Lecture 9.1: **Sequences and convergence**

Sequences

(Infinite) sequence of numbers:

$$1\,,\,2\,,\,3\,,\,4\,,\,\cdots$$

$$1\,,\ \frac{1}{2}\,,\ \frac{1}{3}\,,\ \frac{1}{4}\,,\ \cdots$$

$$1\,,\ 4\,,\ 9\,,\ 16\,,\ \cdots$$

How do we define a sequence?

$\Longrightarrow$ Several ways:

- A function $f : \mathbb{Z}_+ \to \mathbb{R}$

$$\mathbb{Z}_+ \equiv \{1\,,\,2\,,\,3\,,\,4\,,\ \cdots\} \qquad \text{positive integers}$$

$\Longrightarrow$ A sequence will be:

$$\{f(1)\,,\ f(2)\,,\ f(3)\,,\ f(4)\,,\ \cdots\,,\ f(n)\,,\ \cdots\}$$

We call $f(n) = a_n$,

$$\{a_1\,,\ a_2\,,\ a_3\,,\cdots\,,\ a_n\,,\ \cdots\} \equiv \{a_n\}$$

Examples:

- $f(n) = \frac{1}{n} = a_n$. This defines a sequence

$$\left\{\frac{1}{n}\right\} = \left\{1\,,\ \frac{1}{2}\,,\ \frac{1}{3}\,,\ \frac{1}{4}\,,\ \cdots\right\}$$

- $f(n) = \frac{(-1)^n}{n} = a_n$. This defines a sequence

$$\left\{\frac{(-1)^n}{n}\right\} = \left\{-1\,,\ +\frac{1}{2}\,,\ -\frac{1}{3}\,,\ +\frac{1}{4}\,,\ \cdots\right\}$$

- $f(n) = n^3 = a_n$. This defines a sequence

$$\left\{n^3\right\} = \{1\,,\ 8\,,\ 27\,,\ \cdots\}$$

- Another way to define the sequence is to use a recursive formula, *e.g.,*

$$a_{n+1} = \sqrt{1 + a_n}\,, \qquad a_1 = 1$$

$\Longrightarrow$

$$\begin{aligned} a_2 &= \sqrt{1+1} = \sqrt{2} \\ a_3 &= \sqrt{1+a_2} = \sqrt{1+\sqrt{2}} \\ a_4 &= \sqrt{1+a_3} = \sqrt{1+\sqrt{1+\sqrt{2}}} \\ &\cdots \end{aligned}$$

We can calculate any term of this sequence, given the previous terms.

- ▪ A Fibonacci sequence

$$a_{n+1} = a_n + a_{n-1}\,, \qquad a_1 = 1\,, \qquad a_2 = 1$$

$$\{a_n\} = 1\,,\ 1\,,\ 2\,,\ 3\,,\ 5\,,\ 8\,,\ 13\,,\cdots$$

Characterizing sequences

- $\{a_n\}$ is bounded from below by L if $L \le a_n$ for all n
- $\{a_n\}$ is bounded from above by M if $M \ge a_n$ for all n
- $\{a_n\}$ is bounded if it is bounded both from below and from above
- $\{a_n\}$ is positive if it is bounded from below by $L = 0$
  - ▪ Example:

$$\left\{\frac{1}{n}\right\} = \left\{1\,,\ \frac{1}{2}\,,\ \frac{1}{3}\,,\ \frac{1}{4}\,,\ \cdots\right\}$$

is a bounded, positive sequence: $L = 0$ and $M = 1$:

$$\underbrace{0}_{\equiv L} < \frac{1}{n} \le \underbrace{1}_{\equiv M}$$

- $\{a_n\}$ is negative if it is bounded from above by $M = 0$
  - ▪ Example:

$$\left\{-\frac{1}{n^2}\right\} = \left\{-1\,,\ -\frac{1}{4}\,,\ -\frac{1}{9}\,,\ -\frac{1}{16}\,,\ \cdots\right\}$$

is a bounded, negative sequence: $L = -1$ and $M = 0$:

$$\underbrace{-1}_{\equiv L} < -\frac{1}{n^2} \leq \underbrace{0}_{\equiv M}$$

- $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for all n
- $\{a_n\}$ is decreasing if $a_{n+1} \leq a_n$ for all n
- $\{a_n\}$ is monotonic if it is either increasing or decreasing
- $\{a_n\}$ is alternating if $a_{n+1} \cdot a_n < 0$ for all $n \Longrightarrow \quad a_n$ always changes sign as n increases by 1

Examples:

■

$$\{a_n\} = \left\{\frac{1}{n^2}\right\} = \left\{1\,,\ \frac{1}{4}\,,\ \frac{1}{9}\,,\cdots\right\}$$

$\Longrightarrow$ This sequence is

- bounded from above with $M = 1$
- bounder from below with $L = 0$
- is positive
- is monotonic
- is decreasing

■

$$\{a_n\} = \left\{\frac{e^n}{\pi^n}\right\} = \left\{\frac{e}{\pi}\,,\ \frac{e^2}{\pi^2}\,,\ \frac{e^3}{\pi^3}\,,\cdots\right\}$$

$\Longrightarrow$ This sequence is

- bounded from above with $M = \frac{e}{\pi}$ since

$$\frac{e}{\pi} < 1$$

- bounder from below with $L = 0$

- is positive
- is monotonic
- is decreasing

Convergence

Let $\{a_n\}$ be a sequence defined by $a_n = f(n)$.

- If
$$\lim_{n\to\infty} f(n) = L \qquad \Longrightarrow$$
the sequence $\{a_n\}$ converges to L (assuming L is a finite number)

- (alternative definition) $\{a_n\}$ converges to L if any open interval $(L-\epsilon, L+\epsilon)$ leaves out only a finite number of terms of the sequence

- if the sequence is not convergent, it is called divergent.

- For a divergent sequence either:
  - the limit does not exist (L does not exist)
  - $L = \pm\infty$ (in this case we call the sequence divergent to $\pm\infty$)

Examples:

■
$$\left\{\frac{n+1}{n}\right\} = \left\{2\,,\ \frac{3}{2}\,,\ \frac{4}{3}\,,\ \cdots\right\}$$
$\Longrightarrow$
$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \qquad \Longrightarrow$$
the sequence converges to $L = 1$

■
$$\{n^2\} = \{1\,,\ 4\,,\ 9\,,\cdots\}$$
$\Longrightarrow$
$$\lim_{n\to\infty} n^2 = \infty \qquad \Longrightarrow$$

the sequence diverges to ∞

■

$$\left\{\sin\frac{\pi n}{2}\right\} = \{1\,,0\,,\ -1\,,\ 0\,,\ 1\,,\ \cdots\}$$

$\Longrightarrow$

$$\lim_{n\to\infty}\sin\frac{\pi n}{2} = DNE \qquad \Longrightarrow$$

the sequence diverges (L does not exist)

Properties

Because limits of sequences are essentially the limits of functions, same properties will apply. Assume that $\{a_n\}$ and $\{b_n\}$ are convergent sequences. Then

- $$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \ \pm\ \lim_{n\to\infty} b_n$$
- $$\lim_{n\to\infty}(k\cdot a_n) = k\cdot \lim_{n\to\infty} a_n$$
- $$\lim_{n\to\infty}(a_n\cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
- If $a_n \le b_n$, for all $n > N$ (N a fixed number)
$$\lim_{n\to\infty} a_n \ \le\ \lim_{n\to\infty} b_n$$
- squeeze theorem: $a_n \le c_n \le b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \Longrightarrow$
$$\lim_{n\to\infty} c_n = L$$

Example: Find the limit of $\{\sqrt{n^2+2n}-n\}$, if it exists.

$\Longrightarrow$

$$\lim_{n\to\infty}\left(\sqrt{n^2+2n}-n\right) = \lim_{n\to\infty}\frac{(\sqrt{n^2+2n}-n)(\sqrt{n^2+2n}+n)}{(\sqrt{n^2+2n}+n)} = \lim_{n\to\infty}\frac{n^2+2n-n^2}{n\left(\sqrt{1+\frac{2}{n}}+1\right)}$$

$$= \lim_{n\to\infty}\frac{2}{\left(\sqrt{1+\frac{2}{n}}+1\right)} = \frac{2}{2} = 1$$

- Theorem 1 A convergent sequence is bounded
- Theorem 2 A sequence that is bounded and monotonic is convergent

From Theorem 1 $\Longrightarrow$ an unbounded sequence is divergent, *e.g.,*

$$\{n^2 - 1\} = \{0\,,\ 3\,,\ 8\,,\ 15\,,\cdots\}$$

clearly unbounded $\Longrightarrow$ divergent

From Theorem 2 $\Longrightarrow$ A divergent sequence is either

$$\underbrace{\text{unbounded}}_{\text{case A}} \qquad \underbrace{\text{not monotonic}}_{\text{case B}}$$

-
$$\{n^2 - 1\} = \{0\,,\ 3\,,\ 8\,,\ 15\,,\cdots\}$$
is divergent and unbounded (case A);
-
$$\{0\,,\ 1\,,\ 0\,,\ 1\,,\cdots\}$$
is divergent, bounded, but not monotonic (case B)
-
$$\{(-1)^n n\} = \{-1\,,\ 2\,,\ -3\,,\ 4\,,\cdots\}$$
is divergent, unbounded and not monotonic (both case A and case B)

Example: Show that the sequence $\{a_n\}$,

$$a_{n+1} = \sqrt{6 + a_n}\,, \qquad a_1 = 1$$

is convergent and find its limit.

$\Longrightarrow$

- Note that

$$a_2 = \sqrt{6+1} = \sqrt{7} > 1$$

- Suppose that $a_{n+1} \geq a_n$ (for some $n \implies$ we know that it is true at least for $n = 1$) $\implies$

$$a_{n+2} = \sqrt{6 + a_{n+1}} \geq \sqrt{6 + a_n} = a_{n+1}$$

  i.e., , we showed that if $a_{n+1} \geq a_n$ then $a_{n+2} \geq a_{n+1} \implies$ this implies that the sequence is increasing (is monotonic).

- Note that $a_1 < 3$. Suppose that $a_n \leq 3$ (true at least for $n = 1$) $\implies$

$$a_{n+1} = \sqrt{6 + a_n} \leq \sqrt{6+3} = 3$$

  i.e., we showed that if $a_n \leq 3$, then $a_{n+1} \leq 3 \implies$ the sequence is bounded from above with $M = 3$

- thus, by Theorem 2, the sequence is convergent, and L exists:

$$\lim_{n\to\infty} a_n = L > 0$$

  since the sequence is positive

- $\implies$

$$\lim_{n\to\infty} a_{n+1} = L = \lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + L}$$

- $\implies$

$$\boxed{L = \sqrt{6 + L}}$$

- $\implies$

$$L^2 - L - 6 = 0 \implies (L-3)(L+2) = 0 \implies$$

  $L = 3$