PHYS40682: GAUGE THEORIES Prof A Pilaftsis

EXAMPLES SHEET IV: The Standard Model of Electroweak Interactions

1 Spontaneous Symmetry Breaking and the Goldstone Theorem

(i) Show that the unbroken generators Y^c form a subgroup H of G, including the possibility of $H \equiv \mathbb{I}$.

(ii) Show that the vacuum manifold as described by the coset space G/H does *not* necessarily form a group.

(iii) Prove that the Goldstone fields $G^b(x)$, defined as

$$G^b(x) \;=\; \frac{(iX^b\,\mathbf{v})_j}{\|X^b\,\mathbf{v}\|}\,\phi_j(x)\;, \qquad\qquad (\text{with } b = 1,\,2,\,\dots,\,\nu \text{ and } j = 1,\,2,\,\dots,\,n)$$

do not have mass terms in the potential $V(\mathbf{\Phi})$, and hence they are truly massless. Note that X^b are the generators of $\mathrm{SO}(n)$ acting on the n-dimensional scalar field space $\phi_i = (\phi_1,\,\phi_2,\,\dots,\,\phi_n) \in \mathbb{R}$. Moreover, explain why the remaining $(n-\nu)$ scalar fields $H^c(x)$ orthogonal to $G^b(x)$ are in general massive.

(iv) Show that if the SU(2) group breaks spontaneously in its fundamental representation, it then breaks completely to the identity group $\mathbb{I}$: $\mathrm{SU}(2) \xrightarrow{\langle\mathbf{\Phi}\rangle} \mathbb{I}$, where $\mathbf{\Phi} = (\Phi_1,\,\Phi_2)^{\mathsf{T}}$ is an SU(2) doublet consisting of two complex scalar fields.

(v)** **The Coleman–Mermin-Wagner-Hohenberg theorem**. Show that there are *no* Goldstone bosons in theories with two spacetime dimensions. Using a less rigorous approach, you may show that the field generated by a massless Goldstone boson is not well localized in space and time, but it diverges logarithmically at large distances. The latter prevents spontaneous symmetry breaking from occurring about a fixed point of the vacuum manifold.

2 The Higgs-Englert-Brout Mechanism

(i) Prove the electroweak symmetry breaking pattern for the SM Higgs potential:

$$\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y \xrightarrow{\langle\Phi\rangle} \mathrm{U}(1)_{\mathrm{em}} ,$$

where Φ is a colourless SU(2) doublet, with hypercharge quantum number $y_\Phi = 1/2$. [You may find useful the identity: $\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2\eta^{\mu\nu}\,\mathbf{1}_2$, with $\sigma^\mu = (\mathbf{1}_2\,, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (\mathbf{1}_2\,, -\boldsymbol{\sigma})$.]

(ii) The scalar-kinetic term of the SM Lagrangian is

$$\mathcal{L}_\Phi \;=\; (D_\mu\Phi)^\dagger\,(D^\mu\Phi) \;-\; V(\Phi)\,,$$

where $D_\mu\Phi = \left(\partial_\mu + \frac{i}{2}g\,\sigma^i W^i_\mu + \frac{i}{2}g' B_\mu \mathbf{1}_2\right)\Phi$, and W^i_μ and B_μ are the gauge fields of the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ local groups, respectively. Using $\mathcal{L}_\Phi$, show that after SSB the mass eigenstates Z_μ and A_μ are given in terms of the weak-basis fields W^3_μ and B_μ as follows:

$$Z_\mu \;=\; c_w\,W^3_\mu \;-\; s_w\,B_\mu\,, \quad A_\mu \;=\; s_w\,W^3_\mu \;+\; c_w\,B_\mu\,,$$

with $s_w \equiv \sin\theta_w$, $c_w \equiv \cos\theta_w$ and $t_w = s_w/c_w = g'/g$. Moreover, evaluate the masses of the physical $W^\pm$ and Z bosons.

(iii) $\boldsymbol{R_\xi}$ **gauge fixing**. In the SM, we usually adopt the gauge-fixing Lagrangian for the $W^\pm$ bosons,

$$\mathcal{L}^W_{\mathrm{GF}} \;=\; -\frac{1}{\xi}\left(\partial_\mu W^{+,\mu} + i\,\xi\,\frac{gv}{2}\,G^+\right)\left(\partial_\mu W^{-,\mu} - i\,\xi\,\frac{gv}{2}\,G^-\right),$$

where $v = 2M_W/g \simeq 245$ GeV is the vacuum expectation value of the Higgs field, and $G^\pm$ are the would-be Goldstone bosons related to the $W^\pm$ bosons. Show that the choice of $\mathcal{L}^W_{\mathrm{GF}}$ removes mixed terms of the type $W^{+,\mu}\,(\partial_\mu G^-)$, originating from $\mathcal{L}_\Phi$ given in part (ii).

(iv) With the aid of $\mathcal{L}_\Phi$ stated in part (ii), calculate the mass of the Higgs boson H, all its self-interactions, as well as its interactions with the gauge bosons $W^\pm$, Z and γ in the unitary gauge.

3 Quark and Lepton Mixing

(i) Show that the electric charge Q_f of a fermion f is given by the relation:

$$Q_f \;=\; T^3_f \,+\, Y_f \;,$$

where T^3_f is the eigenvalue to the weak isospin operator $T^3 = \sigma^3/2$, i.e. $T^3 f_L = T^3_f f_L$ and $T^3 f_R = 0$, and Y_f is the corresponding hypercharge operator acting on f_L and f_R with $\frac{1}{2}\, y_{f_L}$ and $\frac{1}{2}\, y_{f_R}$ quantum numbers, respectively. In addition, verify that $Q_{f_L} = Q_{f_R}$.

(ii) **Theorem**. Show that any *non*-Hermitian $N \times N$ matrix $\mathbf{M}$ can *always* be brought into a diagonal form $\widehat{\mathbf{M}}$, with *non*-negative diagonal entries, by a bi-unitary transformation: $\mathbf{U}\,\mathbf{M}\,\mathbf{V} = \widehat{\mathbf{M}}$, where $\mathbf{U}, \mathbf{V} \in \mathrm{U}(N)$.

(iii) Using the gauge-kinetic Lagrangian $\mathcal{L}_f$ for quarks, show that in the mass eigenbasis, the interaction of the $W^\pm$ bosons to the up- and down-type quarks, $\hat{u}_i$ and $\hat{d}_j$, is governed by the Lagrangian

$$\mathcal{L}_{W^\pm ud} \;=\; -\,\frac{g}{\sqrt{2}}\, W^+_\mu\, \bar{\hat{u}}_i\, \mathbf{V}_{ij}\, \gamma^\mu P_L\, \hat{d}_j \;+\; \mathrm{H.c.},$$

where $P_L = (\mathbf{1} - \gamma_5)/2$ is the left-handed chirality projection operator, and $\mathbf{V}_{ij}$ is a 3×3 unitary matrix, the so-called Cabbibo–Kobayashi–Maskawa (CKM) matrix describing **quark mixing**.

(iv) Explain why one can add to the SM Lagrangian a Lorentz- and gauge-invariant **Majorana** mass term for the right-handed neutrinos ν_{iR} of the form:

$$\mathcal{L}_M \;=\; -\,\frac{1}{2}\,\bar{\nu}^C_{iR}\,(\mathbf{m}_M)_{ij}\,\nu_{jR} \;+\; \mathrm{H.c.},$$

where C indicates charge conjugation and $\mathbf{m}_M$ is a 3×3 matrix. Show that $\mathcal{L}_M$ violates the lepton number L of the SM by two units, i.e. $\Delta L = 2$, and calculate the neutrino mass spectrum for large Majorana masses. To ease the computation, you may consider a single family of leptons first.

(v)* Taking into account the results derived in part (iv) for the SM augmented with $\mathcal{L}_M$, derive the Lagrangian governing the interactions of the $W^\pm$, Z and Higgs bosons with the Majorana neutrinos and charged leptons.

4 Standard Model Phenomenology

The Standard Model has proven to be a remarkable theory which gives very accurate predictions for all testable processes in low-energy experiments and high-energy colliders. In this exercise you are encouraged to analyze in detail a few of them in the lowest order of perturbation theory. To this end, some basic formulae that you learned during the course of QFT will be very useful.

(i) The effective Lagrangian describing the Fermi theory (the predecessor of the SM) is given by

$$\mathcal{L}_{\rm Fermi} \;=\; 2\sqrt{2}G_F\, J^-_\mu J^{+,\mu}\,,$$

where $G_F = \pi\alpha_w/\big(\sqrt{2}M_W^2\big) \approx 1.17\times 10^{-5}\ \mathrm{GeV}^{-2}$ is the Fermi constant and $J^-_\mu(x) = [J^+_\mu(x)]^\dagger = \bar{f}'(x)\gamma_\mu P_L f(x)$ are the charged currents for two SM fermions with charge difference $Q_{f'} - Q_f = 1$, e.g. $f' = e^-, d$ and $f = \nu_e, u$. Show that the squared amplitude $|\mathcal{M}|^2$ for the process $d\bar{u} \to e^-\nu_e$ at high centre-of-mass (CoM) energies $\sqrt{s}$ grows as s^2, and so it violates unitarity. For the purpose of this computation, you may simplify $J^{\mp}_\mu(x)$ as $J^-_\mu(x) = [J^+_\mu(x)]^\dagger \approx \bar{f}'(x)\gamma_\mu f(x)$. Then, estimate the largest CoM energy $\sqrt{s}_{\rm max}$, for which the inequality $|\mathcal{M}| \le 1$ still holds. What is the physical significance of $\sqrt{s}_{\rm max}$?

(ii) Show that the decay width of the Z boson into two *massless* fermions f is given by

$$\Gamma(Z\to f\bar{f}) \;=\; \frac{N_c^f\sqrt{2}G_F M_Z^3}{6\,\pi}\,\Big[\big(g_L^f\big)^2 + \big(g_R^f\big)^2\Big]\,,$$

where $N_c^f = 1\,(3)$ for leptons (quarks), $g_L^f = T_f^3 - s_w^2 Q_f$ and $g_R^f = -s_w^2 Q_f$.

(iii) Calculate the decay width Γ_t of the top quark due to its dominant decay channel: $t \to W^+ b$. Assuming that one can ignore the b-quark mass, verify that the top width is given by

$$\Gamma_t \;=\; \frac{\sqrt{2}G_F m_t^3}{16\,\pi}\,|V_{tb}|^2\left(1 - \frac{M_W^2}{m_t^2}\right)^2\left(1 + \frac{2M_W^2}{m_t^2}\right).$$

(iv) Show that the decay rate of a new SM heavy Higgs boson H', with mass $m_{H'} \ge 2M_W$, into W^+W^- bosons is given by

$$\Gamma(H'\to W^+W^-) \;=\; \frac{\sqrt{2}G_F M_{H'}^3}{16\,\pi}\left(1 - \frac{4M_W^2}{M_{H'}^2}\right)^{1/2}\left(1 - \frac{4M_W^2}{M_{H'}^2} + \frac{12M_W^4}{M_{H'}^4}\right).$$

Without doing an extensive calculation, determine the decay rate $\Gamma(H'\to ZZ)$.

(v)* The leading decay mode of the muon μ is: $\mu^- \to \nu_\mu e^- \bar{\nu}_e$. Ignoring the electron mass m_e next to the muon mass m_μ as well as mixing effects between light neutrinos, show that the muon lifetime τ_μ is given by

$$\tau_\mu^{-1} \;=\; \Gamma_\mu \;=\; \frac{G_F^2 m_\mu^5}{192\,\pi^3}\,.$$