Lecture 9.2: **Infinite series**

Consider a sequence

$$\{a_n\} = \{a_1\,,\ a_2\,,\ a_3\,,\ \cdots\}$$

$\Longrightarrow$ we can form an infinite series:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \ \equiv \sum_{i=1}^{\infty} a_i$$

Examples:

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$$1 + 2 + 3 + \cdot + n + \cdots = \sum_{i=1}^{\infty} i$$

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$$1 + \frac{1}{2} + \frac{1}{3} + \cdot + \frac{1}{n} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$$

■

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdot + \frac{1}{2^n} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i}$$

Note: the sigma notation form does not have to start at $i = 1$, *e.g.*,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdot + \frac{1}{2^n} + \cdots = \sum_{i=3}^{\infty} \frac{1}{2^{i-2}}$$

$\Longrightarrow$ We need to be careful to define $\sum$, since it involves and infinite number of terms. We proceed as follows:

- Construct partial sums, s_n as

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\cdots$$

$$s_n \equiv \sum_{i=1}^{n} a_i$$

- The sequence $\{s_n\}$ is called the sequence of partial sums

Convergence of a series

Def: a series

$$\sum_{i=1}^{\infty} a_i$$

is convergent to the sum $\mathcal{S}$ if the sequence of partial sums $\{s_n\}$,

$$s_n \equiv \sum_{i=1}^{n} a_i$$

is convergent to $\mathcal{S}$, *i.e.*,

$$\lim_{n\to\infty} s_n = \mathcal{S}$$

Example 1: Find if convergent, and if yes, compute $\mathcal{S}$ of

$$\sum_{i=1}^{\infty} i = 1 + 2 + 3 + \cdots n + \cdots$$

$\Longrightarrow$

- compute partial sums s_n:

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Study the convergence of the sequence $\{s_n\}$:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

- $\Longrightarrow$ the series diverges to ∞

Geometric series

A geometric series is

$$a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^{n-1} + \cdots = \sum_{i=1}^{\infty} a \cdot r^{i-1}$$

- To study its convergence, form partial sums:

$$s_n = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^{n-1} = a\left(1 + r + r^2 + \cdots r^{n-1}\right)$$

- If $r = 1$, $(a \neq 0)$

$$s_n = n \cdot a \qquad \Longrightarrow \qquad \lim_{n\to\infty} a \cdot n = \pm\infty$$

the series diverges to $\pm\infty$

- If $r \neq 1$, $(a \neq 0)$,

$$1 + r + r^2 + \cdots r^{n-1} = \frac{r^n - 1}{r - 1} \qquad \Longrightarrow \qquad s_n = a \cdot \frac{r^n - 1}{r - 1}$$

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$$\lim_{n\to\infty} s_n = a \cdot \lim_{n\to\infty} \frac{r^n - 1}{r - 1} = \begin{cases} \frac{a}{1-r}\,, & |r| < 1, \text{ since } \lim_{n\to 1} r^n = 0 \\ = a \cdot \infty = \pm\infty\,, & r > 1\,, \text{ since } \lim_{n\to 1} r^n = \infty \\ DNE\,, & r \leq -1 \end{cases}$$

- Thus, the geometric series is
  - convergent for $|r| < 1$
  - diverges to $\pm\infty$ (the sign is the same as the sign of a) for $r \geq 1$
  - diverges for $r \leq -1$

Example 2: Find if convergent, and if yes, compute $\mathcal{S}$ of

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots \frac{1}{3^{n-1}} + \cdots$$

$\Longrightarrow$

- Note that

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots \frac{1}{3^{n-1}} + \cdots = \sum_{i=1}^{\infty} 1 \cdot \left(\frac{1}{3}\right)^{i-1}$$

- The series is geometric with

$$a = 1\,, \qquad r = \frac{1}{3}$$

- $\Longrightarrow$ the series is convergent and

$$\mathcal{S} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Telescopic series

$\Longrightarrow$ Consider a series:

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$$

- To study its convergence, consider partial sums:

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \boxed{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) \boxed{-\frac{1}{n+1}}$$

$$= 1 - \frac{1}{1+n}$$

so each partial sum telescopically folds so that only the first and the last term in the sequence survives

- $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{1+n}\right) = 1$$

$\Longrightarrow$ the original series is converges to $\mathcal{S} = 1$

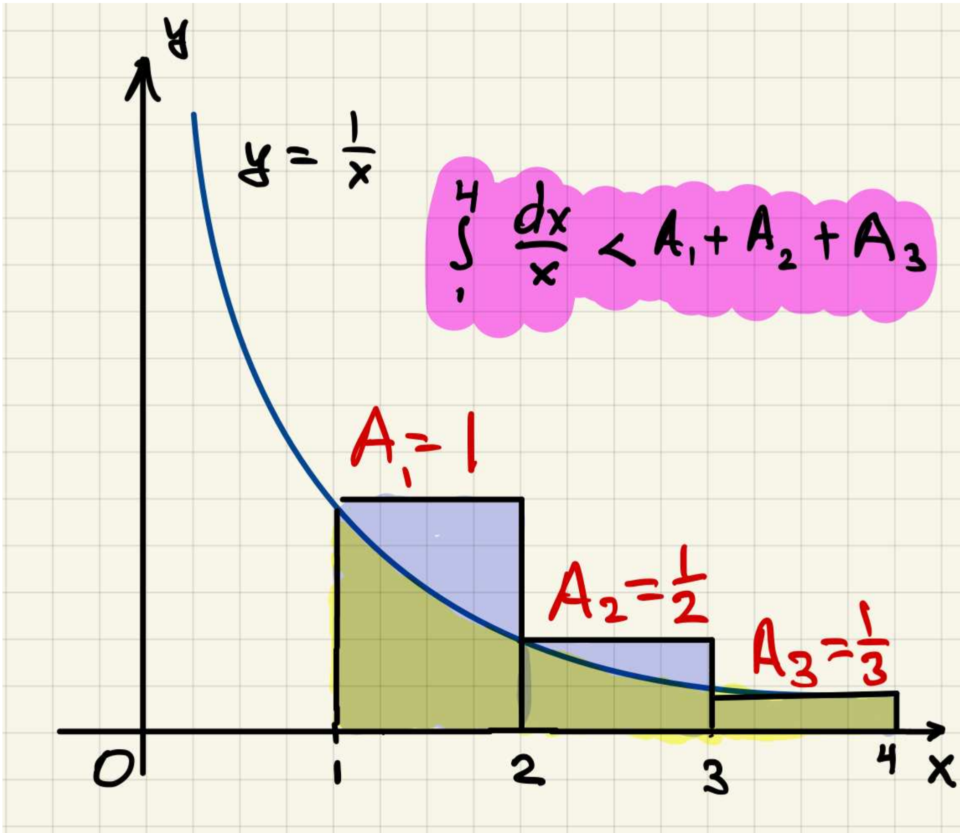
The harmonic series

$\Longrightarrow$ Harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Is it convergent?

- Consider the function $f(x) = \frac{1}{x}$:



- Note from the picture that

$$s_n = \sum_{k=1}^{n} \frac{1}{k} > \int_1^{n+1} \frac{dx}{x} = \ln x \Big|_1^{n+1} = \ln(n+1)$$

- from the comparison of the sequences,

$$\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} \ln(n+1) = \infty$$

- $\implies$ the harmonic series diverges to ∞

Theorem:

- if $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

- if the limit

$$\lim_{n\to\infty} a_n \equiv L$$

exists and $L \neq 0$, or if

$$\lim_{n\to\infty} a_n = DNE$$

$\Longrightarrow$ the series $\sum_{n=1}^{\infty} a_n$ is divergent

Example 3: Determine if convergent:

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

$\Longrightarrow$ Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

the series is divergent

Important: the condition

$$\lim_{n\to\infty} a_n = 0$$

is a necessary **but not a sufficient** condition for the convergence of

$$\sum_{n=1}^{\infty} a_n$$

$\Longrightarrow$ a counterexample is a harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty\,, \qquad \text{while} \qquad \lim_{n=1}^{\infty} \frac{1}{n} = 0$$

Def: Ultimate properties of a sequence are the properties that are true for all terms of a sequence except perhaps for a finite number of them, *e.g.*, consider

$$a_n = \frac{1}{\pi} - \frac{1}{n}$$

$\Longrightarrow$

$$a_1 < 0\,,\ a_2 < 0\,,\ a_3 < 0\,,\ a_n > 0\,, \qquad \text{for all } n \geq 4$$

$\Longrightarrow$ the sequence

$$\left\{\frac{1}{\pi} - \frac{1}{n}\right\}$$

is ultimately positive

Convergence properties

Let the series $\sum_{n=1}^{\infty} a_n$ is convergent to A, and the series $\sum_{n=1}^{\infty} b_n$ is convergent to B. Then:

- $$\sum_{n=1}^{\infty} (a_n \pm b_n) \qquad \text{is convergent to } A \pm B$$
- $$\sum_{n=1}^{\infty} k \cdot a_n \qquad \text{is convergent to } k \cdot A$$
- If $a_n \leq b_n$ for all n, then $A \leq B$

Example 4: Determine if convergent:

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$\Longrightarrow$

- Note that for all n,
$$a_n = \frac{1}{2n-1} > \frac{1}{2n}$$
- $\Longrightarrow$
$$\sum_{n=1}^{\infty} \frac{1}{2n-1} \;\geq\; \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$
since the last series is harmonic $\Longrightarrow$ the original series diverges to ∞ as well