Lecture 6.3: **Inverse substitutions**

$\Longrightarrow$ Direct substitution

$$\int \underbrace{f(g(x))}_{u=g(x)} \cdot \underbrace{g'(x)dx}_{du=g'(x)dx} = \int f(u)du$$

$\Longrightarrow$ Inverse substitution

$$\int \underbrace{f(x)}_{x=h(u)} \underbrace{dx}_{dx=h'(u)du} = \int f(h(u)) \cdot h'(u)du$$

- (1) Inverse sin substitution (assume $a > 0$):

$$x = a\sin\theta\,, \qquad \theta = \sin^{-1}\frac{x}{a}\,, \qquad dx = a\cos\theta\; d\theta$$

is useful to compute integrals containing expressions

$$(a^2 - x^2)\,, \qquad \sqrt{a^2 - x^2}$$

Note, in

$$\sqrt{a^2 - x^2}: \qquad -a \le x \le a \qquad \Longrightarrow \qquad \theta = \sin^{-1}\frac{x}{a} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\Longrightarrow$

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = a|\cos\theta| = a\cos\theta, \qquad \text{since } \cos\theta \ge 0 \text{ for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

Example:

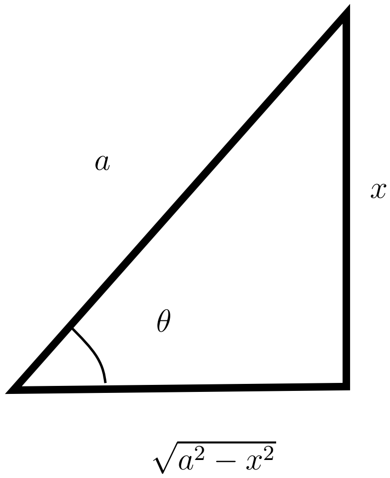
$$I = \int \underbrace{\frac{1}{(3 - x^2)^{3/2}}dx}_{a=\sqrt{3}\,,\; x=\sqrt{3}\sin\theta\,,\; \theta=\sin^{-1}\frac{x}{\sqrt{3}}\,,\; dx=\sqrt{3}\cos\theta d\theta}$$

$\Longrightarrow$

$$(3 - x^2)^{3/2} = (3 - 3\sin^2\theta)^{3/2} = 3^{3/2}\;(\cos^2\theta)^{3/2} = 3^{3/2}\;\cos^3\theta$$

$$I = \int \frac{3^{1/2}\cos\theta}{3^{3/2}\cos^3\theta}\,d\theta = \frac{1}{3}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{3}\tan\theta + C \; \ominus$$

- How do we evaluate $\cos\theta$ and $\tan\theta$ (and other trig functions) if we know $\sin\theta = \frac{x}{a}$?



$$\sin\theta = \frac{x}{a} \qquad \Longrightarrow \qquad \cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$$

$\Longrightarrow$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{x}{a}}{\frac{\sqrt{a^2-x^2}}{a}} = \frac{x}{\sqrt{a^2 - x^2}}$$

$\Longrightarrow$

$$\sin(2\theta) = 2\ \sin\theta\ \cos\theta = 2\ \frac{x}{a}\ \frac{\sqrt{a^2 - x^2}}{a} = \frac{2x\sqrt{a^2 - x^2}}{a^2}$$

$\Longrightarrow$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{a^2 - x^2}{a^2} - \frac{x^2}{a^2} = \frac{a^2 - 2x^2}{a^2}$$

We can now continue with our example

$$= \frac{1}{3}\ \frac{x}{\sqrt{3 - x^2}} + C$$

A warning: leaving the answer as

$$\frac{1}{3}\tan\theta + C \qquad - - - \qquad \text{a mark reduction } 50\%$$

$$\frac{1}{3}\tan\left(\sin^{-1}\frac{x}{\sqrt{3}}\right) + C \qquad - - - \qquad \text{a mark reduction } 25\%$$

- (2) Inverse tan substitution

$$x = a\tan\theta\,, \qquad \theta = \tan^{-1}\frac{x}{a} \ \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\,, \qquad dx = a\sec^2\theta\ d\theta = \frac{a}{\cos^2\theta}\ d\theta$$

is useful in evaluating integrals containing expression

$$(x^2+a^2)\,, \qquad \sqrt{a^2+x^2}$$

Note,

$$\sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2\theta)} = a|\sec\theta| = \frac{a}{|\cos\theta|} = \frac{a}{\cos\theta}, \qquad \text{since } \cos\theta \geq 0 \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Example:

$$I = \int \underbrace{\frac{1}{(1+9x^2)^2}dx}_{3x=\tan\theta\,,\ \theta=\tan^{-1}(3x)\,,\ dx=\frac{1}{3}\sec^2\theta d\theta}$$

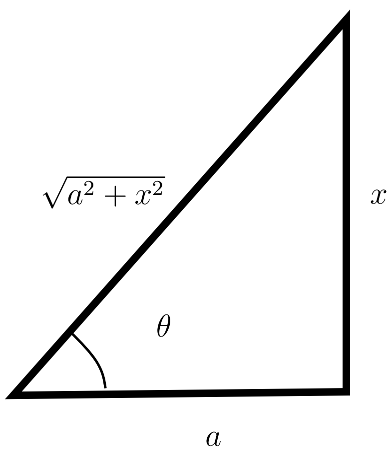
$\Longrightarrow$

$$(1+9x^2)^2 = (1+\tan^2\theta)^2 = (\sec^2\theta)^2 = \sec^4\theta$$

$$I = \int \frac{\frac{1}{3}\sec^2\theta}{\sec^4\theta}\ d\theta = \frac{1}{3}\int\frac{d\theta}{\sec^2\theta} = \frac{1}{3}\int\cos^2\theta\ d\theta = \frac{1}{3}\int\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)\ d\theta$$

$$= \frac{1}{6}\theta + \frac{1}{12}\ \sin(2\theta) + C \ominus$$

- How do we evaluate $\sin\theta$ and $\cos\theta$ (and other trig functions) if we know $\tan\theta = \frac{x}{a}$?



$$\sin\theta = \frac{x}{\sqrt{a^2+x^2}}\,, \qquad \cos\theta = \frac{a}{\sqrt{a^2+x^2}}\,, \qquad \sin(2\theta) = \frac{2xa}{(x^2+a^2)}\,, \qquad \cos(2\theta) = \frac{a^2-x^2}{a^2+x^2}$$

We can now continue with our example noting that

$$\tan\theta = \frac{x}{\frac{1}{3}} \qquad \Longrightarrow \qquad \sin\theta = \frac{3x}{\sqrt{1+9x^2}}\,, \qquad \cos\theta = \frac{1}{\sqrt{1+9x^2}}\,, \qquad \sin(2\theta) = \frac{6x}{1+9x^2}$$

$$= \frac{1}{6}\ \tan^{-1}(3x) + \frac{x}{2(1+9x^2)} + C$$

Example:

$$I = \int \frac{dx}{(4x-x^2)^{3/2}}$$

Complete the square first:

$$4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (2-x)^2 \qquad \Longrightarrow \qquad x - 2 = 2\sin\theta\,,\ dx = 2\cos\theta\ d\theta$$

$\Longrightarrow$

$$(4x-x^2)^{3/2} = (4 - 4\sin^2\theta)^{3/2} = 4^{3/2}\ (\cos^2\theta)^{3/2} = 8\cos^3\theta$$

$$I = \int \frac{2\cos\theta}{8\cos^3\theta}\ d\theta = \frac{1}{4}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\tan\theta + C =$$

Since

$$\sin\theta = \frac{x-2}{2}\,, \qquad \cos\theta = \sqrt{1 - \frac{(x-2)^2}{4}} = \frac{1}{2}\sqrt{4x-x^2}\,, \qquad \tan\theta = \frac{x-2}{\sqrt{4x-x^2}}$$

thus, we continue

$$= \frac{1}{4}\frac{x-2}{\sqrt{4x-x^2}} + C$$

- (3) Other substitutions:

$$I = \int \underbrace{\frac{1}{1+\sqrt{2x}}dx}_{2x=u^2\,,\ dx=udu\,,u=\sqrt{2x}} = \int \frac{udu}{1+u} = \int \frac{(u+1)-1}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right)\,du$$

$$= u - \ln|1+u| + C = \sqrt{2x} - \ln|1+\sqrt{2x}| + C$$

Example:

$$I = \int \frac{1}{x^{1/2}(1+x^{1/3})}\ dx$$

to remove <u>all</u> fractional powers we set

$$x = u^6 \qquad \Longrightarrow \qquad u = x^{1/6}\,, \qquad dx = 6u^5 du$$

$\Longrightarrow$

$$I = \int \frac{6u^5}{u^3(1+u^2)}\ du = 6\int \frac{u^2}{1+u^2}du = 6\int \frac{(u^2+1)-1}{1+u^2}du =$$

$$= 6\int\left(1-\frac{1}{1+u^2}\right)du = 6\left(u-\tan^{-1}u\right)+C = 6x^{1/6}-6\tan^{-1}\left(x^{1/6}\right)+C$$

- (4) Another useful substitution:

$$x = \tan\frac{\theta}{2} \qquad \Longrightarrow \qquad \theta = 2\tan^{-1}x$$

$$\sin\frac{\theta}{2} = \frac{x}{\sqrt{1+x^2}}\,, \qquad \cos\frac{\theta}{2} = \frac{1}{\sqrt{1+x^2}}\,, \qquad dx = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta = \frac{d\theta}{2\cos^2\frac{\theta}{2}}$$

$\Longrightarrow$

$$d\theta = 2\cos^2\frac{\theta}{2}\,dx = \frac{2dx}{1+x^2}\,, \qquad \sin\theta = 2\sin\frac{\theta}{2}\,\cos\frac{\theta}{2} = \frac{2x}{1+x^2}\,, \qquad \cos\theta = \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} = \frac{1-x^2}{1+x^2}$$

Example:

$$I = \int\frac{d\theta}{2+\cos\theta} = \int\frac{1+x^2}{3+x^2}\,\frac{2dx}{1+x^2} = \int\frac{2dx}{3+x^2} = \frac{2}{\sqrt{3}}\,\tan^{-1}\frac{x}{\sqrt{3}}+C$$

where we used

$$2+\cos\theta = 2+\frac{1-x^2}{1+x^2} = \frac{3+x^2}{1+x^2}$$