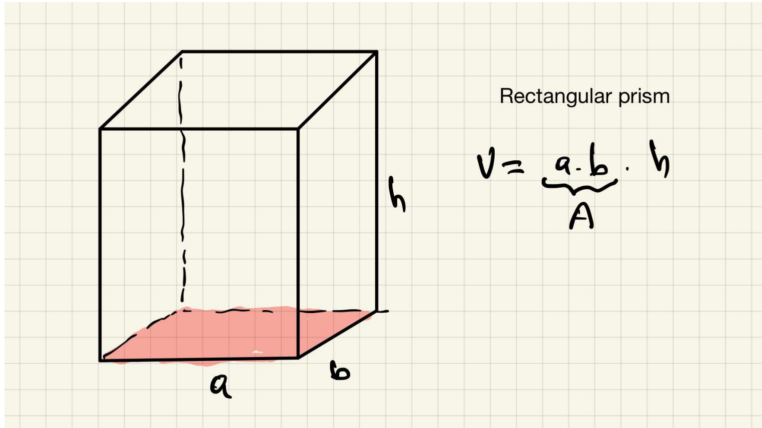
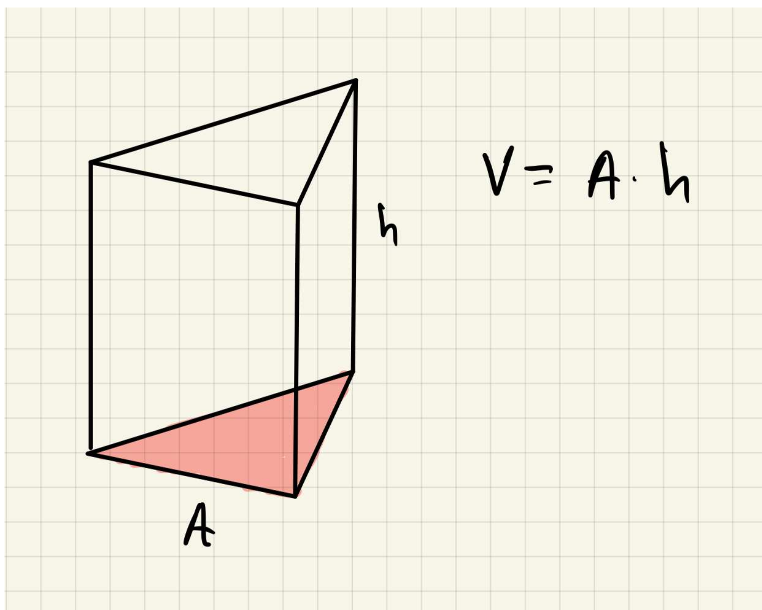
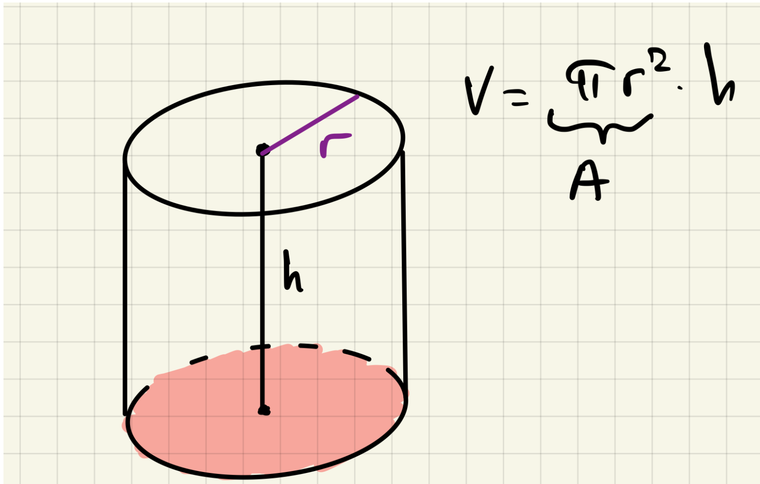
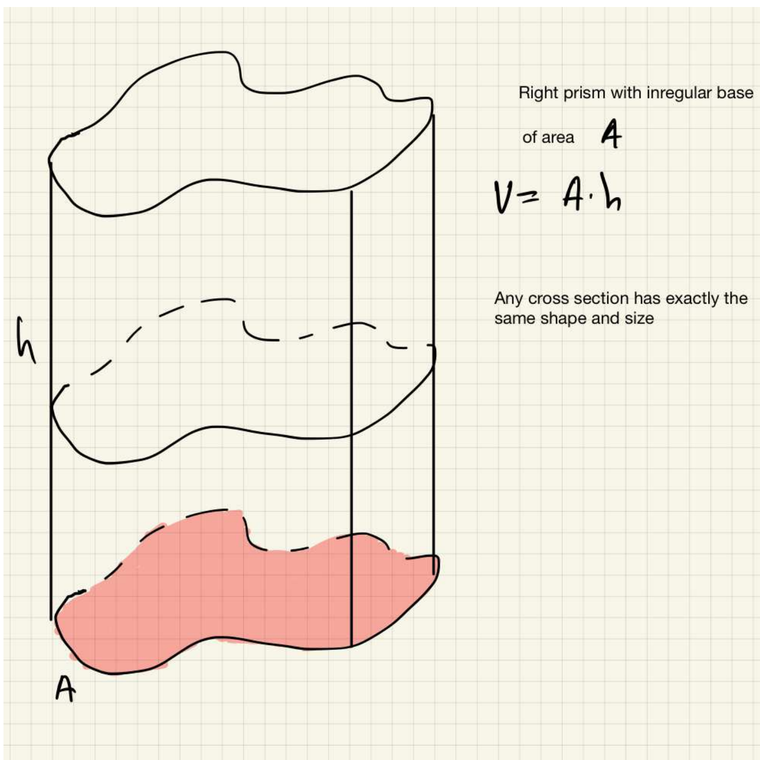
Lecture 7.1: **Volumes by slicing**

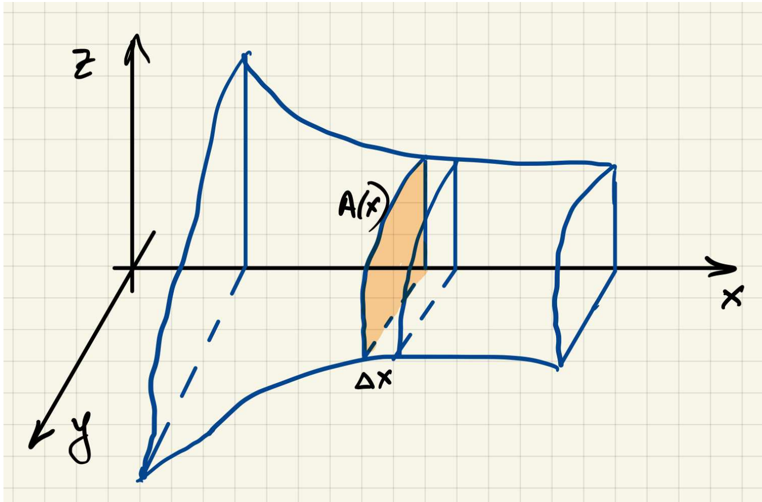





$V = \underbrace{\pi r^2}_{A} \cdot h$
r
h


Right prism with inregular base
of area A
$V = A \cdot h$
Any cross section has exactly the same shape and size
h
A

$\Longrightarrow$ Consider now an irregular shape:



- approximate by slicing into slabs of width $\Delta x \ll 1$
- while the cross-section area is not constant $A = A(x)$, we assume that it is *almost* constant over the width Δx:

$$\Delta V \approx A(x) \cdot \Delta x$$

- total volume is approximated by:

$$V \approx \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} A(x_i)\Delta x_i$$

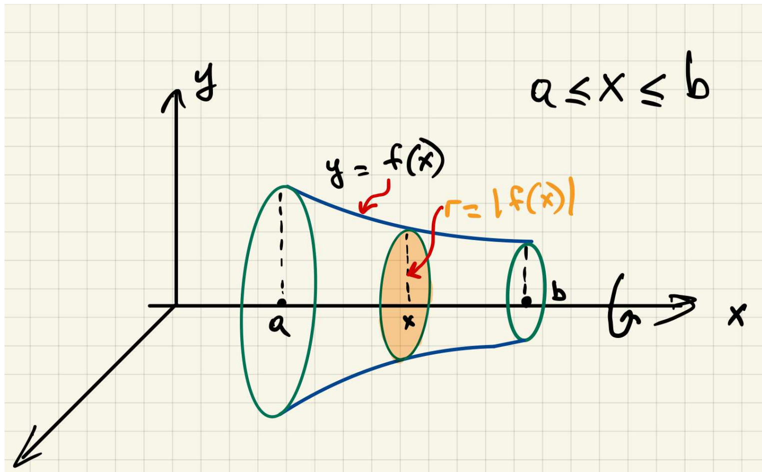
where n is the total number of slices, x_i is a sample point within the width Δx_i

- we take now take a limit $n \to \infty$ and $\Delta x_i \to 0$
- assuming $A(x)$ is a continuous function,

$$V = \lim_{n\to\infty, \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i)\Delta x_i = \int_a^b A(x)dx$$

$$\boxed{V = \int_a^b A(x)dx}$$

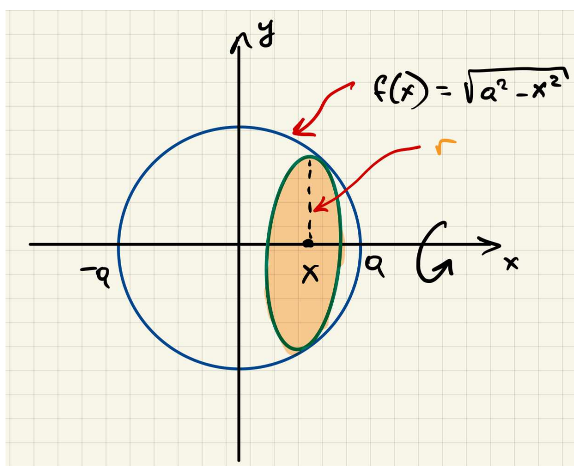
Solid of revolution



$\Longrightarrow$

$$V = \int_a^b A(x)\ dx = \int_a^b \pi r^2\ dx = \int_a^b \pi (f(x))^2\ dx$$

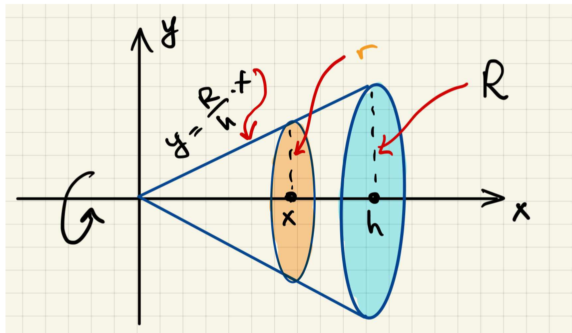
Example 1: Compute the volume of the sphere of radius a



$\Longrightarrow$ Think about the sphere as a solid obtained by rotation of $f(x) = \sqrt{a^2 - x^2}$, $x \in [-a, a]$, about the x-axis

$$V = \int_{-a}^{a} \underbrace{A(x)}_{=\pi f(x)^2}\ dx = \int_{-a}^{a} \pi(a^2 - x^2)\, dx = \pi \left(a^2 x - \frac{1}{3} x^3 \right) \Big|_{-a}^{a} = \pi \left(2a^3 - \frac{2}{3} a^3 \right) = \frac{4}{3} \pi a^3$$

Example 2: Find the volume of a circular cone of base radius R and height h

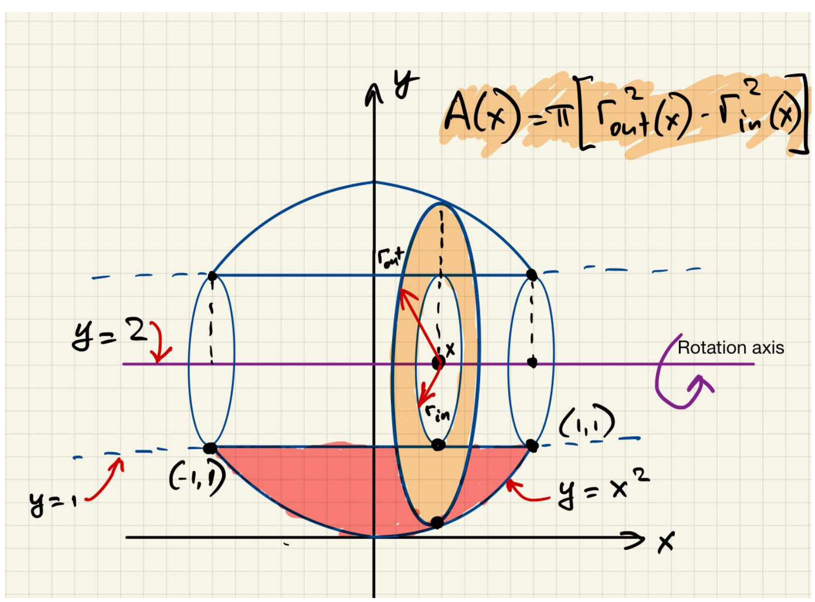


$\Longrightarrow$ Think about the cone as a solid obtained by rotation of $f(x) = \frac{R}{h}x$, $x \in [0, h]$, about the x-axis

$$V = \int_0^h \underbrace{A(x)}_{=\pi f(x)^2} \, dx = \int_0^h \pi \frac{R^2}{h^2} \, x^2 \, dx = \pi \frac{R^2}{h^2} \, \frac{1}{3} x^3 \Big|_0^h = \pi \frac{R^2}{h^2} \, \frac{1}{3} h^3 = \frac{\pi}{3} R^2 h$$

$\Longrightarrow$ What if the axis of rotation is not the x-axis?

Example 3: Find the volume of a solid obtained by rotating the region bounded by $y = x^2$ and $y = 1$ (red) about the line $y = 2$ (purple)



- First, identify the intersection points of $y = x^2$ and $y = 1$:

$$x^2 = 1 \qquad \Longrightarrow \qquad (x, y) = \{(-1, 1)\,,\ (1, 1)\}$$

$\Longrightarrow$

$$x \in [a = -1, b = 1]$$

- At fixed x, the cross-section perpendicular to a rotation axis is a washer (orange) with inner the radius r_{in} and the outer radius r_{out}. Note that

$$r_{in} = r_{in}(x) \qquad \text{and} \qquad r_{out} = r_{out}(x)$$

- The area of the washer is

$$A(x) = \pi \left[r_{out}(x)^2 - r_{in}(x)^2 \right]$$

- From the picture,

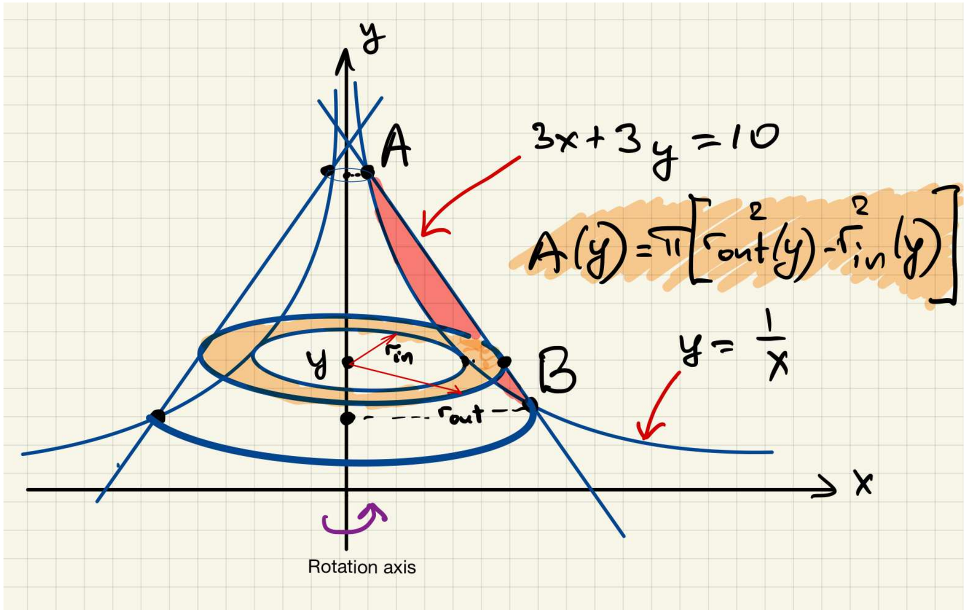
$$r_{in} = 2 - 1 = 1\,, \qquad r_{out} = 2 - \underbrace{x^2}_{y_{bottom} \text{ of the red region}}$$

-

$$V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \pi \Big(\underbrace{(2 - x^2)^2}_{r_{out}^2} - \underbrace{1^2}_{r_{in}^2} \Big)\, dx = \underbrace{2}_{\text{from symmetry}} \pi \int_0^1 \left(3 - 4x^2 + x^4\right)\, dx$$

$$= 2\pi \left(3x - \frac{4}{3}x^3 + \frac{1}{5}x^5 \right) \Bigg|_0^1 = \frac{56\pi}{15}$$

Example 4: Find the volume of a solid obtained by rotating the region bounded by $y = \frac{1}{x}$ and $3x + 3y = 10$ (red) about the y-axis



- First, identify the intersection points of $y = \frac{1}{x}$ and $3x + 3y = 10$ (since rotation is around the y-axis we need to set the integral in dy, thus we need all functions as functions of y) :

$$\frac{3}{y} + 3y = 10 \implies 3y^2 - 10y + 3 = 0 \implies (3y-1)(y-3) = 0$$

$\implies$ the 2 intersection points A and B have coordinates

$$A = \left(\frac{1}{3}, 3\right), \qquad B = \left(3, \frac{1}{3}\right) \qquad \implies \quad y \in \left[\frac{1}{3}, 3\right]$$

- At fixed y, the cross-section perpendicular to a rotation axis is a washer (orange) with inner the radius r_{in} and the outer radius r_{out}. Note that

$$r_{in} = r_{in}(y) \qquad \text{and} \qquad r_{out} = r_{out}(y)$$

- The area of the washer is

$$A(y) = \pi\left[r_{out}(y)^2 - r_{in}(y)^2\right]$$

- From the picture,

$$r_{in} = \frac{1}{y}, \qquad r_{out} = \frac{10-3y}{3} = \frac{10}{3} - y$$

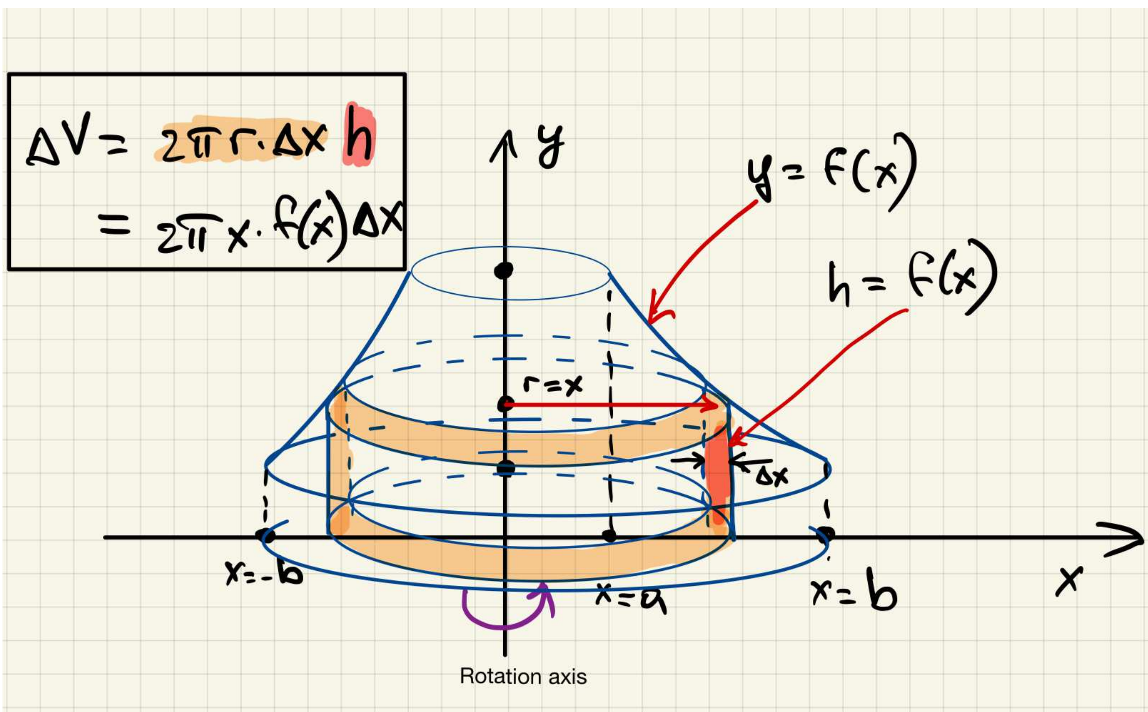
Note: for $r_{in} = x$ we solve for the x for the function $y = \frac{1}{x}$; for $r_{out} = x$ we solve for the x for the function $3x + 3y = 10$

- $$V=\int_{\frac{1}{3}}^{3}A(y)\,dy=\int_{\frac{1}{3}}^{3}\pi\Bigg(\underbrace{\left(\frac{10}{3}-y\right)^2}_{r_{out}^2}-\underbrace{\frac{1}{y^2}}_{r_{in}^2}\Bigg)\,dy=\pi\int_{\frac{1}{3}}^{3}\left(\frac{100}{9}-\frac{20}{3}y+y^2-\frac{1}{y^2}\right)\,dy$$

$$=\pi\left(\frac{100}{9}y-\frac{20}{3}\frac{y^2}{2}+\frac{y^3}{3}+\frac{1}{y}\right)\Bigg|_{\frac{1}{3}}^{3}=\pi\Bigg(\frac{100}{9}\left(3-\frac{1}{3}\right)-\frac{10}{3}\left(9-\frac{1}{9}\right)+\frac{1}{3}\left(27-\frac{1}{27}\right)$$
$$+\left(\frac{1}{3}-3\right)\Bigg)=\frac{512\pi}{81}$$

Volume by cylindrical shells

$\Longrightarrow$ Compute the volume obtained by rotation of region below $y=f(x)>0$, above $y=0$ and $x\in[a,b]$ about y-axis



- approximate by stashing thin cylindrical shells of thickness Δx
- the volume of a single shell at height $y=h=f(x)$

$$\Delta V=\underbrace{A(x)}_{\text{cross}-\text{section area}}\quad h$$

- The cross-section of a shell is a washer of inner and outer radii

$$r_{in} = x\,, \qquad r_{out} = x + \Delta x$$

$\Longrightarrow$

$$A(x) = \pi \left[r_{out}^2 - r_{in}^2\right] = \pi \left((x+\Delta x)^2 - x^2\right) = \pi \left(x^2 + 2x\Delta x + (\Delta x)^2 - x^2\right) \approx 2\pi x\, \Delta x$$

where we neglected $(\Delta x)^2$ term, which is valid in the limit $\Delta x \to 0$

$\Longrightarrow$ thus

$$\Delta V \approx 2\pi x f(x)\ \Delta x$$

- the total volume is approximated as sum of volumes of individual shells

$$V \approx \sum_{i=1}^{n}\ \Delta V_i \approx \sum_{i=1}^{n}\ 2\pi x_i f(x_i)\Delta x_i$$

where n is the total number of slices, x_i is a sample point within the width Δx_i

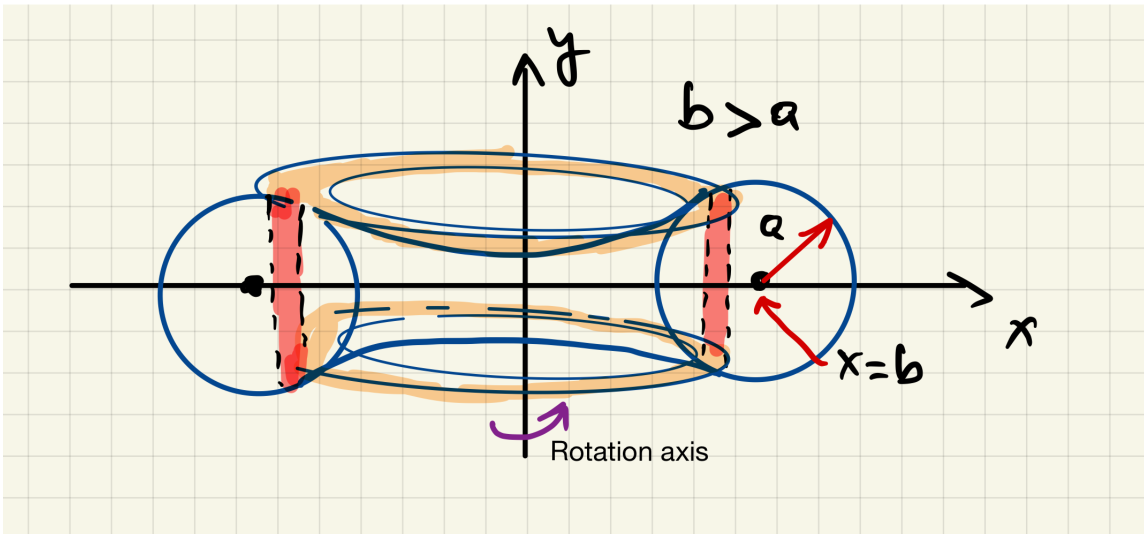
- we take now take a limit $n \to \infty$ and $\Delta x_i \to 0$

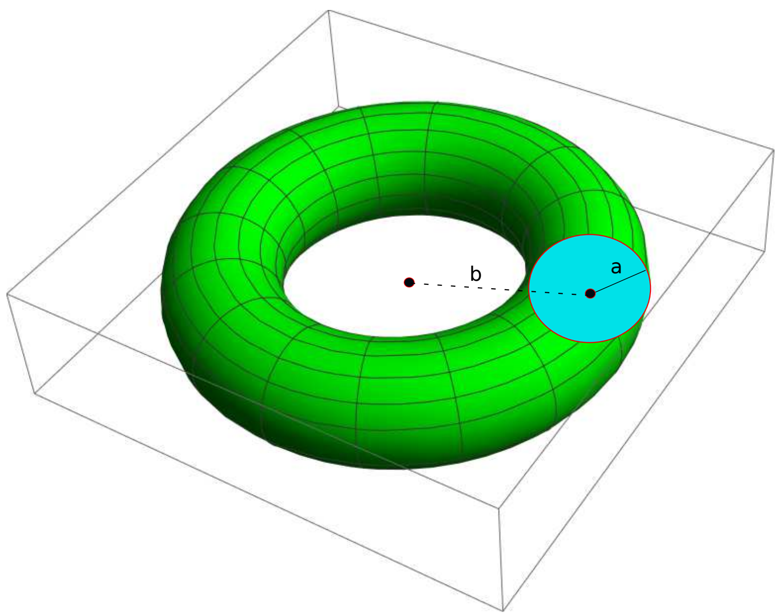
- assuming $f(x)$ is a continuous function,

$$V = \lim_{n\to\infty, \Delta x_i \to 0} \sum_{i=1}^{n}\ 2\pi x_i f(x_i)\ \Delta x_i = \int_a^b 2\pi x f(x) dx$$

$$\boxed{V = 2\pi \int_a^b x f(x) dx}$$

Example 4 A circular disk of radius a is centered on the x-axis at $x = b$ ($b > a$). The disk is rotated around the y-axis to produce a *torus*. Find the volume of this torus





- We use method of cylindrical shells:

$$V = 2\pi \int_{x_{min}}^{x_{max}} x h(x) dx$$

where $x_{min} = b - a$ (the minimum value of $x > 0$ of the solid), $x_{max} = b + a$ (the maximum value of $x > 0$ of the solid), and $h(x)$ is the height of a cylindrical shell

- We need to determine $y = f(x)$. The equation of a disk boundary is

$$(x-b)^2 + y^2 = a^2 \qquad \Longrightarrow \qquad y = f(x) = \sqrt{a^2 - (x-b)^2}$$

From the picture it is clear that

$$h(x) = 2f(x)$$

- Thus,

$$V = 4\pi \int_{b-a}^{b+a} \underbrace{x\sqrt{a^2-(x-b)^2}\,dx}_{u=x-b,du=dx} = 4\pi \int_{-a}^{a} (u+b)\sqrt{a^2-u^2}\,du$$

$$= 4\pi \underbrace{\int_{-a}^{a} u\sqrt{a^2-u^2}\,du}_{=0 \text{ since the integrand is an odd function}} + 4\pi \int_{-a}^{a} b \underbrace{\sqrt{a^2-u^2}\,du}_{u=a\sin\theta,du=a\cos\theta}$$

$$= 4\pi ba^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta = 4\pi a^2 b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta = 4\pi a^2 b\,\frac{1}{2}\,2\left(\frac{\pi}{2} + \underbrace{\left.\frac{\sin(2\theta)}{4}\right|_{-\pi/2}^{\pi/2}}_{=0}\right)$$

$$= 2\pi^2 a^2 b$$