Lecture 7.4: **Mass, moments and center of mass**

$\Longrightarrow$ Consider a homogeneous object of mass m and volume V. We can define a density ρ of the object as

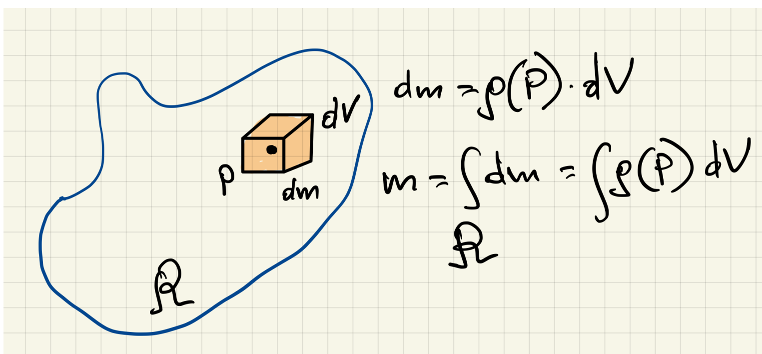
$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{m}{V}$$

For a homogeneous object, the density is constant over its volume.

$\Longrightarrow$ A more realistic situation is when the density changes within the object:

$$\rho = \rho(P)$$

$\Longrightarrow$ Suppose that the density is almost constant within a small element dV (orange box):



$$dm = \rho(P)\ dV$$

$\Longrightarrow$ the total mass is a sum of masses of small elements of almost constant density

$$m = \int_{\mathcal{R}} dm = \int_{\mathcal{R}} \rho(P) dV$$

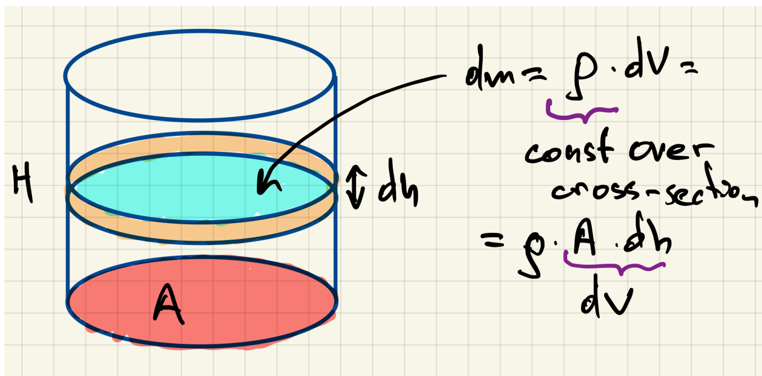
$\Longrightarrow$ In general the mass-integral is a multi-variable (3D) integral; in this course we consider only objects what have some symmetry, so that the mass-integral can be reduced to a single-variable integral

Example 1: Cylinder of height H and base area A has density

$$\rho(h) = \rho_0(1+h)$$

Find its mass

$\Longrightarrow$ Note: the density depends only on height, and not the location at a fixed cross-section



- Consider a cross-section of width dh (the orange band), its mass is

$$dm = \underbrace{\rho}_{=\rho_0(1+h)} \cdot \underbrace{dV}_{=Adh}$$

- the total mass is then

$$M = \int m = \int_0^H \rho_0(1+h)Adh = A\rho_0 \int_0^H (1+h)dh$$

$$= A\rho_0\left(h + \frac{1}{2}h^2\right)\Big|_0^H = \rho_0 A\left(H + \frac{1}{2}H^2\right)$$

Example 2 A ball of radius R has density

$$\rho = \frac{\rho_0}{1+r^2}\,, \qquad r \in [0, R]$$

where r is the distance from the ball center. Find its mass.

$\Longrightarrow$

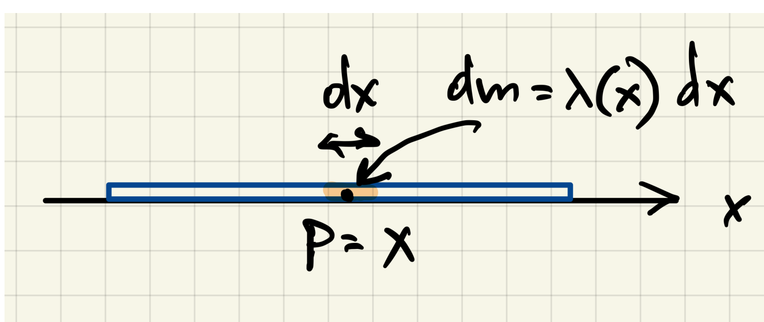
- Let's exploit the symmetry of the problem and break the ball into concentric shells of radius r and thickness dr. Each shell has almost constant density, and the mass

$$dm = \rho \cdot \underbrace{dV_{shell}}_{=A_{shell}\cdot dr} = \frac{\rho_0}{1+r^2} \underbrace{A_{shell}}_{=4\pi r^2} dr = 4\pi\rho_0 \left(1 - \frac{1}{1+r^2}\right) dr$$

- The total mass is the sum of masses of individual shells:

$$M = \int dm = \int_0^R 4\pi\rho_0 \left(1 - \frac{1}{1+r^2}\right) dr = 4\pi\rho_0 \left(r - \tan^{-1} r\right)\Big|_0^R$$
$$= 4\pi\rho_0 \left(R - \tan^{-1} R\right)$$

1-D example: Consider a wire of linear density $\lambda(x)$:



The units of the linear density

$$\lambda = \frac{dm}{dx} = \left[\frac{\text{kg}}{\text{m}}\right]$$

$\Longrightarrow$ Let

$$\lambda(x) = \sin\frac{\pi x}{L}\,, \qquad x \in [0, L]$$

what is the total mass of the wire?

$$M = \int dm = \int_0^L \lambda(x)dx = \int_0^L \sin\frac{\pi x}{L}dx = -\cos\frac{\pi x}{L}\cdot\frac{L}{\pi}\bigg|_0^L = \frac{2L}{\pi}$$

2-D example: Consider a plate of surface density $\sigma(P)$. In principle, the point P is

$$P = (x, y)$$

The units of the surface density

$$\sigma(P) = \frac{dm}{dA} = \left[\frac{\text{kg}}{\text{m}^2}\right]$$

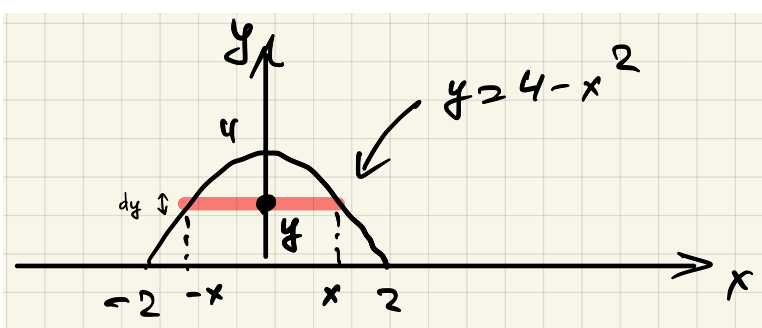
$\Longrightarrow$ Find the mass of a plate occupying the region

$$0 \leq y \leq 4 - x^2 \Longrightarrow x \in [-2, 2]\,, \qquad y \in [0, 4]$$

if the area density is

$$\sigma(x, y) = ky$$

what is the total mass of the plate?



- Note that the surface density is independent of x coordinate $\Longrightarrow$ we will slice the shape with the horizontal stripes (red) of width dy

- A stripe located at y extends for

$$-\sqrt{4-y} \;\leq\; x \;\leq \sqrt{4-y} \qquad \Longrightarrow \qquad \ell_{stripe} = 2\sqrt{4-y}$$

- The area of the stripe dA_{stripe} and the mass dm are correspondingly

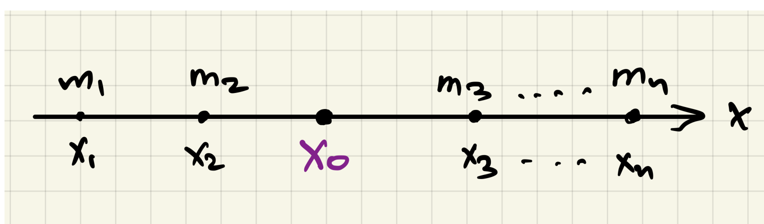
$$dA_{stripe} = \ell_{stripe}\; dy = 2\sqrt{4-y}\; dy, \qquad dm = \sigma \cdot dA_{stripe} = ky \cdot 2\sqrt{4-y}\; dy$$

- The total mass is then

$$M = \int dm = \int_0^4 \underbrace{ky \cdot 2\sqrt{4-y}\; dy}_{u=4-y, du=-dy} = 2k \int_0^4 (4-u)u^{1/2} du$$

$$= 2k\left(4u^{3/2}\frac{2}{3} - u^{5/2}\frac{2}{5}\right)\Bigg|_0^4 = 2k\left(\frac{8}{3}\cdot 8 - \frac{2}{5} \;\cdot 32\right) = \frac{256k}{15}$$

Moments and centers of mass



<u>**Def:**</u> the moment of the 1-D system of masses

$$m_1, m_2, \cdots m_n$$

located along a straight line (the x-axis) at locations

$$x_1, x_2, \cdots x_n$$

relative to a point x_0 (purple) is

$$M_{x=x_0} \equiv m_1(x_1 - x_0) + m_2(x_2 - x_0) + \cdots + m_n(x_n - x_0) \equiv \sum_{k=1}^{n} m_k(x_k - x_0)$$

Def: the center of mass of a 1-D system of masses is a location $\bar{x}$, such that

$$M_{x=\bar{x}} = 0$$

$\Longrightarrow$ From the definition of the moment,

$$0 = M_{x=\bar{x}} = \sum_{k=1}^{n} m_k(x_k - \bar{x}) = \sum_{k=1}^{n} m_k \cdot x_k - \bar{x} \cdot \sum_{k=1}^{n} m_k \implies$$

$$\bar{x} = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k}$$

$\Longrightarrow$ For a continuous mass distribution of linear density $\lambda(x)$, $x \in [a, b]$ we can easily generalize the formulas above:

$$M_{x=0} = \int_a^b x \cdot \lambda(x)\ dx\,, \qquad M = \int_a^b \lambda(x)\ dx \qquad \Longrightarrow \qquad \bar{x} = \frac{M_{x=0}}{M}$$

1-D example: Find the center of mass of a wire of

$$\lambda = kx\,,\ \ x \in [0, L]$$

$\Longrightarrow$

- Compute the total mass

$$M = \int_0^L kxdx = \frac{k}{2}x^2\Big|_0^L = \frac{1}{2}kL^2$$

- Compute the moment relative to $x = 0$

$$M_{x=0} = \int_0^L x \cdot kxdx = \frac{k}{3}x^3\Big|_0^L = \frac{1}{3}kL^3$$

- the center of mass is the ratio of the above quantities:

$$\bar{x} = \frac{M_{x=0}}{M} = \frac{2}{3}L$$

2-D center of mass

$\Longrightarrow$ Consider a 2-D distribution of point masses

$$m_1 \qquad (x_1, y_1)$$

$$m_2 \qquad (x_2, y_2)$$

$$\dots$$

$$m_n \qquad (x_n, y_n)$$

$\Longrightarrow$ by analogy with the 1-D case ,we can define the moments of distribution relative to $x = 0$

$$M_{x=0} = \sum_{k=1}^{n} m_k x_k$$

and $y = 0$

$$M_{y=0} = \sum_{k=1}^{n} m_k y_k$$

$\Longrightarrow$ The center of mass is located at $(\bar{x}, \bar{y})$, such that

$$\bar{x} = \frac{M_{x=0}}{M}\,, \qquad \bar{y} = \frac{M_{y=0}}{M}$$

where as before

$$M = \sum_{k=1}^{n} m_k$$

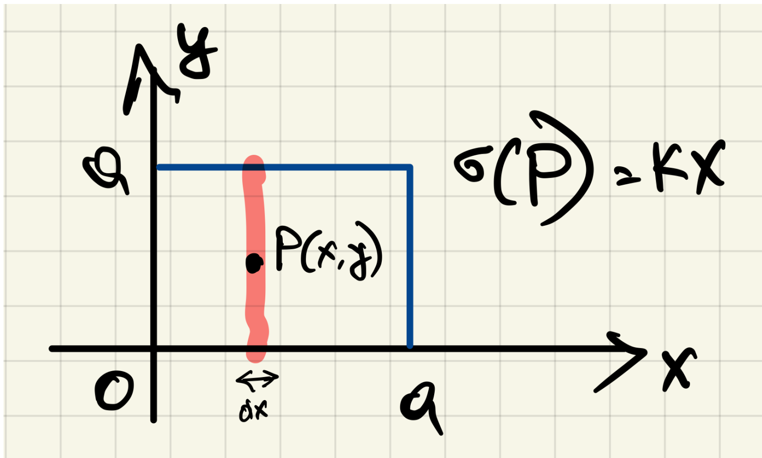
$\Longrightarrow$ For a continuous distribution the sums are replaced with the integrals:

$$M_{x=0} = \int x \cdot \sigma \; dA\,, \qquad M_{y=0} = \int y \cdot \sigma \; dA$$

$$\bar{x} = \frac{\int x \cdot \sigma \; dA}{\int \sigma \; dA}\,, \qquad \bar{y} = \frac{\int y \cdot \sigma \; dA}{\int \sigma \; dA}$$

2-D example A square plate of size a has an area density

$$\sigma = kx$$



Find the total mass and the location of the center of mass.

$\Longrightarrow$

- Note that the area density is independent of y $\Longrightarrow$ it is convenient to break the plate into a constant density vertical stripes of width dx.

- A stripe passing through point $P = (x, y)$ has a mass

$$dm_{stripe} = \sigma \underbrace{a}_{\text{extent along y-direction}} \cdot dx = ka\ x\ dx$$

- The total mass is

$$M = \int dm = \int_0^a ka\ x\ dx = ka\frac{1}{2}x^2\Big|_0^a = \frac{1}{2}ka^3$$

- By symmetry, the center of mass of any stripe, and thus of the full plate, is at

$$\bar{y} = \frac{a}{2}$$

- For $\bar{x}$ we need to compute the moment

$$M_{x=0} = \int x \cdot dm = \int_0^a x \cdot ka\ x\ dx = ka\frac{1}{3}x^3\bigg|_0^a = \frac{1}{3}ka^4$$

- $\Longrightarrow$

$$\bar{x} = \frac{M_{x=0}}{M} = \frac{2}{3}a$$

3-D center of mass

$\Longrightarrow$ Consider a 3-D distribution of point masses

$$\begin{array}{ll} m_1 & (x_1, y_1, z_1) \\ m_2 & (x_2, y_2, z_2) \\ & \dots \\ m_n & (x_n, y_n, z_n) \end{array}$$

$\Longrightarrow$ by analogy with the 1-D or 2-D cases ,we can define the moments of distribution relative to $x = 0$

$$M_{x=0} = \sum_{k=1}^{n} m_k x_k$$

relative to $y = 0$

$$M_{y=0} = \sum_{k=1}^{n} m_k y_k$$

and relative to $z = 0$

$$M_{z=0} = \sum_{k=1}^{n} m_k z_k$$

$\Longrightarrow$ The center of mass is located at $(\bar{x}, \bar{y}, \bar{z})$, such that

$$\bar{x} = \frac{M_{x=0}}{M}\,, \qquad \bar{y} = \frac{M_{y=0}}{M}\,, \qquad \bar{z} = \frac{M_{z=0}}{M}$$

where as before

$$M = \sum_{k=1}^{n} m_k$$

$\Longrightarrow$ For a continuous distribution the sums are replaced with the integrals:

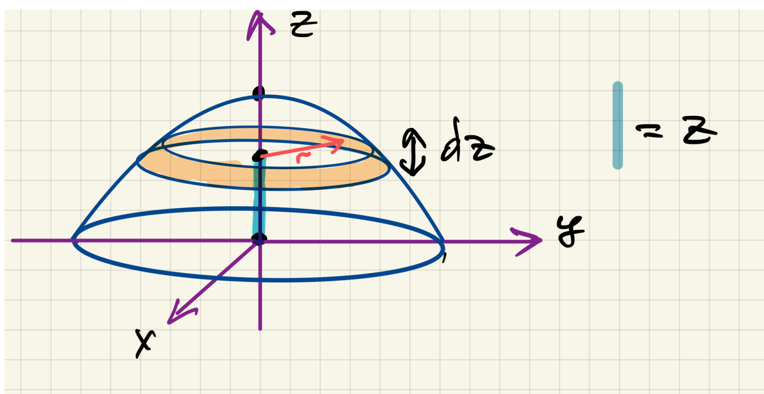
$$M_{x=0} = \int x \cdot \rho \; dV \,, \qquad M_{y=0} = \int y \cdot \rho \; dV \,, \qquad M_{z=0} = \int z \cdot \rho \; dV$$

$$\bar{x} = \frac{\int x \cdot \rho \; dV}{\int \rho \; dV} \,, \qquad \bar{y} = \frac{\int y \cdot \rho \; dV}{\int \rho \; dV} \,, \qquad \bar{z} = \frac{\int z \cdot \rho \; dV}{\int \rho \; dV}$$

Since we do not know (yet) how to do multi-variable calculus, exploiting the symmetries is crucial

3-D example A hemisphere of radius R has density

$$\rho = \rho_0 \; z$$



Find its center of mass.

$\Longrightarrow$

- Note that the density is constant at fixed $z \Longrightarrow$ we slice the sphere by disked of thickness dz (orange)
- A disk centered at z has radius r_{disk}, the volume dV_{disk}, and the mass dm correspondingly

$$r_{disk} = \sqrt{R^2 - z^2} \,, \qquad dV_{disk} = A_{disk} \; dz = \pi \; r_{disk}^2 \; dz = \pi(R^2 - z^2) dz$$

$$dm = \rho \cdot dV_{disk} = \rho_0 z \cdot \pi(R^2 - z^2) dz$$

- The total mass is

$$M = \int dm = \int_0^R \rho_0 z \cdot \pi (R^2 - z^2) dz = \pi \rho_0 \left(\frac{1}{2} z^2 R^2 - \frac{1}{4} z^4 \right) \Bigg|_0^R = \frac{\pi \rho_0 R^4}{4}$$

- By symmetry, any disk, and thus the full hemisphere, has a center of mass

$$\bar{x} = 0\,, \qquad \bar{y} = 0$$

- For $\bar{z}$ we need to compute the moment

$$M_{z=0} = \int z \cdot dm = \int_0^R z \cdot \rho_0 z \cdot \pi (R^2 - z^2) dz = \pi \rho_0 \left(\frac{1}{3} z^3 R^2 - \frac{1}{5} z^5 \right) \Bigg|_0^R = \frac{2\pi \rho_0 R^5}{15}$$

- $\Longrightarrow$

$$\bar{z} = \frac{M_{z=0}}{M} = \frac{8}{15} R$$