Lecture 9.4: **Absolute and conditional convergence**

Def: We call a series $\sum_{n=1}^{\infty} a_n$ absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ converges

Example 1: consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \qquad \Longrightarrow \qquad a_n = \frac{(-1)^n}{n^3}$$

Since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges as a $p = 3 > 1$ series, the original series is absolutely convergent

Example 2: consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \Longrightarrow \qquad a_n = \frac{(-1)^n}{n}$$

Since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges to ∞ (it is a harmonic series), the original series is not absolutely convergent.

Example 3: consider

$$\sum_{n=1}^{\infty} \frac{n \cos \pi n}{2^n} \qquad \Longrightarrow \qquad a_n = \frac{n(-1)^n}{2^n}$$

$\Longrightarrow$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{2^n}$$

From the ratio test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)2^n}{n2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{2} < 1$$

the series $\sum_{n=1}^{\infty} |a_n|$ converges, thus $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

Theorem: if a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent, *i.e.*, if $\sum_{n=1}^{\infty} |a_n|$ converges $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ converges as well

Conditional convergence

Def: We call a series $\sum_{n=1}^{\infty} a_n$ conditionally convergent if it is convergent, but is not absolutely convergent

Alternating series test:

If a series $\sum_{n=1}^{\infty} a_n$ is such that

- $a_n \cdot a_{n+1} < 0$ for all $n \geq N$ (N some integer), *i.e.*, $\{a_n\}$ is ultimately alternating
- $|a_{n+1}| \leq |a_n|$ for all $n \geq N$
- $\lim_{n\to\infty} a_n = 0$

then this series is convergent

Example 4: consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \Longrightarrow \qquad a_n = \frac{(-1)^n}{n}$$

$\Longrightarrow$

- Note that

$$a_{n+1} \cdot a_n = -\frac{1}{n(n+1)} < 0$$

$$|a_{n+1}| = \frac{1}{n+1} \; < \; |a_n| = \frac{1}{n}$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

- $\Longrightarrow$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test

- In example 2 we established that this series is not absolutely convergent $\Longrightarrow$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent

Example 5: Find the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{3x+2}{-5}\right)^n$$

is

- absolutely convergent
- conditionally convergent
- divergent

$\Longrightarrow$ Note

$$a_n = \frac{1}{2n-1} \left(\frac{3x+2}{-5}\right)^n$$

- From the ratio test

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2n-1}{2(n+1)-1} \cdot \frac{|3x+2|}{5} = \frac{|3x+2|}{5} \cdot \lim_{n\to\infty} \frac{2-\frac{1}{n}}{2+\frac{1}{n}} = \frac{|3x+2|}{5}$$

- $\rho < 1$, the original series is absolutely convergent $\Longrightarrow$

$$\frac{|3x+2|}{5} < 1 \qquad \Longrightarrow \qquad -5 < 3x+2 < 5 \qquad \Longrightarrow \qquad -\frac{7}{3} < x < 1$$

- Note for $\rho > 1$,

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{\rho^n}{2n-1} \underbrace{=}_{\text{use LR}} = \lim_{n\to\infty} \frac{\rho^n \ln \rho}{2} = \infty$$

$\Longrightarrow$ $\lim_{n\to\infty} a_n \neq 0 \Longrightarrow$ the original series is divergent:

$$x \in \left(-\infty, -\frac{7}{3}\right) \cup (1, \infty)$$

- Finally, we need to analyze $\rho = 1$ cases:
  - $x = -\frac{7}{3} \Longrightarrow$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n-1} = \infty$$

diverges to ∞ by relation to the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

- $x = 1 \implies$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{2n-1}$$

converges by alternating series test, but is not absolutely convergent $\implies$ the original series is conditionally convergent