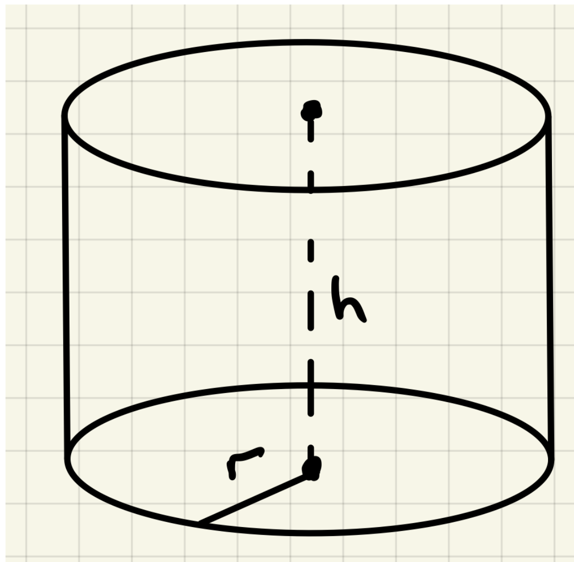
Lecture 13.1: **Functions of several variables**

$\Longrightarrow$ Volume of a cylinder depends on its height h and the radius of its base disk r



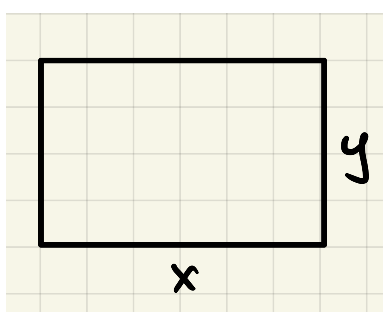
- the volume is

$$V = \pi r^2 h = f(r, h)$$

- $r, h \geq 0$ defines the domain of f:

$$D(f) = \Big\{(r, h) \in \mathbb{R}\,,\ r, h \geq 0\Big\}$$

$\Longrightarrow$ Area of a rectangle depends on its length x and its width y



- the area is

$$A(x, y) = x \cdot y$$

- $x, y \geq 0$ defines the domain of A:

$$D(A) = \left\{ (x, y) \in \mathbb{R}\,,\ x, y \geq 0 \right\}$$

In general, a function of n real variables:

- $f(x_1, x_2, \cdots, x_n)$ assigns a real number to each set $(x_1, x_2, \cdots, x_n)$ in the domain of f:

$$\text{domain :} \qquad D(f) \subseteq \mathbb{R}^n$$

$$\text{range :} \qquad R(f) \subseteq \mathbb{R}$$

- The domain is the largest subset of $\mathbb{R}^n$ for which the expression of $f(x_1, x_2, \cdots, x_n)$ is defined

Example 1: Find the domain of the functions

$$f(x, y) = \sqrt{4 - x^2 - y^2}$$

$\Longrightarrow$

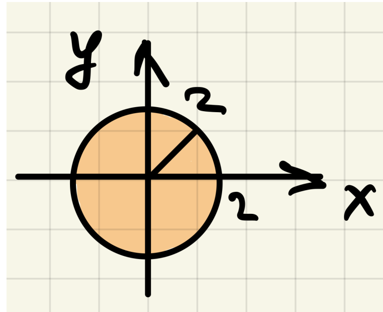
- f is defined when

$$4 - x^2 - y^2 \geq 0 \qquad \Longleftrightarrow \qquad x^2 + y^2 \leq 4$$

- $\Longrightarrow$

$$D(f) = \left\{ (x, y) \in \mathbb{R}\,,\ x^2 + y^2 \leq 2^2 \right\}$$

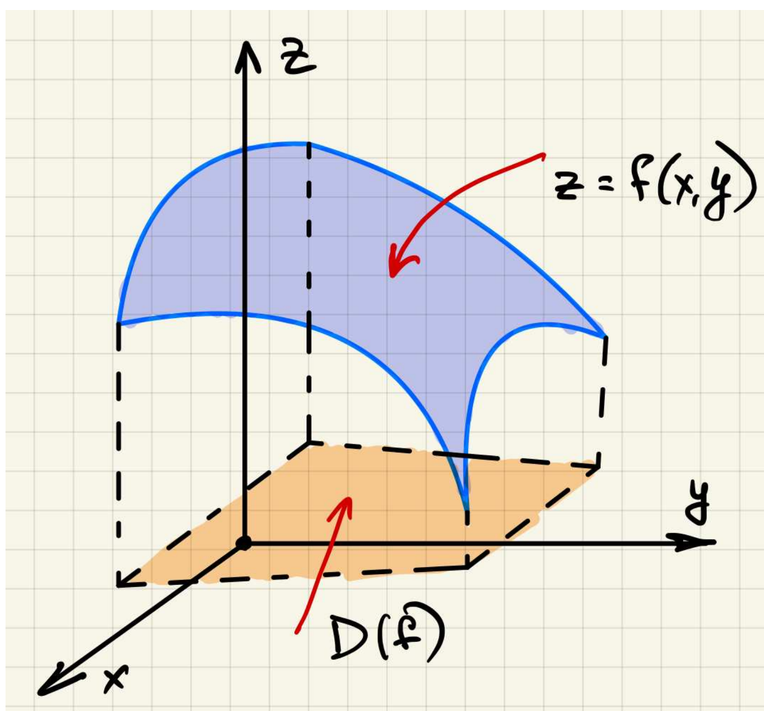
  i.e., a disk of radius 2, centered at the origin $(0, 0)$:



Graphs

Let $z = f(x, y)$ be a function of two variables:

- the domain of the function (orange region of the xy-plane) $D(f) \subseteq \mathbb{R}^2$
- Plot the points
$$(x, y, z) \;=\; (x, y, f(x, y))$$
- the set of such points determine a surface (blue) in $\mathbb{R}^3$:

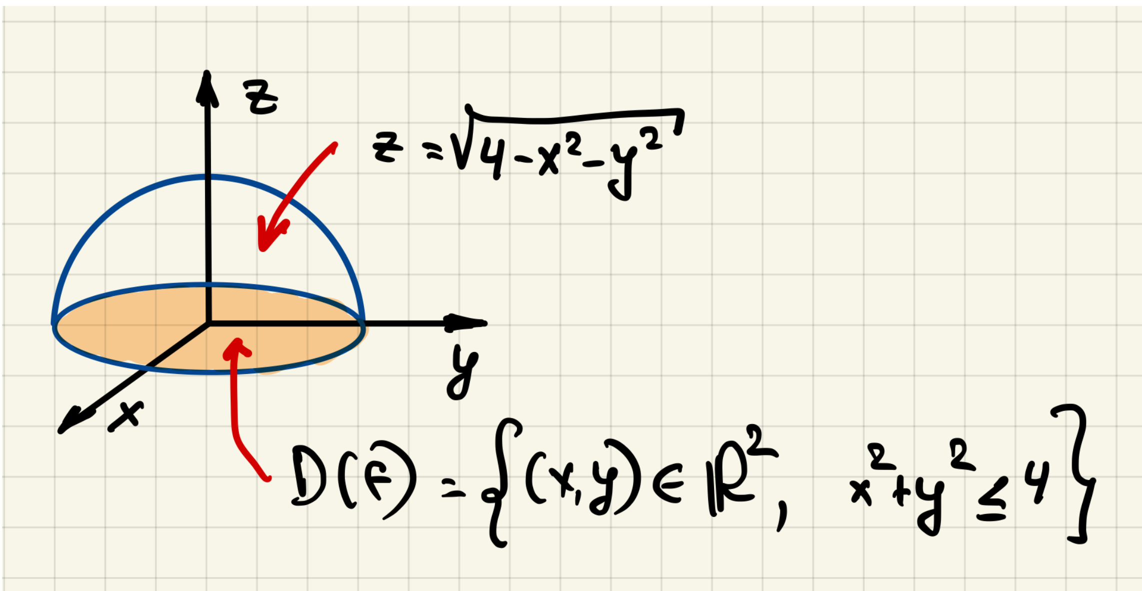


Example 2: Plot

$$z = f(x,y) = \sqrt{4 - x^2 - y^2}\,, \qquad z \geq 0$$

$\Longrightarrow$

- $z^2 = 4 - x^2 - y^2 \qquad \Longrightarrow \qquad x^2 + y^2 + z^2 = 4$
- $\Longrightarrow$ the surface is the upper hemisphere of radius 2:



Level curves

Further consider the function

$$z = f(x,y) = \sqrt{4 - x^2 - y^2}$$

$\Longrightarrow$ Draw in 2D xy-plane the curves that have a constant value of z

- *e.g.*, let $z = 0$, $z = \frac{1}{2}$, $z = 1$, $z = \frac{3}{2}$, $z = 2$
- $z = 2$:

  $$x^2 + y^2 + 2^2 = 4 \implies x^2 + y^2 = 0 \implies x = y = 0$$

  is the max value of z; a single point $(0,0)$ (blue)

- $z=0$:

$$x^2+y^2+0^2=4 \implies x^2+y^2=4 \implies$$

is the min value of z; a circle of radius 2 centered at $(0,0)$ (black)

- $z=\frac{1}{2}$:

$$x^2+y^2+\left(\frac{1}{2}\right)^2=4 \implies x^2+y^2=\frac{15}{4} \implies$$

a circle of radius $\sqrt{15}/2$ centered at $(0,0)$ (red)

- $z=1$:

$$x^2+y^2+1^2=4 \implies x^2+y^2=3 \implies$$

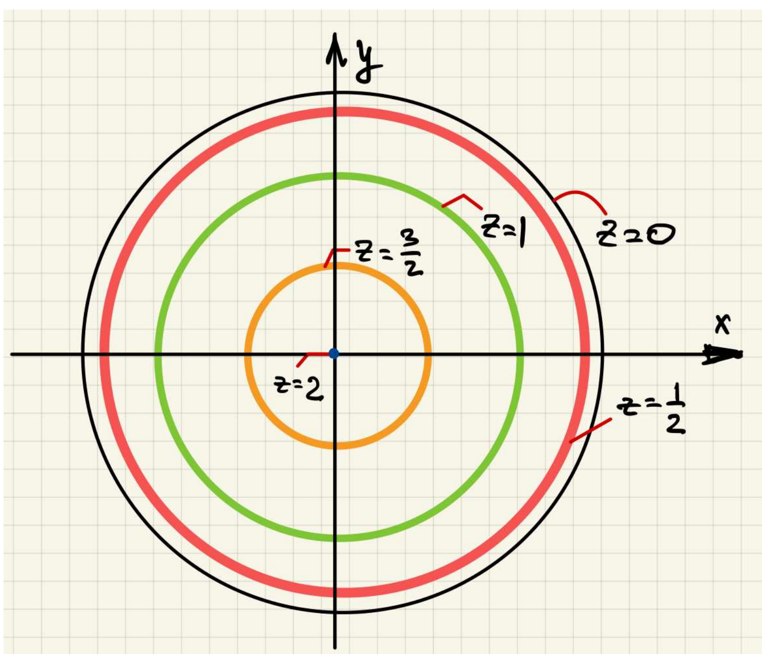
a circle of radius $\sqrt{3}$ centered at $(0,0)$ (green)

- $z=\frac{3}{2}$:

$$x^2+y^2+\left(\frac{3}{2}\right)^2=4 \implies x^2+y^2=\frac{7}{4} \implies$$

a circle of radius $\sqrt{7}/2$ centered at $(0,0)$ (orange)

- $\implies$



represents topographic map of the surface — *i.e.,* the level curves for evenly spaced values of z

Further applications of level curves:

- mountain ranges on maps $\Longrightarrow$ curves of equal altitude at 1000m, 2000m, 3000, ...
  - closely spaces curves $\Longrightarrow$ steep terrain
  - widely spaced curves $\Longrightarrow$ flatter terrain
- ocean maps $\Longrightarrow$ curves of equal depth