Lecture 12.1: **Vector functions of one variable**

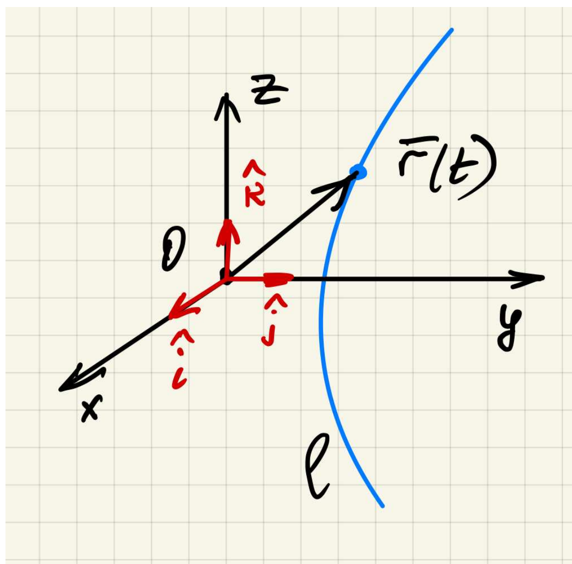
Consider a vector $\vec{u}$ that depends on a parameter t:

$$\vec{u}(t) = u_1(t) \cdot \hat{i} + u_2(t) \cdot \hat{j} + u_3(t) \cdot \hat{k}$$

$\Longrightarrow$ same as three real functions of one variable:

$$(u_1(t)\,,\ u_2(t)\,,\ u_3(t))$$

$\Longrightarrow$ best example is a position vector of a moving object, with t begin the time:



- $$\vec{r}(t) = x(t)\ \hat{i} + y(t)\ \hat{j} + z(t)\ \hat{k}$$

- We can differential with respect to t, keeping $\hat{i}\,,\ \hat{j}\,,\ \hat{k}$ constant, obtaining the velocity vector of a moving object:

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\ \hat{i} + \frac{dy}{dt}\ \hat{j} + \frac{dz}{dt}\ \hat{k} \equiv \vec{v}(t)$$

The absolute value of the velocity vector is the speed:

$$|\vec{v}(t)| = v(t)$$

- The differentiation again produces the acceleration vector

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\ \hat{\imath} + \frac{d^2y}{dt^2}\ \hat{\jmath} + \frac{d^2z}{dt^2}\ \hat{k} \equiv \vec{a}(t)$$

Example 1: Find the velocity, speed and the acceleration and describe the path of an object with the position vector

$$\vec{r} = a\cos(wt)\ \hat{\imath} + b\ \hat{\jmath} + a\sin(wt)\ \hat{k}\,, \qquad \{a, w\} = \text{const} > 0$$

$\Longrightarrow$

- the velocity
$$\vec{v} = \frac{d\vec{r}}{dt} = -aw\sin(wt)\ \hat{\imath} + aw\cos(wt)\hat{k}$$

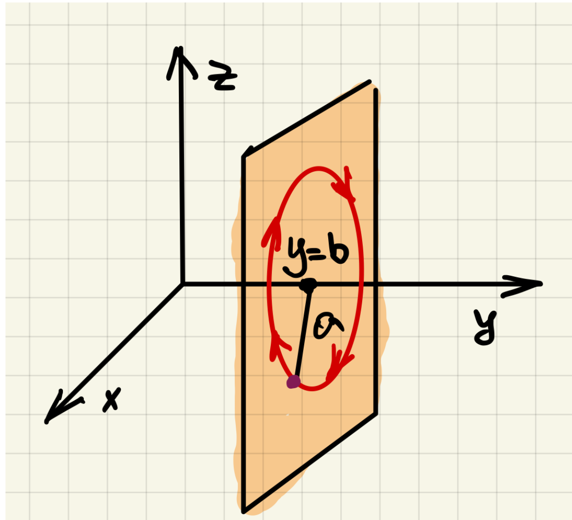
- the speed
$$|\vec{v}| = \sqrt{(-aw\sin(wt))^2 + (aw\cos(wt))^2} = aw$$
Note that the speed is constant

- the acceleration is
$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -aw^2\cos(wt)\ \hat{\imath} - aw^2\sin(wt)\hat{k} = -w^2\cdot\left(\vec{r}(t) - b\ \hat{\jmath}\right)$$

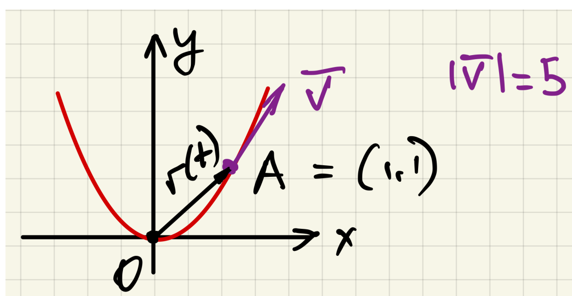
- To determine the path, note that $y(t) = b = const \Longrightarrow$ the motion is in the plane $\|$ to xz-plane exclusively, located at fixed $y = b$

- In this plane, the motion is around the circle, of radius a,
$$x^2 + z^2 = a^2$$

- The motion is clockwise, at constant speed:



Example 2: An object is moving to the right along the plane curve $y = x^2$ with constant speed $v = 5$. Find the velocity and the acceleration of the object when it is at point $A = (1, 1)$:



$\Longrightarrow$

- Assume that

$$\vec{r} = x(t)\hat{i} + y(t)\hat{j}$$

- the velocity vector is then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\ \hat{i} + \frac{dy}{dt}\ \hat{j} = \frac{dx}{dt}\ \hat{i} + \frac{dy}{dx} \cdot \frac{dx}{dt}\ \hat{j} = \frac{dx}{dt}\ \left(\hat{i} + \frac{dy}{dx}\ \hat{j}\right)$$

$$= \frac{dx}{dt}\ \left(\hat{i} + 2x\ \hat{j}\right)$$

- $\Longrightarrow$ the speed is then

$$v = \left|\frac{dx}{dt}\right| \sqrt{(1+4x^2)}$$

- since the speed is constant, and the object moves to the right $\Longrightarrow$ $\frac{dx}{dt} > 0$ and

$$\left|\frac{dx}{dt}\right| \sqrt{(1+4x^2)} = 5 \qquad \Longrightarrow \qquad \frac{dx}{dt} = \frac{5}{\sqrt{1+4x^2}}$$

- $\Longrightarrow$

$$\vec{v} = \frac{5}{\sqrt{1+4x^2}}\ \hat{i} + \frac{10x}{\sqrt{1+4x^2}}\ \hat{j}$$

  and

$$\vec{v}\Big|_A = \frac{5}{\sqrt{1+4\cdot 1^2}}\ \hat{i} + \frac{10\cdot 1}{\sqrt{1+4\cdot 1^2}}\ \hat{j} = \sqrt{5}\ \hat{i} + 2\sqrt{5}\ \hat{j}$$

- the acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{5}{\sqrt{1+4x^2}}\ \hat{i} + \frac{10x}{\sqrt{1+4x^2}}\ \hat{j}\right)$$

$$= 5(1+4x^2)^{-3/2}\cdot\left(-\frac{1}{2}\right)\cdot 8x\cdot\frac{dx}{dt}\cdot\hat{i} + \left(10x(1+4x^2)^{-3/2}\cdot\left(-\frac{1}{2}\right)\cdot 8x + 10(1+4x^2)^{-1/2}\right)\cdot\frac{dx}{dt}\cdot\hat{j}$$

$$= \frac{10}{(1+4x^2)^{3/2}}\cdot\frac{dx}{dt}\cdot(-2x\hat{i}+\hat{j}) = \frac{50}{(1+4x^2)^2}\cdot(-2x\hat{i}+\hat{j})$$

  where in the last equality we substituted the explicit expression for $\frac{dx}{dt}$.

- at point A, the acceleration is then

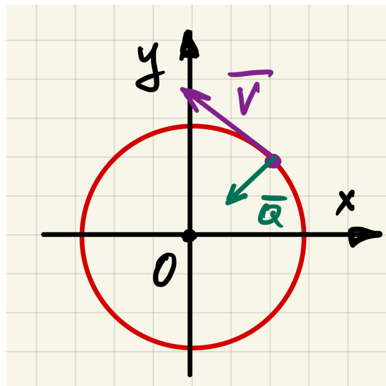
$$\vec{a}\Big|_A = \frac{50}{(1+4x^2)^2}\cdot(-2x\hat{i}+\hat{j})\Big|_{x=1} = 2(-2\hat{i}+\hat{j}) = -4\hat{i}+2\hat{j}$$

Differentiating vector functions

$\Longrightarrow$ Usual rules apply. Let $\vec{u}(t)$, $\vec{v}(t)$ are vector functions and $\lambda(t)$ is a scalar function; then:

- $$\frac{d}{dt}\Big(\vec{u}(t)+\vec{v}(t)\Big) = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt}$$
- $$\frac{d}{dt}\Big(\lambda(t)\cdot\vec{u}(t)\Big) = \frac{d\lambda}{dt}\cdot\vec{u} + \lambda\cdot\frac{d\vec{u}}{dt}$$
- for the dot product of two vectors
$$\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \frac{d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{d\vec{v}}{dt}$$
- for the cross product of two vectors
$$\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$$
- chain rule (recall that a vector function is just three scalar functions):
$$\frac{d}{dt}\Big(\vec{u}(\lambda(t))\Big) = \frac{d\vec{u}}{d\lambda}\cdot\frac{d\lambda}{dt}$$
- for the length of a vector function:
$$\frac{d}{dt}|\vec{u}(t)|^2 = 2|\vec{u}|\cdot\frac{d|\vec{u}|}{dt}$$
$$= \frac{d}{dt}\underbrace{(\vec{u}\cdot\vec{u})}_{=|\vec{u}(t)|^2} = \frac{d\vec{u}}{dt}\cdot\vec{u} + \vec{u}\cdot\frac{d\vec{u}}{dt} = 2\frac{d\vec{u}}{dt}\cdot\vec{u}$$
- $\Longrightarrow$
$$2|\vec{u}|\cdot\frac{d|\vec{u}|}{dt} = 2\frac{d\vec{u}}{dt}\cdot\vec{u}$$
$\Longrightarrow$
$$\boxed{\frac{d|\vec{u}|}{dt} = \frac{\frac{d\vec{u}}{dt}\cdot\vec{u}}{|\vec{u}|}}$$
- Note that if the acceleration $\frac{d\vec{u}}{dt}$ is $\perp$ to the velocity vector $\vec{u} \Longrightarrow$
$$\frac{d\vec{u}}{dt}\cdot\vec{u} = 0 \qquad \Longrightarrow \qquad \frac{d|\vec{u}|}{dt} = 0$$
i.e., the speed is constant

Example 3: Consider a particle moving with a constant speed around a circle trajectory



$\Longrightarrow$

- the position vector of the particle is

$$\vec{r} = a\cos(wt)\ \hat{i} + a\sin(wt)\ \hat{j}$$

- the velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt} = -aw\sin(wt)\ \hat{i} + aw\cos(wt)\ \hat{j}$$

  so that indeed

$$v^2 = \left|\frac{d\vec{r}}{dt}\right|^2 = a^2w^2 = \text{const}$$

- the acceleration vector

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -aw^2\cos(wt)\ \hat{i} - aw^2\sin(wt)\ \hat{j} = -w^2\cdot\vec{r}(t)$$

- indeed

$$\vec{a}\cdot\vec{v} = a^2w^3\sin(wt)\cdot\cos(wt)\cdot\underbrace{(\hat{i}\cdot\hat{i})}_{=1} - a^2w^3\cos(wt)\cdot\sin(wt)\cdot\underbrace{(\hat{j}\cdot\hat{j})}_{=1} = 0$$

  where we also used that $\hat{i}\cdot\hat{j} = 0$, and thus dropped the terms involving that dot product

Example 4: Calculate

$$\frac{d}{dt}\left(\vec{u} \cdot \left(\frac{d\vec{u}}{dt} \times \frac{d^2\vec{u}}{dt^2}\right)\right)$$

which for a moving object (position $\vec{r}$, velocity vector $\vec{v}$ and the acceleration vector is $\vec{a}$) is equivalent

$$\frac{d}{dt}\Big(\vec{r} \cdot (\vec{v} \times \vec{a})\Big)$$

$\Longrightarrow$

$$= \underbrace{\frac{d\vec{u}}{dt} \cdot \left(\frac{d\vec{u}}{dt} \times \frac{d^2\vec{u}}{dt^2}\right)}_{\text{A}} + \underbrace{\vec{u} \cdot \left(\frac{d^2\vec{u}}{dt^2} \times \frac{d^2\vec{u}}{dt^2}\right)}_{\text{B}} + \vec{u} \cdot \left(\frac{d\vec{u}}{dt} \times \frac{d^3\vec{u}}{dt^3}\right)$$

Note that

$$\text{A} = \vec{v} \cdot (\vec{v} \times \vec{a}) = 0$$

because the vector $\vec{v} \times \vec{a}$ is $\perp$ to $\vec{v}$

Note that

$$\text{B} = \vec{r} \cdot (\vec{a} \times \vec{a}) = 0$$

because the vector $\vec{a} \times \vec{a} = \vec{0}$

$\Longrightarrow$

$$\frac{d}{dt}\left(\vec{u} \cdot \left(\frac{d\vec{u}}{dt} \times \frac{d^2\vec{u}}{dt^2}\right)\right) = \vec{u} \cdot \left(\frac{d\vec{u}}{dt} \times \frac{d^3\vec{u}}{dt^3}\right)$$