Lecture 7.9: **First-order differential equations**

Def: An equation for an unknown function $y(x)$ that involves its derivatives is called a differential equation

$\Longrightarrow$ First-order means that a differential equation contains only $y'(x)$ and not higher-order derivatives of $y(x)$:

$$F(x, y, y') = 0$$

■ Separable equations

$$y'(x) = f(x) \cdot g(y) \qquad \Longrightarrow \qquad \frac{dy}{dx} = f(x) \cdot g(y)$$

$$\Longrightarrow \qquad \frac{dy}{g(y)} = f(x)dx$$

$\Longrightarrow$ Integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)dx \qquad \Longrightarrow \qquad \text{solve for } y(x)$$

Example 1: find the most general solution of

$$y' = \frac{x}{y}$$

$\Longrightarrow$

- separate the variables

$$\frac{dy}{dx} = \frac{x}{y} \qquad \Longrightarrow \qquad ydy = xdx$$

- Integrate

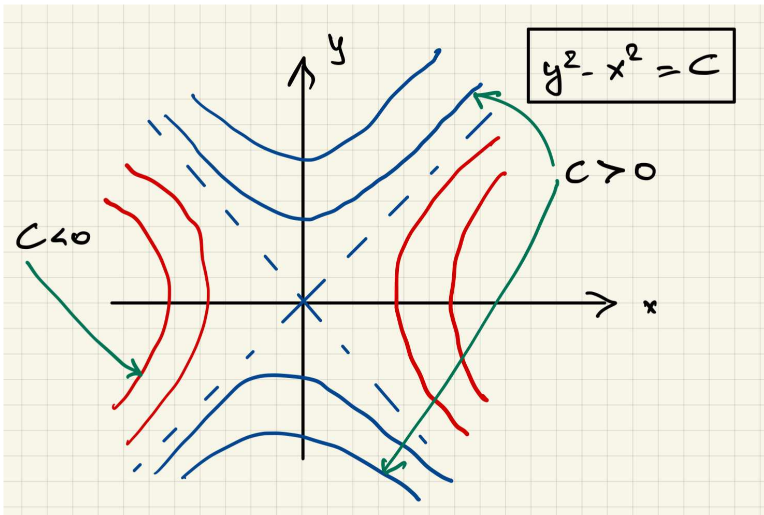
$$\int ydy = \int xdx \qquad \Longrightarrow \qquad \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

where C is an arbitrary integration constant

- We can express the solution as (note that we relabeled the integration constant $C \to \frac{C}{2}$)

$$y^2 - x^2 = C \qquad \Longrightarrow \qquad \text{hyperbola}$$

- Note that there are infinitely many solutions of the differential equation, one for each C:



where *blue* hyperbolas are for $C > 0$ and *red* hyperbolas are for $C < 0$

Example 2: find the most general solution of

$$\frac{dx}{dt} = e^x \sin t$$

$\Longrightarrow$

- separate the variables

$$e^{-x} dx = \sin t \; dt$$

- integrate both sides

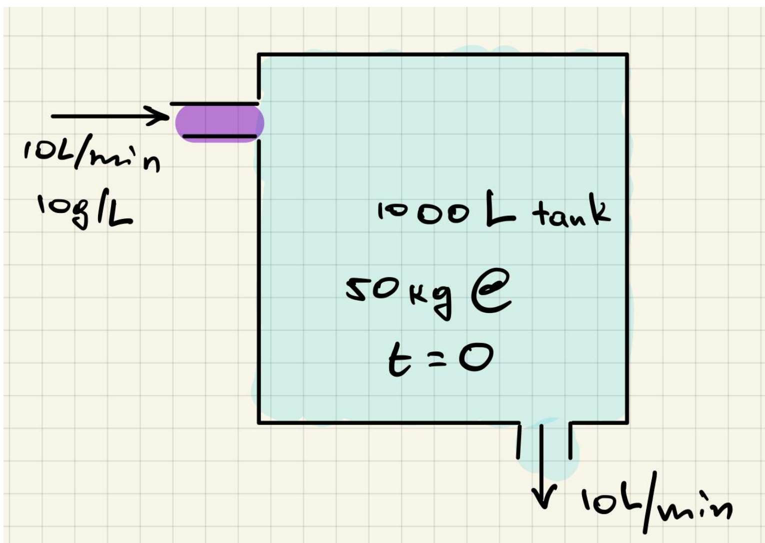
$$\int e^{-x} dx = \int \sin t \; dt \qquad \Longrightarrow \qquad -e^{-x} = -\cos t + C$$

- solve for $x(t)$:

$$x(t) = -\ln(\cos t - C)$$

- Note that solution exists only when $(\cos t - C) > 0$

Example 3: A tank contains $1000L$ of brine with $50kg$ dissolved salt. Brine with $10g$ salt per liter flows into the tank at a rate of $10L/min$. Solution flows out of the tank at $10L/min$. How much salt is in the tank after $40min$?



$\Longrightarrow$

- Let $x(t)$ is the amount of salt in the tank at time t.
- The concentration of the salt in the tank:

$$C_{tank} = \frac{x(t)}{1000}$$

- The amount of salt in the tank changes with time:

$$\frac{dx}{dt} = Rate_{in} - Rate_{out}$$

- The in/out rates are

$$Rate_{in} = 10\frac{L}{min} \cdot 10\frac{g}{L} = 100\ \frac{g}{min} = \frac{1}{10}\ \frac{kg}{min}$$

$$Rate_{out} = 10\frac{L}{min} \cdot C_{tank}\frac{kg}{L} = \frac{10x(t)}{1000}\ \frac{kg}{min} = \frac{1}{100}x(t)\ \frac{kg}{min}$$

- $\Longrightarrow$

$$\frac{dx}{dt} = \frac{1}{10} - \frac{x}{100} = \frac{10-x}{100}$$

- Separate variables:

$$\frac{dx}{10-x} = \frac{dt}{100}$$

- Integrate both sides:

$$\int \frac{dx}{10-x} = \int \frac{dt}{100} \qquad \Longrightarrow \qquad -\ln|10-x| = \frac{t}{100} + C$$

- Solve for x, using the fact that initially $x(0) = 50 > 10$

$$|10 - x(t)| = e^{-\frac{t}{100}-C} \qquad \Longrightarrow \qquad x(t) - 10 = e^{-\frac{t}{100}-C}$$

- To determine C we impose the the initial condition

$$x(0) = 50 \qquad \Longrightarrow \qquad 50 - 10 = e^{-\frac{0}{100}-C} \qquad \Longrightarrow \qquad e^{-C} = 40$$

- $\Longrightarrow$

$$x(t) = 10 + e^{-C}e^{-\frac{t}{100}} = 10 + 40e^{-\frac{t}{100}}$$

- $\Longrightarrow$

$$x(40) = 10 + 40e^{-4/10}$$

- <u>First-order linear equations:</u>

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are continuous functions.

$\Longrightarrow$

- The differential equation is solved introducing an integration factor

$$e^{\mu(x)}$$

  where

$$\mu(x) = \int p(x)dx \qquad \Longrightarrow \qquad \mu'(x) = p(x)$$

- Multiply the original equation with the integration factor

$$e^{\mu(x)} \cdot \left(\frac{dy}{dx} + p(x)y\right) = e^{\mu(x)} \cdot q(x)$$

  $\Longrightarrow$ Note the LHS:

$$e^{\mu(x)} \cdot \left(\frac{dy}{dx} + p(x)y\right) = e^{\mu(x)} \cdot \frac{dy}{dx} + \frac{d\mu}{dx} e^{\mu(x)} \cdot y = e^{\mu(x)} \cdot \frac{dy}{dx} + \frac{d}{dx}\left(e^{\mu(x)}\right) \cdot y$$

$$= \frac{d}{dx}\left(e^{\mu(x)} \cdot y\right)$$

- $\Longrightarrow$

$$\frac{d}{dx}\left(e^{\mu(x)} \cdot y\right) = e^{\mu(x)} \cdot q(x)$$

$$\int d\left(e^{\mu(x)} \cdot y\right) = \int e^{\mu(x)} \cdot q(x)dx$$

$$e^{\mu(x)} \cdot y - C = \int e^{\mu(x)} \cdot q(x)dx$$

- Solve for y:

$$\boxed{y(x) = e^{-\mu(x)} \cdot \int e^{\mu(x)} \cdot q(x)dx + C \cdot e^{-\mu(x)}}$$

Example 4: find the general solution of

$$\frac{dy}{dx} + \frac{2}{x}\, y = \frac{1}{x^2}$$

$\Longrightarrow$

- This is a linear equation with

$$p(x) = \frac{2}{x}\,, \qquad q(x) = \frac{1}{x^2}$$

- Compute the integration factor:

$$\mu = \int p(x)dx = \int \frac{2}{x}dx = 2\ln x \qquad \Longrightarrow \qquad e^{\mu} = x^2$$

- The general solution is

$$\boxed{y(x) = e^{-\mu(x)} \cdot \int e^{\mu(x)} \cdot q(x)dx + C \cdot e^{-\mu(x)}}$$

- $\Longrightarrow$

$$\int e^{\mu} q dx = \int x^2 \cdot \frac{1}{x^2}\, dx = \int dx = x$$

- $\Longrightarrow$

$$y = \frac{1}{x^2} \cdot x + C \cdot \frac{1}{x^2} = \frac{1}{x} + \frac{C}{x^2}$$