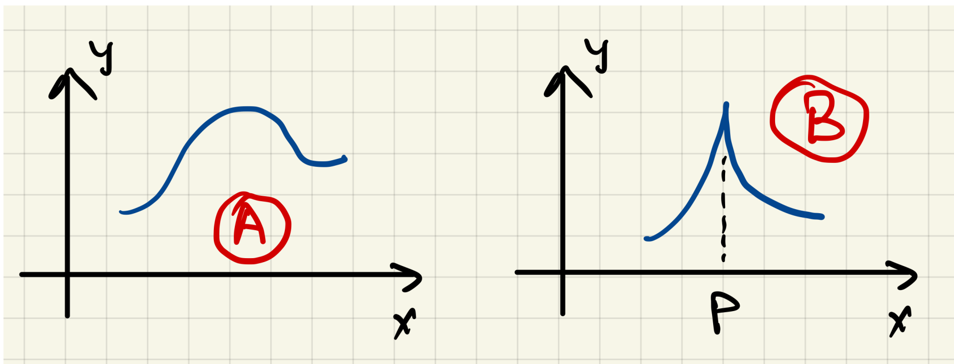
Lecture 8.3: **Smooth parametric curves and their slopes**



- Ⓐ a smooth curve: tangent line at every point; tangent turns smoothly as the point on the curve moves
- Ⓑ a not smooth curve: at point P it does not have a tangent line

Example 1 Consider a parametric curve:

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases} , \qquad t \in \mathbb{R}$$

$$t > 0: \qquad \Longrightarrow \ t = \sqrt{x} \qquad \Longrightarrow \ y = (\sqrt{x})^3 = x^{3/2}$$

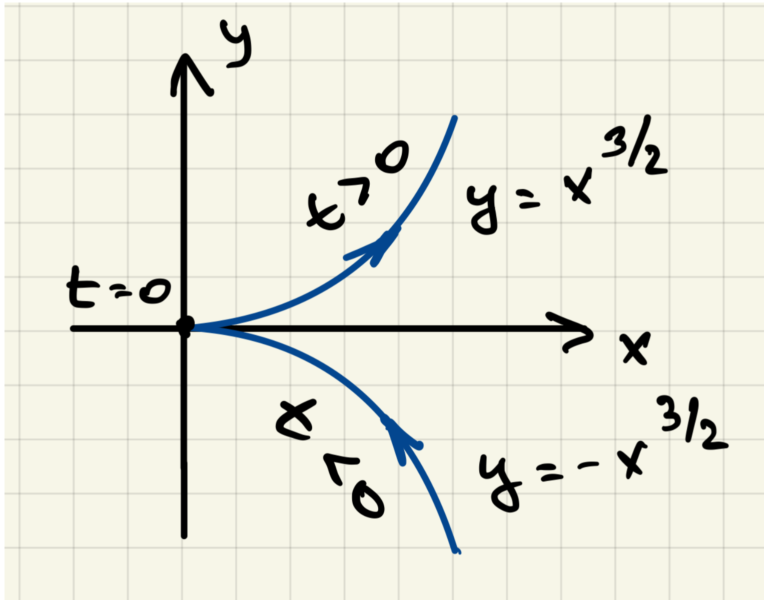
$$t < 0: \qquad \Longrightarrow \ t = -\sqrt{x} \qquad \Longrightarrow \ y = (-\sqrt{x})^3 = -x^{3/2}$$

Although

$$\lim_{t \to 0+} \frac{dy}{dx} = \lim t \to 0+ \frac{3}{2}x^{1/2} = 0$$

$$\lim_{t \to 0-} \frac{dy}{dx} = \lim t \to 0- -\frac{3}{2}x^{1/2} = 0$$

there is no tangent to the curve at $t = 0$:



- The curve has a cusp at $t = 0$, *i.e.*, (x,y)=(0,0)
- The reason for the cusp is the vanishing of both derivatives:

$$\frac{dx}{dt} = 2t \qquad \Longrightarrow \qquad \left.\frac{dx}{dt}\right|_{t=0} = 0$$

$$\frac{dy}{dt} = 3t^2 \qquad \Longrightarrow \qquad \left.\frac{dy}{dt}\right|_{t=0} = 0$$

Theorem. Let ℓ be a parametric curve

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in \mathcal{I}$$

such that f' and g' are continuous on $\mathcal{I}$.

- (1) If $f'(t) \neq 0$ on $\mathcal{I} \implies \ell$ is smooth and has a tangent line at t with the slope

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

- (2) If $g'(t) \neq 0$ on $\mathcal{I} \implies \ell$ is smooth and has a normal line at t with the slope

$$-\frac{dx}{dy} = -\frac{f'(t)}{g'(t)} \qquad -(\text{the reciprocal of the tangent slope})$$

Note: if $f'(t) = 0$ at a point, but $g'(t) \neq 0$ then the curve has a vertical tangent

Example 2: Find the points where the curve

$$\begin{cases} x = t^3 - 3t \\ y = t^3 - 12t \end{cases} \quad , \qquad t \in \mathbb{R}$$

has a horizontal or a vertical tangent.

$\Longrightarrow$

- horizontal tangent:

$$\frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0 : \qquad \Longrightarrow \qquad \frac{dy}{dt} = 3t^2 - 12 = 3(t-2)(t+2) \implies t = \pm 2$$

$$\left.\frac{dx}{dt}\right|_{t=\pm 2} = 3t^2 - 3\Big|_{t=\pm 2} = 9 \neq 0$$

- vertical tangent:

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} \neq 0 : \qquad \Longrightarrow \qquad \frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1) \implies t = \pm 1$$

$$\left.\frac{dy}{dt}\right|_{t=\pm 1} = 3t^2 - 12\Big|_{t=\pm 1} = -9 \neq 0$$

Example 3: Find the slope of the parametric curve

$$\begin{cases} x = t^4 - t^2 \\ y = t^3 - 2t \end{cases} \quad , \qquad t \in \mathbb{R}$$

at $t = -1$.

$\Longrightarrow$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \left.\frac{g'(t)}{f'(t)}\right|_{t=-1} = \left.\frac{3t^2 - 2}{4t^3 - 2t}\right|_{t=-1} = -\frac{1}{2}$$

$\Longrightarrow$ Let ℓ

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in \mathcal{I}$$

be a smooth parametric curve at $t = t_0$, *i.e.*, both f' and g' are continuous and

$$f'(t_0)^2 + g'(t_0)^2 \neq 0$$

- A tangent line to the curve at $t = t_0$ has a parametric equation

$$\begin{cases} x = f(t_0) + f'(t_0) \cdot (t - t_0) \\ y = g(t_0) + g'(t_0) \cdot (t - t_0) \end{cases}$$

  Note: these are the linear approximations to $\{f(t), g(t)\}$ at $t = t_0$

- A normal line to the curve at $t = t_0$ has a parametric equation

$$\begin{cases} x = f(t_0) + g'(t_0) \cdot (t - t_0) \\ y = g(t_0) - f'(t_0) \cdot (t - t_0) \end{cases}$$

Example 4: Find the parametric equations of the tangent and the normal lines to

$$\begin{cases} x = t - \cos t \\ y = 1 - \sin t \end{cases}, \qquad t \in \mathbb{R}$$

at $t = \frac{\pi}{4}$.

$\Longrightarrow$

$$(x, y)\Big|_{t=\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right)$$

$$(x', y')\Big|_{t=\frac{\pi}{4}} = (1 + \sin t, -\cos t)\Big|_{t=\frac{\pi}{4}} = \left(1 + \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

- Tangent line:

$$\begin{cases} x(t) = \frac{\pi}{4} - \frac{1}{\sqrt{2}} + \left(1 + \frac{1}{\sqrt{2}}\right)\left(t - \frac{\pi}{4}\right) \\ y(t) = 1 - \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\left(t - \frac{\pi}{4}\right) \end{cases}$$

- Normal line:

$$\begin{cases} x(t) = \frac{\pi}{4} - \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\left(t - \frac{\pi}{4}\right) \\ y(t) = 1 - \frac{1}{\sqrt{2}} - \left(1 + \frac{1}{\sqrt{2}}\right)\left(t - \frac{\pi}{4}\right) \end{cases}$$

Concavity of a parametric curve

Consider a parametric curve ℓ

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \qquad t \in \mathcal{I}$$

Concavity of the curve is determined by the sign of $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} > 0 : \qquad \text{concave up}$$

$$\frac{d^2y}{dx^2} < 0 : \qquad \text{concave down}$$

At $\frac{d^2y}{dx^2} = 0$ the curve has an inflection point.

Note:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{g'}{f'}\right) \cdot \frac{dt}{dx} = \frac{g''f' - f''g'}{(f')^2} \cdot \frac{1}{f'} = \frac{g''f' - f''g'}{(f')^3}$$

Example 5: Sketch the graph of

$$\begin{cases} x = t^3 - 3t \\ y = t^2 \end{cases}, \qquad t \in [-2, 2]$$

$\Longrightarrow$

- $x = 0$:

$$t(t^2 - 3) = 0 \implies t = \{0, \pm\sqrt{3}\} \qquad \text{or} \qquad (x, y) = \Big\{(0,0), (0,3)\Big\}$$

- $y = 0$:

$$t^2 = 0 \implies t = 0 \qquad \text{or} \qquad (x, y) = (0, 0)$$

- Horizontal tangent:

$$g' = 0 \implies 2t = 0 \qquad \implies \qquad t = 0 \qquad \text{or} \qquad (x, y) = (0, 0)$$

- Vertical tangent:

$$f' = 0 \implies 3(t-1)(t+1) = 0 \qquad \implies \qquad t = \pm 1 \qquad \text{or} \qquad (x,y) = \Big(\mp 2, 1\Big)$$

- Any cusps? $\implies$ NO

- Concavity:

$$\frac{d^2y}{dx^2} = \frac{g''f' - f''g'}{(f')^3} = -\frac{2(1+t^2)}{9(t^2-1)^3}$$

  - concave up:

$$\frac{d^2y}{dx^2} > 0 \qquad \implies \qquad t \in (-1, 1)$$

  - concave down:

$$\frac{d^2y}{dx^2} < 0 \qquad \implies \qquad t \in [-2, -1) \ \cup \ (1, 2]$$

  - inflection points? $\implies$ NO

- as $t \to +\infty$:

$$x \approx t^3\,, \qquad y \approx t^2 \qquad \implies y \approx (x^2)^{1/3}$$

- as $t \to -\infty$:

$$x \approx t^3\,, \qquad y \approx t^2 \qquad \implies y \approx (x^2)^{1/3}$$

- there is a symmetry $(x, y) \leftrightarrow (-x, y)$

$\implies$