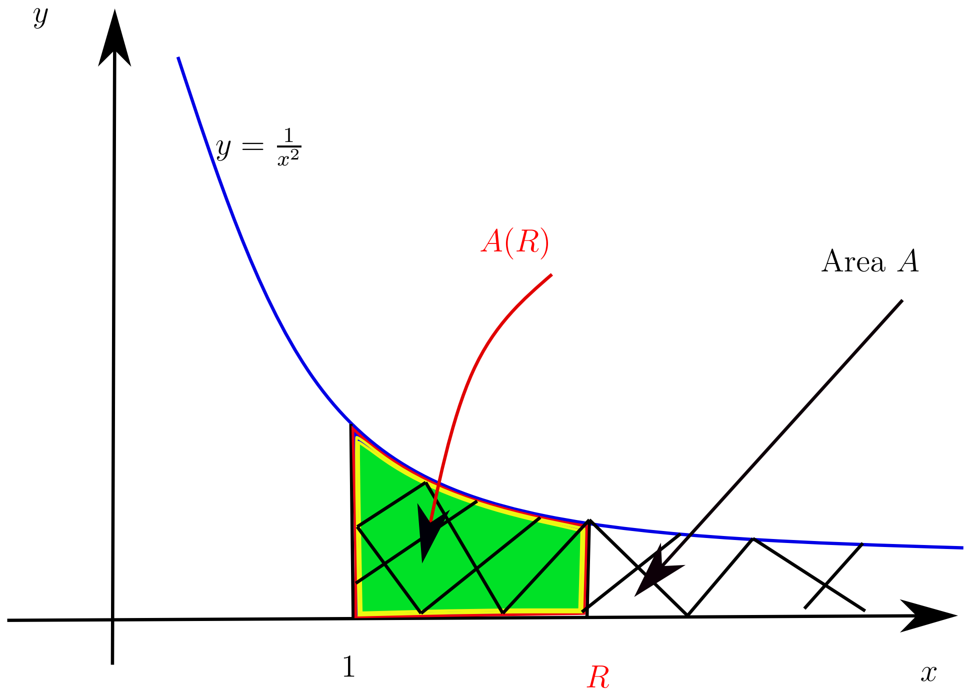
Lecture 6.5: **Improper integrals**



- Find the are A of the shaded area: above $y = 0$, below $f(x) = \frac{1}{x^2}$ and to the right of $x = 1$.
- The shaded region has infinite extend in the x direction
- Instead,
  - consider first the finite region,
  $$x \in [1, R]$$
  - compute the area $A(R)$ of the green region
  $$A(R) = \int_1^R \frac{1}{x^2}\, dx$$
  - Define area A as a limit $R \to +\infty$, if the limit exists,
  $$A = \lim_{R\to\infty} A(R) = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\, dx \equiv \int_1^{+\infty} \frac{1}{x^2}\, dx$$

$\boxed{\text{Improper integral of type I : one or both integration limits are infinite}}$

$\Longrightarrow$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \lim_{R\to+\infty} \left(-\frac{1}{x}\right)\Bigg|_1^R = \lim_{R\to+\infty} \left(-\frac{1}{R} + 1\right) = 1$$

$\Longrightarrow$ We say that the improper integral is convergent to 1.

An example with a different $f(x) = \frac{1}{x}$:

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \lim_{R\to+\infty} \ln x\Big|_1^R = \lim_{R\to+\infty} [\ln R - \ln 1] = \infty$$

$\Longrightarrow$ We say that the improper integral is divergent

<u>In general:</u>

- $f(x)$ is continuous for $x \in [a, \infty)$

<u>Def.</u>
$$\int_a^{\infty} f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx$$

- $f(x)$ is continuous for $x \in (-\infty, a]$

<u>Def.</u>
$$\int_{-\infty}^{a} f(x)dx = \lim_{R\to-\infty} \int_R^a f(x)dx$$

  - if limit is finite $\Longrightarrow$ integral converges
  - if limit is $\pm\infty$ $\Longrightarrow$ diverges to $\pm\infty$
  - if limit does not exist $\Longrightarrow$ diverges

- $f(x)$ is continuous for $x \in (-\infty, \infty)$

<u>Def.</u>
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

for convergence of the LHS integral, both RHS integrals must converge

Example 1:

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \underbrace{\int_{-\infty}^{0} \frac{dx}{1+x^2}}_{\equiv I_1} + \underbrace{\int_{0}^{\infty} \frac{dx}{1+x^2}}_{\equiv I_2}$$

$\Longrightarrow$

$$I_1 = \int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{R\to-\infty} \int_{R}^{0} \frac{dx}{1+x^2} = \lim_{R\to-\infty} \tan^{-1} x \Big|_{R}^{0} = \lim_{R\to-\infty} \Big(0 - \tan^{-1} R\Big) = -\left(-\frac{\pi}{2}\right)$$

$$I_2 = \int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_{0}^{R} \frac{dx}{1+x^2} = \lim_{R\to\infty} \tan^{-1} x \Big|_{0}^{R} = \lim_{R\to\infty} \Big(\tan^{-1} R - 0\Big) = \frac{\pi}{2}$$

$\Longrightarrow$ both convergent, so I is convergent and

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Example 2:

$$I = \int_{0}^{\infty} \sin x dx = \lim_{R\to\infty} \int_{0}^{R} \sin x dx = \lim_{R\to\infty} (-\cos x)\Big|_{0}^{R} = \lim_{R\to\infty} (-\cos R + 1) = D.N.E$$

$\Longrightarrow$ the integral is divergent

Example 3:

$$I = \int_{-\infty}^{\infty} \sin x dx = \int_{-\infty}^{0} \sin x dx + \underbrace{\int_{0}^{\infty} \sin x dx}_{\text{is divergent from above}}$$

$\Longrightarrow$ thus I is also divergent — this is the case even though for any R

$$\int_{-R}^{R} \sin x dx = 0$$

because of the symmetry (odd integrand)

$$\sin(-x) = -\sin x$$

Example 4:

$$I = \int_0^\infty xe^{-x}dx = \lim_{R\to\infty}\int_0^R xe^{-x}dx \qquad =$$

Recall the integration by parts:

$$\int_0^R \underbrace{xe^{-x}\ \ dx}_{u=x,dv=e^{-x}dx,v=-e^{-x},du=dx} = -xe^{-x}\Big|_0^R - \int_0^R \left(-e^{-x}\right)dx = -Re^{-R} + \int_0^R e^{-x}\ dx$$

$$= -Re^{-R} - e^{-x}\Big|_0^R = -Re^{-R} - e^{-R} + 1$$

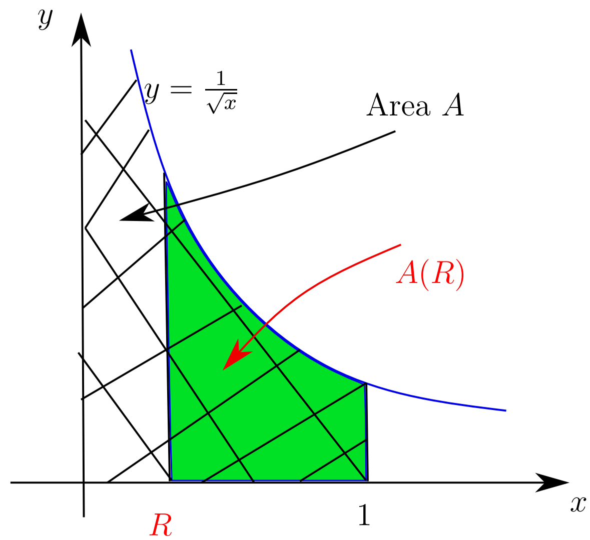
$\Longrightarrow$ thus, we continue with the original integral as

$$= \qquad \lim_{R\to\infty}\left(-Re^{-R} - e^{-R} + 1\right) = 1$$

where we used

$$\lim_{R\to\infty} Re^{-R} = \lim_{R\to\infty}\frac{R}{e^R} \qquad \underbrace{=}_{\frac{\infty}{\infty}\ \text{useL.R.}} \qquad \lim_{R\to\infty}\frac{1}{e^R} = 0$$

Improper integrals of type II : $f(x)$ is divergent as x approaches the integration limits



- Find the are A of the shaded area: above $y = 0$, below $f(x) = \frac{1}{\sqrt{x}}$ and to the right of $x = 0$ and to the left of $x = 1$.
- The shaded region has a finite extend in the x direction, but

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} \frac{1}{\sqrt{x}} = \infty$$

- the area A is represented by the improper integral of type II

$$A = \int_0^1 \frac{1}{\sqrt{x}}\, dx$$

- We proceed as follows:
  - consider first the reduced region,

$$x \in [R, 1]\,, \qquad R > 0$$

- compute the area $A(R)$ of the green region

$$A(R) = \int_R^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_R^1 = 2(1 - \sqrt{R})$$

- Define area A as a limit $R \to 0_+$, if the limit exists,

$$A = \lim_{R \to 0_+} A(R) = \lim_{R \to 0_+} 2(1 - \sqrt{R}) = 2$$

$\Longrightarrow$ integral for A converges to 2.

Example:

$$I = \int_0^1 \frac{1}{x}\, dx = \lim_{R \to 0_+} \int_R^1 \frac{dx}{x} = \lim_{R \to 0_+} (\ln 1 - \ln R) = \infty$$

$\Longrightarrow$ the integral I diverges to ∞

Example:

$$I = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^1 \frac{dx}{\sqrt{x - x^2}}$$

Note that $f(x) = \frac{1}{\sqrt{x - x^2}}$ diverges both as $x \to 0_+$ and $x \to 1_-$ — this is an improper integral of type II

- complete the square for $x - x^2$:

$$x - x^2 = \frac{1}{4} - \left(\frac{1}{4} - x + x^2\right) = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$$

- set

$$u = x - \frac{1}{2}, \qquad du = dx \implies \qquad x - x^2 = \frac{1}{4} - u^2$$

- $$\int_0^1 \frac{dx}{\sqrt{x - x^2}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{du}{\sqrt{\frac{1}{4} - u^2}} = \underbrace{\int_{-\frac{1}{2}}^{0} \frac{du}{\sqrt{\frac{1}{4} - u^2}}}_{\equiv I_1} + \underbrace{\int_0^{\frac{1}{2}} \frac{du}{\sqrt{\frac{1}{4} - u^2}}}_{\equiv I_2}$$

- $$I_1 = \int_{-\frac{1}{2}}^{0} \frac{du}{\sqrt{\frac{1}{4} - u^2}} = \lim_{R \to -\frac{1}{2}_+} \int_R^0 \frac{du}{\sqrt{\frac{1}{4} - u^2}} = \lim_{R \to -\frac{1}{2}_+} \sin^{-1}(2u)\Big|_R^0 = \lim_{R \to -\frac{1}{2}_+} \left(-\sin^{-1}(2R)\right)$$
$$= -\sin^{-1}(-1) = \frac{\pi}{2}$$
$\Longrightarrow$ I_1 converges

- $$I_2 = \int_0^{\frac{1}{2}} \frac{du}{\sqrt{\frac{1}{4} - u^2}} = \lim_{R \to \frac{1}{2}_-} \int_0^R \frac{du}{\sqrt{\frac{1}{4} - u^2}} = \lim_{R \to \frac{1}{2}_-} \sin^{-1}(2u)\Big|_0^R = \lim_{R \to \frac{1}{2}_-} \sin^{-1}(2R)$$
$$= \sin^{-1} 1 = \frac{\pi}{2}$$
$\Longrightarrow$ I_2 converges

- thus I converges as well and
$$I = I_1 + I_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

<u>p-integrals</u>

$\Longrightarrow$ assume: a is finite, $p > 0$

- Type I:
$$\int_a^\infty x^{-p}\, dx : \qquad \begin{cases} \text{converges to } \frac{a^{1-p}}{p-1}\,, p > 1 \\ \text{diverges to } \infty\,, p \le 1 \end{cases}$$

Indeed, assume $p \neq 1$,
$$\int_a^\infty x^{-p}\, dx = \lim_{R \to \infty} \int_a^R x^{-p}\, dx = \lim_{R \to \infty} \left(\frac{x^{1-p}}{1-p}\right)\Big|_a^R = \lim_{R \to \infty} \left(\frac{R^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right)$$
$$= \begin{cases} \frac{a^{1-p}}{p-1}\,,\ p > 1 \\ \infty\,,\ 0 < p < 1 \end{cases}$$

The last equality came from
$$\lim_{R \to \infty} R^{1-p} = \begin{cases} 0\,,\ p > 1 \\ \infty\,,\ 0 < p < 1 \end{cases}$$

Now take $p = 1$:

$$\int_a^\infty \frac{dx}{x} = \lim_{R\to\infty} \int_a^R \frac{dx}{x} = \lim_{R\to\infty} \ln x \Big|_a^R = \lim_{R\to\infty} (\ln R - \ln a) = \infty$$

thus the $p = 1$ integral is divergent as well

- Type II:

$$\int_0^a x^{-p}\, dx : \qquad \begin{cases} \text{converges to } \frac{a^{1-p}}{1-p}\,, p < 1 \\ \text{diverges to } \infty\,, p \geq 1 \end{cases}$$

Indeed, assume $p \neq 1$,

$$\int_0^a x^{-p}\, dx = \lim_{R\to 0_+} \int_R^a x^{-p}\, dx = \lim_{R\to 0_+} \left(\frac{x^{1-p}}{1-p}\right)\Big|_R^a = \lim_{R\to 0_+} \left(\frac{a^{1-p}}{1-p} - \frac{R^{1-p}}{1-p}\right)$$

$$= \begin{cases} \frac{a^{1-p}}{1-p}\,, \;\; 0 < p < 1 \\ \infty\,, \;\; p > 1 \end{cases}$$

The last equality came from

$$\lim_{R\to 0_+} R^{1-p} = \begin{cases} 0\,, \;\; 0 < p < 1 \\ \infty\,, \;\; p > 1 \end{cases}$$

Now take $p = 1$:

$$\int_0^a \frac{dx}{x} = \lim_{R\to 0_+} \int_R^a \frac{dx}{x} = \lim_{R\to 0_+} \ln x \Big|_R^a = \lim_{R\to 0_+} (\ln a - \ln R) = \infty$$

thus the $p = 1$ integral is divergent as well

Theorem (comparison of integrals)

Let
$$-\infty \leq a < b \leq \infty$$
and (f, g) are continuous for $x \in (a, b)$ and
$$0 \leq f(x) \leq g(x) \qquad \text{for} \qquad x \in (a, b)$$
$\Longrightarrow$
$$0 \leq \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

- (1) If $\int_a^b g(x)dx$ improper integral converges, then $\int_a^b f(x)dx$ converges as well
- (2) If $\int_a^b f(x)dx$ improper integral diverges, then $\int_a^b g(x)dx$ diverges as well

Example: Establish whether
$$I = \int_0^\infty \frac{dx}{\sqrt{x + x^3}}$$
is convergent or divergent

$\Longrightarrow$ I is improper on both ends, so we split it
$$I = \underbrace{\int_0^1 \frac{dx}{\sqrt{x + x^3}}}_{\equiv I_1} + \underbrace{\int_1^\infty \frac{dx}{\sqrt{x + x^3}}}_{\equiv I_2}$$

▪ for I_1:
$$0 < \underbrace{\frac{1}{\sqrt{x + x^3}}}_{\equiv f(x)} < \underbrace{\frac{1}{\sqrt{x}}}_{\equiv g(x)} \equiv x^{-\frac{1}{2}}\,, \qquad x \in (0, 1)$$
$\Longrightarrow$ $g(x)$ converges as type II improper integral since $p = \frac{1}{2}$, thus I_1 is convergent is well

▪ for I_2:
$$0 < \underbrace{\frac{1}{\sqrt{x + x^3}}}_{\equiv f(x)} < \underbrace{\frac{1}{\sqrt{x^3}}}_{\equiv g(x)} \equiv x^{-\frac{3}{2}}\,, \qquad x \in (1, \infty)$$
$\Longrightarrow$ $g(x)$ converges as type I improper integral since $p = \frac{3}{2}$, thus I_2 is convergent is well

$\Longrightarrow$ I is convergent