Lecture 9.5: **Power series**

Def: a series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 \cdot (x-c) + a_2 \cdot (x-c)^2 + \cdots$$

is called a power series around c.

- c — the center of convergence of the series
- a_n — the coefficients of the power series

Note: the power series may or may not converge depending on the value of x. But if always converges for $x = c$:

$$\sum_{n=0}^{\infty} a_n (c-c)^n = a_0$$

Example 1: A geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

absolutely converges to

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \text{if } |x| < 1$$

Theorem: Let $\sum_{n=0}^{\infty} a_n (x-c)^n$ be a power series. One of the following is true:

- series converges (absolutely) only at $x = c$
- series converges (absolutely) for all $x \in \mathbb{R}$
- there is a positive, nonzero number R, such that the series is absolutely convergent for $|x-c| < R$, *i.e.*, $-R+c < x < R+c$ and diverges for all x such that $|x-c| > R$, *i.e.*, $x \in (-\infty, -R+c) \cup (R+c, \infty)$. The series may or may not converge when $|x-c| = R$, *i.e.*, $x = \{-R+c, R+c\}$

▪ R — the radius of convergence of the power series

▪ if the series is only converges at $x = c \Longrightarrow \qquad R = 0$

▪ if the series is convergent for all $x \in \mathbb{R} \Longrightarrow \qquad R = \infty$

▪ The set of values of x for which the series converges is called the interval of convergence. This could be one of the following:

$$\{c\}\,,\ \mathbb{R}\,,\ (c-R, c+R)\,,\ [c-R, c+R)\,,\ (c-R, c+R]\,,\ [c-R, c+R]$$

<u>How do we find the radius of convergence of $\sum_{n=0}^{\infty} a_n (x-c)^n$?</u>

$\Longrightarrow$ Use the ratio test for the **absolute** convergence:

- Recall that

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}(x-c)^{n+1}|}{|a_n(x-c)^n|} = |x-c| \cdot \underbrace{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}_{\text{assume that the limit exists and is finite nonzero}} = |x-c| \cdot L$$

- $\Longrightarrow$ the power series is absolutely convergent if

$$\rho = |x-c| \cdot L < 1 \qquad \Longrightarrow \qquad |x-c| < \frac{1}{L} \qquad \Longrightarrow R = \frac{1}{L}$$

- if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$$

$\Longrightarrow$ the power series is absolutely convergent for $x \in \mathbb{R} \Longrightarrow \qquad R = \infty$

- if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = DNE \qquad \text{or} \qquad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$$

the power series is convergent only for $x = c \Longrightarrow \qquad R = 0$

Example 2: Find the center, the radius and the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}$$

$\Longrightarrow$

- from the ratio test for absolute convergence:

$$\rho = \lim_{n\to\infty} \frac{|2x+5|^{n+1}(n^2+1)3^n}{((n+1)^2+1)3^{n+1}|2x+5|^n} = \frac{|2x+5|}{3} \cdot \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2}$$

$$= \frac{|2x+5|}{3} \cdot \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} = \frac{|2x+5|}{3} \cdot 1 = \frac{|2x+5|}{3}$$

- from $\rho < 1 \Longrightarrow$

$$\frac{|2x+5|}{3} < 1 \implies |2x+5| < 3 \implies \left| x - \frac{-5}{2} \right| < \frac{3}{2}$$

- $\Longrightarrow$ we identify the center c and the radius R of convergence as

$$c = -\frac{5}{2}\,, \qquad R = \frac{3}{2}$$

- we need to the check the boundaries, *i.e.*, $|2x+5| = 3 \Longrightarrow$ $x = -4$ or $x = -1$.
  - $x = -4$:
  $$\sum_{n=0}^{\infty} \frac{(-8+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n^2+1)}$$
  $\Longrightarrow$ the series converges absolutely
  - $x = -1$:
  $$\sum_{n=0}^{\infty} \frac{(-2+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \frac{1}{(n^2+1)}$$
  $\Longrightarrow$ the series converges absolutely

- $\Longrightarrow$ the interval of convergence is

$$x \in [-4, -1]\,, \qquad \text{or} \qquad -4 \le x \le -1$$

Differentiation of a power series

Theorem: if $\sum_{n=0}^{\infty} a_n x^n$ converges to $f(x)$ for $|x| < R$, then $f(x)$ is differentiable for $x \in (-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

i.e., we can differentiate the series term-by-term

Example 3: Find the power series representation for

$$f(x) = \frac{1}{(1-x)^2}$$

$\Longrightarrow$

- These type of problems are always solved finding a relation to a geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,, \qquad |x| < 1$$

- Note that

$$f(x) = \frac{d}{dx}\,\frac{1}{1-x}$$

- $\Longrightarrow$ from the differentiation theorem, for $|x| < 1$

$$f(x) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx}x^n = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{k=0}^{\infty}(k+1)x^k$$

  where we dropped $n = 0$ term from the series since it vanishes; and in the last equality introduced a new summation index $k = n-1$ ($n = k+1$)

Integrating a power series

Theorem: if $\sum_{n=0}^{\infty} a_n x^n$ converges to $f(x)$ for $|x| < R$, then $f(x)$ is integrable, and for $x \in (-R, R)$

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

i.e., we can integrate the series term-by-term

Example 4: Find the power series representation for

$$f(x) = \ln(1+x)$$

$\Longrightarrow$

- Once again, we need to find a relation to a geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,, \qquad |x| < 1$$

- Note that

$$f(x) - f(0) = \int_0^x \frac{dt}{1+t} = \int_0^x \frac{dt}{1-(-t)}$$

- $\Longrightarrow$ from the integration theorem, for $|x| < 1$

$$f(x) - f(0) = \int_0^x dt \cdot \frac{1}{1-(-t)} = \int_0^x dt \cdot \sum_{n=0}^{\infty}(-t)^n = \sum_{n=0}^{\infty}(-1)^n \cdot \int_0^x dt \cdot t^n$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

- $\Longrightarrow$

$$f(x) = f(0) + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \ln(1) + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

Example 5: Find the power series representation for

$$f(x) = \frac{1}{2+x}$$

$\Longrightarrow$

- Once again, we need to find a relation to a geometric series:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n\,, \qquad |u| < 1$$

- Note that

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1 - \frac{-x}{2}} \underbrace{=}_{\text{set } u = -\frac{x}{2}} \frac{1}{2} \cdot \frac{1}{1-u}$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{\infty} u^n = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

- Above series representation is convergent for

$$|u| < 1 \qquad \Longrightarrow \qquad \left| -\frac{x}{2} \right| < 1 \qquad \Longrightarrow \qquad |x| < 2$$

Example 6: Find the series representation for

$$f(x) = \frac{1}{x}$$

in powers of $(x-1)$

$\Longrightarrow$

- Once again, we need to find a relation to a geometric series:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \,, \qquad |u| < 1$$

- Note that

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \frac{1}{1-(-(x-1))} \underbrace{=}_{\text{set } u=-(1-x)} \frac{1}{1-u}$$

$$= \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (-(x-1))^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

- Above series representation is convergent for

$$|u| < 1 \qquad \Longrightarrow \qquad \Big| -(x-1) \Big| < 1 \qquad \Longrightarrow \qquad |x-1| < 1$$