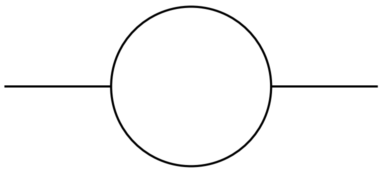
PHYS40481: QUANTUM FIELD THEORY Prof A Pilaftsis

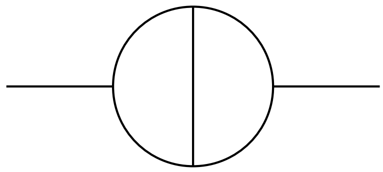
EXAMPLES SHEET IV: RENORMALIZATION

1 Superficial Degree of Divergence and Renormalizability

(a) What is the superficial degree of divergence D_Γ of a graph Γ containing L loops and P scalar propagators, in d dimensions? Show that 2-dimensional ϕ^n theories are renormalizable for any power of $n > 0$.

(b) Find the superficial degree of divergence in a scalar ϕ^3-theory for the one- and two-loop self-energy graphs:





(c)* Determine the overall superficial degree of divergence D_Γ in a scalar ϕ^4-theory for the *non-planar* One-Particle-Irreducible (1PI) graph Γ_{ϕ^4} which has the form of a tetrahedron:

2 Loop Effects on the Vacuum

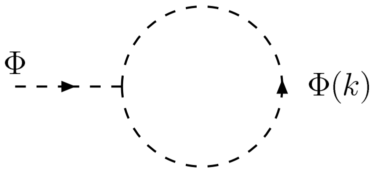
The Lagrangian density describing the scalar–fermion sector of a theory is given by

$$\mathcal{L} \;=\; \frac{1}{2}\Big[(\partial_\mu\Phi)(\partial^\mu\Phi) \;-\; M^2\Phi^2\Big] \;+\; \bar{f}\left(i\,\partial\!\!\!/ - m\right) f \;-\; \frac{1}{6}\lambda M\Phi^3 \;-\; h\,\Phi\bar{f}\,f\;,$$

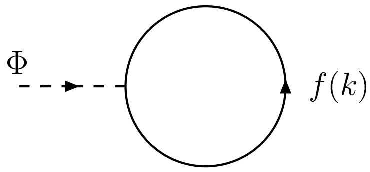
where Φ is a real scalar field and f is a Dirac fermion field.

(i) Use $\mathcal{L}$ to read off the Feynman rules for the Φ-scalar propagator, the f-fermion propagator and their interactions, Φ^3 and Φ-$\bar{f}$-f.

(ii) The loop effects on the vacuum in this theory are given by the following graphs:



(a)



(b)

Use the Feynman rules derived in part (i) to write down the amplitudes $T_\Phi^{(\mathrm{a})}$ and $T_\Phi^{(\mathrm{b})}$ for the one-loop graphs drawn above.

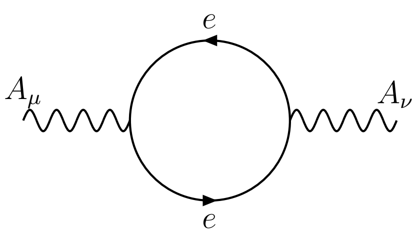
(iii) What are the superficial degrees of divergence D_Γ of graphs **(a)** and **(b)**?
[Note that $\mathrm{Tr}(\not{k}) = 0$.]

(iv) Without performing the loop integration, derive a set of relations that the *non-zero* kinematic parameters M^2, m, λ and h have to satisfy, such that the loop effects on the vacuum vanish.

3 The Vacuum Polarization of the Photon

(a) Write down the Feynman rules for the electron propagator and the electron-photon interaction.

(b) Use these rules to write down the amplitude $\Pi_{\mu\nu}(p)$ (drawn below) for the one-loop vacuum polarization of a photon with momentum p (do not perform the loop integral).



(c) Show that $\Pi_{\mu\nu}(p)$ satisfies the property:

$$p^\mu \, \Pi_{\mu\nu}(p) \;=\; 0 \, .$$

What is the physical significance of the above property?

(d) From (c), we know that $\Pi_{\mu\nu}(p)$ may be expressed as:

$$\Pi_{\mu\nu}(p) \;=\; \Big(\eta_{\mu\nu}p^2 \;-\; p_\mu p_\nu\Big)\, \Pi(p^2) \, ,$$

where $\Pi(p^2)$ is a regular function in the limit $p^2 \to 0$, for a fixed given ultra-violet cut-off Λ. Use this fact to show that the photon vacuum polarization amplitude $\Pi_{\mu\nu}(p)$ scales logarithmically as a function of Λ.

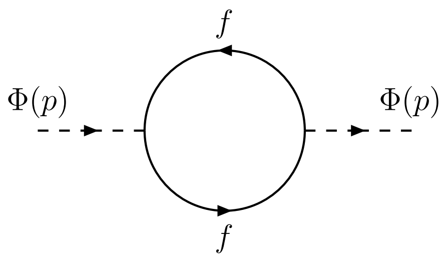
4 Renormalization of a Scalar Field Theory with a Fermion

(a) The Lagrange density of a theory with a real scalar field Φ and a Dirac fermion f is given by

$$\mathcal{L} \;=\; \frac{1}{2}\Big[(\partial_\mu\Phi)(\partial^\mu\Phi) \;-\; M^2\Phi^2\Big] \;+\; \bar{f}\,\Big(i\,\partial\!\!\!/ - m\Big)\, f \;-\; h\,\Phi\bar{f}\,f \, .$$

Use $\mathcal{L}$ to read off the Feynman rules for the Φ-scalar propagator, the f-fermion propagator and the interaction Φ-$\bar{f}$-f.

(b) Use these rules to write down the amplitude $\Pi_{\Phi\Phi}(p^2)$ for the one-loop scalar self-energy drawn below.



(c) Use a finite ultra-violet (UV) cut-off regulator Λ to perform the loop integral in $\Pi_{\Phi\Phi}(p^2)$, in the limit that $p_\mu \to 0$ and $m \to 0$. Given that $d\Pi_{\Phi\Phi}(p^2)/dp^2$ is a non-singular function as $p^\mu \to 0$, explain why $d\Pi_{\Phi\Phi}(p^2)/dp^2$ can at most diverge logarithmically with the UV cut-off Λ. How are the UV divergences in $\Pi_{\Phi\Phi}(p^2)$ removed in this scalar field theory?

[*Hint:* You may find useful the formula for Wick rotation: $\int d^4k = i\pi^2 \int_0^{\Lambda^2} k_E^2 dk_E^2$, where $k_E^2 = (k_E^0)^2 + |\mathbf{k}|^2$.]

5 The Wess–Zumino Model

The Lagrangian density describing the Wess–Zumino (WZ) model is given by

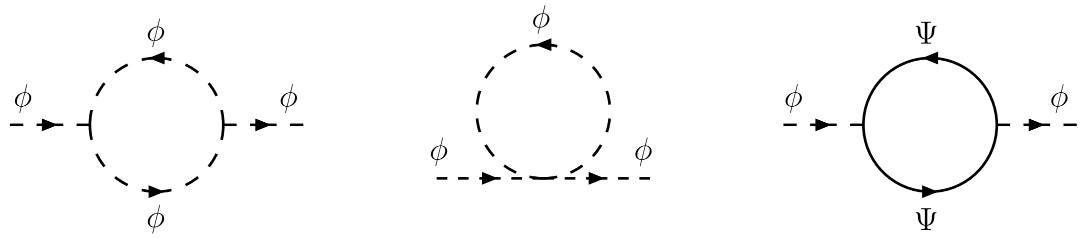
$$\begin{aligned}\mathcal{L}_{\rm WZ} &= (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2 \phi^\dagger \phi + \frac{1}{2}\,\overline{\Psi}\, i\gamma^\mu \partial_\mu \Psi - \frac{1}{2}\, m\, \overline{\Psi}\, \Psi \\ &\quad - \frac{mh}{2}\,(\phi^\dagger \phi^2 + \phi^{\dagger 2}\phi) - \frac{h^2}{4}\,(\phi^\dagger\phi)^2 - \frac{h}{2}\,\phi\,\overline{\Psi}\, P_L\, \Psi - \frac{h}{2}\,\phi^\dagger \overline{\Psi}\, P_R\, \Psi\,,\end{aligned}$$

where ϕ is a complex field, Ψ is a Majorana fermion, and $P_L = \frac{1}{2}\,(\mathbf{1}_4 - \gamma_5)$ and $P_R = \frac{1}{2}\,(\mathbf{1}_4 + \gamma_5)$ are the left- and right-chirality projection operators.

(i) Write down the defining equation for the Majorana fermion field Ψ.

(ii) Deduce from $\mathcal{L}_{\rm WZ}$ the Feynman rules for the ϕ- and Ψ-propagators and their interactions.

(iii) Use the Feynman rules derived in part (ii) to write down the amplitudes for the following three self-energy graphs (do not perform the loop integrals):



(a) **(b)** **(c)**

(iv) In the WZ model the *total* self-energy $\Pi(p^2)$ for the ϕ field is given by the sum of the three self-energy graphs (a), (b) and (c) shown in part (iii). Without performing the loop integrals, show that $\Pi(p^2)$ vanishes in this model at zero external momentum, i.e., $p^\mu \to 0$.

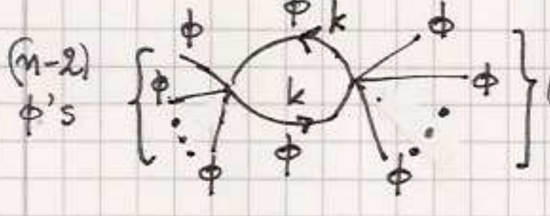
SOLUTIONS TO EXAMPLES SHEET (IV)

1.

(a) $D_\Gamma = dL - 2P$

A φ^n-theory (with $0 < n \leq 2$, i.e. $n=1,2$) is a free theory (without interactions), so it is trivially renormalizable for any $d \geq 2$.

For $\varphi^{n>2}$ theories, it is easy to check that for $d=2$,



$$\propto \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2)^2}$$

$\mapsto D_\Gamma = -2$; graph is UV convergent.

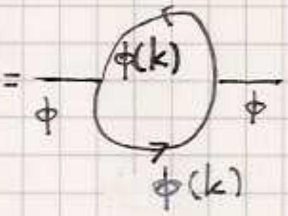
Higher loops do not change the value of $D_\Gamma = 2(L-P) = -2$, as one extra loop would require one extra ϕ-propagator.

Hence, ϕ^n theories are renormalizable for any $n>0$ in $d=2$ dimensions.

$\longleftrightarrow$

(b) Set external momenta of ϕ-particles to zero:

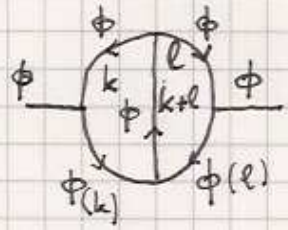
$\Gamma^{(1)}_{\phi^2} =$



$$\propto \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} \rightarrow D_\Gamma = 4-4 = 0$$

log-divergent, i.e. $\propto \log \Lambda$

$\Gamma^{(2)}_{\phi^2} =$



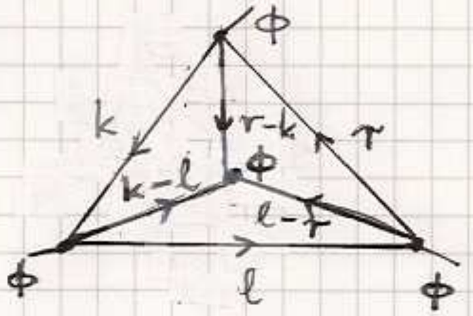
$$\propto \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(k^2)^2} \frac{1}{(\ell^2)^2} \frac{1}{(k+\ell)^2}$$

$\longmapsto D_\Gamma = 4\times 2 - 2\times 5 = -2 < 0$

$\longmapsto \Gamma^{(2)}_{\phi^2}$ is UV finite or UV convergent.

(c)* (non-examinable) All external momenta are set to zero.

Γ_{ϕ^4} =



The arrows show the flow of the momenta

L=3

$$\propto \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4 l}{(2\pi)^4} \int \frac{d^4 r}{(2\pi)^4} \frac{1}{k^2 l^2 r^2 (k-l)^2 (l-r)^2 (r-k)^2}$$

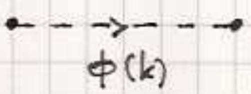
$D_\Gamma = 4 \times \overset{L}{\overset{\shortparallel}{3}} - 2 \times \overset{P}{\overset{\shortparallel}{6}} = 0$; the non-planar 1PI graph is log-divergent w.r.t. the UV cutoff Λ, i.e. $\propto \log\Lambda$

Notice that all possible subgraphs of $\Gamma^{(3)}_{\phi^4}$ are UV finite (WHY?)

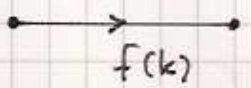
$\longleftrightarrow$

2.

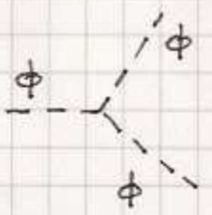
(i) Feynman rules:



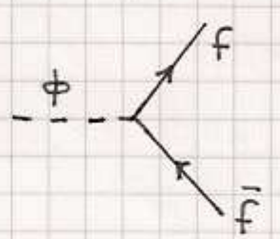
: $\dfrac{i}{k^2 - M^2 + i\varepsilon}$



: $\dfrac{i}{\not{k} - m\mathbb{1}_4 + i\varepsilon} = \dfrac{i(\not{k} + m\mathbb{1}_4)}{k^2 - m^2 + i\varepsilon}$



: $-i\lambda M \, \frac{1}{6} \times 3! = -i\lambda M$



: $-ih$

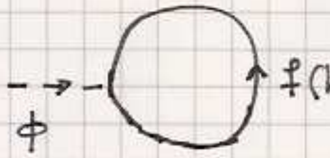
(ii)

combinatorics

$$: \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{(-i\lambda M)\,i}{k^2-M^2+i\varepsilon} \equiv iT_\phi^{(a)}$$

(a)

closed fermion loop



$$: (-1)\int\frac{d^4k}{(2\pi)^4}\,(-ih)\,\mathrm{Tr}\left(\frac{i}{\not k - m + i\varepsilon}\right) \equiv iT_\phi^{(b)}$$

(b)

(iii)

$$T_\phi^{(a)} \propto \int^{\Lambda}\frac{d^4k}{k^2-M^2} \propto \Lambda^2 \;\mapsto\; D_T^{(a)} = 2$$

$$T_\phi^{(b)} \propto -\int^{\Lambda} d^4k\,\mathrm{Tr}\left(\frac{1}{\not k - m}\right) = -\int^{\Lambda} d^4k\,\frac{\mathrm{Tr}(\not k + m\mathbb{1}_4)}{k^2-m^2}$$

$$= -\int^{\Lambda}\frac{d^4k}{k^2-m^2}\,4m \sim \Lambda^2 \;\mapsto\; D_T^{(b)} = 2$$

(iv) From (ii) and (iii), we have

$$iT_\phi^{(a)} = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{\lambda M}{k^2-M^2}$$

$$iT_\phi^{(b)} = -4\int\frac{d^4k}{(2\pi)^4}\,\frac{hm}{k^2-m^2}.$$

Obsere that $T_\phi^{(a)}$ (bosonic loop) is negative to $T_\phi^{(a)}$ (fermionic loop).

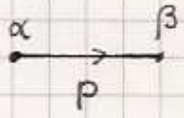
Requiring that $T_\phi^{(a)} + T_\phi^{(b)} = 0$, or $T_\phi^{(a)} = -T_\phi^{(b)}$

implies that
$$\boxed{\begin{array}{l} m = M \\ \frac{\lambda}{2} = 4h \rightsquigarrow \lambda = 8h \end{array}}$$

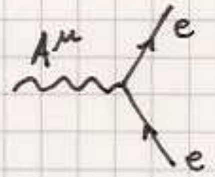
Such relations between couplings and masses occur in Supersymmetric theories*

3.

(a) Feynman rules:

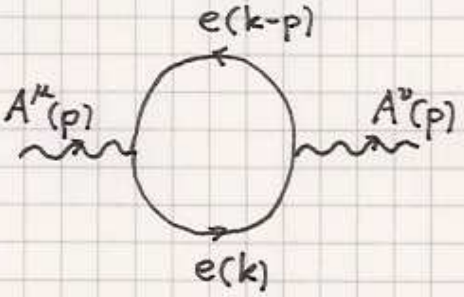


$$: \quad \frac{i}{\not p - m_e}$$



$$: \quad ie\gamma_\mu$$

(b)



$$\equiv i\Pi_{\mu\nu}(p)$$

$$i\Pi_{\mu\nu}(p) = (-1)\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[(ie\gamma_\mu)\frac{i}{\not k - \not p - m_e}(ie\gamma_\nu)\frac{i}{\not k - m_e}\right]$$

Closed fermionic loop

$$= -\,e^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\mu\frac{1}{\not k - \not p - m_e}\gamma_\nu\frac{1}{\not k - m_e}\right]$$

(c) $$p^\mu\Pi_{\mu\nu}(p) = ie^2\int\frac{d^4k}{(2\pi)^4}\,\underbrace{\mathrm{Tr}\left[\not p\,\frac{1}{\not k - \not p - m_e}\gamma_\nu\frac{1}{\not k - m_e}\right]}$$

$$= \mathrm{Tr}\left[\frac{1}{\not k - m_e}\,\not p\,\frac{1}{\not k - \not p - m_e}\gamma_\nu\right]$$

Due to cyclicity of trace

We may write $\not p$ as $\not p = (\not k - m_e) - (\not k - \not p - m_e)$, i.e. as the <u>difference</u> of two <u>inverse</u> electon-propagators.

$$p^\mu\Pi_{\mu\nu}(p) = ie^2\int\frac{d^4k}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not k - \not p - m_e}\gamma_\nu\right] - \mathrm{Tr}\left[\frac{1}{\not k - m_e}\gamma_\nu\right]\right\}$$

Shift the integration variable (and limits) for the first trace term in the integrand:

$(k-p)^\mu \mapsto k^\mu$, and $d^4(k-p) = d^4k$, since p^μ is a <u>finite</u> constant.

Hence, we get

after the shift

$$p^\mu \Pi_{\mu\nu}(p) = ie^2 \int \frac{d^4k}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not{k}-m_e}\gamma_\nu\right] - \mathrm{Tr}\left[\frac{1}{\not{k}-m_e}\gamma_\nu\right] \right\}$$

$$= 0 \qquad \underline{\text{q.e.d.}}$$

<u>Physical significance:</u> Transversality or gauge invariance of $\Pi_{\mu\nu}(p)$.

<u>(d)</u> From <u>(c)</u>, we have that $\Pi_{\mu\nu}(p)$ has the form:

$$\Pi_{\mu\nu}(p) = \left(\eta_{\mu\nu} p^2 - p_\mu p_\nu\right) \Pi(p^2) ,$$

with $\Pi(p^2)$ regular at $p^2 \mapsto 0$

Then, the superficial degree of divergence D_Π for the 1PI graph in (<u>b</u>) is:

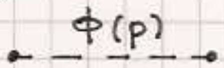
$$D_\Pi = 4 \times \underset{L=1}{1} \quad \leftarrow \text{\#loop momenta in the numerator}$$
$$- 2 \quad \leftarrow \text{\#loop momenta in the denominator}$$
$$- 2 \quad \leftarrow \text{mometa } p_\mu, p_\nu \text{ due to gauge symmetry}$$
$$= 0$$

$\therefore$ $\Pi_{\mu\nu}(p)$ scales logarithmically with the UV cut-off Λ

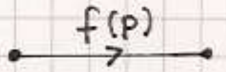
<u>q.e.d.</u>

4.

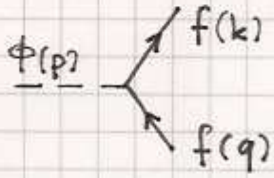
(a)



: $\dfrac{i}{p^2-M^2+i\varepsilon}$

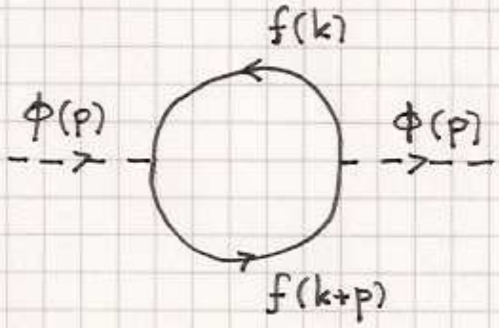


: $\dfrac{i}{\not p - m\mathbb{1}_4+i\varepsilon} = \dfrac{i(\not p + m\mathbb{1}_4)}{p^2-m^2+i\varepsilon}$



: $-ih$

(b)



$\equiv i\Pi_{\phi\phi}(p^2)$

$$i\Pi_{\phi\phi}(p^2) = (-1)\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[(-ih)\frac{i}{\not k - m}(-ih)\frac{i}{\not k+\not p - m}\right]$$

$$= -h^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}\left[(\not k + m)(\not k + \not p + m)\right]}{(k^2-m^2)\left[(k+p)^2-m^2\right]}$$

(c) In the limit $p_\mu \to 0$ and $\frac{m}{M}\to 0$, $\Pi_{\phi\phi}(p^2)$ simplifies to

$$i\Pi_{\phi\phi}(p^2=0) = -h^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\not k(\not k+\not p)]}{k^2(k+p)^2}\Bigg|_{p_\mu\to 0} = -h^2\int\frac{d^4k}{(2\pi)^4}\frac{4k^2}{k^2(k+p)^2}$$

$$= -\frac{h^2}{4\pi^2}\int\frac{d^4k}{\pi^2}\frac{1}{k^2} \underset{(\text{rotation})}{\overset{(\text{Wick})}{=}} -\frac{h^2}{4\pi^2}\, i\int_0^{\Lambda^2} k_E^2\,\frac{dk_E^2}{-k_E^2}$$

$$= i\frac{h^2}{4\pi^2}\Lambda^2$$

Selfenergy diverges qudratically with the UV cut-off Λ.

We may now rewrite $\Pi_{\phi\phi}(p^2)$ as

$$\Pi_{\phi\phi}(p^2) = \Pi_{\phi\phi}(0) + p^2\,\Pi'_{\phi\phi}(0) + \tilde{\Pi}_{\phi\phi}(p^2)$$

It is easy to check that $\tilde{\Pi}_{\phi\phi}(0) = 0$ and $\tilde{\Pi}'_{\phi\phi}(0) = \frac{d}{dp^2}\tilde{\Pi}_{\phi\phi}\Big|_{p^2=0} = 0$.

Given that $\Pi_{\phi\phi}(0) \propto \Lambda^2$ and $\Pi_{\phi\phi}(p^2)$, we must have that on dimensional grounds $\Pi'_{\phi\phi}(p^2) = \frac{d\Pi_{\phi\phi}(p^2)}{dp^2} \propto \Lambda^0$ or $\underline{\ln\Lambda}$

The UV divergences Λ^2 and $\ln\Lambda$ in $\Pi_{\phi\phi}(p^2)$ can be removed by a <u>mass renormalization</u> of Φ:

$$\delta M^2 = \Pi_{\phi\phi}(0)$$

and a <u>wave-function renormalization</u>:

$$\delta Z_\phi = -\Pi'_{\phi\phi}(0) \;.$$

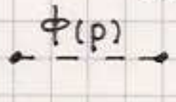
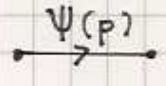
$\longleftrightarrow$

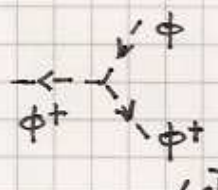
5.

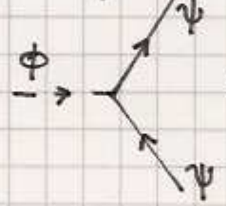
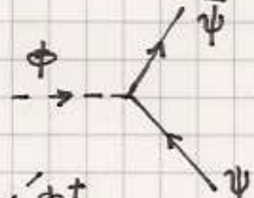
<u>(i)</u> $\psi = \psi^c = C\bar{\psi}^T$, where C is the charge conjugation operator.

<u>or</u> $\psi = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot\alpha} \end{pmatrix}$, with ξ_α and $\bar{\xi}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}(\xi_\beta)^\dagger$;
ξ_α is Weyl spinor.

$\longleftrightarrow$

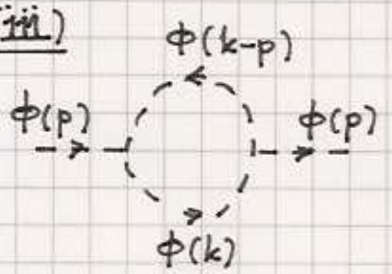
<u>(ii)</u>  : $\dfrac{i}{p^2 - m^2 + i\varepsilon}$  : $\dfrac{i(\not p + m\mathbf{1}_4)}{p^2 - m^2 + i\varepsilon}$

 : $-imh$  : $-imh$

 : $-ihP_L$  : $-ihP_R$

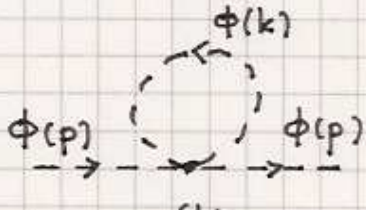
 : $-ih^2$

(iii)



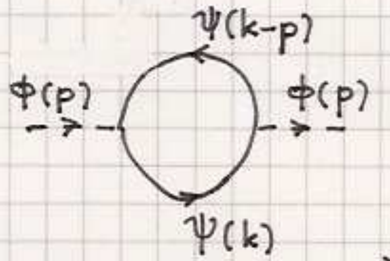
(a)

$$: \; i\Pi^{(a)}(p^2) = \int \frac{d^4k}{(2\pi)^4} (-imh) \frac{i}{k^2-m^2+i\varepsilon} \times (-imh) \frac{i}{(k-p)^2-m^2+i\varepsilon}$$



(b)

$$: \; i\Pi^{(b)}(p^2) = \int \frac{d^4k}{(2\pi)^4} (-ih^2) \frac{i}{k^2-m^2+i\varepsilon}$$



(c)

because $\psi = \psi^c$ in the loop

$$: \; i\Pi^{(c)}(p^2) = (-1)\,\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \times \mathrm{Tr}\left[(-ih)P_R \frac{i(\not k+m)}{k^2-m^2+i\varepsilon}(-ih)P_L \frac{i(\not k-\not p+m)}{(k-p)^2-m^2+i\varepsilon}\right]$$

(iv) At $p^\mu \mapsto 0$, we have

$$\Pi(0) = \Pi^{(a)}(0) + \Pi^{(b)}(0) + \Pi^{(c)}(0)$$

$$= \int \frac{d^4k}{(2\pi)^4 i}\left[\frac{m^2h^2}{(k^2-m^2)^2} + \frac{h^2}{k^2-m^2} - \frac{h^2}{2}\frac{\mathrm{Tr}[P_R(\not k+m)P_L(\not k+m)]}{(k^2-m^2)^2}\right]$$

Since $\{\gamma_\mu, \gamma_5\} = 0 \;\mapsto\; \not k P_L = P_R \not k$. In addition, $P_R^2 = P_R$ and $P_R P_L = 0$

Consequently,

$$\mathrm{Tr}\left[P_R(\not k+m)P_L(\not k+m)\right] = \mathrm{Tr}\left[P_R \underset{\overset{\parallel}{k^2}}{\not k^2}\right] = k^2\,\mathrm{Tr}\left[\frac{\mathbf{1}_4+\gamma_5}{2}\right] = 2k^2$$

$\boxed{\mathrm{Tr}\,\gamma_5 = 0}$

Hence,

$$\Pi(0) = \int \frac{d^4k}{(2\pi)^4 i} h^2 \left[\frac{m^2}{(k^2-m^2)^2} + \underbrace{\frac{1}{k^2-m^2} - \frac{k^2}{(k^2-m^2)^2}}_{= \frac{k^2-m^2-k^2}{(k^2-m^2)^2} = -\frac{m^2}{(k^2-m^2)^2}}\right]$$

$$= \int \frac{d^4k}{(2\pi)^4 i} h^2 \left[\frac{m^2}{(k^2-m^2)^2} - \frac{m^2}{(k^2-m^2)^2}\right] = 0 \quad \underline{\text{q.e.d.}}$$