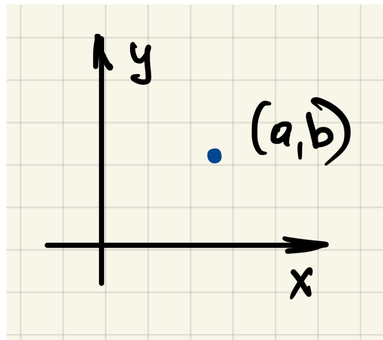
Lecture 13.2: **Limits and continuity**

$\Longrightarrow$ Let $f(x, y)$ has a domain $D(f) \subseteq \mathbb{R}^2$ and $(a, b) \in D(f)$

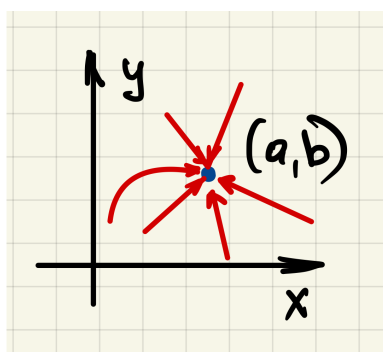


Def:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if $f(x, y)$ approaches L as (x, y) approaches (a, b)

Note that the limit of a function of several variables is more restrictive than that for a function of a single variable $\Longrightarrow$ there are infinitely many ways as we can approach (a, b) in $D(f)$:



Limit laws:

- for the sum:

$$\lim_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = \lim_{(x,y)\to(a,b)} f(x, y) + \lim_{(x,y)\to(a,b)} g(x, y)$$

- for the product:

$$\lim_{(x,y)\to(a,b)} (f(x,y) \cdot g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y)$$

- for the ratio:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$$

  if the denominator is $\neq 0$

Def: $f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Note three conditions must be satisfied:

- $(a,b) \in D(f)$
- the limit exist
- the limit equals the value of the function

Example 1: $f(x,y) = x^2y^3$ is continuous in $\mathbb{R}^2$:

$$\lim_{(x,y)\to(a,b)} x^2y^3 = a^2b^3$$

Example 2: $f(x,y) = \sin\frac{x}{y}$. Is it continuous at $(\pi, 2)$?

$\Longrightarrow$

$$\lim_{(x,y)\to(\pi,2)} \sin\frac{x}{y} \underbrace{=}_{DS} \sin\frac{\pi}{2}$$

is continuous.

In general, above function is continuous at any point where $y \neq 0$

Example 3: Calculate the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$

if it exists

$\Longrightarrow$

- Let's approach $(x, y) \to (0, 0)$ along the path $y = k \cdot x$, for fixed k
- $\implies$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} \underbrace{=}_{y=k\cdot x} \lim_{x\to 0} \frac{2x \cdot kx}{x^2+(kx)^2} = \lim_{x\to 0} \frac{2k}{1+k^2} = \frac{2k}{1+k^2}$$

- We established that the limit depends on the path of approach to $(0, 0) \implies$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = DNE$$