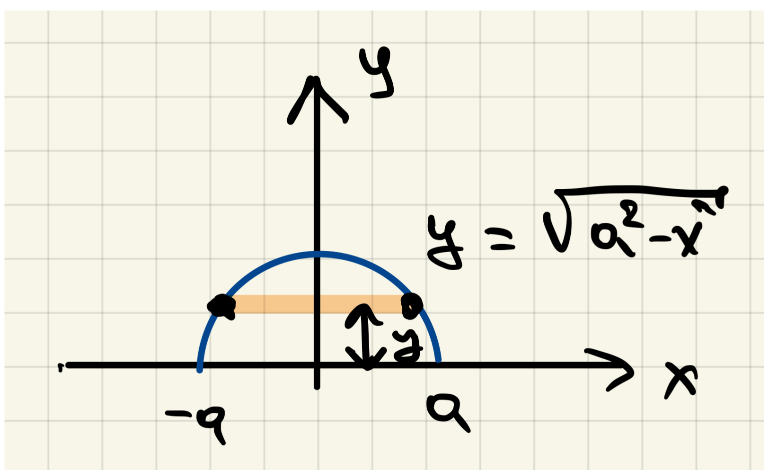
Find the centroid $(\bar{x}, \bar{y})$ of the shape

$\Longrightarrow$

- It is convenient to split the shape into strips (orange) of width dx.

- Note
$$A = \int_a^b f(x)\ dx$$
- The moments of the strip $dM_{x=0}$ and $dM_{y=0}$ are correspondingly
$$dM_{x=0} = x \underbrace{dA}_{=f(x)dx} = xf(x)\ dx$$
$$dM_{y=0} = dx \cdot \int_0^{f(x)} ydy = dx \cdot \frac{1}{2}y^2\Big|_0^{f(x)} = dx \cdot \frac{1}{2}(f(x))^2$$
- The total moments are simply the sums (the integrals) of the individual strip moments
$$M_{x=0} = \int dM_{x=0} = \int_a^b xf(x)\ dx\,, \qquad M_{y=0} = \int dM_{y=0} = \frac{1}{2}\int_a^b (f(x))^2\ dx$$
- $\Longrightarrow$
$$\bar{x} = \frac{\int_a^b xf(x)\ dx}{\int_a^b f(x)\ dx}\,, \qquad \bar{y} = \frac{\frac{1}{2}\int_a^b (f(x))^2\ dx}{\int_a^b f(x)\ dx}$$

Example: Find the centroid of a half-disk of radius a centered at the origin:



$\Longrightarrow$

- By symmetry

$$\bar{x} = 0$$

- Note that

$$A = \frac{1}{2}\pi a^2$$

- To compute $\bar{y}$,

$$A\bar{y} = M_{y=0} = \frac{1}{2}\int_{-a}^{a}(a^2 - x^2)dx = \frac{1}{2}\left(a^2x - \frac{1}{3}x^2\right)\Big|_{-a}^{a} = \frac{2}{3}a^3$$

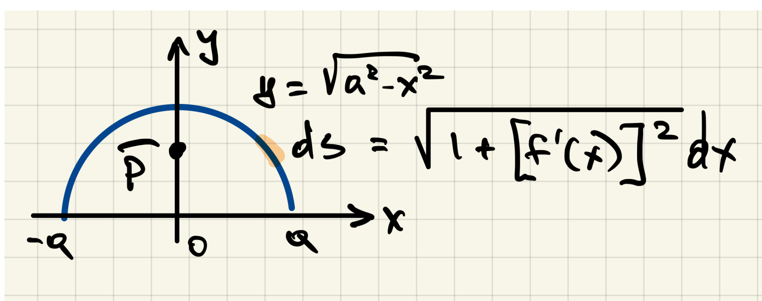
$$\bar{y} = \frac{1}{A}\,\frac{2}{3}a^3 = \frac{4a}{3\pi}$$

- Thus

$$(\bar{x}, \bar{y}) = \left(0, \frac{4a}{3\pi}\right)$$

1-D example: Find the position of the centroid $\bar{P}$ for half-circle

$$y = \sqrt{a^2 - x^2}\,, \qquad x \in [-a, a]$$



$\Longrightarrow$

- The location of the centroid is

$$\bar{x} = \frac{M_{x=0}}{\ell}\,, \qquad \bar{y} = \frac{M_{y=0}}{\ell}$$

where ℓ is the length of the wire

- Consider an element of length ds (orange) located at

$$(x, y) = (x, f(x))$$

contributing to the moments as

$$dM_{x=0} = x \cdot ds = x\sqrt{1 + (f'(x))^2}dx\,, \qquad dM_{y=0} = y \cdot ds = f(x)\sqrt{1 + (f'(x))^2}dx$$

- Using

$$\sqrt{1 + (f'(x))^2} = \left(1 + \left(\frac{-2x}{2\sqrt{a^2 - x^2}}\right)^2\right)^{1/2} = \left(1 + \frac{x^2}{a^2 - x^2}\right)^{1/2} = \frac{a}{\sqrt{a^2 - x^2}}$$

$$dM_{x=0} = \frac{ax}{\sqrt{a^2 - x^2}}\,dx\,, \qquad dM_{y=0} = a\ dx$$

- $\Longrightarrow$

$$M_{x=0} = \int dM_{x=0} = \int_{-a}^{a} \frac{ax}{\sqrt{a^2 - x^2}}\,dx = 0$$

by symmetry; and

$$M_{y=0} = \int dM_{y=0} = \int_{-a}^{a} a\ dx = 2a^2$$

- Since

$$\ell = \pi a$$

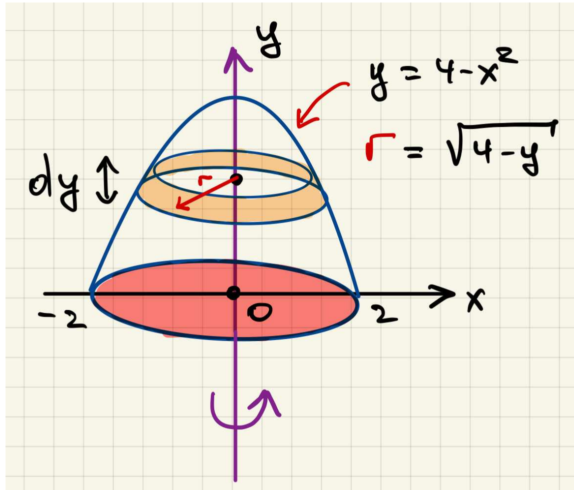
we have

$$\bar{x} = 0\,, \qquad \bar{y} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

3-D example: $\mathcal{R}$ is the region in the first quadrant bounded by

$$y = 4 - x^2$$

Rotate $\mathcal{R}$ around the y-axis (purple). Find the centroid of the solid obtained.



$\Longrightarrow$

- The centroid

$$\bar{P} = (\bar{x}, \bar{y}, \bar{z})$$

  By symmetry,

$$\bar{x} = 0\,, \qquad \bar{z} = 0$$

- To compute $\bar{y}$, slice the solid with disks of width dy and volume $dV_{disk} = A_{disk}dy$ (orange); the moment of each disk is

$$dM_{y=0} = y \cdot dV_{disk} = y \cdot \underbrace{A_{disk}}_{=\pi r^2} dy = y\pi \Big(\underbrace{r}_{=\sqrt{4-y}} \Big)^2 \, dy = y\pi(4-y)\; dy$$

- $\Longrightarrow$

$$M_{y=0} = \int dM_{y=0} = \int_0^4 y\pi(4-y)\; dy = \pi \left(4\frac{y^2}{2} - \frac{1}{3}y^3 \right) \Bigg|_0^4 = \frac{32\pi}{3}$$

-

$$\bar{y} = \frac{M_{y=0}}{V}$$

- To compute the volume V:

$$V = \int dV_{disk} = \int_0^4 \pi(4-y)\; dy = \pi \left(4y - \frac{1}{2}y^2 \right) \Bigg|_0^4 = 8\pi$$

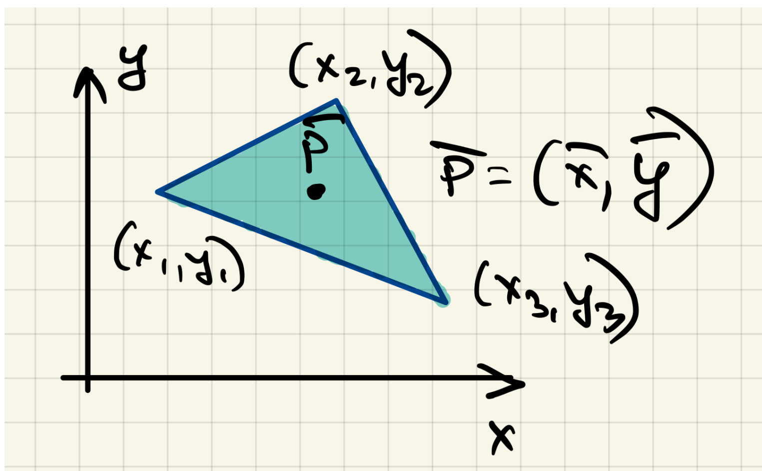
- $\implies$

$$\bar{y} = \frac{\frac{32\pi}{3}}{8\pi} = \frac{4}{3}$$

- The centroid of a triangle

$\implies$ (no proof) Consider a triangle with vertices

$$(x_1, y_1), \qquad (x_2, y_2), \qquad (x_3, y_3)$$



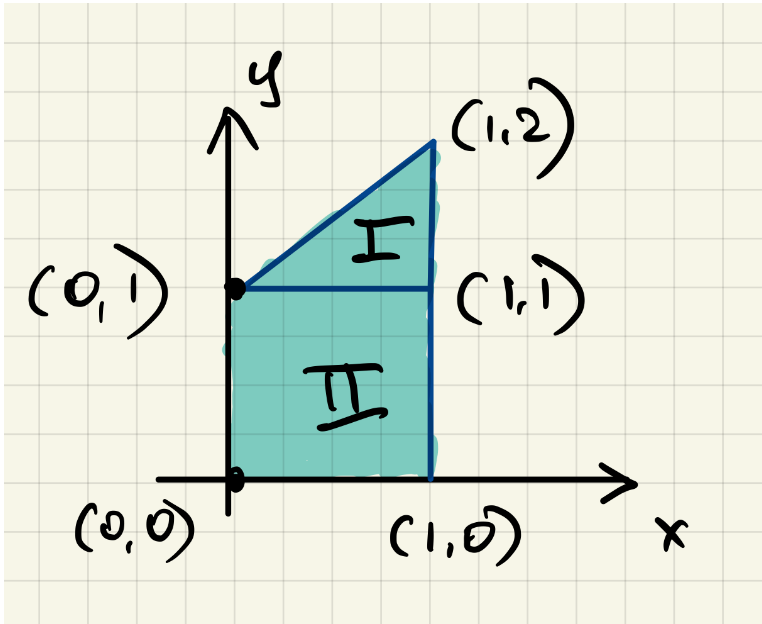
its centroid $\bar{P}$ is at

$$\bar{P} = (\bar{x}, \bar{y}), \qquad \bar{x} = \frac{x_1 + x_2 + x_3}{3}, \qquad \bar{y} = \frac{y_1 + y_2 + y_3}{3}$$

$\implies$ More results from symmetry:

- The centroid of a rectangle is at the center
- The centroid of a circular disk is at the center
- The centroid of a sphere is at the center

Example: Find the centroid of the trapezoid



$\Longrightarrow$

- Break the trapezoid into 2 pieces
  - I-triangle
  - II-square

- region-I:

$$(\bar{x}_1, \bar{y}_1) = \left(\frac{0+1+1}{3}, \frac{1+1+2}{3}\right) = \left(\frac{2}{3}, \frac{4}{3}\right)$$

  its area is

$$A_1 = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

- we can compute the moments of the region-I:

$$M_{1,x=0} = \bar{x}_1 \cdot A_1 = \frac{1}{3}, \qquad M_{1,y=0} = \bar{y}_1 \cdot A_1 = \frac{2}{3}$$

- region-II:

$$(\bar{x}_2, \bar{y}_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

its area is

$$A_2 = 1$$

- we can compute the moments of the region-II:

$$M_{2,x=0} = \bar{x}_2 \cdot A_2 = \frac{1}{2}, \qquad M_{2,y=0} = \bar{y}_2 \cdot A_2 = \frac{1}{2}$$

- We can now compute moments of the trapezoid:

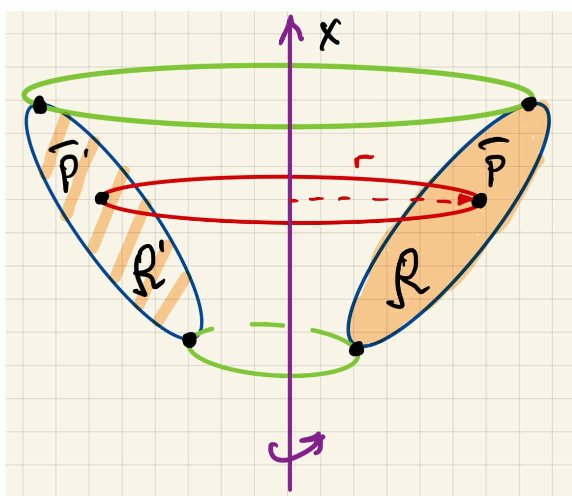
$$M_{x=0} = M_{1,x=0} + M_{2,x=0} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}, \qquad M_{y=0} = M_{1,y=0} + M_{2,y=0} = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$$

- $\Longrightarrow$

$$\bar{x} = \frac{M_{x=0}}{A_1 + A_2} = \frac{\frac{5}{6}}{\frac{1}{2} + 1} = \frac{5}{9}, \qquad \bar{y} = \frac{M_{y=0}}{A_1 + A_2} = \frac{\frac{7}{6}}{\frac{1}{2} + 1} = \frac{7}{9}$$

Pappus's theorem

▪ (1) Plane region $\mathcal{R}$ (orange) is rotated around the axis x (purple) outside of $\mathcal{R}$ $\Longrightarrow$ solid of revolution. Let $\bar{P}$ be the centroid of $\mathcal{R}$. Let r be the distance from the centroid to the axis of rotation.

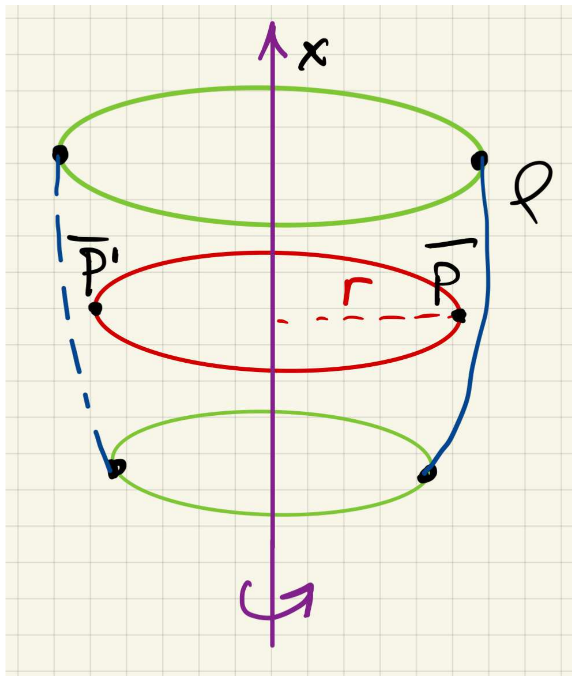


$\Longrightarrow$ The volume V of the revolution solid is

$$\boxed{V = 2\pi r \cdot A}$$

where A is the area of the region $\mathcal{R}$

■ (2) Curve ℓ (blue) is rotated around the axis x (purple) outside of ℓ. Let $\bar{P}$ be the centroid of ℓ — note that the centroid is **NOT** necessarily on the curve! Let r be the distance from the centroid to the axis of rotation.

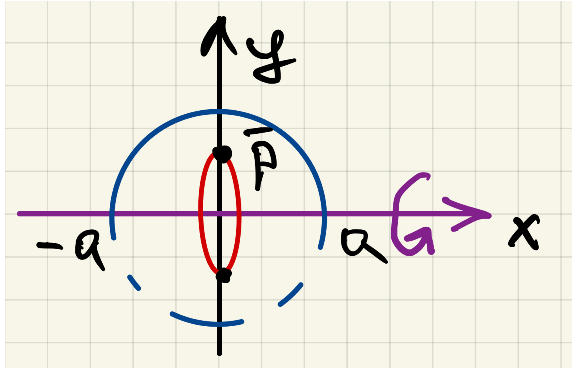


$\Longrightarrow$ The surface area A of the revolution solid is

$$\boxed{A = 2\pi r \cdot s}$$

where s is the length of the curve ℓ

Example 1: Find the centroid of the half-circle of radius a.



- Let's rotate the half-circle around x-axis (purple) The resulting shape is a sphere.
- We know that

$$A = 4\pi a^2\,, \qquad s = \pi a$$

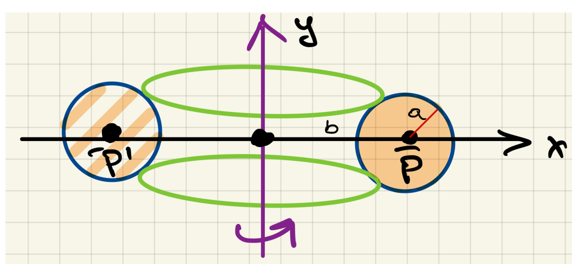
- From the Pappus's theorem:

$$A = 2\pi\bar{y}\cdot s \qquad \Longrightarrow \qquad \bar{y} = \frac{A}{2\pi s} = \frac{4\pi a^2}{2\pi\pi a} = \frac{2a}{\pi}$$

Example 2: Rotate the disk

$$(x-b)^2 + y^2 \leq a^2$$

about the y-axis (purple). We obtain the torus. Find the volume of this torus.



- The area of $\mathcal{R}$ (orange) is
$$A = \pi a^2$$
- Centroid of $\mathcal{R}$ is
$$(\bar{x}, \bar{y}) = (b, 0)$$
- The distance from the centroid to the axis of rotation is
$$r = b$$
- From the Pappus's theorem:
$$V = 2\pi r \cdot A = 2\pi b \cdot \pi a^2 = 2\pi^2 a^2 b$$