Lecture 13.7: **Gradients and directional derivatives**

Def. Let $z = f(x, y)$ be a function of two variables, such that

$$\frac{\partial z}{\partial x} = f_1(x, y)\,, \qquad \frac{\partial z}{\partial y} = f_2(x, y)$$

exits. Then

$$\vec{\nabla} f(x, y) \equiv f_1(x, y)\ \hat{i} + f_2(x, y)\ \hat{j}$$

is called the gradient of f at (x, y)

- The gradient of $f(x, y)$ is a vector function of two variable x and y
- a differential vector operator

  $$\vec{\nabla} \equiv \hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y}$$

  can be applied to a differentiable function f:

  $$\vec{\nabla} f \equiv \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} \right) f = \hat{i} \cdot \frac{\partial f}{\partial x} + \hat{j} \cdot \frac{\partial f}{\partial y}$$

Example 1: compute $\vec{\nabla} f$ for $f(x, y) = x^2 + y^2$

$\Longrightarrow$

- Note that

  $$\vec{r} = x\ \hat{i} + y\ \hat{j}$$

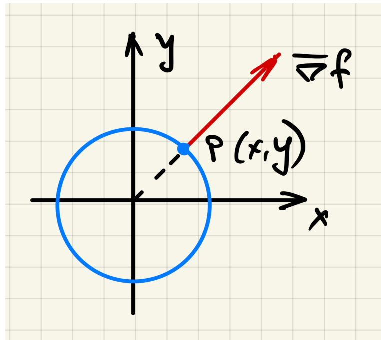
- $$\vec{\nabla} f = \hat{i} \cdot \frac{\partial f}{\partial x} + \hat{j} \cdot \frac{\partial f}{\partial y} = 2x\ \hat{i} + 2y\ \hat{j} = 2\vec{r}$$

- the level curves of the function $f(x, y)$

  $$x^2 + y^2 = k$$

  are circles of radius $\sqrt{k}$ centered at the origin

- Note that $\vec{\nabla} f$ at point P (red) is orthogonal to the level curve of f passing through this point (blue):



Theorem: if $f(x,y)$ is differentiable at (x,y), $\vec{\nabla} f(x,y)$ is a normal vector (orthogonal) to the level curve passing through (x,y).

$\Longrightarrow$ The similar statement applied to functions of more than two variables:

- let $w = f(x,y,z)$

- $\Longrightarrow$
$$\vec{\nabla} f = \hat{i} \cdot \frac{\partial f}{\partial x} + \hat{j} \cdot \frac{\partial f}{\partial y} + \hat{k} \cdot \frac{\partial f}{\partial z}$$

- $\vec{\nabla} f(x,y,z)$ is a normal vector to the level surface
$$f(x,y,z) = k$$
passing through (x,y,z)

Example 2: Find an equation of the tangent plane to the level surface of
$$f(x,y,z) = x^2y + y^2z + z^2x$$
at $P = (1,-1,1)$

$\Longrightarrow$

- the normal is

$$\vec{n} = \vec{\nabla} f\Big|_{(1,-1,1)} = (2xy + z^2)\hat{i} + (x^2 + 2zy)\hat{j} + (y^2 + 2xz)\hat{k}\Big|_{(1,-1,1)}$$

$$= -\hat{i} - \hat{j} + 3\hat{k} = (-1, -1, 3)$$

- the equation for the plane passing through $(x_0, y_0, z_0) = (1, -1, 1)$ is

$$(-1) \cdot (x - 1) + (-1) \cdot (y + 1) + 3 \cdot (z - 1) = 0$$

or

$$-x - y + 3z = 3$$

Directional derivatives

$\Longrightarrow$ Let $z = f(x, y)$ be a differentiable function of two variables.

▪ the rate of change of f is the direction of the positive x-axis is

$$\frac{\partial z}{\partial x} = f_1(x, y)$$

▪ the rate of change of f is the direction of the positive y-axis is

$$\frac{\partial z}{\partial y} = f_2(x, y)$$

$\Longrightarrow$ How do we determine the rate of change of f in the direction defined by a unit vector $\hat{u}$?

$$\hat{u} = u_1 \ \hat{i} + u_2 \ \hat{j}\,, \qquad ||\hat{u}|| = \sqrt{\hat{u} \cdot \hat{u}} = \sqrt{u_1^2 + u_2^2} = 1$$

Def. The directional derivative of f in the direction $\hat{u}$ at point (x, y) is

$$D_{\hat{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + u_1 \cdot h, y + u_2 \cdot h) - f(x, y)}{h}$$

$\Longrightarrow$ We can compute $D_{\hat{u}} f(x, y)$ using the LR:

$$D_{\hat{u}} f(x, y) = \lim_{h \to 0} \frac{\frac{d}{dh}\left(f(x + u_1 \cdot h, y + u_2 \cdot h) - f(x, y)\right)}{\frac{d}{dh} h} = \lim_{h \to 0} \frac{f_1 \cdot u_1 + f_2 \cdot u_2}{1} = f_1 \cdot u_1 + f_2 \cdot u_2$$

$$= \hat{u} \cdot \vec{\nabla} f(x, y)$$

$$\boxed{D_{\hat{u}} f(x,y) = \hat{u} \cdot \vec{\nabla} f(x,y)}$$

Example 3: find the rate change of $f(x,y) = y^4 + 2xy^3 + x^2y^2$ at $(0,1)$ in the direction of $\hat{i} + 2\hat{j}$

$\Longrightarrow$

- we need to find $\hat{u} \;\parallel (\hat{i} + 2\hat{j})$:

$$\hat{u} = \frac{1}{\sqrt{5}}\,\hat{i} + \frac{2}{\sqrt{5}}\,\hat{j}$$

- Note that

$$\vec{\nabla} f\Big|_{(0,1)} = (2y^3 + 2xy^2)\,\hat{i} + (4y^3 + 6xy^2 + 2x^2y)\,\hat{j}\Big|_{(0,1)} = 2\hat{i} + 4\hat{j}$$

- $\Longrightarrow$

$$D_{\hat{u}} f(x,y) = \hat{u} \cdot \vec{\nabla} f(x,y) = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

Properties of directional derivatives:

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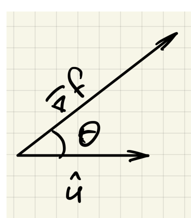
$$D_{\hat{u}} f(x,y) = \hat{u} \cdot \vec{\nabla} f(x,y) = |\vec{\nabla} f(x,y)| \cdot \underbrace{|\hat{u}|}_{=1} \cdot \cos\theta = |\vec{\nabla} f(x,y)| \cdot \cos\theta$$



- the maximal rate of change (increase) is when $\theta = 0 \Longrightarrow \qquad \hat{u}_1 \;\parallel \vec{\nabla} f \Longrightarrow$

$$\vec{u}_1 = \frac{1}{|\vec{\nabla} f|}\,\vec{\nabla} f$$

$$\max\Big\{D_{\hat{u}_1} f(x,y)\Big\} = \frac{1}{|\vec{\nabla} f|}\,\vec{\nabla} f \cdot \vec{\nabla} f = |\vec{\nabla} f|$$

- if $\theta = \frac{\pi}{2}$, *i.e.*, $\hat{u}_2 \perp \vec{\nabla} f(x,y) \Longrightarrow$

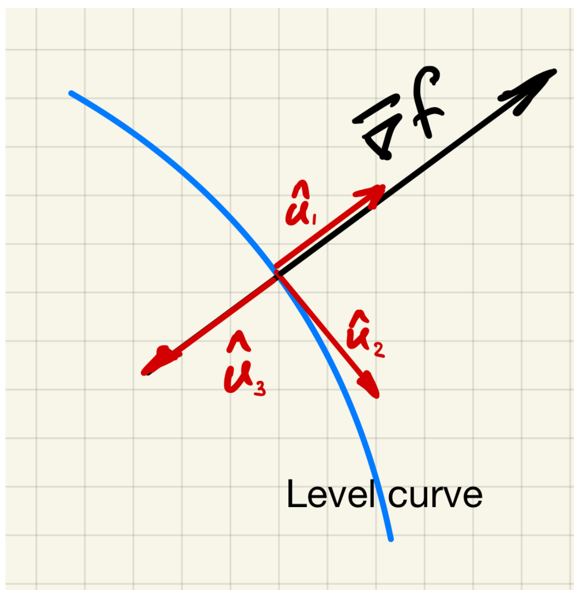
$$D_{\hat{u}_2} f(x,y) = 0$$

i.e., the rate of change vanishes in the direction orthogonal to the direction of gradient

- the maximal rate of decrease is then $\theta = \pi \Longrightarrow \quad \hat{u}_3 \parallel -\vec{\nabla} f \Longrightarrow$

$$\vec{u}_3 = -\frac{1}{|\vec{\nabla} f|}\, \vec{\nabla} f$$

$$D_{\hat{u}_3} f(x,y) = -\frac{1}{|\vec{\nabla} f|}\, \vec{\nabla} f \cdot \vec{\nabla} f = -|\vec{\nabla} f|$$



Example 4: Let $T(x,y) = x^2 e^{-y}$. In what direction at $(2,1)$ does T increase most rapidly? What is the rate of change in that direction?

$\Longrightarrow$

- Note

$$\vec{\nabla} T\Big|_{(2,1)} = 2xe^{-y}\ \hat{\imath} - x^2 e^{-y}\ \hat{\jmath}\Big|_{(2,1)} = \frac{4}{e}\ \hat{\imath} - \frac{4}{e}\hat{\jmath} = \frac{4}{e}(\hat{\imath} - \hat{\jmath})$$

- from $\vec{\nabla} T$, the max rate of change is for

$$\hat{u} = \frac{1}{\sqrt{2}}\ \hat{\imath} - \frac{1}{\sqrt{2}}\ \hat{\jmath}$$

- the rate of change in this direction is

$$D_{\hat{u}} f = |\vec{\nabla} T| = \frac{4}{e}\ \sqrt{2} = \frac{4\sqrt{2}}{e}$$

Rates of change perceived by a moving observer

$\Longrightarrow$ Let $w = f(x, y, z)$. The observer is moving according to

$$\vec{r}(t) = x(t)\ \hat{\imath} + y(t)\ \hat{\jmath} + z(t)\ \hat{k}$$

What is the change of w along the observer path?

$\Longrightarrow$

$$\frac{dw}{dt} = \frac{d}{dt} w(x(t), y(t), z(t)) = w_1 \cdot x' + w_2 \cdot y' + w_3 \cdot z' = \vec{\nabla} w \cdot \vec{v}$$

where $\vec{v}$ is the velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\ \hat{\imath} + \frac{dy}{dt}\ \hat{\jmath} + \frac{dz}{dt}\ \hat{k}\,, \qquad \vec{v} = |\vec{v}| \cdot \hat{v}$$

$$\boxed{\frac{dw}{dt} = \vec{\nabla} w \cdot \vec{v} = |\vec{v}| \cdot \vec{\nabla} w \cdot \hat{v} = |v| \cdot D_{\hat{v}} f}$$

Equation of the tangent plane to a surface revisited

- Let $z = f(x, y)$

- Introduce

$$G(x, y, z) \equiv= z - f(x, y)$$

- Note that points on $z = f(x,y)$ are points on the level surface

$$G(x,y,z) = 0$$

$\Longrightarrow$ We can now compute the tangent plane to this level surface:

$$\vec{\nabla} G\Big|_{(a,b,f(a,b))} = -f_1\ \hat{i} - f_2\ \hat{j} + \hat{k}\Big|_{(a,b,f(a,b))} = (-f_1(a,b), -f_2(a,b), 1)$$

- the equation for the tangent plane is then

$$-f_1 \cdot (x-a) - f_2 \cdot (y-b) + 1 \cdot (z - f(a,b)) = 0$$

or

$$\boxed{z = f(a,b) + f_1(a,b) \cdot (x-a) + f_2(a,b) \cdot (y-b)}$$