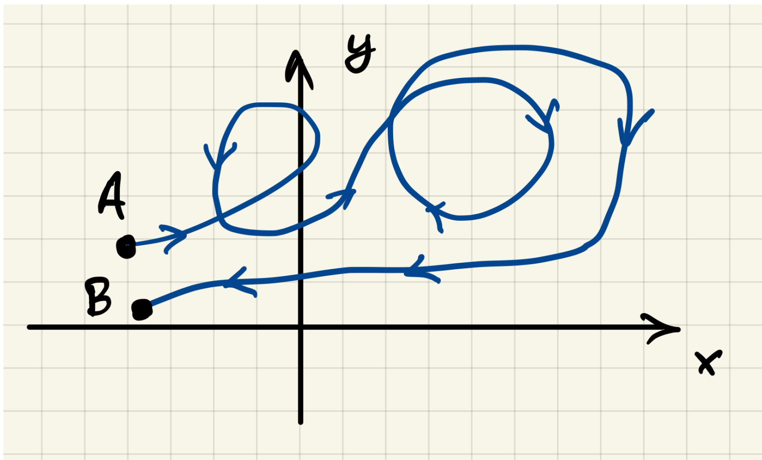
Lecture 8.2: **Parametric curves**

$\Longrightarrow$ Consider a curve ℓ from A to B:



- ℓ is not a graph of any function $y = f(x)$; neither it is a graph of any function $x = g(y)$. (Why?)

- We need a way to describe this curve. Imagine this is a trajectory of a moving ant, as a function of time t.
  - At time $t_A \to A$
  - At time t_B $(t_B > t_A)$ $\to B$

- A trajectory is a function

$$[t_A, t_B] \implies \mathbb{R}^2$$

  In fact, we have two real functions:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in \mathcal{I} = [t_A, t_B]$$

Parametric equation of ℓ implies a direction along the curve $\Longrightarrow$ a direction of increasing t

Example 1:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \qquad t \in [0, \pi]$$

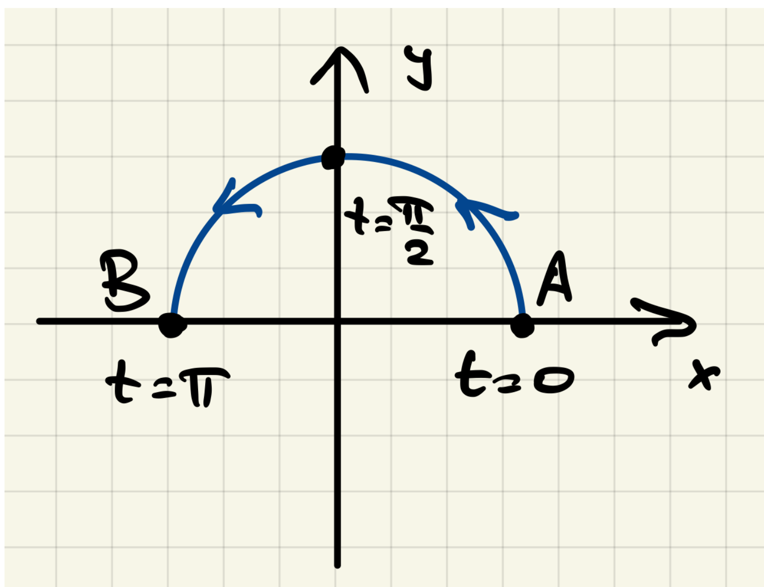
What does this curve look like?

$\Longrightarrow$

- Note that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \qquad \Longrightarrow \qquad x^2 + y^2 = 1$$

- $\Longrightarrow$ Portion of the circle of radius 1 centered at 0:



Example 2: Sketch and identify

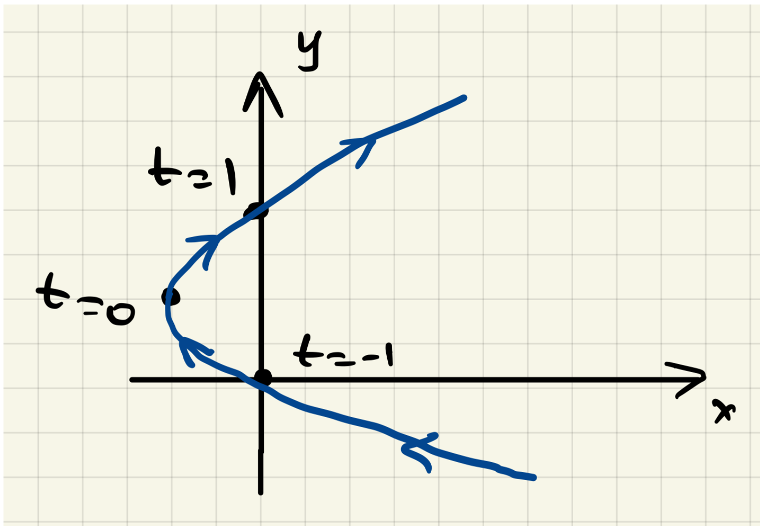
$$\begin{cases} x = t^2 - 1 \\ y = t + 1 \end{cases}, \qquad t \in (-\infty, \infty)$$

$\Longrightarrow$

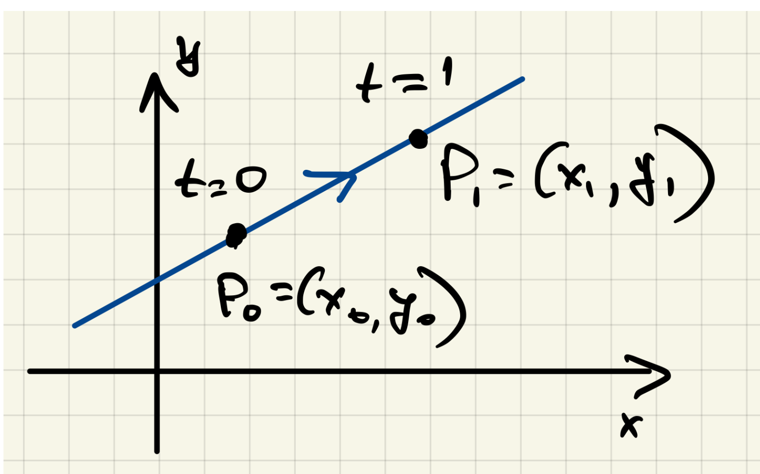
- It is useful always to eliminate t:

$$t = y - 1 \implies x = (y-1)^2 - 1 \implies x = y^2 - 2y$$

- The resulting equation is a parabola; direction of increasing t is indicated with arrows:



Example 3: Find a parameterization of a straight line passing through the points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ such that $t = 0$ at P_0 and $t = 1$ at P_1:



$\Longrightarrow$

- Straight line $\Longrightarrow$ the linear equation $y(x)$
- The parameterization is

$$\begin{cases} x = f(t) = m_1 t + b_1 \\ y = g(t) = m_2 t + b_2 \end{cases}$$

where both functions $(f(t), g(t))$ are linear

- $t = 0 \leftrightarrow P_0$; $t = 1 \leftrightarrow P_1$;

$$\begin{cases} f(0) = x_0 = b_1 \\ g(0) = y_0 = b_2 \\ f(1) = x_1 = m_1 \ \cdot 1 + b_1 \\ g(1) = y_1 = m_2 \ \cdot 1 + b_2 \end{cases}$$

- Solving above system of linear equations we find

$$\begin{cases} x = (x_1 - x_0)t + x_0 \\ y = (y_1 - y_0)t + y_0 \end{cases} , \qquad t \in \mathbb{R}$$

Example 4: Sketch and identify the curve

$$\begin{cases} x = a \cos t \\ y = b \sin t \end{cases} , \qquad 0 \le t \le 2\pi , \qquad a > b > 0$$

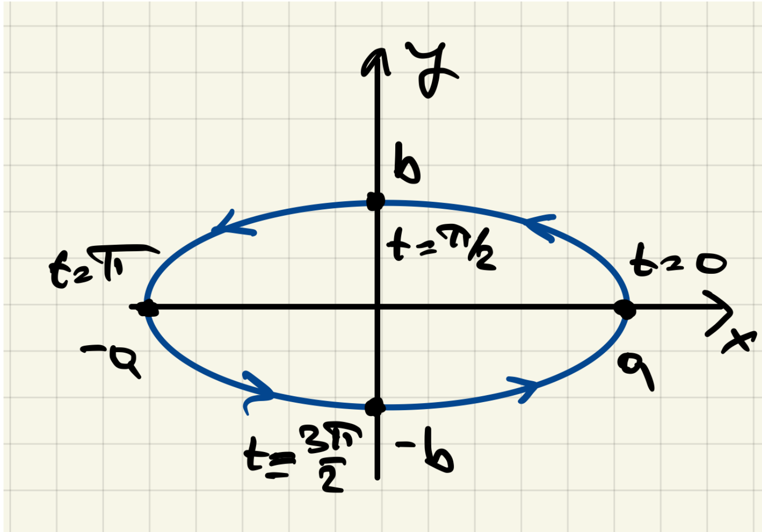
$\Longrightarrow$

- Eliminate t:

$$\cos t = \frac{x}{a} , \qquad \sin t = \frac{y}{b}$$

$$\sin^2 t + \cos^2 t = 1 \qquad \Longrightarrow \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- $\Longrightarrow$ the curve is an ellipse:



Plane curves

Def: ℓ is a plane curve provided

$$\ell = \Big\{ (x,y) \ \in \ \mathbb{R}^2 \text{ such that } x = f(t), \ y = g(t), \qquad t \in \mathcal{I} \text{ interval} \Big\}$$

for some continuous functions $(f(t), g(t))$

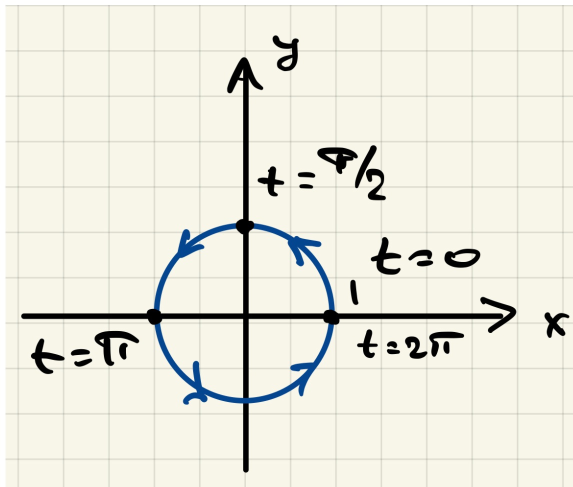
A choice of $f(t)$, $g(t)$, an interval $\mathcal{I}$ realize a parameterization of the curve ℓ

Note: A curve has many possible parameterizations

Example 5:

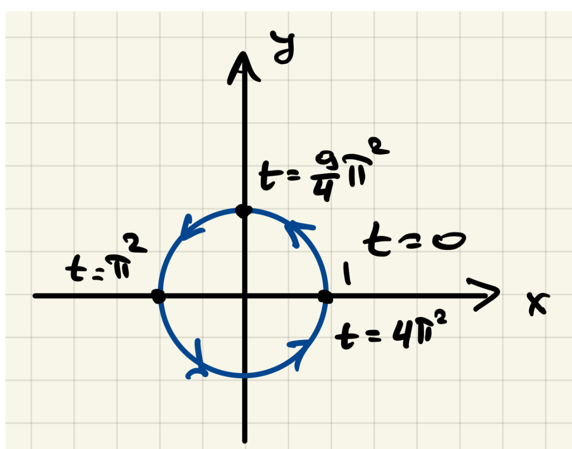
- Consider a curve — a circle of radius 1:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \qquad t \in [0, 2\pi]$$



- The same curve, *i.e.*, $x^2 + y^2 = 1$, can be represented as

$$\begin{cases} x = \cos \sqrt{t} \\ y = \sin \sqrt{t} \end{cases}, \qquad t \in [0, 4\pi^2]$$



Example 6: Sketch and identify the parametric curve

$$\begin{cases} x = \frac{1}{1+t^2} \\ y = \frac{t}{1+t^2} \end{cases}, \qquad t \in (-\infty, \infty)$$

$\Longrightarrow$

- let's eliminate t:

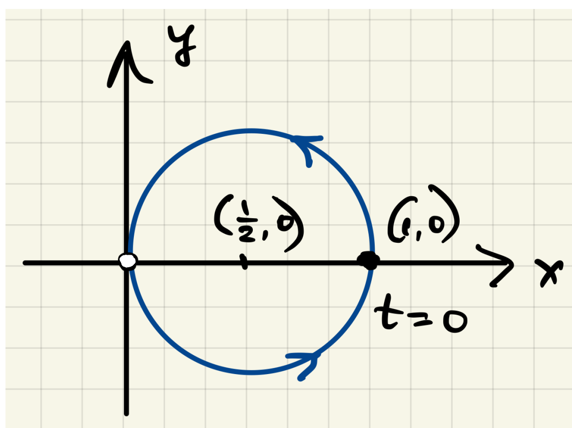
$$x^2 + y^2 = \frac{1+t^2}{(1+t^2)^2} = \frac{1}{1+t^2} = x$$

- $\Longrightarrow$

$$x^2 + y^2 = x \implies \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

- $\Longrightarrow$ the curve is a circle of radius $R = \frac{1}{2}$, centered at

$$(x_O, y_O) = \left(\frac{1}{2}, 0\right)$$



- Note that

$$t \to -\infty : \qquad (x, y) \to (0_+, 0_-)$$
$$t = 0 : \qquad (x, y) = (1, 0)$$
$$t \to +\infty : \qquad (x, y) \to (0_+, 0_+)$$

- $\Longrightarrow$ the circle is traced counterclockwise, starting from $(0,0)$

- Note that the point $(0,0)$ (empty circle) does not below to the curve