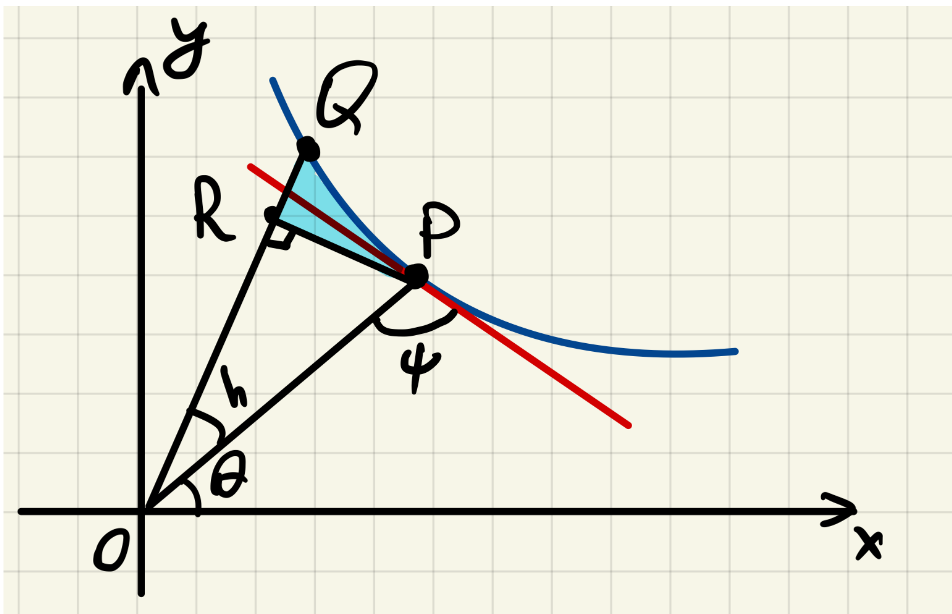
Lecture 8.6: **Slopes, areas and arc lengths for polar curves**

Slopes

Pick a point $P = (r, \theta)$ on a polar curve (blue)

$$r = f(\theta)$$

Def: ψ is the angle between the tangent line at P (red) and OP (black):



How do we compute ψ?

- Let $h \to 0$.
- In a right triangle RQP,

$$\lim_{h\to 0} \measuredangle RQP = \psi$$

- $\Longrightarrow$

$$\tan\psi = \lim_{h\to 0} \frac{PR}{QR}$$

- Since

$$PQ \approx OP \cdot h = f(\theta) \cdot h$$

$$QR = QO - RO = f(\theta + h) - OP\cos h \approx f(\theta) + h \cdot f'(\theta) - f(\theta) = h \cdot f'(\theta)$$

- $\implies$

$$\boxed{\tan\psi = \lim_{h\to 0}\frac{f(\theta)\cdot h}{h\cdot f'(\theta)} = \frac{f(\theta)}{f'(\theta)}}$$

- Note: assuming

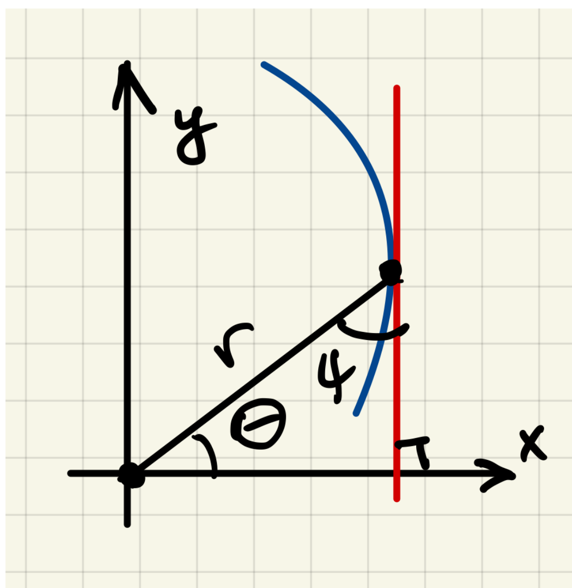
$$f'(\theta) = 0 \text{ and } f(\theta) \neq 0 \qquad \implies \qquad \psi = \frac{\pi}{2}$$

To understand this statement, imagine a circle:

$$r = a \implies f' = 0 \text{ for any } \theta \qquad \implies \qquad \psi = \frac{\pi}{2}$$

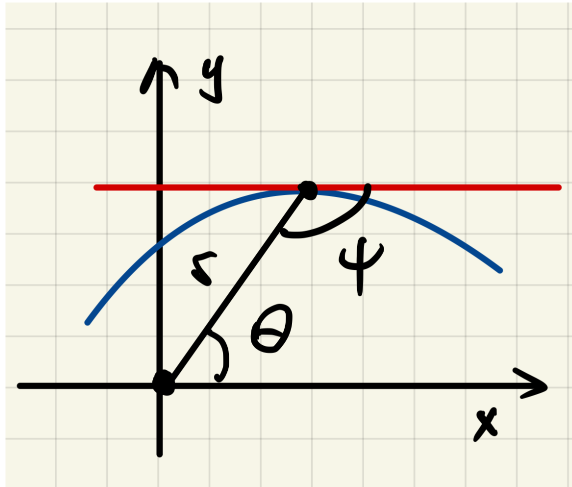
as expected

- Condition for a tangent line to be vertical:



$$\theta + \psi = \frac{\pi}{2} \qquad \implies \qquad \tan\psi = \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

- Condition for a tangent line to be horizontal:



$$\theta + \psi = \pi \qquad \Longrightarrow \qquad \tan\psi = \tan(\pi - \theta) = -\tan\theta$$

Example 1: For a cardioid

$$r = 1 + \cos\theta$$

find the points where the tangent line is horizontal/vertical

$\Longrightarrow$

- Recall

$$\tan\psi = \frac{f(\theta)}{f'(\theta)} = \frac{1+\cos\theta}{-\sin\theta}$$

- Tangent is vertical when:

$$\tan\psi = \cot\theta \;\Longrightarrow\; \frac{1+\cos\theta}{-\sin\theta} = \frac{\cos\theta}{\sin\theta}$$

  - One solution is

$$\sin\theta = 0 \;\&\; 1 + \cos\theta \neq 0 \qquad \Longrightarrow \qquad \theta = 0$$

- The other solutions are

$$1 + \cos\theta = -\cos\theta \implies \cos\theta = -\frac{1}{2} \implies \theta = \pm\frac{2}{3}\pi$$

- Tangent is horizontal when:

$$\tan\psi = -\tan\theta \implies \frac{1+\cos\theta}{-\sin\theta} = -\frac{\sin\theta}{\cos\theta}$$

- Solving above we find

$$\cos^2\theta + \cos\theta = \sin^2\theta \implies$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

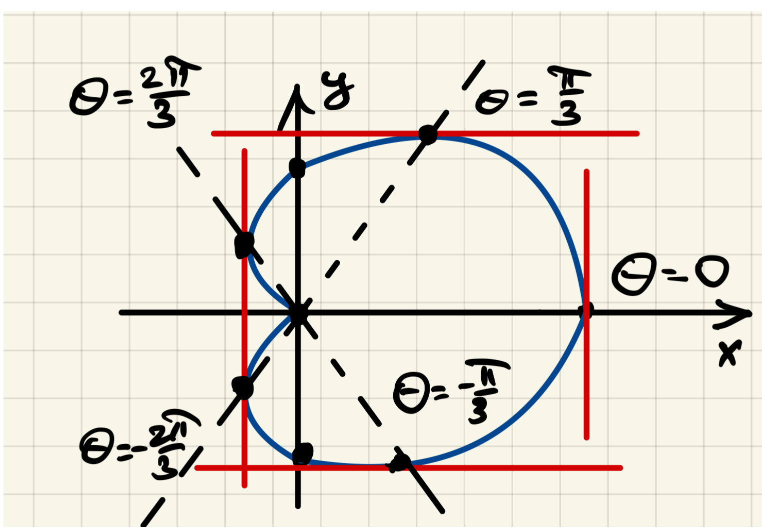
Set $\cos\theta = u \implies$

$$2u^2 + u - 1 = 0 \implies (2u-1)(u+1) = 0 \implies u = \{-1, 1/2\}$$

- $$\cos\theta = -1 \implies \theta = \pi$$

- $$\cos\theta = \frac{1}{2} \implies \theta = \pm\frac{1}{3}\pi$$

- Results are collected in the fig.:



- Note that at $\theta = \pi$:

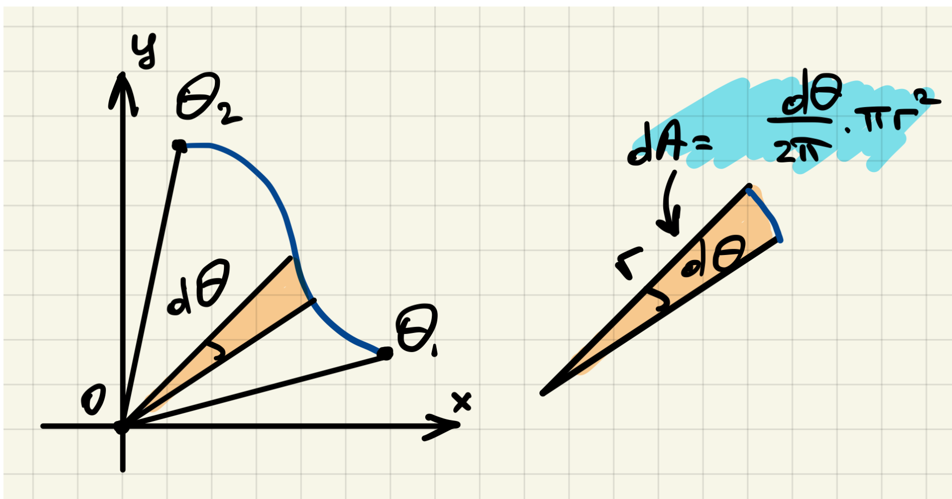
$$f'(\theta) = 0 \qquad \text{and} \qquad f(\theta) = 0$$

$\Longrightarrow$ the tangent does not exist; the polar curve has a cusp

Areas

$\Longrightarrow$ Consider a region bounded by the rays $\theta = \theta_1$, $\theta = \theta_2$ and the polar curve

$$r = f(\theta)$$



How do we compute the area of the region?

- Cover the region with sectors of opening angle $d\theta$ (orange)
- Each orange sector has an area

$$dA = \underbrace{\frac{d\theta}{2\pi}}_{\text{angular fraction of a disk}} \cdot \underbrace{\pi r^2}_{\text{disk area}} = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} f(\theta)^2 \, d\theta$$

- The total area is the sum (an integral) of all the sectors

$$A = \int dA = \frac{1}{2} \int_{\theta_1}^{\theta_2} f(\theta)^2 \, d\theta$$

Example 2: Find the area of one leaf of the curve

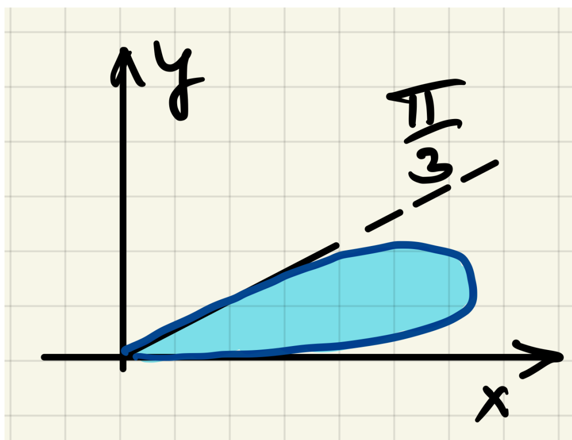
$$r = \sin(3\theta)$$

$\Longrightarrow$

- Leaves start/end when $r = 0 \Longrightarrow$

$$\sin(3\theta) = 0 \implies \theta \in \left[0, \frac{1}{3}\pi\right]$$

is one leaf:



- $\Longrightarrow$

$$A = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)d\theta = \frac{1}{2}\int_0^{\pi/3}\left(\frac{1}{2} - \frac{1}{2}\cos(6\theta)\right)d\theta$$

$$= \frac{\theta}{4} - \frac{1}{24}\sin(6\theta)\Big|_0^{\pi/3} = \frac{1}{4}\frac{\pi}{3} = \frac{\pi}{12}$$

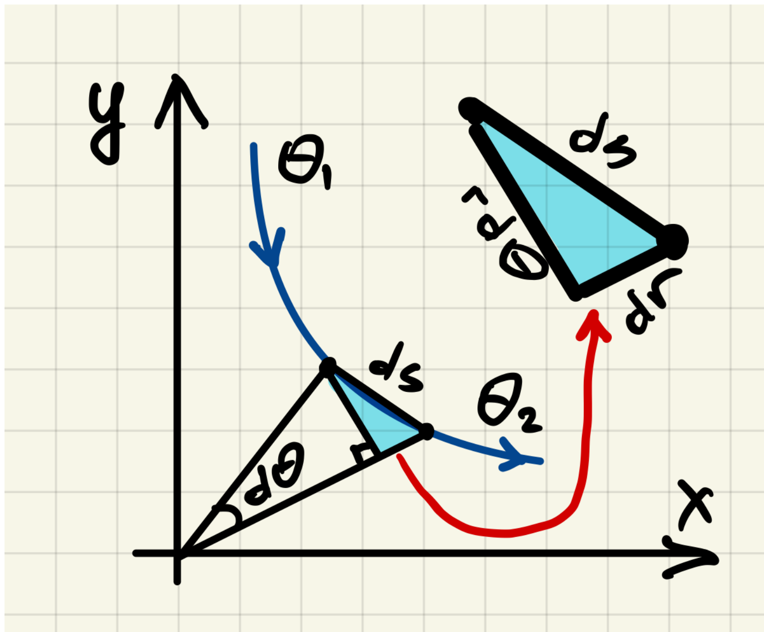
since

$$\sin(6 \cdot \pi/3) = \sin(2\pi) = 0\,, \qquad \sin(0) = 0$$

Arc length

$\Longrightarrow$ Consider a polar curve

$$r = f(\theta)$$



How do we compute the arc length of the curve for $\theta \in [\theta_1, \theta_2]$?

- Consider an element ds of the curve.
- From Pythagorean theorem

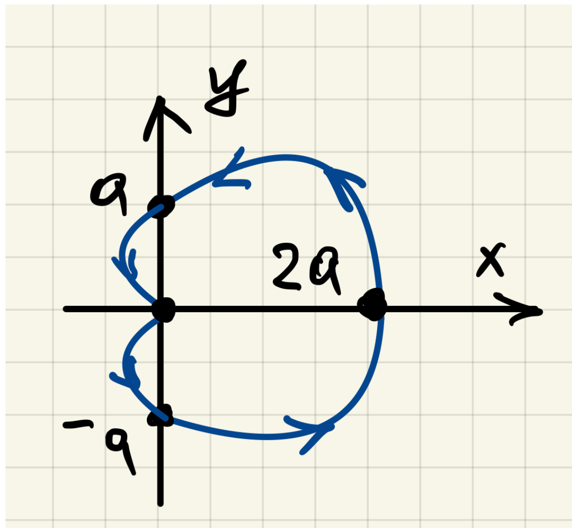
$$ds^2 = (rd\theta)^2 + (dr)^2 \implies ds^2 = \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)(d\theta)^2 = ((f(\theta))^2 + (f'(\theta))^2)d\theta^2$$

- The total arc length is the sum of lengths of individual segments:

$$L = \int ds = \int_{\theta_1}^{\theta^2} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

Example 3: Find the arc length of the cardioid

$$r = a(1+\cos\theta)\,, \qquad \theta \in [0, 2\pi]$$



$\Longrightarrow$

- $$ds^2 = (f^2 + (f')^2)d\theta^2 = a^2\left((1+\cos\theta)^2 + (-\sin\theta)^2\right)d\theta^2$$
  $$= a^2\,(2 + 2\cos\theta)\,d\theta^2 = 2a^2(1+\cos\theta)d\theta^2 = 2a^2\cdot 2\cos^2\frac{\theta}{2}d\theta^2$$
- $\Longrightarrow$
  $$ds = 2a\left|\cos\frac{\theta}{2}\right|d\theta$$
- $\Longrightarrow$
  $$L = \int ds = \int_0^{2\pi} 2a\left|\cos\frac{\theta}{2}\right|d\theta = 2\cdot\int_0^{\pi} 2a\cos\frac{\theta}{2}d\theta = 4a\cdot 2\sin\frac{\theta}{2}\bigg|_0^{\pi} = 8a$$
  where we used the symmetry of the curve (hence twice the integral $\theta \in [0, \pi]$), and the fact that on the interval
  $$\left|\cos\frac{\theta}{2}\right| = \cos\frac{\theta}{2}$$