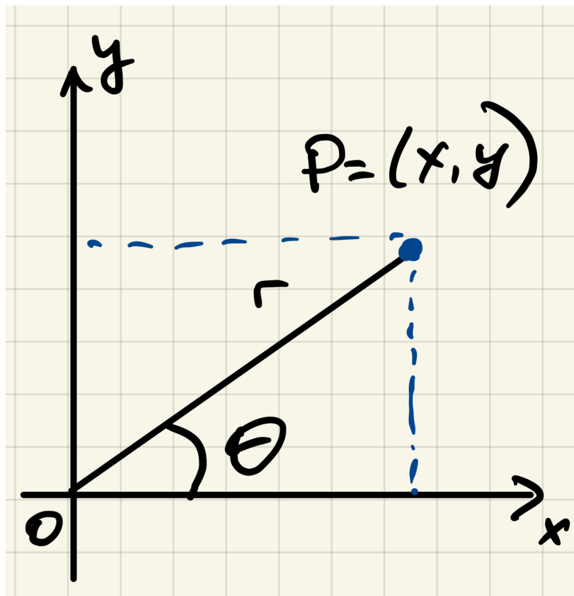
Lecture 8.5: **Polar coordinates, polar curves**

Consider a point $P \in \mathbb{R}^2$ (on the plane):



- Cartesian coordinates:

$$P = (x, y)$$

- polar coordinates:

$$P = (r, \theta)$$

  - r - the distance from P to O
  - θ - the angle of OP with the positive x axis

- The correspondence Cartesian⟷polar

$$\begin{cases} x = r \ \cos\theta \\ y = r \ \sin \ \theta \end{cases} , \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

- Note that

$$(r, \theta)\,, \qquad (r, \theta + 2\pi)\,, \qquad (-r, \theta + \pi)$$

represent the same point on the plane

Example 1: Write the equation of the straight line

$$3x + y = 4$$

in polar coordinates

$\Longrightarrow$

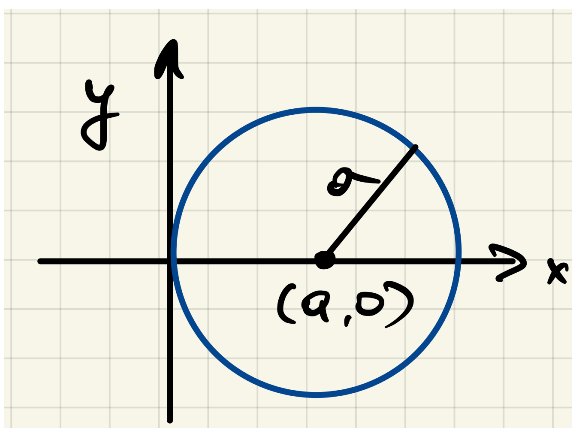
$$3\underbrace{x}_{=r\cos\theta} + \underbrace{y}_{=r\sin\theta} = 4$$

$\Longrightarrow$

$$r(3\cos\theta + \sin\theta) = 4 \qquad \Longrightarrow \qquad r = \frac{4}{3\cos\theta + \sin\theta}$$

Example 2: Write the equation of the circle

$$(x-a)^2 + y^2 = a^2$$



in polar coordinates.

$\Longrightarrow$

$$(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$$

$\Longrightarrow$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) - 2ar\cos\theta + a^2 = a^2$$

$\Longrightarrow$ (in general $r \not\equiv 0$)

$$r^2 = 2ar\cos\theta \qquad \Longrightarrow \qquad r = 2a\cos\theta$$

- A curve in Cartesian coordinates can be written as

$$y = f(x)$$

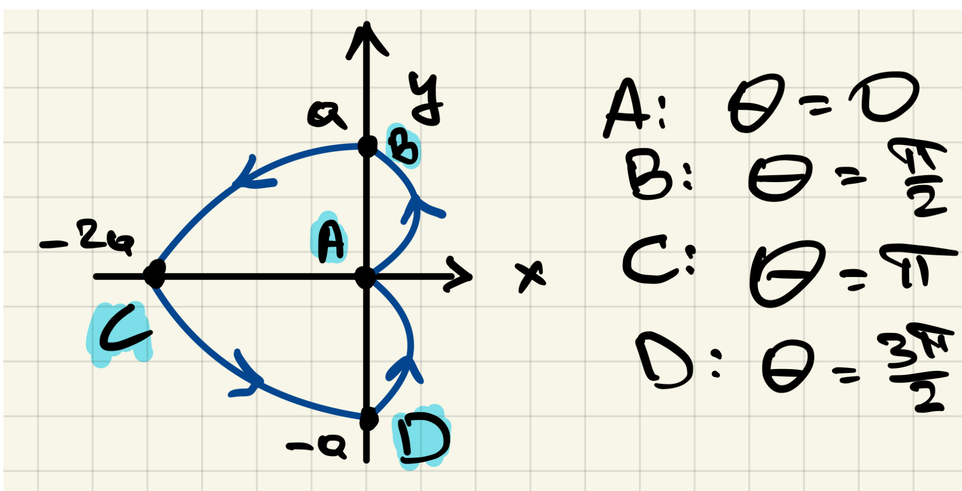
- A curve in polar coordinates (called a polar curve) can be written as

$$r = f(\theta)$$

Example 3: Sketch the cardioid curve

$$r = a(1-\cos\theta)$$

$\Longrightarrow$ Nothing fancy: pick some points, and plot:

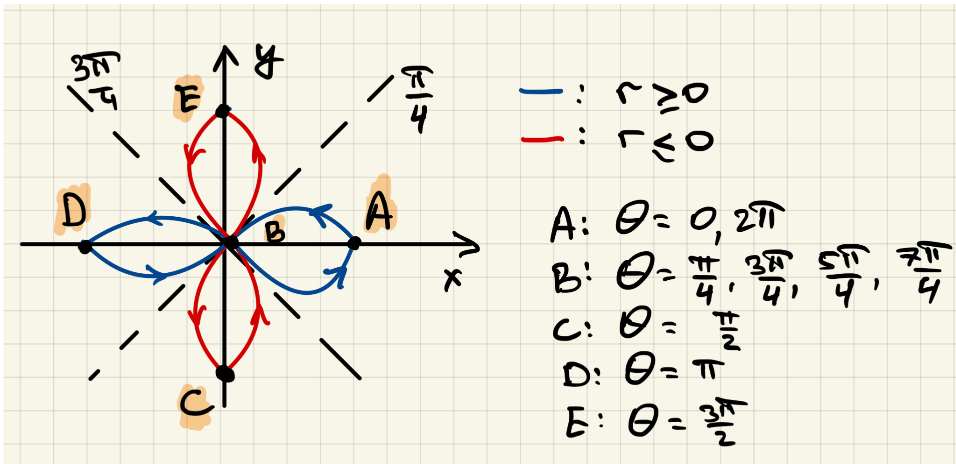


$$A: \ \theta = 0 \implies (r,\theta) = (0,0) \qquad \text{and} \qquad (x,y) = (0,0)$$
$$B: \ \theta = \frac{\pi}{2} \implies (r,\theta) = (a,\pi/2) \qquad \text{and} \qquad (x,y) = (0,a)$$
$$C: \ \theta = \pi \implies (r,\theta) = (2a,\pi) \qquad \text{and} \qquad (x,y) = (-2a,0)$$
$$D: \ \theta = \frac{3}{2}\pi \implies (r,\theta) = (a,3\pi/2) \qquad \text{and} \qquad (x,y) = (0,-a)$$

Example 4: Sketch the curve

$$r = \cos\ 2\theta$$

$\Longrightarrow$ Nothing fancy: pick some points, and plot:



$$A: \ \theta = 0 \implies (r, \theta) = (1, 0) \qquad \text{and} \qquad (x, y) = (1, 0)$$

$$B: \ \theta = \frac{\pi}{4} \implies (r, \theta) = (0, \pi/4) \qquad \text{and} \qquad (x, y) = (0, 0)$$

$$C: \ \theta = \frac{\pi}{2} \implies (r, \theta) = (-1, \pi/2) \qquad \text{and} \qquad (x, y) = (0, -1)$$

$$D: \ \theta = \pi \implies (r, \theta) = (1, \pi) \qquad \text{and} \qquad (x, y) = (-1, 0)$$

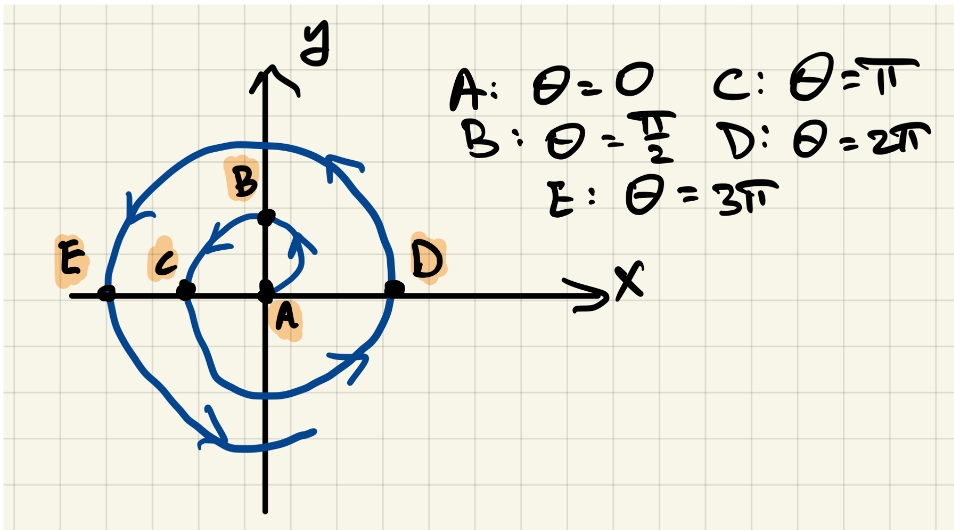
$$E: \ \theta = 3\pi/2 \implies (r, \theta) = (-1, 3\pi/2) \qquad \text{and} \qquad (x, y) = (0, 1)$$

- we use ▬ for leaves with $r \geq 0$
- we use ▬ for leaves with $r \leq 0$

Example 5: Sketch the curve

$$r = \theta$$

$\Longrightarrow$ Nothing fancy: pick some points, and plot:



$$A: \ \theta = 0 \implies (r,\theta) = (0,0) \qquad \text{and} \qquad (x,y) = (0,0)$$

$$B: \ \theta = \frac{\pi}{2} \implies (r,\theta) = (\pi/2, \pi/2) \qquad \text{and} \qquad (x,y) = (0,\pi/2)$$

$$C: \ \theta = \pi \implies (r,\theta) = (\pi,\pi) \qquad \text{and} \qquad (x,y) = (-\pi,0)$$

$$D: \ \theta = 2\pi \implies (r,\theta) = (2\pi,0) \qquad \text{and} \qquad (x,y) = (2\pi,0)$$

$$E: \ \theta = 3\pi \implies (r,\theta) = (3\pi,\pi) \qquad \text{and} \qquad (x,y) = (-3\pi,0)$$