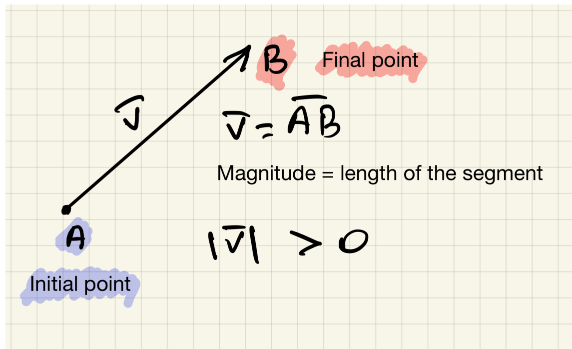
Lecture 10.2: **Vectors**

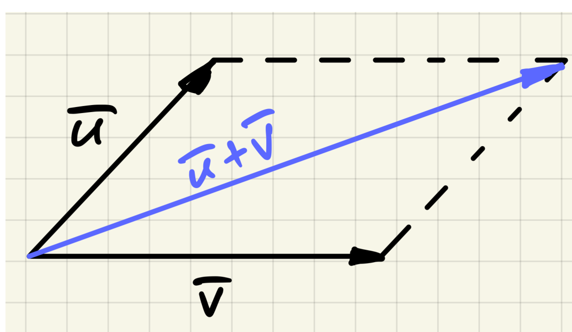
A vector is characterized by the magnitude and direction:



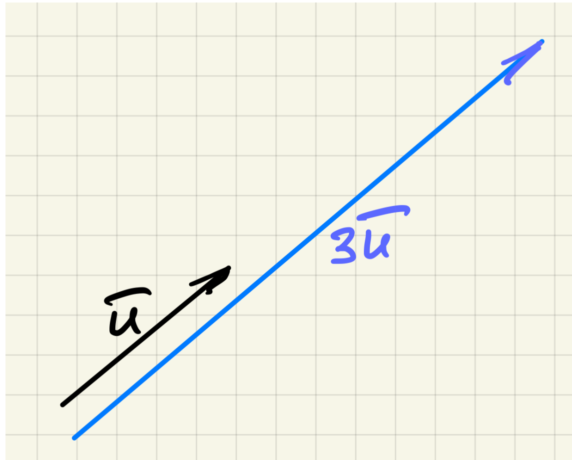
- magnitude: the length of the line segment $length(AB) = |\vec{v}|$
- direction: from the initial point A to the final point B

Operation with vectors:

- addition (parallelogram rule):

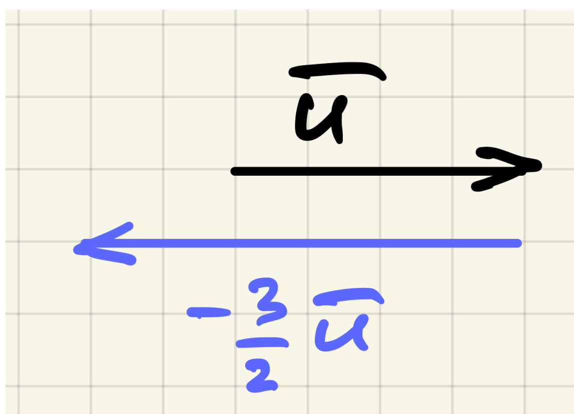


- multiplication by a scalar:



$\Longrightarrow$ $k\vec{u}$ is a vector (in the picture above $k = 3$) pointing in the same direction (for $k > 0$) as vector $\vec{u}$, with the length $k \cdot |\vec{u}|$

$\Longrightarrow$ if $k < 0$, the $k\vec{u}$ points in the direction opposite to that of $\vec{u}$, and having the length $|k| \cdot |\vec{u}|$ — in the picture below $k = -\frac{3}{2}$:

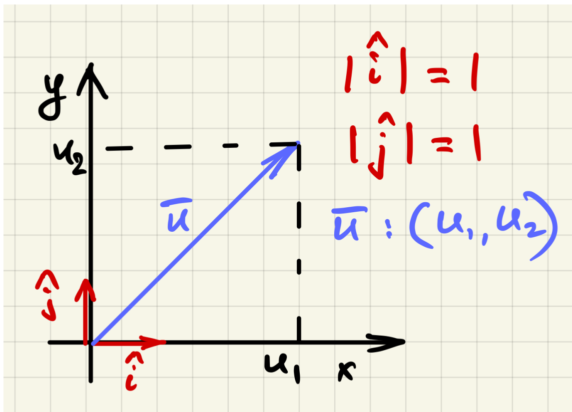


Properties of the vector operations:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
- there is a zero vector $\vec{0}$ such that $\vec{u} + \vec{0} = \vec{u}$

- there is a vector $-\vec{u}$ for each vector $\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$
- $k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$
- $k \cdot (\ell \cdot \vec{u}) = (k \cdot \ell) \cdot \vec{u}$

Vectors in 2D:



- $\hat{\imath} = (1, 0)$ and $\hat{\jmath} = (0, 1)$ are unit (length 1) vectors in the direction of x and y axes correspondingly
- Any vector $\vec{u} = (u_1, u_2)$ in the plane can be written as

$$\vec{u} = u_1 \cdot \hat{\imath} + u_2 \cdot \hat{\jmath}$$

  where u_1 and u_2 are the projections of $\vec{u}$ on the axes

- if $\vec{u} = u_1 \cdot \hat{\imath} + u_2 \cdot \hat{\jmath}$ and $\vec{v} = v_1 \cdot \hat{\imath} + v_2 \cdot \hat{\jmath}$ then

$$\vec{u} + \vec{v} = (u_1 + v_1) \cdot \hat{\imath} + (u_2 + v_2) \cdot \hat{\jmath}$$

$$k \cdot \vec{u} = (k \cdot u_1) \cdot \hat{\imath} + (k \cdot u_2) \cdot \hat{\jmath}$$

Example 1: Let

$$\vec{u} = -3\hat{\imath} + 4\hat{\jmath}\,, \qquad \vec{v} = 2\hat{\imath} + 7\hat{\jmath}$$

$\Longrightarrow$

- $\vec{u} + \vec{v} = (-3+2)\hat{i} + (4+7)\hat{j} = -\hat{i} + 11\hat{j}$
- from Pythagorean theorem

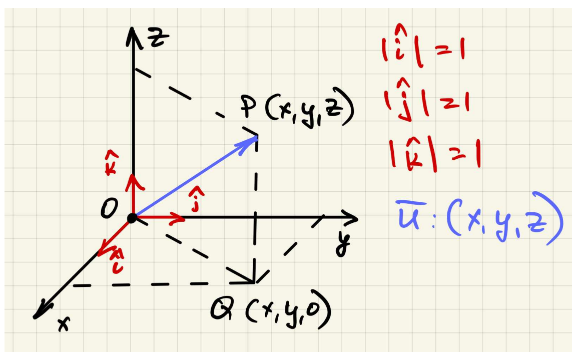
$$|\vec{u}| = \sqrt{u_1^2 + u_2^2} = \sqrt{9+16} = 5$$

- From any vector $\vec{u}$ we can make a unit length vector that has the same direction as $\vec{u}$ and the magnitude 1:

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} \qquad \Longrightarrow \qquad \hat{u} = \frac{1}{5}\left(-3\hat{i} + 4\hat{j}\right) = -\frac{3}{5}\,\hat{i} + \frac{4}{5}\,\hat{j}$$

$$|\hat{u}| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9+16}{25}} = 1$$

Vectors in 3D



- $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$ and $\hat{k} = (0,0,1)$ are unit (length 1) vectors in the direction of x, y and z axes correspondingly
- Any vector $\vec{r} = (x, y, z)$ can be written as

$$\vec{r} = x \cdot \hat{i} + y \cdot \hat{j} + z \cdot \hat{k}$$

where x, y and z are the projections of $\vec{r}$ on the axes

- Let $P_1 = (x_1, y_1, z_1)$ be an initial point of the vector $\overline{P_1P_2}$ and $P_2 = (x_2, y_2, z_2)$ be the final point, then

$$\overline{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = \vec{r}_2 - \vec{r}_1$$

- Since

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

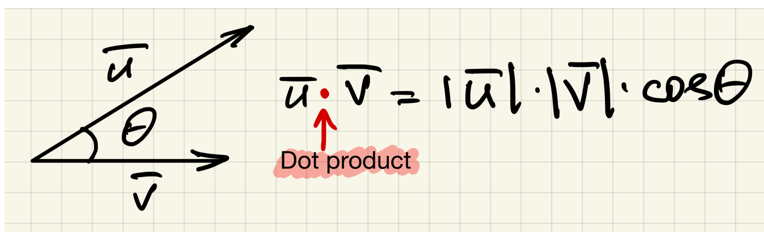
i.e., the distance from O to P $\Longrightarrow$

$$|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

is the distance from P_1 to P_2

The dot product

aka the scalar product of two vectors:



- if two vectors are $\perp$, *i.e.*, $\theta = \frac{\pi}{2} \Longrightarrow \qquad \vec{u} \cdot \vec{v} = 0$

- The other way around: if $\vec{u} \cdot \vec{v} = 0$, where $\vec{u}$ and $\vec{v}$ are both nonzero vectors $\Longrightarrow$

$$|\vec{u}|\ |\vec{v}|\ \cos\theta = 0 \qquad \Longrightarrow \qquad \cos\theta = 0 \Longrightarrow \theta = \frac{\pi}{2}$$

i.e., the two vectors are $\perp$

- in 2D, if $\vec{u} = u_1 \cdot \hat{i} + u_2 \cdot \hat{j}$ and $\vec{v} = v_1 \cdot \hat{i} + v_2 \cdot \hat{j}$ then

$$\vec{u} \cdot \vec{v} = u_1\ v_1 + u_2\ v_2$$

- in 3D, if $\vec{u} = u_1 \cdot \hat{i} + u_2 \cdot \hat{j} + u_3 \cdot \hat{k}$ and $\vec{v} = v_1 \cdot \hat{i} + v_2 \cdot \hat{j} + v_3 \cdot \hat{k}$ then

$$\vec{u} \cdot \vec{v} = u_1\ v_1 + u_2\ v_2 + u_3\ v_3$$

- since $\vec{u} \cdot \vec{v} = |\vec{u}|\ |\vec{v}|\ \cos\theta \Longrightarrow$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|}$$

Properties of the dot product

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- Note that the basis vectors are mutually perpendicular

$$\hat{i} \cdot \hat{j} = 0\,, \qquad \hat{i} \cdot \hat{k} = 0\,, \qquad \hat{j} \cdot \hat{k} = 0$$

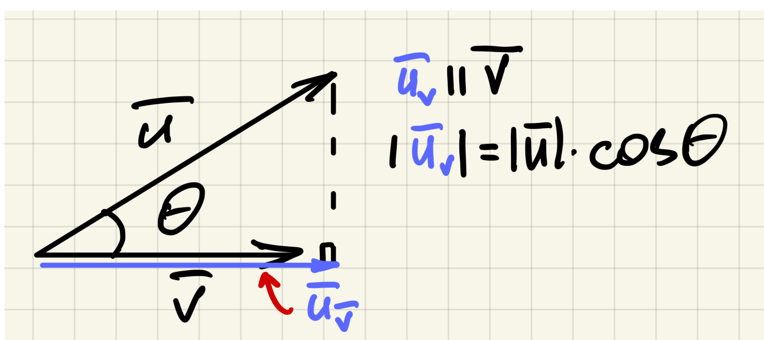
Example 2: for what value of t is the vector $\vec{u} = 2t\,\hat{i} + 4\,\hat{j} - (10+t)\,\hat{k}$ is $\perp$ to the vector $\vec{v} = \hat{i} + t\,\hat{j} + \hat{k}$?

$\Longrightarrow$

$$0 = \vec{u} \cdot \vec{v} = 2t \cdot 1 + 4 \cdot t - (10+t) \cdot 1 = 5t - 10 \qquad \Longrightarrow \qquad t = 2$$

Projections

A projection of $\vec{u}$ on $\vec{v}$, with an angle θ between them, is a vector $\vec{u}_{\vec{v}}$, which is parallel to $\vec{v}$ and has length $|\vec{u}|\cos\theta$:



- consider a unit vector in the direction of $\vec{v}$:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

- Note that
$$\vec{u}_{\vec{v}} = |\vec{u}| \cdot \cos\theta = |\vec{u}| \cdot \underbrace{\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}}_{=\cos\theta}$$

- $\Longrightarrow$
$$\vec{u}_{\vec{v}} = |\vec{u}_{\vec{v}}|\ \hat{v} = |\vec{u}| \cdot \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} \cdot \frac{\vec{v}}{|\vec{v}|} = \vec{v} \cdot \underbrace{\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}}_{\text{a scalar (a real number)}} = \vec{v} \cdot \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

- And now the reason $\vec{u}_{\vec{v}}$ is called a projection. Take the vector
$$\vec{w} \equiv \vec{u} - \vec{u}_{\vec{v}}$$
from the picture we expect $\vec{w} \perp \vec{v}$ — is that true?
$$\vec{w} \cdot \vec{v} == \left(\vec{u} - \vec{v} \cdot \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \cdot \vec{v} = \vec{u} \cdot \vec{v} - (\vec{v} \cdot \vec{v})\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = 0$$
$\Longrightarrow$ indeed $\vec{w} \perp \vec{v}$

- Note
$$\vec{u} = \vec{u}_{\vec{v}} + \vec{w}\,, \qquad \text{where} \qquad \vec{u}_{\vec{v}} \parallel \vec{v} \qquad \text{and} \qquad \vec{w} \perp \vec{v}$$

Example 3: Express the vector $\vec{w} = 3\hat{i} + \hat{j}$ as a sum of vectors $\vec{u} + \vec{v}$, where $\vec{u} \parallel \hat{i} + \hat{j}$ and $\vec{v} \perp \hat{i} + \hat{j}$

$\Longrightarrow$

- $\vec{u}$ is the projection of $\vec{w}$ on $\hat{i} + \hat{j}$:
$$\vec{u} = (\hat{i} + \hat{j})\ \frac{\vec{w} \cdot (\hat{i} + \hat{j})}{|\hat{i} + \hat{j}|^2} = (\hat{i} + \hat{j})\ \frac{3 \cdot 1 + 1 \cdot 1}{1 \cdot 1 + 1 \cdot 1} = 2\hat{i} + 2\hat{j}$$

- $\Longrightarrow$
$$\vec{v} = \vec{w} - \vec{u} = \hat{i} - \hat{j}$$

- Check:
$$\vec{v} \cdot (\hat{i} + \hat{j}) = (\hat{i} - \hat{j}) \cdot (\hat{i} + \hat{j}) = 1 \cdot 1 - 1 \cdot 1 = 0$$
i.e., $\vec{v} \perp \hat{i} + \hat{j}$

Note: in 2D or in 3D, $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$

Vectors in n dimensions:

- a vector is an ordered set of n real numbers $v_1, \cdots v_n$, the components of a vector,

$$\vec{v} = (v_1\,,\, v_2\,,\, \cdots\,,\, v_n)$$

- the basis vectors

$$\hat{e}_1 = (1, 0, \cdots, 0)$$
$$\hat{e}_2 = (0, 1, \cdots, 0)$$
$$\cdots$$
$$\hat{e}_n = (0, 0, \cdots, 1)$$

- $\Longrightarrow$

$$\vec{v} = v_1 \cdot \hat{e}_1 + v_2 \cdot \hat{e}_2 + \cdots + v_n \cdot \hat{e}_n$$

- given two vectors $\vec{u}$ and $\vec{v}$,

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

- $$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

- We can define the angle θ between two nonzero n-vectors $\vec{u}$ and $\vec{v}$ as

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta \qquad \Longrightarrow \qquad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$