Master Generators: A Novel Approach to Construct and Solve Ordinary Differential Equations

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Abstract

Ordinary differential equations (ODEs) play a crucial role in applied mathematics, engineering, and various other fields. With a rich history of study and countless applications, they have attracted the attention of researchers who have developed numerous methods for solving linear and non-linear ODEs. However, despite considerable progress, there remain many unsolved equations and challenges in finding appropriate solutions. This research introduces a groundbreaking method for generating an infinite number of generators for both linear and non-linear ODEs, based on the use of master improper integrals and a set of novel theorems. The proposed method enables the creation of infinite ODEs using a single generator, providing a versatile and powerful approach to tackling various types of differential equations. This research not only presents several specific generators derived from the theorems, but also establishes general forms for linear and non-linear generators, highlighting their adaptability to a wide range of scenarios. In addition, we demonstrate the practical utility of the generators through real-world examples and applications across diverse fields such as quantum field theory, astrophysics, and fluid dynamics. These examples showcase the potential of the proposed method in solving complex problems and its far-reaching impact on multiple disciplines. Through a comprehensive presentation of the theoretical underpinnings, derivation of generators, and application examples, this research aims to provide a thorough understanding of the infinite generators for linear and non-linear ODEs, as well as to inspire further research and development in this fascinating and vital area of applied mathematics. The introduction of infinite generators not only addresses the existing challenges in solving ODEs but also opens up new avenues for exploring novel solutions and approaches, ultimately contributing to the advancement of various fields that rely on differential equations.

1 Introduction

Solving ordinary differential equations (ODEs) has been a central topic in applied mathematics for centuries, with a rich history of study and a plethora of applications across various disciplines. From the groundbreaking work of ancient mathematicians to the cutting-edge research of contemporary scholars, the pursuit of finding solutions to ODEs has been a persistent and ever-evolving challenge, resulting in the development of a diverse array of methods and approaches. Numerous researchers have dedicated their careers to the study of ODEs, authoring books, presenting equations both with and without solutions, and proposing innovative techniques to tackle these complex problems [1-3].

In recent times, rapid advancements in applied mathematics, engineering, and other fields have invigorated research into novel techniques for solving ODEs. As the significance of ODEs transcends the realm of mathematics, the demand for new and more effective methods continues to grow, particularly in areas such as quantum field theory, astrophysics, fluid dynamics, and non-linear equations [4-6].

Finding appropriate methods to solve specific ODEs can be a daunting task. While some equations allow for the derivation of exact solutions with relative ease, others require the application of more sophisticated approximate methods [7-13]. However, a significant number of these equations remain unsolved or prove to be intractable using existing techniques. This research presents an innovative method that generates an infinite number of generators for a wide array of linear and non-linear ordinary differential equations (ODEs). By utilizing a single generator, one can generate an infinite variety of ODEs, harnessing the remarkable power of master improper integrals as detailed in [14-17].

In the context of existing methods, the Differential Transform Method (DTM) is a well-established and widely used technique for solving differential equations, including non-linear ones and systems of differential equations. The DTM has been successfully applied in various studies, demonstrating its versatility and effectiveness[18-19]. However, our proposed method offers a groundbreaking approach that can be employed to solve the pantograph equation, a specific type of functional differential equation characterized by a proportional delay. Moreover, it can be utilized in the evaluation and construction of a diverse collection of linear, non-linear, and multi-order complex differential equations, some of which are otherwise unattainable using traditional methods or even advanced approximate techniques such as DTM.

While acknowledging the proven capabilities of the DTM, this research aims to pave the way for a new era in the study of ODEs. The proposed method holds immense potential for a vast range of applications across a multitude of scientific domains, offering a novel perspective in the ongoing exploration of differential equations.

2 Background and Notations

To fully comprehend the proposed method and its implications, it is essential to establish a thorough understanding of the background concepts and notation that will be used throughout this research. This section aims to provide a concise overview of the fundamentals of ODEs, linear and non-linear equations, and master improper integrals, thereby laying a solid foundation for the forthcoming discussions.

I. **Ordinary Differential Equations (ODEs)** [4-7]: An *ordinary differential equation (ODE)* is a relation that contains functions of only one independent variable, and one or more of their derivatives with respect to that variable. The general n-th order ODE can be represented as:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

where:

- x is the independent variable.
- y is the unknown function (dependent variable) of x.
- The terms $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$ represent the first, second, ..., up to n-th derivatives of y with respect to x.
- F is a given function of its arguments, and the equation is of order n if n is the highest derivative present.

ODEs are used to model various real-world phenomena, such as population growth, fluid flow, and heat transfer. The order of an ODE is determined by the highest derivative present in the equation. The primary goal in studying ODEs is to find a function that satisfies the given equation.

II. **Linear and Non-linear Differential Equations** [4-7]: ODEs can be broadly classified into two categories: linear and non-linear. An n-th order ODE is said to be linear if it can be written in the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where:

- x is the independent variable.
- y is the unknown function (dependent variable) of x.
- The coefficients $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ are continuous functions of x over the interval of interest.
- $g(x)$ is a given function of x.

A non-linear ODE, on the other hand, contains higher powers or products of the dependent variable and its derivatives. Generally, linear ODEs are more straightforward to solve than their non-linear counterparts, as they can often be addressed using established techniques, such as the method of integrating factors or the Laplace transform.

III. **Master Improper Integrals**: Master improper integrals refer to a set of theorems in solving improper integrals and generating infinite improper integrals with their exact solutions. An improper integral can be simply defined as an integral whose integrand may not be defined or bounded over the entire interval of integration, or the interval of integration itself may be unbounded. Specifically, the improper integral of a function $f(x)$ over an interval $[a, b]$ is defined by limits:

$$\int_a^b f(x)\,dx = \lim_{t\to b}\int_a^t f(x)\,dx + \lim_{t\to a^+}\int_t^b f(x)\,dx$$

These theorems, provided and proved by M. Abu-Ghuwaleh et al. in 2022 [14-17], serve as a powerful tool in the study of ODEs. By harnessing the power of master improper integrals, one can derive an infinite number of generators for linear and non-linear ODEs, as demonstrated in this research.

With this background and notation in place, the following sections will examine into the details of the proposed method for generating an infinite number of generators for linear and non-linear ODEs. The innovative approach outlined in this research has the potential to significantly advance the field of applied mathematics and open new avenues for solving challenging ODEs in various scientific domains.

3 Theoretical FrameWork: Master Improper Integrals and Master Theorems

This section will present the relevant theorems and master improper integrals that form the basis for generating linear and non-linear ODEs. We will discuss the properties of these integrals, their connections to the theorems, and how they enable the creation of infinite generators.
Assume that f is an analytical function in disc D centered at α. Moreover, let α, β and θ be real constants. Then, using Taylor's expansion we have [14-17]

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k. \tag{3.1}$$

By substituting $z = \alpha + \beta e^{i\,\theta\,x}$ in $f(z)$, where β is smaller than the radius of D, we obtain:

$$f(\alpha+\beta e^{i\theta x}) = \sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\,\beta^k e^{i\theta k x}\,, \quad x\in\mathbb{R}, \tag{3.2}$$

and lastly, by using

$$e^{i\theta x} + e^{-i\theta x} = 2\cos(\theta x) \tag{3.3}$$

$$, e^{i\theta x} - e^{-i\theta x} = 2i\,\sin(\theta x), \tag{3.4}$$

we can obtain the following two formulas:

$$\begin{aligned}\frac{1}{2}\Big(f\left(\alpha+\beta e^{i\theta x}\right)+f\left(\alpha+\beta e^{i\theta x}\right)\Big) &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\,\beta^k\left(e^{i\theta k x}+e^{-i\theta k x}\right)\\ &= \sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\,\beta^k\cos(k\theta x).\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{1}{2i}\Big(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{i\theta x}\right)\Big) &= \frac{1}{2i}\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\,\beta^k\left(e^{i\theta k x}-e^{-i\theta k x}\right)\\ &= \sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}\,\beta^k\sin(k\theta x).\end{aligned} \tag{3.6}$$

Moreover, the two mathematical facts below are essential to our proofs:

$$e^{-xk\sin(\omega)}\sin\left(xk\cos(\omega)\right)=\frac{1}{2i}\left(e^{ixk\cos(\omega)-xk\sin(\omega)}-e^{-xk\cos(\omega)-xk\sin(\omega)}\right), \tag{3.7}$$

$$e^{-xk\sin(\omega)}\cos\left(xk\cos(\omega)\right)=\frac{1}{2}\left(e^{ixk\cos(\omega)-xk\sin(\omega)}+e^{-ixk\cos(\omega)-xk\sin(\omega)}\right). \tag{3.8}$$

Moreover, according to a research article by M. Abu-Ghuwaleh et al. in 2022, the following results were proven.

Theorem 3.1. *Let $f(z)$ be an analytic function around α, where $\alpha\in\mathbb{R}$. Then, we have the following results [16]:*

$$I=\int_0^\infty \frac{f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)}{ix\left(1+x^{2n}\right)^r}dx=$$
$$\left(\frac{(-1)^{r-1}}{\Gamma(r)}\right)\frac{\partial^{r-1}}{\partial u^{r-1}}\left(\frac{\pi}{n}u^{-1}\sum_{s=1}^{n}\left(f(\alpha+\beta)-\frac{1}{2}\left(\psi+\phi\right)\right)\right)\Bigg|_{u=1}, \tag{3.9}$$

where $n\in\mathbb{N}$, $\theta>0$, $r\in\mathbb{R}$, $w=w(s)=\frac{(2s-1)\pi}{2n}$, $\psi=\psi(s)=f\left(\alpha+\beta e^{\left(i\theta\sqrt[2n]{u}\cos(\omega)-\theta\sqrt[2n]{u}\sin(\omega)\right)}\right)$, and $\phi=\phi(s)=f\left(\alpha+\beta e^{\left(-i\theta\sqrt[2n]{u}\cos(\omega)-\theta\sqrt[2n]{u}\sin(\omega)\right)}\right)$.

$$I=\int_0^\infty \frac{x^{m-1}\left(f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)\right)}{i\left(1+x^{2n}\right)^r}dx=$$
$$\left(\frac{(-1)^{r-1}}{\Gamma(r)}\right)\frac{\partial^{r-1}}{\partial u^{r-1}}\left(u^{\frac{m-2n}{2n}}\left(\frac{-\pi}{n}\right)\sum_{s=1}^{n}\left(\frac{\cos(m\omega)}{2}\left(\psi+\phi-2f(\alpha)\right)+\frac{\sin(m\omega)}{2i}\left(\psi-\phi\right)\right)\right)\Bigg|_{u=1}, \tag{3.10}$$

where m is even, $n\in\mathbb{N}$, $0<m<2n$, $\theta>0$, $r\in\mathbb{R}$, $w=w(s)=\frac{(2s-1)\pi}{2n}$, and ψ and ϕ as defined in Equation (3.9).

$$I=\int_0^\infty \frac{x^{m-1}f\left(\alpha+\beta e^{i\theta x}\right)-f\left(\alpha+\beta e^{-i\theta x}\right)}{\left(1+x^{2n}\right)^r}dx=$$
$$\left(\frac{(-1)^{r-1}}{\Gamma(r)}\right)\frac{\partial^{r-1}}{\partial u^{r-1}}\left((u)^{\frac{r-2n}{2n}}\frac{\pi}{n}\sum_{s=1}^{n}\left(\frac{1}{2}\sin(m\omega)(\psi+\phi)+\frac{1}{2i}\cos(m\omega)(\psi-\phi)\right)\right)\Bigg|_{u=1}, \tag{3.11}$$

where $\theta>0$, m is odd, $n,m\in\mathbb{N}$, $0<m<2n$, $r\in\mathbb{R}$, and all of ω,ψ and ϕ as defined previously in Equation (3.9).

The proof of these theorems can be found in the mentioned research article, where detailed mathematical arguments are presented to support the validity of these statements. The theorems presented above provide a powerful tool for generating linear and non-linear ODEs. The integrals involved in the theorems are improper integrals, and their solutions are not easily obtained through traditional methods. It is important to note that the theorems are valid for a wide range of functions. The properties of these integrals and their connections to the theorems enable the creation of infinite generators, which provide a powerful tool for solving ODEs.

4 Fundamental Theorems

In this section, we introduce two fundamental theorems that are essential in constructing our generators.

Theorem 4.1. *Let f be an analytic function in disc D centred at $\alpha \in \mathbb{R}$, if we have:*

$$y(x) = \frac{\pi}{2n}\sum_{s=1}^{n}\Big(2f(\alpha+\beta) - (\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big), \tag{4.1}$$

then,

$$\begin{aligned} y'\left(x\right) = \frac{\pi}{2n}\sum_{s=1}^{n}\beta e^{-x\sin(\omega)}\Big(&\frac{1}{i}\cos\left(x\cos\left(\omega\right)+\omega\right)\frac{\partial}{\partial\alpha}\left(\psi\left(\alpha,\omega,x\right)-\phi\left(\alpha,\omega,x\right)\right) \\ &+\sin\left(x\cos\left(\omega\right)+\omega\right)\frac{\partial}{\partial\alpha}\left(\psi\left(\alpha,\omega,x\right)+\phi\left(\alpha,\omega,x\right)\right)\Big), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} y''\left(x\right) = \frac{\pi}{2n}\sum_{s=1}^{n}\beta e^{-x\sin(\omega)}&\Bigg(\cos\left(x\cos\left(\omega\right)+2\omega\right)\frac{\partial}{\partial\alpha}\left(\psi\left(\alpha,\omega,x\right)+\phi\left(\alpha,\omega,x\right)\right) \\ &+\frac{1}{i}\sin\left(x\cos\left(\omega\right)+2\omega\right)\frac{\partial}{\partial\alpha}\left(\psi\left(\alpha,\omega,x\right)-\phi\left(\alpha,\omega,x\right)\right)\Bigg) \\ +\beta^2 e^{-2x\sin(\omega)}&\Bigg(\cos\left(2x\cos\left(\omega\right)+2\omega\right)\frac{\partial^2}{\partial\alpha^2}\left(\psi\left(\alpha,\omega,x\right)+\phi\left(\alpha,\omega,x\right)\right) \\ &+\frac{1}{i}\sin\left(2x\cos\left(\omega\right)+2\omega\right)\frac{\partial^2}{\partial\alpha^2}\left(\psi\left(\alpha,\omega,x\right)-\phi\left(\alpha,\omega,x\right)\right)\Bigg), \end{aligned} \tag{4.3}$$

where $n \in \mathbb{N}, x \leq 0$, $\omega = \omega(s) = \frac{(2s-1)\pi}{2n}$, $\psi\left(\alpha,\omega,x\right) = f\left(\alpha+\beta e^{ix\cos(\omega)-x\sin(\omega)}\right)$, and $\phi\left(\alpha,\omega,x\right) = f\left(\alpha+\beta e^{-ix\cos(\omega)-x\sin(\omega)}\right)$.

Proof. Using Equation (3.9) in Theorem 3.1, and setting $r=1$, the equation can be written as:

$$y(x) \ = \int_0^\infty \frac{\left(f\left(\alpha+\beta e^{ixt}\right)-f\left(\alpha+\beta e^{-ixt}\right)\right)}{it\left(1+t^{2n}\right)}dt.$$

Since $f(z)$ is an analytic function around α, we obtain:

$$\begin{aligned} y\left(x\right) &= 2\sum_{k=1}^{\infty}\frac{f^{(k)}\left(\alpha\right)\beta^k}{k!}\int_0^\infty\frac{\sin\left(kxt\right)}{t\left(1+t^{2n}\right)}dt \\ &= 2f'\left(\alpha\right)\beta\int_0^\infty\frac{\sin\left(xt\right)}{t\left(1+t^{2n}\right)}dt + 2\sum_{k=2}^{\infty}\frac{f^{(k)}\left(\alpha\right)\beta^k}{k!}\int_0^\infty\frac{\sin\left(kxt\right)}{t\left(1+t^{2n}\right)}dt. \end{aligned} \tag{4.4}$$

Differentiating both sides of Equation (4.4) with respect to x, as follows

$$\begin{aligned} y\prime(x) &= \frac{\partial}{\partial x}\left(2f'\left(\alpha\right)\beta\int_0^\infty\frac{\sin\left(xt\right)}{t\left(1+t^{2n}\right)}dt \ +\sum_{k=2}^{\infty}\frac{2f^{(k)}\left(\alpha\right)}{k!}\beta^k\int_0^\infty\frac{\sin\left(kxt\right)}{t\left(1+t^{2n}\right)}dt\right) \\ &= 2f'\left(\alpha\right)\beta\int_0^\infty\frac{\cos\left(xt\right)}{\left(1+t^{2n}\right)}dt \ +\sum_{k=2}^{\infty}\frac{2f^{(k)}\left(\alpha\right)}{k!}\beta^k k\int_0^\infty\frac{\cos\left(kxt\right)}{\left(1+t^{2n}\right)}dt\ . \end{aligned} \tag{4.5}$$

Using Formula (3.11) in Theorem 3.1 and setting $m=1$, $r=1$, $f(z)=z$, we obtain:

$$\int_0^\infty\frac{\cos(xt)}{1+t^{2n}}\,dt = \frac{\pi}{n}\sum_{s=1}^{n} e^{-x\sin\left(\frac{(2s-1)\pi}{2n}\right)}\sin\left(\frac{(2s-1)\pi}{2n}+x\cos\left(\frac{(2s-1)\pi}{2n}\right)\right). \tag{4.6}$$

Substitute Equation (4.6) in Equation (4.5) to obtain:

$$y'(x) = \frac{\pi}{n} f'(\alpha)\beta \sum_{s=1}^{n} e^{-x\sin\left(\frac{(2s-1)\pi}{2n}\right)} \sin\left[\left(\frac{(2s-1)\quad\pi}{2n}\right) + x\cos\left(\frac{(2s-1)\pi}{2n}\right)\right]$$
$$+ \frac{\pi}{n}\sum_{k=2}^{\infty}\frac{f^{(k)}(\alpha)}{(k-1)!}\beta^{k}\sum_{s=1}^{n} e^{-xk\sin\left(\frac{(2s-1)\pi}{2n}\right)} \sin\left[\left(\frac{(2s-1)\quad\pi}{2n}\right) + xk\cos\left(\frac{(2s-1)\pi}{2n}\right)\right]. \tag{4.7}$$

By shifting the index of $k=2$ to $k=1$, we can write Equation (4.7) as:

$$y'(x) = \frac{\pi}{n} f'(\alpha)\beta \sum_{s=1}^{n} e^{-x\sin\left(\frac{(2s-1)\pi}{2n}\right)} \sin\left[\left(\frac{(2s-1)\quad\pi}{2n}\right) + x\cos\left(\frac{(2s-1)\pi}{2n}\right)\right]$$
$$+ \frac{\pi}{n}\sum_{k=1}^{\infty}\frac{f^{(k+1)}(\alpha)}{(k)!}\beta^{k+1}\sum_{s=1}^{n} e^{-x(k+1)\sin\left(\frac{(2s-1)\pi}{2n}\right)} \sin\left[\left(\frac{(2s-1)\quad\pi}{2n}\right) + x(k+1)\cos\left(\frac{(2s-1)\pi}{2n}\right)\right]. \tag{4.8}$$

To simplify Equation (4.8), let $\omega = \frac{(2s-1)\pi}{2n}$. Then, we can write the equation as:

$$y'(x) = \frac{\pi}{n} f'(\alpha)\beta \sum_{s=1}^{n} e^{-x\sin(\omega)} \sin\left[(\omega) + x\cos(\omega)\right]$$
$$+ \frac{\pi}{n}\sum_{k=1}^{\infty}\frac{f^{(k+1)}(\alpha)}{(k)!}\beta^{k+1}\sum_{s=1}^{n} e^{-x(k+1)\sin(\omega)} \sin\left[(\omega) + x(k+1)\cos\omega\right] . \tag{4.9}$$

Now, observe that:

$$e^{x(k+1)\sin(\omega)}\sin\left[(\omega) + x(k+1)\cos\omega\right]$$
$$= e^{-x\sin(\omega)}\left(e^{-xk\sin(\omega)}\sin\left(xk\cos(\omega)\right)\cos\left(x\cos(\omega)+\omega\right)\right.$$
$$\left. + e^{-x\sin(\omega)}e^{-xk\sin(\omega)}\left(\cos\left(x\ k\cos(\omega)\right)\sin\left(x\cos(\omega)+\omega\right)\right)\right). \tag{4.10}$$

By substituting Equation (4.10) into Equation (4.9) we get:

$$y'(x) = \frac{\pi}{n} f'(\alpha)\beta \sum_{s=1}^{n} e^{-x\sin(\omega)} \sin\left[(\omega) + x\cos(\omega)\right]$$
$$+ \frac{\pi}{n}\beta\sum_{s=1}^{n} e^{-x\sin(\omega)}\cos\left(x\cos(\omega)+\omega\right)\frac{\partial}{\partial\alpha}\sum_{k=1}^{\infty}\frac{f^{(k)}(\alpha)}{(k)!}\beta^{k}\left(e^{-xk\sin(\omega)}\sin(xk\cos(\omega)\right)$$
$$+ \frac{\pi}{n}\beta\sum_{s=1}^{n} e^{-x\sin(\omega)}\sin\left(x\cos(\omega)+\omega\right)\frac{\partial}{\partial\alpha}\sum_{k=1}^{\infty}\frac{f^{(k)}(\alpha)}{(k)!}\beta^{k}\left(e^{-xk\sin(\omega)}\left(\cos\left(xk\cos(\omega)\right)\right)\right). \tag{4.11}$$

Finally, by substituting equations (3.7) and (3.8) and using the fact mentioned in Equation (3.2), we can write Equation (4.11) as:

$$y\prime(x) = \frac{\pi}{n} f'(\alpha)\beta \sum_{s=1}^{n} e^{-x\sin(\omega)} \sin\left[(\omega) + x\cos(\omega)\right]$$
$$+ \frac{\pi}{n}\beta\sum_{s=1}^{n}\left[\frac{1}{2i}e^{-x\sin(\omega)}\cos\left(xcos(\omega)+\omega\right)\right.$$
$$\left.\frac{\partial}{\partial\alpha}\left(f\left(\alpha+\beta e^{ix\cos(\omega)-x\sin(\omega)}\right) - f\left(\alpha+\beta e^{-ix\cos(\omega)-x\sin(\omega)}\right)\right)\right]$$
$$+ \frac{1}{2}e^{-x\sin(\omega)}\sin\left(xcos(\omega)+\omega\right)$$
$$\left(\frac{\partial}{\partial\alpha}\left(f\left(\alpha+\beta e^{ix\cos(\omega)-x\sin(\omega)}\right) + f\left(\alpha+\beta e^{-ix\cos(\omega)-x\sin(\omega)}\right) - 2f(\alpha)\right)\right). \tag{4.12}$$

By letting $\psi(\alpha,\omega,x) = f\left(\alpha + \beta e^{ix\cos(\omega) - x\sin(\omega)}\right)$ and $\phi(\alpha,\omega,x) = f\left(\alpha + \beta e^{-ix\cos(\omega) - x\sin(\omega)}\right)$, as well as performing some simplifying calculations, we obtain:

$$y'(x) = \frac{\pi}{2n}\sum_{s=1}^{n}\beta e^{-x\sin(\omega)}\Big(\frac{1}{i}\cos(xcos(\omega)+\omega)\frac{\partial}{\partial\alpha}(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x)) \\ +\sin(x\cos(\omega)+\omega)\frac{\partial}{\partial\alpha}(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big).$$

Hence, Equation (4.2) is proved. □

Proof. In order to prove (4.3), we differentiate Equation (4.4) twice with respect to x as follows:

$$y''(x) = -2f'(\alpha)\beta\int_0^{\infty}\frac{t\,sin(xt)}{(1+t^{2n})}dt - \sum_{k=2}^{\infty}\frac{2f^{(k)}(\alpha)}{(k-1)!}\beta^k k\int_0^{\infty}\frac{t\sin(kxt)}{(1+t^{2n})}dt\ . \tag{4.13}$$

Using Formula (3.10) in Theorem 3.1 and setting $m=1$, $r=1$, $f(z)=z$, we get:

$$\int_0^{\infty}\frac{t\sin(xt)}{1+t^{2n}}dt = \frac{-\pi}{2n}\sum_{s=1}^{n}e^{-x\sin\left(\frac{(2s-1)\pi}{2n}\right)}\cos\left(\frac{2\pi(2s-1)}{2n}+x\cos\left(\frac{(2s-1)\pi}{2n}\right)\right). \tag{4.14}$$

As in the previous proof, let $\omega = \frac{(2s-1)\pi}{2n}$. additionally, substitute Equation (4.14) in Equation (4.13) to obtain:

$$\begin{aligned} y''(x) &= \frac{\pi}{n}f'(\alpha)\beta\sum_{s=1}^{n}e^{-x\sin(\omega)}\cos((2\omega)+x\cos(\omega)) \\ &\quad + \frac{\pi}{n}\sum_{k=2}^{\infty}\frac{f^{(k)}(\alpha)}{(k-1)!}\beta^k k\sum_{s=1}^{n}e^{-xk\sin(\omega)}\cos((2\omega)+xk\cos(\omega)) \\ &= \frac{\pi}{n}\Bigg(f'(\alpha)\beta\sum_{s=1}^{n}e^{-x\sin(\omega)}\cos((2\omega)+x\cos(\omega)) \\ &\quad + \sum_{k=2}^{\infty}\frac{f^{(k)}(\alpha)}{(k-1)!}\beta^k(k-1+1)\ \sum_{s=1}^{n}e^{-xk\sin(\omega)}\cos((2\omega)+xk\cos(\omega))\Bigg) \\ &= \frac{\pi}{n}\Bigg(f'(\alpha)\beta\sum_{s=1}^{n}e^{-x\sin(\omega)}\cos((2\omega)+x\cos(\omega)) \\ &\quad + \sum_{k=2}^{\infty}\frac{f^{(k)}(\alpha)}{(k-1)!}\beta^k\sum_{s=1}^{n}e^{-xk\sin(\omega)}\cos((2\omega)+xk\cos(\omega)) \\ &\quad + \sum_{k=2}^{\infty}\frac{f^{(k)}(\alpha)}{(k-2)!}\beta^k\sum_{s=1}^{n}e^{-xk\sin(\omega)}\cos((2\omega)+xk\cos(\omega))\Bigg). \end{aligned} \tag{4.15}$$

As in the previous proof, by shifting the index, the equation above becomes:

$$\begin{aligned} &= \frac{\pi}{n}\Bigg(f'(\alpha)\beta\sum_{s=1}^{n}e^{-x\sin(\omega)}\cos((2\omega)+x\cos(\omega)) \\ &\quad + \sum_{k=1}^{\infty}\frac{f^{(k+1)}(\alpha)}{(k)!}\beta^{k+1}\sum_{s=1}^{n}e^{-x(k+1)\sin(\omega)}\cos((2\omega)+x(k+1)\cos(\omega)) \\ &\quad + \sum_{k=0}^{\infty}\frac{f^{(k+2)}(\alpha)}{(k)!}\beta^{k+2}\ \sum_{s=1}^{n}e^{-x(k+2)\sin(\omega)}\cos((2\omega)+x(k+2)\cos(\omega))\Bigg). \end{aligned} \tag{4.16}$$

Observe that we can expand the following expressions as follows:

$$\begin{aligned} &e^{-x(k+1)\sin(\omega)}\cos((2\omega)+x(k+1)\cos(\omega)) \\ &= e^{-x\sin(\omega)}e^{-xk\sin(\omega)}\cos(xk\cos(\omega))\cos(x\cos(\omega)+2\omega) \\ &\quad - e^{-x\sin(\omega)}e^{-xk\sin(\omega)}\sin(x\ k\cos(\omega))\sin(x\cos(\omega)+2\omega) \end{aligned} \tag{4.17}$$

and

$$
\begin{aligned}
&e^{-x(k+2)\sin(\omega)}\cos\left((2\omega)+x\left(k+2\right)\cos\left(\omega\right)\right)\\
&\quad=e^{-2x\sin(\omega)}e^{-xk\sin(\omega)}\cos\left(xk\cos\left(\omega\right)\right)\cos\left(2x\cos\left(\omega\right)+2\omega\right)\\
&\qquad\qquad-e^{-2x\sin(\omega)}e^{-xk\sin(\omega)}\sin\left(x\ k\cos\left(\omega\right)\right)\sin\left(2x\cos\left(\omega\right)+2\omega\right).
\end{aligned}
\tag{4.18}
$$

Substitute the equations above in Equation (4.16) results:

$$
\begin{aligned}
y''(x)=\frac{\pi}{n}\Bigg(&f'(\alpha)\,\beta\sum_{s=1}^{n}e^{-x\sin(\omega)}\cos\left((2\omega)+x\cos\left(\omega\right)\right)\\
&+\sum_{k=1}^{\infty}\frac{f^{(k+1)}(\alpha)}{(k)!}\beta^{k+1}\sum_{s=1}^{n}e^{-x\sin(\omega)}e^{-x\ k\sin(\omega)}\cos\left(xk\cos\left(\omega\right)\right)\cos\left(x\cos\left(\omega\right)+2\omega\right)-\\
&\qquad e^{-x\sin(\omega)}e^{-xk\sin(\omega)}\sin\left(x\ k\cos\left(\omega\right)\right)\sin\left(x\cos\left(\omega\right)+2\omega\right)\\
&+\sum_{k=0}^{\infty}\frac{f^{(k+2)}(\alpha)}{(k)!}\beta^{k+2}\sum_{s=1}^{n}e^{-2x\sin(\omega)}e^{-xk\sin(\omega)}\cos\left(xk\cos\left(\omega\right)\right)\cos\left(2x\cos\left(\omega\right)+2\omega\right)-\\
&\qquad e^{-2x\sin(\omega)}e^{-xk\sin(\omega)}\sin\left(x\ k\cos\left(\omega\right)\right)\sin\left(2x\cos\left(\omega\right)+2\omega\right)\Bigg).
\end{aligned}
\tag{4.19}
$$

By applying the linearity property of finite series, we obtain:

$$
\begin{aligned}
y''(x)=\frac{\pi}{n}\Bigg(&f'(\alpha)\,\beta\sum_{s=1}^{n}e^{-x\sin(\omega)}\cos\left((2\omega)+x\cos\left(\omega\right)\right)+\\
&\sum_{s=1}^{n}e^{-x\sin(\omega)}\cos\left(x\cos\left(\omega\right)+2\omega\right)\sum_{k=1}^{\infty}\frac{f^{(k+1)}(\alpha)}{(k)!}\beta^{k+1}e^{-x\ k\sin(\omega)}\cos\left(xk\cos\left(\omega\right)\right)-\\
&\sum_{s=1}^{n}e^{-x\sin(\omega)}\sin\left(x\cos\left(\omega\right)+2\omega\right)\sum_{k=1}^{\infty}\frac{f^{(k+1)}(\alpha)}{(k)!}\beta^{k+1}e^{-xk\sin(\omega)}\sin\left(x\ k\cos\left(\omega\right)\right)+\\
&\sum_{s=1}^{n}e^{-2x\sin(\omega)}\cos\left(2x\cos\left(\omega\right)+2\omega\right)\sum_{k=0}^{\infty}\frac{f^{(k+2)}(\alpha)}{(k)!}\beta^{k+2}e^{-xk\sin(\omega)}\cos\left(xk\cos\left(\omega\right)\right)-\\
&\sum_{s=1}^{n}e^{-2x\sin(\omega)}\sin\left(2x\cos\left(\omega\right)+2\omega\right)\sum_{k=0}^{\infty}\frac{f^{(k+2)}(\alpha)}{(k)!}\beta^{k+2}e^{-xk\sin(\omega)}\sin\left(xk\cos\left(\omega\right)\right)\Bigg).
\end{aligned}
\tag{4.20}
$$

Now, by using the Equations (3.7), (3.8), and the mathematical fact in Equation (3.2), Equation (4.20) becomes:

$$
\begin{aligned}
y''(x) = \frac{\pi}{n}\Bigg[& f'(\alpha)\,\beta \sum_{s=1}^{n} e^{-x\sin(\omega)} \cos\left((2\omega) + x\cos(\omega)\right) \\
& + \beta \sum_{s=1}^{n} \Bigg(\frac{1}{2} e^{-x\sin(\omega)} \cos\left(x\cos(\omega) + 2\omega\right) \\
& \qquad \frac{\partial}{\partial\alpha} \left(f\left(\alpha + \beta e^{ix\cos(\omega) - x\sin(\omega)}\right) + f\left(\alpha + \beta e^{-ix\ \cos(\omega) - x\sin(\omega)}\right) - 2f(\alpha) \right) \Bigg) \\
& - \sum_{s=1}^{n} \Bigg(\frac{1}{2i} e^{-x\sin(\omega)} \sin\left(x\cos(\omega) + 2\omega\right) \\
& \qquad \frac{\partial}{\partial\alpha} \left(f\left(\alpha + \beta e^{ix\cos(\omega) - x\sin(\omega)}\right) - f\left(\alpha + \beta e^{-ix\cos(\omega) - x\sin(\omega)}\right) \right) \Bigg) \\
& + \beta^2 \sum_{s=1}^{n} \Bigg(\frac{1}{2} e^{-2x\sin(\omega)} \cos\left(2x\cos(\omega) + 2\omega\right) \\
& \qquad \frac{\partial^2}{\partial\alpha^2} \left(f\left(\alpha + \beta e^{ix\cos(\omega) - x\sin(\omega)}\right) + f\left(\alpha + \beta e^{-ix\cos(\omega) - x\sin(\omega)}\right) - 2f(\alpha) \right) \Bigg) \\
& - \beta^2 \sum_{s=1}^{n} \Bigg(\frac{1}{2i} e^{-2x\sin(\omega)} \sin\left(2x\cos(\omega) + 2\omega\right) \\
& \qquad \frac{\partial^2}{\partial\alpha^2} \left(f\left(\alpha + \beta e^{ix\cos(\omega) - x\sin(\omega)}\right) - f\left(\alpha + \beta e^{-ix\cos(\omega) - x\sin(\omega)}\right) \right) \Bigg) \Bigg]. \qquad (4.21)
\end{aligned}
$$

As in the previous proof, we let $\psi(\alpha,\omega,x) = f\left(\alpha + \beta e^{ix\cos(\omega) - x\sin(\omega)}\right)$ and $\phi(\alpha,\omega,x) = f\left(\alpha + \beta e^{-ix\cos(\omega) - x\sin(\omega)}\right)$, as well as performing some simplifying calculations, we obtain:

$$
\begin{aligned}
y''(x) = \frac{\pi}{2n}\Bigg[& \sum_{s=1}^{n} \beta e^{-x\sin(\omega)} \frac{\partial}{\partial\alpha} \Big(\cos\left(x\cos(\omega) + 2\omega\right) \left(\psi(\alpha,\omega,x) + \phi(\alpha,\omega,x)\right) \\
& \qquad - \frac{1}{i} \sin\left(x\cos(\omega) + 2\omega\right) \left(\psi(\alpha,\omega,x) - \phi(\alpha,\omega,x)\right) \Big) \\
& + \beta^2 e^{-2x\sin(\omega)} \frac{\partial^2}{\partial\alpha^2} \Big(\cos\left(2x\cos(\omega) + 2\omega\right) \left(\psi(\alpha,\omega,x) + \phi(\alpha,\omega,x)\right) \\
& \qquad - \frac{1}{i} \sin\left(2x\cos(\omega) + 2\omega\right) \left(\psi(\alpha,\omega,x) - \phi(\alpha,\omega,x)\right) \Big) \Bigg], \qquad (4.22)
\end{aligned}
$$

and this completes the proof of Equation (4.3). □

Now we introduce the theorem below, a general rule to find higher-order derivatives.

Theorem 4.2. *Let f be an analytic function in a disc D centered at $\alpha \in \mathbb{R}$ if we have $y(x)$ to be defined as in Equation (4.1). Then,*

$$
\begin{aligned}
y^{(2m)}(x) = \sum_{s=1}^{n} \Bigg(& \beta e^{-x\sin(\omega)} \frac{\partial}{\partial\alpha} \Big(\cos\left(x\cos(\omega) + 2m\ \omega\right) \left(\psi(\alpha,\omega,x) + \phi(\alpha,\omega,x)\right) \\
& \qquad - \frac{1}{i} \sin\left(x\cos(\omega) + 2m\ \omega\right) \left(\psi(\alpha,\omega,x) - \phi(\alpha,\omega,x)\right) \Big) \\
& \beta^{2m} e^{-2m\ x\sin(\omega)} \frac{\partial^{2m}}{\partial\alpha^{2m}} \Big(\cos\left(\ 2m\,x\cos(\omega) + 2m\omega\right) \left(\psi(\alpha,\omega,x) + \phi(\alpha,\omega,x)\right) \\
& \qquad - \frac{1}{i} \sin\left(2m\ x\cos(\omega) + 2m\ \omega\right) \left(\psi(\alpha,\omega,x) - \phi(\alpha,\omega,x)\right) \Big) \\
& + \sum_{i=2}^{2m-1} (a_j)\, \beta^j e^{-jx\sin(\omega)} \frac{\partial^j}{\partial\alpha^j} \Big(\cos\left(jx\cos(\omega) + 2m\ \omega\right) \left(\psi(\alpha,\omega,x) + \phi(\alpha,\omega,x)\right) \\
& \qquad - \frac{1}{i} \sin\left(j\ x\cos(\omega) + 2m\ \omega\right) \left(\psi(\alpha,\omega,x) - \phi(\alpha,\omega,x)\right) \Big) \Bigg) \qquad (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
y^{(2m-1)}(x) = \frac{(-1)^{m+1}\pi}{2n}\sum_{s=1}^{n}\Bigg(&\beta e^{-x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\frac{1}{i}\cos\left(x\cos(\omega)+(2m-1)\omega\right)\left(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x)\right)\\
&\qquad\qquad+\sin\left(x\cos(\omega)+(2m-1)\omega\right)\left(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x)\right)\Big)\\
&+\beta^{2m-1}e^{-(2m-1)x\sin(\omega)}\frac{\partial^{2m-1}}{\partial\alpha^{2m-1}}\Big(\frac{1}{i}\cos\left((2m-1)x\cos(\omega)+(2m-1)\omega\right)\left(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x)\right)\\
&\qquad\qquad+\sin\left((2m-1)x\cos(\omega)+(2m-1)\omega\right)\left(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x)\right)\Big)\\
&+\sum_{j=2}^{2m-2}(a_j)\beta^{j}e^{-jx\sin(\omega)}\frac{\partial^{j}}{\partial\alpha^{j}}\Big(\frac{1}{i}\cos\left(jx\cos(\omega)+(2m-1)\omega\right)\left(\psi(\alpha,\omega,x)\right)\\
&\qquad\qquad+\sin\left(jx\cos(\omega)+(2m-1)\omega\right)\left(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x)\right)\Big)\Bigg),
\end{aligned}
\tag{4.24}
$$

where $n\in\mathbb{N}$, $m\in\mathbb{Z}_+$, $m\le n, x,\beta>0$, $\omega=\frac{(2s-1)\pi}{2n}$, $\psi(\alpha,\omega,x)=f\left(\alpha+\beta e^{ix\cos(\omega)-x\sin(\omega)}\right)$, *and* $\phi(\alpha,\omega,x)=f\left(\alpha+\beta e^{-ix\ \cos(\omega)-x\sin(\omega)}\right)$. *The coefficient* a_j *can be found using the coefficients of an auxiliary equation and sub-equations that are illustrated in Appendix 1.*

Proof. The proof of the theorem is done by repeating the differentiation process $2m$ and $2m-1$ times and using Theorem 3.1. □

Corollary 4.2.1. *Let* f *be an analytic function around* α, *where* $\alpha\in\mathbb{R}$. *If we have*

$$
y(x)=\frac{\pi}{2n}\sum_{s=1}^{n}\left(2f(\alpha+\beta)-\left(f\left(\alpha+\beta e^{ix\cos(\omega)-x\sin(\omega)}\right)+f\left(\alpha+\beta e^{-ix\cos(\omega)-x\sin(\omega)}\right)\right)\right),
$$

then,

$$
\begin{aligned}
y'''(x)=\frac{-\pi}{2n}\sum_{s=1}^{n}\Bigg(&\beta e^{-x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\frac{1}{i}\cos\left(x\cos(\omega)+3\omega\right)\left(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x)\right)\\
&\qquad+\sin\left(x\cos(\omega)+3\omega\right)\left(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x)\right)\Big)\\
&+\beta^{3}e^{-3x\sin(\omega)}\frac{\partial^{3}}{\partial\alpha^{3}}\Big(\frac{1}{i}\cos\left(3x\cos(\omega)+3\omega\right)\left(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x)\right)\\
&\qquad+\sin\left(3x\cos(\omega)+3\right)\left(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x)\right)\Big)\\
&+3\beta^{2}e^{-2x\sin(\omega)}\frac{\partial^{2}}{\partial\alpha^{2}}\Big(\frac{1}{i}\cos\left(2x\cos(\omega)+3\omega\right)\left(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x)\right)\\
&\qquad+\sin\left(2x\cos(\omega)+3\omega\right)\left(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x)\right)\Big)\Bigg)
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
y^{(4)} = \frac{-\pi}{2n}\sum_{s=1}^{n}\Bigg(&\beta e^{-\ x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\cos\left(x\cos\left(\omega\right)+4\omega\right)\left(\psi\left(\alpha,\omega,x\right)+\phi\left(\alpha,\omega,x\right)\right)\\
&\qquad\qquad -\frac{1}{i}\sin\left(x\cos\left(\omega\right)+4\omega\right)\left(\psi\left(\alpha,\omega,x\right)-\phi\left(\alpha,\omega,x\right)\right)\Big)\\
&+\beta^4 e^{-4\ x\sin(\omega)}\frac{\partial^4}{\partial\alpha^4}\Big(\cos\left(4x\cos\left(\omega\right)+4\omega\right)\left(\psi\left(\alpha,\omega,x\right)+\phi\left(\alpha,\omega,x\right)\right)\\
&\qquad\qquad -\frac{1}{i}\sin\left(2mx\cos\left(\omega\right)+2m\omega\right)\left(\psi\left(\alpha,\omega,x\right)-\phi\left(\alpha,\omega,x\right)\right)\Big)\\
&+7\beta^2 e^{-2\ x\sin(\omega)}\frac{\partial^2}{\partial\alpha^2}\Big(\cos\left(2\ x\cos\left(\omega\right)+4\ \omega\right)\left(\psi\left(\alpha,\omega,x\right)+\phi\left(\alpha,\omega,x\right)\right)\\
&\qquad\qquad -\frac{1}{i}\sin\left(2\ x\cos\left(\omega\right)+4\ \omega\right)\left(\psi\left(\alpha,\omega,x\right)-\phi\left(\alpha,\omega,x\right)\right)\Big)\\
&+6\beta^3 e^{-3\ x\sin(\omega)}\frac{\partial^3}{\partial\alpha^3}\Big(\cos\left(3\ x\cos\left(\omega\right)+4\ \omega\right)\left(\psi\left(\alpha,\omega,x\right)+\phi\left(\alpha,\omega,x\right)\right)\\
&\qquad\qquad -\frac{1}{i}\sin\left(3\ x\cos\left(\omega\right)+4\ \omega\right)\left(\psi\left(\alpha,\omega,x\right)-\phi\left(\alpha,\omega,x\right)\right)\Big)\Bigg),
\end{aligned}
\tag{4.26}
$$

where $n\in\mathbb{N}$, $\beta>0$, $\omega=\omega(s)=\frac{(2s-1)\pi}{2n}$, $\psi\left(\alpha,\omega,x\right)=f\left(\alpha+\beta e^{ix\cos(\omega)-x\sin(\omega)}\right)$, *and* $\phi\left(\alpha,\omega,x\right)=f\left(\alpha+\beta e^{-ix\ \cos(\omega)-x\sin(\omega)}\right)$.

Proof. The results were obtained directly by setting $m=2$ in Theorem 4.1 and using *Appendix 1* to find a_j. □

5 Infinite Generators: Linear and Non-linear Cases

In this section, we will present a set of generators that are derived from the theorems discussed earlier. These generators are used for constructing and solving ODEs, and they possess unique characteristics that distinguish them from other methods of ODE solving.

The generators that we will showcase in this section are applicable to both linear and non-linear cases of ODEs. They are based on the fundamental theorems discussed in section (3), and they provide solutions to a variety of ODE problems. The generators we present are limited to third-order derivatives for simplicity, but it should be noted that the main theorems can be used to generate an infinite number of generators for derivatives of order $2m$ and $2m-1$.

To create these generators, we can follow the algorithm described below:

i. Identify the problem: Understand the nature of the problem and the type of generator needed for its solution.

ii. Select a suitable theorem or mathematical framework: Choose a theorem or framework that provides a foundation for constructing the generator, considering the problem's context and requirements.

iii. Define the generator parameters and functions: Determine the order of differentiation n, the functions $f(x)$, $\psi(x)$, and $\phi(x)$, and any additional parameters ($\alpha,\ \beta,\ M,\ q,\ v,\ a,\ \omega$) required by the chosen theorem or framework.

iv. Apply the theorem or mathematical framework: Use the chosen theorem or framework to derive a functional form for $y(x)$ and its derivatives $(y\left(x\right),\ y'\left(x\right),\ y''\left(x\right),\ ...,\ y^{(n)}\left(x\right))$ based on the defined parameters and functions.

v. Combine the functional forms: Create the generator by combining the derived functional forms using appropriate coefficients or operations, as dictated by the chosen theorem or framework.

vi. Validate the generator: Ensure that the constructed generator satisfies the requirements of the problem and the constraints imposed by the chosen theorem or framework.

vii. Adjust and refine the generator (if necessary): If the initial generator does not adequately address the problem, modify the parameters, functions, or functional forms and repeat steps *iv* - *vi* until a suitable generator is obtained.

The general form of the generator can be expressed as:

$$G\left(x,\ y\left(x\right),\ y'\left(x\right),\ y''\left(x\right),\ ...,\ y^{(n)}\left(x\right)\right)\ =\ F(\alpha,\ \beta,\ M,\ x,\ n,\ q,\ v,\ a,\ \omega,\ f(x),\ \psi(x),\ \phi(x)).$$

Where:

- G is the general generator function that takes x, y, and n derivatives of y as its input arguments.
- F is a function that combines the parameters and functions to generate the desired output.
- x is the independent variable.
- y is the dependent variable.
- $y\left(x\right),\ y'\left(x\right),\ y''\left(x\right),\ ...,\ y^{(n)}\left(x\right)$ are the first, second, ..., and n^{th} derivatives of y with respect to x.
- $\alpha,\ \beta,\ M,\ n,\ q,\ v,\ a$, and ω are parameters for the generator, which might represent constants or coefficients in the generator function.
- $f(x),\ \psi(x)$ and $\phi(x)$ are functions that may appear in the generator, with x as their input.

For each generator, we will provide its initial conditions, solution, and unique characteristics. The initial conditions are the starting values for the dependent variable and its derivatives, which are used to produce the solution. The solution represents the standard version of the ODE that can be addressed by the generator. The unique characteristics of the generator refer to its specific properties, such as its range of applicability, its stability, and its accuracy.

We have provided a table below showcasing some of the generators that can be created through theorems, bearing in mind that an infinite number of these generators can be created using the theorems. The table includes the generator's functional form, initial conditions, solution, and any unique characteristics that may be of interest. By selecting suitable functional forms and setting appropriate values for n, α, β, and other parameters, specific generators can be created for a variety of situations.

6 Master Generators:

In this section, we present some master generators to construct and solve ODE which depends on the fundamental theorems in section (3). An infinite number of generators could be obtained using the main theorems, and different kinds of equations could be solved using these generators, in addition, using these generators can create unlimited numbers of ODE that cannot be solved analytically using software or by other methods and techniques. For simplicity in dealing with the main theorems, we will take generators up to the third derivative, and some special cases, knowing that using the main theorems it is possible to produce an unlimited number of generators for derivatives of order $2m$ and $2m-1$.

Table 1: Some of the Linear Generators

Gen. #	Generator	Initial Condition/s	Solution	Notes
1	$y''(x)+y(x)=\pi\Big(f(\alpha+\beta)-f(\alpha+\beta e^{-x})+M-\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x}))-\beta^2e^{-2x}\frac{\partial^2}{\partial\alpha^2}(f(\alpha+\beta e^{-x}))\Big)$	$y(0)=\pi M$, $y'(0)=\pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x)=\pi\left(f(\alpha+\beta)-f(\alpha+\beta e^{-x})+M\right)$	Using theorem (4.2), setting $n=1$
2	$y''(x)+y'(x)=-\pi\beta^2e^{-2x}\frac{\partial^2}{\partial\alpha^2}f(\alpha+\beta e^{-x})$	$y(0)=\pi M$, $y'(0)=\pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x)=\pi\left(f(\alpha+\beta)-f(\alpha+\beta e^{-x})+M\right)$	Using theorem (4.2), setting $n=1$
3	$y(x)+y'(x)=\pi\left(f(\alpha+\beta)-f(\alpha+\beta e^{-x})+M+\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x}))\right)$	$y(0)=\pi M$	$y(x)=\pi\left(f(\alpha+\beta)-f(\alpha+\beta e^{-x})+M\right)$	Using theorem (4.2), setting $n=1$
4	$y''(x)+\frac{y(x)}{a}-y(x)=\pi\Big(f(\alpha+\beta e^{-x})-f(\alpha+\beta e^{-x/a})-\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x}))-\beta^2e^{-2x}\frac{\partial^2}{\partial\alpha^2}(f(\alpha+\beta e^{-x}))\Big)$	$y(0)=\pi M$, $y'(0)=\pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x)=\pi\left(f(\alpha+\beta)-f(\alpha+\beta e^{-x})+M\right)$	Using theorem (4.2), setting $n=1$
5	$y(x/a)+y'(x)=\pi\left(f(\alpha+\beta)-f(\alpha+\beta e^{-x/a})+M+\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x}))\right)$	$y(0)=\pi M$	$y(x)=\pi\left(f(\alpha+\beta)-f(\alpha+\beta e^{-x})+M\right)$	Using theorem (4.2), setting $n=1$
6	$y'''(x)+y(x)=-\frac{\pi}{4}\sum_{s=1}^{2}\Bigg(\Big(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x)-2f(\alpha+\beta)\Big)$ $+\beta e^{-x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\frac{1}{i}\cos(x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)$ $+\beta^3e^{-3x\sin(\omega)}\frac{\partial^3}{\partial\alpha^3}\Big(\frac{1}{i}\cos(3x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(3x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)$ $+3\beta^2e^{-2x\sin(\omega)}\frac{\partial^2}{\partial\alpha^2}\Big(\frac{1}{i}\cos(2x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(2x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)\Bigg)$	$y(0)=0$ $y'(0)=\frac{\pi}{\sqrt{2}}\beta\frac{\partial}{\partial\alpha}f(\alpha+\beta)$ $y''(0)=0$	$y(x)=\pi(f(\alpha+\beta)-f(\alpha+\beta e^{-x}))$	Using corollary (4.2.1), setting $n=2$. $\omega=\omega(s)=(2s-1)\pi/4$, $\psi(\alpha,\omega,x)=f(\alpha+\beta e^{i(x\cos(\omega)-x\sin(\omega))})$, $\phi(\alpha,\omega,x)=f(\alpha+\beta e^{-i(x\cos(\omega)+x\sin(\omega))})$
7	$y'''(x)+y'(x)=-\frac{\pi}{4}\sum_{s=1}^{2}\Bigg(-\beta e^{-x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\frac{1}{i}\cos(x\cos(\omega)+\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(x\cos(\omega)+\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)+\beta e^{-x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\frac{1}{i}\cos(x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)+\beta^3e^{-3x\sin(\omega)}\frac{\partial^3}{\partial\alpha^3}\Big(\frac{1}{i}\cos(3x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(3x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)+3\beta^2e^{-2x\sin(\omega)}\frac{\partial^2}{\partial\alpha^2}\Big(\frac{1}{i}\cos(2x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(2x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)\Bigg)$	$y(0)=0$ $y'(0)=\frac{\pi}{\sqrt{2}}\beta\frac{\partial}{\partial\alpha}f(\alpha+\beta)$ $y''(0)=0$	$y(x)=\pi(f(\alpha+\beta)-f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n=2$. and corollary (4.2.1), setting$n=2$ $\omega=\omega(s)=(2s-1)\pi/4$, $\psi(\alpha,\omega,x)=f(\alpha+\beta e^{i(x\cos(\omega)-x\sin(\omega))})$, $\phi(\alpha,\omega,x)=f(\alpha+\beta e^{-i(x\cos(\omega)+x\sin(\omega))})$
8	$y'''(x)+y''(x)=-\frac{\pi}{4}\sum_{s=1}^{2}\Bigg(-\beta e^{-x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\cos(x\cos(\omega)+2\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))$ $+\frac{1}{i}\sin(x\cos(\omega)+2\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))\Big)+\beta^2e^{-2x\sin(\omega)}\frac{\partial^2}{\partial\alpha^2}\Big(\cos(2x\cos(\omega)+2\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))$ $+\frac{1}{i}\sin(2x\cos(\omega)+2\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))\Big)+\beta e^{-x\sin(\omega)}\frac{\partial}{\partial\alpha}\Big(\frac{1}{i}\cos(x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)+\beta^3e^{-3x\sin(\omega)}\frac{\partial^3}{\partial\alpha^3}\Big(\frac{1}{i}\cos(3x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(3x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)+3\beta^2e^{-2x\sin(\omega)}\frac{\partial^2}{\partial\alpha^2}\Big(\frac{1}{i}\cos(2x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)-\phi(\alpha,\omega,x))$ $+\sin(2x\cos(\omega)+3\omega)(\psi(\alpha,\omega,x)+\phi(\alpha,\omega,x))\Big)\Bigg)$	$y(0)=0$ $y'(0)=\frac{\pi}{\sqrt{2}}\beta\frac{\partial}{\partial\alpha}f(\alpha+\beta)$ $y''(0)=0$	$y(x)=\pi(f(\alpha+\beta)-f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n=2$. and corollary (4.2.1), setting$n=2$ $\omega=\omega(s)=(2s-1)\pi/4$, $\psi(\alpha,\omega,x)=f(\alpha+\beta e^{i(x\cos(\omega)-x\sin(\omega))})$, $\phi(\alpha,\omega,x)=f(\alpha+\beta e^{-i(x\cos(\omega)+x\sin(\omega))})$

*This table shows 8 linear generators that were created from the Master Theorem. We have mentioned some generators that can be created through theories, bearing in mind that an infinite number of these generators can be created using theorems.

Table 2: Some of the Non-Linear Generators

Gen.#	Generator	Initial conditions	Solution	Notes
1	$(y''(x))^q + y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x})) + M + \pi^q(-\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x}) - \beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}f(\alpha+\beta e^{-x}))^q$	$y(0) = \pi M$, $y'(0) = \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
2	$(y''(x))^q + (y'(x))^v = \pi^q(-\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x}) - \beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}f(\alpha+\beta e^{-x}))^q + \pi^v(\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x}))^v$	$y(0) = \pi M$, $y'(0) = \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
3	$y(x) + (y'(x))^v = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x})) + M + \pi^v(\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x}))^v$	$y(0) = \pi M$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
4	$(y''(x))^q + y(x/a) - y(x) = \pi(f(\alpha+\beta e^{-x}) - f(\alpha+\beta e^{-x/a})) - \pi^q(\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x})) - \beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}(f(\alpha+\beta e^{-x})))^q$	$y(0) = \pi M$, $y'(0) = \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
5	$y(x/a) + (y'(x))^v = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x/a})) + \pi^v(\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x})))^v$	$y(0) = \pi M$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
6	$\sin(y''(x)) + y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x})) + \sin(\pi(-\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x}) - \beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}f(\alpha+\beta e^{-x})))$	$y(0) = \pi M$, $y'(0) = \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
7	$e^{y''(x)} + e^{y'(x)} = \exp(\pi(-\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x})) - \beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}(f(\alpha+\beta e^{-x})))) + \exp(\pi(\beta e^{-x}\frac{\partial}{\partial\alpha}(f(\alpha+\beta e^{-x}))))$	$y(0) = \pi M$, $y'(0) = \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
8	$y(x) + e^{y'(x)} = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x})) + \exp(\pi(\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x})))$	$y(0) = \pi M$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
9	$e^{y''(x)} + y(x/a) - y(x) = \pi(f(\alpha+\beta e^{-x}) - f(\alpha+\beta e^{-x/a}) + 2M + \exp(\pi(-\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x}) - \beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}f(\alpha+\beta e^{-x}))))$	$y(0) = \pi M$, $y'(0) = \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$
10	$y(x/a) + \ln(y'(x)) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x/a}) + M + \ln(\beta e^{-x}\frac{\partial}{\partial\alpha}f(\alpha+\beta e^{-x})))$	$y(0) = \pi M$, $y'(0) = \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha+\beta))$	$y(x) = \pi(f(\alpha+\beta) - f(\alpha+\beta e^{-x}))$	Using theorem (4.1), setting $n = 1$

*This Table shows non-linear generators obtained from the theorems, knowing that an unlimited number of non-linear generators could be obtained using the theorems.

7 Generators Code

In this section, we intend to demonstrate the potential applications of these generators in software programs like Mathematica and Python. These programs often struggle to solve a significant number of the ODEs generated by the generators listed in tables (1) and (2).
The code takes several input parameters including function $f(z)$, a represented by α, b represented by β, variables x and y, complex numbers z, a sum counter denoted by s, and powers p and q for the non-linear differential equations. The output of the code is the generator itself.
A sample of the code is presented below, while the complete code can be accessed via the provided GitHub link: Master Generators Code.

7.1 Python Sample Code

The code below takes any $f(z)$ and uses the second Non-linear Generator in Table (2) to generate a differential equation and its solution.

Input:

```
from sympy import *
import sympy as sm
import math
import numpy as np
from IPython.display import display, Markdown, Latex
a,b,x,y,M,z,i,s ,c, v ,q, pi= sm.symbols('a b x y M z i s c v q pi')
function = "cos(z)"
def Complex_Function(z):
    F_ofZ=eval(function)
    return F_ofZ
F_ofAandB = Complex_Function(a+b)
F_ofAandBe = Complex_Function(a+b*exp(-x))
F_ofandbexa = Complex_Function(a+b*exp(-x/c))
DerivativeN1_Func = sm.diff(F_ofAandBe , a)
DerivativeN2_Func = sm.diff(DerivativeN1_Func , a)
print("The function you entered is")
display(Complex_Functi on(z))
print("\033[1m The Non-Linear Generator Number 2 will be \033[0m")
if DerivativeN1_Func ==0:
```

```
    display(Latex(r"$(y''(x))^q + (y'(x))^v = 0$"))
elif DerivativeN2_Func ==0:
    display(Latex(r"$(y''(x))^q + (y'(x))^v = {\pi}^q ($"), (-b*exp(-x)*
    DerivativeN1_Func)**q , Latex(r"$+\,{\pi}^v$") ,
    (b*exp(-x)*DerivativeN1_Func)**v)
else:
    display(Latex(r"$(y''(x))^q + (y'(x))^v = (\pi)^q ($"),
    (-b*exp(-x) * DerivativeN1_Func - b**2 * exp(-2*x) *
    DerivativeN2_Func)**q
    ,Latex(r"$+\,{\pi}^v$"), (b*exp(-x)* DerivativeN1_Func)**v )
```

Output:

```
    The function you entered is
```

$\cos(z)$

```
    The Non-Linear Generator Number 2 will be
```

$$(y''(x))^q + (y'(x))^v = (\pi)^q \left(\left(b^2 e^{-2x} \cos\left(a + be^{-x}\right) + be^{-x} \sin\left(a + be^{-x}\right) \right)^q + \pi^v \left(-be^{-x} \sin\left(a + be^{-x}\right) \right)^v \right.$$

7.2 Mathematica

This is a sample code for any generator that includes a sum with complex numbers.

Input

```
f[z_] = z^2;
omega = ((2 s - 1)Pi)/4 ;
psi = f[a + b E[I x Cos[omega] - x Sin[omega] ]];
phi = f[a + b E[-I x Cos[omega] - x Sin[omega] ]];
psiplusphi = (psi + phi);
psiminusphi = (psi - phi);
FofAandB = f[a + b];
DerivativeN1Plus = D[psiplusphi , {a, 1}];
DerivativeN1Minus = D[psiminusphi , {a, 1}];
DerivativeN2Plus = D[psiplusphi , {a, 2}];
DerivativeN2Minus = D[psiminusphi , {a, 2}];
DerivativeN3Plus = D[psiplusphi , {a, 3}];
DerivativeN3Minus = D[psiminusphi , {a, 3}];
FunctionInTheSum =
  psiplusphi - 2 FofAandB +
   b Exp[-x Sin[omega]] * -I Cos[x Cos[omega] + 3 omega] *
    DerivativeN1Minus +
   b Exp[-x Sin[omega]] * Sin[x Cos[omega] + 3 omega] *
    DerivativeN1Plus +
   b^3 Exp[-3 x Sin[omega]] (-I Cos[3 x Cos[omega] + 3 omega] *
      DerivativeN3Minus ) +
   3 b^2 Exp[-2 x Sin[
       omega]] ( (-I Cos[
           2 x Cos[omega] + 3 omega]) DerivativeN2Minus +
      Sin[2 x Cos[omega] + 3 omega] DerivativeN2Plus);
TheSum = Sum[FunctionInTheSum, {s, 1, 2}];
cc = Re[TheSum];
FullSimplify[cc]
```

Output

$$2\Bigg(3Im\Bigg[b^2e^{\frac{-x}{\sqrt{2}}}\left(e^{-(-1)^{\frac{1}{4}}x}-e^{(-1)^{\frac{3}{4}}x}\right)\sin\left(\frac{\pi}{4}+\frac{x}{\sqrt{2}}\right)\Bigg]$$
$$Re\Bigg[-2(a+b)^2+12b^2e^{-\sqrt{2}x}\cos\left(\frac{\pi}{4}+\sqrt{2}x\right)+\left(a+be^{-(-1)^{\frac{1}{4}}x}\right)^2+\left(a+be^{-(-1)^{\frac{3}{4}}x}\right)^2$$
$$+2be^{\frac{-x}{\sqrt{2}}}\cos\left(\frac{\pi}{4}+\frac{x}{\sqrt{2}}\right)\left(2a+b\left(e^{-(-1)^{\frac{1}{4}}x}+e^{(-1)^{\frac{3}{4}}x}\right)\right)\Bigg]\Bigg).$$

8 Applications and Examples

The Master generators are powerful tools for solving a wide variety of differential equations that cannot be solved using traditional methods. In this section, we will explore various examples of how these generators can be applied to real-world problems in physics, engineering, and other fields.
The examples presented here demonstrate the versatility and usefulness of these non-linear generators in solving complex problems that would otherwise be difficult, if not impossible, to solve. By identifying the appropriate generator for a given problem, one can easily obtain a solution to the corresponding differential equation, bypassing the need for lengthy and often complicated analytical techniques.
From solving differential equations in fluid mechanics to modelling chaotic systems in physics and engineering, the generators in Table (1) and Table (2) have a wide range of applications. Whether in academia or industry, these generators are powerful tools for researchers and practitioners alike, enabling them to solve problems that were once considered unsolvable.
In the following examples, we will see how these generators can be used to solve a variety of differential equations and demonstrate their practical applications in solving real-world problems.
The generators derived from the theorems in the presented work can be applied to a variety of problems in mathematics, physics, engineering, and other disciplines involving ordinary differential equations (ODEs). Their primary purpose is to provide a systematic approach to constructing and solving linear and nonlinear ODEs. Some specific applications of these generators include [20-26]:

i. Mathematical modeling: The generators can be used to model a wide range of phenomena, such as population dynamics, chemical reactions, and the spread of diseases. By using the generators, researchers can create ODEs that describe the behavior of the systems under study and solve them to understand the underlying processes.

ii. Physics: In physics, ODEs are commonly used to describe the motion of particles, wave propagation, heat transfer, fluid dynamics, and other physical processes. The generators can help physicists construct and solve ODEs that describe these phenomena and analyze their behavior.

iii. Engineering: Engineers often rely on ODEs to model and analyze various systems, such as mechanical vibrations, electrical circuits, and control systems. The generators can be employed to develop and solve ODEs that provide insights into the behavior of these systems, which can be used for design, optimization, and control purposes.

iv. Economics: ODEs play a significant role in economic modeling, including growth models, consumer behavior, and market dynamics. The generators can be used to develop and analyze economic models to better understand market forces, make predictions, and inform policy decisions.

v. Numerical methods: The generators can also be used to develop and analyze numerical methods for solving ODEs. By providing a framework for constructing ODEs and their solutions, the generators can help researchers develop more efficient and accurate numerical algorithms for solving ODEs in various fields.

vi. Education and research: The generators offer a valuable tool for teaching and research in mathematics and other disciplines. They provide a systematic approach to constructing and solving ODEs, which can help students and researchers develop a deeper understanding of the theory and applications of ODEs.

Example (1): Consider a simple pantograph system, which consists of a mechanical arm attached to an electric power line. The pantograph arm moves up and down as the train moves, making contact with the power line to supply electrical power to the train. The movement of the pantograph arm is described by the equation of motion:

$$y''\left(x\right)+\ y\left(\frac{x}{a}\right)\ -\ y\left(x\right)=\ \pi\,\beta e^{-\frac{x}{a}}, \tag{8.1}$$

where $y(x)$ is the height of the pantograph arm at time $x, y''(x)$ is the acceleration of the arm at time x, and $\frac{x}{a}$ is the damping coefficient. The terms on the right-hand side of the equation represent external forces acting on the arm, such as air resistance, friction, and tension in the power line. The initial conditions for the pantograph system are:

$$y(0)=\pi M$$
$$y'(0)=\pi\beta,$$

where M is the height of the arm when it is stationary, and β is the initial velocity of the arm.

Solution: Using *Generator 4* in Table (1), and setting $f\left(z\right)=z$, we get, the desired ODE, under the given initial conditions.
Using the given exact solution:

$$y(x)\ =\ \pi\left(f\left(\alpha\ +\ \beta\right)-\ f\left(\alpha\ +\ \beta e^{-x}\right)+\ M\right).$$

Setting $f(z)=z$, we get

$$y\left(x\right)=\pi\left(\beta-\beta e^{-x}+M\right). \tag{8.2}$$

The exact solution to the given pantograph ODE does not contain the damping coefficient (α), it indicates that the damping effect is somehow absorbed or canceled out in the process of solving the equation. This could mean that the impact of damping on the motion of the pantograph arm is not explicitly visible in the final solution.

As we can see, the solution shows the behavior of the pantograph arm's height, $y(x)$, as a function of time x. The solution contains the initial height M and the initial velocity β, but the damping coefficient (α) is absent.

This might suggest that the pantograph system, under the given conditions, experiences a type of motion that is less sensitive to the damping coefficient or that the effect of damping has been implicitly incorporated into other terms of the solution. Keep in mind that the absence of the damping coefficient in the exact solution does not necessarily mean that damping has no effect on the system's motion. The relationship between the damping coefficient and the system's response might be more complex or hidden within the given solution.

To compare the exact solution with the numerical solution in Mathematica. The numerical solution was obtained using the NDSolve function in Mathematica, with the parameters $a=1$ and $\beta=1$, The resulting numerical solution was then compared to the exact solution, which was obtained by solving the ODE analytically for $M=0$.

Table (3) presents the absolute error between the exact and numerical solutions for various values of x, where it can be seen that the error is relatively small, indicating a good agreement between the two solutions. Additionally, Figure (1) shows a plot of the exact and numerical solutions for different M values, where it can be observed that the two curves overlap, further supporting the accuracy of the numerical solution.

Table (3): The absolute error between the exact and numerical solutions for various values of x for the pantograph system in Example (1).

x	Absolute error	x	Absolute error
0.	0.	1.5	$1.0825 * 10^{-8}$
0.1	$1.81707 * 10^{-8}$	1.6	$1.10577 * 10^{-8}$
0.2	$5.25248 * 10^{-9}$	1.7	$1.20443 * 10^{-8}$
0.3	$2.00906 * 10^{-8}$	1.8	$1.37763 * 10^{-8}$
0.4	$2.10757 * 10^{-8}$	1.9	$1.58073 * 10^{-8}$
0.5	$2.114 * 10^{-8}$	2.0	$1.81466 * 10^{-8}$
0.6	$2.87205 * 10^{-8}$	2.1	$2.04738 * 10^{-8}$
0.7	$1.15812 * 10^{-8}$	2.2	$2.27117 * 10^{-8}$
0.8	$3.41048 * 10^{-9}$	2.3	$2.38078 * 10^{-8}$
0.9	$6.35243 * 10^{-10}$	2.4	$2.5148 * 10^{-8}$
1.0	$4.06428 * 10^{-9}$	2.5	$2.96193 * 10^{-8}$
1.1	$7.25498 * 10^{-9}$	2.6	$3.3629 * 10^{-8}$
1.2	$1.3708 * 10^{-8}$	2.7	$3.87451 * 10^{-8}$
1.3	$1.88562 * 10^{-8}$	2.8	$4.22415 * 10^{-8}$
1.4	$1.48517 * 10^{-8}$	2.9	$4.69401 * 10^{-8}$
1.5	$1.0825 * 10^{-8}$	3.0	$4.96176 * 10^{-8}$

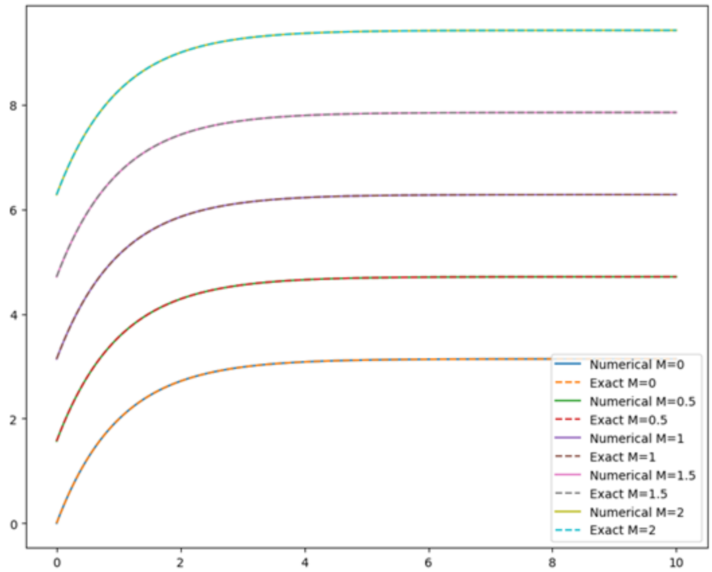

Figure (1): The exact solution compared to the numerical solution, for various M values.

We can see that the absolute error is generally small for all values of x. This suggests that the numerical solution agrees with the exact solution when $M = 0$. However, we can also see some oscillations in the absolute error values, particularly at $x = 0.7,\ 1.3$, and 1.9. These oscillations could be due to the behavior of the exact solution at these points, or they could be artifacts of the numerical method used to solve the ODE.

Example (2): Consider a harmonic oscillator with a time-dependent forcing term described by the following ODE:

$$\begin{aligned} y''(x) + y(x) &= \pi\big(\sin(\alpha+\beta) - \sin\left(\alpha+\beta e^{-x}\right) + M - \beta e^{-x}\cos\left(\alpha+\beta e^{-x}\right) \\ &\quad + \beta^2\, e^{-2x}\sin\left(\alpha+\beta e^{-x}\right)\big), \end{aligned} \tag{8.3}$$

where $y(x)$ is the displacement of the oscillator at time x, $y''(x)$ is the acceleration of the oscillator at time x, and α and β are real constants representing the amplitude and frequency of the forcing term, respectively. M is a constant representing the mean displacement of the oscillator. Under the initial conditions $y(0) = \pi M$ and $y'(0) = \pi\beta\ \cos(\alpha+\beta)$.

Solution: Using *Generator 1* in Table (1) and setting $f(z) = \sin z$, we obtain the desired ODE, and using the given exact solution we have:

$$y(x) \;=\; \pi(\sin(\alpha+\beta) \;-\sin(\alpha+\beta e^{-x}) \;+\; M). \tag{8.4}$$

We can study the behaviour of the oscillator under different values of α, β, and M, as well as the effect of the forcing term on the motion of the oscillator. We can also investigate the role of the initial conditions on the solution and how they influence the motion of the oscillator over time.
The following table provides the absolute error for different x values that were calculated using Mathematica.

Table (4): The absolute error between the exact and numerical solutions for various values of x for the system in Example (2).

x	**Absolute error**	x	**Absolute error**
0.0	0.0	1.1	1.12505×10^{-8}
0.1	4.05517×10^{-9}	1.2	9.76376×10^{-9}
0.2	1.95984×10^{-9}	1.3	9.57145×10^{-9}
0.3	4.74736×10^{-9}	1.4	9.65439×10^{-9}
0.4	5.01838×10^{-9}	1.5	9.94743×10^{-9}
0.5	5.17229×10^{-9}	1.6	9.80167×10^{-9}
0.6	1.03314×10^{-8}	1.7	1.14×10^{-8}
0.7	1.24176×10^{-8}	1.8	1.27268×10^{-8}
0.8	1.40478×10^{-8}	1.9	1.38973×10^{-8}
0.9	1.94133×10^{-8}	2.0	1.41356×10^{-8}
1.0	1.56988×10^{-8}		

The table above shows the absolute error between the exact value and the numerical values for the differential equation in *Example (2)*, with initial conditions $y(0) \;= \pi M$ and $y' = \pi\beta\cos(\alpha+\beta)$. The results reveal that for $M = 0$, the numerical values are very close to the exact value for most values of x. This supports the research's primary goal of finding the exact value of the differential equation. The small absolute error indicates that the numerical solution can be relied upon to accurately approximate the exact solution for this differential equation. It is worth noting that for large values of x, the absolute error increases slightly, which could be attributed to the accumulation of rounding errors in the numerical computations. Overall, the numerical values provide valuable support for the research's goal of finding the exact value.

Additionally, Figure (2) shows a plot of the exact and numerical solutions for different M values, $\beta = 0.5$ and $\alpha = 1$,.

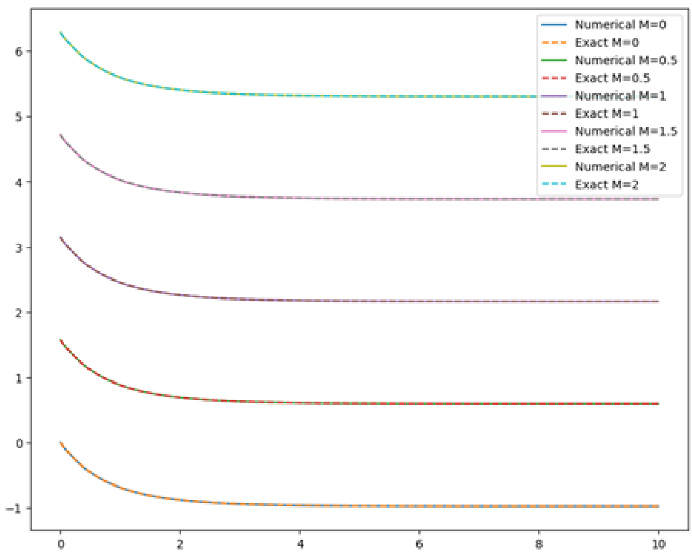

Figure (2): Exact Solution compared to the Numerical solution, for various M values, $\beta = 0.5$ and $\alpha = 1$.

Example (3): Consider a system where a pendulum is attached to a spring with a varying spring constant. The motion of the pendulum is described by the following non-linear ODE:

$$\begin{aligned}(y''(t))^2 + (y'(t))^3 = {} & \pi^2 \left(-\beta e^{-t} \sinh\left(\alpha + \beta e^{-t}\right) - \beta^2 e^{-2t} \cosh\left(\alpha + \beta e^{-t}\right)\right)^2 \\ & + \pi^3 \left(\beta e^{-t} \sinh\left(\alpha + \beta e^{-t}\right)\right)^3, \end{aligned} \tag{8.5}$$

where $y(t)$ is the displacement of the pendulum from its equilibrium position at time $t, y''(t)$ is its acceleration, and $y'(t)$ is its velocity. The function $\cosh(\alpha + \beta e^{-t})$ describes the varying spring constant, where α and β are constants that determine its shape. The system is subject to an initial displacement $y(0) = \pi M$ and initial velocity $y'(0) = \pi\beta \sinh(\alpha + \beta)$.

<u>Solution:</u> Using *generator 2* in Table (2), and setting $f(z) = \cosh(z)$, The exact solution to the ODE is given by:

$$y(t) = \pi\left(\cosh(\alpha + \beta) - \cosh\left(\alpha + \beta e^{-t}\right) + M\right). \tag{8.6}$$

This example could be made more complicated, let's consider a scenario where the pendulum is attached to the spring, but there is also an external force acting on the pendulum. This external force could represent, for example, a gust of wind or an earthquake.
In this case, the ODE would be modified to include the external force term, which could be a function of time, position, velocity, or any combination of these. We could also consider more complex forms of the function $\cosh(\alpha + \beta e^{-t})$ that vary in a more intricate way over time. Furthermore, we could study how the system behaves under different initial conditions, such as varying the initial displacement or velocity, or under different values of the constants α, β, M, or the external force function.
Figure 3 and Table 5 present a comparison between the exact and numerical solutions. The table reveals that the exact and numerical solutions begin in close proximity, but as time progresses, the numerical solution significantly diverges. This divergence suggests that the numerical method may not accurately represent the system's dynamics over an extended period.
To address this discrepancy, we could consider employing a more advanced numerical method or solver, particularly one that can better manage stiff or complex equation systems. However, this approach may demand more computational resources and does not always guarantee improved results.
Alternatively, we could reassess the physical model and its underlying assumptions. Simplifying the model or adopting different assumptions could lead to a system of equations that is more amenable to numerical solutions.
It's also important to note that the system's initial conditions and parameters (such as the values of α, β, and M) can significantly influence the solution's behavior. If these values are inaccurately known or estimated, they could be contributing to the challenges in deriving a numerical solution.
Therefore, the exact solution provides a crucial benchmark for evaluating the accuracy of the numerical method and pinpointing its limitations.

Table (5): A comparison between the Exact and the Numerical solution of Example (3).

t	**Exact Solution**	**Numerical Solution**
0.000000	3.141593	3.141593
0.055556	3.740374	3.795815
0.111111	4.276602	4.457095
0.166667	4.758126	5.119937
0.222222	5.191646	5.784361
0.277778	5.582905	6.450466
0.333333	5.936844	7.118214
0.388889	6.257727	7.787477
0.444444	6.549250	8.458263
0.500000	6.814624	9.130592

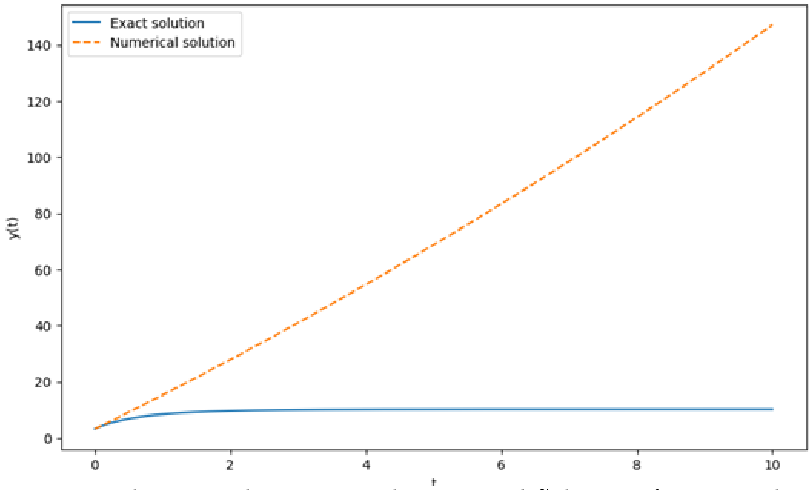

Figure (3): A comparison between the Exact and Numerical Solutions for Example 3 over the Time Interval $t \in [0, 10]$.

Example (4): Let's consider the pantograph equation with a delay and a non-linear term:

$$y\left(\frac{x}{a}\right) + [y'(x)]^v = \pi[f(\alpha+\beta) - f(\alpha+\beta e^{-\frac{x}{a}}) + M] + \pi^v\left[\beta e^{-x}\frac{\partial}{\partial\alpha}f\left(\alpha+\beta e^{-x}\right)\right]^v, \tag{8.7}$$

where v is a positive exponent, $f(z) = e^z$ is the exponential function, and α, β, and M are constants that describe the initial conditions and parameters of the system. The function $y(x)$ represents the displacement of one of the bars in the pantograph as a function of its position along a certain axis, and a is a scaling parameter that determines the overall size of the system.
The term $y\left(\frac{x}{a}\right)$ represents a scaling factor that accounts for the overall size of the system, while the term $[y'(x)]^v$ represents a non-linear term that captures the rotational motion of the bars. The non-linear term is raised to the power of v, which is a positive exponent that determines the degree of non-linearity in the system.
The right-hand side of Equation (8.7) contains two terms: the first term represents a constant force or displacement that is applied to the system, while the second term represents a time-delayed force or displacement that depends on the previous state of the system. The time delay is introduced through the exponential function $e^{-\frac{x}{a}}$, where x is the current position of the bar and a is a scaling parameter that determines the time constant of the delay.
In order to understand the physical implications of the pantograph equation, we can think of the system as a network of interconnected bars that can move and rotate relative to each other. The equation describes the motion and deformation of these bars in response to external forces or displacements.
The exponential function $f(z) = e^z$ can be interpreted as a scaling factor that determines the magnification or reduction factor applied to the original image or drawing. In a pantograph, this factor would correspond to the ratio between the movements of two different points in the system.
The initial conditions for the system, $y(0)$ and $y'(0)$, represent the starting position and velocity of one of the bars in the pantograph. The constants M and β can be related to the mass and stiffness of the bar, respectively, and the exponent v determines the degree of non-linearity in the system.
Let's consider the pantograph equation with a delay and a non-linear term, but with the sinusoidal function $f(z) = \sin(z)$ instead of the exponential function. The equation becomes:

$$\begin{aligned} y\left(\frac{x}{a}\right) + [y'(x)]^v = \pi\left[\sin(\alpha+\beta) - \sin\left(\alpha+\beta e^{-\frac{x}{a}}\right)\right] + M\pi \\ +\pi^v\left[\beta e^{-x}\frac{\partial}{\partial\alpha}\left[\sin\left(\alpha+\beta e^{-x}\right)\right]\right]^v, \end{aligned} \tag{8.8}$$

where α, β, and M are constants that describe the initial conditions and parameters of the system.
The choice of $f(z) = \sin(z)$ is interesting because it introduces a periodicity into the system, which means that the magnification or reduction factor applied to the original image or drawing varies sinusoidally with time. This could be useful in certain applications where a dynamic or time-varying scaling factor is desired, such as in animation or music.

Let's discuss the given exact solution for the pantograph equation with a delay and a non-linear term for each case of the function $f(z)$.

1. $f(z) = e^z$: In this case, the exact solution for the pantograph equation is:

$$y(x) = \pi(e^{\alpha+\beta} - e^{\alpha+\beta e^{-x}} + M). \tag{8.9}$$

   This solution represents the displacement of one of the bars in the pantograph as a function of its position along a certain axis, given the initial conditions and parameters of the system. The exponential function e^z represents a scaling factor that determines the magnification or reduction factor applied to the original image or drawing.
   The term $e^{\alpha+\beta}$ represents the original size or magnification factor of the image, while the term $e^{\alpha+\beta e^{-x}}$ represents the size or magnification factor of the image after it has been scaled or reduced by the pantograph. The constant M represents an additional displacement or force that is applied to the system, and π is a scaling factor that accounts for the overall size of the system.

2. $f(z) = z$: In this case, the exact solution for the pantograph equation is:

$$y(x) = \pi\left(\alpha + \beta - \left(\alpha + \beta e^{-x}\right) + M\right). \tag{8.10}$$

   This solution represents the displacement of one of the bars in the pantograph as a function of its position along a certain axis, given the initial conditions and parameters of the system. The linear function z represents a constant scaling factor that does not depend on the position or displacement of the bars in the pantograph.
   The term $\alpha + \beta$ represents the original size or magnification factor of the image, while the term $\alpha + \beta e^{-x}$ represents the size or magnification factor of the image after it has been scaled or reduced by the pantograph. The constant M represents an additional displacement or force that is applied to the system, and π is a scaling factor that accounts for the overall size of the system.

3. $f(z) = \sin(z)$: In this case, the exact solution for the pantograph equation is:

$$y(x) = \pi\left(\sin(\alpha + \beta) - \sin(\alpha + \beta e^{-x}) + M\right). \tag{8.11}$$

   This solution represents the displacement of one of the bars in the pantograph as a function of its position along a certain axis, given the initial conditions and parameters of the system. The sinusoidal function $\sin(z)$ introduces a periodicity into the system, which means that the magnification or reduction factor applied to the original image or drawing varies sinusoidally with time.
   The choice of the sinusoidal function $f(z) = sin(z)$ in the pantograph equation introduces a periodicity into the system, which can be useful in applications where a time-varying scaling factor is desired. The exact solution for this case is given by $y(x) = \pi(sin(\alpha+\beta) - \ sin(\alpha + \beta e^{-x}) + M)$, where the first term represents the original size or magnification factor of the image, the second term represents the size or magnification factor of the image after it has been scaled or reduced by the pantograph, and the constant M represents an additional displacement or force that is applied to the system. The scaling factor π accounts for the overall size of the system.

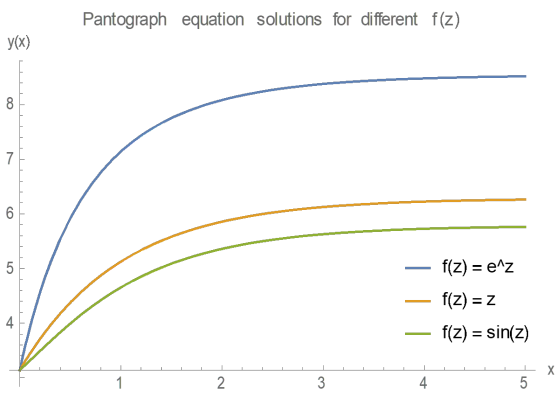

Figure (4): Pantograph equation solutions for different $f(z)$.

Example (5): Let's consider an economics application using *Generator 3* in Table (1).

$$y(x) + y'(x) = \pi\left(f\left(\alpha+\beta\right) - f\left(\alpha+\beta e^{-x}\right) + M + \beta e^{-x}\frac{\partial}{\partial\alpha}\left(f\left(\alpha+\beta e^{-x}\right)\right)\right), \tag{8.12}$$

with the initial condition: $y(0) = \pi M$.
In economics, assume we have a market where the demand function is represented by $f(\alpha+\beta)$ and the supply function is represented by $f\left(\alpha+\beta e^{-x}\right)$. α and β are parameters that characterize the functions, and M is an external factor that affects the market. The variable x represents time.
The given generator describes a dynamic system where the equilibrium between demand and supply is constantly changing due to various factors like time-varying demand, supply shocks, and external factors M. The term $\beta e^{-x}\frac{\partial}{\partial\alpha}\ \left(f\left(\alpha+\beta e^{-x}\right)\right)$ represents the rate of change of the supply function with respect to the parameter α, which could be influenced by various external factors.
To find the market equilibrium over time, we can solve the given ODE:

$$y(x) = \pi\left(\alpha+\beta-\left(\alpha+\beta e^{-x}\right)+M\right). \tag{8.13}$$

This equation shows the equilibrium price level $(y(x))$ as a function of time x. As time progresses, the equilibrium price level will adjust according to the changing demand and supply functions, and the external factor M. By analyzing this solution, economists and policymakers can better understand how the market responds to shocks and changes in various factors, allowing them to make informed decisions and implement appropriate policies.
The figure below shows the market equilibrium price level over time with different functions, where $0 \le x \le 5$, $\alpha = 1$, $\beta = 2$, and $M = 3$.

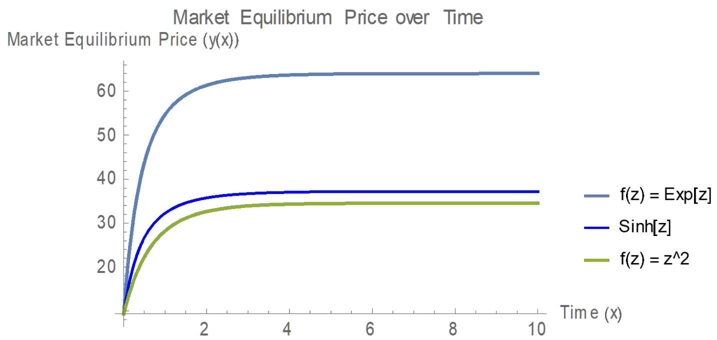

Figure (5): Market equilibrium price over time for different functions.

Example (6): To illustrate the use of one of the linear generators in developing a numerical method for solving ODEs, let's choose *Generator 2* from Table (1):
Generator 2:

$$y''(x)+y'(x) = -\pi\beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}f\left(\alpha\ +\ \beta e^{-x}\right). \tag{8.14}$$

Initial Conditions:

$$\begin{aligned} y\left(0\right) &= \pi M \\ y'\left(0\right) &= \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha\ +\beta)). \end{aligned} \tag{8.15}$$

Solution:

$$y(x) = \pi(f(\alpha\ +\beta)-f(\alpha\ +\beta e^{-x})+M). \tag{8.16}$$

To solve this ODE numerically, we can use a standard method such as the Runge-Kutta method. The 4^{th}-*order Runge-Kutta* method is a widely used numerical integration technique for solving ODEs. However, we first need to rewrite the second-order ODE as a system of two first-order ODEs. Let's introduce a new function $u(x) = y'(x)$, then the system becomes:

$$\begin{aligned} y'(x) &= u(x) \\ u'\left(x\right) &= -\pi\beta^2 e^{-2x}\frac{\partial^2}{\partial\alpha^2}f\left(\alpha\ +\beta e^{-x}\right)-u(x), \end{aligned} \tag{8.17}$$

with initial conditions:

$$\begin{aligned} y(0) &= \pi M \\ u\left(0\right) &= \pi\beta\frac{\partial}{\partial\alpha}(f(\alpha\ +\ \beta)). \end{aligned}$$

Now we can apply the 4^{th}-*order Runge-Kutta* method to this system of first-order ODEs:

1. Define the step size, h, and the number of steps, N, for the numerical integration.
2. Initialize the values of y and u at $x = 0$ using the initial conditions.
3. For each step i from 1 to N:
   (a) Compute $k1_y = h * u(x)$.
   (b) Compute $k1_u = h * (-\pi\beta^2 e^{-2x} \frac{\partial^2}{\partial\alpha^2} f(\alpha + \beta e^{-x}) - u(x))$.
   (c) Update the intermediate values y_1 and u_1 using $k1_y$ and $k1_u$.
   (d) Compute $k2_y$, $k2_u$, $k3_y$, $k3_u$, $k4_y$, and $k4_u$ similarly, using the updated intermediate values.
   (e) Update y and u using the weighted average of the computed k values.
   (f) Increment x by the step size, h.
4. After all steps are completed, the approximate solution for $y(x)$ is obtained.

This numerical method can be adapted and optimized for the specific problem at hand, depending on the properties of the ODE and the desired level of accuracy. By analyzing the structure and properties of the ODEs generated by the linear generator, researchers can develop more efficient and accurate numerical algorithms for solving ODEs in various fields.

Example (7): Consider an electrical circuit that consists of a nonlinear inductor, a resistor, and an external voltage source. The nonlinear inductor exhibits a varying inductance related to the even powers of the current. The equation for the current (i) in the circuit can be modeled by the given nonlinear ODE:

$$y(x) + (-1)^n \, y^{(2n)}(x) = \pi\beta, \tag{8.18}$$

where $y(x)$ represents the current i in the circuit, and β is a constant related to the external voltage source.
Initial conditions are given as:

$$y(0) = 0, \qquad y'(0) = \frac{\pi\beta}{n}\csc\left(\frac{\pi}{2n}\right), \qquad y''(0) = 0, \qquad y'''(0) = -\frac{\pi\beta}{n}\csc\left(\frac{3\pi}{2n}\right),$$
$$y^{(4)}(0) = 0, \qquad y^{(2r)}(0) = 0, \qquad y^{(2r-1)}(0) = (-1)^{r+1}\,\frac{\pi\beta}{n}\csc\left(\frac{\pi(2r-1)}{2n}\right),$$

where $r \in \mathbb{Z}$, $r < n$, and $x > 0$.
Solution: The equation can be found using Theorems 4.1 and 4.2, and setting $f(z) = z$. The solution for the current in the circuit is given by:

$$y(x) = \frac{\pi\beta}{n}\sum_{s=1}^{n} 1 - e^{-x\sin\left(\frac{(2s-1)\pi}{2n}\right)}\cos\left(x\cos\left(\frac{(2s-1)\pi}{2n}\right)\right). \tag{8.19}$$

This solution helps analyze the response of the electrical circuit under the influence of nonlinear inductance and external voltage. Engineers can use this information to understand the circuit's behaviour and improve its performance.
Using this information, engineers can make informed decisions about how to optimize the circuit's performance. They can design circuits that are more efficient, consume less power, and have better noise immunity. They can also troubleshoot existing circuits to diagnose and fix performance issues. Overall, the solution provides a valuable tool for engineers working with nonlinear electrical circuits.
A different approach to solving the ODE using the master theorems of integrals could be described as follows:
The integral that satisfies the ODE and its initial conditions are chosen as follows:
Let $y(x) = 2\beta\int\limits_0^\infty \frac{\sin(x\,t)}{t\,(1+t^{2n})}dt$, and clearly, $y(0) = 0$.

By differentiating both sides with respect to x, we obtain:

$$y'(x) = 2\beta \int_0^\infty \frac{\cos(xt)}{(1+t^{2n})}dt \quad \longrightarrow \quad y'(0) = \int_0^\infty \frac{1}{(1+t^{2n})}dt = \frac{\pi\beta}{n}\csc\left(\frac{\pi}{2n}\right).$$

$$y''(x) = 2\beta \int_0^\infty \frac{-t\sin(xt)}{(1+t^{2n})}dt \quad \longrightarrow \quad y''(0) = 0.$$

$$y'''(x) = 2\beta \int_0^\infty \frac{-t^2\cos(xt)}{(1+t^{2n})}dt \quad \longrightarrow \quad y'''(0) = \int_0^\infty \frac{-t^2}{(1+t^{2n})}dt = \frac{-\pi\beta}{n}\csc\left(\frac{3\pi}{2n}\right).$$

$$y^{(2r)}(x) = 2\beta \int_0^\infty \frac{(-1)^r \; t^{2r-1}\sin(x\,t)}{(1+t^{2n})}dt \quad \longrightarrow \quad y^{(2r)}(0) = 0, \tag{8.20}$$

$$y^{(2r-1)}(x) = 2\beta \int_0^\infty \frac{(-1)^{r+1}\,t^{2r-2}\cos(x\,t)}{(1+t^{2n})}dt, \tag{8.21}$$

$$y^{(2r-1)}(0) = (-1)^{r+1}\,\frac{\pi\beta}{n}\csc\left(\frac{\pi(2r-1)}{2n}\right). \tag{8.22}$$

where r is an integer and $r < n$.

$$y^{(2n)}(x) = \int_0^\infty \frac{(-1)^n \; t^{2n-1}\sin(xt)}{(1+t^{2n})}dt. \tag{8.23}$$

Therefore,

$$\begin{aligned}(-1)^n y^{2n}(x) + y(x) &= 2\beta \int_0^\infty \frac{t^{2n}\sin(xt)}{t(1+t^{2n})}dt + 2\beta \int_0^\infty \frac{\sin(xt)}{t(1+t^{2n})}dt \\ &= 2\beta \int_0^\infty \frac{\sin(x\,t)}{t}dt = \pi\beta \end{aligned} \tag{8.24}$$

where $x > 0$, and thus,

$$y(x) = 2\beta \int_0^\infty \frac{\sin(xt)}{t(1+t^{2n})}dt. \tag{8.25}$$

This is the solution to this ODE. Thus, by using Equation (4.9) and setting $\alpha = 0$, $\beta = 1$, $r = 1$, and $f(z) = z$ we have:

$$2\int_0^\infty \frac{\sin(xt)}{t(1+t^{2n})}dt = \frac{\pi}{n}\sum_{k=1}^{n} 1 - e^{-x\sin\left(\frac{(2k-1)\pi}{2n}\right)}\cos\left[x\cos\left(\frac{(2k-1)\pi}{2n}\right)\right], \tag{8.26}$$

implies,

$$y(x) = 2\beta\int_0^\infty \frac{\sin(xt)}{t(1+t^{2n})}dt = \frac{\pi\beta}{n}\sum_{k=1}^{n} 1 - e^{-x\sin\left(\frac{(2k-1)\pi}{2n}\right)}\cos\left[x\cos\left(\frac{(2k-1)\pi}{2n}\right)\right]. \tag{8.27}$$

The figure below shows the solution $y(x)$, with setting $\beta = 1$ and different n values.

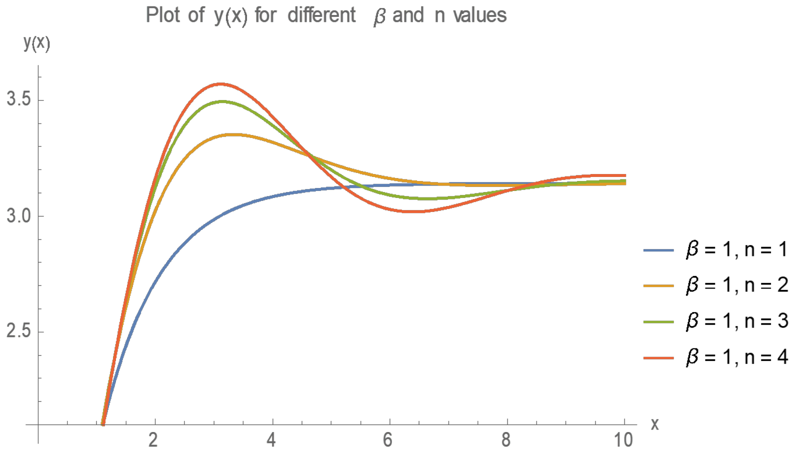

Figure (6): The plot of $y(x)$ for different β and n values.

9 Conclusion

In conclusion, this research introduces a groundbreaking approach to constructing and solving both linear and non-linear Ordinary Differential Equations (ODEs) using master theorems of integrals. The development of this novel method offers a systematic way to generate master generators for ODEs, which leads to exact solutions when choosing the appropriate function f(z). This advancement has the potential to revolutionize the field of applied mathematics, addressing long-standing challenges and opening new opportunities for the study of ODEs in various scientific domains, including physics and engineering.
Harnessing the power of master improper integrals, we have derived an infinite number of generators for ODEs, providing a versatile and powerful tool for tackling various types of differential equations. This proposed method enables the creation of infinite ODEs using a single generator, showcasing its adaptability to a wide range of scenarios and applications across diverse fields such as quantum field theory, astrophysics, and fluid dynamics. Notably, this adaptability is a significant strength of the proposed method, as it allows researchers to address a variety of previously unsolvable or intractable problems using a single, unified approach.
A comprehensive analysis of the theoretical underpinnings behind the proposed method, including a detailed examination of master improper integrals and the derivation of generators for linear and non-linear ODEs, has been presented in this research. Through this in-depth exploration, we have established a solid foundation for understanding the infinite generators and their potential applications in various disciplines.
We also demonstrated the practical utility of infinite generators through several real-world examples. The robustness of our method, particularly when dealing with non-linear ODEs and higher-order ODEs, was emphasized, filling a critical gap in the field. However, it is also important to note that despite providing exact solutions, numerical implementations can sometimes lead to divergences, especially when dealing with non-linear and complex problems. Hence, refining these solutions is sometimes required to minimize errors.
The value of this research lies not only in its theoretical and practical contributions but also in its potential to inspire further investigation and development in this vital area of applied mathematics. The introduction of infinite generators has the potential to drive continued progress and innovation in the study of ODEs and their applications, fostering collaboration and knowledge-sharing among researchers from diverse fields. Our contributions have been further refined based on the valuable feedback received from our esteemed reviewer. We appreciate the insightful comments and constructive criticisms, which have significantly enhanced the quality of our work. The advancements in this research contribute to our collective understanding and ability to solve complex mathematical problems, opening new avenues for research in the realm of applied mathematics.

Conflict of Interests:
The authors declare that there is no conflict of interests regarding the publication of this paper.

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Appendix 1: Coefficient a_j

	c	k	k^2	k^3	k^4	k^5	$\ldots$	$k^{t-1},\ \ t\in\mathbb{N}, 2<t$ $m-t=1$	$k^t \quad m-t=1$, m is the order of the derivative	$a_j,\quad$ where $\ 2\le j<m$
Coef. of $(k+1)^t$	$\binom{t}{0}$	$\binom{t}{1}$	$\binom{t}{2}$	$\binom{t}{3}$	$\binom{t}{4}$	$\binom{t}{5}$	$\ldots$	$\binom{t}{t-1}$	$\binom{t}{t}$	$\sum_{k=0}^{t}\binom{t}{k}$
Aux Equation$(k+1)^{t-1}$		$\binom{t}{1}$	$\binom{t}{2}(k+1)$	$\binom{t}{3}(k+1)^2=\binom{t}{3}(k^2+2k+1)$	$\binom{t}{4}(k+1)^3=\binom{t}{4}(k^3+3k^2+3k+1)$	$\binom{t}{5}(k+1)^4=\binom{t}{5}(k^4+4k^3+6k^2+4k+1)$	$\ldots$	$\binom{t}{t-1}(k+1)^{t-2}=\binom{t}{t-1}\left(1+\binom{t-2}{1}k+\binom{t-2}{2}k^2+\ldots+\binom{t-2}{t-2}k^{t-3}+\binom{t-1}{t-1}k^{t-2}\right)$	$\binom{t}{t}(k+1)^{t-1}=\binom{t}{t}\left(1+\binom{t-1}{1}k+\binom{t-1}{2}k^2+\ldots+\binom{t-1}{t-2}k^{t-2}+\binom{t-1}{t-1}k^{t-1}\right).$	
Sub Equations				Equation 1: $\binom{t}{3}.(k+1)$	Equation (1): $3.\binom{t}{4}.(k+1),$ Equation (2): $\binom{t}{4}.(k^2+2k+1)$	Equation (1): $6.\binom{t}{5}.(k+1),$ Equation (2): $4.\binom{t}{5}.(k^2+2k+1)$ Equation (3): $\binom{t}{5}.(k^3+3k^2+3k+1)$	$\ldots$	Equation (1): $\binom{t}{t-1}.\binom{t-2}{2}.(k+1),$ Equation (2): $\binom{t}{t-1}.\binom{t-2}{2}.(k^2+2k+1)$ Equation (3) : $\binom{t}{t-1}.\binom{t-2}{3}.(k^3+3k^2+3k+1)$ $\vdots$ Equation (t-3): $\binom{t}{t-1}.((1+\binom{t-3}{1}.k$ $+\binom{t-3}{2}.k^2+\ldots+\binom{t-3}{t-4}.k^{t-4}$ $+\binom{t-3}{t-4}.k^{t-3}))$	Equation (1): $\binom{t-1}{2}.(k+1),$ Equation (2): $\binom{t-1}{3}.(k^2+2k+1)$ Equation (3): $\binom{t-1}{4}.(k^3+3k^2+3k+1)$ $\vdots$ Equation (t-2): $((1+\binom{t-2}{1}k+\binom{t-2}{2}k^2+\ldots+\binom{t-2}{t-3}k^{t-3}+\binom{t-2}{t-2}k^{t-2}))$	
Coef. of $\frac{\partial^2}{\partial\alpha^2}$ * which is the coefficient of the constant terms	0	$\binom{t}{1}$	$\binom{t}{2}$	$\binom{t}{3}$	$\binom{t}{4}$	$\binom{t}{5}$	$\ldots$	$\binom{t}{t-1}$	$\binom{t}{t}$	$\sum_{s=1}^{t}\binom{t}{s}$
Coef. of $\frac{\partial^3}{\partial\alpha^3}$ * which is the coefficient of k	0	0	$1.\binom{t}{2}$	$2\binom{t}{3}+\binom{t}{3}=3\binom{t}{3}$	$3\binom{t}{4}+3\binom{t}{4}+2\binom{t}{4}=8\binom{t}{4}$	$4\binom{t}{5}+6\binom{t}{5}+8\binom{t}{5}+3\binom{t}{5}=21\binom{t}{5}$	$\ldots$	$\left(\binom{t}{t-1}\right)\left(\binom{t-2}{1}+\binom{t-2}{2}+2\binom{t-2}{3}+3\binom{t-2}{4}+4\binom{t-2}{5}+\cdots+\binom{t-3}{1}\binom{t-2}{t-2}\right)$	$\binom{t-1}{1}+\binom{t-1}{2}+2\binom{t-1}{3}+3\binom{t-1}{4}+4\binom{t-1}{5}+\cdots+\binom{t-2}{1}\binom{t-1}{t-1}$	$\binom{t}{2}+3\binom{t}{3}+8\binom{t}{4}+21\binom{t}{5}+\cdots+\binom{t-1}{1}+\binom{t-1}{2}+2\binom{t-1}{3}+3\binom{t-1}{4}+4\binom{t-1}{5}+\cdots+\binom{t-2}{1}\binom{t-1}{t1}$
Coef. of $\frac{\partial^4}{\partial\alpha^4}$ * which is the coefficient of k^2	0	0	0	$\binom{t}{3}$	$3\binom{t}{4}+\binom{t}{4}=4\binom{t}{4}$	$6\binom{t}{5}+4\binom{t}{5}+3\binom{t}{5}=13\binom{t}{5}$	$\ldots$	$\binom{t}{t-1}\left(\binom{t-2}{2}+\binom{t-2}{3}+3\binom{t-2}{4}+6\binom{t-2}{5}+10\binom{t-2}{6}+15\binom{t-2}{7}+\cdots+\binom{t-3}{2}\binom{t-2}{t-2}\right)$	$\left(\binom{t-1}{2}+\binom{t-1}{3}+3\binom{t-1}{4}+6\binom{t-1}{5}+10\binom{t-1}{6}+15\binom{t-1}{7}+\cdots+\binom{t-2}{2}\binom{t-1}{t-1}\right)$	$\binom{t}{3}+4\binom{t}{4}+13\binom{t}{5}+\cdots+\left(\binom{t-1}{2}+\binom{t-1}{3}+3\binom{t-1}{4}+6\binom{t-1}{5}+10\binom{t-1}{6}+15\binom{t-1}{7}+\cdots+\binom{t-2}{2}\binom{t-1}{t-1}\right)$
Coef. of $\frac{\partial^5}{\partial\alpha^5}$ * which is the coefficient of k^3	0	0	0	0	$\binom{t}{4}$	$4\binom{t}{5}+\binom{t}{5}=5\binom{t}{5}$	$\ldots$	$\binom{t}{t-1}\binom{t-2}{3}+\binom{t-2}{4}+4\binom{t-2}{5}+10\binom{t-2}{6}+20\binom{t-2}{7}+\cdots+\binom{t-3}{3}\binom{t-2}{t-2}$	$\binom{t-1}{3}+\binom{t-1}{4}+4\binom{t-1}{5}+10\binom{t-1}{6}+20\binom{t-1}{7}+\cdots+\binom{t-2}{3}\binom{t-1}{t-1}$	$\binom{t}{4}+5\binom{t}{5}+\cdots+\binom{t-1}{3}+\binom{t-1}{4}+4\binom{t-1}{5}+10\binom{t-1}{6}+20\binom{t-1}{7}+\cdots+\binom{t-2}{3}\binom{t-1}{t-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
Coef. of $\frac{\partial^t}{\partial\alpha^t}$ * which is the coefficient of k^{t-2}	0	0	0	0	0	0	$\ldots$	$\binom{t}{t-1}$	$\binom{t-1}{t-2}+\binom{t-2}{t-2}$	$\binom{t}{t-1}+\binom{t-1}{t-2}+\binom{t-2}{t-2}$

*The table above can be used to find the coefficients as follows:

1. Auxiliary equation: $c\,(k+1)^{t-1}$ where $t\in\mathbb{N}$, $t=m-1$, $m>2$, m is the order of the derivative, and c is the coefficient of $(k+1)^t$.
2. Sub equations: The equations derived from the auxiliary equations which are all the equations in the form $b\,(k+1)^q$, where $q\le t-2$ and b is the coefficient of k^q in the auxiliary equations.
3. The number of sub-equations derived from the auxiliary equations is $t-2$.
4. The coefficient a_j: which is the sum of the coefficients of the auxiliary and the sub equations for $\frac{\partial^j}{\partial\alpha^j}$, where $2\le j<m$

10 References

1. Y. Muroya, E. Ishiwata and H. Brunner, On the attainable order of collocation methods for pantograph integrodifferential equations, J. Comput. Appl. Math. 152 (2003), pp. 347–366 (2014) 485–493.
2. E.H.Doha,A.H.Bhrawy,D.Baleanu,andR.M.Hafez, Anew Jacobi rational—Gauss collocation method for numerical solution of generalized pantograph equations, Applied Numerical Mathematics, vol.77, pp.43–54, 2014.
3. A. El-Safty and S.M. Abo-Hasha, On the application of spline functions to initial value problems with retarded argument, Int. J. Comput. Math. 32 (1990), pp. 173–179.
4. Coddington, E. A. (1989). Ordinary Differential Equations: An Introduction to the Theory and Its Applications. New York: Dover Publications.
5. trogatz, S. H. (2015). Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering. 2nd ed. Boca Raton, FL: CRC Press.
6. Hubbard, J. H., & Hubbard, B. B. (1996). Differential Equations: A Dynamical Systems Approach: Part 2: Higher-Dimensional Systems. Princeton, NJ: Princeton University Press.
7. Bainov, D. D., & Simeonov, P. S. (1993). Nonlinear Differential Equations: Analysis and Applications. Chichester, UK: Ellis Horwood Limited.Atkinson, K. (1989). An Introduction to Numerical Analysis (2nd ed.). John Wiley & Sons.
8. Boyce, W. E., & DiPrima, R. C. (2017). Elementary Differential Equations and Boundary Value Problems (11th ed.). John Wiley & Sons.
9. Burden, R. L., & Faires, J. D. (2011). Numerical Analysis (9th ed.). Brooks/Cole, Cengage Learning.
10. Butcher, J. C. (2008). Numerical Methods for Ordinary Differential Equations (2nd ed.). John Wiley & Sons.
11. Hairer, E., Nørsett, S. P., & Wanner, G. (1993). Solving Ordinary Differential Equations I: Nonstiff Problems. Springer.
12. Hairer, E., & Wanner, G. (1996). Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems. Springer.
13. Lambert, J. D. (1991). Numerical Methods for Ordinary Differential Systems: The Initial Value Problem. John Wiley & Sons.
14. Abu Ghuwaleh, M.; Saadeh, R.; Burqan, A. New Theorems in Solving Families of Improper Integrals. Axioms 2022, 11, 301. https://doi.org/10.3390/axioms11070301
15. Abu-Ghuwaleh, M.; Saadeh, R.; Qazza, A. A Novel Approach in Solving Improper Integrals. Axioms 2022, 11, https://doi.org/10.3390/axioms11100572
16. Abu-Ghuwaleh, M.; Saadeh, R.; Qazza, A. General Master Theorems of Integrals with Applications. Mathematics 2022, 10, 3547. https://doi.org/10.3390/math10193547
17. Saadeh, R., Abu-Ghuwaleh, M., Qazza, A., & Kuffi, E. (2022). A Fundamental Criteria to Establish General Formulas of Integrals. Journal of Applied Mathematics, 2022, Article 6049367. https://doi.org/10.1155/2022/6049367
18. Grzymkowski, R., Pleszczyński, M. (2018). Application of the Taylor Transformation to the Systems of Ordinary Differential Equations. Springer, Cham. https://doi.org/10.1007/978-3-319-99972
19. Hetmaniok, E.; Pleszczyński, M. Comparison of the Selected Methods Used for Solving the Ordinary Differential Equations and Their Systems. Mathematics 2022, 10, 306. https://doi.org/10.3390/math10030306
20. Arnold, V. I. (2006). Ordinary Differential Equations. 2nd ed. Berlin: Springer.
21. Polyanin, A. D., & Zaitsev, V. F. (2003). Handbook of Exact Solutions for Ordinary Differential Equations. 2nd ed. Boca Raton, FL: CRC Press.

22. Zill, D. G., & Cullen, M. R. (2012). Differential Equations with Boundary-Value Problems. 8th ed. Boston, MA: Brooks/Cole Cengage Learning.

23. Ince, E. L. (1956). Ordinary Differential Equations. New York: Dover Publications.

24. Butkovsky, O., & Scheutzow, M. (2014). Ordinary Differential Equations and Dynamical Systems. Graduate Studies in Mathematics, 140. Providence, RI: American Mathematical Society.

25. Tenenbaum, M., & Pollard, H. (1985). Ordinary Differential Equations: An Elementary Textbook for Students of Mathematics, Engineering, and the Sciences. New York: Dover Publications.

26. Bender, C. M., & Orszag, S. A. (1999). Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory. New York: Springer.