Lecture 13.4: **Higher-order derivatives**

$\Longrightarrow$ Consider the function of $z = f(x, y)$:

- $$\frac{\partial z}{\partial x} = f_1(x, y)$$
  $$\frac{\partial z}{\partial y} = f_2(x, y)$$
  are in itself the functions of (x, y)

- Now differentiate further (assuming we can):
  $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \equiv f_{11}(x, y)$$
  $$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \equiv f_{22}(x, y)$$
  and for the mixed second order partial derivatives:
  $$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \equiv f_{12}(x, y)$$
  $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \equiv f_{21}(x, y)$$

- Works in a similar way for functions of more than two variables

- We can also keep differentiating $\Longrightarrow$ 3rd, 4th order, *etc.*

Example 1: Compute all second order derivatives of

$$z = f(x, y) = \sqrt{3x^2 + y^2}$$

$\Longrightarrow$

- the first-order derivatives:

$$f_1 = \frac{\partial z}{\partial x} = \frac{1}{2(3x^2+y^2)^{1/2}} \cdot 6x = \frac{3x}{(3x^2+y^2)^{1/2}}$$

$$f_2 = \frac{\partial z}{\partial y} = \frac{1}{2(3x^2+y^2)^{1/2}} \cdot 2y = \frac{y}{(3x^2+y^2)^{1/2}}$$

- the second-order derivatives:

$$f_{11} = \frac{\partial f_1}{\partial x} = \frac{3}{(3x^2+y^2)^{1/2}} - \frac{3x}{2(3x^2+y^2)^{3/2}} \cdot 6x = \frac{3(3x^2+y^2)-9x^2}{(3x^2+y^2)^{3/2}} = \frac{3y^2}{(3x^2+y^2)^{3/2}}$$

$$f_{22} = \frac{\partial f_2}{\partial y} = \frac{1}{(3x^2+y^2)^{1/2}} - \frac{y}{2(3x^2+y^2)^{3/2}} \cdot 2y = \frac{(3x^2+y^2)-y^2}{(3x^2+y^2)^{3/2}} = \frac{3x^2}{(3x^2+y^2)^{3/2}}$$

$$f_{12} = \frac{\partial f_1}{\partial y} = -\frac{3x}{2(3x^2+y^2)^{3/2}} \cdot 2y = \frac{-3xy}{(3x^2+y^2)^{3/2}}$$

$$f_{21} = \frac{\partial f_2}{\partial x} = -\frac{y}{2(3x^2+y^2)^{3/2}} \cdot 6x = \frac{-3xy}{(3x^2+y^2)^{3/2}}$$

IMPORTANT : note that the mixed derivatives are equal, *i.e.*, $f_{12} = f_{21}$ or

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

- This is always true: mixed second order derivatives are equal, regardless of the order of differentiation

- Also works for mixed higher order derivatives, *e.g.*, for $w = f(x, y, z)$,

$$\frac{\partial^3 w}{\partial y \partial x^2} = \frac{\partial^3 w}{\partial x \partial y \partial x} = \frac{\partial^3 w}{\partial x^2 \partial y}$$

or

$$f_{112} = f_{121} = f_{211}$$

Laplace equation

Def. A function $z = f(x, y)$ satisfies the (2D) Laplace equation if

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

A function that satisfies the Laplace equation is called harmonic.

Example 1: Show that $z = e^{kx}\cos(ky)$, where k =const, is harmonic.

$\Longrightarrow$

- for the first derivatives:

$$f_1 = \frac{\partial z}{\partial x} = ke^{kx}\cos(ky)\,, \qquad f_2 = \frac{\partial z}{\partial y} = -ke^{kx}\sin(ky)$$

- for the second derivatives:

$$f_{11} = \frac{\partial^2 z}{\partial x^2} = k^2 e^{kx}\cos(ky) = k^2 z\,, \qquad f_{22} = \frac{\partial^2 z}{\partial y^2} = -k^2 e^{kx}\cos(ky) = -k^2 z$$

- $\Longrightarrow$

$$f_{11} + f_{22} = k^2\ z - k^2\ z = 0$$

- You can check, repeating above steps, that $z = e^{kx}\sin(ky)$ is also harmonic.

$\Longrightarrow$ Laplace equation can be defined in any number of dimensions, *e.g.,* in 3D, a function $w = f(x, y, z)$ is harmonic if,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

The wave equation

Def. A function $u = f(x, t)$, with x being the position in space and t being the time, satisfies the (2D) wave equation if

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where c is a constant

Example 2: Show that $u(x,t) = g(x - ct) + h(x + ct)$, where g and h are arbitrary (differentiable) functions of a single variable

$\Longrightarrow$

- for the first derivatives:

$$f_1 = \frac{\partial u}{\partial x} = g' + h' \,, \qquad f_2 = \frac{\partial u}{\partial t} = -c \cdot g' + c \cdot h'$$

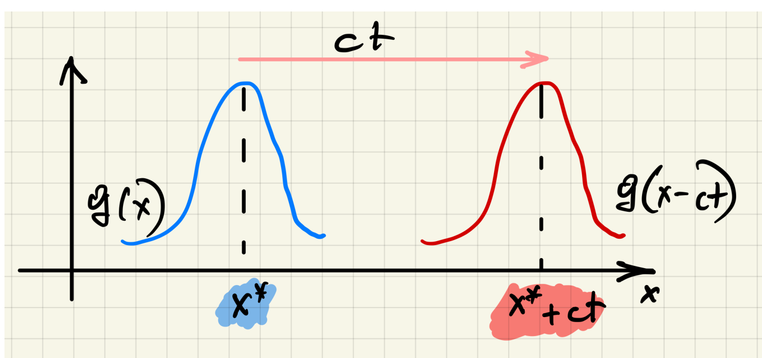
- for the second derivatives:

$$f_{11} = \frac{\partial^2 u}{\partial x^2} = g'' + h'' \,, \qquad f_{22} = (-c)^2 \cdot g'' + c^2 \cdot h'' = c^2 \; f_{11}$$

- $\Longrightarrow$

$$f_{22} = c^2 \; f_{11}$$

- To simplify things, let's assume that $h \equiv 0$. What is the physical meaning of g functions and the constant c?
  - Let $t = 0 \Longrightarrow$ the profile is given by the function $g(x)$
  - at $t \neq 0$, the same profile shifts to the right by distance ct
  - $\Longrightarrow$ $g(x - ct)$ describes a travelling wave (travelling signal) to the right with velocity c, with the wave (signal) profile determined by g



- Similarly, $h(x + ct)$ describes a travelling wave (travelling signal) to the left with velocity c, with the wave (signal) profile determined by h