Lecture 6.1: **Integration by parts**

$\Longrightarrow$ Start with the product rule:

$$u(x)\,, \qquad v(x) \qquad \text{differentiable}$$

$$\frac{d}{dx}\Big(u(x)\cdot v(x)\Big) = u(x)\cdot\frac{dv}{dx} + \frac{du}{dx}\cdot v(x)$$

$\Longrightarrow$ Integrate both sides:

$$\int \frac{d}{dx}\Big(u(x)\cdot v(x)\Big)\; dx = \int u(x)\cdot\frac{dv}{dx}\; dx + \int \frac{du}{dx}\cdot v(x)\; dx$$

or

$$\int u(x)\cdot\frac{dv}{dx}\; dx = \int \frac{d}{dx}\Big(u(x)\cdot v(x)\Big)\; dx - \int \frac{du}{dx}\cdot v(x)\; dx$$

$\Longrightarrow$

$$\int u(x)\cdot\frac{dv}{dx}\; dx = u(x)\cdot v(x) - \int \frac{du}{dx}\cdot v(x)\; dx$$

Using differentials

$$dv = dx\cdot\frac{dv}{dx}\,, \qquad du = dx\cdot\frac{du}{dx}$$

we find the

Integration by parts formular (IBP):

$$\boxed{\int u\cdot dv = u\cdot v - \int v\cdot du}$$

Example 1

$$\int xe^x\ dx \qquad \text{interpret as} \qquad \int u \cdot dv$$

Choose

$$u = x \qquad \Longrightarrow \qquad du = dx$$

$$dv = e^x dx \qquad \Longrightarrow \qquad v = \int e^x dx = e^x$$

Note that we did not add any constant in computing v from dv

$$\int \underbrace{u}_{x} \cdot \underbrace{dv}_{e^x dx} = \underbrace{u \cdot v}_{xe^x} - \int \underbrace{v}_{e^x} \cdot \underbrace{du}_{dx}$$

$\Longrightarrow$

$$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x + C$$

Rule of thumb:

$$\boxed{e^x \qquad \text{combine into} \qquad dv}$$

Example 2

$$\int \ln x\ dx$$

Set $u = \ln x$, $dv = dx \Longrightarrow \qquad v = x$ and $du = \frac{dx}{x}$,

$$\int \ln x\ dx = x \ln x - \int x \cdot \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + C$$

Sometimes it is useful to verify your answer:

$$\frac{d}{dx} \int \ln x\ dx = \frac{d}{dx}\Big(x \ln x - x + C \Big) = \ln x + x \cdot \frac{1}{x} - 1 + 0 = \ln x$$

Rule of thumb:

$$\boxed{\ln x \qquad \text{combine into} \qquad u}$$

Example 3

$$\int x^2 \cos x\ dx$$

Set $u = x^2$, $dv = \cos x\ dx \Longrightarrow \qquad v = \sin x$ and $du = 2x dx$,

$$\int x^2 \cos x\ dx\ dx = x^2 \sin x - \underbrace{\int 2x \sin x\ dx}_{\text{use IBP again}}$$

Set $u = 2x$, $dv = \sin x\ dx \implies \quad v = -\cos x$ and $du = 2dx \implies$

$$\int x^2 \cos x\ dx\ dx = x^2 \sin x - \left(2x(-\cos x) - \int (-\cos x) 2dx\right) = x^2 \sin x + 2x\cos x - 2\int \cos x dx$$

$$= x^2 \sin x + 2x \cos x - 2\sin x + C$$

Rule of thumb:

$$\boxed{\sin x, \cos x \qquad \text{combine into} \qquad dv}$$

Example 4

$$\int x \tan^{-1} x\ dx$$

Set $u = \tan^{-1} x$, $dv = x\ dx \implies \quad v = \frac{1}{2}x^2$ and $du = \frac{dx}{1+x^2}$,

$$\int x \tan^{-1} x\ dx = \frac{1}{2}x^2\ \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\ dx = \frac{1}{2}x^2\ \tan^{-1} x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\ dx$$

$$= \frac{1}{2}x^2\ \tan^{-1} x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{1+x^2} = \frac{1}{2}x^2\ \tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

Example 5

$$I = \int e^{ax} \cos(bx)\ dx\,, \qquad \{a, b\} - \text{constants}$$

Set $u = e^{ax}$, $dv = \cos(bx)dx \implies \quad v = \frac{1}{b}\sin(bx)$ and $du = ae^{ax}dx$

$$I = \frac{1}{b}e^{ax}\sin(bx) - \frac{a}{b}\underbrace{\int e^{ax}\sin(bx)dx}_{\text{repeat IBP}}$$

Again, we set $u = e^{ax}$, $dv = \sin(bx)dx \implies \quad v = -\frac{1}{b}\cos(bx)$ and $du = ae^{ax}dx$

$$I = \frac{1}{b}e^{ax}\sin(bx) - \frac{a}{b}\left(-\frac{1}{b}e^{ax}\cos(bx) + \frac{a}{b}\int e^{ax}\cos(bx)\ dx\right)$$

$$= \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx) - \frac{a^2}{b^2}\int e^{ax}\cos(bx)\ dx = \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx) - \frac{a^2}{b^2}\ I$$

$\implies$

$$\left(1 + \frac{a^2}{b^2}\right)I = \frac{1}{b^2}\left(be^{ax}\sin(bx) + ae^{ax}\cos(bx)\right) \implies \boxed{I = \frac{1}{a^2+b^2}\left(be^{ax}\sin(bx) + ae^{ax}\cos(bx)\right) + C}$$

Definite integrals:

$$\int_a^b u(x)v'(x)\ dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx$$

were

$$u(x)v(x)\Big|_a^b = u(b)v(b) - u(a)v(a)$$

Example 6

$$I = \int_1^e x^3 (\ln x)^2\, dx$$

Set $u = (\ln x)^2$, $dv = x^3 dx \implies \qquad v = \frac{1}{4}x^4$ and $du = \frac{2\ln x}{x}dx$

$$I = \frac{1}{4}x^4(\ln x)^2\Big|_1^e - \frac{1}{2}\int_1^e x^3 \ln x\ dx = \frac{e^4}{4} - \frac{1}{2}\underbrace{\int_1^e x^3 \ln x\ dx}_{\text{repeat IBP}}$$

where we used

$$\ln e = 1\,, \qquad \ln 1 = 0$$

Set $u = \ln x$, $dv = x^3 dx \implies \qquad v = \frac{1}{4}x^4$ and $du = \frac{1}{x}dx$

$$I = \frac{e^4}{4} - \frac{1}{2}\left(\frac{1}{4}x^4 \ln x\Big|_1^e - \frac{1}{4}\int_1^e x^3\ dx\right) = \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\int_1^e x^3 dx$$

$$= \frac{e^4}{8} + \frac{1}{32}x^4\Big|_1^e = \frac{e^4}{8} + \frac{e^4 - 1}{32} = \frac{5e^4}{32} - \frac{1}{32}$$

Example 7

$$I_4 = \int (\ln x)^4\ dx$$

$\implies \qquad$ Let's derive the reduction formular

$$\boxed{I_n \equiv \int (\ln x)^n\ dx\,, \qquad I_n = x(\ln x)^n - nI_{n-1}}$$

In computing I_n, set $u = (\ln x)^n$, $dv = dx \implies \qquad v = x$ and $du = \frac{n(\ln x)^{n-1}}{x}dx$

$$I_n = x(\ln x)^n - \int x\frac{n(\ln x)^{n-1}}{x}dx = x(\ln x)^n - n\int (\ln x)^{n-1}\ dx = x(\ln x)^n - nI_{n-1}$$

We can use the reduction formular to compute I_4:

- $n = 1$ was computed in **Example 2**:

$$I_1 = \int \ln x = x \ln x - x + A$$

- $n = 2$,

$$I_2 = x(\ln x)^2 - 2I_1 = x(\ln x)^2 - 2\left(x \ln x - x + C\right) = x(\ln x)^2 - 2x \ln x + 2x + B$$

- $n = 3$,

$$I_3 = x(\ln x)^3 - 3I_2 = x(\ln x)^3 - 3\left(x(\ln x)^2 - 2x \ln x + 2x + B\right)$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$$

- finally, $n = 4$,

$$I_4 = x(\ln x)^4 - 4I_3 = x(\ln x)^4 - 4\left(x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C\right)$$

$$= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x \ln x + 24x + D$$