Lectures on Quantum Field Theory

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1. Preliminaries

– Literature

Recommended Texts:

- F. Mandl and G. Shaw, *Quantum Field Theory*, Wiley, 1992.
- M. E. Peskin and D. V. Schröder, *Quantum Field Theory*, Perseus Books Group, 1995.
- L. H. Ryder, *Quantum Field Theory*, Cambridge University Press.

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Advanced Texts:

- T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press, 1984.
- S. Pokorski, *Gauge Field Theories*, Cambridge University Press, 2000, Second Edition.
- P. Ramond, *Field Theory: A Modern Primer*, Addison Wesley, 1990.
- J. Zinn–Justin, *Quantum Field Theory and Critical Phenomena*, Oxford Science Publications, 2002, Fourth Edition.

– Classical Lagrangian Dynamics

Variational Principle and Equation of Motion

The **Lagrangian** for an n-particle system is

$$L(q_i, \dot{q}_i) \;=\; T(\dot{q}_i, q_j \dot{q}_i) \;-\; V(q_i)\,,$$

where $q_{1,2,\dots,n}$ are the **generalized coordinates** describing the n particles, and $\dot{q}_{1,2,\dots,n}$ are their respective time derivatives.

T and V denote the total kinetic and potential energies.

The **action** S of the n-particle system is given by

$$S[q_i(t)] \;=\; \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i)\,.$$

Note that S is a *functional* of $q_i(t)$.

The **conjugate momenta** are defined as $p_i \;=\; \dfrac{\partial L}{\partial \dot{q}_i}$.

The **Hamiltonian** H is defined by the Legendre transform:

$$H(q_i, p_i) \;=\; \sum_i p_i\, \dot{q}_i \;-\; L(q_i, \dot{q}_i)\,,$$

where p_i satisfy **Hamilton's equations**:

$$\dot{q}_i \;=\; \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i \;=\; -\,\frac{\partial H}{\partial q_i}\,.$$

The Hamiltonian describes the total energy of the system:

$$H \;=\; T + V\,.$$

Hamilton's principle

Hamilton's principle states that the actual motion of the system is determined by the stationary behaviour of S under small variations $\delta q_i(t)$ of the ith particle's generalized coordinate $q_i(t)$, with $\delta q_i(t_1) = \delta q_i(t_2) = 0$, i.e.

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} dt \left(\delta q_i \frac{\partial L}{\partial q_i} + \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) \\ &= \int_{t_1}^{t_2} dt\, \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) = 0\,.\end{aligned}$$

The Euler–Lagrange equation of motion for the ith particle is

$$\therefore \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,.$$

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Exercise: Show that the Euler–Lagrange equations of motion for a particle system described by a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i)$ are

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial q_i} = 0\,.$$

[Hint: Consider only variations with $\delta q_i(t_{1,2}) = \delta \dot{q}_i(t_{1,2}) = 0$.]

– Lagrangian Field Theory

In Field Theory (FT), a (scalar) particle is described by a field $\phi(x)$, whose Lagrangian has the functional form:

$$L = \int d^3\mathbf{x}\; \mathcal{L}(\phi(x)\,, \partial_\mu \phi(x))\,,$$

where $\mathcal{L}$ is the so-called **Lagrangian density**, often termed Lagrangian in FT.

In FT, the action S is given by

$$S[\phi(x)] = \int_{-\infty}^{+\infty} d^4x\; \mathcal{L}(\phi(x)\,, \partial_\mu \phi(x))\,.$$

By analogy, the Euler–Lagrange equations can be obtained by determining the stationary points of S, under variations $\phi(x) \to \phi(x) + \delta\phi(x)$:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0\,.$$

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Exercise: Derive the above Euler–Lagrange equation for a scalar particle by extremizing $S[\phi(x)]$, i.e. $\delta S = 0$.

Lagrangian for the Klein–Gordon equation

$$\mathcal{L}_{\rm KG} \;=\; \frac{1}{2}\,(\partial_\mu\phi)\,(\partial^\mu\phi) \;-\; \frac{1}{2}\,m^2\,\phi^2\,,$$

where $\phi(x)$ is a real scalar field describing one dynamical degree of freedom.

The Euler–Lagrange equation of motion is the Klein–Gordon equation

$$(\partial_\mu\partial^\mu \;+\; m^2)\,\phi(x) \;=\; 0\,.$$

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Lagrangian for the Electromagnetic Field A^μ

$$\mathcal{L}_{\rm ED} \;=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \;-\; J_\mu A^\mu\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and J_μ is the 4-vector current satisfying charge conservation: $\partial_\mu J^\mu = 0$.

A_μ describes a spin-1 particle, e.g. a photon, with 2 physical degrees of freedom.

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Exercise: Use the Euler-Lagrange equations for $\mathcal{L}_{\rm ED}$ to show that $\partial_\mu F^{\mu\nu} \;=\; J^\nu$, as is expected in relativistic electrodynamics (with $\mu_0 = \varepsilon_0 = c = 1$).

– Global and Local Symmetries

Consider the Lagrangian (density) for a complex scalar:

$$\mathcal{L} \;=\; (\partial^\mu\phi^*)\,(\partial_\mu\phi) \;-\; m^2\,\phi^*\phi \;-\; \lambda(\phi^*\phi)^2\;.$$

$\mathcal{L}$ is invariant under a U(1) rotation of the field ϕ:

$$\phi(x) \;\rightarrow\; \phi'(x) \;=\; e^{i\theta}\,\phi(x)\;,$$

where θ does not depend on $x \equiv x^\mu$.

A transformation in which the fields are rotated about x-independent angles is called a **global transformation**. If the angles of rotation depend on x, the transformation is called a **local** or a **gauge transformation**.

Infinitesimal global or local trans of fields ϕ_i:

$$\phi_i(x) \;\rightarrow\; \phi_i'(x) \;=\; \phi_i(x) + \delta\phi_i(x)\,,$$

where $\delta\phi_i(x) \;=\; i\,\theta^a(x)\;(T^a)_i^{\,j}\,\phi_j(x)$, and T^a are the generators of the Lie Group. Note that the angles or group parameters θ^a are x-independent for a global trans.

If a Lagrangian $\mathcal{L}$ is invariant under a global or local trans, it is said that $\mathcal{L}$ has a **global** or **local (gauge) symmetry**.

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Exercise: Show that the above Lagrangian for a complex scalar is *not* invariant under a U(1) gauge trans.

– Noether's Theorem

If a Lagrangian $\mathcal{L}$ is (up to a total derivative) invariant under a given transformation of fields and spacetime, then there is a conserved current $J^\mu(x)$ and a conserved charge $Q = \int d^3\mathbf{x}\, J^0(x)$, associated with this symmetry, such that

$$\partial_\mu J^\mu \;=\; 0 \quad \text{and} \quad \frac{dQ}{dt} \;=\; 0\,.$$

Proof for a global symmetry:

Consider a Lagrangian $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ to be invariant under the infinitesimal global trans:

$$\delta\phi_i \;=\; i\,\theta^a (T^a)_i^{\,j}\,\phi_j\,,$$

where T^a are the generators of some group G.

Hence, the change of $\mathcal{L}$ is vanishing, i.e.

$$\delta\mathcal{L} \;=\; \frac{\partial\mathcal{L}}{\partial\phi_i}\,\delta\phi_i \;+\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\partial_\mu(\delta\phi_i) \;=\; 0\,.$$

This last equation can be rewritten as

$$\delta\mathcal{L} \;=\; \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\delta\phi_i\right] + \left[\frac{\partial\mathcal{L}}{\partial\phi_i} \;-\; \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right]\delta\phi_i \;=\; 0\,.$$

With the aid of the equations of motions for ϕ_i, the last equation implies that

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\delta\phi_i\right] \;=\; \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \;-\; \frac{\partial\mathcal{L}}{\partial\phi_i}\right]\delta\phi_i \;=\; 0\,.$$

The conserved current (or currents) is

$$J^{a,\,\mu} \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\frac{\partial\delta\phi_i}{\partial\theta^a} \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,i\,(T^a)_i^{\,j}\,\phi_j\,.$$

The corresponding conserved charges are

$$Q^a(t) \;=\; \int d^3\mathbf{x}\, J^{a,\,0}(x)\,.$$

Indeed, it is easy to check that

$$\frac{dQ^a}{dt} \;=\; \int d^3\mathbf{x}\;\partial_0\,J^{a,\,0}(x) \;=\; -\int d^3\mathbf{x}\;\nabla\cdot\mathbf{J}^a(x)$$
$$\;=\; -\int d\mathbf{s}\cdot\mathbf{J}^a \;\to\; 0\,,$$

assuming that surface terms vanish at infinity, or requiring that $\lim_{\mathbf{x}\to\pm\infty}\phi(x) = 0$.

Exercises:

Derive the conserved currents and charges for a scalar theory, whose action $S = \int d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i)$ is invariant under:

(i) the infinitesimal spacetime translations

$$x^\mu \;\to\; x'^\mu \;=\; x^\mu \,+\, \epsilon^\mu \;.$$

(ii)* the infinitesimal Lorentz transformations (LTs)

$$x^\mu \;\to\; x'^\mu \;\equiv\; \Lambda^\mu_{\ \nu}(\omega) x^\nu \;\approx\; x^\mu \,+\, \omega^\mu_{\ \nu}\, x^\nu \;,$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu} \ll 1$.

2. Canonical Quantization

– From Classical to Quantum Mechanics

- Physical states are vectors $|A\rangle$ in the Hilbert space $\mathcal{H}$.
- If a system is in state $|A\rangle$, the probability to observe it in another state $|B\rangle$ is $P_{A\to B} = |\langle B|A\rangle|^2$.
- Observables are mapped into Hermitian operators $\widehat{O}$:

$$\langle\alpha|\widehat{O}|\beta\rangle \;=\; \langle\alpha|\widehat{O}^\dagger|\beta\rangle \;=\; \langle\beta|\widehat{O}|\alpha\rangle^* \,.$$

- Free particles are described by plane waves:

$$\phi_{\mathbf{p}}(t,\mathbf{x}) = \langle \mathbf{x}; t|\mathbf{p}\rangle = \sqrt{2E_p}\, e^{-ip_\mu x^\mu}; \quad E_p = \sqrt{\mathbf{p}^2 + m^2}\;.$$

- Completeness and orthogonality of the Hilbert space:

$$\sum_{\mathbf{p}} |\mathbf{p}\rangle\,\langle\mathbf{p}| \;\equiv\; \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_p}\,|\mathbf{p}\rangle\,\langle\mathbf{p}| \;=\; \hat{\mathbf{1}}\;,$$

$$\langle\mathbf{k}|\mathbf{p}\rangle \;=\; (2\pi)^3\, 2E_p\, \delta^{(3)}(\mathbf{k}-\mathbf{p})\;.$$

Exercises:

(i) Use the 4-momentum operator $\hat{P}^\mu$, defined such that $\hat{P}_\mu|\mathbf{x}; t\rangle \;=\; i\partial/\partial x^\mu|\mathbf{x}; t\rangle$, to derive the Klein-Gordon equation from special relativity.

(ii) Show that $\langle\mathbf{k}|\mathbf{p}\rangle$ is Lorentz invariant.

– Quantum Fields and Causality

Particle States are vectors in the Fock space $\mathcal{F} \sim \prod_{\otimes} \mathcal{H}$:

$$\begin{aligned} |0; t_0\rangle &: \text{ state with zero particle at time } t_0 \\ |\mathbf{p}; t_0\rangle &: \text{ state with one particle of momentum } \mathbf{p} \text{ at } t_0 \\ |\mathbf{p}, \mathbf{q}; t_0\rangle &: \text{ state with two particles of momenta } \mathbf{p}, \mathbf{q} \text{ at } t_0 \end{aligned}$$

Creation operator $a^\dagger(\mathbf{p})$ adds a particle of momentum $\mathbf{p}$ to the Fock state:

$$a^\dagger(\mathbf{p})\,|\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N\rangle \;\equiv\; |\mathbf{p}, \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N\rangle\,.$$

Field operator for a free scalar theory:

$$\Phi(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \Big(a(\mathbf{k})\, e^{-ik\cdot x} \;+\; a^\dagger(\mathbf{k})\, e^{ik\cdot x} \Big)$$

$\cdots$

Exercises:

(i) Show that $[a(\mathbf{p}),\, a^\dagger(\mathbf{q})] \;=\; (2\pi)^3\, 2E_p\, \delta^{(3)}(\mathbf{p} - \mathbf{q})$.

(ii) Show that the Hermitian adjoint $a(\mathbf{p})$ is the **annihilation operator** removing a particle from a Fock state as follows:

$$a(\mathbf{p})\,|\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N\rangle \;=\; \sum_{i=1}^{N} (2\pi)^3 2E_p\, \delta^{(3)}(\mathbf{p} - \mathbf{q}_i) \times |\mathbf{q}_1, \ldots, \mathbf{q}_{i-1}, \mathbf{q}_{i+1}, \ldots, \mathbf{q}_N\rangle\,.$$

In particular, $a(\mathbf{p})|0\rangle \;=\; 0$ and $\langle 0|0\rangle \;=\; 1$ defines the **vacuum** or the **ground state** of the system.

Causality of the field operator $\Phi(x)$:

$$[\Phi(x),\, \Phi(y)] \;=\; 0\,, \quad \text{for } (x-y)^2 < 0.$$

This means that the field strength Φ is simultaneously measurable at points with space-like separations.

Straightforward calculation of the above commutator gives

$$[\Phi(x)\,,\; \Phi(y)] \;=\; \Delta(x-y)\,,$$

with

$$\Delta(x-y) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \Big(e^{-ik\cdot(x-y)} \;-\; e^{ik\cdot(x-y)} \Big)\,.$$

$\cdots$

Exercises:

(i) Derive the above expression for $\Delta(x-y)$.

(ii) Show that $\Delta(x-y) = 0$, *only* for $(x-y)^2 < 0$.

(iii) Verify that $\Delta(x-y)$ satisfies the Klein–Gordon eqn:

$$(\partial_\mu \partial^\mu \;+\; m^2)\,\Delta(x-y) \;=\; 0\,.$$

(iv) Show that the integral measure

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \;=\; \int \frac{d^4k}{(2\pi)^4}\, 2\pi\, \theta(k_0)\, \delta(k^2 - m^2)$$

is Lorentz invariant, where $\theta(z)$ is the step function.

– Canonical Quantization of Scalar Field Theory

The **conjugate momentum operator $\mathbf{\Pi}(x)$** is defined via the so-called **equal-time commutation** relations:

$$\begin{aligned} \big[\Phi(t,\mathbf{x}),\, \Pi(t,\mathbf{y})\big] \;&=\; i\delta^{(3)}(\mathbf{x}-\mathbf{y})\;, \\ \big[\Phi(t,\mathbf{x}),\, \Phi(t,\mathbf{y})\big] \;&=\; \big[\Pi(t,\mathbf{x}),\, \Pi(t,\mathbf{y})\big] \;=\; 0\;. \end{aligned}$$

The **quantized Hamiltonian (density) $\mathcal{H}$** is defined as

$$\mathcal{H}(\Phi,\Pi) \;=\; \Pi\,\partial_0\Phi \;-\; \mathcal{L}(\Phi,\partial_\mu\Phi)$$

$$\ldots$$

Exercises:

(i) Show that $\Pi(t,\mathbf{x}) = \partial_0\Phi(t,\mathbf{x})$ satisfies the equal-time commutation relations given above.

(ii) Show that the definition of $\Pi(x)$ given in (i) is identical to the classical relation:

$$\Pi(x) \;=\; \frac{\partial\mathcal{L}(x)}{\partial\big(\partial_0\Phi(x)\big)}\;,$$

where $\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}m^2\Phi^2$.

(iii) Generalize the above canonical quantization rules for a free scalar theory with a number n of field operators $\Phi_i(t,\mathbf{x})$, where $i = 1, 2, \ldots, n$.

(iv) Show that $\mathcal{H}$ is positive definite for $m^2 > 0$.

– Complex Fields and Anti-Particles

Lagrangian: $\mathcal{L} \;=\; (\partial^\mu\Phi^\dagger)\,(\partial_\mu\Phi) \;-\; m^2\,\Phi^\dagger\Phi\;.$

Complex field operators:

$$\begin{aligned} \Phi(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2E_k}\,\Big(a(\mathbf{k})\,e^{-ik\cdot x} \;+\; b^\dagger(\mathbf{k})\,e^{ik\cdot x}\Big)\;, \\ \Phi^\dagger(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2E_k}\,\Big(a^\dagger(\mathbf{k})\,e^{ik\cdot x} \;+\; b(\mathbf{k})\,e^{-ik\cdot x}\Big)\;. \end{aligned}$$

Conjugate momenta: $\Pi(x) = \partial_0\Phi^\dagger(x)$, $\quad\Pi^\dagger(x) = \partial_0\Phi(x)$.

Equal-time commutators:

$\big[\Phi(t,\mathbf{x}),\, \Pi(t,\mathbf{y})\big] \;=\; \big[\Phi^\dagger(t,\mathbf{x}),\, \Pi^\dagger(t,\mathbf{y})\big] \;=\; i\delta^{(3)}(\mathbf{x}-\mathbf{y})\;,$
whereas all other commutators vanish.

Conserved current (*Why?*): $J^\mu \;=\; \Phi^\dagger(i\partial^\mu\Phi) \;-\; \Phi(i\partial^\mu\Phi^\dagger)\;.$

Conserved charge: $Q = \int d^3\mathbf{x}\; : J_0 :\;.$

The double dots $:\cdots:$ indicate **normal or Wick's ordering** that ignore vacuum effects according to the prescription

$$:\frac{1}{2}\Big(a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})\Big): \,=\, a^\dagger(\mathbf{k})a(\mathbf{k})\,,$$

where the infinite term $(2\pi)^3 2E_k\delta^{(3)}(\mathbf{0})$ has been neglected.

Exercises:

(i) Show that the equal-time commutators given on the previous page hold true, provided

$$[a(\mathbf{k}),\, a^\dagger(\mathbf{k}')] = [b(\mathbf{k}),\, b^\dagger(\mathbf{k}')] = (2\pi)^3 2E_k\, \delta^{(3)}(\mathbf{k}-\mathbf{k}')\ ,$$

and all other commutators vanish.

(ii) Show that

$$Q = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \Big(a^\dagger(\mathbf{k}) a(\mathbf{k}) \;-\; b^\dagger(\mathbf{k}) b(\mathbf{k}) \Big)$$

and hence that

$$Q a^\dagger(\mathbf{k})|0\rangle = a^\dagger(\mathbf{k})|0\rangle\ , \qquad Q b^\dagger(\mathbf{k})|0\rangle = -b^\dagger(\mathbf{k})|0\rangle\ .$$

(iii) What is the physical meaning of $a^\dagger$ and $b^\dagger$ creation operators in terms of **particle** and **anti-particle states**.

(iv) Show the field-particle duality relations:

$$\begin{aligned} \langle 0|b(\mathbf{k})\Phi(x)|0\rangle &= e^{ik\cdot x} = N_\mathbf{k}\,\langle \mathbf{x}; t|b;\mathbf{k}\rangle^*\ ,\\ \langle 0|a(\mathbf{k})\Phi^\dagger(x)|0\rangle &= e^{ik\cdot x} = N_\mathbf{k}\,\langle \mathbf{x}; t|a;\mathbf{k}\rangle^*\ , \end{aligned}$$

where $N_\mathbf{k} = (2E_\mathbf{k})^{-1/2}$ is a normalization factor. Give a physical interpretation for these relations.

3. The S-Matrix in Quantum Field Theory

– Time Evolution of Quantum States and the S-Matrix

The Hamiltonian H is the sum of the free H_0 plus the interaction $H_{\rm int}$ Hamiltonian: $H = H_0 + H_{\rm int}$.

Time evolution of the state $|A(t);t_0\rangle$ in the **Schrödinger picture** (SP):

$$i\frac{\partial}{\partial t}|A(t);t_0\rangle_S \;=\; H\,|A(t);t_0\rangle_S\ ,$$

where t_0 is the initial time imposed on the state or system.

Formal solution: $|A(t);t_0\rangle_S \;=\; e^{-iH(t-t_0)}\,|A(t_0);t_0\rangle_S$.

In the **Heisenberg picture** (HP), states do not change with time, i.e. $|A;t_0\rangle_H \equiv |A(t_0);t_0\rangle_S$, but operators do:

$$\widehat{O}^H(t) \;=\; e^{iH(t-t_0)}\,\widehat{O}^S\,e^{-iH(t-t_0)}\ ,$$

so that ${}_S\langle B(t);t_0|\widehat{O}^S|A(t);t_0\rangle_S \;=\; {}_H\langle B;t_0|\widehat{O}^H(t)|A;t_0\rangle_H$.

Interaction picture (IP): $\widehat{O}^I(t) \;=\; e^{iH_0(t-t_0)}\widehat{O}^S e^{-iH_0(t-t_0)}$, and

$$|A(t);t_0\rangle_I \;=\; e^{iH_0(t-t_0)}\,|A(t);t_0\rangle_S \;=\; U(t,t_0)\,|A;t_0\,\rangle_H\ ,$$

where $U(t,t_0) = e^{iH_0^S(t-t_0)}\,e^{-iH^S(t-t_0)}$ is the **time evolution operator**.

Exercises:

(i) Prove the relation: $|A(t);t_0\rangle_I \;=\; U(t,t_0)\,|A;t_0\rangle_H$.

(ii) Show that

$$i\,\frac{\partial}{\partial t}|A(t);t_0\rangle_I \;=\; H^I_{\rm int}(t)\,|A(t);t_0\rangle_I\;,$$

and likewise that

$$i\,\frac{\partial}{\partial t}U(t,t_0) \;=\; H^I_{\rm int}(t)\,U(t,t_0)\;.$$

(iii) Show that

$$\begin{aligned} U(t,t_0) \;&=\; \hat{\mathbf{1}} \,+\, (-i)\int_{t_0}^{t} dt_1\, H^I_{\rm int}(t_1) \\ &+\, \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\Big\{H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\Big\} \,+\, \dots \\ &=\; T\Big\{\exp\Big[-i\int_{t_0}^{t} dt'\, H^I_{\rm int}(t')\Big]\Big\}\;, \end{aligned}$$

where $T\{\dots\}$ is the **time-ordered product** of the operators $H^I_{\rm int}(t)$ in order of decreasing time when they are written from left to right, e.g.

$$T\{H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\} \;=\; \begin{cases} H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\,, & \text{if } t_1 > t_2 \\ H^I_{\rm int}(t_2)H^I_{\rm int}(t_1)\,, & \text{if } t_2 > t_1 \end{cases}\;.$$

The S-Matrix

Transition amplitude in the IP: ${}_I\langle f;t_f|U(t_f,t_i)|i;t_i\rangle_I$.

In a scattering process $i\to f$, we are interested in times of $t_{i/f}\to\mp\infty$. In this case, the relevant U-operator is the ***S*-matrix operator** defined by

$$S \;=\; \lim_{t_{i/f}\to\mp\infty} U(t_f,t_i)\;.$$

The S-matrix elements are

$$S_{i\to f} \;=\; {}_I\langle f;+\infty|S|i;-\infty\rangle_I\;.$$

Using the approximation of adiabatically switching-off the interactions, i.e. $\lim_{t\to\pm\infty} H_{\rm int}(t)=0$, we may write

$$|\mathbf{p},\mp\infty\rangle_I \;=\; Z^{1/2}\,|\mathbf{p},\mp\infty\rangle_H \;=\; Z^{1/2}\,|\mathbf{p}\rangle\;,$$

where $Z^{1/2}$ is a wave-function normalization of the asymptotic state $|\mathbf{p}\rangle$.

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Exercise:

Show that the S-matrix operator in QFT is given by

$$S \;=\; T\Big\{\exp\Big[-i\int_{-\infty}^{+\infty} d^4x\, \mathcal{H}^I_{\rm int}(x)\Big]\Big\}\;.$$

– Feynman Propagator and Wick's Theorem

Calculation of the T-ordered product:

$$T\big[\Phi(x)\,\Phi(y)\big] \;=\; \theta(x^0-y^0)\,\Phi(x)\,\Phi(y)+\theta(y^0-x^0)\,\Phi(y)\,\Phi(x)$$
$$=\; :\Phi(x)\,\Phi(y):\;+\,i\,\Delta_F(x-y)\;,$$

where $\Delta_F(x-y)$ is the **Feynman propagator**:

$$i\Delta_F(x-y) \;=\; \langle 0|T\big[\Phi(x)\,\Phi(y)\big]|0\rangle = \int\frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-ik\cdot(x-y)}}{k^2-m^2+i\varepsilon}\,.$$

Wick's Theorem (for even n):

$$T\big[\Phi(x_1)\,\Phi(x_2)\ldots\Phi(x_n)\big] \;\equiv\; T\big[\Phi_1\Phi_2\ldots\Phi_n\big]$$
$$=\; :\Phi_1\Phi_2\ldots\Phi_n:$$
$$+\,D_{12}:\Phi_3\Phi_4\ldots\Phi_n:\,+\,D_{13}:\Phi_2\Phi_4\ldots\Phi_n:\,+\,\ldots$$
$$+\,D_{12}D_{34}:\Phi_5\Phi_6\ldots\Phi_n:\,+\,\ldots$$
$$+\,\ldots$$
$$+\,D_{12}D_{34}\ldots D_{n-1,n}\,+\,\text{permutations,}$$

where $D_{ij}=i\Delta_F(x_i-x_j)$.

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Exercises:

(i) Show that $\Delta_F(x-y)$ is the Green function of the Klein-Gordon eqn: $(\partial_\mu\partial^\mu+m^2)\,\Delta_F(x-y)=-\delta^{(4)}(x-y)$.

(ii) Prove Wick's theorem for $n=2,3$ and for odd n.

– Transition Amplitudes and Feynman Rules

Toy model:

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\Phi)\,(\partial^\mu\Phi)-\frac{1}{2}\,M^2\Phi^2+(\partial_\mu\chi^\dagger)\,(\partial^\mu\chi)-m^2\chi^\dagger\chi-g\,\chi^\dagger\chi\Phi\;,$$

with $\mathcal{H}_{\rm int}(x) \;=\; g\,\chi^\dagger(x)\chi(x)\Phi(x)$.

S-matrix amplitude for the decay $\Phi(q)\to\chi^+(p)\chi^-(k)$:

$$S_{fi} \;=\; \langle\chi^+(p),\chi^-(k)|\,T\Big\{\exp\Big[-i\int d^4x\,\mathcal{H}_{\rm int}(x)\Big]\Big\}\,|\Phi(q)\rangle$$
$$=\; \langle 0|a(\mathbf{p})b(\mathbf{k})\,T\Big\{\exp\Big[-i\int d^4x\,\mathcal{H}_{\rm int}(x)\Big]\Big\}\,a^\dagger_\Phi(\mathbf{q})|0\rangle$$
$$=\; -ig\int d^4x\;\langle 0|a(\mathbf{p})b(\mathbf{k})\,\chi^\dagger(x)\chi(x)\Phi(x)\,a^\dagger_\Phi(\mathbf{q})|0\rangle$$
$$+\,\mathcal{O}(g^2)\,.$$

...

Exercise:

Calculate the S-matrix element for $\Phi\to\chi^+\chi^-$ to obtain

$$S_{fi} \;=\; (-ig)\,(2\pi)^4\delta^{(4)}(p+k-q)\;+\;\mathcal{O}(g^2)\,.$$

The amplitude for $\chi^+(p_1)\chi^-(p_2) \to \chi^+(k_1)\chi^-(k_2)$**:**

The **transition amplitude** is defined by $iT_{fi} \equiv (S - \hat{\mathbf{1}})_{fi}$.

The lowest order contribution to iT_{fi} is

$$
\begin{aligned}
iT_{fi} &= \langle 0| a(\mathbf{k}_1) b(\mathbf{k}_2) \frac{(-ig)^2}{2!} \\
&\times \int d^4x\, d^4y\, T\Big\{\chi^\dagger(x)\chi(x)\Phi(x)\chi^\dagger(y)\chi(y)\Phi(y)\Big\}\, a^\dagger(\mathbf{p}_1) b^\dagger(\mathbf{p}_2)\, |0\rangle\, .
\end{aligned}
$$

...

Exercises:

(i) Calculate the transition amplitude T_{fi} for the scattering process $\chi^+\chi^- \to \chi^+\chi^-$ to obtain

$$
\begin{aligned}
iT_{fi} &= (-ig)^2 \left(\frac{i}{s - M^2} + \frac{i}{t - M^2} \right) \\
&\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, ,
\end{aligned}
$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 - k_1)^2$.

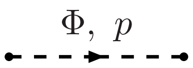
(ii) Calculate the S-matrix element for the same scattering process $\chi^+\chi^- \to \chi^+\chi^-$, but for a theory whose interaction Lagrangian is given by

$$
\mathcal{L}_{\rm int} = -\, g\, \chi^\dagger \chi \Phi - \frac{\lambda}{4} (\chi^\dagger \chi)^2\, .
$$

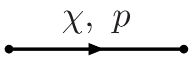
Feynman Rules

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} M^2 \Phi^2 + (\partial_\mu \chi^\dagger)(\partial^\mu \chi) - m^2 \chi^\dagger \chi \\
&\quad - g\, \chi^\dagger \chi \Phi - \frac{\lambda}{4} (\chi^\dagger \chi)^2\, .
\end{aligned}
$$

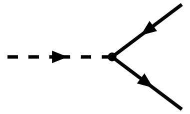
Feynman rules derived from $\mathcal{L}$ for calculating iT_{fi}:

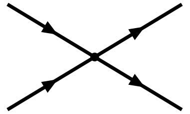


: $\dfrac{i}{p^2 - M^2}$



: $\dfrac{i}{p^2 - m^2}$


: $-ig$


: $-i\lambda$

Additional rule for loops: Include an integral factor of

$$
+(-) \int \frac{d^4k}{(2\pi)^4}
$$

for a bosonic (fermionic) loop with momentum k.

– Particle Decays and Cross Sections

Decay of particle a: $a(p) \to b_1(k_1) + \cdots + b_n(k_n)$

The decay rate Γ_a of the unstable particle a is conventionally defined at its rest frame. The total **decay rate** Γ_a can be calculated from the differential form:

$$\begin{aligned} d\Gamma_a &= \frac{1}{2m_a} |\mathcal{M}_{fi}|^2 \\ &\times (2\pi)^4 \, \delta^{(4)}(k_1 + \cdots + k_n - p) \prod_{i=1}^{n} \frac{d^3\mathbf{k}_i}{(2\pi)^3 \, 2E_{k_i}} \, S \,, \end{aligned}$$

where $\mathcal{M}_{fi}$ is related to T_{fi} via

$$T_{fi} = (2\pi)^4 \delta^{(4)}(k_1 + \cdots + k_n - p) \, \mathcal{M}_{fi}$$

and $S = 1/m!$ is a combinatorial factor that removes multiple-counting if there are m identical particles in the final state.

...

Exercises:

(i) Derive the analytical expression for the total decay rate Γ_Φ of the particle Φ to lowest order in perturbation theory.

(ii) Use the formula derived in (i) to estimate Γ_Φ for model parameters: $M = 120$ GeV, $m = 50$ GeV and $g/M = 0.5$.

Scattering process: $a_1(p_1) + a_2(p_2) \to b_1(k_1) + \cdots + b_n(k_n)$

Definition of **cross section:**

$$\sigma = \frac{\text{Transition rate: } P_{fi}/T}{\text{Flux of incoming particles: } F}$$

Differential form:

$$\begin{aligned} d\sigma &= \frac{dP_{fi}/T}{F} \\ &= \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|} \left(\frac{1}{2E_{p_1}}\right) \left(\frac{1}{2E_{p_2}}\right) |\mathcal{M}_{fi}|^2 \\ &\times (2\pi)^4 \, \delta^{(4)}(k_1 + \cdots + k_n - p_1 - p_2) \prod_{i=1}^{n} \frac{d^3\mathbf{k}_i}{(2\pi)^3 \, 2E_{k_i}} \, S \,, \end{aligned}$$

where $|\mathbf{v}_1 - \mathbf{v}_2|$ is the modulus of the relative velocity of the incoming $a_{1,2}$ particles.

The number N of scattering events per unit time is obtained by multiplying σ $[\mathrm{cm}^2]$ with the luminosity L $[\mathrm{cm}^{-2}\,\mathrm{s}^{-1}]$:

$$N = L \times \sigma \,.$$

Units of measuring cross sections: $1 \text{ barn} = 10^{-24} \text{ cm}^2$, $1 \text{ nb} = 10^{-9} \text{ barn}$, $\text{GeV}^{-2} \approx 0.39 \text{ mb}$.

Exercises:

(i) Show that

$$\frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\left(\frac{1}{2E_{p_1}}\right)\left(\frac{1}{2E_{p_2}}\right) \;=\; \frac{1}{2\,\lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)}\ ,$$

where $\lambda(x, y, z) = (x - y - z)^2 - 4yz$ and $s = (p_1 + p_2)^2$, and so that σ is Lorentz invariant.

(ii) Calculate analytically the cross section σ for the scattering processes: (a) $\chi^+\chi^- \to \chi^+\chi^-$ and (b) $\chi^+\chi^- \to \Phi\Phi$.

(iii) Assuming that the scattering processes in (ii) take place at centre-of-mass energies $\sqrt{s} = 300$ GeV, estimate the value of σ in both GeV^{-2} and nb, for $M = 120$ GeV, $m = 50$ GeV and $\lambda = g/M = 0.5$.

(iv) How many scatterings do you expect to detect per year at a collider such as TEVATRON with $L = 10^{31}\,\mathrm{cm}^{-2}s^{-1}$?

– Unitarity and the Optical Theorem

Unitarity of the S-matrix: $S^\dagger S \;=\; \hat{\mathbf{1}}$

This is equivalent to:

$$\langle i'|i\rangle \;=\; \langle i'|S^\dagger S|i\rangle \;=\; \sum_f \langle f|S|i'\rangle^* \,\langle f|S|i\rangle\ .$$

In terms of the transition operator $iT \;=\; S - \hat{\mathbf{1}}$, this implies

$$\frac{1}{2i}\Big(\langle i'|T|i\rangle \;-\; \langle i|T|i'\rangle^*\Big) \;=\; \frac{1}{2}\sum_f \langle f|T|i'\rangle^* \,\langle f|T|i\rangle\ .$$

Optical Theorem for $i = i'$:

$$\Im m(\mathcal{M}_{ii}) \;=\; \lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)\,\sigma_{\rm tot}(i \to X)\ ,$$

where $|i\rangle = |a_1(\mathbf{p}_1)a_2(\mathbf{p}_2)\rangle$.

...

Exercise:

Prove the optical theorem stated above and describe a possible experimental test that can probe its validity.

4. Quantum Electrodynamics

– Weyl and Dirac Spinors

Weyl spinors, represented by $\xi_\alpha(x)$ and $\bar\eta^{\dot\alpha}(x)$ $(\alpha = 1, 2)$, are 2-dim complex vectors whose components anti-commute. Under a Lorentz transformation (LT) $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, they transform as:

$$\xi'_\alpha = M_\alpha{}^\beta\,\xi_\beta\,, \qquad \bar\eta'_{\dot\alpha} = M^{\dagger\,\dot\beta}{}_{\dot\alpha}\,\bar\eta_{\dot\beta}\,,$$
$$\xi'^\alpha = M^{-1}{}_\beta{}^\alpha\,\xi^\beta\,, \quad \bar\eta'^{\dot\alpha} = M^{\dagger-1\,\dot\alpha}{}_{\dot\beta}\,\bar\eta^{\dot\beta}\,.$$

$M \in \mathrm{SL}(2,\mathbb{C})$ is a 2×2 complex matrix with $\det M = 1$.

Duality relations among Weyl spinors:

$$(\xi^\alpha)^\dagger = \bar\xi^{\dot\alpha}\,, \quad (\xi_\alpha)^\dagger = \bar\xi_{\dot\alpha}\,, \quad (\bar\eta_{\dot\alpha})^\dagger = \eta_\alpha\,, \quad (\eta^\alpha)^\dagger = \bar\eta^{\dot\alpha}$$

Lowering and raising spinor indices:

$$\xi_\alpha = \varepsilon_{\alpha\beta}\xi^\beta\,, \quad \xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta\,, \quad \bar\eta_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\eta^{\dot\beta}\,, \quad \bar\eta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\eta_{\dot\beta}\,,$$

with $\varepsilon^{\alpha\beta} \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha\beta}$ and $\varepsilon^{\dot\alpha\dot\beta} \equiv i\sigma_2 = -\varepsilon_{\dot\alpha\dot\beta}$.
Lorentz-invariant spinor contractions:

$$\xi\eta \equiv \xi^\alpha\eta_\alpha = \xi^\alpha\varepsilon_{\alpha\beta}\eta^\beta = -\eta^\beta\varepsilon_{\alpha\beta}\xi^\alpha = \eta^\beta\varepsilon_{\beta\alpha}\xi^\alpha = \eta^\beta\xi_\beta = \eta\xi$$

Likewise, $\bar\xi\bar\eta \equiv (\eta\xi)^\dagger = \xi^\dagger_\alpha\eta^{\alpha\dagger} = \bar\xi_{\dot\alpha}\bar\eta^{\dot\alpha} = \bar\eta_{\dot\alpha}\bar\xi^{\dot\alpha} = \bar\eta\bar\xi$.

The Dirac spinor is a 4-dim complex vector made of 2 Weyl spinors:

$$\psi(x) = \begin{pmatrix} \xi_\beta(x) \\ \bar\eta^{\dot\beta}(x) \end{pmatrix}.$$

The Dirac equation:

$$(i\,\gamma^\mu\partial_\mu - m)\,\psi = 0\,,$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix},$$

$\sigma^\mu = (\mathbf{1}_2,\ \boldsymbol{\sigma})$, $\bar\sigma^\mu = (\mathbf{1}_2,\ -\boldsymbol{\sigma})$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

Lagrangian for Dirac fermions:

$$\mathcal{L}_{\mathrm{D}} = \bar\psi\,(i\,\gamma^\mu\partial_\mu - m)\,\psi\,,$$

with $\bar\psi \equiv (\eta^\alpha,\ \bar\xi_{\dot\alpha})$.

...

Exercises:

(i) Derive the Euler–Lagrange equations with respect to the Dirac fields $\bar\psi(x)$ and $\psi(x)$.

(ii) Given that $M\sigma_\mu M^\dagger = \Lambda^\nu{}_\mu\,\sigma_\nu$ and $M^{\dagger-1}\bar\sigma_\mu M^{-1} = \Lambda^\nu{}_\mu\,\bar\sigma_\nu$, show that $\mathcal{L}_{\mathrm{D}}$ is invariant under a LT.

– Quantization of the Fermion Field

The Dirac field operators:

$$\Psi(x) = \sum_{s=\pm 1/2} \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \Big[b(\mathbf{k},s)\, u(\mathbf{k},s)\, e^{-ik\cdot x} + d^\dagger(\mathbf{k},s)\, v(\mathbf{k},s)\, e^{ik\cdot x} \Big] ,$$

$$\overline{\Psi}(x) = \sum_{s=\pm 1/2} \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \Big[b^\dagger(\mathbf{k},s)\, u^\dagger(\mathbf{k},s)\gamma_0\, e^{ik\cdot x} + d(\mathbf{k},s)\, v^\dagger(\mathbf{k},s)\gamma_0\, e^{-ik\cdot x} \Big] .$$

Conjugate momentum operator Π_Ψ for Ψ:

$$\Pi_\Psi(x) = -\frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = i\overline{\Psi}(x)\,\gamma_0 .$$

Quantization via equal-time anti-commutators:

$$\big\{ \Psi_\alpha(t,\mathbf{x})\,,\; i[\overline{\Psi}(t,\mathbf{y})\,\gamma_0]_\beta \big\} = i\delta_{\alpha\beta}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}) ,$$

where $\alpha,\beta = 1,2,3,4$ are Dirac spinor indices.

...

Exercises:

(i) Show that the equal-time anti-commutators are satisfied provided that

$$\{b(\mathbf{k},s)\,,\, b^\dagger(\mathbf{k}',s')\} = \{d(\mathbf{k},s)\,,\, d^\dagger(\mathbf{k}',s')\} = \delta_{ss'}\,(2\pi)^3 2E_k \delta^{(3)}(\mathbf{k}-\mathbf{k}') .$$

(ii) What is the physical significance of $b^\dagger(\mathbf{k},s)$ and $d^\dagger(\mathbf{k},s)$?

– Gauge Symmetry

$\mathcal{L}_{\mathrm{D}}$ is invariant under the U(1) global trans:

$$\psi(x) \to \psi'(x) = e^{i\theta}\,\psi(x) ,$$

but it is *not* invariant under a U(1) gauge transformation, when $\theta = \theta(x)$. Instead, we find the residual term

$$\delta\mathcal{L}_{\mathrm{D}} = -\,(\partial_\mu\theta(x))\,\bar{\psi}\gamma^\mu\psi .$$

To compensate for this term, we introduce a vector field A^μ in the theory, the so-called **photon**, and add to $\mathcal{L}_{\mathrm{D}}$ the term:

$$\mathcal{L}_{A\bar{\psi}\psi} = -\,e\,A_\mu\,\bar{\psi}\gamma^\mu\psi .$$

We demand that A_μ transforms under a local U(1) as

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\,\partial_\mu\theta(x) .$$

Then, $\mathcal{L}_\psi = \mathcal{L}_{\mathrm{D}} + \mathcal{L}_{A\bar{\psi}\psi}$ is invariant under a U(1) gauge transformation of ψ and A^μ.

The **gauge symmetry** gives rise to a conserved current and charge according to Noether's theorem.

Otherwise, gauge symmetry alone guarantees neither the *masslessness* of the photon nor even its *existence*!

The Lagrangian of the electron and photon

The Lagrangian of Quantum Electrodynamics (QED) includes the interaction of the photon with the electron:

$$\mathcal{L}_{\rm QED} \;=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \;+\; \bar{\psi}\,(i\,\not{\partial} \;-\; m \;-\; e\,\not{A})\,\psi\,,$$

where we used the convention: $\not{a} \equiv \gamma_\mu a^\mu$.

...

Exercises:

(i) Derive the equation of motions with respect to the photon and electron fields.

(ii) Derive the conserved current and charge from $\mathcal{L}_{\rm QED}$.

(iii) How should the Lagrangian describing a complex scalar field $\phi(x)$,

$$\mathcal{L} \;=\; (\partial^\mu\phi)^*\,(\partial_\mu\phi) \;-\; m^2\,\phi^*\phi\,,$$

be extended so as to become gauge symmetric under a U(1) local trans?

(iv)* A Lorentz-invariant photon mass term is described by the Lagrangian $\mathcal{L}_{\rm mass} \;=\; m_A^2\,A^\mu A_\mu$. Find a renormalizable gauge-symmetric extension of $\mathcal{L}_{\rm mass}$. Likewise, find a gauge-symmetric non-renormalizable extension of $\mathcal{L}_{\rm D}$ without the need of introducing a vector field A^μ.

– Quantization of the Electromagnetic Field

Electromagnetic field operator:

$$A_\mu(x) \;=\; \sum_{\lambda=0}^{3}\int\frac{d^3\mathbf{k}}{(2\pi)^3\,2E_k}\Big[a(\mathbf{k},\lambda)\,\varepsilon_\mu(\mathbf{k},\lambda)\,e^{-ik\cdot x} + a^\dagger(\mathbf{k},\lambda)\,\varepsilon^*_\mu(\mathbf{k},\lambda)\,e^{ik\cdot x}\Big]$$

and the conjugate momentum operator is $\partial_t A_\mu$.

Quantization proceeds via the equal-time commutators:

$$\big[A_\mu(t,\mathbf{x})\,,\;\partial_t A_\nu(t,\mathbf{y})\big] \;=\; -i\eta_{\mu\nu}\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,$$

where $\eta_{\mu\nu} \;=\; {\rm diag}\,(+1,\;-1,\;-1,\;-1)$.

The polarization vectors $\varepsilon_\mu(\mathbf{k},\lambda)$ satisfy the relations:

$$\sum_{\lambda=0}^{3}(-1)^{\delta_{\lambda 0}}\,\varepsilon_\mu(\mathbf{k},\lambda)\,\varepsilon^*_\nu(\mathbf{k},\lambda) \;=\; -\,\eta_{\mu\nu}\,,$$

and

$$\begin{aligned} k^\mu\,\varepsilon_\mu(\mathbf{k},\lambda=1,2,3) \;&=\; 0\,,\\ \varepsilon_\mu(\mathbf{k},\lambda)\,\varepsilon^{\mu\,*}(\mathbf{k},\lambda') \;&=\; -(-1)^{\delta_{\lambda 0}}\,\delta_{\lambda\lambda'}\,. \end{aligned}$$

But, the photon possesses **two physical polarizations**: the transvere degrees $\lambda = 1,2$. The unphysical degrees $\lambda = 0,3$ may be removed by **gauge fixing** which we discuss next.

– The Photon Propagator and Gauge Fixing

We add to $\mathcal{L}_{\rm QED}$ the **covariant gauge-fixing term**:

$$\mathcal{L}_{\rm GF} \;=\; -\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2\,.$$

The Euler-Lagrange equation for the photon becomes:

$$\left[\eta_{\mu\nu}\,\partial_\kappa\partial^\kappa \;-\; \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right] A^\nu \;=\; 0\,.$$

The photon propagator $\Delta_{\mu\nu}(x-y)$ is the Green-function of the above differential operator:

$$\left[\eta^{\mu\nu}\frac{\partial}{\partial x^\kappa}\frac{\partial}{\partial x_\kappa}-\left(1-\frac{1}{\xi}\right)\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right]\Delta_{\nu\lambda}(x-y) \;=\; \delta^\mu_\lambda\,\delta^{(4)}(x-y)\,.$$

...

Exercises:

(i) Derive the Euler-Lagrange equation of the photon in the presence of $\mathcal{L}_{\rm GF}$.

(ii) Show that the photon propagator is given by the Green function:

$$\Delta_{\mu\nu}(x-y) \;=\; \int\frac{d^4k}{(2\pi)^4}\left(-\,\eta_{\mu\nu} + (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right)\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}\,.$$

(iii) Use the equal-time commutators to show that

$$\langle 0|T[A_\mu(x)A_\nu(y)]|0\rangle \;=\; i\Delta_{\mu\nu}(x-y)$$

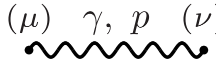
in the Feynman gauge $\xi = 1$.

– Feynman Rules for Quantum Electrodynamics

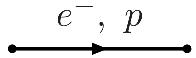
From the Lagrangian,

$$\mathcal{L}_{\rm QED} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \bar\psi\,(i\,\partial\!\!\!/ \;-\; m \;-\; e\,A\!\!\!/)\,\psi\,,$$

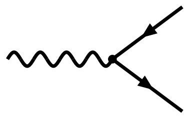
the following Feynman rules may be derived:

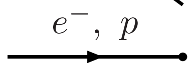


: $\dfrac{-i\,\eta_{\mu\nu}}{p^2+i\varepsilon}$

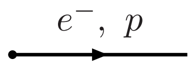


: $\dfrac{i}{p\!\!\!/-m+i\varepsilon}$

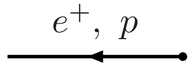

: $-ie\,\gamma_\mu$



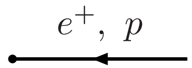
: $u(p)$ for an e^- in the initial state



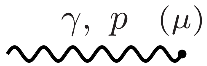
: $\bar u(p)$ for an e^- in the final state



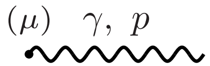
: $\bar v(p)$ for an e^+ in the initial state



: $v(p)$ for an e^+ in the final state



: $\varepsilon^\mu(\mathbf{p},\lambda)$ for a γ in the initial state



: $\varepsilon^{\mu\,*}(\mathbf{p},\lambda)$ for a γ in the final state

Exercises:

(i) Show that

(a) $\mathrm{Tr}(\gamma_\mu\gamma_\nu) \;=\; 4\,\eta_{\mu\nu}$,
(b) $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) \;=\; 4\,(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma})$,
(c) $\mathrm{Tr}(\gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_{2n+1}}) \;=\; 0$
(Hint: you may use the properties: $\{\gamma_5\,,\,\gamma_\mu\} = 0$ and $\gamma_5^2 = \mathbf{1}_4$, where $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$.),
(d) $\sum_{s=\pm 1/2} \bar{u}(p,s) M u(p,s) \;=\; \mathrm{Tr}\,[M\,(\not{p} + m)]$, where M is any arbitrary 4×4 matrix.

(ii) Use the Feynman rules for QED to write down the matrix element $\mathcal{M}_{fi}$ for the reaction $e^-(p_1)e^+(p_2) \to \mu^-(k_1)\mu^+(k_2)$.

(iii) With the aid of trace techniques given in (i), calculate $\overline{|\mathcal{M}_{fi}|^2}$, where the long bar indicates averaging over the spins of the electrons in the initial state.

(iv) Calculate analytically the differential cross section $d\sigma/d\Omega$ for $e^-e^+ \to \mu^-\mu^+$ which was taking place at the CERN LEP collider at CMS energies $\sqrt{s} = M_Z = 90$ GeV. Draw an accurate graph of $d\sigma/d\Omega$ as a function of $\cos\theta$.

(v) Supersymmetry predicts that in addition to muons $\mu^\pm$ there should be scalar muons $\tilde{\mu}^\pm$. Calculate $d\sigma/d\Omega$ for the process $e^-e^+ \to \tilde{\mu}^-\tilde{\mu}^+$. Plot $d\sigma/d\Omega$ as a function of $\cos\theta$ and comment on your results.

5. Renormalization

– Renormalizability

Superficial degree of divergence D_Γ of an **one-particle-irreducible** (1PI) loop graph Γ is defined as
$D_\Gamma = \#$ loop momenta in the numerator
$\qquad - \#$ loop momenta in the denominator.

For a ϕ^4 theory in d spacetime dimensions, we obtain that $D_\Gamma = d\,L - 2P$, where L: # loops and P: # propagators.

Examples in a ϕ^4 theory in 4 dimensions:

The 1PI 1-loop self-energy graph $\Gamma^{(1)}_{\phi^2}(p) \;\propto\; \Lambda^2 \;\to\; D = 2$, where Λ is an ultra-violet (UV) cut-off regulator.

The 1PI 1-loop coupling graph $\Gamma^{(1)}_{\phi^4} \;\propto\; \ln(\Lambda) \;\to\; D = 0$.

Weinberg's theorem on renormalizability: An 1PI loop graph Γ is UV finite, if D_Γ is negative ($D_\Gamma < 0$) and the superficial degree of all possible subgraphs is negative as well.

Possible caveats in Weinberg's theorem may result from additional symmetries acting on the theory, such as gauge symmetry, chiral symmetry, supersymmetry etc.

Exercises:

(i) Use the gauge symmetry to show that the 1-loop photon selfenergy graph has $D_\Gamma = 0$, instead of $D_\Gamma = 2$.
(ii) Use the chiral symmetry to show that the 1-loop electron selfenergy has $D_\Gamma = 0$, instead of $D_\Gamma = 1$.

General remarks on renormalizability of ϕ^n-theories

A theory is called **renormalizable by power counting** if all UV infinities induced by loop effects can be absorbed by a redefinition of field wavefunctions and the parameters of the Lagrangian.

If only a finite number of 1PI graphs is UV infinite, such a theory is called **super-renormalizable**, e.g. ϕ^3-theory in 4-dimensions.

d dimensions	Renormalizable	Non-renormalizable
6	ϕ^3	$\phi^{n>3}$
4	ϕ^4	$\phi^{n>4}$
3	ϕ^6	$\phi^{n>6}$
2	$\phi^{n\geq 2}$	–

Exercises:

(i) Show by power counting that ϕ^5 is non-renormalizable in 4 dimensions, but it is *not* so in 3.

(ii) Show that 2-dimensional ϕ^n theories are renormalizable for any power of $n > 0$.

– Renormalization of a Scalar Theory

Introduce the **bare Lagrangian**:

$$\mathcal{L}_\mathrm{B} \;=\; \frac{1}{2}\,(\partial^\mu\phi_0)\,(\partial_\mu\phi_0) \;-\; \frac{1}{2}\,m_0^2\,\phi_0^2 \;-\; \frac{\lambda_0}{4!}\,\phi_0^4 \;+\; \Lambda_c^0\,.$$

Since the ϕ^4-theory is renormalizable, all UV infinities of the 1PI loop graphs can be absorbed into the bare Lagrangian parameters: ϕ_0, m_0^2, λ_0 and Λ_c^0.

To this end, we introduce the UV finite (renormalized) parameters through the relations:

$$\phi_0 = Z_\phi^{1/2}\,\phi_R\,,\quad m_0^2 = Z_{m^2}\,m_R^2\,,\quad \lambda_0 = Z_\lambda\,\lambda_R\,,\quad \Lambda_c^0 = Z_{\Lambda_c}\Lambda_c^R\,.$$

where $Z_x = 1 + \delta Z_x$ (with $x \in \{m^2, \lambda, \Lambda_c\}$) and $Z_\phi^{1/2} = 1 + \frac{1}{2}\delta Z_\phi$ are the so-called **renormalization factors**.

The renormalization constants δZ_ϕ, δZ_x are UV infinite to be determined by **renormalization conditions**.

On-shell (OS) scheme of mass and wavefunction renormalization:

$$\Gamma_{\phi^2}(p^2 = m_R^2) \;=\; 0\,,\qquad \left.\frac{-i\,d\Gamma_{\phi^2}(p^2)}{dp^2}\right|_{p^2=m_R^2} \;=\; 1\,.$$

Exercise:

Use cut-off regularization to calculate the 1-loop renormalization constants $\delta Z_\phi^{(1)}$ and $\delta Z_{m^2}^{(1)}$ in the OS scheme.

Coupling-constant renormalization:

Different renormalization conditions are possible.

(i) infra-red (IR) renormalization:

$$\Gamma_{\phi^4}(p_1, p_2, p_3, p_4)|_{p_i=0} \;=\; -i\lambda_R\,.$$

(ii) OS renormalization:

$$\Gamma_{\phi^4}(p_1, p_2, p_3, p_4)|_{p_i^2=m_R^2} \;=\; -i\lambda_R\,.$$

(iii) symmetric renormalization:

$$\Gamma_{\phi^4}(p_1, p_2, p_3, p_4)|_{s=t=u=4m_R^2/3} \;=\; -i\lambda_R\,.$$

(iv) the minimal subtraction (MS) scheme:

$$\Gamma_{\phi^4}(p_1, p_2, p_3, p_4)|_{\mathrm{UV-part}} \;=\; 0\,.$$

Exercise:

Use cut-off regularization to calculate the 1-loop renormalization constants δZ_ϕ, δZ_{m^2}, δZ_λ and δZ_{Λ_c} in the IR, OS, MS and symmetric renormalization schemes.

The 1-loop matrix element for $\phi(p_1)\phi(p_2) \to \phi(k_1)\phi(k_2)$

$$i\mathcal{T} \;=\; -iZ_\lambda\, Z_\phi^2\, \lambda_R \;+\; \Gamma^{(1)}_{\phi^4}(p_1, p_2, p_3, p_4)\,,$$

where

$$\Gamma^{(1)}_{\phi^4}(p_1, p_2, p_3, p_4) \;=\; \tilde{\Gamma}(s) \;+\; \tilde{\Gamma}(t) \;+\; \tilde{\Gamma}(u)\,,$$

and

$$\tilde{\Gamma}(p^2) \;=\; \frac{i\,\lambda_R^2}{32\pi^2}\,\ln\left(\frac{\Lambda^2}{m_R^2}\right) \;+\; \frac{i\,\lambda_R^2}{32\pi^2}\int_0^1 \frac{dx\,(1-x)\,(1-2x)\,p^2}{m_R^2 \;-\; x(1-x)\,p^2 - i\varepsilon}\,,$$

in cut-off regularization.

Had we used dimensional regularization (DR) where the 4 dimensions are analytically continued to $4-2\epsilon$, we would have got:

$$\tilde{\Gamma}(p^2) \;=\; \frac{i\,\lambda_R^2}{32\pi^2}\left[\frac{1}{\epsilon} \;-\; \gamma_E \;-\; \ln\left(\frac{4\pi m_R^2}{\mu^2}\right) \right.$$
$$\left. -\int_0^1 dx\,\ln\left(\frac{m_R^2 - x(1-x)p^2 - i\varepsilon}{m_R^2}\right)\right].$$

Exercises:

(i) Calculate the renormalized 1-loop transition amplitude and show that it is UV finite.

(ii) Show that up to a constant, the UV finite part of $\tilde{\Gamma}(p^2)$ is identical both in cut-off and DR regularization methods.

Concluding remarks:

- The UV finite part of 1PI loop graphs is (up to momentum-independent constants) independent of the regularization method.

- The values of the renormalized quantities λ_R, m_R^2 and ϕ_R depend on the conditions of renormalization. However, **physical observables**, such as S-matrix elements, are **renormalization-scheme independent**.

- Finally, even a UV finite theory would have needed renormalization! (*Why?*)

Exercises:*

Consider a ϕ^4-theory that includes a massive Dirac fermion f of mass m_f. The interaction of the fermion f with the scalar field ϕ is described by $\mathcal{L}_{\phi\bar{f}f} = h\,\phi\,\bar{f}f$.

(i) Use both cut-off and dimensional regularization methods to calculate the 1-loop renormalization constants δZ_ϕ, δZ_f, δZ_{m^2}, δZ_{m_f}, δZ_λ and δZ_h in the OS scheme.

(ii) On the basis of the above results, derive the correspondence between the cut-off and the DR method.

(iii) Calculate the 1-loop renormalized transition amplitudes for the processes: (a) $\phi\phi \to f\bar{f}$ and (b) $f\bar{f} \to f\bar{f}$.