

PHYS40682: GAUGE THEORIES Prof A Pilaftsis

EXAMPLES SHEET I: Preliminaries on Quantum Electrodynamics

1 Basic Aspects of Electrodynamics

The Lagrangian (density) of Electrodynamics reads:

$$\mathcal{L}_{\text{ED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and $J^\mu = (\rho, \mathbf{J})$ is the 4-vector current satisfying charge conservation: $\partial_\mu J^\mu = 0$. Use the Euler–Lagrange equations of motion for A_μ to obtain the first and the fourth Maxwell equations:

$$\partial_\mu F^{\mu\nu} = J^\nu ,$$

with $\mu_0 = \varepsilon_0 = c = 1$. How could one obtain the remaining second and third Maxwell equations?

2 Quantization of Dirac Fermion Fields

- (i) If $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s)$ are the positive and negative energy solutions of the Dirac equation at the rest frame $p_{\text{rest}}^\mu = (m, \mathbf{0})$ in momentum representation, show that

$$u(\mathbf{p}, s) = \frac{\not{p} + m}{\sqrt{2m(E_{\mathbf{p}} + m)}} u(\mathbf{0}, s) , \quad v(\mathbf{p}, s) = \frac{-\not{p} + m}{\sqrt{2m(E_{\mathbf{p}} + m)}} v(\mathbf{0}, s)$$

are the general solutions in an arbitrary Lorentz frame, with $p^\mu = \Lambda^\mu_{\nu} p_{\text{rest}}^\nu$.

[*Hint:* More details may be found in Appendix A of the textbook by S. Pokorski (see page 3 of lecture notes).]

- (ii) Given that $\Pi_\Psi(x) = i\bar{\Psi}(x)\gamma^0$ is the momentum operator conjugate to the Dirac field operator $\Psi(x)$, prove the equal-time anti-commutation relation:

$$\left\{ \Psi_\alpha(t, \mathbf{x}), [i\bar{\Psi}(t, \mathbf{y})\gamma_0]_\beta \right\} = i\delta_{\alpha\beta} \delta^{(3)}(\mathbf{x} - \mathbf{y}) ,$$

where $\alpha, \beta = 1, 2, 3, 4$ are Dirac spinor indices.

[*Hint:* You may find useful to know that

$$\{b(\mathbf{k}, s), b^\dagger(\mathbf{k}', s')\} = \{d(\mathbf{k}, s), d^\dagger(\mathbf{k}', s')\} = \delta_{ss'} (2\pi)^3 2E_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k}') . \quad]$$

(iii) Show that the Feynman propagator for a Dirac fermion field is given by

$$i S_F(x-y) \equiv \langle 0|T\{\Psi(x)\bar{\Psi}(y)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k} + m) e^{-ik \cdot (x-y)}}{k^2 - m^2 + i\varepsilon}.$$

(iv)* Why does the quantization of fermions in (ii) require equal-time anti-commutators, rather than commutators? What would go wrong in the quantization of the theory?

3 Gauge Symmetry and Photons

As discussed in the lectures, the Lagrange density of Quantum Electrodynamics includes the interaction of the photon with the electron and is given by:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \not{\partial} - m - e \not{A}) \psi,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and we used the convention: $\not{a} \equiv \gamma_\mu a^\mu$.

- (i) Derive the equation of motions with respect to the photon and electron fields.
- (ii) Derive the conserved current and charge from \mathcal{L}_{QED} .
- (iii) How should the Lagrangian describing a complex scalar field $\phi(x)$,

$$\mathcal{L} = (\partial^\mu \phi)^* (\partial_\mu \phi) - m^2 \phi^* \phi,$$

be extended so as to become gauge symmetric under a U(1) local transformations? Derive the Feynman rules for this extended theory of Scalar Quantum Electrodynamics.

- (iv) **The Stueckelberg Model.** A Lorentz-invariant photon mass term is described by the Lagrangian $\mathcal{L}_{\text{mass}} = \frac{1}{2} m_A^2 A^\mu A_\mu$. Find a renormalizable gauge-symmetric extension of $\mathcal{L}_{\text{mass}}$.
- (v)* Like in (iv), find a gauge-symmetric (non-renormalizable) extension of the Dirac Lagrangian \mathcal{L}_D , *without* introducing a vector field A^μ . What are the physical consequences of such an extension?

4 The Photon Propagator and Gauge Fixing

As discussed in the lectures, the gauge-fixing part of the QED Lagrange density reads:

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2 ,$$

where ξ is the so-called gauge-fixing parameter.

(i) Derive the Euler-Lagrange equation of the photon in the presence of \mathcal{L}_{GF} .

(ii) Show that the photon propagator is given by the Green function:

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \left(-\eta_{\mu\nu} + (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \frac{e^{-ik \cdot (x-y)}}{k^2 + i\varepsilon} .$$

(iii) Use the equal-time commutators to show that

$$\langle 0|T[A_\mu(x)A_\nu(y)]|0\rangle = i\Delta_{\mu\nu}(x-y)$$

in the Feynman gauge $\xi = 1$.

5 COURSEWORK I: Scattering Processes in Quantum Electrodynamics

(i) Show that

(a) $\text{Tr}(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu},$

(b) $\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma}),$

(c) $\text{Tr}(\gamma_{\alpha_1} \gamma_{\alpha_2} \cdots \gamma_{\alpha_{2n+1}}) = 0$
 (Hint: you may use the properties: $\{\gamma_5, \gamma_\mu\} = 0$ and $\gamma_5^2 = \mathbf{1}_4$, where $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$.),

(d) $\sum_{s=\pm 1/2} \bar{u}(p, s) M u(p, s) = \text{Tr}[M(\not{p} + m)],$ where M is any arbitrary 4×4 matrix.

(ii) Use the Feynman rules for QED to write down the matrix element \mathcal{M}_{fi} for the reaction $e^-(p_1)e^+(p_2) \rightarrow \mu^-(k_1)\mu^+(k_2)$.

(iii) With the aid of trace techniques given in (i), calculate $\overline{|\mathcal{M}_{fi}|^2}$, where the long bar indicates averaging over the spins of the electrons in the initial state.

(iv) Calculate analytically the differential cross section $d\sigma/d\Omega$ for $e^-e^+ \rightarrow \mu^-\mu^+$ which was taking place at the CERN LEP collider at CMS energies $\sqrt{s} = M_Z = 90$ GeV. Draw an accurate graph of $d\sigma/d\Omega$ as a function of $\cos\theta$.

(v) Supersymmetry predicts that in addition to muons μ^\pm there should be scalar muons $\tilde{\mu}^\pm$. Calculate $d\sigma/d\Omega$ for the process $e^-e^+ \rightarrow \tilde{\mu}^-\tilde{\mu}^+$. Plot $d\sigma/d\Omega$ as a function of $\cos\theta$ and comment on your results.