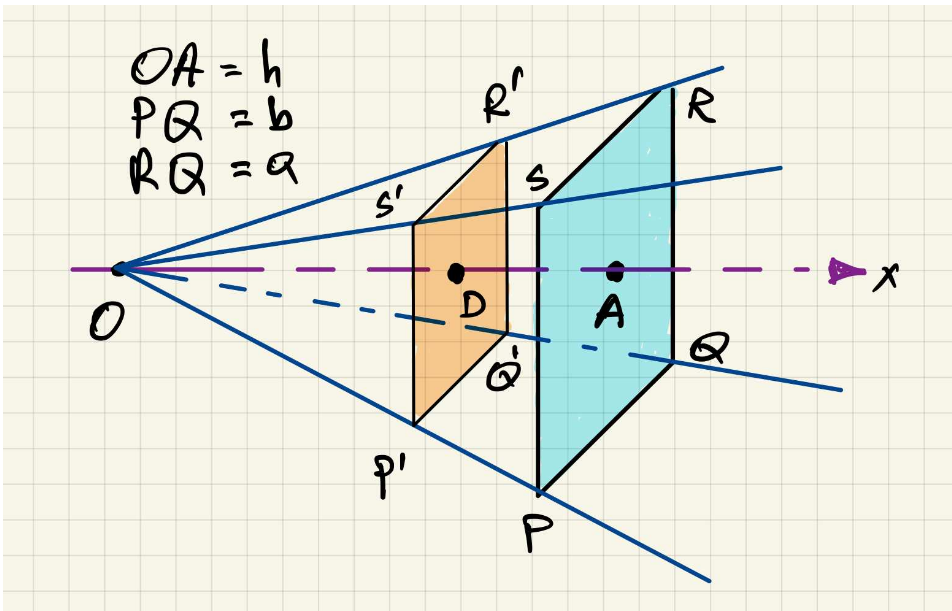
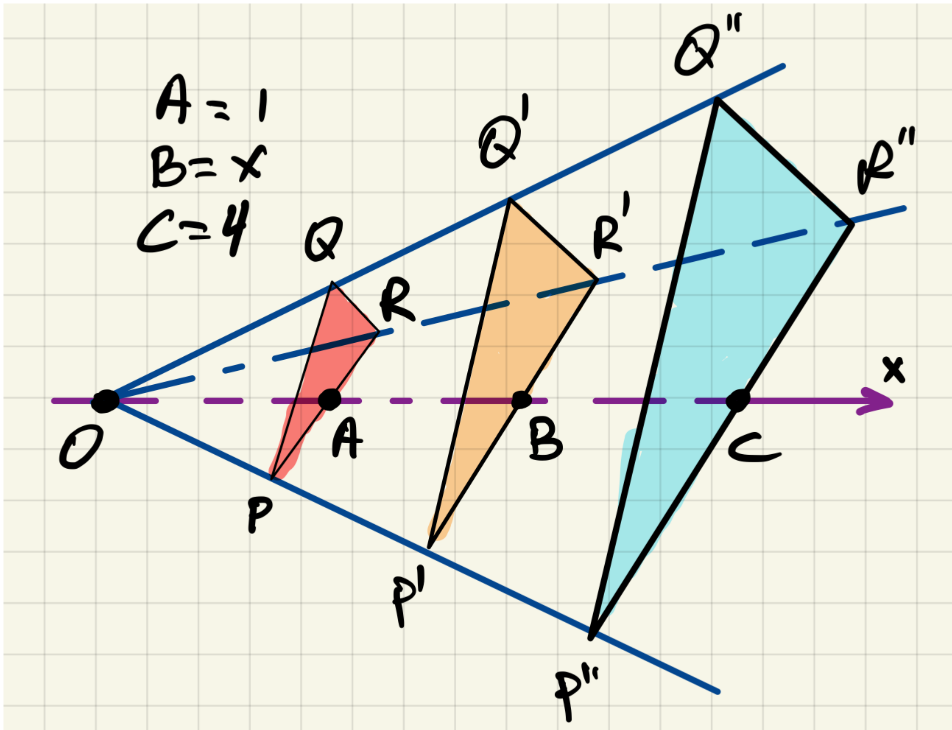
Lecture 7.2: **More volumes by slicing**

Problem: compute volume of a pyramid with rectangular base PSRQ with a height $OA = h$:



- Let
$$OD = x\,, \qquad x \in [0, h]$$
- triangles $\Delta OR'S'$ and ΔORS are similar $\Longrightarrow$
$$S'R' = \frac{OD}{OA}\ SR = \frac{x}{h}\ b$$
- triangles $\Delta OS'P'$ and ΔOSP are similar $\Longrightarrow$
$$S'P' = \frac{OD}{OA}\ SP = \frac{x}{h}\ a$$
- The cross-section $S'R'Q'P'$ has area $A(x)$,
$$A(x) = S'R' \cdot S'P' = \frac{x^2}{h^2}\ ab$$
- $\Longrightarrow$
$$V = \int_0^h A(x)\ dx = \int_0^h \frac{x^2}{h^2}\ ab\ dx = \frac{x^3}{3h^2}\ ab\Big|_0^h = \frac{1}{3}abh$$

Example 1: A solid extends on the x-axis from $x = 1$ (point A) to $x = 4$ (point C). A cross-section at x (point B, the triangle $P'Q'R'$) is an equilateral triangle of side $\sqrt{x}$. Find the volume of the solid.



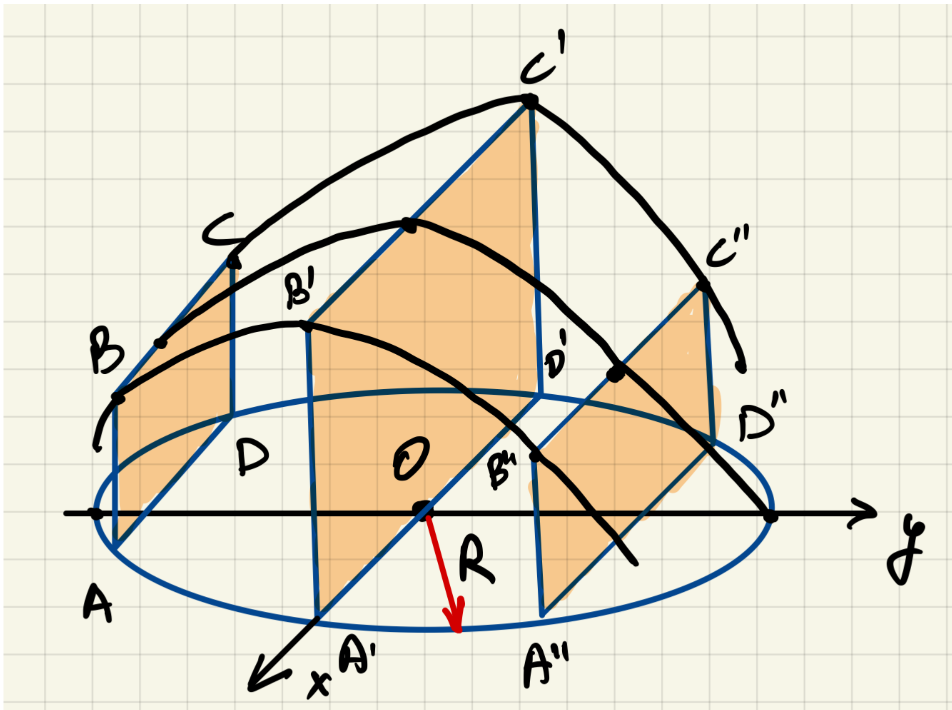
- Area $A(x)$ of the cross-section $P'Q'R'$ is

$$A(x) = \frac{1}{2} \underbrace{P'Q'}_{=\sqrt{x}} \cdot \underbrace{P'R'}_{=\sqrt{x}} \cdot \sin\frac{\pi}{3} = \frac{x}{2}\,\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\,x$$

- $\Longrightarrow$

$$V = \int_1^4 A(x)\,dx = \int_1^4 \frac{\sqrt{3}}{4}\,x\,dx = \frac{\sqrt{3}}{4}\,\frac{x^2}{2}\bigg|_1^4 = \frac{\sqrt{3}}{4}\,\frac{15}{2} = \frac{15\sqrt{3}}{8}$$

Example 2: A solid has a circular base of radius R. A cross section of the solid perpendicular to a diameter of the base is a square. Find the volume of the solid



- Cross-sections $ABCD$, etc, are squares arranged along

$$y \in [-R, R]$$

- At fixed y, the size of the square $ABCD$ is

$$AD = 2\sqrt{R^2 - y^2}$$

- The area $A(y)$ of a typical cross-section $ABCD$ is

$$A(y) = AD^2 = 4(R^2 - y^2)$$

- $\Longrightarrow$

$$V = \int_{-R}^{R} A(y)\; dy = \int_{-R}^{R} 4(R^2 - y^2)\; dy = 4\left(R^2 y - \frac{1}{3}y^3\right)\Big|_{-R}^{R} = \frac{16}{3}\; R^3$$