Lecture 13.3: **Partial derivatives**

$\Longrightarrow$ Consider the function of n variables $f(x_1, x_2, \cdots, x_n)$. We can differentiate with respect to one variable at a time, keeping the other variables constant — there will be n partial derivatives; *e.g.*, for $f(x, y)$:

- $$f_1(x,y) \equiv \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \equiv \frac{\partial f}{\partial x} \equiv D_1 f(x,y)$$
- $$f_2(x,y) \equiv \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \equiv \frac{\partial f}{\partial y} \equiv D_2 f(x,y)$$

Example 1: Compute partial derivatives of $f(x,y) = x^2y - y + 2x$

$\Longrightarrow$

- $$\frac{\partial f}{\partial x} = 2xy + 2$$
- $$\frac{\partial f}{\partial y} = x^2 - 1$$

Example 2: Compute partial derivatives of $f(x,y) = \sin\left(\frac{x^2}{y}\right)$

$\Longrightarrow$

- $$\frac{\partial f}{\partial x} = \cos\left(\frac{x^2}{y}\right)\cdot\frac{2x}{y}$$
- $$\frac{\partial f}{\partial y} = \cos\left(\frac{x^2}{y}\right)\cdot\left(-\frac{x^2}{y^2}\right)$$

Example 4: Compute partial derivatives of

$$f(x,y,z) = \frac{xy}{1+xz+yz}$$

$\Longrightarrow$

- $$\frac{\partial f}{\partial x} = \frac{y}{1+xz+yz} - \frac{xy\cdot z}{(1+xz+yz)^2} = \frac{y+xyz+y^2z-xyz}{(1+xz+yz)^2} = \frac{y(1+yz)}{(1+xz+yz)^2}$$

- $$\frac{\partial f}{\partial y} = \frac{x}{1+xz+yz} - \frac{xy\cdot z}{(1+xz+yz)^2} = \frac{x+x^2z+xyz-xyz}{(1+xz+yz)^2} = \frac{x(1+xz)}{(1+xz+yz)^2}$$

- $$\frac{\partial f}{\partial z} = -\frac{xy(x+y)}{(1+xz+yz)^2}$$

Standard differentiation rules apply

Example 5: If $z = f(x/y)$, prove that

$$x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

$\Longrightarrow$

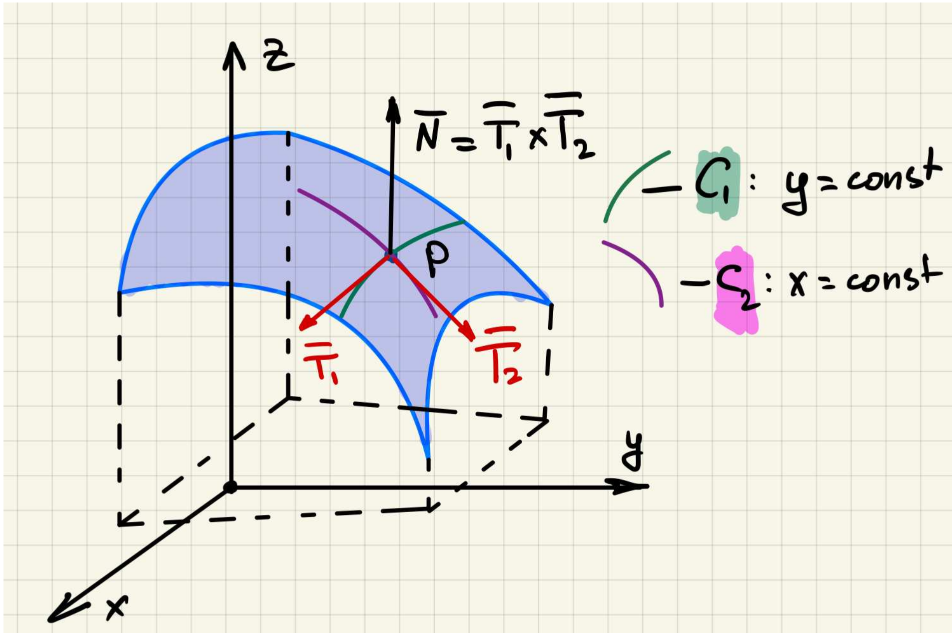
- $$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\ f\left(\frac{x}{y}\right) = f'\left(\frac{x}{y}\right)\cdot\frac{1}{y}$$

- $$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\ f\left(\frac{x}{y}\right) = f'\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right)$$

- $\Longrightarrow$
$$x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x}{y}\cdot f' - \frac{x}{y}\ f' = 0$$

Tangent planes and normal lines

Let graph the function $z = f(x,y)$, $\mathcal{G}$:



- let P be a point on the graph $\mathcal{G}$:

$$P = (a, b, f(a, b))$$

- consider the curves on $\mathcal{G}$ passing through P:
  - (green) $\mathcal{C}_1 :\ y =$constant, *i.e.*,

$$\mathcal{C}_1 : \qquad (x, b, f(x, b))$$

  - (purple) $\mathcal{C}_2 :\ x =$constant, *i.e.*,

$$\mathcal{C}_2 : \qquad (a, y, f(a, y))$$

- tangent vector $\vec{T}_1$ to $\mathcal{C}_1$ at P:

$$\vec{T}_1 = \frac{d}{dx}\left(x\hat{i} + b\hat{j} + f(x, b)\hat{k}\right)\Big|_{x=a} = \hat{i} + f_1(a, b)\hat{k}$$

- tangent vector $\vec{T}_2$ to $\mathcal{C}_2$ at P:

$$\vec{T}_2 = \frac{d}{dy}\left(a\hat{i} + y\hat{j} + f(a, y)\hat{k}\right)\Big|_{y=b} = \hat{j} + f_2(a, b)\hat{k}$$

- The normal $\vec{n}$ to the surface is

$$\vec{n} = \vec{T}_1 \times \vec{T}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{vmatrix} = \hat{i} \cdot (-f_1) - \hat{j} \cdot f_2 + \hat{k} \cdot 1 = (-f_1, -f_2, 1)$$

- $\Longrightarrow$ the equation for the tangent plane is

$$-f_1(a,b) \cdot (x-a) - f_2(a,b) \cdot (y-b) + (z - f(a,b)) = 0$$

or

$$\boxed{z = f(a,b) + f_1(a,b) \cdot (x-a) + f_2(a,b) \cdot (y-b)}$$

$\Longrightarrow$ Note: for a function of one variable, $y = f(x)$, a similar formula produces the equation of the tangent line at $(a, f(a))$:

$$y = f(a) + f'(a)(x-a)$$

- Equation for a normal line in the standard form:

$$\frac{x-a}{-f_1(a,b)} = \frac{y-a}{-f_2(a,b)} = \frac{z - f(a,b)}{1}$$

Example 6: Find the equations of the tangent line and the normal line to $z = \sin(xy)$ at $(x,y) = (\frac{\pi}{3}, -1)$

$\Longrightarrow$

- $f(\frac{\pi}{3}, -1) = -\frac{\sqrt{3}}{2}$

- $$f_1 = \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(\frac{\pi}{3},-1)} = y \cdot \cos(xy)\Big|_{(x,y)=(\frac{\pi}{3},-1)} = -\frac{1}{2}$$
$$f_2 = \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(\frac{\pi}{3},-1)} = x \cdot \cos(xy)\Big|_{(x,y)=(\frac{\pi}{3},-1)} = \frac{\pi}{6}$$

- the tangent plane:

$$z = f(a,b) + f_1(a,b)\cdot(x-a) + f_2(a,b)\cdot(y-b)\Big|_{(a,b)=(\frac{\pi}{3},-1)}$$

$$= -\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\pi}{6}(y+1)$$

- the normal line

$$\frac{x-a}{-f_1(a,b)} = \frac{y-a}{-f_2(a,b)} = \frac{z-f(a,b)}{1}\Big|_{(a,b)=(\frac{\pi}{3},-1)}$$

$\Longrightarrow$

$$\frac{x-\frac{\pi}{3}}{\frac{1}{2}} = \frac{y+1}{-\frac{\pi}{6}} = \frac{z+\frac{\sqrt{3}}{2}}{1}$$