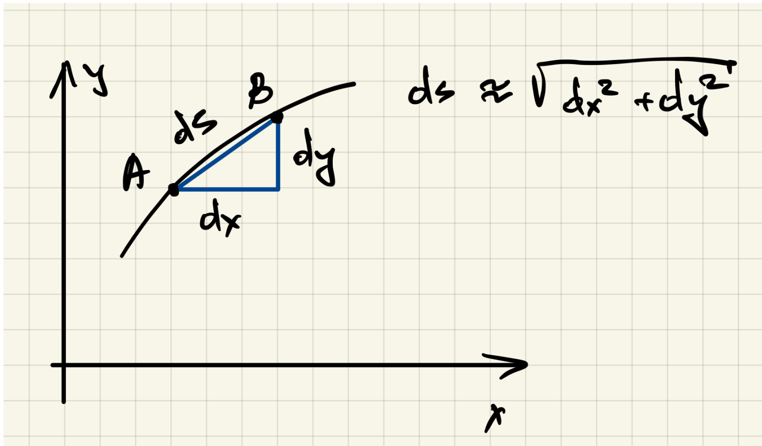
Lecture 7.3: **Arc length and surface area**

Problem: compute the arc length of $y = f(x)$ $(x = g(y))$ as $x \in [a, b]$ $(y \in [c, d])$



- Consider a curve segment between points $A = (a, c)$ and $B = (b, d)$:

$$ds^2 = dx^2 + dy^2 \qquad \Longrightarrow \qquad ds = \sqrt{dx^2 + dy^2}$$

- The total length L is simply the sum of the segments:

$$L = \int_A^B ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

- If the curve is given as $y = f(x) \Longrightarrow$

$$\frac{dy}{dx} = f'(x) \qquad \Longrightarrow$$

$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

- If the curve is given as $x = g(y) \Longrightarrow$

$$\frac{dx}{dy} = g'(y) \qquad \Longrightarrow$$

$$\boxed{L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy}$$

Example 1: Find the arc length of

$$y = x^4 + \frac{1}{32x^2}\,, \qquad x \in [1,2]$$

$\Longrightarrow$ Note that the curve is represented as $y = f(x)$, so, using

$$1 + f'^2 = 1 + \left(4x^3 - \frac{2}{32x^3}\right)^2 = 1 + \left(4x^3 - \frac{1}{16x^3}\right)^2 = \left(4x^3 + \frac{1}{16x^3}\right)^2$$

we find

$$L = \int_1^2 \sqrt{\left(4x^3 + \frac{1}{16x^3}\right)^2}\; dx = \int_1^4 \left(4x^3 + \frac{1}{16x^3}\right)\; dx = \left(x^4 - \frac{1}{32x^2}\right)\Bigg|_1^2$$

$$= 16 - 1 - \frac{1}{32 \cdot 4} + \frac{1}{32} = 15 + \frac{3}{128}$$

Example 2: Find the arc length of the hyperbolic cosine function

$$y = \cosh x \equiv \frac{e^x + e^{-x}}{2}\,, \qquad x \in [0,a]$$

$\Longrightarrow$ Note that the curve is represented as $y = f(x)$, so, using

$$1 + f'^2 = 1 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2$$

we find

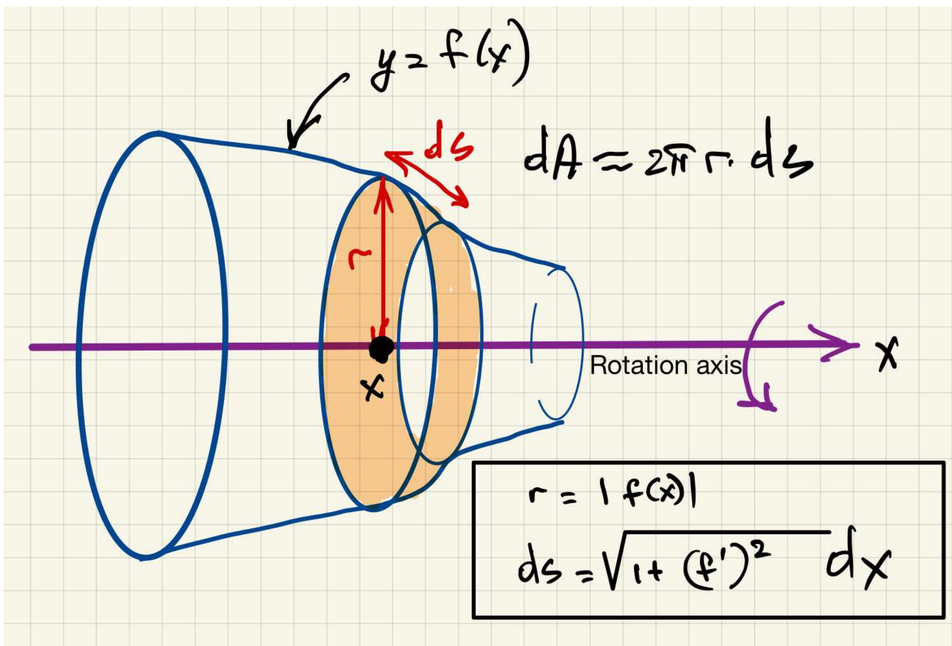
$$L = \int_0^a \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\; dx = \int_0^a \left(\frac{e^x + e^{-x}}{2}\right)\; dx = \left(\frac{e^x - e^{-x}}{2}\right)\Bigg|_0^a = \frac{e^a - e^{-a}}{2}$$

Areas of surfaces of revolution

Problem: find a surface area obtained by rotation of $y = f(x)$, $x \in [a, b]$ about

- case 1: x-axis
- case 2: y-axis

Case 1:



- Let's break the surface into stripes (orange) of small width ds

- The area dA of each stripe is

$$dA = 2\pi \; r \; ds$$

  where r is the distance from the axis of rotation (purple) to the stripe surface

- For a stripe centered at location x on the rotation axis

$$r = |f(x)| \,, \qquad \text{and} \qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (f'(x))^2} \; dx$$

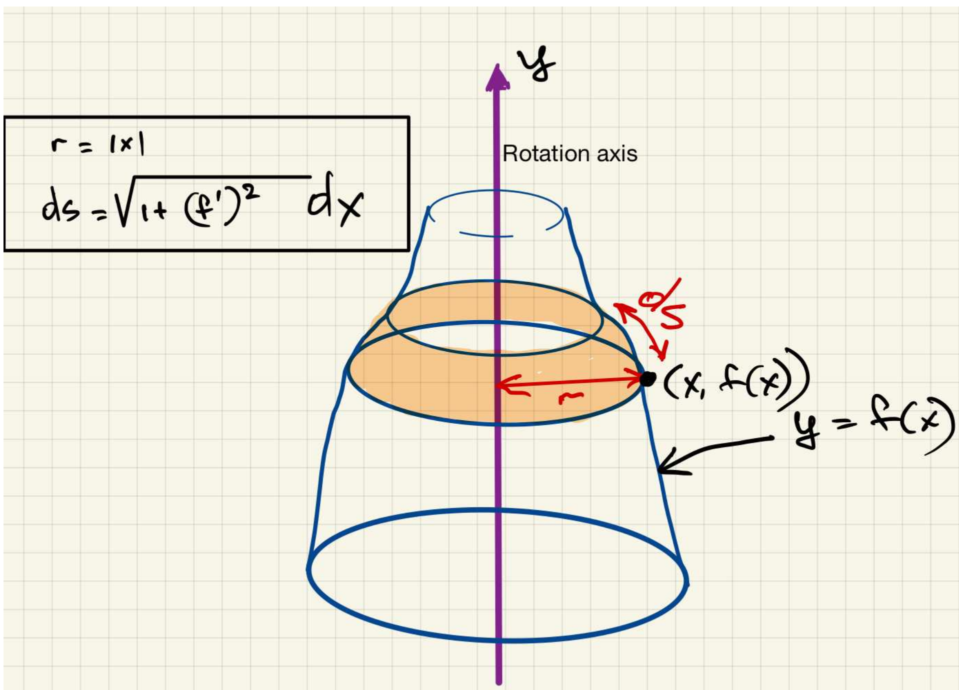
- $\Longrightarrow$

$$dA = 2\pi |f(x)| \sqrt{1 + (f'(x))^2}\ dx$$

- The total area is a sum (an integral) of all the infinitesimal stripes

$$A = \int dA = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2}\ dx$$

<u>**Case 2:**</u>



- Let's break the surface into stripes (orange) of small width ds

- The area dA of each stripe is

$$dA = 2\pi\ r\ ds$$

where r is the distance from the axis of rotation (purple) to the stripe surface

- For a stripe with a point $(x, f(x))$ on the surface,

$$r = |x|\,, \qquad \text{and} \qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (f'(x))^2}\; dx$$

- $\Longrightarrow$

$$dA = 2\pi |x| \sqrt{1 + (f'(x))^2}\; dx$$

- The total area is a sum (an integral) of all the infinitesimal stripes

$$A = \int dA = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2}\; dx$$

<u>Case 3:</u> (similar to case 2) A curve $x = g(y)$, $y \in [c, d]$, is rotated about y-axis

$\Longrightarrow$ Here we have

$$r = |x| = |g(y)|\,, \qquad \text{and} \qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (g'(y))^2}\; dy$$

$\Longrightarrow$

$$A = \int dA = 2\pi \int_c^d |g(y)| \sqrt{1 + (g'(y))^2}\; dy$$

<u>Case 4:</u> (similar to case 1) A curve $x = g(y)$, $y \in [c, d]$, is rotated about x-axis

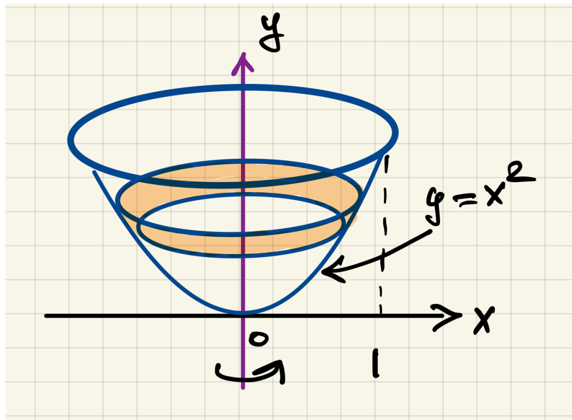
$\Longrightarrow$ Here we have

$$r = y\,, \qquad \text{and} \qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (g'(y))^2}\; dy$$

$\Longrightarrow$

$$A = \int dA = 2\pi \int_c^d |y| \sqrt{1 + (g'(y))^2}\; dy$$

Example 3: Calculate the area obtained by rotating $y = x^2$, $0 \leq x \leq 1$ around y-axis

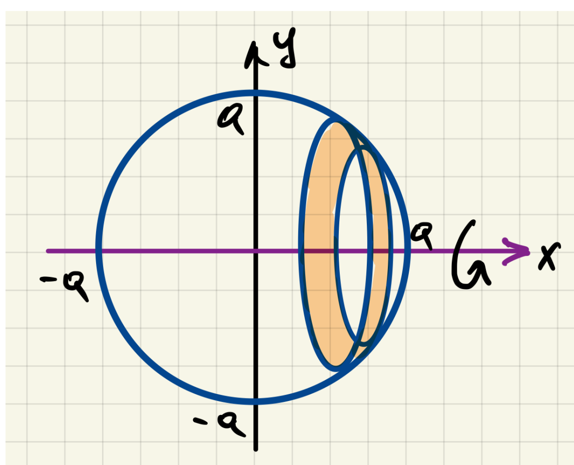


$\Longrightarrow$ This is case 2:

$$A = \int_0^1 2\pi x \left(1 + (\underbrace{2x}_{=f'(x)})^2\right)^{1/2} dx = 2\pi \int_0^1 \underbrace{x(1+4x^2)^{1/2} dx}_{u=1+4x^2, du=8xdx}$$

$$= 2\pi \frac{1}{8} \int u^{1/2} du = 2\pi \frac{1}{8} \frac{2}{3} u^{3/2} = \frac{\pi}{6}(1+4x^2)^{3/2}\Big|_0^1 = \frac{\pi}{6}(5^{3/2} - 1)$$

Example 4: Calculate the surface area of a sphere of radius a



$\Longrightarrow$ A sphere is a surface of revolution of a half-circle of radius a

$$y = \sqrt{a^2 - x^2}, \qquad x \in [-a, a]$$

around x-axis.

$\Longrightarrow$ This is case 1:

$$A = \int_{-a}^{a} 2\pi \underbrace{\sqrt{a^2 - x^2}}_{=|f(x)|} \Bigg(1 + \Bigg(\underbrace{\frac{-2x}{2\sqrt{a^2 - x^2}}}_{=f'(x)}\Bigg)^2\Bigg)^{1/2} dx = 2\pi \int_{-a}^{a} a\, dx = 2\pi a x \Big|_{-a}^{a} = 2\pi a\, 2a = 4\pi a^2$$

We used:

$$\left(1 + \left(\frac{-2x}{2\sqrt{a^2 - x^2}}\right)^2\right)^{1/2} = \sqrt{1 + \left(\frac{x}{\sqrt{a^2 - x^2}}\right)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}}$$

$$= \sqrt{\frac{a^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}$$