Lecture 9.3: **Convergence tests for positive series**

Consider a positive series

$$\sum_{n=1}^{\infty} a_n\,, \qquad a_n \geq 0\,, \qquad \text{for all } n$$

■ The integral test

Theorem: let $\sum_{n=1}^{\infty} a_n$ be a positive series where $a_n = f(n)$ for a positive continuous and non-increasing on $[N, \infty)$ function $(N > 0)$. Then

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_N^{\infty} f(x)dx$$

either both diverge or converge

Example 1: When is the p-series convergent?

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\,, \qquad p > 0$$

$\Longrightarrow$

- Since
  $$a_n = f(n) \implies f(x) = \frac{1}{x^p}$$
  the function $f(x)$ is continuous, positive and non increasing on $[1, \infty)$. Indeed, if $x_2 > x_1$, then
  $$\frac{f(x_2)}{f(x_1)} = \left(\frac{x_1}{x_2}\right)^p < 1$$
- $\Longrightarrow$ convergence/divergence is correlated for
  $$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \Longleftrightarrow \qquad \int_1^{\infty} \frac{dx}{x^p}$$
- Recall the p-integral is convergent if $p > 1$ and is divergent if $0 < p \leq 1 \Longrightarrow$ the same is true for the p-series

■ Comparison test

Theorem: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive series such that there is a constant k for which $0 \le a_n \le k b_n$ for all $n \ge N$ (N is some integer). Then

- if $\sum_{n=1}^{\infty} b_n$ is convergent $\Longrightarrow$ $\quad\sum_{n=1}^{\infty} a_n$ is convergent as well
- if $\sum_{n=1}^{\infty} a_n$ diverges to ∞ $\Longrightarrow$ $\quad\sum_{n=1}^{\infty} b_n$ diverges to ∞ as well

Example 2: Is the series

$$\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$$

convergent?

$\Longrightarrow$

- We are going to show that

  $$x > \ln x\,, \qquad \text{for } x > 1$$

  Indeed, consider the function

  $$f(x) \equiv x - \ln x \qquad \Longrightarrow \quad f'(x) = 1 - \frac{1}{x} > 0\,,\ \ x \in (1, +\infty)$$

  i.e., $f(x)$ is an increasing function. Since

  $$f(1) = 1 - \ln 1 = 1 > 0 \ \Longrightarrow \ f(x) > 1 > 0 \text{ for all } x > 1$$

  Thus,

  $$x > \ln x$$

- $\Longrightarrow$

  $$\frac{1}{x} < \frac{1}{\ln x} \qquad \Longrightarrow \qquad \frac{1}{(3n)} < \frac{1}{\ln(3n)}$$

- Since

  $$\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \cdot \sum_{n=1}^{\infty} \frac{1}{n}$$

  diverges to infinity (as a harmonic series), so does the original series by the comparison test

Example 3: Is the series
$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$
convergent?

$\Longrightarrow$

- Note that
$$\frac{1}{2^n+1} < \frac{1}{2^n}$$
- The series
$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$
is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2} < 1 \Longrightarrow$ it converges to
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$
- By comparison test, the original series is convergent as well, and
$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} \leq 1$$

■ Limit comparison test

Theorem: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive series and
$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \geq 0 \qquad \text{or} \qquad \text{infinity}$$
Then

- if $L < \infty$, *i.e.*, is finite, and $\sum_{n=1}^{\infty} b_n$ is convergent $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent as well
- if $L > 0$ or L is ∞, and $\sum_{n=1}^{\infty} b_n$ diverges to ∞ $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ diverges to ∞ as well
- if a finite $L > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge

Example 4: Is the series

$$\sum_{n=1}^{\infty} \frac{n}{n^4+3}$$

convergent?

$\Longrightarrow$

- Consider a $p = 3$ series

$$b_n = \frac{1}{n^3}$$

- Note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^4+3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4}{n^4+3} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n^4}} = 1$$

- Since

$$\sum_{n\to\infty} b_n = \sum_{n\to\infty} \frac{1}{n^3}$$

is convergent ($p = 3 > 1$), by the limit comparison test so does the original series

Example 5: Is the series

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$$

convergent?

$\Longrightarrow$

- Consider a $p = \frac{1}{2}$ series

$$b_n = \frac{1}{\sqrt{n}}$$

- Note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1$$

- Since

$$\sum_{n\to\infty} b_n = \sum_{n\to\infty} \frac{1}{n^{1/2}}$$

is divergent to ∞ ($p = \frac{1}{2} < 1$), by the limit comparison test so does the original series

■ The ratio test

Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a positive series and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \geq 0 \qquad \text{or} \qquad \text{infinity}$$

Then

- if $0 \leq \rho < 1$, the $\sum_{n=1}^{\infty} a_n$ is convergent
- if $\rho > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges to ∞
- if $\rho = 1 \Longrightarrow$ the test is useless

Example 6: Is the series

$$\sum_{n=1}^{\infty} \frac{n^5}{2^n}$$

convergent?

$\Longrightarrow$

- From the ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5 2^n}{n^5 2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^5 = \frac{1}{2}$$

- Since $\rho < 1$, the original series is convergent

Example 7: Is the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

convergent?

$\Longrightarrow$

- From the ratio test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!(n)^n}{(n+1)^{n+1}n!} = \lim_{n\to\infty} \underbrace{\frac{(n+1)!}{n!}}_{=n+1} \cdot \frac{n^n}{(1+n)^n} \cdot \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}\right)^{-1} = e^{-1} = \frac{1}{e}$$

- Since $\rho < 1$, the original series is convergent

■ The root test

Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a positive series and

$$\lim_{n\to\infty} {a_n}^{1/n} = \sigma \geq 0 \qquad \text{or} \qquad \text{infinity}$$

Then

- if $0 \leq \sigma < 1$, the $\sum_{n=1}^{\infty} a_n$ is convergent
- if $\sigma > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges to ∞
- if $\sigma = 1 \Longrightarrow$ the test is useless

Example 8: Is the series

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$$

convergent?

$\Longrightarrow$

- From the root test

$$\sigma = \lim_{n\to\infty} {a_n}^{1/n} = \lim_{n\to\infty} \frac{2^{1+1/n}}{n} = 2 \cdot \lim_{n\to\infty} \frac{1}{n} = 0$$

- Since $\sigma < 1$, the original series is convergent