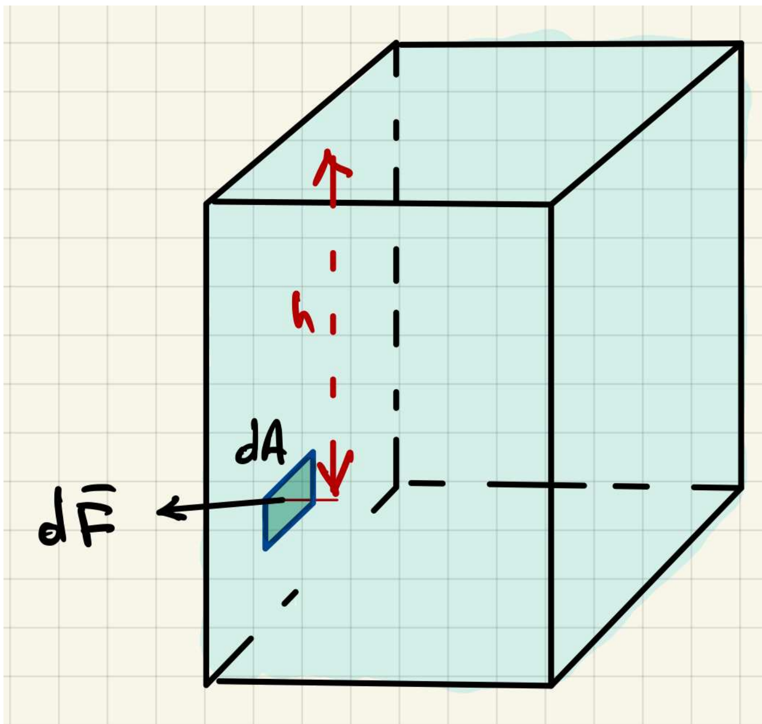
Lecture 7.6: **Other physical applications**

- (1) Hydrostatic pressure

A container is filled with water (density ρ, $[\rho] = \frac{kg}{m^3}$)



- Consider an element dA at depth h on the left-side wall of the container.
- The force $d\bar{F}$ acting on the element, in the direction of the outward unit normal $\bar{n}$ to the element, is

$$d\bar{F} = \bar{n} \cdot dF\,, \qquad dF = P \cdot dA$$

  where P is the hydrostatic pressure

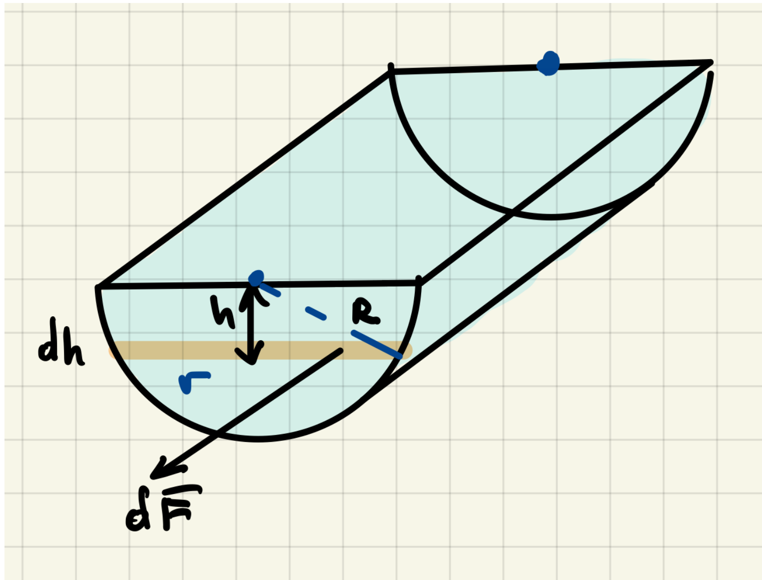
- According to Pascal's law,

$$P = \rho g h$$

  where g is the gravity (acceleration due to gravity on Earth surface)

$$g \approx 9.8\ \frac{m}{s^2}$$

Example 1. A water trough is filled to the top. The cross section is a half-disk of radius R. Find the total force F acting on the front.



$\Longrightarrow$

- Consider a horizontal strip of the front at depth h and width dh.
- The pressure is constant (for small dh) along the strip:

$$P = \rho g h$$

- The force acting on the strip (out of plane) is

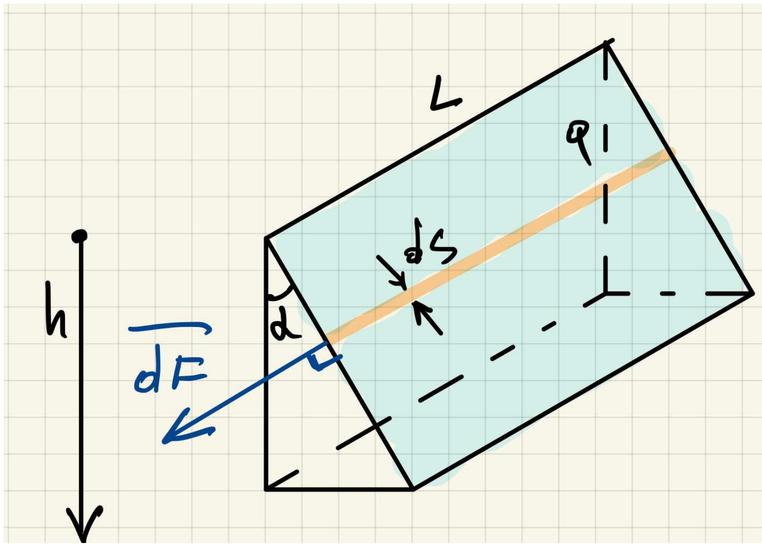
$$dF = P \cdot \underbrace{dA}_{=2r dh} = P \cdot 2\sqrt{R^2 - h^2} dh = 2\rho g \cdot h\sqrt{R^2 - h^2} dh$$

- The total force is the sum (an integral) of the forces from all the strips

$$F = \int dF = \int_0^R 2\rho g \cdot h\sqrt{R^2 - h^2} dh = \rho g \left(-\frac{2}{3}\right) (R^2 - h^2)^{3/2} \Big|_0^R$$

$$= \frac{2}{3}\rho g R^3$$

Example 2. Find the total force exerted by the water on the face of the dam of height a.



$\Longrightarrow$

- Consider a horizontal strip (orange) distance h from the top of the dam,

$$0 \leq h \leq a$$

  and the width ds.

- Note that

$$dh = ds \cos\alpha \qquad \Longrightarrow \qquad ds = \sec\alpha \cdot dh$$

- The pressure is constant along the strip:

$$P = \rho g h$$

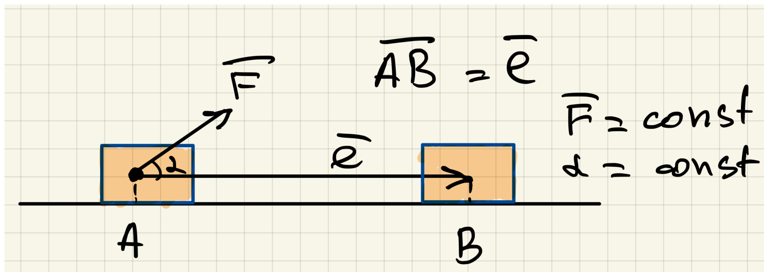
  resulting in the force

$$dF = P \cdot dA = P \cdot L ds = \rho g h L \sec\alpha dh$$

- The total force is

$$F = \int dF = \int_0^a \rho g h L \sec\alpha dh = \rho g L \sec\alpha \int_0^a h dh = \rho g L \sec\alpha \left. \frac{1}{2} h^2 \right|_0^a = \frac{1}{2} \rho g L \sec\alpha a^2$$

- (2) Work



- Suppose the box (orange) is pulled with a constant force $\bar{F}$, at angle α to the surface, from A to B.

- The total displacement is

$$\overline{AB} = \bar{\ell}$$

- The work W, performed on the object is

$$W = \bar{F} \cdot \bar{\ell} = F\ell \ \cos\alpha\,, \qquad |\bar{F}| = F\,, \qquad |\bar{\ell}| = \ell$$

- If the force is in the direction of the displacement, *i.e.*, $\alpha = 0$

$$W = F\ell$$

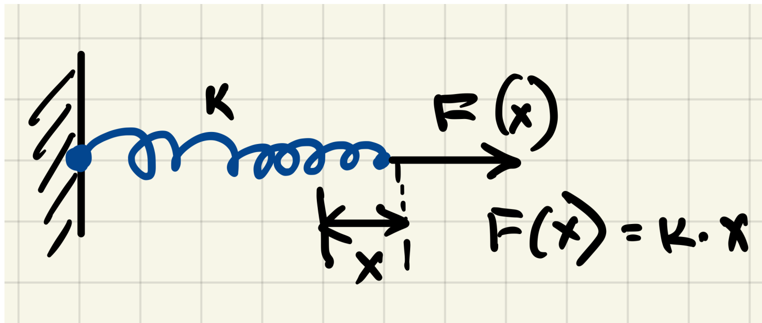
- If the force (assume $\alpha = 0$) is not constant $\Longrightarrow$ we can calculate the amount of work dW due to a small displacement dx:

$$dW = F(x)\ dx$$

- The total work is the sum (an integral) of the works on each small displacement

$$W = \int dW = \int_0^\ell F(x)dx$$

Example 3. Suppose a spring requires a force of 2000N to extend is 4cm longer than normal. What is the work needed to extend the spring from normal state to 5cm longer?



$\Longrightarrow$

- The force needed to extend the spring to x is

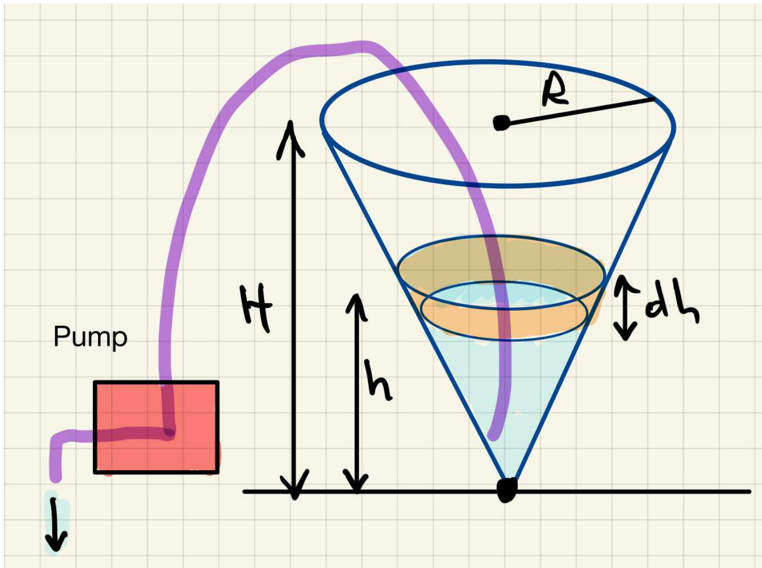
$$F(x) = kx \qquad \Longrightarrow \qquad k = \frac{2000N}{4 \cdot 10^{-2} m} = 5 \cdot 10^4 \ \frac{N}{m}$$

- The work needed to extend the spring to length ℓ is

$$W = \int dW = \int_0^\ell kx dx = \frac{k}{2} x^2 \bigg|_0^\ell = \frac{k}{2} \ \ell^2 = 5 \cdot 10^4 \ \frac{1}{2} \ (5 \cdot 10^{-2})^2$$

$$= \frac{1}{2} 5^3 N \cdot m = 62.5J$$

Example 4. A water tank has the shape of an inverted cone of height H, and top radius R. Initially the tank is full. What work is done by a pump to completely empty the tank?



$\Longrightarrow$

- Suppose the water level is h. Consider a small slab of water (orange) at the top, with depth dh. Its volume is

$$dV = A \cdot dh = \pi r^2 \; dh = \pi \underbrace{\left(R\frac{h}{H}\right)}_{=r}^2 \; dh = \frac{\pi R^2}{H^2} \; h^2 dh$$

- The water in the slab has a mass dm

$$dm = \rho dV = \frac{\pi \rho R^2}{H^2} \; h^2 dh$$

and has to be lifted to the height $(H - h)$, resulting in the work done dW:

$$dW = dmg(H-h) = \frac{\pi \rho g R^2}{H^2} \; (H-h)h^2 dh$$

- The total work is

$$W = \int dW = \frac{\pi \rho g R^2}{H^2} \; \int_0^H (H-h)h^2 dh \frac{\pi \rho g R^2}{H^2} \left(\frac{1}{3}Hh^3 - \frac{1}{4}h^4\right)\Big|_0^H = \frac{\pi \rho g R^2}{H^2} \; \frac{H^4}{12}$$

$$= \frac{1}{12}\pi \rho g R^2 H^2$$