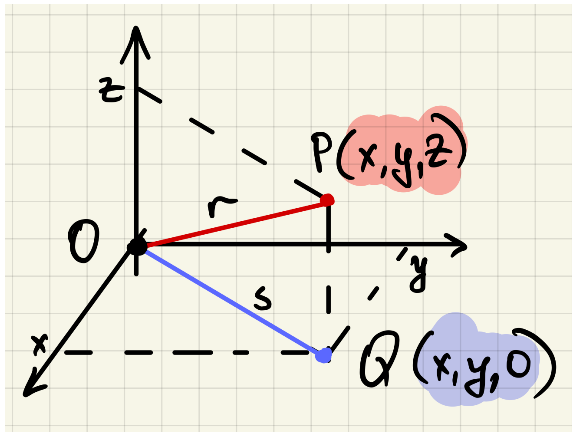
Lecture 10.1: **Analytic geometry in 3D**

Let (x, y, z) be Cartesian coordinates:



- $P = (x, y, z)$ and $Q = (x, y, 0)$ is a projection of P on the xy-plane; $O = (0, 0, 0)$ is the origin
- distance between O and Q

$$OQ = \sqrt{x^2 + y^2} \equiv s$$

- distance between O and P

$$r = \sqrt{s^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

Distance between two points:

$$P_1 = (x_1, y_1, z_1)\,, \qquad P_2 = (x_2, y_2, z_2)$$

$\Longrightarrow$

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Properties:

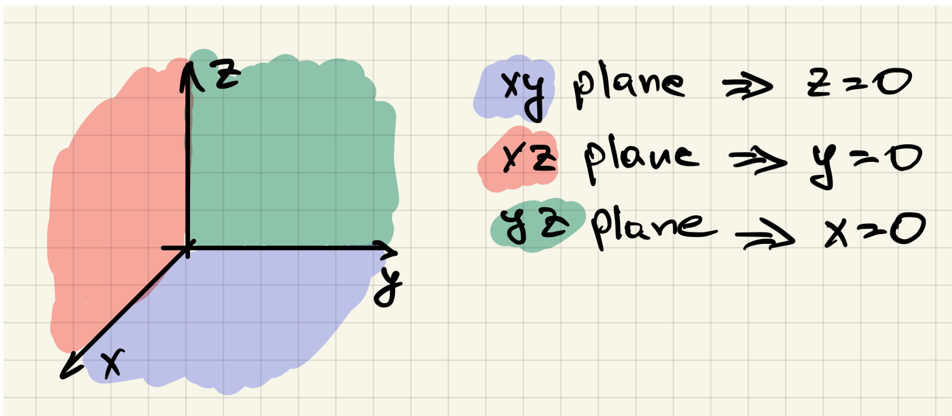
- if $d(P_1, P_2) = 0 \Longrightarrow \qquad P_1 = P_2$

- $d(P_1, P_2) = d(P_2, P_1)$
- $d(P_1, P_2) \leq d(P_1, P_3) + d(P_3, P_2)$

$\Longrightarrow$ Definition of the distance determines the geometry of the space

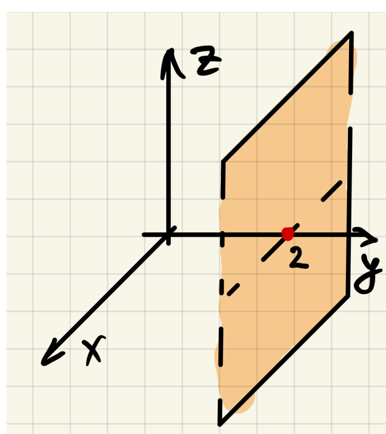
$\Longrightarrow$ Distance defined as above determines the Euclidean geometry

Example 1: coordinate planes in 3D Euclidean geometry



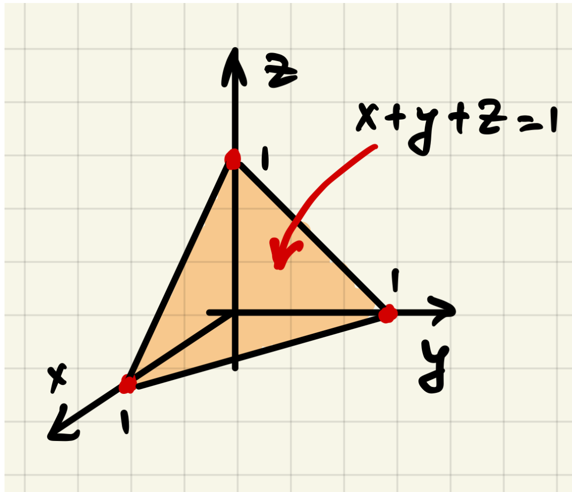
Example 2: other surfaces in 3D Euclidean geometry:

- $y = 2$ — the plane $\perp$ y-axis, intersecting it $y = 2$ (the red dot)



This plane is parallel to xz-plane ($y = 0$)

- $x + y + z = 1$:



Axis intercept (red dots):

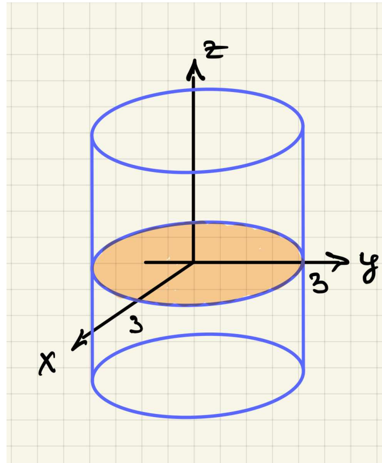
- $y = z = 0 \implies x = 1$
- $x = y = 0 \implies z = 1$
- $x = z = 0 \implies y = 1$

Note: planes indefinitely extend in all directions

Example 3: Consider the equation

$$x^2 + y^2 = 9$$

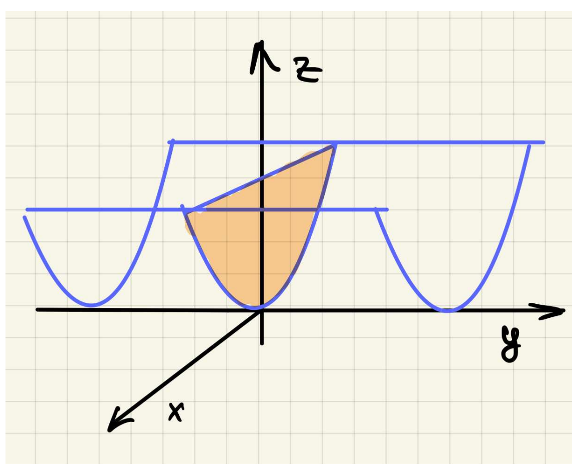
- In 2D (Euclidean geometry) $\implies$ a circle of radius 3 centered at 0
- In 3D (Euclidean geometry) $\implies$ a cylinder of radius 3, with the axis being z-axis



Example 4: Consider the equation

$$z = x^2$$

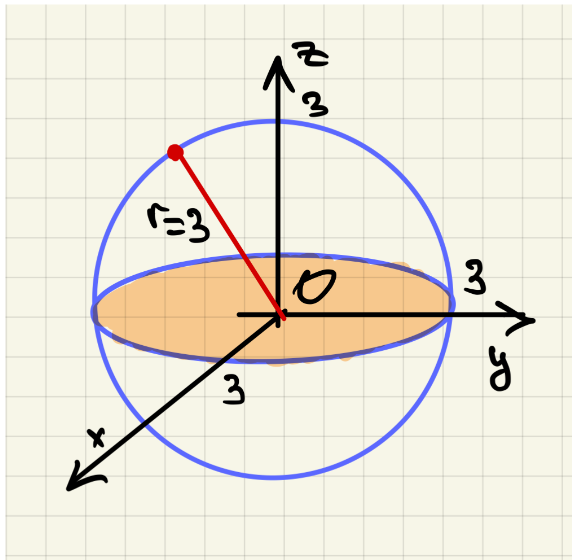
- In 2D (Euclidean geometry) on xz-plane $\Longrightarrow$ a parabola
- In 3D (Euclidean geometry) $\Longrightarrow$ parabolic cylinder



Example 5: Consider the equation

$$x^2 + y^2 + z^2 = 9$$

- In 3D (Euclidean geometry) $\Longrightarrow$ sphere of radius 3, centered at the origin



Example 6: Consider the equation

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 0$$

- sphere of radius 0, centered at $(1,2,3)$ $\Longrightarrow$ equivalently a single point

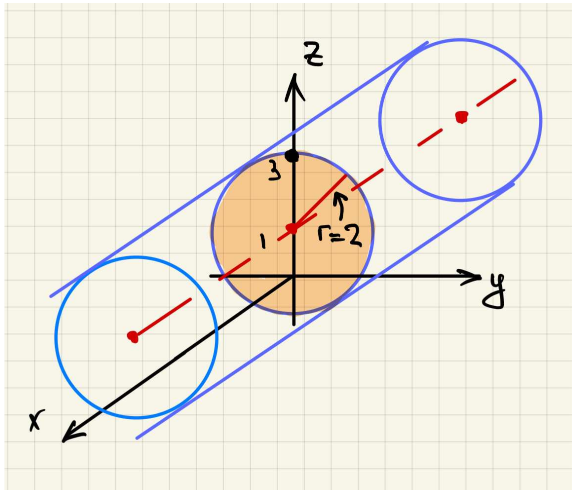
$$P = (1,2,3)$$

Example 7: Consider the equation

$$y^2 + (z-1)^2 = 4$$

- In 2D (Euclidean geometry) on yz-plane $\Longrightarrow$ a circle of radius 2 centered at $(0,1)$

- In 3D (Euclidean geometry) $\Longrightarrow$ a cylinder with axis parallel to x-axis, passing through the point $(0, 0, 1)$:



Regions in 3D space can be specified:

- by inequalities
- by systems of equations

"Inequality" example: Consider a region defined as

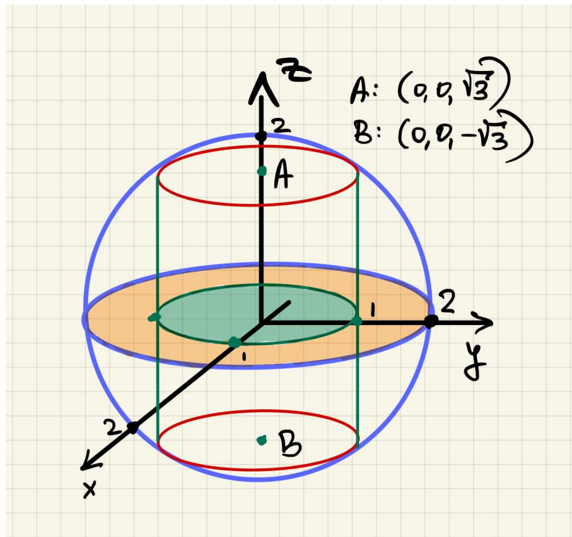
$$x^2 + y^2 \geq 4$$

- In 2D (Euclidean geometry) on xy-plane $\Longrightarrow$ this is set of all points including and outside the circle of radius 2 centered at the origin
- In 3D (Euclidean geometry) $\Longrightarrow$ points outside, and including the surface, of the cylinder $x^2 + y^2 = 4$

"System of equations" example: Consider a region defined as

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 + z^2 = 4 \end{cases}$$

- The first equation defines a cylinder $x^2 + y^2 = 1$
- The second equation defined a sphere of radius 2, centered at the origin
- The system $\Longrightarrow$ the intersection of the cylinder and the sphere:



these are the two red circles, parallel to xy-plane, each centered about the z-axis and is of radius 1. The planes of the circles intersect the z-axis at

$$eq.2 - eq.1 \quad \Longrightarrow \quad z^2 = 3 \quad \Longrightarrow \quad z = \pm\sqrt{3}$$