Lecture 7

Time Evolution of Quantum States

Hamiltonian: $H = \underbrace{H_0}_{\text{free theory}} + \underbrace{H_{int}}_{\text{interaction part, e.g. } \lambda\phi^4}$

Time evolution of the state $|A(t);t_0\rangle$ in the Schrödinger picture (SP):

$$i\frac{\partial}{\partial t}|A(t);t_0\rangle_S = H|A(t);t_0\rangle_S$$

t_0 is the initial or boundary condition imposed on the system.

Formal solution:

$$|A(t);t_0\rangle_S = e^{-iH(t-t_0)}|A(t_0);t_0\rangle_S ,$$

given that $\frac{d}{dt}H = 0$

Heisenberg picture (HP): $|A;t_0\rangle_H = |A(t_0);t_0\rangle_S$

states are fixed in time, whereas

Operators are time-dependent:

$${}_S\langle B(t);t_0|\underbrace{\hat{O}^S}_{\text{time independendent}}|A(t);t_0\rangle_S = {}_H\langle B;t_0|\hat{O}^H(t)|\hat{A};t_0\rangle_H$$

$$\rightsquigarrow \quad \hat{O}^H(t) = e^{iH(t-t_0)}\,\hat{O}^S\,e^{-iH(t-t_0)}$$

Interaction picture (IP):

$$\hat{O}^I(t) = e^{iH_0(t-t_0)}\,\hat{O}^S\,e^{-iH_0(t-t_0)}$$

Time evolution of the state in the IP

Imposing the condition

$$ {}_I\langle B(t);t_0|\hat{O}^I(t)|A(t);t_0\rangle_I = {}_S\langle B(t);t_0|O^S|A(t);t_0\rangle_S , $$

$$ \hat{O}^I(t) = e^{iH_0(t-t_0)}\hat{O}^S e^{-iH_0(t-t_0)} \rightarrow \boxed{H_0^S = H_0^I = H_0} $$

we get

$$ e^{-iH_0(t-t_0)}|A(t);t_0\rangle_I = |A(t);t_0\rangle_S = e^{-iH(t-t_0)}\underbrace{|A(t_0);t_0\rangle}_{|A;t_0\rangle_H} $$

$$ \rightsquigarrow |A(t);t_0\rangle_I = \underbrace{e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)}}_{\hat{=}\ \mathcal{U}(t,t_0)}|A;t_0\rangle_H \qquad \boxed{H^S = H^H \neq H^I} $$

$\hat{=}\ \mathcal{U}(t,t_0)$ is the time evolution operator

Exercise (ii)

$$ i\frac{\partial}{\partial t}|A(t);t_0\rangle_I = \Big[-H_0^S e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)} + e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)}H^S\Big]|A;t_0\rangle_H $$

$$ = e^{iH_0^S(t-t_0)}\underbrace{(-H_0^S+H^S)}_{H^S_{int}}\,\boxed{e^{-iH_0^S(t-t_0)}e^{iH_0^S(t-t_0)}}\,e^{-iH^S(t-t_0)}|A;t_0\rangle_H $$

$$ = H^I_{int}\,\mathcal{U}(t,t_0)|A;t_0\rangle_H = H^I_{int}|A(t);t_0\rangle_I $$

Likewise, we obtain

$$ i\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = H^I_{int}(t)\,\mathcal{U}(t,t_0) $$

Iterative solution:

$$ \mathcal{U}(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t}dt_1\, \mathcal{H}^I_{int}(t_1)\,\mathcal{U}(t_1,t_0) $$

$$ = \mathbb{1} + (-i)\int_{t_0}^{t}dt_1\, H^I_{int}(t_1) + (-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\, H^I_{int}(t_1)\,H^I_{int}(t_2) \qquad (t_1 > t_2) $$

$$ + \ldots $$

$$\mathcal{U}(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1\, H^I_{int}(t_1) + \cdots \frac{(-i)^2}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\{H^I_{int}(t_1)\, H^I_{int}(t_2)$$

$$+ \cdots + \frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\, T\{H^I_{int}(t_1)\cdots H^I_{int}(t_n)\}\cdots$$

there are $n!$
permutations to devide out

$$\rightsquigarrow \mathcal{U}(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt'\, H^I_{int}(t')\right]\right\}$$

$T\{\cdots\}$ indicates time-ordering, e.g.

$$T(H_{int}(t_1)\, H_{int}(t_2)) = \begin{cases} H^I_{int}(t_1)\, H^I_{int}(t_2)\,; & t_1 > t_2 \\ H^I_{int}(t_2)\, H^I_{int}(t_1)\,; & t_2 > t_1 \end{cases}$$

<u>S-matrix operator:</u> $S = \lim_{t_{i/f} \to \mp\infty} \mathcal{U}(t_f, t_i)$

$$= T\left\{\exp\left[-i\int_{-\infty}^{+\infty} d^4x\, \mathcal{H}^I_{int}(x)\right]\right\}$$

<u>S-matrix elements:</u>

$$S_{i\to f} = {}_I\langle f; +\infty | S | i, -\infty\rangle_I$$

Using the adiabatic approximation

$$\lim_{t\to\pm\infty} H_{int}(t) = 0\,,$$

we may write $|p, \mp\infty\rangle_I = Z^{1/2}\, |p, \mp\infty\rangle_H = Z^{1/2}\, |p\rangle$,

where $Z^{1/2}$ is a wave-function normalization of the asymptotic state $|p\rangle$

Lecture 8

Feynman Propagator

Calculation of the T-ordered product:

$$T\left[\phi(x)\phi(y)\right] = \theta(x^0-y^0)\,\phi(x)\phi(y) + \theta(y^0-x^0)\,\phi(y)\,\phi(x)$$

First, observe that $\Phi(x) = \underset{\propto a^+ e^{ipx}}{\phi^+(x)} + \underset{\propto a\, e^{-ipx}}{\phi^-(x)}$

and $[\phi^+(x), \phi^+(y)] = [\phi^-(x), \phi^-(y)] = 0$;

$$[\phi(x), \phi(y)] = \underbrace{[\phi^+(x), \phi^-(y)]}_{\equiv \Delta^-(x-y) = -\int \frac{d^3p\, e^{ip(x-y)}}{(2\pi)^3 2E_p}} + \underbrace{[\phi^-(x), \phi^+(y)]}_{\equiv \Delta^+(x-y) = \int \frac{d^3p\, e^{-ip(x-y)}}{(2\pi)^3 2E_p}}$$

N.B. $\Delta^+(x-y) = -\Delta^-(y-x)$

Introduce short-hand notation: $\begin{cases} \phi^\pm_x = \phi^\pm(x)\,;\ \phi^\pm_y = \phi^\pm(y) \\ \theta_> = \theta(x^0-y^0)\,;\ \theta_< = \theta(y^0-x^0) \end{cases}$

$$T[\phi_x\phi_y] = \theta_>\left(\phi^+_x + \phi^-_x\right)\left(\phi^+_y + \phi^-_y\right) + \theta_<\,[x\leftrightarrow y]$$

$$= \left\{\theta_>\left[\phi^+_x\phi^+_y + \phi^-_x\phi^-_y\right] + \theta_<\,[x\leftrightarrow y]\right\}$$

$$+\, \theta_>\Big[\underbrace{\phi^-_x\phi^+_y}_{\phi^+_y\phi^-_x + [\phi^-_x, \phi^+_y]} + \phi^+_x\phi^-_y\Big] + \theta_<\Big[\underbrace{\phi^-_y\phi^+_x}_{\phi^+_x\phi^-_y + [\phi^-_y, \phi^+_x]} + \phi^+_y\phi^-_x\Big]$$

$$= \underbrace{\phi^+_x\phi^+_y + \phi^-_x\phi^-_y + \phi^+_y\phi^-_x + \phi^+_x\phi^-_y}_{:\phi\phi:\ \leftarrow \text{normal ordering}} + \theta_>\underbrace{[\phi^-_x, \phi^+_y]}_{\Delta^+(x-y)} - \theta_<\underbrace{[\phi^+_x, \phi^-_y]}_{\Delta^-(x-y)}$$

Since $\langle 0|:\phi\phi:|0\rangle = 0$, the VEV of $T(\phi_x\phi_y)$ is

$$\langle 0|T[\phi(x)\phi(y)]|0\rangle = \theta(x^0-y^0)\,\Delta^+(x-y) - \theta(y^0-x^0)\,\Delta^-(x-y)$$
$$\equiv i\Delta_F(x-y)$$

Calculation of $i\Delta_F(x-y)$

$$i\Delta_F(x-y) = \theta(x^0-y^0)\,\Delta^+(x-y) - \theta(y^0-x^0)\,\Delta^-(x-y)$$

$$= \int \frac{d^3\underline{k}}{(2\pi)^3 2E_k}\left[e^{-ik(x-y)}\theta(x^0-y^0) + e^{ik(x-y)}\theta(y^0-x^0)\right]$$

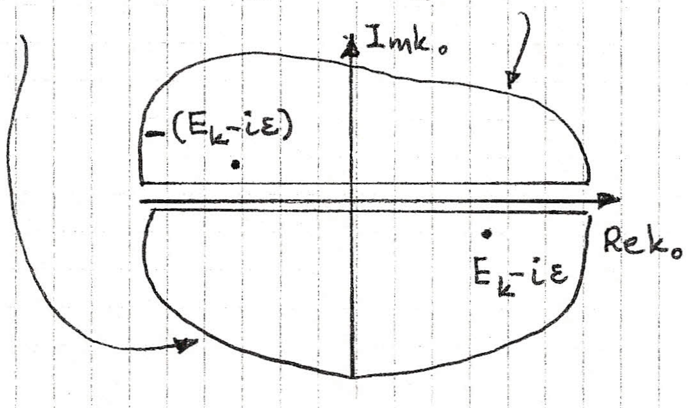
$$= \int \frac{d^3\underline{k}}{(2\pi)^3 2E_k}\, e^{i\underline{k}(\underline{x}-\underline{y})}\left[e^{-iE_k(x^0-y^0)}\theta(x^0-y^0) + e^{iE_k(x^0-y^0)}\theta(y^0-x^0)\right]$$

N.B. The integral is symmetric under $\underline{k} \to -\underline{k}$

N.B. $E_k^2 = \underline{k}^2 + m^2$

$$= \int_{-\infty}^{+\infty} \frac{d^3\underline{k}\; e^{i\underline{k}(\underline{x}-\underline{y})}}{(2\pi)^3 2E_k}\left[-\int_{-\infty}^{+\infty}\frac{dk_0}{2\pi i}\,\frac{e^{-ik_0(x^0-y^0)}}{k^0 - E_k + i\varepsilon} + \int_{-\infty}^{+\infty}\frac{dk_0}{2\pi i}\,\frac{e^{-ik_0(x^0-y^0)}}{k^0 + E_k - i\varepsilon}\right]$$

Complex integration:



$$\rightsquigarrow i\Delta_F(x-y) = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\, e^{-ik_\mu(x-y)^\mu}\,\frac{i}{k_0^2 - \underbrace{(E_k - i\varepsilon)^2}_{E_k^2 - 2i\varepsilon E_k \mapsto \underline{k}^2 + m^2 - i\varepsilon}}$$

$$= \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon} \quad \leftarrow \text{Feynman propagator}$$

$\Delta_F(x-y)$ is the Green function of the Klein-Gordon eqn: $(-\partial_\mu\partial^\mu - m^2)\,\Delta_F(x-y) = \delta^{(4)}(x-y)$.

Wick's Theorem: (see electronic notes).

Lectures 9 + 10:

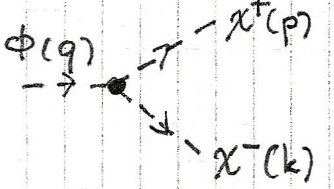
Transition Amplitudes

Consider the toy model:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}M^2\phi^2 + (\partial_\mu \chi^+)(\partial^\mu \chi) - m^2 \chi^+ \chi \underline{- g\chi^+\chi\Phi}$$

with $\mathcal{H}_{int} = g\,\chi^+(x)\chi(x)\Phi(x)$; $[g] = \text{energy}$

Calculation of the S-matrix amplitude:



$$: \quad S_{fi} = \langle \chi^+(p)\chi^-(k)|T\{e^{-i\int d^4x\, \mathcal{H}_{int}}\}|\phi(q)\rangle$$

$i \xrightarrow{\text{time}} f$

$$= \langle 0|a_\chi(\underline{p})\, b_\chi(\underline{k})\, T\underbrace{\{e^{-i\int d^4x\, \mathcal{H}_{int}}\}}_{\mathbb{1} - i\int d^4x\, \mathcal{H}_{int} + \cdots}\, a^+_\phi(\underline{q})|0\rangle$$

To lowest order $\mathcal{O}(g)$, we have — $\mathbb{1}$: $\boxed{\text{it does not contribute}}$

$$S_{fi} = -ig\int d^4x\, \langle 0|\, a_\chi(\underline{p})\, b_\chi(\underline{k})\, \chi^+(x)\, \chi(x)\, \Phi(x)\, a^+_\phi(\underline{q})|0\rangle + \mathcal{O}(g^2)$$

a^+_χ, b_χ a_χ, b^+_χ a_ϕ, a^+_ϕ

We are just interested in all non-vanishing commutators, resulting from Wick contraction

Given that

$$\langle 0|a_\chi(\underline{p})\,\chi^+(x)\,|0\rangle = e^{ipx}$$

$$\langle 0|\,b_\chi(\underline{k})\,\chi(x)\,|0\rangle = e^{ikx}$$

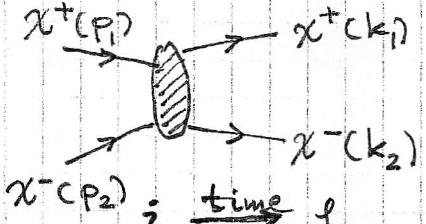
$$\langle 0|\,\phi(x)\,a^+_\phi(\underline{q})|0\rangle = e^{-iqx}$$

$$S_{fi} = -ig\int d^4x\; e^{ix(p+k-q)} = (-ig)\,(2\pi)^4\,\delta^{(4)}(p+k-q)$$

energy-momentum conservation

The amplitude for $\chi^+(p_1)\,\chi^-(p_2) \mapsto \chi^+(k_1)\,\chi^-(k_2)$

Transition amplitude: $iT_{fi} \equiv (S-\hat{\mathbb{1}})_{fi}$



$$: \langle \underline{k}_1, \underline{k}_2 | \left(e^{-i\int d^4x\, \mathcal{H}_{int}} - \mathbb{1} \right) | \underline{p}_1, \underline{p}_2 \rangle = iT_{fi}$$

To lowest order $\mathcal{O}(g^2)$, we get

$$iT_{fi} = \frac{(-ig)^2}{2!} \langle 0 | a_{\underline{k}_1} b_{\underline{k}_2} \int d^4x\, d^4y\, T\{ \chi^+(x)\chi(x)\underbrace{\Phi(x)\chi^+(y)\chi(y)\Phi(y)}_{\text{the only non-zero contraction}} \}$$

$$\times\, a^+_{\underline{p}_1} b^+_{\underline{p}_2} |0\rangle$$

Φ - propagator : $\langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle = i\Delta_F(x-y)$

<u>Wick contractions:</u>

$$\langle 0 | a_{\underline{k}_1} b_{\underline{k}_2}\ \chi^+(x)\chi(x)\ \chi^+(y)\chi(y)\ a^+_{\underline{p}_1} b^+_{\underline{p}_2} | 0 \rangle$$

$$= \langle 0 | a_{\underline{k}_1} b_{\underline{k}_2}\ \chi^+(x)\chi(x)\ \chi^+(y)\chi(y)\ a^+_{\underline{p}_1} b^+_{\underline{p}_2} | 0 \rangle$$

$$+ \langle 0 | a_{\underline{k}_1} b_{\underline{k}_2}\ \chi^+(x)\chi(x)\ \chi^+(y)\chi(y)\ a^+_{\underline{p}_1} b^+_{\underline{p}_2} | 0 \rangle$$

$$+ \langle 0 | a_{\underline{k}_1} b_{\underline{k}_2}\ \chi^+(x)\chi(x)\ \chi^+(y)\chi(y)\ a^+_{\underline{p}_1} b^+_{\underline{p}_2} | 0 \rangle$$

$$+ \langle 0 | a_{\underline{k}_1} b_{\underline{k}_2}\ \chi^+(x)\chi(x)\ \chi^+(y)\chi(y)\ a^+_{\underline{p}_1} b^+_{\underline{p}_2} | 0 \rangle$$

$$= e^{ik_1x}\, e^{ik_2x}\, e^{-ip_2y}\, e^{-ip_1y}$$

$$+ e^{ik_1x}\, e^{ik_2y}\, e^{-ip_1x}\, e^{-ip_2y}$$

$$+ e^{ik_1y}\, e^{ik_2x}\, e^{-ip_2x}\, e^{-ip_1y}$$

$$+ e^{ik_1y}\, e^{ik_2y}\, e^{-ip_2x}\, e^{-ip_1x}$$

$$iT_{fi} = -\frac{g^2}{2}\int d^4x\, d^4y \underbrace{\int \frac{d^4q}{(2\pi)^4}\frac{i\, e^{-iq(x-y)}}{q^2-M^2+i\varepsilon}}_{\Delta_F(x-y)}$$

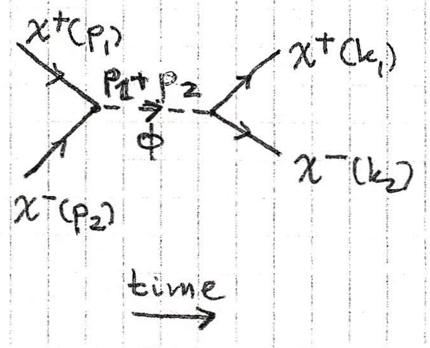
$$\times \Big\{ e^{i(k_1+k_2)x}\, e^{-i(p_1+p_2)y} + e^{i(k_1-p_1)x}\, e^{i(k_2-p_2)y}$$
$$+ e^{i(k_2-p_2)x}\, e^{i(k_1-p_1)y} + e^{-i(p_1+p_2)x}\, e^{i(k_1+k_2)y}\Big\}$$

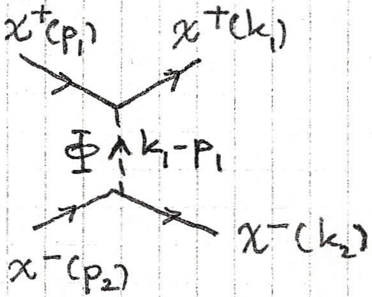
$$= -\frac{g^2}{2}(2\pi)^4 \int d^4q \frac{i}{q^2-M^2+i\varepsilon}$$

$$\times \Big\{ \underline{\delta^{(4)}(k_1+k_2-q)\, \delta^{(4)}(q-p_1-p_2)} + \delta^{(4)}(k_1-p_1-q)\,\delta^{(4)}(q+k_2-p_2)$$
$$+ \delta^{(4)}(k_2-p_2-q)\,\delta^{(4)}(q+k_1-p_1) + \underline{\delta^{(4)}(q-p_1-p_2)\,\delta^{(4)}(q+k_1+k_2)}\Big\}$$

$$\rightsquigarrow\ iT_{fi} = (-ig)^2\left[\frac{i}{(p_1+p_2)^2-M^2} + \frac{i}{(k_1-p_1)^2-M^2}\right]$$
$$\times (2\pi)^4\, \delta^{(4)}(p_1+p_2-k_1-k_2)$$

Feynman graphs

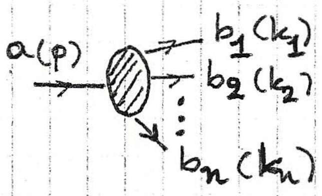




Feynman rules (see lecture notes)

Lecture 11

Particle decay



The decay rate Γ_a of an unstable particle is defined at its rest frame.

probability rate

Decay rate: $$d\Gamma_a = \frac{dP_{a\to b_i}}{T}$$

unit time

volume of normalization

$$d\Gamma_a = \frac{1}{2m_a}\left(\frac{1}{V}\right)\frac{1}{T}\left[(2\pi)^4\,\delta^{(4)}(k_1+\cdots+k_n-p)\right]^2 |\mathcal{M}_{fi}|^2$$
$$\times \prod_{i=1}^{n}\frac{d^3\underline{k}_i}{(2\pi)^3\,2E_{k_i}}$$

phase space: summation over all possible $|\underline{k}_i\rangle$ states

Fermi's trick:

$$\left[(2\pi)^4\,\delta^{(4)}(k_1+k_2+\cdots+k_n-p)\right]^2 =$$
$$= \int_{VT} d^4x\; e^{i(k_1+\cdots+k_n-p)\cdot x}\cdot(2\pi)^4\,\delta^{(4)}(k_1+\cdots k_n-p)$$

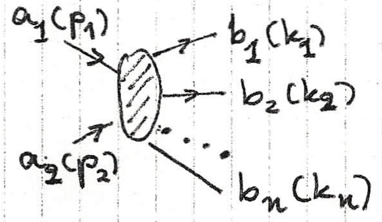
energy-momentum conservation

$$= VT\,(2\pi)^4\,\delta^{(4)}(k_1+\cdots+k_n-p)$$

$$\rightsquigarrow\; d\Gamma_a = \frac{1}{2m_a}|\mathcal{M}_{fi}|^2\,(2\pi)^4\,\delta^{(4)}(k_1+\cdots+k_n-p)$$
$$\times\prod_{i=1}^{n}\frac{d^3\underline{k}_i}{(2\pi)^3\,2E_{k_i}}$$

Units of Γ_a: energy or $(\text{time})^{-1}$, e.g. GeV, sec^{-1}

Scattering process



Definition of cross section:

$$\sigma = \frac{\text{Transition rate: } P_{fi}/T}{\text{Flux of incoming particles: } F}$$

$1\,\text{barn} = 10^{-24}\,\text{cm}^2$

$[\text{cm}^2]$ or $[E^{-2}]$

energy

$$F = \frac{|\underline{v}_1 - \underline{v}_2|}{V} \leftarrow \boxed{\text{relative velocity of incoming particles}} = \frac{\Delta x/\Delta t}{\Delta x \Delta y \Delta z} = \frac{1 \text{ particle}}{\Delta t\, \Delta y\, \Delta z}$$

$$\frac{dP_{fi}}{T} = \frac{1}{2E_{p_1}} \frac{1}{2E_{p_2}} \left(\frac{1}{V}\right)^2 \frac{1}{T} \underbrace{\left[(2\pi)^4 \delta^{(4)}(k_1 + \dots + k_n - p_1 - p_2)\right]^2}_{VT\,(2\pi)^4 \delta^{(4)}(k_1 + \dots + k_n - p_1 - p_2)}$$

$$\times\, |\mathcal{M}_{fi}|^2 \prod_{i=1}^{n} \frac{d^3\underline{k}_i}{(2\pi)^3\, 2E_{k_i}}$$

Fermi's trick

$$d\sigma = \frac{dP_{fi}/T}{F} = \frac{1}{|\underline{v}_1 - \underline{v}_2|} \left(\frac{1}{2E_{p_1}}\right)\left(\frac{1}{2E_{p_2}}\right) |\mathcal{M}_{fi}|^2$$

$$\times\, (2\pi)^4 \delta^{(4)}(k_1 + \dots + k_n - p_1 - p_2) \prod_{i=1}^{n} \frac{d^3\underline{k}_i}{(2\pi)^3\, 2E_{k_i}}$$

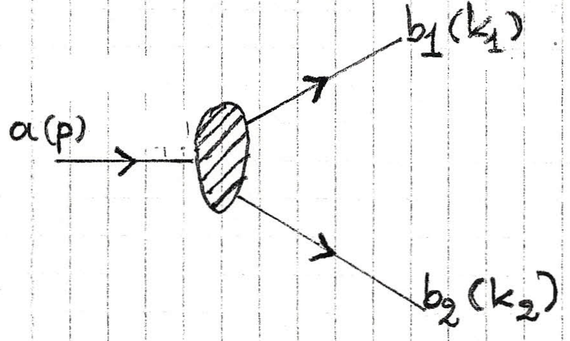
Number of scattering events:

$$N = \sigma \times L,$$

where $L[\text{cm}^{2}\,\text{s}^{-1}]$ is the luminosity of the collider, e.g. $L \cong 10^{31}\,\text{cm}^{-2}\,\text{s}^{-1}$ for TEVATRON, $10^{34}\,\text{cm}^{-2}\,\text{s}^{-1}$ for LHC (final stages).

Lecture 12

Kinematics



in CMS

$$p^\mu = k_1^\mu + k_2^\mu = (m_a, 0, 0, 0)$$

$$p^2 = m_a^2\ ,\quad k_1^2 = m_1^2\ ,\quad k_2^2 = m_2^2$$

in CMS $\rightsquigarrow$ $\underline{k}_1 = -\underline{k}_2\ ,\ \underline{p} = 0$

CMS:

$$2k_1 \cdot k_2 = (k_1+k_2)^2 - k_1^2 - k_2^2 = p^2 - k_1^2 - k_2^2 = m_a^2 - m_1^2 - m_2^2$$

$$2p\cdot k_1 = 2m_a E_1 \quad \underline{\text{or}} \quad 2(k_1+k_2)\cdot k_1 = 2m_1^2 + 2k_1\cdot k_2$$

$$= m_a^2 + m_1^2 - m_2^2$$

$$\rightsquigarrow \quad E_1 = \frac{m_a^2 + m_1^2 - m_2^2}{2m_a}$$

Likewise: $2p\cdot k_2 = m_a^2 + m_2^2 - m_1^2 = 2m_a E_2$

$$\rightsquigarrow \quad E_2 = \frac{m_a^2 + m_2^2 - m_1^2}{2m_a}$$

$\boxed{\text{N.B. } E_1 + E_2 = m_a \text{ in CMS}}$

$$|\underline{k}_1| = \sqrt{E_1^2 - m_1^2} = \frac{1}{2m_a}\lambda^{1/2}(m_a^2, m_1^2, m_2^2)$$

where $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

$$= (x-y-z)^2 - 4yz$$

$$\boxed{\begin{aligned}\lambda(x,y,0) &= (x-y)^2\\ \lambda(x,y,y) &= x(x-4y)\end{aligned}}$$

Finally, $|\underline{k}_2| = |\underline{k}_1|$

Decay width for $a \to b_1 b_2$

$$\Gamma_a = \frac{1}{2m_a} (2\pi)^{4-2\times 3} \int \frac{d^3\underline{k}_1}{2E_1} \frac{d^3\underline{k}_2}{2E_2} \underbrace{|\mathcal{M}_{fi}|^2}_{\text{const.}} \delta^{(4)}(p-k_1-k_2)$$

$$= \frac{1}{2m_a} \frac{|\mathcal{M}_{fi}|^2}{(2\pi)^2} \int d^4k_1 \int d^4k_2 \quad \delta_+(k_1^2-m_1^2)\, \delta_+(k_2^2-m_2^2)\, \delta^{(4)}(p-k_1-k_2)$$

$$\boxed{\delta_+(k^2-m^2) \equiv \theta(k_0)\,\delta(k^2-m^2)}$$

$$\rightsquigarrow \Gamma_a = \frac{1}{2m_a} \frac{|\mathcal{M}_{fi}|^2}{(2\pi)^2} \int d^4k_1 \quad \delta_+(k_1^2-m_1^2)\, \delta_+((p-k_1)^2-m_2^2)$$

$$= \frac{1}{2m_a} \frac{|\mathcal{M}_{fi}|^2}{(2\pi)^2} \int \frac{d^3\underline{k}_1}{2E_1} \underbrace{\delta_+\left(m_a^2 - 2m_aE_1 + m_1^2 - m_2^2\right)}_{= \frac{1}{2m_a}\delta\left(E_1 - \frac{m_a^2+m_1^2-m_2^2}{2m_a}\right)}$$

$$\int \frac{|\underline{k}_1|^2\, d|\underline{k}_1|\, d\Omega}{2E_1} = \int \frac{|\underline{k}_1|\, E_1\, dE_1\, d\Omega}{2E_1},$$

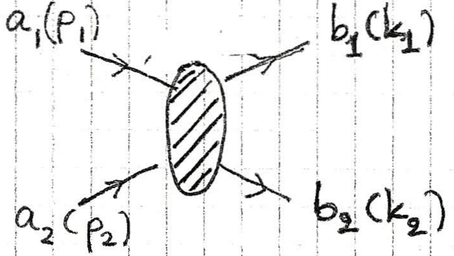
because $|\underline{k}_1|\, d|\underline{k}_1| = E_1\, dE_1$

$$\rightsquigarrow \Gamma_a = \frac{1}{8m_a^2} \frac{|\mathcal{M}_{fi}|^2}{4\pi^2} |\underline{k}_1^*| \,(4\pi)$$

$$\Gamma_a = \frac{|\underline{k}_1^*|}{8\pi m_a^2} |\mathcal{M}_{fi}|^2 = \frac{\lambda^{1/2}(m_a^2, m_1^2, m_2^2)}{16\pi\, m_a^3} |\mathcal{M}_{fi}|^2$$

Lecture 13

Kinematics for $2 \mapsto 2$ scattering



$$p_1 + p_2 = k_1 + k_2$$

$$s = (p_1+p_2)^2 = (k_1+k_2)^2$$
$$t = (p_1-k_1)^2 = (p_2-k_2)^2$$
$$u = (p_1-k_2)^2 = (p_2-k_1)^2$$

$$s+t+u = m_{a_1}^2 + m_{a_2}^2 + m_{b_1}^2 + m_{b_2}^2$$

In CMS:

$$|\underline{p}_1| = |\underline{p}_2| = \frac{\lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)}{2\sqrt{s}}$$

$$|\underline{k}_1| = |\underline{k}_2| = \frac{\lambda^{1/2}(s, m_{b_1}^2, m_{b_2}^2)}{2\sqrt{s}}$$

$\boxed{\text{Just replace } m_a^2 \text{ with } s}$

$$E_{a_1} = \frac{s + m_{a_1}^2 - m_{a_2}^2}{2\sqrt{s}}, \quad E_{a_2} = \frac{s + m_{a_2}^2 - m_{a_1}^2}{2\sqrt{s}}$$

$$E_{b_1} = \frac{s + m_{b_1}^2 - m_{b_2}^2}{2\sqrt{s}}, \quad E_{b_2} = \frac{s + m_{b_2}^2 - m_{b_1}^2}{2\sqrt{s}}$$

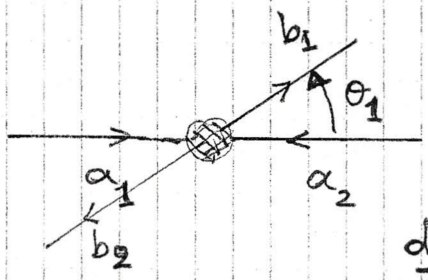
Differential cross section (CMS) $\leftarrow$ $\boxed{\text{Try to derive all these relations}}$

$$\frac{d\sigma}{d\Omega_1} = \frac{1}{64\pi^2 s} \frac{|\underline{k}_1|}{|\underline{p}_1|} |\mathcal{M}_{fi}|^2 \quad ; \quad d\Omega_1 = \sin\theta_1 \, d\theta_1 \, d\phi_1$$

$$dt = \frac{|\underline{p}_1||\underline{k}_1|}{\pi} d\Omega_1,$$

where $\theta_1 = \sphericalangle(\underline{p}_1, \underline{k}_1)$

CMS:



$$\frac{d\sigma}{dt} = \frac{|\mathcal{M}_{fi}|^2}{16\pi \lambda(s, m_{a_1}^2, m_{a_2}^2)}$$

Limits: $t^{\pm} = m_{a_1}^2 + m_{b_1}^2 - \frac{1}{2s}\left[(s + m_{a_1}^2 - m_{a_2}^2)(s + m_{b_1}^2 - m_{b_2}^2) \mp \lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)\lambda^{1/2}(s, m_{b_1}^2, m_{b_2}^2)\right]$

Unitarity of the S-matrix: $S^\dagger S = \mathbb{1}$

$$\langle i'|i\rangle = \langle i'|S^\dagger S|i\rangle = \sum_f \langle i'|S^\dagger|f\rangle\langle f|S|i\rangle$$

$$= \sum_f \langle f|S|i'\rangle^* \langle f|S|i\rangle$$

$$S = \mathbb{1} + i\mathcal{T} \rightsquigarrow (\mathbb{1} - i\mathcal{T}^\dagger)(\mathbb{1}+i\mathcal{T}) = \mathbb{1}$$

$$\rightsquigarrow i(\mathcal{T}-\mathcal{T}^\dagger) + \mathcal{T}^\dagger\mathcal{T} = 0$$

$$\rightsquigarrow \frac{1}{2i}(\mathcal{T}-\mathcal{T}^\dagger) = \frac{1}{2}\mathcal{T}^\dagger\mathcal{T}$$

or equivalently:

$$\frac{1}{2i}\left(\langle i'|\mathcal{T}|i\rangle - \langle i|\mathcal{T}|i'\rangle^*\right) = \frac{1}{2}\sum_f \langle f|\mathcal{T}|i'\rangle^*\langle f|\mathcal{T}|i\rangle$$

$$\mathcal{T}_{fi} = (2\pi)^4 \delta^{(4)}(k_f - p_i)\, \mathcal{M}_{fi} \leftarrow \boxed{\text{Remember this relation}}$$

For $i=i'$,

generic phase space

$$(2\pi)^4 \delta^{(4)}(k_i - p_i)\ \mathrm{Im}(\mathcal{M}_{ii})$$

$$= (2\pi)^4 \delta^{(4)}(k_i - p_i)\ \frac{1}{2} \underbrace{(2\pi)^4 \delta^{(4)}(p_i - k_f)\, |\mathcal{M}_{if}|^2 \frac{d^3 k_f}{(2\pi)^3 2E_f}}_{2\lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)\,\sigma(a_1 a_2 \to f)}$$

if $i = a_1 a_2$ $\longrightarrow$ $2\lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)\,\sigma(a_1 a_2 \to f)$

Optical theorem: $|i\rangle = |a_1(p_1)\, a_2(p_2)\rangle$

$$\mathrm{Im}\,\mathcal{M}_{ii} = \lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)\,\sigma_{tot}(a_1 a_2 \to X)$$

where X indicates all possible final states that come from $a_1 + a_2$ scatterings.