Lecture 13.5: **The chain rule**

$\Longrightarrow$ Recall the chain rule for a function of a single variable:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

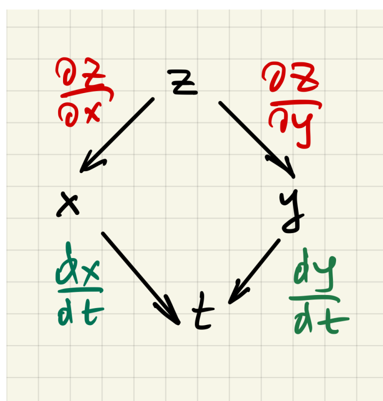
or if $y = f(x)$ and $x = g(u)$

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$\Longrightarrow$ the chain rule is straightforwardly generalized for a function of several variables:

- let $z = f(x(t), y(t))$, then

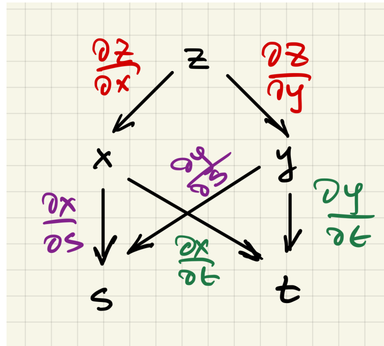
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$



- let $z = f(x(s,t), y(s,t))$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$



Example 1 Given $u = \sqrt{x^2 + y^2}$, $x = e^{st}$, $y = 1 + s^2 \cos t$, compute $\frac{\partial u}{\partial t}$

$\Longrightarrow$

- <u>first method</u>: use the chain rule formula

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} = \frac{x}{\sqrt{x^2 + y^2}} \cdot se^{st} + \frac{y}{\sqrt{x^2 + y^2}} \cdot (-s^2 \sin t)$$

$$= \frac{se^{2st} - s^2 \sin t(1 + s^2 \cos t)}{\sqrt{e^{2st} + (1 + s^2 \cos t)^2}}$$

- <u>second method</u>: do the substitution, and then compute the partial derivative
  - $$u(t, s) = \sqrt{e^{2st} + (1 + s^2 \cos t)^2}$$
  - compute $\frac{\partial u}{\partial t}$ as usual:

$$\frac{\partial u}{\partial t} = \frac{1}{2\sqrt{e^{2st} + (1 + s^2 \cos t)^2}} \left(2se^{2st} + 2(1 + s^2 \cos t)(-s^2 \sin t) \right)$$

$$= \frac{se^{2st} - s^2 \sin t(1 + s^2 \cos t)}{\sqrt{e^{2st} + (1 + s^2 \cos t)^2}}$$

- note that both results are the same, as it must be

Example 2: Compute $\frac{\partial}{\partial x} f(x^2y, x+2y)$ and $\frac{\partial}{\partial y} f(x^2y, x+2y)$, assuming f is differentiable and has partial derivatives $f_1(x^2y, x+2y)$ and $f_2(x^2y, x+2y)$

$\Longrightarrow$ using the chain rule

$$\frac{\partial}{\partial x} f = f_1 \cdot 2xy + f_2 \cdot 1\,, \qquad \frac{\partial}{\partial y} f = f_1 \cdot x^2 + f_2 \cdot 2$$

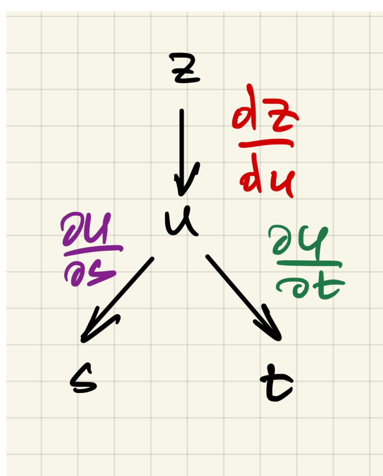
Example 3: Compute partial derivatives of $z = f(g(s,t))$

$\Longrightarrow$

- set $u = g(s,t)$
- using the chain rule,

$$\frac{\partial z}{\partial s} = \frac{df}{du} \cdot \frac{\partial u}{\partial s} = f'(g(s,t)) \cdot g_1(s,t)$$

$$\frac{\partial z}{\partial t} = \frac{df}{du} \cdot \frac{\partial u}{\partial t} = f'(g(s,t)) \cdot g_2(s,t)$$

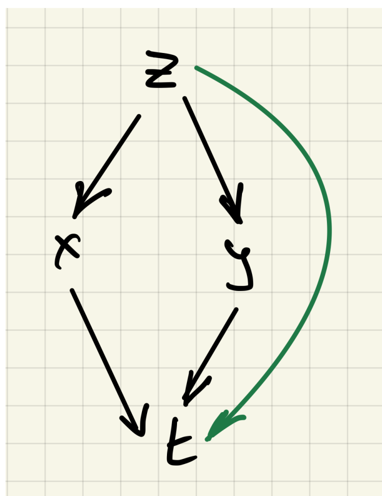


Example 4: Compute $\frac{d}{dt}$ of $z = f(x, y, t)$, where $x = g(t)$ and $y = h(t)$

$\Longrightarrow$

- using the chain rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial t}$$
$$= f_1 \cdot g' + f_2 \cdot h' + f_3$$



- note that

$$\frac{df}{dt} \qquad \neq \qquad \frac{\partial f}{\partial t} = f_3$$

Note: sometimes when writing the partial derivatives, we need to indicate what variables are fixed, *e.g.*,

- let $z = f(x, y, s, t)$ and $x = g(s, t)$, y=h(s,t)

- varying s and keeping t fixed:

$$\left(\frac{\partial z}{\partial s}\right)_t = f_1 \cdot g_1 + f_2 \cdot h_1 + f_3$$

where

$$f_3 = \left(\frac{\partial z}{\partial s}\right)_{x,y,t}$$

i.e., we compute the partial s-derivative keeping all x, y, t fixed

- An applied example: consider a thermometer attached to a balloon moving along the trajectory

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

What is the variation of the temperature T with time?

  - $$T = T\left[\underbrace{x(t), y(t), z(t)}_{\text{changes due to ballon motion}}, \underbrace{t}_{\text{variation due to weather at fixed position}}\right]$$

  - $\Longrightarrow$

$$\frac{dT}{dt} = T_1 \ \cdot x' + T_2 \ \cdot y' + T_3 \ \cdot z' + T_4$$

where

$$T_4 = \left(\frac{\partial T}{\partial t}\right)\Bigg|_{x,y,z}$$

Higher-order derivatives

Example 5: compute

$$\frac{\partial^2}{\partial x \partial y} f(x^2 - y^2, xy)$$

$\Longrightarrow$

- Note

$$\frac{\partial^2}{\partial x \partial y} f(x^2 - y^2, xy) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

- set $u = x^2 - y^2$ and $v = xy$, so that

$$f_1 = \frac{\partial f}{\partial u} \qquad f_2 = \frac{\partial f}{\partial v}$$

- using the chain rule:

$$\frac{\partial f}{\partial y} = f_1 \cdot (-2y) + f_2 \cdot x = -2y f_1(u,v) + x f_2(u,v)$$

- $\Longrightarrow$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -2y \cdot \frac{\partial f_1}{\partial x} + f_2 + x \cdot \frac{\partial f_2}{\partial x}$$

$$= -2y\left(\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x}\right) + f_2 + x\left(\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x}\right)$$

$$= -2y\Bigg(f_{11} \cdot 2x + f_{12} \cdot y\Bigg) + f_2 + x\Bigg(f_{21} \cdot 2x + f_{22} \cdot y\Bigg)$$

which, using $f_{12} = f_{21}$, becomes

$$= f_2 - 4xy \cdot f_{11} + (2x^2 - 2y^2) \cdot f_{12} + xy \cdot f_{22}$$