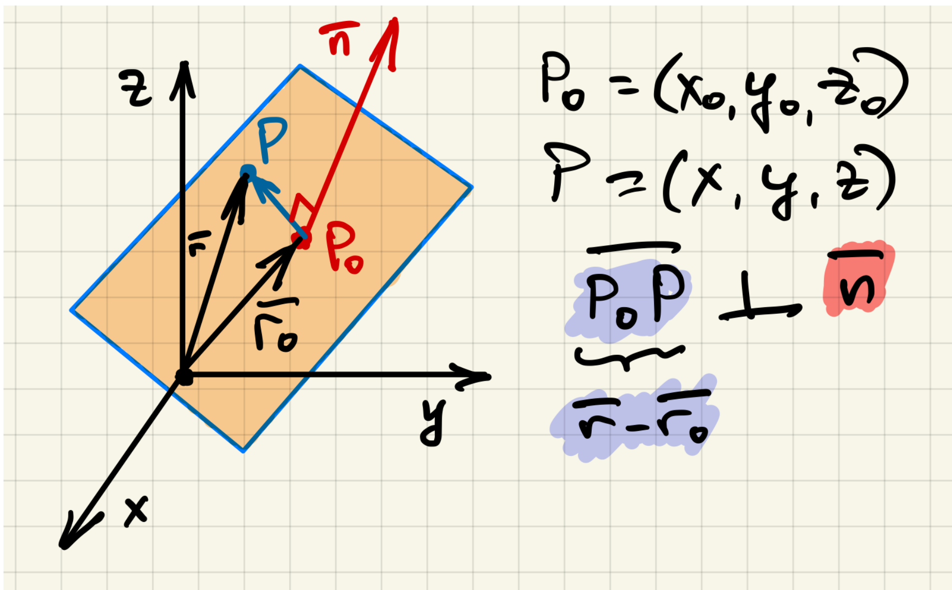
Lecture 10.4: **Planes and lines**

Planes

$\Longrightarrow$ To fully determine a plane (orange) we need

- a point in the plane, $P_0 = (x_0, y_0, z_0)$ (red)
- a normal vector to the plane, $\vec{n} = A \cdot \hat{i} + B \cdot \hat{j} + C \cdot \hat{k}$ (red)



- Suppose the point $P = (x, y, z)$ (blue) belongs to the plane.
- $\Longrightarrow$ the vector $\overline{P_0P} = (x - x_0, y - y_0, z - z_0)$ must belong to the plane as well
- $\Longrightarrow$

$$\overline{P_0P} \perp \vec{n} \qquad \overline{P_0P} \cdot \vec{n} = 0$$

or

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

- $\Longrightarrow$ the equation for the plane:

$$\boxed{A \cdot (x - x_0) + B \cdot (y - y_0) + C \cdot (z - z_0) = 0}$$

alternatively:

$$Ax + By + Cz = D\,, \qquad \text{where} \qquad D \equiv Ax_0 + By_0 + Cz_0$$

Example 1: Find the equation of the plane containing a point $(1, 0, 5)$ and having a normal vector $\vec{n} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$

$\Longrightarrow$ From $\vec{n}$:

$$A = 3\,, \qquad B = 1\,, \qquad C = -1$$

$\Longrightarrow$

$$0 = 3 \cdot (x - 1) + 1 \cdot (y - 0) + (-1) \cdot (z - 5)$$

or

$$3x + y - z = -2$$

Example 2: Find the equation of the plane containing the points $P_1 = (1, 1, 0)$, $P_2 = (2, 0, 2)$ and $P_3 = (0, 3, 3)$

$\Longrightarrow$

- If P_1, P_2, P_3 below to the plane, then so do the vectors

$$\overline{P_1P_2} = (2 - 1, 0 - 1, 2 - 0) = (1, -1, 2)$$

and

$$\overline{P_1P_3} = (0 - 1, 3 - 1, 3 - 0) = (-1, 2, 3)$$

- The normal to the plane is orthogonal to both $\overline{P_1P_2}$ and $\overline{P_1P_3} \Longrightarrow$

$$\vec{n} = \overline{P_1P_2} \times \overline{P_1P_3} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ -1 & 2 & 3 \end{vmatrix} = \hat{\imath}(-3 - 4) - \hat{\jmath}(3 + 2) + \hat{k}(2 - 1)$$

$$= -7\hat{\imath} - 5\hat{\jmath} + \hat{k} \qquad A = -7\,,\ B = -5\,,\ C = 1$$

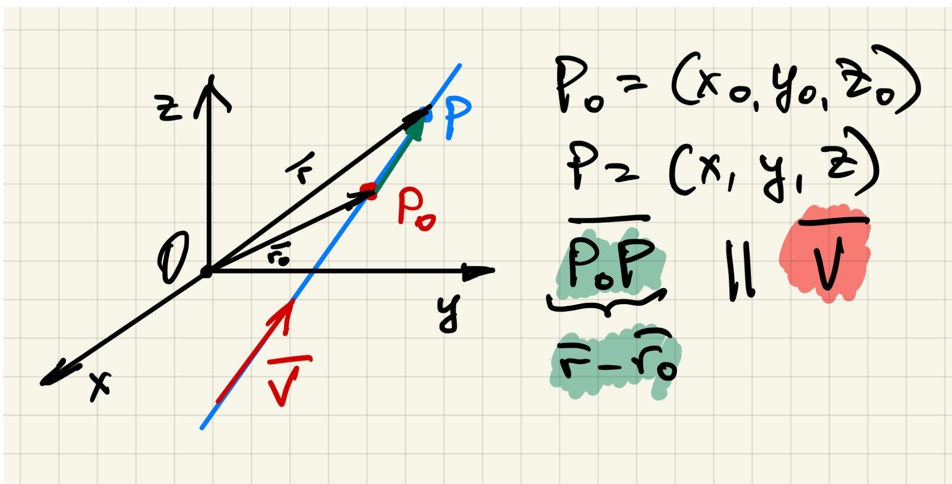
- $\implies$

$$-7 \cdot (x-1) - 5 \cdot (y-1) + 1 \cdot (z-0) = 0 \qquad \implies \qquad -7x - 5y + z = -12$$

Lines

$\implies$ To fully determine a line (blue) we need

- a point on the line, $P_0 = (x_0, y_0, z_0)$ (red)
- a vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ (red) in the direction of the line



- Suppose the point $P = (x, y, z)$ (blue) belongs to the line.

- $\implies$ the vector $\overline{P_0P} = (x - x_0, y - y_0, z - z_0)$ (green) must belong to the line as well

- $\implies$

$$\overline{P_0P} \parallel \vec{v} \qquad \implies \qquad \overline{P_0P} = t \cdot \vec{v}$$

for some scalar parameter t; or

$$(\vec{r} - \vec{r}_0) = t \cdot \vec{v}$$

- $\Longrightarrow$ the vector equation for the line:

$$\boxed{\vec{r} = \vec{r}_0 + t \cdot \vec{v}}$$

  - alternatively, in components,

$$\begin{cases} x = x_0 + a \cdot t \\ y = y_0 + b \cdot t \\ z = z_0 + c \cdot t \end{cases}$$

  - yet another representation — *aka* the standard form — (obtained by elimination of t):

    – assume $abc \neq 0 \Longrightarrow$

$$t = \frac{x - x_0}{a}, \qquad t = \frac{y - y_0}{b}, \qquad t = \frac{z - z_0}{c}$$

$\Longrightarrow$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

    – let $a = 0$, but $bc \neq 0 \Longrightarrow$

$$\frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad \& \qquad x = x_0$$

Example 3: Find in standard form the equation for the line of intersection of the planes $x + y - z = 0$ and $y + 2z = 6$

$\Longrightarrow$

- the normal to the first plane is

$$\vec{n}_1 = (1, 1, -1)$$

- the normal to the second plane is

$$\vec{n}_2 = (0, 1, 2)$$

- intersection line belongs to both planes $\Longrightarrow$ its directional vector $\vec{v}$ must be orthogonal both to $\vec{n}_1$ and $\vec{n}_2$

- $\Longrightarrow$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \hat{i}(2+1) - \hat{j}(2+0) + \hat{k}(1-0) = 3\hat{i} - 2\hat{j} + \hat{k}$$

or

$$\vec{v} = (3, -2, 1)$$

- We need to find a point P_0, which is a solution (there are infinitely many solutions!) of

$$\begin{cases} x_0 + y_0 - z_0 = 0 \\ y_0 + 2z_0 = 6 \end{cases}$$

$\Longrightarrow$ one such solution is

$$P_0 = (3, 0, 3)$$

- $\Longrightarrow$ the equation for the line is then

$$\frac{x-3}{3} = \frac{y}{-2} = \frac{z-3}{1}$$

- The answer is not unique: choosing

$$P_0 = (0, 2, 2)$$

$\Longrightarrow$ a different representation of the same line

$$\frac{x}{3} = \frac{y-2}{-2} = \frac{z-2}{1}$$