Lecture 14

Weyl and Dirac spinors

Weyl spinors, $\xi_\alpha(x)$ and $\bar\eta^{\dot\alpha}$ ($\alpha = 1,2$), are 2-dim complex vectors whose components anti-commute, e.g. $\xi_1\xi_2 = -\xi_2\xi_1$, $\bar\eta^{\dot 1}\xi_2 = -\xi_2\bar\eta^{\dot 1}$, $\bar\eta^{\dot 1}\bar\eta^{\dot 2} = -\bar\eta^{\dot 2}\bar\eta^{\dot 1}$ etc.
Hence, $\xi_1^2 = 0$, $\xi_2^2 = 0$ etc, i.e. they are Grassman numbers.

Under a LT, $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$, they transform as

$$\xi'_\alpha(x') = M_\alpha{}^\beta \xi_\beta(x), \quad \bar\eta'_{\dot\alpha}(x') = M^{*\dot\beta}{}_{\dot\alpha}\bar\eta_{\dot\beta}(x),$$

$$\xi'^\alpha(x') = (M^{-1})_\beta{}^\alpha \xi^\beta(x), \quad \bar\eta'^{\dot\alpha}(x') = (M^{*-1})^{\dot\alpha}{}_{\dot\beta}\bar\eta^{\dot\beta},$$

where $M \in SL(2,\mathbb{C})$, $M = e^{i(\theta^i + i\phi^i)\frac{\sigma^i}{2}}$

$\boxed{\text{complexification of } SU(2): e^{i\theta^i\sigma^i/2}}$

2-independent LT transf: $\underbrace{M}_{(\frac{1}{2},0)}, \underbrace{M^*}_{(0,\frac{1}{2})}$

Duality relations:

$$(\xi^\alpha)^\dagger = \bar\xi^{\dot\alpha}, \quad (\xi_\alpha)^\dagger = \bar\xi_{\dot\alpha}, \quad (\bar\eta_{\dot\alpha})^\dagger = \eta_\alpha, \quad (\eta^\alpha)^\dagger = \bar\eta^{\dot\alpha}$$

Lowering and raising spinor indices: $\boxed{\text{like } x_\mu = \eta_{\mu\nu}x^\nu}$

$$\xi_\alpha = \varepsilon_{\alpha\beta}\xi^\beta, \quad \xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta, \quad \bar\eta_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\eta^{\dot\beta}, \quad \bar\eta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\eta_{\dot\beta},$$

with $\varepsilon^{\alpha\beta} \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \varepsilon^{\dot\alpha\dot\beta}$, $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma \rightsquigarrow \varepsilon_{\alpha\beta} = -i\sigma_2$,

$\boxed{\text{2-dim Levi-Civita}}$ likewise: $\varepsilon_{\dot\alpha\dot\beta} = -i\sigma_2$

Lorentz-invariant spinor contractions:

$$\xi\eta \equiv \xi^\alpha\eta_\alpha = \xi^\alpha\varepsilon_{\alpha\beta}\eta^\beta = -\eta^\beta\varepsilon_{\alpha\beta}\xi^\alpha = \eta^\beta\varepsilon_{\beta\alpha}\xi^\alpha = \eta^\beta\xi_\beta = \eta\xi$$

Likewise, $\bar\xi\bar\eta \equiv (\eta\xi)^\dagger = \xi^\dagger_\alpha\eta^{\alpha\dagger} = \bar\xi_{\dot\alpha}\bar\eta^{\dot\alpha}$

$\boxed{\text{Notice direction of contraction}}$

The Dirac spinor

4-dim complex vector made of 2 Weyl spinors:

$$\psi(x) = \begin{pmatrix} \xi_\beta(x) \\ \bar{\eta}^{\dot\beta}(x) \end{pmatrix} \begin{matrix} \longleftarrow (\tfrac{1}{2},0) \\ \longleftarrow (0,\tfrac{1}{2}) \end{matrix}$$

spinorial representations of $\underbrace{SO(1,3)^\uparrow}_{\boxed{\text{with } \Lambda^0{}_0>0}} \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$ ⊛

Dirac eqn.: $(i\gamma^\mu\partial_\mu - m)\psi = 0$

where $\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix}$, $\sigma^\mu = (\mathbf{1}_2, \sigma_{1,2,3})$

$\bar\sigma^\mu = (\mathbf{1}_2, -\sigma_{1,2,3})$

Clifford algebra:

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1}_2 , \quad \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1}_2$$

and $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}_4$

Lagrangian for Dirac fermions

$$\mathcal{L}_D = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi , \quad \text{with } \bar\psi \equiv (\eta^\alpha, \bar\xi_{\dot\alpha})$$

$$= \eta^\alpha\, i(\sigma^\mu)_{\alpha\dot\beta}(\partial_\mu\bar\eta^{\dot\beta}) + \bar\xi_{\dot\alpha}\, i(\bar\sigma^\mu)^{\dot\alpha\beta}(\partial_\mu\xi_\beta)$$

$$- m(\eta^\alpha\xi_\alpha + \bar\xi_{\dot\alpha}\bar\eta^{\dot\alpha})$$

$\mathcal{L}_D$ is invariant under LTs

Charge conjugation: $\psi^c = C\bar\psi^T = \begin{pmatrix} \eta_\beta \\ \bar\xi^{\dot\beta} \end{pmatrix}$; $C = i\gamma^0\gamma^2 = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix}$

Majorana spinor: $\psi_M = \psi_M^c$, $\psi_M = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}$

⊛ Relation between group elements:

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(M^\dagger\bar\sigma^\mu M\sigma_\nu) , \quad \text{with } \Lambda^\mu{}_\nu \in SO(1,3)^\uparrow \ (\Lambda^0{}_0>0)$$

$$M \in SL(2,\mathbb{C})$$

MATHEMATICAL SUPPLEMENT *

-M.1-

LORENTZ-GROUP : SO(1,3)

A point in the space-time-manifold is denoted by $(x^\mu) = (x^0, x^1, x^2, x^3)$ where $x^0 = t$ and x^1, x^2, x^3 are the space-components of the four-vector x^μ. The laws of relativity are invariant under the LORENTZ-group. Transformations of this group are linear transformations acting on four-vectors:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.1}$$

leaving the quadratic form

$$\begin{aligned} x^2 &= x^\mu x_\mu \\ &= \eta_{\mu\nu} x^\mu x^\nu \\ &= (x^0)^2 - (\vec{x})^2 \end{aligned} \tag{1.2}$$

invariant, i.e.:

$$\begin{aligned} x'^2 &= x'^\mu x'_\mu \\ &= \eta_{\mu\nu} x'^\mu x'^\nu \\ &= \eta_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\tau x^\tau \\ &\overset{!}{=} \eta_{\rho\tau} x^\rho x^\tau \end{aligned}$$

Hence:

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau = \eta_{\rho\tau} \tag{1.3}$$

Here $\eta_{\mu\nu} =$ diag (+1, -1, -1, -1) is the metric tensor, it lowers indices and its inverse, $\eta^{\mu\nu}$ raises them.

Proposition:
The constraints

$$\det \Lambda = \pm 1 , \quad |\Lambda^0{}_0| \geq +1 \tag{1.4}$$

define four disconnected pieces in the parameter space.

-M.2-

Proof: The determinant of a product of matrices is the product of the determinants. Hence taking the determinant of (1.3), we have

$$[\det(\Lambda)]^2 = 1$$

$\det(AB) = \det A \cdot \det B$

or

$$\det \Lambda = \pm 1$$

Taking the o - o - component of (1.3) we obtain:

$$\begin{aligned} \eta_{00} &= \eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 \\ &= \eta_{00} \Lambda^0{}_0 \Lambda^0{}_0 + \eta_{ii} \Lambda^i{}_0 \Lambda^i{}_0 \\ &= (\Lambda^0{}_0)^2 - (\Lambda^k{}_0)^2 \\ &\overset{!}{=} 1 \end{aligned}$$

or:

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^k{}_0)^2$$

hence:

$$(\Lambda^0{}_0)^2 \geq 1$$

The second constraint in (1.4) distinguishes socalled orthochronous LORENTZ-transformations with $\Lambda^0{}_0 \geq +1$ and nonorthochronous LORENTZ-transformations with $\Lambda^0{}_0 \leq -1$.

The matrices $(\Lambda^\mu{}_\nu)$, satisfying (1.3) form a non-compa[ct] LIE-group, the LORENTZ-group

$$L := O(1,3) = \{\Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta\}^{\circledast}$$

with LIE-Algebra

$$o(1,3) := \{\mathfrak{a} \in M_{4\times4}(\mathbb{R}) \mid \mathfrak{a}^T = -\eta \mathfrak{a} \eta\}$$

where GL $(4, \mathbb{R})$ denotes the set of all invertible 4 x 4-matrices

⊛ Here η is the short-hand notation for $\eta_{\mu\nu}$, Λ for $\Lambda^\mu{}_\nu$ etc.

with real compoments and $M_{4x4}(\mathbb{R})$ is the set of all 4 x 4-matrices with real elements.

Proof:

From LIE-algebra-theory we know that each $\Lambda \in O(1,3)$ can be written in the form:

$$\Lambda(t) = \exp(t\alpha)$$

where t is a real parameter and $\alpha \in o(1,3)$ is an element of the LIE-algebra. Matrices of O (1,3) are subject to the condition

$$\Lambda^T(t)\,\eta\,\Lambda(t) = \eta$$

Inserting the above expression, we obtain:

$$[\exp(t\alpha)]^T\eta\,[\exp(t\alpha)] = \eta$$

and considering

$$\frac{d}{dt}\left\{[\exp(t\alpha)]^T\eta\,[\exp(t\alpha)]\right\}\Big|_{t=0} = 0$$

we obtain the condition for LIE-algebra elements, since the LIE-algebra of any LIE-group is isomorphic to the tangent-space at the identity of the group. It follows:

$$\frac{d}{dt}[\exp(t\alpha)]^T\eta\,\exp(t\alpha)\Big|_{t=0} + [\exp(t\alpha)]^T\eta\,\frac{d}{dt}\exp(t\alpha)\Big|_{t=0} = 0$$

which leads to

$$\alpha^T\eta + \eta\alpha = 0$$

or

$$\alpha^T = -\eta\alpha\eta \qquad \forall \alpha \in o(1,3)$$

In summary we have the following classification: Let Λ be any invertible 4 x 4 Matrix with real elements, i. e. $\Lambda \in GL(4,\mathbb{R})$, then:

(i) The full LORENTZ-group is:

$$L := O(1,3) = \{\Lambda \in GL(4,\mathbb{R})\,|\,\Lambda^T\eta\Lambda = \eta\}$$

(ii) Proper LORENTZ-transformations are:

$$L_+ := SO(1,3) = \{\Lambda \in O(1,3)\,|\,\det\Lambda = +1\}$$

L_+ is a subgroup of L.

(iii) Improper LORENTZ-transformations are:

$$L_- = \{\Lambda \in O(1,3)\,|\,\det\Lambda = -1\}$$

L- is not a subgroup of L, since the identity-element is not an element of L-. Note, however, that discrete transformations such as time-or space-reflection are elements of L-.

(iv) Orthochronous LORENTZ-transformations:

$$L^{\uparrow} := \{\Lambda \in O(1,3)\,|\,\Lambda^0{}_0 \geq +1\}$$

$L^{\uparrow}$ is a subgroup of L

(v) Nonorthochronous LORENTZ-transformations:

$$L^{\downarrow} := \{\Lambda \in O(1,3)\,|\,\Lambda^0{}_0 \leq -1\}$$

(vi) The restricted LORENTZ-group is:

$$L^{\uparrow}_+ = L^{\uparrow} \cap L_+$$
$$= \{\Lambda \in O(1,3)\,|\,\det\Lambda = +1,\ \Lambda^0{}_0 \geq +1\}$$

This subgroup of L is also called proper orthochronous LORENTZ-group, and does not contain any time-or space-reflection.

$\underline{SL(2,\mathbb{C})\ \text{group}} := \{ M \in GL(2,\mathbb{C}) \,/\, \det M = 1 \}$

Generators

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}\left(\sigma^\mu_{\alpha\dot\gamma}\bar\sigma^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\bar\sigma^{\mu\dot\gamma\beta}\right)$$

and

$$(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha} = \frac{i}{4}\left(\bar\sigma^{\mu\dot\beta\gamma}\sigma^\nu_{\gamma\dot\alpha} - \bar\sigma^{\nu\dot\beta\gamma}\sigma^\mu_{\gamma\dot\alpha}\right)$$

<u>Proof</u>:

For an infinitesimal LORENTZ-transformation (ω_{ij} infinitesimal)

$$M \propto \Lambda^\mu{}_\nu \sigma_\mu \bar\sigma^\nu = \Lambda_{\mu\nu}\sigma^\mu\bar\sigma^\nu$$

$$= \begin{pmatrix} 1 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & -1 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & -1 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & -1 \end{pmatrix}_{\mu\nu} \sigma^\mu\bar\sigma^\nu$$

$$= \begin{pmatrix} 4 - 2(\omega_{03} + i\omega_{12}) & -2(\omega_{01} + i\omega_{23}) + 2i(\omega_{02} + i\omega_{31}) \\ -2(\omega_{01} + i\omega_{23}) - 2i(\omega_{02} + i\omega_{31}) & 4 + 2(\omega_{03} + i\omega_{12}) \end{pmatrix}$$

This is a 2x2-matrix

Hence

$$\det\left(\Lambda^\mu{}_\nu \sigma_\mu \bar\sigma^\nu\right) = 16 + O(\omega^2)$$

and (omitting terms of $o(w^2)$)

$$N = +\left[\det\left(\Lambda^\mu{}_\nu \sigma_\mu \bar\sigma^\nu\right)\right]^{1/2} = 4$$

The relationship between element $\Lambda^\rho{}_\tau$ of the LIE-group SO (3,1) and element $a^{\mu\nu}$ of the algebra so (3,1) is $\Lambda^\rho{}_\tau(\omega) = \left[\exp\left(-\frac{i}{2}\omega_{\mu\nu}a^{\mu\nu}\right)\right]^\rho{}_\tau$.

For infinitesimal $\omega_{\mu\nu}$, an element $M(\Lambda) \in SL(2,\mathbb{C})$ can be written down as:

$$M(\Lambda) = \frac{1}{N}\Lambda^\rho{}_\tau(\omega)\sigma_\rho\bar\sigma^\tau$$

$$= \frac{1}{N}\left(\eta_{\rho\tau} - \frac{i}{2}\omega_{\mu\nu}(a^{\mu\nu})_{\rho\tau}\right)\sigma^\rho\bar\sigma^\tau$$

$$= \underbrace{\frac{1}{N}\sigma^\mu\bar\sigma_\mu}_{=1_2} - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}$$

where

$$\sigma^{\mu\nu} := \frac{1}{N}(a^{\mu\nu})_{\rho\tau}\sigma^\rho\bar\sigma^\tau$$

Since also:

$$x'_\rho = \Lambda_{\rho\tau}x^\tau = (\eta_{\rho\tau} + \omega_{\rho\tau})x^\tau$$

we have

$$-\frac{i}{2}\omega_{\mu\nu}(a^{\mu\nu})_{\rho\tau} = \omega_{\rho\tau}$$

This equation is satisfied for:

$$(a^{\mu\nu})^\rho{}_\tau = i\left(\eta^{\mu\rho}\delta^\nu{}_\tau - \eta^{\nu\rho}\delta^\mu{}_\tau\right)$$

Of course, from (1.11) we know that the generators of LORENTZ-transformations have this form.

Hence

$$\sigma^{\mu\nu} = \frac{i}{N}\left(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma\right)\sigma_\rho\bar\sigma^\sigma$$

$$= \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)$$

In a similar way $\bar\sigma^{\mu\nu}$ is obtained by considering M^* as a function of Λ.

The $\sigma^{\mu\nu}$, $\bar\sigma^{\mu\nu}$ are the generators of the $SL(2,\mathbb{C})$ group:

$$M = \left(e^{-i\frac{\omega_{\mu\nu}}{2}\sigma^{\mu\nu}}\right)_\alpha{}^\beta \; ; \quad \boxed{\text{with } \omega_{\mu\nu} = -\omega_{\nu\mu}}$$

$$M^+ = \left(e^{+i\frac{\omega_{\mu\nu}}{2}\bar\sigma^{\mu\nu}}\right)^{\dot\beta}{}_{\dot\alpha}$$

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† Discussion not fully covered in the lectures.

Lecture 15

Quantization of the fermion field

$$\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\underline{k}}{(2\pi)^3 2E_k} \Big[b(\underline{k},s)\, u(\underline{k},s)\, e^{-ik\cdot x} + d^+(\underline{k},s)\, v(\underline{k},s)\, e^{ik\cdot x} \Big]$$

N.B. $k^0 = E_k$

$$\bar{\psi}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\underline{k}}{(2\pi)^3 2E_k} \Big[b^+(\underline{k},s)\, u^+(\underline{k},s)\, \gamma_0\, e^{ik\cdot x} + d(\underline{k},s)\, v^+(\underline{k},s)\gamma_0\, e^{-ikx} \Big]$$

$$b^+(\underline{k},s)|0\rangle = |e^-(\underline{k},s)\rangle \quad , \quad d^+(\underline{k},s)|0\rangle = |e^+(\underline{k},s)\rangle$$

electron positron

Dirac spinors: $u(\underline{k},s)$ and $v(\underline{k},s)$ ← 4-dim vectors

defined as solutions of ($\not{a} \equiv a^\mu \gamma_\mu$):

$$(\not{p} - m)\, u(\underline{p},s) = 0 \quad \text{and} \quad (\not{p} + m)\, v(p,s) = 0$$

General solutions:

$$u(\underline{p},s) = \frac{\not{p}+m}{\sqrt{2m(E_p+m)}}\, u(\underline{0},s)$$

$$v(\underline{p},s) = \frac{-\not{p}+m}{\sqrt{2m(E_p+m)}}\, v(\underline{0},s)$$

← spinors at rest frame

Normalization properties:

$$\bar{u}(\underline{p},s)\, u(\underline{p},s') = \delta_{ss'}(2m)$$

$$\bar{v}(\underline{p},s)\, v(\underline{p},s') = -\delta_{ss'}(2m)$$

$$\boxed{\bar{u} = u^+\gamma^0 \quad \bar{v} = v^+\gamma^0}$$

$$\sum_{s=\pm\frac{1}{2}} u(\underline{p},s)\, \bar{u}(\underline{p},s) = \not{p} + m$$

$$\sum_{s=\pm\frac{1}{2}} v(\underline{p},s)\, \bar{v}(\underline{p},s) = \not{p} - m$$

Further properties on Dirac spinors may be found in textbooks!

Conjugate momentum:

$$\Pi_\psi = -\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\,\overline{\Psi}(x)\,\gamma_0 .$$

Quantization via equal-time anti-commutators:

$$\left\{ \Psi_\alpha(t,\underline{x}),\, \left[i\overline{\Psi}(t,\underline{y})\,\gamma_0 \right]_\beta \right\} = i\,\delta_{\alpha\beta}\,\delta^{(3)}(\underline{x}-\underline{y}) ,$$

where $\alpha,\beta = 1,2,3,4$ are Dirac spinor indices.
The equal-time anti-commutators are satisfied provided:

$$\left\{ b(\underline{k},s), b^\dagger(\underline{k}',s') \right\} = \left\{ d(\underline{k},s), d^\dagger(\underline{k}',s') \right\} = \delta_{ss'}\,(2\pi)^3\, 2E_k\, \delta^{(3)}(\underline{k}-\underline{k}')$$

This is left as exercise

The fermion propagator:

Time-ordered product for fermions:

$$T\left\{ \Psi(x)\overline{\Psi}(x') \right\} = \theta(t-t')\,\Psi(x)\,\overline{\Psi}(x') - \theta(t'-t)\,\overline{\Psi}(x')\,\Psi(x)$$

$\ominus$ because fermion fields are Grassman numbers

Performing a similar calculation as for scalars, we get

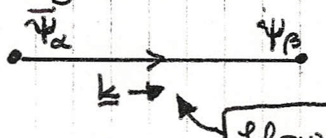
$$\langle 0 | T\left\{ \Psi(x)\overline{\Psi}(x') \right\} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4}\, \frac{i\, e^{-ik(x-x')}}{\not{k} - m + i\varepsilon} \equiv i\,S_F(x-x')$$

$$\rightsquigarrow \quad i\,S_F(x-x') = \int \frac{d^4k}{(2\pi)^4}\, \frac{i(\not{k}+m)\, e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon}$$

$$= (i\not{\partial} + m)\, \underline{i\Delta_F(x-x')}$$

Feynman scalar propagator

Feynman rule



$$: \left(\frac{i}{\not{k} - m + i\varepsilon} \right)_{\beta\alpha} = \left(\frac{i(\not{k}+m)}{k^2 - m^2 + i\varepsilon} \right)_{\beta\alpha}$$

flow of fermion number

Lecture 16

Gauge symmetry

$\mathcal{L}_D = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$ is invariant under the global $U(1)$ transf:

$$\psi(x) \mapsto \psi'(x) = e^{i\theta}\psi(x),$$
$$\bar\psi(x) \mapsto \bar\psi'(x) = e^{-i\theta}\bar\psi(x),$$

where θ is independent of x^μ.

If $\theta = \theta(x)$, we then have under a local $U(1)$ transf:

$$\mathcal{L}_D(\bar\psi,\psi) \longmapsto \mathcal{L}'_D(\bar\psi',\psi') = \bar\psi e^{-i\theta}(i\gamma^\mu\partial_\mu - m)e^{i\theta}\psi$$
$$= \underbrace{\bar\psi(i\gamma^\mu\partial_\mu - m)\psi}_{=:\ \mathcal{L}_D(\bar\psi,\psi)} \underbrace{-(\partial_\mu\theta)\bar\psi\gamma^\mu\psi}_{+\ \delta\mathcal{L}_D\ \leftarrow\ \boxed{\text{extra term}}} \neq \mathcal{L}_D$$

To compensate for $\delta\mathcal{L}_D$, we add the term

$$\mathcal{L}_{A\bar\psi\psi} = -eA_\mu\bar\psi\gamma^\mu\psi,$$

e.g. the photon

where A_μ is a vector field that transforms as

$$A_\mu \xmapsto{U(1)} A'_\mu = A_\mu - \frac{1}{e}(\partial_\mu\theta)$$

Then, $\mathcal{L}_{A\bar\psi\psi} \xmapsto{U(1)} \mathcal{L}'_{A\bar\psi\psi} = \mathcal{L}_{A\bar\psi\psi} + (\partial_\mu\theta)\bar\psi\gamma^\mu\psi$

Consequently $\mathcal{L}_\psi = \mathcal{L}_D + \mathcal{L}_{A\bar\psi\psi} = \bar\psi(i\not\partial - e\not A - m)\psi$ is gauge invariant.

Gauge symmetry gives rise to a conserved (electric) current and charge according to Noether's theorem.

But, gauge symmetry $\not\Rightarrow$ massless photon or $\not\Rightarrow$ its existence

Quantization of the Electromagnetic Field

QED

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi$$

how is this quantized?

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{d^3\underline{k}}{(2\pi)^3 2E_k} \Big[a(\underline{k},\lambda)\, \varepsilon_\mu(\underline{k},\lambda)\, e^{-ik\cdot x} + a^+(\underline{k},\lambda)\, \varepsilon^*_\mu(\underline{k},\lambda)\, e^{ik\cdot x} \Big],$$

with conjugate momentum operator: $\Pi_\mu = \partial_t A_\mu$.

— — — — — — — —

$A_\mu(x)$ carries $\underbrace{\text{4 degrees of freedom}}_{\text{described by } \varepsilon_\mu(\underline{k},\lambda)}$, but only $\underbrace{\text{2 are physical}}_{\lambda = 1,2}$

$\lambda = 0, 1, 2, 3$

time-like photon ($\lambda = 0$)

polarization vectors

longitudinal photon ($\lambda = 3$)

transverse photons ($\lambda = 1, 2$)

Properties of $\varepsilon_\mu(\underline{k},\lambda)$:

$$\sum_{\lambda=0}^{3} (-1)^{\delta_{\lambda 0}}\, \varepsilon_\mu(\underline{k},\lambda)\, \varepsilon^*_\nu(\underline{k},\lambda) = -\eta_{\mu\nu},$$

and $k^\mu \varepsilon_\mu(\underline{k},\lambda) = 0$, $\boxed{1,2,3}$ $\quad \varepsilon_\mu(\underline{k},\lambda)\, \varepsilon^{*\mu}(\underline{k},\lambda') = -(-1)^{\delta_{\lambda 0}}\delta_{\lambda\lambda'}$

Assuming a fictitious photon mass $m_\gamma \mapsto 0$, one possible solution is

A_μ $\quad \underline{k} \parallel \hat{\underline{z}}$

$$\varepsilon^\mu(\underline{k},0) = \frac{k^\mu}{m_\gamma}$$

$$\varepsilon^\mu(\underline{k},1(2)) = \frac{1}{\sqrt{2}}(0, 1, \pm(-) i, 0)$$

if photon moves $\parallel \hat{\underline{z}}$ $\quad \boxed{k^\mu = (k^0, 0, 0, k_z)}$

$$\varepsilon^\mu(\underline{k},3) = \frac{1}{m_\gamma}\left(|\underline{k}|, \frac{k^0}{|\underline{k}|}\underline{k}\right)$$

$\boxed{\text{N.B. } k^2 = m_\gamma^2 \mapsto 0}$

Physical observables, such as S-matrix elements, do not have singularities in the limit $m_\gamma \mapsto 0$.

Lecture 17

The photon propagator and gauge fixing

Canonical quantization of the photon field A^μ:

$$[A_\mu(t,\underline{x}), \underbrace{\Pi_\nu(t,\underline{y})}_{\partial_t A_\nu(t,\underline{y})}] = -i\eta_{\mu\nu}\,\delta^{(3)}(\underline{x}-\underline{y})$$

$\boxed{\text{All other equal-time commutators vanish}}$

This equal-time commutation relation is fullfilled, if

$$[a(\underline{k},\lambda), a^\dagger(\underline{k}',\lambda')] = (-1)^{\delta_{\lambda 0}}\,\delta_{\lambda\lambda'}\,(2\pi)^3\,2E_k\,\delta^{(3)}(\underline{k}-\underline{k}'),$$

whilst all other commutators vanish.

Then, the photon propagator may be calculated by means of the time-ordered product: $\mathbb{T}(A_\mu(x)A_\nu(y))$

$$= \Theta(x^0-y^0)A_\mu(x)A_\nu(y) + \Theta(y^0-x^0)A_\nu(y)A_\mu(x)$$

$$i\Delta_{\mu\nu}(x-y) = \langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = \int\frac{d^4k}{(2\pi)^4}\,\frac{-i\eta_{\mu\nu}\,e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}$$

Alternative approach

Add $\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to $\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\dots$

$\boxed{\text{gauge fixing term}}$

E-L eqn for the photon: $\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right] - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0$

$$\rightsquigarrow \left[\eta_{\mu\nu}\partial_k\partial^k - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^\nu(x) = 0$$

Photon propagator $\Delta^{(\xi)}_{\mu\nu}$ as the Green function:

$$\left[\eta^{\mu\nu}\partial_k\partial^k - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]\Delta^{(\xi)}_{\nu\lambda}(x-y) = \delta^\mu{}_\lambda\,\delta^{(4)}(x-y)$$

$$\rightsquigarrow \Delta^{(\xi)}_{\mu\nu}(x-y) = \int\frac{d^4k}{(2\pi)^4}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}$$

Remarks:

- The gauge-fixing term breaks explicitly the gauge symmetry. Such a breaking is necessary to remove the unphysical degrees of freedom of the photon $A_\mu(x)$.

- Notice that $\Delta_{\mu\nu}(x-y)$ calculated using the canonical quantization method is equal to $\Delta^{(\xi=1)}_{\mu\nu}(x-y)$, i.e. the photon propagator in the Feynman gauge.

- $\mathcal{L}_{QED} + \mathcal{L}_{GF} + \mathcal{L}_{FP}$, where $\mathcal{L}_{FP}$ is the so-called Faddeev-Popov Lagrangian term, is invariant under the Becchi-Rouet-Stora transformations. $\mathcal{L}_{FP} = -\bar{c}(x)\partial_\mu\partial^\mu c(x)$, where $c(x)$, $\bar{c}(x)$ are unphysical anti-commuting (Grassman) fields designed to remove the unphysical degrees of freedom of A^μ. More details may be found in the textbook by Peskin and Schröder.

- Feynman rules (see lecture notes) for QED