Lectures on Gauge Theories

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1. Preliminaries

– Literature

Recommended Texts:

- T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press, 1984.
- S. Pokorski, *Gauge Field Theories*, Cambridge University Press, 2000, Second Edition.
- M. E. Peskin and D. V. Schröder, *Quantum Field Theory*, Perseus Books Group, 1995.
- H. F. Jones, *Groups, Representations and Physics*, Institute of Physics, 1998 (Second edition).
- L. H. Ryder, *Quantum Field Theory*, Cambridge University Press.

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Advanced Texts:

- P. Ramond, *Field Theory: A Modern Primer*, Addison Wesley, 1990.
- J. Zinn–Justin, *Quantum Field Theory and Critical Phenomena*, Oxford Science Publications, 2002, Fourth Edition.
- R. Slansky, *Group Theory for Unified Model Building*, Phys. Rept. **79** (1981) 1.

– Lagrangian Field Theory

In Quantum Field Theory (QFT), a (scalar) particle is described by a field $\phi(x)$, whose Lagrangian has the functional form:

$$L \;=\; \int d^3x \; \mathcal{L}(\phi(x)\,, \partial_\mu\phi(x))\,,$$

where $\mathcal{L}$ is the so-called *Lagrangian density*, often termed Lagrangian in QFT.

In QFT, the action S is given by

$$S[\phi(x)] \;=\; \int_{-\infty}^{+\infty} d^4x \; \mathcal{L}(\phi(x)\,, \partial_\mu\phi(x))\,,$$

with $\lim\limits_{x\to\pm\infty} \phi(x) = 0$.

By analogy, the Euler–Lagrange equations of motion (EoMs) can be obtained by determining the stationary points of S, under variations $\phi(x) \to \phi(x) + \delta\phi(x)$:

$$-\,\frac{\delta S}{\delta\phi} \;=\; \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \;-\; \frac{\partial\mathcal{L}}{\partial\phi} \;=\; 0\,.$$

. . .

Exercise: Derive the above Euler–Lagrange EoM for a scalar particle by extremizing $S[\phi(x)]$, i.e. $\delta S = 0\,.$

Lagrangian for a free real scalar field ϕ:

$$\mathcal{L}_{\rm KG} \;=\; \frac{1}{2}\,(\partial_\mu\phi)\,(\partial^\mu\phi) \;-\; \frac{1}{2}\,m^2\,\phi^2\,,$$

where $\phi(x)$ is a real scalar field describing one dynamical degree of freedom.

The Euler–Lagrange EoM is the Klein–Gordon equation

$$(\partial_\mu\partial^\mu \;+\; m^2)\,\phi(x) \;=\; 0\,.$$

...

Lagrangian for the electromagnetic field A_μ:

$$\mathcal{L}_{\rm em} \;=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \;-\; J_\mu A^\mu\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and J_μ is the 4-vector current satisfying charge conservation: $\partial_\mu J^\mu = 0$.

A_μ describes a spin-1 particle, e.g. a photon, with 2 physical degrees of freedom.

Exercise: Derive the Euler-Lagrange EoMs from $\mathcal{L}_{\rm em}$ and show that $\partial_\mu F^{\mu\nu} \;=\; J^\nu$, as is expected in relativistic electrodynamics (with $\mu_0 = \varepsilon_0 = c = 1$).

Lagrangian for a Dirac fermion field ψ:

$$\mathcal{L}_{\rm D} \;=\; \bar{\psi}\,(i\,\gamma^\mu\partial_\mu \;-\; m)\,\psi\,,$$

where

$$\psi(x) \;=\; \begin{pmatrix} \xi_\beta(x) \\ \bar{\eta}^{\dot{\beta}}(x) \end{pmatrix}\,, \quad \gamma^\mu \;=\; \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix}$$

and $\bar{\psi}(x) \equiv (\eta^\alpha(x),\; \bar{\xi}_{\dot{\alpha}}(x)\,)$, with $\sigma^\mu = (\mathbf{1}_2,\; \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (\mathbf{1}_2,\; -\boldsymbol{\sigma})$.

The ξ_α and $\bar{\eta}^{\dot{\alpha}}$ are 2-dim complex vectors (also called Weyl spinors) whose components anti-commute: $\xi_1\xi_2 = -\xi_2\xi_1$, $\bar{\eta}^{\dot{1}}\bar{\eta}^{\dot{2}} = -\bar{\eta}^{\dot{2}}\bar{\eta}^{\dot{1}}$, $\xi_1\bar{\eta}^{\dot{2}} = -\bar{\eta}^{\dot{2}}\xi_1$ etc.

The Euler–Lagrange EoM derived by differentiating $\mathcal{L}_{\rm D}$ with respect to $\bar{\psi}(x)$ is the Dirac equation:

$$\frac{\partial\mathcal{L}_{\rm D}}{\partial\bar{\psi}} \;=\; 0 \;\Rightarrow\; (i\,\gamma^\mu\partial_\mu \;-\; m)\,\psi \;=\; 0\,.$$

The 4-component Dirac spinor $\psi(x)$ that satisfies the Dirac equation describes 4 dynamical degrees of freedom.

Exercises:
(i) Derive the Euler–Lagrange equation with respect to the Dirac field $\psi(x)$;
(ii) Show that up to a total derivative term, $\mathcal{L}_{\rm D}$ is Hermitian, i.e. $\mathcal{L}_{\rm D} = \mathcal{L}_{\rm D}^\dagger \;+\; \partial^\mu j_\mu$, with $j_\mu = \bar{\psi}\,i\gamma_\mu\,\psi$.

Weyl and Dirac spinors$^{(*)}$

The Dirac spinor ψ is the direct sum of two Weyl spinors ξ and $\bar{\eta}$ with Lorentz trans properties:

$$\xi'_\alpha = M_\alpha{}^\beta \xi_\beta \,, \qquad \bar{\eta}'_{\dot\alpha} = M^{\dagger\,\dot\beta}{}_{\dot\alpha}\, \bar{\eta}_{\dot\beta} \,,$$
$$\xi'^\alpha = M^{-1}{}_\beta{}^\alpha \xi^\beta \,, \qquad \bar{\eta}'^{\dot\alpha} = M^{\dagger -1\,\dot\alpha}{}_{\dot\beta}\, \bar{\eta}^{\dot\beta} \,.$$

with $M \in \mathrm{SL}(2, \mathbb{C})$.

Duality relations among 2-spinors:

$$(\xi^\alpha)^\dagger = \bar{\xi}^{\dot\alpha} \,, \quad (\xi_\alpha)^\dagger = \bar{\xi}_{\dot\alpha} \,, \quad (\bar{\eta}_{\dot\alpha})^\dagger = \eta_\alpha \,, \quad (\eta^\alpha)^\dagger = \bar{\eta}^{\dot\alpha}$$

Lowering and raising spinor indices:

$$\xi_\alpha = \varepsilon_{\alpha\beta}\xi^\beta \,, \quad \xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta \,, \quad \bar{\eta}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar{\eta}^{\dot\beta} \,, \quad \bar{\eta}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{\eta}_{\dot\beta} \,,$$

with $\varepsilon^{\alpha\beta} \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha\beta}$ and $\varepsilon^{\dot\alpha\dot\beta} \equiv i\sigma_2 = -\varepsilon_{\dot\alpha\dot\beta}$.
Lorentz-invariant spinor contractions:

$$\xi\eta \equiv \xi^\alpha\eta_\alpha = \xi^\alpha\varepsilon_{\alpha\beta}\eta^\beta = -\eta^\beta\varepsilon_{\alpha\beta}\xi^\alpha = \eta^\beta\varepsilon_{\beta\alpha}\xi^\alpha = \eta^\beta\xi_\beta = \eta\xi$$

Likewise, $\bar{\xi}\bar{\eta} \equiv (\eta\xi)^\dagger = \xi^\dagger_\alpha \eta^{\alpha\dagger} = \bar{\xi}_{\dot\alpha}\bar{\eta}^{\dot\alpha} = \bar{\eta}_{\dot\alpha}\bar{\xi}^{\dot\alpha} = \bar{\eta}\bar{\xi}$.

Exercise: Given that $M\sigma_\mu M^\dagger = \Lambda^\nu{}_\mu\,\sigma_\nu$ and $M^{\dagger -1}\bar{\sigma}_\mu M^{-1} = \Lambda^\nu{}_\mu\,\bar{\sigma}_\nu$, show that $\mathcal{L}_D$ is invariant under Lorentz trans.

– Global and Local Symmetries

Consider the Lagrangian (density) for a complex scalar:

$$\mathcal{L} = (\partial^\mu\phi^*)\,(\partial_\mu\phi) - m^2\,\phi^*\phi - \lambda(\phi^*\phi)^2 \,.$$

$\mathcal{L}$ is invariant under a U(1) rotation of the field ϕ:

$$\phi(x) \to \phi'(x) = e^{i\theta}\,\phi(x) \,,$$

where θ does not depend on $x \equiv x^\mu$.

A transformation in which the fields are rotated about x-independent angles is called a **global transformation**. If the angles of rotation depend on x, the transformation is called a **local** or a **gauge transformation**.

Infinitesimal global or local trans of fields ϕ_i:

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \delta\phi_i(x) \,,$$

where $\delta\phi_i(x) = i\,\theta^a(x)\,(T^a)_i^{\,j}\,\phi_j(x)$, and T^a are the generators of the Lie Group. Note that the angles or group parameters θ^a are x-independent for a global trans.

If a Lagrangian $\mathcal{L}$ is invariant under a global or local trans, it is said that $\mathcal{L}$ has a **global** or **local (gauge) symmetry**.

...

Exercise: Show that the above Lagrangian for a complex scalar is *not* invariant under a U(1) gauge trans.

Noether's Theorem

If a Lagrangian $\mathcal{L}$ is (up to a total derivative) invariant under a given transformation of fields and spacetime, then there is a conserved current $J^\mu(x)$ and a conserved charge $Q = \int d^3\mathbf{x}\, J^0(x)$, associated with this symmetry, such that

$$\partial_\mu J^\mu \;=\; 0 \quad \text{and} \quad \frac{dQ}{dt} \;=\; 0\,.$$

$$\ldots$$

Proof as a revision exercise:

Show that if the Lagrangian $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ is invariant under the infinitesimal global trans:

$$\delta\phi_i \;=\; i\,\theta^a (T^a)_i^{\;j}\,\phi_j\,,$$

then the conserved currents are

$$J^{a,\,\mu} \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\,\frac{\partial \delta\phi_i}{\partial \theta^a} \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\; i\,(T^a)_i^{\;j}\,\phi_j\,.$$

The corresponding conserved charges are

$$Q^a(t) \;=\; \int d^3\mathbf{x}\, J^{a,\,0}(x)\,.$$

– Quantum Electrodynamics (QED)

Consider first the Lagrangian for a Dirac field ψ:

$$\mathcal{L}_{\rm D} \;=\; \bar\psi\,(i\gamma^\mu \partial_\mu \;-\; m)\,\psi\,.$$

$\mathcal{L}_{\rm D}$ is invariant under the U(1) global trans:

$$\psi(x) \;\to\; \psi'(x) \;=\; e^{i\theta}\,\psi(x)\,,$$

but it is *not* invariant under a U(1) gauge trans, when $\theta = \theta(x)$. Instead, we find the residual term

$$\delta\mathcal{L}_{\rm D} \;=\; -\,(\partial_\mu \theta(x))\;\bar\psi\gamma^\mu\psi\,.$$

To cancel this term, we introduce a vector field A^μ in the theory, the so-called photon, and add to $\mathcal{L}_D$ the extra term:

$$\mathcal{L}_\psi \;=\; \mathcal{L}_{\rm D} \;-\; e\,A_\mu\,\bar\psi\gamma^\mu\psi\,.$$

We demand that A_μ transforms under a local U(1) as

$$A_\mu \;\to\; A'_\mu \;=\; A_\mu \;-\; \frac{1}{e}\,\partial_\mu\theta(x)\,.$$

$\mathcal{L}_\psi$ is invariant under a U(1) gauge trans of ψ and A^μ.

The Lagrangian of the electron and the photon

The Lagrangian of Quantum Electrodynamics (QED) includes the interaction of the photon with the electron:

$$\mathcal{L}_{\rm QED} \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;+\; \bar{\psi}\,(i\,\partial\!\!\!/ \;-\; m \;-\; e\,A\!\!\!/)\,\psi\,,$$

where we used the convention: $a\!\!\!/ \equiv \gamma_\mu a^\mu$.

...

Exercises:

(i) Derive the equation of motions with respect to the photon and electron fields.

(ii) Derive the conserved current and charge from $\mathcal{L}_{\rm QED}$.

(iii) How should the Lagrangian describing a complex scalar field $\phi(x)$,

$$\mathcal{L} \;=\; (\partial^\mu\phi)^*\,(\partial_\mu\phi) \;-\; m^2\,\phi^*\phi\,,$$

be extended so as to become gauge symmetric under a U(1) local trans?

(iv) A Lorentz-invariant photon mass term is described by the Lagrangian $\mathcal{L}_{\rm mass} \;=\; m_A^2\,A^\mu A_\mu$. Find a renormalizable gauge-symmetric extension of $\mathcal{L}_{\rm mass}$. *Likewise, find a gauge-symmetric non-renormalizable extension of $\mathcal{L}_{\rm D}$ without the need of introducing a vector field A^μ.

The Photon Propagator and Gauge Fixing

We add to $\mathcal{L}_{\rm QED}$ the **covariant gauge-fixing term**:

$$\mathcal{L}_{\rm GF} \;=\; -\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2\,.$$

The Euler-Lagrange equation for the photon becomes:

$$\left[\eta_{\mu\nu}\,\partial_\kappa\partial^\kappa \;-\; \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right] A^\nu \;=\; 0\,.$$

The photon propagator $\Delta_{\mu\nu}(x-y)$ is the Green's function of the above differential operator:

$$\left[\eta^{\mu\nu}\,\frac{\partial}{\partial x^\kappa}\frac{\partial}{\partial x_\kappa} - \left(1-\frac{1}{\xi}\right)\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right]\Delta_{\nu\lambda}(x-y) \;=\; \delta^\mu_\lambda\,\delta^{(4)}(x-y)\,.$$

...

Exercises:

(i) Derive the Euler-Lagrange equation of the photon in the presence of $\mathcal{L}_{\rm GF}$.

(ii) Show that the photon propagator is given by the Green's function:

$$\Delta_{\mu\nu}(x-y) \;=\; \int\frac{d^4k}{(2\pi)^4}\left(-\,\eta_{\mu\nu} \,+\, (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right)\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}\,.$$

(iii) Use the equal-time commutators to show that

$$\langle 0|T[A_\mu(x)A_\nu(y)]|0\rangle \;=\; i\Delta_{\mu\nu}(x-y)$$

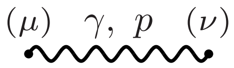
in the Feynman gauge $\xi = 1$.

– QED Feynman Rules

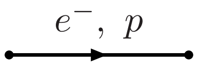
From the Lagrangian,

$$\mathcal{L}_{\rm QED} \; = \; -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} \; + \; \bar{\psi}\,(i\,\partial\!\!\!/ \; - \; m \; - \; e\, A\!\!\!/)\,\psi\,,$$

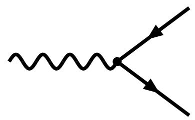
the following Feynman rules may be derived:

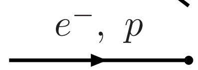


: $\dfrac{-i\,\eta_{\mu\nu}}{p^2 + i\varepsilon}$

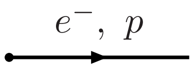


: $\dfrac{i}{p\!\!\!/ - m + i\varepsilon}$

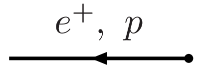

: $-ie\,\gamma_\mu$



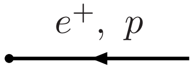
: $u(p)$ for an e^- in the initial state



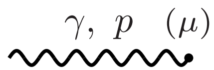
: $\bar{u}(p)$ for an e^- in the final state



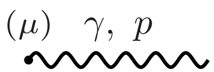
: $\bar{v}(p)$ for an e^+ in the initial state



: $v(p)$ for an e^+ in the final state



: $\varepsilon^\mu(\mathbf{p},\lambda)$ for a γ in the initial state



: $\varepsilon^{\mu\,*}(\mathbf{p},\lambda)$ for a γ in the final state

Revision exercises:

(i) Show that

(a) $\mathrm{Tr}(\gamma_\mu\gamma_\nu) \;=\; 4\,\eta_{\mu\nu}$,
(b) $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) \;=\; 4\,(\eta_{\mu\nu}\eta_{\rho\sigma} \;+\; \eta_{\mu\sigma}\eta_{\nu\rho} \;-\; \eta_{\mu\rho}\eta_{\nu\sigma})$,
(c) $\mathrm{Tr}(\gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_{2n+1}}) \;=\; 0$
(Hint: you may use the properties: $\{\gamma_5\,,\,\gamma_\mu\} = 0$ and $\gamma_5^2 = \mathbf{1}_4$, where $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$.),
(d) $\sum_{s=\pm 1/2} \bar{u}(p,s) M u(p,s) \;=\; \mathrm{Tr}\,[M\,(p\!\!\!/ + m)]$, where M is any arbitrary 4×4 matrix.

(ii) Use the Feynman rules for QED to write down the matrix element $\mathcal{M}_{fi}$ for the reaction $e^-(p_1)e^+(p_2) \to \mu^-(k_1)\mu^+(k_2)$.

(iii) With the aid of trace techniques given in (i), calculate $\overline{|\mathcal{M}_{fi}|}^2$, where the long bar indicates averaging over the spins of the electrons in the initial state.

(iv) Calculate analytically the differential cross section $d\sigma/d\Omega$ for $e^-e^+ \to \mu^-\mu^+$ which was taking place at the CERN LEP collider at CMS energies $\sqrt{s} = M_Z = 90$ GeV. Draw an accurate graph of $d\sigma/d\Omega$ as a function of $\cos\theta$.

(v) Supersymmetry predicts that in addition to muons $\mu^\pm$ there should be scalar muons $\tilde{\mu}^\pm$. Calculate $d\sigma/d\Omega$ for the process $e^-e^+ \to \tilde{\mu}^-\tilde{\mu}^+$. Plot $d\sigma/d\Omega$ as a function of $\cos\theta$ and comment on your results.

2. Group Theory

– Definition of a Group G

A *group* $(G, \cdot)$ is a set of elements $\{a, b, c \ldots\}$ endowed with a composition law $\cdot$ that has the following properties:

(i) *Closure*. $\forall\, a, b \in G$, the element $c = a \cdot b \ \in G$.

(ii) *Associativity*. $\forall\, a, b, c \in G$, it holds $a \cdot (b \cdot c) \ = \ (a \cdot b) \cdot c$

(iii) *The identity element* e. $\exists\, e \in G$: $e \cdot a = a \cdot e = a$, $\forall\, a \in G$.

(iv) *The inverse element* a^{-1} *of* a. $\forall\, a \in G, \quad \exists\, a^{-1} \in G$: $a \cdot a^{-1} = a^{-1} \cdot a = e$.

If $a \cdot b = b \cdot a$, $\forall\, a, b \in G$, the group G is called *Abelian*.

Examples of Discrete Groups: S_n, Z_n and C_n

Group G	Multiplication	Order	Remarks
S_n: permutation of n objects	Successive operation	$n!$	Non-Abelian in general
Z_n: integers modulo n	Addition mod n	n	Abelian
C_n: cyclic group $\{e, a, \ldots a^{n-1}\}$ with $a^n = e = 1$	Unspecified $\cdot$ product	n	$C_n \cong Z_n$

Coset. Let $H = \{h_1, h_2, \ldots, h_r\}$ be a *proper* (i.e. $H \neq G$ and $H \neq I = \{e\}$) subgroup of G.
For a given $g \in G$, the sets

$$gH = \{gh_1, gh_2, \ldots, gh_r\}\,, \quad Hg = \{h_1 g, h_2 g, \ldots, h_r g\}$$

are called the *left* and *right cosets* of H.

Lagrange's Theorem. If $g_1 H$ and $g_2 H$ are two (left) cosets of H, then *either* $g_1 H = g_2 H$ *or* $g_1 H \cap g_2 H = \varnothing$.

Coset Decomposition. If H is a proper subgroup of G, then G can be decomposed into a sum of (left) cosets of H:

$$G \ = \ H \cup g_1 H \cup g_2 H \cdots \cup g_{\nu-1} H\,,$$

where $g_{1,2,\ldots} \in G$, $g_1 \notin H$; $g_2 \notin H$, $g_2 \notin g_1 H$, etc.

The number ν is called the index of H in G.

The set of all distinct cosets, $\{H, g_1 H, \ldots, g_{\nu-1} H\}$, is called *the coset space*, and is denoted by G/H.

...

Exercise: Prove Lagrange's Theorem.

Morphisms between Groups

Group Homomorphism. If $(A,\cdot)$ and $(B,\star)$ are two groups, then *group homomorphism* is a *functional* mapping f from the set A *into* the set B, i.e. each element of $a \in A$ is mapped into a single element of $b = f(a) \in B$, such that the following multiplication law is preserved:

$$f(a_1 \cdot a_2) = f(a_1) \star f(a_2)\,.$$

In general, $f(A) \neq B$, i.e. $f(A) \subset B$.

Group Isomorphism. Consider a $1:1$ mapping f of $(A,\cdot)$ *onto* $(B,\star)$, such that each element of $a \in A$ is mapped into a single element of $b = f(a) \in B$, and conversely, each element of $b \in B$ is the image resulting from a single element of $a \in A$. If this bijective $1:1$ mapping f satisfies the composition law:

$$f(a_1 \cdot a_2) = f(a_1) \star f(a_2)\,,$$

it is said to define an *isomorphism* between the groups A and B, and is denoted by $A \cong B$.

A group homomorphism of A into itself is called *endomorphism*.

A group isomorphism of A into itself is called *automorphism*.

– Continuous Groups

SL$(N,\mathbb{C})$, SO(N), SU(N), and SO(N,M)

Group	Properties	No. of indep. parameters	Remarks
GL$(N,\mathbb{C})$	det$M \neq 0$	$2N^2$	General rep
SL$(N,\mathbb{C})$	det$M = 1$	$2(N^2-1)$	SL$(N,\mathbb{C})$ $\subset$ GL$(N,\mathbb{C})$
O$(N,\mathbb{R})$	$\sum_{i=1}^N (x^i)^2$ $= \sum_{i=1}^N (x'^i)^2$	$\frac{1}{2}N(N-1)$	$O^T = O^{-1}$
SO$(N,\mathbb{R})$	as above + det$O = 1$	$\frac{1}{2}N(N-1)$	as above
SU(N)	$\sum_{i=1}^N \lvert x^i\rvert^2$ $= \sum_{i=1}^N \lvert x'^i\rvert^2$ det$U = 1$	N^2-1	$U^\dagger = U^{-1}$
SO(N,M)	$\sum_{i,j=1}^{N+M} x^i\eta_{ij}x^j$ $= \sum_{i,j=1}^{N+M} x'^i\eta_{ij}x'^j$ $\eta_{ij} =$ diag $(\underbrace{1,\ldots,1}_{N-\text{times}},\ \underbrace{-1,\ldots,-1}_{M-\text{times}})$	?	$\Lambda^T\eta\Lambda = \eta$ det $\Lambda = 1$

Useful Matrix Relations in GL$(N, \mathbb{C})$

Definitions:

$$
\begin{aligned}
\text{(i) } e^M \quad &\equiv \quad \sum_{n=0}^{\infty} \frac{M^n}{n!}\,; \\
\text{(ii) } \ln M \quad &\equiv \quad \sum_{n=1}^{\infty} (-1)^{n+1}\, \frac{(M-\mathbf{1})^n}{n} \\
&= \quad \int_0^1 du\,(M-\mathbf{1})\,\left[u(M-\mathbf{1})+\mathbf{1}\right]^{-1}\,,
\end{aligned}
$$

where $M \in$ GL$(N, \mathbb{C})$, i.e. $\det M \neq 0$.

Basic properties: If $[M_1,\ M_2] = 0$ and $M_{1,2} \in$ GL$(N, \mathbb{C})$, then the following relations hold:

$$
\text{(i) } e^{M_1}\, e^{M_2} \;=\; e^{M_1+M_2}\,, \quad \text{(ii) } \ln(M_1 M_2) \;=\; \ln M_1 + \ln M_2\,.
$$

Useful identity:

$$
\ln(\det M) \;=\; \mathrm{Tr}\,(\ln M)\,.
$$

This identity can be proved more easily if M can be diagonalized through a similarity trans: $S^{-1} M S = \widehat{M}$, where $\widehat{M}$ is a diagonal matrix, and noticing that $\ln M = S \ln \widehat{M}\, S^{-1}$. (*Question*: How?)

Generators and Exponential rep of Groups [Examples: SO(2), U(1), SO(3), and SU(2)]

SO(2): Transf. of a point $P(x, y)$ under a rotation through ϕ about z axis:

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} \;=\; \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}}_{\equiv\, O(\phi)} \begin{pmatrix} x \\ y \end{pmatrix}.
$$

Note that $O^T(\phi) O(\phi) = \mathbf{1}_2$ and hence $x^2 + y^2 = x'^2 + y'^2$, i.e. $O(\phi)$ is an orthogonal matrix, with det$O$=1.

SO(2) is an Abelian group, since $O(\phi)O(\phi') = O(\phi+\phi') = O(\phi')O(\phi)$.

Taylor expansion of $O(\phi)$ about $\mathbf{1}_2 = O(0)$:

$$
O(\delta\phi) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{:\ \mathbf{1}_2} - i\,\delta\phi \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{:\ \sigma_2 \;=\; i\frac{\partial O(\phi)}{\partial \phi}|_{\phi=0}} + \mathcal{O}[(\delta\phi)^2]\,,
$$

with $\sigma_2^2 = \mathbf{1}_2$ and $\sigma_2 = \sigma_2^\dagger$.

Exponential rep for finite ϕ:

$$
O(\phi) \;=\; \lim_{N\to\infty} [O(\phi/N)]^N \;=\; \exp[-i\phi\,\sigma_2]\,.
$$

The Pauli matrix σ_2 is the *generator* of the SO(2) group.

U(1): The 2-dim rep of SO(2) in $(V, \mathbb{R})$ can be reduced in $(V, \mathbb{C})$, by means of the trans:

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix},$$

i.e.

$$M^{-1}\, O(\phi)\, M = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = D^{(1)}(\phi) \oplus D^{(-1)}(\phi)\,.$$

Both reps, $D^{(1)}(\phi) = e^{i\phi}$ and $D^{(-1)}(\phi) = e^{-i\phi}$, are *faithful* irreps of U(1).

A general irrep of U(1) is

$$D^{(m)}(\phi) = e^{im\,\phi},$$

where $m \in \mathbb{Z}$. (*Question:* What is the generator of U(1)?)

SO(3): Group of proper rotations in 3-dim about a given unit vector $\mathbf{n} = (n_x, n_y, n_z) = (n_1, n_2, n_3)$, with $\mathbf{n}^2 = 1$.

Rotations about x, y, z-axes:

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix},$$

$$R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The *generators* $X_i = i \left.\frac{dR_i(\phi)}{d\phi}\right|_{\phi=0}$ of SO(3) are

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Equivalently, they can be represented as

$$(X_k)_{ij} = -i\,\varepsilon_{ijk}\,; \quad \varepsilon_{ijk} = \begin{cases} 1 & \text{for } (i,j,k) = (1,2,3) \\ & \text{and even permutations,} \\ -1 & \text{for odd permutations,} \\ 0 & \text{otherwise} \end{cases}$$

where ε_{ijk} is the Levi-Civita antisymmetric tensor.

General rep of a Group element of SO(3):

$$R(\phi, \mathbf{n}) = \exp(-i\phi\,\mathbf{n}\cdot\mathbf{X})\,,$$

with $\mathbf{X} = (X_1, X_2, X_3)$.

SU(2): Rotation of a *complex* 2-dim vector $\mathbf{v} = (v_1, v_2)$ (with $v_{1,2} \in \mathbb{C}$) through angle θ about $\mathbf{n}$:

$$\mathbf{v}' \;=\; U(\theta,\mathbf{n})\,\mathbf{v}\,; \qquad \mathbf{v}^* \cdot \mathbf{v} \;=\; \mathbf{v}'^* \cdot \mathbf{v}'\,,$$

with $\det U = 1$ and

$$U(\theta,\mathbf{n}) = \exp(-i\theta\mathbf{n}\cdot\tfrac{1}{2}\boldsymbol{\sigma}) \;=\; \mathbf{1}_2 \cos\tfrac{1}{2}\theta \;-\; i\boldsymbol{\sigma}\cdot\mathbf{n}\sin\tfrac{1}{2}\theta\,,$$

where $\mathbf{n}^2 = 1$ and $\boldsymbol{\sigma} = (\sigma_1,\ \sigma_2,\ \sigma_3)$ are the Pauli matrices.

$\therefore\ \frac{1}{2}\sigma_i$ are the *generators* of SU(2), with

$$\sigma_1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Properties: (i) $\mathrm{Tr}\,\sigma_i = 0$; (ii) $\sigma_i\sigma_j = \delta_{ij}\mathbf{1}_2 + i\,\varepsilon_{ijk}\,\sigma_k$.

Commutation relation: $\left[\frac{1}{2}\sigma_i\,,\,\frac{1}{2}\sigma_j\right] \;=\; i\,\varepsilon_{ijk}\,\frac{1}{2}\sigma_k$.

Exact Relation between SO(3) and SU(2) Groups:

Since $R(0)$ and $R(2\pi)$ [with $R(0) = R(2\pi) = \mathbf{1}_3$] map into different elements $U(0) = \mathbf{1}_2$ and $U(2\pi) = -\mathbf{1}_2$, a *faithful* $1:1$ isomorphic mapping is

$$\mathrm{SO}(3) \;\cong\; \mathrm{SU}(2)/Z_2\,,$$

where $Z_2 \;=\; \{\mathbf{1}_2,\ -\mathbf{1}_2\}$ is a subgroup of SU(2).

...

Exercises:

Verify that the generators of the SO(3) and SU(2) groups satisfy:

(i) the *commutation relation*:

$$[X_i,\ X_j] \;\equiv\; X_iX_j - X_jX_i \;=\; i\varepsilon_{ijk}X_k\,.$$

(Need to use that $(X_k)_{ij} = -i\varepsilon_{ijk}$ and $\varepsilon_{ijm}\varepsilon_{klm} \;=\; \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.)

(ii) the *Jacobi identity*:

$$[X_1,\ [X_2,\ X_3]] + [X_3,\ [X_1,\ X_2]] + [X_2,\ [X_3,\ X_1]] \;=\; 0\,.$$

– Lie Algebra and Lie Groups

A **Lie algebra** L is defined by a set of a number $d(G)$ of *generators* T_a closed under commutation:

$$[T_a,\ T_b]\ =\ T_a\cdot T_b\ -\ T_b\cdot T_a\ =\ if^c_{ab}T_c\,,$$

where f^c_{ab} are the so-called *structure constants* of L.
In addition, the generators T_a's satisfy the **Jacobi identity**:

$$[T_a,\ [T_b,\ T_c]]\ +\ [T_c,\ [T_a,\ T_b]]\ +\ [T_b,\ [T_c,\ T_a]]\ =\ 0\,.$$

The set T_a of generators define a basis of a $d(G)$-dimensional vector space $(V,\mathbb{C})$.

In the **fundamental rep**, T_a are represented by $d(F)\times d(F)$ matrices, where $d(F)$ is the *least* number of dimensions needed to generate the Lie algebra L and the *respective* continuous group G.

Ex: (i) SO(3): $T_a = X_a$; (ii) SU(2): $T_a = \frac{1}{2}\sigma_a$; (iii) U(1): ?

Exponentiation of T_a generates the group elements of the corresponding continuous Lie group G:

$$G(\theta,\mathbf{n}) = \exp[-i\theta\mathbf{n}\cdot\mathbf{T}]\ \in\ G,$$

with $\theta\in\mathbb{R}$ and $\mathbf{n}^2 = 1$.

– Group Representations

The Lie algebra commutator $[T_c,\ \ ]$ (for fixed T_c) defines a linear homomorphic mapping from L to L over $\mathbb{C}$:

$$[T_c,\ \lambda_1 T_a + \lambda_2 T_b]\ =\ \lambda_1[T_c,T_a]\ +\ \lambda_2[T_c,\ T_b]\,,$$

$\forall\ T_a, T_b \subset L$.

For every given $T_a\in L$, $[T_a,\ \ ]$ may be represented in the vector space L by the structure constants themselves:

$$[D_{\mathcal{A}}(T_a)]^c{}_b\ =\ if^c_{ab}\quad (=\ -if^c_{ba})\,.$$

Such a rep of T_a is called the **adjoint representation**, denoted by $\mathcal{A}$.

The Killing product form is defined as

$$g_{ab}\ \equiv\ (T_a,T_b)_{\mathcal{A}}\ \equiv\ \mathrm{Tr}[D_{\mathcal{A}}(T_a)D_{\mathcal{A}}(T_b)]\quad (\equiv\ \mathrm{Tr}_{\mathcal{A}}(T_aT_b)\,)\,.$$

$g_{ab} = -f^d_{ac}f^c_{bd}$ is called the *Cartan metric*.

The Cartan metric g_{ab} can be used to lower the index of f^c_{ab}:

$$f_{abc}\ =\ f^d_{ab}\,g_{dc}\,.$$

Exercise: Show that $f_{abc}\ =\ -i\,\mathrm{Tr}_{\mathcal{A}}([T_a,\ T_b]\,T_c)$, and that f_{abc} is totally antisymmetric under the permutation of a,b,c: $f_{abc} = -f_{bac} = f_{bca}$ etc.

Normalization of Generators and Casimir operators

The generators of a Lie group $D_R(T_a)$ of a given rep R are normalized as

$$\mathrm{Tr}\,[D_R(T_a)\,D_R(T_b)] \;=\; T_R\,\delta_{ab}\,.$$

For example, for SU(N), $T_F = \frac{1}{2}$ for the fundamental rep and $T_{\mathcal{A}} = N$ for the adjoint rep.

Casimir operators $\mathbf{T}_R^2$ of a Lie algebra of a rep R are matrix reps that commute with all generators of L in rep R.

A construction of a Casimir operator $\mathbf{T}_R^2$ in a given rep R of SU(N) [or SO(N)] may be obtained by

$$(\mathbf{T}_R^2)_{ij} = T_{\mathcal{A}} \sum_{a,b=1}^{d(G)} \sum_{k=1}^{d(R)} [D_R(T_a)]_{ik}\, g^{ab}\, [D_R(T_b)]_{kj} \;=\; \delta_{ij}\, C_R\,,$$

where g^{ab} is the inverse Cartan metric satisfying: $g^{ab}\, g_{bc} = \delta^a_c$.

Exercises:

Show that

(i) $g_{ab} = T_{\mathcal{A}}\,\delta_{ab}$, for SU($N$) and SO($N$) theories;
(ii) $[\mathbf{T}_F^2,\ T_a] = 0$;
(iii) $T_R\, d(G) \;=\; C_R\, d(R)$;
(iv) $C_F = \frac{N^2-1}{2N}$ and $C_{\mathcal{A}} = N$ in SU(N).

3. Quantum Chromodynamics

– Non-Abelian Gauge Invariance

The Lagrangian of an SU(N) Yang–Mills (non-Abelian) theory is

$$\mathcal{L}_{\mathrm{YM}} \;=\; -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a,\mu\nu}\,,$$

where

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g\, f^{abc}\, A^b_\mu\, A^c_\nu\,,$$

and $f^{abc} = f_{abc}$ are the structure constants of the SU(N) Lie algebra.

Examples of SU(N) theories:

The SU$(2)_L$ group of the SM predicting 3 weak bosons W^i_μ (with $i = 1, 2, 3$) responsible for the electroweak force.

Quantum Chromodynamics (QCD) based on the SU$(3)_c$ group predicts 8 gluons $A^a_\mu \equiv G^a_\mu$ (with $a = 1, 2, \ldots, 8$) mediating the strong force between quarks.

Gauge bosons of Yang–Mills (YM) theories self-interact! (*How and Why?*)

$\cdots$

Exercise: Show that $\mathcal{L}_{\mathrm{YM}}$ is invariant under the infinitesimal SU(N) local trans:

$$\delta A^a_\mu \;=\; -\frac{1}{g}\, \partial_\mu \theta^a \;-\; f^{abc}\, \theta^b\, A^c_\mu\,.$$

Interaction between Quarks q_i and Gluons G^a_μ in SU(3)$_c$

If $q_i = (q_{\rm red},\ q_{\rm green},\ q_{\rm blue})$ are the 3 colours of the quark q, the interaction of q_i with the 8 gluons G^a_μ is described by the Lagrangian:

$$\mathcal{L}_q \;=\; \bar{q}_i \Big[\, i\,\not{\partial}\,\delta_{ij} \;-\; m_q\,\delta_{ij} \;-\; g_s\,\not{G}^a (T^a)_{ij} \Big]\, q_j\,.$$

Exercise: Show that $\mathcal{L}_q$ is invariant under the SU(3) gauge transformation:

$$\delta G^a_\mu \;=\; -\frac{1}{g_s}\,\partial_\mu\theta^a \;-\; f^{abc}\,\theta^b\,G^c_\mu\,,\qquad \delta q_i \;=\; i\theta^a\,(T^a)_{ij}\,q_j\,,$$

where $T^a = \frac{1}{2}\,\lambda^a$ are the generators of SU(3) and λ^a are the Gell-Mann matrices:

$$\lambda^{1,2,3} \;=\; \begin{pmatrix} \sigma^{1,2,3} & 0 \\ 0 & 0 \end{pmatrix},\qquad \lambda^4 \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 \;=\; \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\qquad \lambda^6 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\qquad \lambda^8 \;=\; \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.$$

– Gauge Fixing in Yang–Mills Theories

Exactly as in QED (see p. 12), to obtain a *non*-singular gauge-field propagator $\Delta^{ab}_{\mu\nu}(x-y)$ in YM theories, we must add to $\mathcal{L}_{\rm YM}$ a **covariant gauge-fixing term**:

$$\mathcal{L}_{\rm GF} \;=\; -\frac{1}{2\xi}\,(\partial_\mu A^{a,\mu})\,(\partial_\nu A^{a,\nu})\,.$$

The Euler-Lagrange equation for a *free* YM gauge field A^a_μ $(g=0)$ is

$$\Big[\eta_{\mu\nu}\,\partial_\kappa\partial^\kappa \;-\; \Big(1-\frac{1}{\xi}\Big)\,\partial_\mu\partial_\nu\Big]\,A^{a,\nu} \;=\; 0\,.$$

The gauge-field propagator $\Delta^{ab}_{\mu\nu}(x-y)$ is the Green's function of the above linear differential operator:

$$\Big[\eta^{\mu\nu}\,\frac{\partial}{\partial x^\kappa}\frac{\partial}{\partial x_\kappa} - \Big(1-\frac{1}{\xi}\Big)\,\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\Big]\,\Delta^{ab}_{\nu\lambda}(x-y) \;=\; \delta^{ab}\delta^\mu_\lambda\,\delta^{(4)}(x-y)\,.$$

...

Exercises:

(i) Derive the Euler-Lagrange equation of motion for the free YM field A^a_μ in the presence of $\mathcal{L}_{\rm GF}$.

(ii) Show that the gauge-field propagator is given by the Green's function:

$$\Delta^{ab}_{\mu\nu}(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\,\Big(-\eta_{\mu\nu} + (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\Big)\,\frac{\delta^{ab}\,e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}\,.$$

– Fadeev–Popov Ghosts and BRS Symmetry

The gauge-fixing term $\mathcal{L}_{\rm GF}$ violates the local SU(N) symmetry of $\mathcal{L}_{\rm YM}$. To restore this symmetry, we first introduce in the theory new Grassman-valued complex fields c^a and $\bar{c}^a$, the so-called **Fadeev–Popov** (FP) **ghosts**. This results in a new Lagrangian term for the FP ghosts:

$$\mathcal{L}_{\rm FP} \;=\; -\,\bar{c}^a\,\partial^\mu\left[\delta^{ab}\partial_\mu \,+\, g f^{abc}\,A^c_\mu\right] c^b\;.$$

As shown in 1974 by Becchi, Rouet and Stora (BRS), the extended Lagrangian $\mathcal{L} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$ is invariant under the **BRS transformations**:

$$\begin{aligned}
\delta A^a_\mu \;&\equiv\; \omega\, s A^a_\mu \;=\; \omega\left[\delta^{ab}\partial_\mu \,+\, g f^{abc}\,A^c_\mu\right] c^b\;,\\
\delta c^a \;&\equiv\; \omega\, s c^a \;=\; \omega\,\frac{1}{2}\,g f^{abc}\,c^b\,c^c\;,\\
\delta \bar{c}^a \;&\equiv\; \omega\, s \bar{c}^a \;=\; -\,\omega\,\frac{1}{\xi}\,\partial^\mu A^a_\mu\;,
\end{aligned}$$

with $\omega^2 = 0$.

Remark: The BRS symmetry plays an important for ensuring unitarity and renormalizability of non-Abelian gauge theories, including spontaneously broken gauge theories, such as the Standard Model (see next section).

...

Exercises:

(i) Show that $\mathcal{L}_{\rm GF}$ is invariant under *global* SU(N) gauge transformations: $\delta A^a_\mu \;=\; f^{abc}\,\theta^b\,A^c_\mu$, for which $\partial_\mu\theta^a = 0$. What happens if $\partial_\mu\theta^a \neq 0$?

(ii) Show that $\mathcal{L} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$ is invariant under BRS transformations.

(iii) Show that the quark–gauge field Lagrangian $\mathcal{L}_q$ given on p. 29 is also invariant under BRS transformations, provided the quark field q_i transforms as follows:

$$\delta q_i \;\equiv\; \omega\, s q_i \;=\; -\,\omega\, ig\,(T^a)_{ij}\,c^a\,q_j\;.$$

(iv) Show that $s^2 A^a_\mu \;=\; s^2 q_i \;=\; s^2 c^a \;=\; 0$, but $s^2\bar{c}^a = -\frac{1}{\xi}\,\partial^\mu\big[\delta^{ab}\partial_\mu \,+\, g f^{abc}\,A^c_\mu\big]\,c^b$. What should one impose upon the ghost fields to also get $s^2\bar{c}^a = 0$?

(v) **The θ term in YM theories.** Show that the term,

$$\mathcal{L}_\theta \;=\; -\,\frac{\theta}{4}\,F^a_{\mu\nu}\,\widetilde{F}^{a,\mu\nu}\;,$$

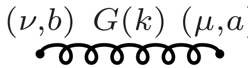
is gauge- and BRS-invariant, and so it can be added to $\mathcal{L}_{\rm YM}$, where $\widetilde{F}^{a,\mu\nu} \;=\; \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}F^a_{\rho\sigma}$ (with the convention $\varepsilon^{0123} = +1$). Verify that $\mathcal{L}_\theta$ is a total derivative.

– QCD Feynman Rules

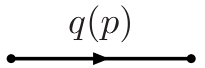
The Feynman rules are derived from the Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm QCD} \;&=\; -\frac{1}{4}\,G^a_{\mu\nu}\,G^{a,\mu\nu} + \bar{q}_i\left[\,i\,\partial\!\!\!/\,\delta_{ij} - m_q\,\delta_{ij} - g_s G\!\!\!/^a (T^a)_{ij}\right] q_j \\ &\quad -\frac{1}{2\xi}\,(\partial_\mu G^{a,\mu})\,(\partial_\nu G^{a,\nu}) - \bar{c}^a\,\partial^\mu\left[\delta^{ab}\partial_\mu + g_s f^{abc}\,G^c_\mu\right] c^b\,.\end{aligned}$$

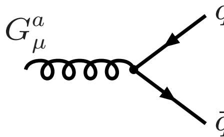
All momenta flow into the 3-gluon vertex: $k+p+q=0$.



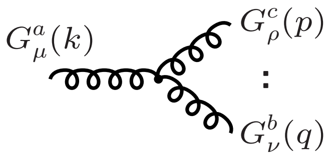
: $$\frac{i\delta^{ab}\left(-\eta_{\mu\nu} + (1-\xi)\dfrac{k_\mu k_\nu}{k^2}\right)}{k^2 + i\varepsilon}$$



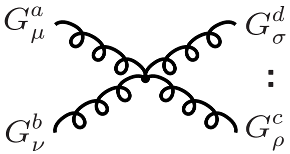
: $$\frac{i}{p\!\!\!/ - m_q + i\varepsilon}$$



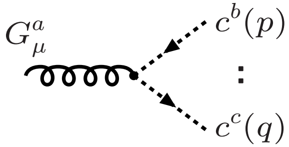
: $$-ig_s\,\gamma_\mu\,\frac{(\lambda^a)_{ij}}{2}$$



: $$-g_s\,f^{abc}\Big[\eta^{\mu\nu}(k-q)^\rho + \eta^{\nu\rho}(q-p)^\mu + \eta^{\rho\mu}(p-k)^\nu\Big]$$



: $$-ig_s^2\left\{\begin{array}{l} f^{xab}f^{xcd}\big(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\big) \\ +f^{xac}f^{xdb}\big(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\big) \\ +f^{xad}f^{xbc}\big(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}\big)\end{array}\right\}$$



: $$-g_s\,f^{abc}\,q_\mu$$

– Asymptotic Freedom and Confinement

The Renormalization Group (RG)

To *all orders* in perturbation theory, the renormalized effective action Γ does not depend on the UV cut-off scale Λ or the 't Hooft mass scale μ in the Minimal Subtraction (MS) scheme.

For a scalar theory with $\mathcal{L}_{\rm int} = \frac{1}{4!}\lambda\phi^4$, we have in MS scheme

$$\phi^n(\mu)\,\Gamma^{(n)}[\lambda(\mu), m(\mu), \mu] \;=\; \phi^n(\mu_0)\,\Gamma^{(n)}[\lambda(\mu_0), m(\mu_0), \mu_0]\,,$$

where $\Gamma^{(n)}$ is the n-point One-Particle-Irreducible (1PI) Green's function, m is the mass of the real scalar field ϕ, and μ and μ_0 are two arbitrary renormalization scales, with μ_0 being some reference scale.

...

Exercise: Given the relations between bare and renormalized quantities: $\phi_0 = Z_\phi^{1/2}\,\phi\,,\;\; m_0 = Z_{m^2}\,m^2\,,\;\; \lambda_0 = Z_\lambda\,\lambda$, show that the μ-dependence of the latter are determined by the differential equations

$$\begin{aligned}\gamma_\phi \;&\equiv\; \mu\,\frac{d\ln\phi(\mu)}{d\mu} \;=\; -\frac{1}{2}\,\mu\frac{d\ln Z_\phi}{d\mu}\,,\\ \beta_\lambda \;&\equiv\; \mu\,\frac{d\lambda(\mu)}{d\mu} \;=\; -\,\mu\frac{d\ln Z_\lambda}{d\mu}\,\lambda,\\ \gamma_{m^2} \;&\equiv\; \mu\,\frac{d\ln m^2(\mu)}{d\mu} \;=\; -\,\mu\frac{d\ln Z_{m^2}}{d\mu}\,.\end{aligned}$$

The relation between two Green's functions renormalized at two different scales μ and μ_0 is given by

$$\Gamma^{(n)}(\mu) \;=\; R^{-n}(\mu;\mu_0)\,\Gamma^{(n)}(\mu_0)$$

where $R(\mu;\mu_0) = \exp\left[\int_{\mu_0}^{\mu}\gamma_\phi(\mu')\,d\ln\mu'\right]$ (*Why?*).

The successive renormalizations from one scale μ_0 to another μ with composition law

$$R(\mu;\mu_0) \;\equiv\; R(\mu;\mu_I)\,R(\mu_I;\mu_0)\;,$$

where μ_I is an arbitrary intermediate scale, form a group (*Why?*), the so-called **Renormalization Group** (RG).

Remark. The above result is general and holds true for any other scheme of renormalization and/or regularization, e.g. cut-off regularization, Pauli-Villars regularization, lattice regularization etc.

The differential equations given on the previous page, which determine the running of the parameters λ and m, and the field ϕ, as functions of μ, are called the **Renormalization Group Equations** (RGEs).

...

Exercise: Use the RGE for the field ϕ to show that

$$\phi(\mu) \;=\; \exp\left[\int_{\mu_0}^{\mu}\gamma_\phi(\mu')\,d\ln\mu'\right]\phi(\mu_0)\;.$$

If the RG scale μ is identified with the typical energy of a scattering process, one then observes that the parameters λ and m change with energy, as determined by their RGEs.

Theories, for which $\lambda(\mu) \to 0$ as $\mu \to \infty$, are said to be **asymptotically free**, or they possess **asymptotic freedom**. The only known examples of such theories are pure YM theories, such as QCD, for which $g_s(\mu) \to 0$ as $\mu \to \infty$, where g_s is the strong coupling constant.

In all known asymptotically free theories, such as YM theories, the gauge coupling $g(\mu)$ becomes non-perturbative ($g \gg 1$) below some scale $\mu < \Lambda_{\rm YM}$. The scale $\Lambda_{\rm YM}$ is called the **confinement scale**, below which the perturbative theory is no longer applicable, and new phenomena due to quark *and* gluon bound states take place. This non-perturbative phase of the theory for energies below $\Lambda_{\rm YM}$ is called **confinement**.

In QCD, the value of the confinement scale $\Lambda_{\rm QCD}$ is around 300 MeV, close to the neutral pion mass $m_{\pi^0} \simeq 134$ MeV. Below this scale, quarks *and* gluons confine to produce mesons and hadrons, e.g. p, n, π^0, $\pi^\pm$ etc. Also, pions become effectively the mediators of the nuclear force.

Remark. QCD still remains the fundamental theory of strong interactions, even for energies beyond the confinement scale. Based on the QCD Lagrangian, **lattice field theories** give remarkable predictions for the mass spectrum of hadrons and mesons consistent with experimental observations, within the level of the achieved theoretical accuracy.

4. The Standard Model for Electroweak Interactions

– Spontaneous Symmetry Breaking

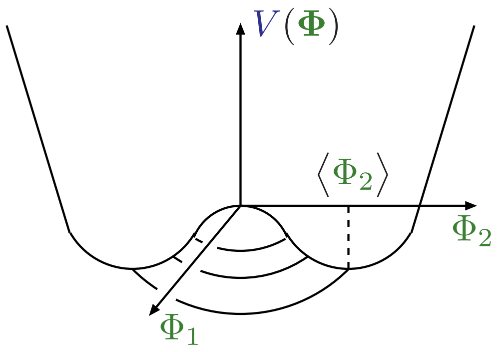
Consider the Lagrangian of a theory with an SO(2)-invariant scalar sector

$$\mathcal{L} \;=\; \frac{1}{2}\,(\partial_\mu \Phi_i)\,(\partial^\mu \Phi_i) \;-\; V(\mathbf{\Phi})\;,$$

with $\mathbf{\Phi} \equiv \{\Phi_i\} = (\Phi_1,\, \Phi_2)$ and

$$V(\mathbf{\Phi}) \;=\; \frac{m^2}{2}\,(\Phi_1^2 + \Phi_2^2) \;+\; \frac{\lambda}{4}\,(\Phi_1^2 + \Phi_2^2)^2\;.$$

For $m^2 < 0$, the scalar potential has the following shape:



$\boldsymbol{m^2 < 0}$ **and** $\boldsymbol{\lambda > 0}$

...

Exercise: Find the shape of the scalar potential for $m^2 > 0$.

The extrema of the potential $V(\mathbf{\Phi})$ for homogeneous fields $\Phi_{1,2}(x) = \text{const.}$ are determined by the *minimization* or *vacuum* equations:

$$\frac{\partial V}{\partial \Phi_1} \;=\; \Phi_1\left[m^2 \,+\, \lambda\big(\Phi_1^2 + \Phi_2^2\big)\right] \;=\; 0\,,$$
$$\frac{\partial V}{\partial \Phi_2} \;=\; \Phi_2\left[m^2 \,+\, \lambda\big(\Phi_1^2 + \Phi_2^2\big)\right] \;=\; 0\,.$$

There are now two distinct cases (*always* assuming $\lambda > 0$):

(i) For $m^2 > 0$, the only real solution is

$$\Phi_1^2 \,+\, \Phi_2^2 \;=\; 0 \quad\Longrightarrow\quad \langle\Phi_1\rangle \;=\; \langle\Phi_2\rangle \;=\; 0\,.$$

No breaking of the SO(2) symmetry by the ground state $\langle\mathbf{\Phi}\rangle = \mathbf{0}$.

(ii) For $m^2 < 0$, there are infinitely many *vacuum* solutions determined by

$$\Phi_1^2 \,+\, \Phi_2^2 \;=\; v^2 \;=\; -\,\frac{m^2}{\lambda} \;>\; 0\,.$$

Spontaneous breaking of the SO(2) symmetry by the ground state $\langle\mathbf{\Phi}\rangle \neq \mathbf{0}$. The vacuum solutions are all *degenerate* in energy. They form a manifold $\mathcal{M}$ in $\mathbf{\Phi}$-*space* homeomorphic to circle S^1, which is called the *vacuum manifold*.

Physical mass spectrum

To determine the physical spectrum for case (ii), we first pick one point from $\mathcal{M} \sim S^1$, e.g.

$$\langle \Phi_1 \rangle \; = \; 0 \; , \qquad \langle \Phi_2 \rangle \; = \; v \; ,$$

and expand $\Phi_{1,2}$ linearly about this vacuum solution as follows:

$$\Phi_1(x) \; = \; \pi(x) \; , \qquad \Phi_2(x) \; = \; v \, + \, \sigma(x) \; ,$$

where $\pi(x)$ and $\sigma(x)$ are the physical fields.

In terms of the new fields $\pi(x)$ and $\sigma(x)$, the Lagrangian $\mathcal{L}$ reads

$$\mathcal{L} \; = \; \frac{1}{2}\Big[\partial_\mu \pi)\,(\partial^\mu \pi) + (\partial_\mu \sigma)(\partial^\mu \sigma)\Big] \; - \; \lambda\, v^2 \sigma^2$$
$$- \, \lambda\, v\, \sigma \left(\pi^2 + \sigma^2\right) \; - \; \frac{\lambda}{4}\,(\pi^2 + \sigma^2)^2 \; .$$

Note that there is no quadratic mass term $\propto \pi^2$ in $\mathcal{L}$. This implies that the field $\pi(x)$ is massless, i.e. it is a *massless Goldstone boson* ($m_\pi = 0$). The field $\sigma(x)$ is massive with mass $m_\sigma = \sqrt{2\lambda}\, v$ (*Why?*).

...

Exercise: Use the Lagrangian $\mathcal{L}$ to derive the Feynman rules for all interactions between π and σ.

– The Goldstone Theorem

Spontaneous breakdown of a continuous global symmetry implies the existence of massless particles in theories with more than $1+1$ dimensions.

Goldstone's theorem: If a Lagrangian $\mathcal{L}$ of a theory possesses a global symmetry group G which breaks *spontaneously* to a smaller symmetry group $H \subset G$, *then* there exists one *massless* Goldstone boson for each *broken* generator X^b of G.

The *broken* generators $\{X^b\} = (T^1, T^2, \dots, T^\nu)$ of G create a vacuum manifold $\mathcal{M}$ given by the *coset space*: $\mathcal{M} = G/H$.

Proof (in the tree approximation):

Consider the Lagrangian

$$\mathcal{L} \; = \; \frac{1}{2}\,(\partial_\mu \Phi_i)\,(\partial^\mu \Phi_i) \; - \; V(\mathbf{\Phi}) \; + \; \dots \; ,$$

where the ellipses denote other interaction terms irrelevant to our proof, and $\mathbf{\Phi} = \{\Phi_i\} = (\Phi_1,\, \Phi_2,\, \dots, \Phi_n)$ represents n real scalar fields.

The Lagrangian $\mathcal{L}$ is invariant under the symmetry group G, which acts on Φ_i as follows:

$$\Phi_i \quad \to \quad \Phi'_i \; = \; \Phi_i \, + \, i\theta^a T^a_{ij}\, \Phi_j \; ,$$

where T^a are the generators of G, e.g. $G = \mathrm{SO}(n)$.

Given that the potential $V(\Phi_i)$ is also invariant under the action of G, i.e. $V(\mathbf{\Phi}) = V(\mathbf{\Phi}')$, we have

$$\delta V \;\equiv\; V(\mathbf{\Phi}) - V(\mathbf{\Phi}') \;=\; 0 \quad\Longrightarrow\quad \frac{\partial V}{\partial \Phi_i}\, T^a_{ij}\, \Phi_j \;=\; 0\,. \qquad \textbf{(A)}$$

If $\mathbf{v} = \{v_i\} = (v_1,\, v_2,\, \ldots,\, v_n)$ is one solution to the *vacuum equation*: $\partial V/\partial \Phi_i\big|_{\mathbf{\Phi}\,=\,\mathbf{v}} = 0$, *then* $\mathbf{v}' = \{v'_i\} = \exp\big(i\theta^a T^a\big)\mathbf{v}$ is another equivalent solution. The complete set of all vacuum solutions forms a manifold $\mathcal{M}$, called the **vacuum manifold**.

We now expand $\mathbf{\Phi}$ about its physical vacuum $\mathbf{v}$ as

$$\mathbf{\Phi} \;=\; \boldsymbol{\phi} + \mathbf{v} \quad\Longleftrightarrow\quad \Phi_i \;=\; \phi_i \,+\, v_i\;,$$

where $\boldsymbol{\phi} = \{\phi_i\} = (\phi_1,\, \phi_2,\, \ldots,\, \phi_n)$ represents the physical fields. The potential V can be rewritten as

$$V(\mathbf{\Phi}) \;=\; V(\mathbf{v}) \,+\, \frac{1}{2} M^2_{ij}\, \phi_i \phi_j \,+\, \ldots\;,$$

where $V(\mathbf{v})$ is a constant and

$$M^2_{ij} \;=\; \frac{\partial^2 V}{\partial \Phi_i \Phi_j}\bigg|_{\mathbf{\Phi}\,=\,\mathbf{v}}$$

is the mass matrix for the physical scalar fields ϕ_i.

Differentiating (**A**) w.r.t. Φ_k and then setting $\Phi_k = v_k$ yields

$$M^2_{ki}\, T^a_{ij}\, v_j \;=\; 0\,. \qquad \textbf{(B)}$$

From (**B**), we see that there are two categories for the generators $T^a = (T^1, T^2, \ldots, T^{n_G}) = (X^b,\, Y^c)$ of the group G:

(i) The *broken* generators X^b of G, for which $X^b\mathbf{v} \neq \mathbf{0}$, with $\{X^b\} = (T^1, T^2, \ldots, T^\nu)$ and $\nu \leq n_G$.

(ii) The *unbroken* generators Y^c of G, for which $Y^c\mathbf{v} = \mathbf{0}$, with $\{Y^c\} = (T^{\nu+1}, \ldots, T^{n_G})$. These generators also form a *little* group $H \subset G$.

Only the *broken* generators give rise to *non-null* eigenvectors in (**B**), such that $\|X^b\,\mathbf{v}\| \neq 0$, which correspond to the *massless Goldstone bosons*:

$$G^b(x) \;=\; \frac{(iX^b\,\mathbf{v})_j}{\|X^b\,\mathbf{v}\|}\,\phi_j(x)\;, \qquad \textbf{(C)}$$

with $b = 1,\, 2,\, \ldots,\, \nu$.

Exceptions to the Goldstone theorem:

(i) For local gauge symmetries, the Goldstone bosons can be gauged away via the Higgs mechanism and so be removed from the physical spectrum.

(ii) There are no Goldstone bosons in theories with $1+1$ dimensions.

...

Exercises:

(i) Show that the unbroken generators Y^c form a subgroup H of G, including the possibility of $H \equiv \mathbb{I}$.

(ii) Show that the vacuum manifold $\mathcal{M}$ is given by the coset space: $\mathcal{M} = G/H$.

(iii) Prove that the Goldstone fields $G^b(x)$ as defined in (**C**) do not have mass terms in the potential $V(\mathbf{\Phi})$, and hence they are truly massless. Likewise, explain why all other scalar fields $H^c(x)$ orthogonal to $G^b(x)$ are in general massive.

(iv) Show that if the SU(2) group breaks spontaneously in its fundamental representation, it then breaks completely to the identity group $\mathbb{I}$: SU(2) $\xrightarrow{\langle\mathbf{\Phi}\rangle} \mathbb{I}$, where $\mathbf{\Phi} = (\Phi_1,\, \Phi_2)^{\mathsf{T}}$ is an SU(2) doublet consisting of two complex scalar fields.

– The Higgs Mechanism [P. W. Higgs '64; F. Englert, R. Brout '64.]

Consider the Abelian U(1) Higgs model described by the Lagrangian:

$$\mathcal{L}_\Phi \;=\; (D_\mu\Phi)^\dagger\,(D^\mu\Phi) \;-\; V(\Phi)\;,$$

where $D_\mu\Phi = \left(\partial_\mu + \frac{i}{2}e\,A_\mu\right)\Phi$, A_μ is the gauge field of the U(1) local group, Φ is a complex scalar charged under U(1), and $V(\Phi)$ is the scalar potential:

$$V(\Phi) \;=\; -\,\mu^2\,\Phi^\dagger\Phi \;+\; \lambda\,(\Phi^\dagger\Phi)^2\;,$$

with $\mu^2 > 0$ and $\lambda > 0$.

We expand Φ about its physical vacuum $\langle\Phi\rangle = v/\sqrt{2}$ as

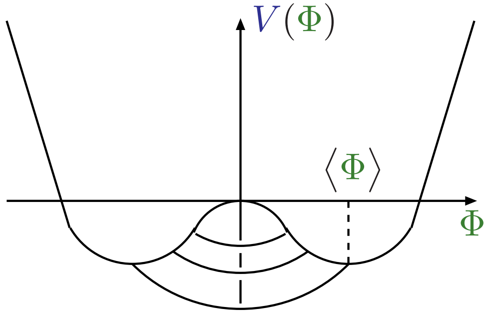
$$\Phi \;=\; \frac{1}{\sqrt{2}}\Big(v \,+\, H \,+\, iG\Big)\;.$$

From $\mathcal{L}_\Phi$, we find that the field A_μ receives a mass given by $M_A = ev/2$, whereas G becomes the longitudinal polarization for the massive A_μ boson in the unitary gauge. This mass generation for A_μ is called the Higgs–Englert–Brout mechanism, or in short the **Higgs mechanism**.

The Higgs mechanism also predicts a massive scalar boson, the so-called **Higgs boson**, with mass $M_H = \sqrt{2\lambda}v$ (*Why?*).

The Higgs Mechanism in the Standard Model

The SM Higgs potential: $V(\Phi) \; = \; -\mu^2\,\Phi^\dagger\Phi \; + \; \lambda\,(\Phi^\dagger\Phi)^2$.



Pattern of Spontaneous Symmetry Breaking (SSB):
$\mathrm{SU(2)}_{\boldsymbol{L}} \otimes \mathrm{U(1)}_{\boldsymbol{Y}} \xrightarrow{\langle\Phi\rangle} \mathrm{U(1)}_{\rm em}$, where the ground state:

$$\langle\Phi\rangle \; = \; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad \text{with} \quad v \; = \; \sqrt{\frac{\mu^2}{\lambda}}\,,$$

carries weak charge, but no electric charge *or* colour.

$\Rightarrow$ $W^\pm$, Z gauge bosons interact with $\langle\Phi\rangle$ and become massive, but not γ and g^a, e.g. $M_W = \frac{1}{\sqrt{2}}\,g\langle\Phi\rangle$.

$\Rightarrow$ Matter fermions $f \; = \; \nu_e, \nu_\mu, \nu_\tau, e, \mu, \tau, u, d, s, c, b, t$ also interact with $\langle\Phi\rangle$ and become massive, via the so-called Yukawa interactions, e.g. $m_f = Y_f\,\langle\Phi\rangle$.

$\Rightarrow$ Quantum excitations of $\Phi = \langle\Phi\rangle \, + \, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ H \end{pmatrix}$,
where H is the Higgs boson observed in 2012,
with spin $= 0$.

Exercises:

(i) Prove the electroweak symmetry breaking pattern for the SM Higgs potential:

$$\mathsf{SU(2)}_L \otimes \mathsf{U(1)}_Y \;\xrightarrow{\langle\Phi\rangle}\; \mathsf{U(1)}_{\rm em}\,,$$

where Φ is a colourless SU(2) doublet, with hypercharge quantum number $Y(\Phi) = y_\Phi = 1/2$.

(ii) The scalar-kinetic term of the SM Lagrangian is

$$\mathcal{L}_\Phi \;\; = \;\; (D_\mu\Phi)^\dagger\,(D^\mu\Phi) \; - \; V(\Phi)\,,$$

where $D_\mu\Phi \; = \; \big(\partial_\mu + \frac{i}{2} g\,\sigma^i W^i_\mu + \frac{i}{2} g' B_\mu\big)\,\Phi$, and W^i_μ and B_μ are the gauge fields of the $\mathsf{SU(2)}_L$ and $\mathsf{U(1)}_Y$ local groups, respectively. Using $\mathcal{L}_\Phi$, show that after SSB the mass eigenstates Z_μ and A_μ are given in terms of the weak-basis fields W^3_μ and B_μ as follows:

$$Z_\mu \;\; = \;\; c_w\,W^3_\mu \; - \; s_w\,B_\mu\,, \qquad A_\mu \;\; = \;\; s_w\,W^3_\mu \; + \; c_w\,B_\mu\,,$$

with $s_w \equiv \sin\theta_w$, $c_w \equiv \cos\theta_w$ and $t_w = s_w/c_w = g'/g$. Moreover, evaluate the masses of the physical $W^\pm$ and Z bosons.

(iii) With the aid of $\mathcal{L}_\Phi$, calculate the mass of the Higgs boson H, all its self-interactions, as well as its interactions with the gauge bosons $W^\pm$, Z, γ in the unitary gauge.

– Fermions in the SM

The gauge-kinetic Lagrangian for a SM fermion f is generically given by

$$\mathcal{L}_f \;=\; \bar{f}_L\, i\gamma^\mu D^L_\mu f_L \;+\; \bar{f}_R i\gamma^\mu D^R_\mu f_R\,,$$

with $f = \nu_e, \nu_\mu, \nu_\tau, e, \mu, \tau, u, d, s, c, b, t$ and D^L_μ (D^R_μ) are the left (right) covariant derivatives acting on (left-) right-handed *chiral* fermions. For example, for colourless fermions, such as $f = \nu_e, \nu_\mu, \nu_\tau, e, \mu, \tau$,

$$\begin{aligned} D^L_\mu\, f_L &= \left(\partial_\mu + \frac{i}{2} g\,\sigma^i W^i_\mu + \frac{i}{2} g' y_{f_L} B_\mu\right) f_L\,,\\ D^R_\mu\, f_R &= \left(\partial_\mu + \frac{i}{2} g' y_{f_R} B_\mu\right) f_R\,, \end{aligned}$$

where y_{f_L} (y_{f_R}) is the hypercharge quantum number for the chiral fermion f_L (f_R).

The hypercharge quantum numbers for the SM fermions are as follows:

$$\begin{aligned} y_{L_L} &= 1/2\,, & y_{Q_L} &= 1/3\,,\\ y_{\nu_R} &= 0\,, & y_{l_R} &= -2\,,\\ y_{d_R} &= -2/3\,, & y_{u_R} &= 4/3\,. \end{aligned}$$

. . .

– Yukawa Interactions

The Higgs mechanism also gives rise to fermion masses via the **Yukawa** Lagrangian

$$\begin{aligned} -\mathcal{L}_{\rm Y} \;=\;& \bar{Q}_{iL}\,\mathbf{Y}^d_{ij}\,\Phi\, d_{jR} \,+\, \bar{Q}_{iL}\,\mathbf{Y}^u_{ij}\,\widetilde{\Phi}\, u_{jR}\\ &+ \bar{L}_{iL}\,\mathbf{Y}^l_{ij}\,\Phi\, l_{jR} \,+\, \bar{L}_{iL}\,\mathbf{Y}^\nu_{ij}\,\widetilde{\Phi}\,\nu_{jR} \,+\, \text{H.c.}, \end{aligned}$$

where $\widetilde{\Phi} \equiv i\sigma^2\Phi^*$, $Q_{iL} = (u_{iL}\,, d_{iL})^{\mathsf{T}}$, $L_{iL} = (\nu_{iL}\,, l_{iL})^{\mathsf{T}}$ (with $i = 1,2,3$), and $u_{1,2,3} = (u,c,t)$, $d_{1,2,3} = (d,s,b)$, $l_{1,2,3} = (e,\mu,\tau)$ and $\nu_{1,2,3} = (\nu_e,\nu_\mu,\nu_\tau)$.

$\mathbf{Y}^{d,u,l,\nu}$ are 3×3 Yukawa-coupling matrices describing the **mixing** between the **three families of quarks and leptons**.

After SSB, the following 3×3 mass matrices for quarks and leptons are generated:

$$\mathbf{M}^u = \frac{v}{\sqrt{2}}\mathbf{Y}^u,\;\; \mathbf{M}^d = \frac{v}{\sqrt{2}}\mathbf{Y}^d,\;\; \mathbf{M}^l = \frac{v}{\sqrt{2}}\mathbf{Y}^l,\;\; \mathbf{M}^\nu = \frac{v}{\sqrt{2}}\mathbf{Y}^\nu.$$

These matrices describe the masses and the mixing between the three family species.

. . .

Exercise: **Theorem**. Show that any $N\times N$ *non*-Hermitian matrix $\mathbf{M}$ can *always* be brought into a diagonal form $\widehat{\mathbf{M}}$, with *non*-negative diagonal entries, by a bi-unitary transformation: $\mathbf{U}\,\mathbf{M}\,\mathbf{V} = \widehat{\mathbf{M}}$, where $\mathbf{U}, \mathbf{V} \in \mathrm{U}(N)$.

Exercises:

(i) Show that the electric charge Q_f of a fermion f is given by the relation: $Q_f = T^3_f + \frac{1}{2}y_f$, where T^3_f is the eigenvalue to the weak isospin operator T^3, i.e. $T^3 f_L = T^3_f f_L$ and $T^3 f_R = 0$. In addition, verify that $Q_{f_L} = Q_{f_R}$.

(ii) Using the gauge-kinetic Lagrangian $\mathcal{L}_f$ for quarks, show that in the mass eigenbasis, the interaction of the $W^\pm$ bosons to the up- and down-type quarks, $\hat{u}_i$ and $\hat{d}_j$, is governed by the Lagrangian

$$\mathcal{L}_W \;=\; -\frac{g}{\sqrt{2}} W^+_\mu \hat{u}_i \mathbf{V}_{ij}\, \gamma^\mu P_L\, \hat{d}_j \;+\; \text{H.c.},$$

where $P_L = (\mathbf{1}_4 - \gamma_5)/2$ is the left chirality projection operator, and $\mathbf{V}_{ij}$ is a 3×3 unitary matrix, the so-called Cabbibo–Kobayashi–Maskawa (CKM) matrix describing **quark mixing**.

(iii) Explain why one can add to the SM Lagrangian a Lorentz- and gauge-invariant **Majorana** mass term for the right-handed neutrinos ν_{iR} of the form:

$$\mathcal{L}_M \;=\; -\frac{1}{2}\,\bar{\nu}^C_{iR}\,(\mathbf{m}_M)_{ij}\,\nu_{jR} \;+\; \text{H.c.},$$

where C indicates charge conjugation and $\mathbf{m}_M$ is a 3×3 matrix. Show that $\mathcal{L}_M$ violates the lepton number L of the SM by two units, i.e. $\Delta L = 2$, and calculate the neutrino mass spectrum for large Majorana masses.

– SM Feynman Rules

In the *unitary gauge*, the Feynman rules may be derived from the following Lagrangian:

$$\mathcal{L}_{\rm SM} \;=\; \mathcal{L}_{\rm G} \,+\, \mathcal{L}_f \,+\, \mathcal{L}_\Phi \,+\, \mathcal{L}_{\rm Y}\,,$$

with

$$\begin{aligned}
\mathcal{L}_{\rm G} \;&=\; -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \;-\; \frac{1}{4} W^i_{\mu\nu} W^{i,\mu\nu} \;-\; \frac{1}{4} B_{\mu\nu} B^{\mu\nu}\,,\\
\mathcal{L}_f \;&=\; \bar{f}_L\, i\gamma^\mu D^L_\mu f_L \;+\; \bar{f}_R\, i\gamma^\mu D^R_\mu f_R\,,\\
\mathcal{L}_\Phi \;&=\; (D_\mu\Phi)^\dagger\,(D^\mu\Phi) \;-\; V(\Phi)\,,\\
\mathcal{L}_{\rm Y} \;&=\; -\bar{Q}_{iL}\,\mathbf{Y}^d_{ij}\,\Phi\, d_{jR} \;-\; \bar{Q}_{iL}\,\mathbf{Y}^u_{ij}\,\widetilde{\Phi}\, u_{jR}\\
&\quad -\bar{L}_{iL}\,\mathbf{Y}^l_{ij}\,\Phi\, l_{jR} \;-\; \bar{L}_{iL}\,\mathbf{Y}^\nu_{ij}\,\widetilde{\Phi}\,\nu_{jR} \;+\; \text{H.c.}
\end{aligned}$$

Here, $G^a_{\mu\nu}$ is the field-strength tensor of the SU(3)$_c$ gluon field G^a_μ, $W^i_{\mu\nu}$ is the respective tensor for the SU(2)$_L$ weak fields W^i_μ, and $B_{\mu\nu}$ for the U(1)$_Y$ B_μ field.

$\cdots$

A complete list of Feynman rules in the R_ξ gauge is given in the textbook by S. Pokorski, *Gauge Field Theories*, Appendix C (see also page 3).

5. Beyond the Standard Model

– Grand Unified Theories

SU(5) Unification

One generation of quarks and leptons in the SM=$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ has 15 degrees of freedom:

$$\begin{pmatrix} u_L^{r,g,b} \\ d_L^{r,g,b} \end{pmatrix}, \quad \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad u_R^{r,g,b} = \bar{u}_L^{r,g,b}, \quad d_R^{r,g,b} = \bar{d}_L^{r,g,b}, \quad e_R = \bar{e}_L \,.$$

In SU(5), the SM fermions are assigned as follows:

$$\mathbf{5}: \quad \psi_i \;=\; \begin{pmatrix} \bar{d}^r \\ \bar{d}^g \\ \bar{d}^b \\ \nu \\ e \end{pmatrix}_L ,$$

and

$$\mathbf{10}: \quad \chi_{ij} \;=\; \begin{pmatrix} 0 & \bar{u}^b & -\bar{u}^g & u^r & d^r \\ -\bar{u}^b & 0 & \bar{u}^r & u^g & d^g \\ \bar{u}^g & -\bar{u}^r & 0 & u^b & d^b \\ -u^r & -u^g & -u^b & 0 & \bar{e} \\ -d^r & -d^g & -d^b & -\bar{e} & 0 \end{pmatrix}_L .$$

Exercise: Given that ψ_i belongs to the fundamental rep $\mathbf{5}$ of SU(5), find the irreducible tensor rep of $\mathbf{10}$ representing the remaining fermions of the SM.

Spontaneous Symmetry Breaking in SU(5)

To break SU(5) down to $SU(3)_c \otimes U(1)_{\rm em}$, we need to introduce two scalar multiplets: (i) Δ_i^j in the adjoint rep **24** of SU(5) and (ii) Φ_i in the fundamental rep **5** of SU(5). The pattern of symmetry breaking is as follows:

$$SU(5) \xrightarrow{\langle\Delta\rangle} SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \xrightarrow{\langle\Phi\rangle} SU(3)_c \otimes U(1)_{\rm em},$$

with $\langle\Delta\rangle \sim 10^{15}$ GeV and $\langle\Phi\rangle \sim v_{\rm SM} \approx 250$ GeV.

The minimal SU(5)-invariant scalar potential is given by

$$V(\Delta,\Phi) = V(\Delta) + V(\Phi) + \lambda_4\,{\rm Tr}(\Delta^2)\,\Phi^\dagger\Phi + \lambda_5\,\Phi^\dagger\Delta^2\Phi\,,$$

with

$$\begin{aligned} V(\Delta) &= -m_1^2\,{\rm Tr}(\Delta^2) + \lambda_1\,{\rm Tr}^2(\Delta^2) + \lambda_2\,{\rm Tr}(\Delta^4)\,, \\ V(\Phi) &= -m_2^2\,\Phi^\dagger\Phi + \lambda_3\,(\Phi^\dagger\Phi)^2\,. \end{aligned}$$

$$\cdots$$

Exercise: Explain why the potential $V(\Delta,\Phi)$ is SU(5) invariant, as well as invariant under $\Delta \to -\Delta$ and $\Phi \to -\Phi$. Derive the matrix form of $\langle\Delta_i^j\rangle$ consistent with SU(5) breaking and evaluate its entries in terms of the potential parameters. Likewise, determine the form of $\langle\Phi_i\rangle$ and its accurate relation to the SM VEV $v_{\rm SM} \approx 250$ GeV.

– Gauge Coupling Unification

The SU(5) theory has 24 gauge bosons, whose masses are determined from the covariant derivatives

$$
\begin{aligned}
D_\mu \Delta &= \partial_\mu \Delta + i g_5 \left[A_\mu \,, \Delta \right] , \\
D_\mu \Phi &= \partial_\mu \Phi + i g_5 \, y_\Phi \, \Phi ,
\end{aligned}
$$

via the kinetic terms $\mathrm{Tr}[D_\mu \Delta D^\mu \Delta]$ and $D_\mu \Phi^\dagger D^\mu \Phi$. (*How?*)

Predictions from SU(5) gauge coupling unification:

Given that $\alpha_s(M_Z) \sim 0.12$ and $\alpha_{\rm em}(M_Z) \sim 1/128$ (which increases from the value $\alpha_{\rm em}(m_e) \sim 1/137$), one predicts $\sin^2\theta_w(M_Z) \sim 0.20$ and $M_X \sim 10^{15}$ GeV, to be compared with the present value $\sin^2\theta_w(M_Z) \sim 0.23$. The low value of $M_X \ll 10^{16}$ GeV is also in tension with experimental constraints on the GUT-predicted proton decay $p \to e^+\pi^0$, which require a proton lifetime $\tau_p > 1.4 \times 10^{34}$ years.

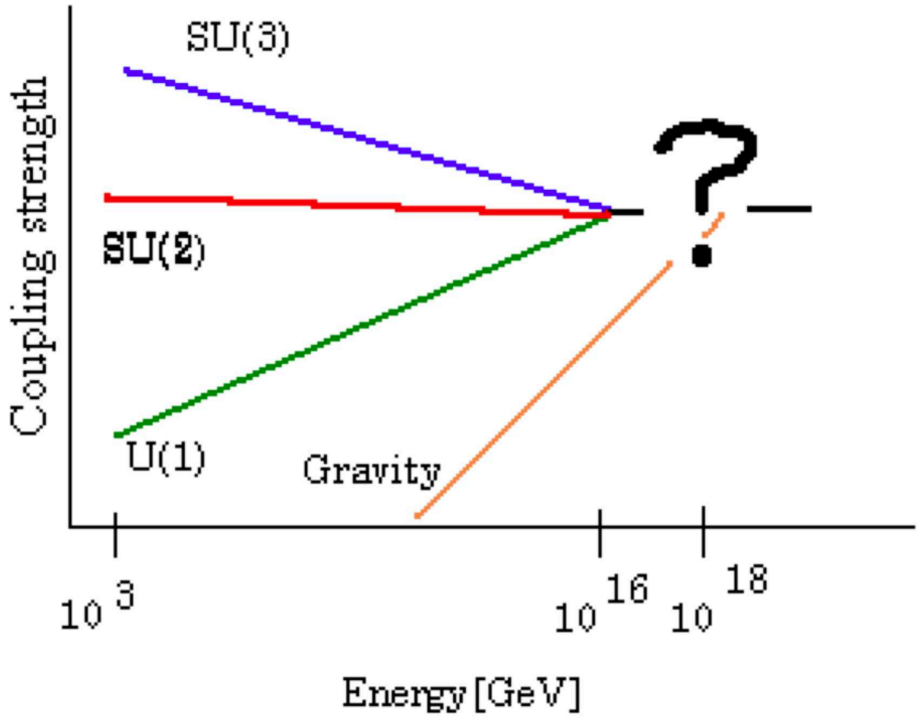
...

Exercise: Ignoring the Φ contribution to the masses of the GUT-scale gauge bosons X and Y, show that

$$
M_X = M_Y = \sqrt{\frac{25}{8}} \, g_5 \, v_\Delta ,
$$

where $v_\Delta \equiv \langle \Delta^1_1 \rangle = \sqrt{m_1^2/(240\lambda_1 + 56\lambda_2)}$.

Super-Grand Unification?



– Supersymmetry*

SUperSYmmetry introduces a new quantum dimension $\Rightarrow$ doubling of the particle spectrum of the SM:

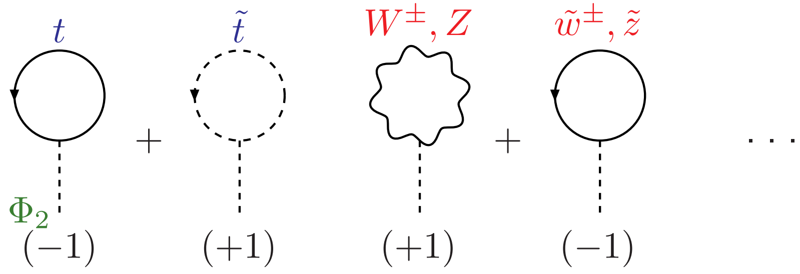
Matter particles, spin $= 1/2$	$\Rightarrow$	SUSY-partners, spin $= 0$
$e^-,\ \mu^-,\ u,\ d,\ \ldots,\ t$		$\tilde{e},\ \tilde{\mu},\ \tilde{u},\ \tilde{d},\ \ldots,\ \tilde{t}$
Anti-Matter, spin $= 1/2$	$\Rightarrow$	SUSY-partners, spin $= 0$
$e^+,\ \mu^+,\ \bar{u},\ \bar{d},\ \ldots,\ \bar{t}$		$\tilde{e}^*,\ \tilde{\mu}^*,\ \tilde{u}^*,\ \tilde{d}^*,\ \ldots,\ \tilde{t}^*$
Force carriers, spin $= 1$	$\Rightarrow$	SUSY-partners, spin $= 1/2$
$\gamma,\ W^+,\ W^-,\ Z,\ g$		$\tilde{\gamma},\ \tilde{w}^+,\ \tilde{w}^-,\ \tilde{z},\ \tilde{g}$
Higgs bosons, spin $= 0$	$\Rightarrow$	SUSY-partners, spin $= 1/2$
2 Higgs doublets: $\Phi_1,\ \Phi_2$		$\tilde{h}_1^0,\ \tilde{h}_1^+,\ \tilde{h}_2^0,\ \tilde{h}_2^+$

No SUSY-partners have been observed yet

$\Rightarrow\ \widetilde{\mathrm{Mass}} - \mathrm{Mass} = M_{\mathrm{SUSY}} \gtrsim 1000$ GeV (from LHC)

Remark. A formal discussion of SUSY theories may be found in specialized textbooks, such as by J. Wess and J. Bagger, *Supersymmetry and Supergravity*, (Princeton University Press, Princeton NJ, 1992).

Quantum fluctuations of the ground state:



$$= \begin{cases} 0, & \text{if SUSY is exact: } M_{\mathrm{SUSY}} = 0 \\ (0.1-1)\ \mathrm{TeV}^3, & \text{if SUSY is softly broken, with } M_{\mathrm{SUSY}} = 1\ \text{TeV} \end{cases}$$

Accurate unification of couplings !