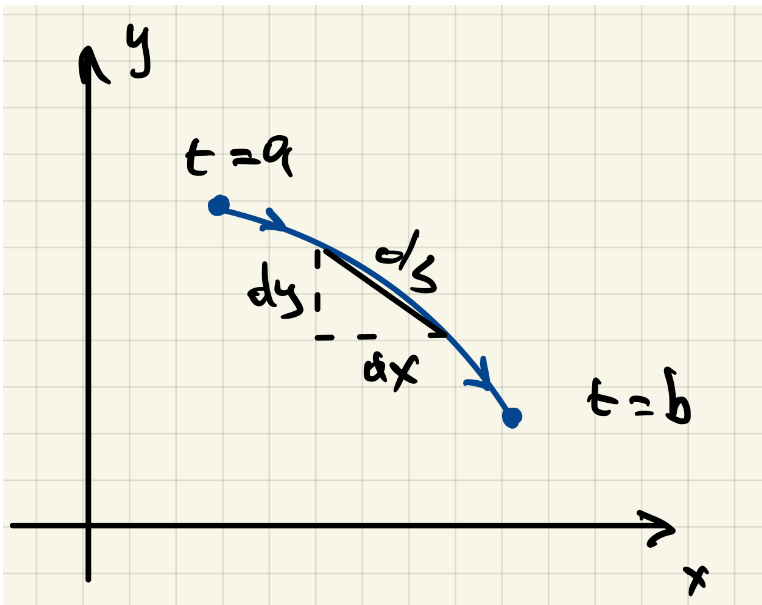
Lecture 8.4: **Arc length**

Consider a parametric curve ℓ

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in [a, b]$$



How do we compute the arc length L of ℓ?

$\Longrightarrow$

$$L = \int ds = \int \sqrt{dx^2 + dy^2}$$

$$dx = f'(t)\ dt\,, \qquad dy = g'(t)\ dt$$

$\Longrightarrow$

$$ds = \sqrt{(f'(t))^2 + (g'(t))^2}\ dt$$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\ dt$$

Example 1: Compute the arc length of ℓ

$$\begin{cases} x = e^t \ \cos t \\ y = e^t \ \sin t \end{cases}, \qquad t \in [0, a], \qquad a > 0$$

$\Longrightarrow$

$$dx = e^t(\cos t - \sin t)\ dt, \qquad dy = e^t(\sin t + \cos t)\ dt$$

$$ds^2 = dx^2 + dy^2 = e^{2t}\left[(\cos t - \sin t)^2 + (\cos t + \sin t)^2\right] dt^2 = 2e^{2t}\ dt^2$$

$$L = \int_0^a \sqrt{2}e^t dt = \sqrt{2}e^t\Big|_0^a = \sqrt{2}\,(e^a - 1)$$

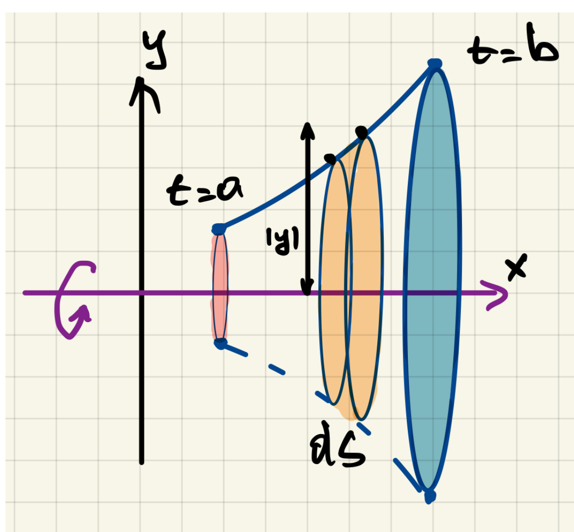
Surface area

Consider a parametric curve ℓ

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in [a, b]$$

Lets compute the surface area of the revolution of ℓ about the

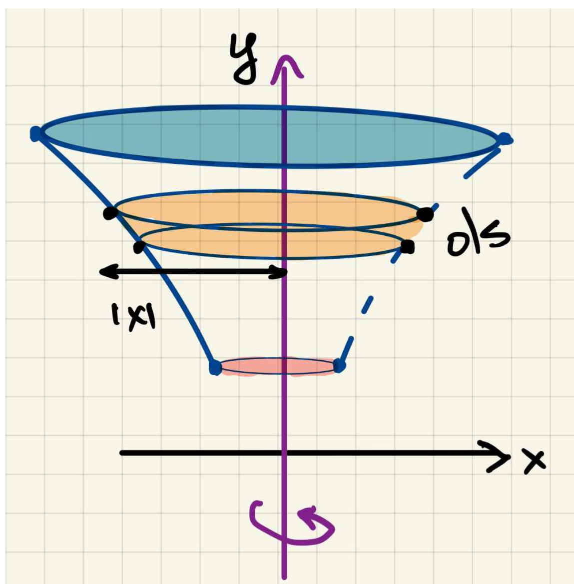
- the x-axis:



- The total area A is
$$A = \int dA$$
where dA is the surface area of the orange slice, centered at $(f(t), 0)$:
$$dA = 2\pi|y|\ ds = 2\pi|g(t)|\ \sqrt{(f'(t))^2 + (g'(t))^2}\ dt$$

- $\Longrightarrow$
$$A = 2\pi \int_a^b |g(t)|\ \sqrt{(f'(t))^2 + (g'(t))^2}\ dt$$

▪ the y-axis:



- The total area A is
$$A = \int dA$$
where dA is the surface area of the orange slice, centered at $(0, g(t))$:
$$dA = 2\pi|x|\ ds = 2\pi|f(t)|\ \sqrt{(f'(t))^2 + (g'(t))^2}\ dt$$

- $\Longrightarrow$
$$A = 2\pi \int_a^b |f(t)|\ \sqrt{(f'(t))^2 + (g'(t))^2}\ dt$$

Example 2: Find the area obtained by rotating the curve

$$\begin{cases} x = 3t^2 \\ y = 2t^3 \end{cases} , \qquad t \in [0,1]$$

about the y axis.

$\Longrightarrow$

- Note that

$$dx = 6t dt \,, \qquad dy = 6t^2 dt$$

$\Longrightarrow$

$$ds = \sqrt{dx^2 + dy^2} = 6t\sqrt{1+t^2}\ dt$$

- $\Longrightarrow$

$$dA = 2\pi |x|\ ds = 2\pi\ 3t^2\ 6t\sqrt{1+t^2}\ dt = 36\pi\ t^3\sqrt{1+t^2}\ dt$$

- The total area A is

$$A = \int dA = \int_0^1 36\pi\ \underbrace{t^3\sqrt{1+t^2}\ dt}_{u=1+t^2, du=2tdt} 18\pi \int (u-1)\sqrt{u} du = 18\pi \int (u^{3/2} - u^{1/2}) du$$

$$= 18\pi \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) = 18\pi \left(\frac{2}{5}(1+t^2)^{5/2} - \frac{2}{3}(1+t^2)^{3/2}\right)\Bigg|_0^1$$

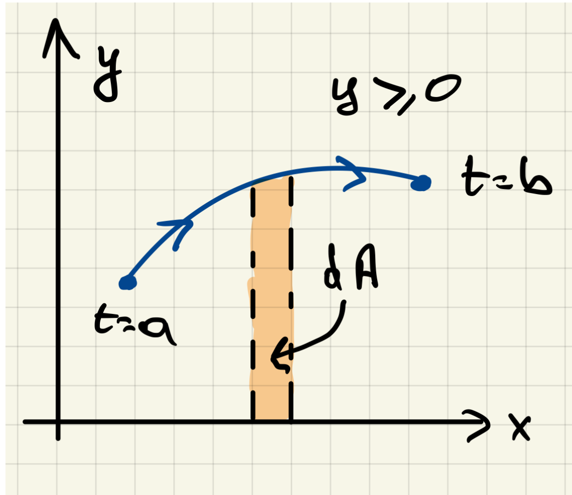
$$= 18\pi \left(\frac{2}{5}(2^{5/2} - 1) - \frac{2}{3}(2^{3/2} - 1)\right) = \frac{24}{5}\pi(\sqrt{2} + 1)$$

Areas bounded by parametric curves

Consider a parametric curve ℓ

$$\begin{cases} x = f(t) \\ y = g(t) > 0 \end{cases} , \qquad t \in [a,b]$$

with $f'(t) > 0$, *i.e.*, $dx > 0$ as the curve moves to the right:



We want to compute the area above x-axis and below the parametric curve.

$\Longrightarrow$

$$A = \int dA$$

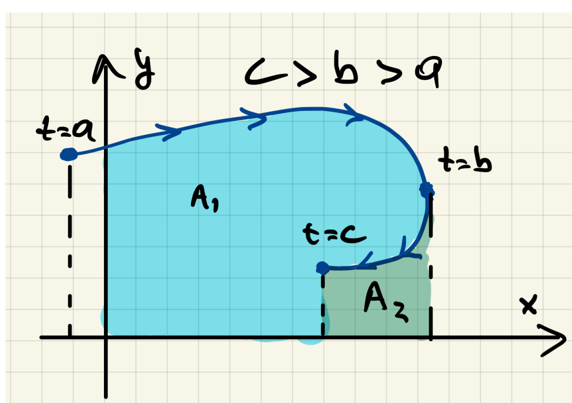
where dA is the area of the orange strip:

$$dA = y dx = \underbrace{g(t) \cdot f'(t)}_{>0} \; dt$$

$\Longrightarrow$ the total area is

$$A = \int dA = \int_a^b g(t) \cdot f'(t) \; dt$$

What if $f'(t)$ changes sign?



Note:

$$f'(t) \geq 0\,, \qquad t \in [a,b]$$
$$f'(t) < 0\,, \qquad t \in [b,c]$$

and

$$g(t) > 0\,, \qquad t \in [a,c]$$

$\Longrightarrow$

- $$\int_a^c g(t)\cdot f'(t)\;dt = \underbrace{\int_a^b g(t)\cdot f'(t)\;dt}_{\equiv I_1} + \underbrace{\int_b^c g(t)\cdot f'(t)\;dt}_{\equiv I_2}$$

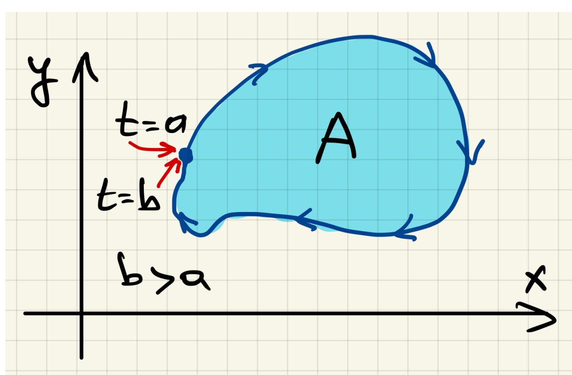
- $$I_1 = A_1 + A_2$$

- $$I_2 = \int_b^c g(t)\cdot f'(t)\;dt = -\int_b^c g(t)\cdot |f'(t)| = -A_2$$

- $\Longrightarrow$
$$\int_a^c g(t)\cdot f'(t)\;dt = (A_1 + A_2) + (-A_2) = A_1$$

i.e., the blue area.

From above, if ℓ is a closed curve, $(f(a), g(a)) = (f(b), g(b))$, with the clockwise direction:



$$\boxed{A = \int_a^b g(t) f'(t) dt}$$

Example 3: Find the area inside the ellipse:

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, \qquad t \in [0, 2\pi]$$

$\Longrightarrow$

- The ellipse is traced counterclockwise, thus

$$A = -\int_0^{2\pi} g(t) f'(t) dt =$$

- $\Longrightarrow$

$$A = -\int_0^{2\pi} b\sin t \cdot (-a\sin t)\ dt = ab\ \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)\ dt$$

$$= ab\left(\frac{t}{2} - \frac{1}{4}\sin(2t)\right)\Bigg|_0^{2\pi} = ab\ \frac{2\pi}{2} = \pi ab$$