Lecture 6.2: **Integrals of rational functions**

Def:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ amd $Q(x)$ are polynomials, is called a rational function.

We will assume that

$$\text{degree of } P(x) \ < \ \text{degree of } Q(x)$$

If

$$\text{degree of } P(x) \ \geq \ \text{degree of } Q(x)$$

we can do the long division and write

$$f(x) = M(x) + \frac{R(x)}{Q(x)}\,, \qquad \deg(R) < \deg(Q)$$

where $M(x)$ and $R(x)$ are also polynomials.

Example:

$$f(x) = \frac{x^3+3x^2}{x^2+1}\,, \qquad P(x) = x^3+3x^2\,, \qquad Q(x) = x^2+1$$

$$\deg(P) = 3\,, \qquad \deg(Q) = 2$$

$$\begin{array}{rr}
 & x+3 \\
x^2+1 \big) & \overline{x^3+3x^2} \\
(-) & \underline{x^3+x} \\
 & 3x^2-x \\
(-) & \\
 & \underline{3x^2+3} \\
 & -x-3
\end{array}$$

$\Longrightarrow$

$$\frac{x^3+3x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1}$$

$$M(x) = x+3\,, \qquad R(x) = -x-3\,, \qquad \deg(R) = 1 < \deg(Q)$$

Suppose we need to compute:

$$I = \int \frac{x^3 + 3x^2}{x^2 + 1}\, dx$$

$$I = \int (x+3)\, dx - \boxed{\int \frac{x+3}{x^2+1}\, dx} = \frac{1}{2}x^2 + 3x - \underbrace{\int \frac{x}{x^2+1}\, dx}_{u=x^2+1, du=2xdx} - 3\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}x^2 + 3x - \int \frac{\frac{1}{2}du}{u} - 3\tan^{-1} x = \frac{1}{2}x^2 + 3x - \frac{1}{2}\ln(1+x^2) - 3\tan^{-1} x + C$$

$\Longrightarrow$ Focus on

$$\boxed{\int \frac{P(x)}{Q(x)}\, dx}, \qquad \deg(P) < \deg(Q)$$

- Case 1: $Q(x) = bx + c$,

$$\int \underbrace{\frac{a}{bx+c}}_{u=bx+c, du=bdx} dx = \int \frac{\frac{a}{b}du}{u} = \frac{a}{b}\ln|bx+c| + C$$

- Case 2: $Q(x) = x^2 + a^2$,

$$\int \frac{bx+c}{x^2+a^2}\, dx = b\underbrace{\int \frac{x}{x^2+a^2}\, dx}_{\equiv I} + c\underbrace{\int \frac{1}{x^2+a^2}\, dx}_{\equiv J} \quad \ominus$$

$$\boxed{I = \int \underbrace{\frac{x}{x^2+a^2}}_{u=x^2+a^2, du=2xdx} dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(x^2+a^2)}$$

$$\boxed{J = \int \underbrace{\frac{dx}{x^2+a^2}}_{u=\frac{x}{a}, du=\frac{dx}{a}} = \frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C}$$

$$\ominus\ \frac{b}{2}\ln(x^2+a^2) + \frac{c}{a}\tan^{-1}\frac{x}{a} + C$$

<u>Partial fractions</u>

Assume:

$$f(x) = \frac{P(x)}{Q(x)}\,, \qquad \deg(P) < \deg(Q)\,, \qquad \underline{\textbf{and}}$$

$$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$$

such that $a_i \neq a_j$ for $i \neq j$

Partial fraction decomposition of $f(x)$:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

where $A_1, A_2, \cdots, A_n$ constants to be determined.

Example 1:

$$f(x) = \frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

- Method-1

$$\frac{A}{x-2}+\frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{(A+B)\;x + (-3A-2B)}{(x-2)(x-3)} = \frac{1\;x+4}{(x-2)(x-3)}$$

$$\begin{cases} A+B=1 \\ -3A-2B=4 \end{cases} \qquad \Longrightarrow \qquad B = 1-A,\ -3A-2(1-A)=4 \implies A=-6, B=7$$

$$\frac{x+4}{(x-2)(x-3)} = \frac{-6}{x-2} + \frac{7}{x-3}$$

- Method-2

$$A_j = \lim_{x\to a_j} \frac{(x-a_j)P(x)}{Q(x)}$$

$$A_1 \equiv A = \lim_{x\to 2} \frac{(x-2)(x+4)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{(x+4)}{(x-3)} \underbrace{=}_{\text{DS}} \frac{6}{-1} = -6$$

$$A_2 \equiv B = \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-2)(x-3)} = \lim_{x\to 3} \frac{(x+4)}{(x-2)} \underbrace{=}_{\text{DS}} \frac{7}{1} = 7$$

$\Longrightarrow$

$$\int \frac{x+4}{(x-2)(x-3)}\,dx = \int \frac{-6}{x-2}\,dx + \int \frac{7}{x-3}\,dx = -6\ln|x-2| + 7\ln|x-3| + C$$

Example 2:

$$I = \int \frac{2+3x+x^2}{x(x^2+1)}\,dx$$

although the integrand is

$$f(x) = \frac{P(x)}{Q(x)}$$

$$Q = x(x^2+1)$$

can not be factorized into degree-1 polynomials (linear functions)

$\Longrightarrow$ In this case we seek decomposition

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)}$$

$$\begin{cases} A+B=1 \\ C=3 \\ A=2 \end{cases} \qquad \Longrightarrow \qquad (A, \underbrace{B=1-A}_{\text{from eq. one}}, C) = (2,-1,3)$$

$\Longrightarrow$

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{2}{x} + \frac{-x+3}{x^2+1}$$

Thus,

$$I = \int \frac{2}{x}\,dx + \int \frac{-x+3}{x^2+1}\,dx = 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C$$

Completing the square

$$\int \frac{dx}{x^3+2x^2+2x} = \int \frac{dx}{x(x^2+2x+2)}$$

$\Longrightarrow$ Note that x^2+2x+2 does not factorize since

$$x^2+2x+2=0$$

does not have real roots.

$\Longrightarrow$ In cases such as this,

$$x^2+bx+c=\left(x+\frac{b}{2}\right)^2+\left(c-\frac{b^2}{4}\right)$$

i.e.,

$$x^2+2x+2=(x+1)^2+2-1=(x+1)^2+1$$

$\Longrightarrow$ we seek decomposition as

$$\frac{1}{x(x^2+2x+2)}=\frac{A}{x}+\frac{Bx+C}{x^2+2x+2}=\frac{A(x^2+2x+2)+x(Bx+C)}{x(x^2+2x+2)}=\frac{(A+B)x^2+(2A+C)x+2A}{x(x^2+2x+2)}$$

$\Longrightarrow$

$$\begin{cases} A+B=0 \\ 2A+C=0 \\ 2A=1 \end{cases} \qquad \Longrightarrow \qquad (A,B,C)=\left(\frac{1}{2},-\frac{1}{2},-1\right)$$

$$\int \frac{1}{x(x^2+2x+2)}\,dx=\int \frac{\frac{1}{2}}{x}\,dx+\int \frac{-\frac{1}{2}x-1}{(x+1)^2+1}\,dx=\frac{1}{2}\int \frac{dx}{x}-\frac{1}{2}\int \underbrace{\frac{x+2}{(x+1)^2+1}\,dx}_{u=x+1,du=dx}$$

$$=\frac{1}{2}\ln|x|-\frac{1}{2}\int \frac{u+1}{u^2+1}du=\frac{1}{2}\ln|x|-\frac{1}{4}\ln(u^2+1)-\frac{1}{2}\tan^{-1}u+C$$

$$=\frac{1}{2}\ln|x|-\frac{1}{4}\ln((x+1)^2+1)-\frac{1}{2}\tan^{-1}(x+1)+C=\frac{1}{2}\ln|x|-\frac{1}{4}\ln(x^2+2x+2)-\frac{1}{2}\tan^{-1}(x+1)+C$$

Repeated factors

$$\frac{1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$$

In general:

- linear repeated factors:

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \cdots + \frac{B_m}{(x-b)^m}$$

- quadratic repeated factors:

$$\frac{1}{(a^2x^2+b^2)^k} = \frac{B_1x+C_1}{a^2x^2+b^2} + \frac{B_2x+C_2}{(a^2x^2+b^2)^2} + \cdots + \frac{B_kx+C_k}{(a^2x^2+b^2)^k}$$

Example:

$$I = \int \frac{x^2+1}{x^3+8}\,dx$$

$$Q(x) = x^3+8 = (x+2)(x^2-2x+4) = (x+2)((x-1)^2+3)$$

$\Longrightarrow$

$$\frac{x^2+1}{x^3+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4} = \frac{A(x^2-2x+4)+(x+2)(Bx+C)}{(x+2)(x^2-2x+4)}$$

$$= \frac{(A+B)x^2+(-2A+2B+C)x+(4A+2C)}{(x+2)(x^2-2x+4)}$$

$\Longrightarrow$

$$\begin{cases} A+B=1 \\ -2A+2B+C=0 \\ 4A+2C=1 \end{cases}$$

$\Longrightarrow$ Let's recall solving system of linear equations using augmented matrix approach (remember that there is always a solution for the partial fraction decomposition, and is unique!)

- augmented matrix $m \times (n+1) \longrightarrow 3 \times 4$

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & b \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 2 & 1 & 0 \\ 4 & 0 & 2 & 1 \end{bmatrix} \equiv \mathcal{M} \end{array}$$

- $\mathcal{M} \to \mathcal{M}_1$:

$$r_3 \to r_3 + 2r_2\,, \qquad r_2 \to r_2 + 2r_1$$

$$\mathcal{M}_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 4 & 4 & 1 \end{bmatrix}$$

- $\mathcal{M}_1 \to \mathcal{M}_2$:

$$r_3 \to r_3 - r_2\,, \qquad r_2 \to \frac{1}{4} r_2$$

$$\mathcal{M}_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

- $\mathcal{M}_2 \to \mathcal{M}_3$:

$$r_3 \to \frac{1}{3} r_3$$

$$\mathcal{M}_3 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

$\Longrightarrow$ from $\mathcal{M}_3$:

$$C = -\tfrac{1}{3}$$
$$B = \tfrac{1}{2} - \tfrac{1}{4}C = \tfrac{1}{2} + \tfrac{1}{12} = \tfrac{7}{12}$$
$$A = 1 - B = 1 - \tfrac{7}{12} = \tfrac{5}{12}$$

$\Longrightarrow$

$$\frac{x^2+1}{x^3+8} = \frac{\frac{5}{12}}{x+2} + \frac{\frac{7}{12}x - \frac{1}{3}}{x^2-2x+4} = \frac{5}{12}\,\frac{1}{x+2} + \frac{\frac{7}{12}x - \frac{1}{3}}{(x-1)^2+3} = \frac{5}{12}\,\frac{1}{x+2} + \frac{\frac{7}{12}(x-1) + \frac{7}{12} - \frac{1}{3}}{(x-1)^2+3}$$

$$= \frac{5}{12}\,\frac{1}{x+2} + \frac{7}{12}\,\frac{x-1}{(x-1)^2+3} + \frac{1}{4}\,\frac{1}{(x-1)^2+3}$$

$\Longrightarrow$

$$I = \frac{5}{12}\int \frac{1}{x+2}\,dx + \frac{7}{12}\int \frac{x-1}{(x-1)^2+3}dx + \frac{1}{4}\int \frac{1}{(x-1)^2+3}\,dx$$
$$= \frac{5}{12}\ln|x+2| + \frac{7}{24}\ln(x^2-2x+4) + \frac{1}{4\sqrt{3}}\,\tan^{-1}\frac{x-1}{\sqrt{3}} + D$$