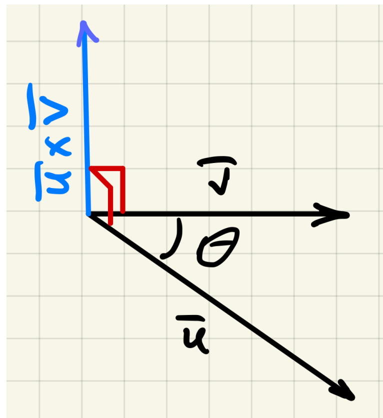
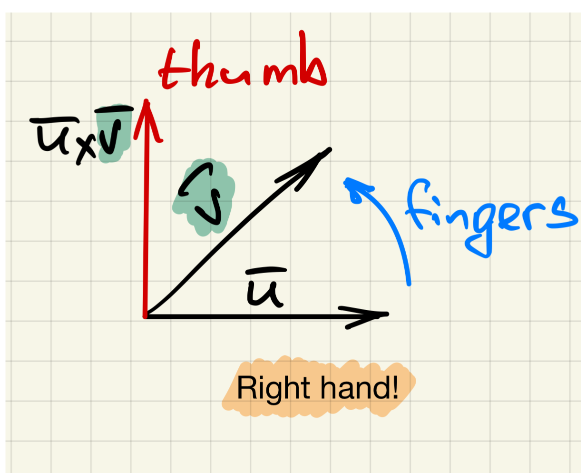
Lecture 10.3: **The cross produced in 3D**

The cross product is defined in 3D space only:



$\Longrightarrow \qquad \vec{u}\times\vec{v}$ is a vector such that

- $\vec{u}\times\vec{v}\perp\vec{u}$ and $\vec{u}\times\vec{v}\perp\vec{v}$
- $|\vec{u}\times\vec{v}| \ = |\vec{u}|\cdot\|\vec{v}|\cdot\sin\theta \Longrightarrow$
  - if $\vec{u}\parallel\vec{v}$, *i.e.*, $\theta=0$ or $\theta=\pi \Longrightarrow \qquad \vec{u}\times\vec{v}=\vec{0}$
- The orientation of the vector $\vec{u}\times\vec{v}$ is determined by the right hand rule:

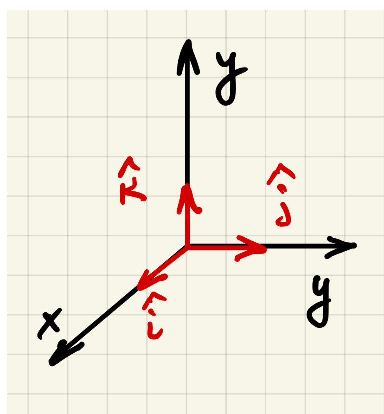


Curl your right hand such that your fingers indicate the direction in which $\vec{u}$ can be rotated to $\vec{v}$ by the shortest angle $\Longrightarrow$ your thumb will point the orientation of $\vec{u} \times \vec{v}$

Properties of the cross product:

- $\vec{u} \times \vec{u} = \vec{0}$
- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $(k \cdot \vec{u}) \times \vec{v} = k \cdot (\vec{u} \times \vec{v})$
- $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = \vec{0}$, since $\vec{u} \times \vec{v} \perp \vec{u}$ and $\vec{u} \times \vec{v} \perp \vec{v}$

Example 1: Consider the basis vectors



- $\hat{i} \times \hat{j} = \hat{k}\,, \qquad \hat{j} \times \hat{i} = -\hat{k}$
- $\hat{j} \times \hat{k} = \hat{i}\,, \qquad \hat{k} \times \hat{j} = -\hat{i}$
- $\hat{k} \times \hat{i} = \hat{j}\,, \qquad \hat{i} \times \hat{k} = -\hat{j}$
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

Example 2: Compute $(3\hat{\imath} - \hat{k}) \times (\hat{\imath} + \hat{\jmath})$:

$$(3\hat{\imath} - \hat{k}) \times (\hat{\imath} + \hat{\jmath}) = 3\hat{\imath} \times (\hat{\imath} + \hat{\jmath}) - \hat{k} \times (\hat{\imath} + \hat{\jmath}) = 3 \cdot \hat{\imath} \times \hat{\imath} + 3\hat{\imath} \times \hat{\jmath} - \hat{k} \times \hat{\imath} - \hat{k} \times \hat{\jmath} =$$

$$= 3 \cdot \vec{0} + 3\hat{k} - \hat{\jmath} + \hat{\imath} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$$

Theorem: if $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$ and $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ then

- following the same procedure as in example 2,

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\ \hat{\imath} + (u_3 v_1 - u_1 v_3)\ \hat{\imath} + (u_1 v_2 - u_2 v_1)\ \hat{k}$$

- better to compute the cross product using determinant formula:

$$\vec{u} \times \vec{v} \ = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

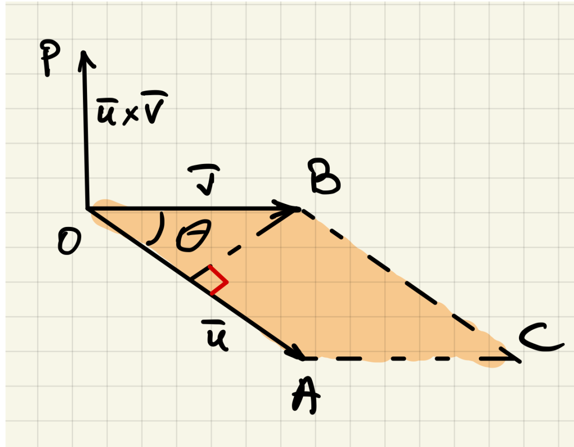
Example 3: Compute $\underbrace{(3\hat{\imath} - \hat{\jmath} + 2\hat{k})}_{\vec{u}} \times \underbrace{(\hat{\imath} + \hat{\jmath} - 3\hat{k})}_{\vec{v}}$:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 2 \\ 1 & 1 & -3 \end{vmatrix} = \hat{\imath} \cdot \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} - \hat{\jmath} \cdot \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} + \hat{k} \cdot \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= \hat{\imath}(3 - 2) - \hat{\jmath}(-9 - 2) + \hat{k}(3 + 1) = \hat{\imath} + 11\hat{\jmath} + 4\hat{k}$$

Area of a parallelogram from the cross product:

$\Longrightarrow$ Consider a parallelogram $OBCA$ with formed by vectors $\vec{v}$ and $\vec{u}$:



- the area $A_{OBCA} = OA \times \underbrace{h}_{=OB\sin\theta} = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta = |\vec{u} \times \vec{v}|$
- area $A_{OBA} = \frac{1}{2}OA \times \underbrace{h}_{=OB\sin\theta} = \frac{1}{2}|\vec{u}| \cdot |\vec{v}| \cdot \sin\theta = \frac{1}{2}|\vec{u} \times \vec{v}|$

The scalar triple product

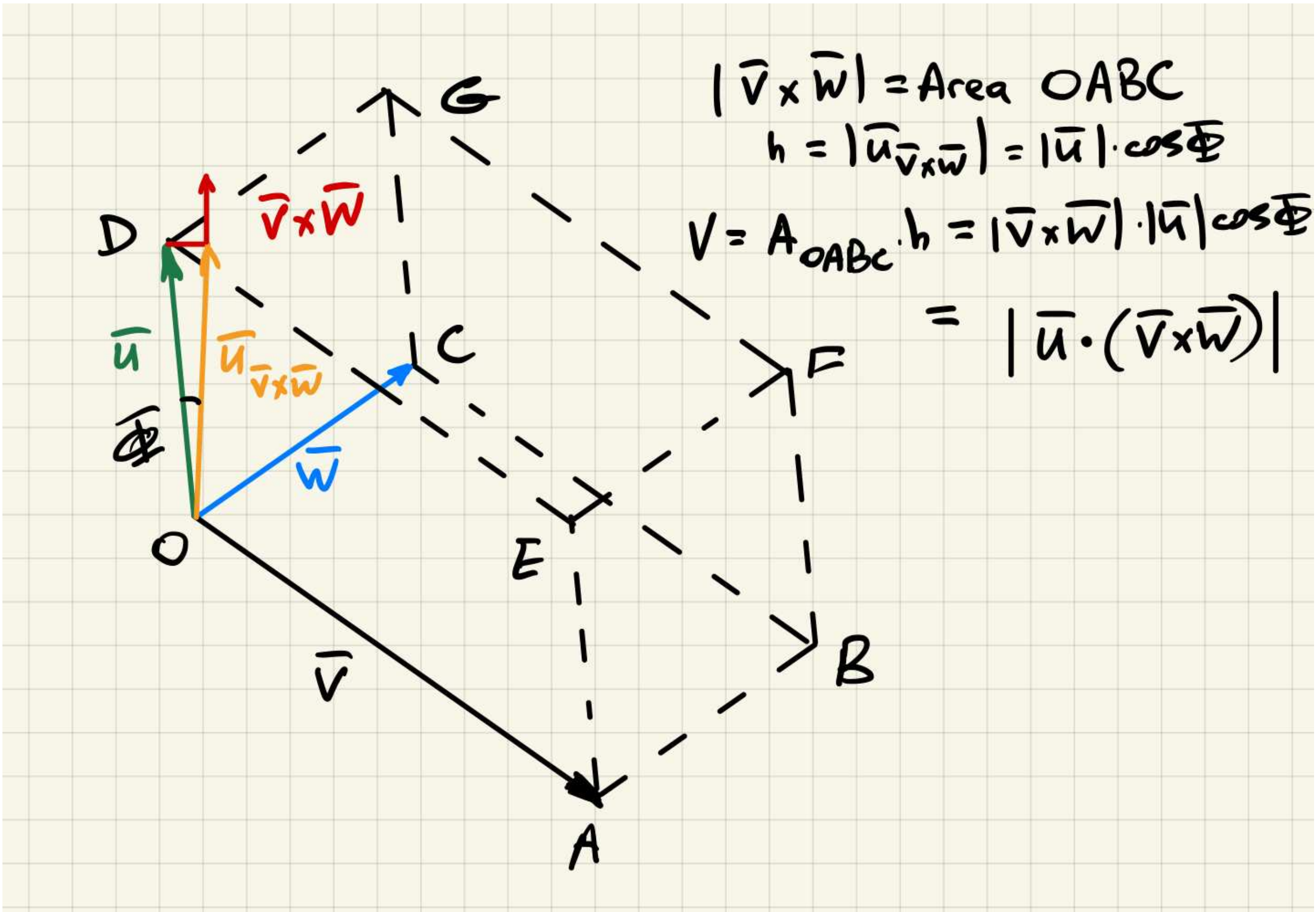
$\Longrightarrow$ Given three vectors $\vec{u}\,,\ \vec{v}\,,\ \vec{w}$ the scalar triple product is

$$\vec{u} \cdot (\vec{v} \times \vec{w}) \;=\; \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Using properties of the determinants:

- $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$
- if any two of these vectors are collinear, *i.e.*, $\parallel$ to each other $\Longrightarrow$ $\vec{u}\cdot(\vec{v}\times\vec{w}) = 0$ (since determinant will have two rows that are proportional)
- if one of the vectors is a linear combination of the other two, the triple product vanishes $\Longrightarrow$ again from the properties of the determinants: one row is a linear combination of the other two
- Three vectors are called coplanar if they are in the same plane. They are necessarily linear dependent $\Longrightarrow$ $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

Triple product in 3D geometry:



$\Longrightarrow$ the volume of the parallelepiped formed by vectors $\vec{v}$, $\vec{w}$, $\vec{u}$ is

$$V_{OABCDEFG} = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

Further applications in engineering/physics:

- angular momentum

$$\vec{L} = \vec{r} \times m \cdot \vec{v}$$

- electromagnetic force

$$\vec{F} = q \cdot \vec{v} \times \vec{B}$$

- torque

$$\vec{\tau} = \vec{r} \times \vec{F}$$