Lecture 9.6: **Taylor and Maclaurin series**

Let a power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 \cdot (x-c) + a_2 \cdot (x-c)^2 + \cdots$$

has a radius of convergence $R > 0$. Then the power series is convergent if $x \in (c - R, c + R)$ and the sum of the series is a function $f(x)$,

$$f(x) \equiv \sum_{n=0}^{\infty} a_n (x-c)^n \,, \qquad |x-c| < R$$

$\Longrightarrow$ We now determine the coefficients a_n directly from $f(x)$:

- set $x = c \Longrightarrow$

$$f(c) = a_0 \ \Longrightarrow a_0 = f(c)$$

- differentiate once (the series remains convergent for $|x-c| < R$!) and set $x = c$:

$$f'(c) = 1 \cdot a_1 + 2a_2 \cdot (x-c) + 3a_3 \cdot (x-c)^2 + \cdots \Big|_{x=c} = 1 \cdot a_1$$

$$\Longrightarrow \qquad a_1 = \frac{f'(c)}{1}$$

- differentiate twice set $x = c$:

$$f''(c) = 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 \cdot (x-c) + \cdots \Big|_{x=c} = 2 \cdot 1 \cdot a_2 = 2! \cdot a_2$$

$$\Longrightarrow \qquad a_2 = \frac{f''(c)}{2!}$$

- and so on $\Longrightarrow$

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Def: if $f(x)$ has derivatives of all orders at $x = c$ then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

- is called the Taylor series of $f(x)$ about $x = c$, or in powers of $(x-c)$;
- if $c = 0$, is called the Maclaurin series

Example 1: Find the Taylor series of

$$f(x) = e^x$$

around $x = c$.

$\Longrightarrow$

- $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$
- $$f(c) = e^c\,, \qquad f'(c) = (e^x)'\Big|_{x=c} = e^x\Big|_{x=c} = e^c$$
- $\Longrightarrow$
$$f^n(c) = \frac{d^n}{dx^n}e^x\Big|_{x=c} = e^x\Big|_{x=c} = e^c$$
- $\Longrightarrow$
$$f(x) = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n$$
- What is R for this Taylor series? $\Longrightarrow$
$$\rho = \lim_{n\to\infty} \frac{\frac{e^c}{(n+1)!}|x-c|^{n+1}}{\frac{e^c}{n!}|x-c|^n} = |x-c| \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0$$

  $\Longrightarrow$ $R = \infty$, and the Taylor series is convergent for

$$x \in \mathbb{R}$$

- Remember the Maclaurin series for e^x:

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \qquad x \in \mathbb{R}}$$

Def: If $f(x)$ has a Taylor series expansion around c such that the Taylor series converges to $f(x)$ on an open interval containing $c \Longrightarrow$ $f(x)$ is called analytic at c. $f(x)$ is analytic on an interval $\mathcal{I}$ if it is analytic at any $x \in \mathcal{I}$.

$\Longrightarrow$ From the example 1, e^x is analytic for $x \in \mathbb{R}$

Example 2: Find the Maclaurin series of

$$f(x) = \frac{1}{1-x}$$

and its analyticity interval

$\Longrightarrow$

- Recall the geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,, \qquad |x| < 1$$

- $\Longrightarrow$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,, \qquad |x| < 1}$$

Example 3: Find the Maclaurin series of

$$f(x) = \sin(x)$$

and its analyticity interval

$\Longrightarrow$

- Note: $\sin(0) = 0$ and

$$\frac{d}{dx}\sin(x)\bigg|_{x=0} = \cos x\bigg|_{x=0} = 1$$
$$\frac{d^2}{dx^2}\sin(x)\bigg|_{x=0} = -\sin x\bigg|_{x=0} = 0$$
$$\frac{d^3}{dx^3}\sin(x)\bigg|_{x=0} = -\cos x\bigg|_{x=0} = -1$$
$$\frac{d^4}{dx^4}\sin(x)\bigg|_{x=0} = \sin x\bigg|_{x=0} = 0$$

- $\Longrightarrow$ for any integer n

$$f^{(2n)}(0) = (\sin(x))^{(2n)}\bigg|_{x=0} = 0\,, \qquad f^{(2n+1)}(0) = (\sin(x))^{(2n+1)}\bigg|_{x=0} = (-1)^n$$

- $\Longrightarrow$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

- apply the ratio test directly to the power series above:

$$\rho=\lim_{n\to\infty}\frac{\frac{1}{(2(n+1)+1)!}|x|^{2(n+1)+1}}{\frac{1}{(2n+1)!}|x|^{2n+1}}=|x|^2\cdot\lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!}=|x|^2\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)}=0$$

$\Longrightarrow \qquad R=\infty$

- $\Longrightarrow$

$$\boxed{\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\,, \qquad x\in\mathbb{R}}$$

Example 4: Find the Maclaurin series of

$$f(x)=e^{-x^2/2}$$

and its analyticity interval

$\Longrightarrow$

- set

$$u=-\frac{x^2}{2}$$

- $\Longrightarrow$

$$f=e^u=\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\underbrace{u}_{u=-x^2/2}\Big)^n, \qquad u\in\mathbb{R}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x^2}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n}\,, \qquad x\in\mathbb{R}$$

Example 5: Find the Taylor series of

$$f(x)=\ln x$$

in terms of $(x-2)$ and its analyticity interval

$\Longrightarrow$

- WARNING! Do no attempt to derive the general formula for $f^{(n)}$ — this approach works only for simple functions: $\{\sin x, \cos x, e^x\}$

- set $u = x - 2 \Longrightarrow$
$$f = \ln(u+2)$$
and we are after the Maclaurin series for the above function

- Note
$$\frac{d}{du}\ln(u+2) = \frac{1}{2+u} = \frac{1}{2}\cdot\frac{1}{1-\underbrace{(-u/2)}_{=w}} = \frac{1}{2}\,\frac{1}{1-w}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty}\frac{1}{2}\left(-\frac{u}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\,u^n$$

- above manipulations are valid when
$$|w| < 1 \quad\Longrightarrow\quad \left|\frac{u}{2}\right| < 1 \quad\Longrightarrow\quad |u| < 2 \quad\Longrightarrow\quad |x-2| < 2$$

- at this stage we have
$$\frac{d}{dt}\ln(t+2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\,t^n$$

- integrate both sides:
$$\int_0^u dt\,\frac{d}{dt}\ln(t+2) = \int_0^u dt\,\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\,t^n$$

  - LHS:
$$\ln(t+2)\Big|_0^u = \ln(u+2) - \ln 2$$

  - RHS:
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\int_0^u dt\,t^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\,\frac{t^{n+1}}{n+1}\Big|_0^u = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}\,\frac{u^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}(n+1)}u^{n+1}$$

- LHS=RHS $\Longrightarrow$
$$\ln(u+2) - \ln 2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}(n+1)}u^{n+1}$$

or

$$\ln(u+2) = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}(n+1)} u^{n+1}$$

or

$$\ln x = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}(n+1)} (x-2)^{n+1}\,, \qquad |x-2| < 2$$

- redefine the summation index in the last formula $n+1=k$

$$\ln x = \ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k k} (x-2)^k\,, \qquad 0 < x < 4$$