Quantum Field Theory

Apostolos Pilaftsis

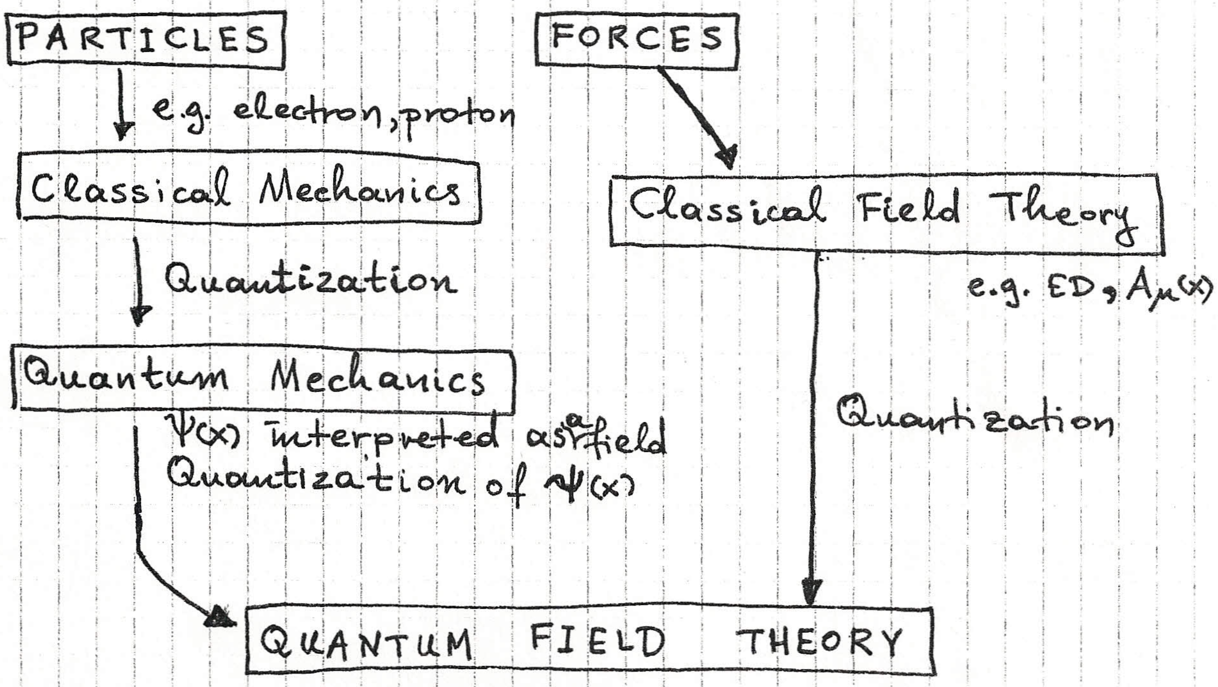
Lecture 1:

Administrative issues:

- sign-up sheet
- Electronic access of lecture notes
- Structure of the course
  (Pre-requisites: Lagrangian Dynamics, Electrodynamics
  Relativistic Quantum Mechanics.)
  (Desirable: Advanced QM, symmetries in Physics.)

- Literature

Why QFT?



→ QFT unifies particles and forces

Many founders: Schwinger, Yukawa, Feynman,
Tomonoga, Weinberg, Veltmann
't-Hooft ... (partial list)

Lagrangian Dynamics

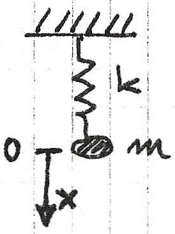
Lagrangian: $L(q_i, \dot q_i) = T(\dot q_i, q_j \dot q_i) - V(q_i)$

kinetic   potential
terms

$q_{1,2,\dots,n}$: generalized coords.

$\dot q_{1,2,\dots,n} = \frac{d}{dt} q_{1,2,\dots,n}$

Example: The Lagrangian of a spring



$$L(x, \dot x) = \frac{1}{2} m \dot x^2 - \frac{1}{2} k x^2$$

Hamiltonian defined by the Legendre transform:

$$H(q_i, p_i) = \sum_i p_i \dot q_i - L(q_i, \dot q_i),$$

where $p_i = \dfrac{\partial L(q_j, \dot q_j)}{\partial \dot q_i}$ are the conjugate momenta.

$\xrightarrow{\text{invert}}$ $\dot q_j = \dot q_j(q_i, p_i)$

Hamilton's equations:

$$\dot q_i = \frac{\partial H}{\partial p_i}, \qquad \dot p_i = -\frac{\partial H}{\partial q_i}$$

q_i need to satisfy the eqns of motion, see next page

Proof:
$$\frac{\partial H}{\partial p_i} = \dot q_i + p_j \frac{\partial \dot q_j}{\partial p_i} - \underbrace{\frac{\partial L(q_j, \dot q_j)}{\partial p_i}}_{\frac{\partial \dot q_j}{\partial p_i} \frac{\partial L}{\partial \dot q_j} = \frac{\partial \dot q_j}{\partial p_i} p_j} = \dot q_i$$

$H = T + V \leftarrow$ total energy of the system

For the spring: $H = \frac{1}{2} m \dot x^2 + \frac{1}{2} k x^2 = \frac{p^2}{2m} + \frac{1}{2} k x^2$

Lecture 2:

Action S

$$S[q_i(t)] = \int_{t_1}^{t_2} dt \; L(q_i(t), \dot{q}_i(t))$$

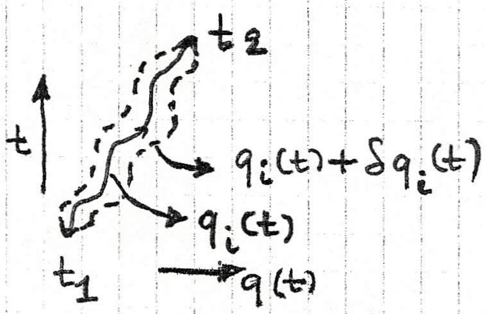
$$\mathcal{F} \ni q_i(t) \xrightarrow[\text{mapping}]{\text{functional}} S[q_i(t)] \in \mathbb{R}$$

space of functions

Differentiation: $\dfrac{\delta q_j(t')}{\delta q_i(t)} = \delta_{ij}\, \delta(t-t')$

Hamilton's principle: Actual motion of system is determined by the stationary points of S: $\dfrac{\delta S}{\delta q_i(t)} = 0$

(extension of Fermat's principle)



$$\delta q_i(t_1) = \delta q_i(t_2) = 0$$

$$S[q_i+\delta q_i] = S[q_i] + \underbrace{\int_{t_1}^{t_2} dt'\, \delta q_j(t') \frac{\delta S}{\delta q_j(t')}}_{\delta S}$$

$$\delta S = \int_{t_1}^{t_2} dt \left(\delta q_i \frac{\partial L}{\partial q_i} + \underbrace{\delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}}_{= \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right) - \delta q_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \right)$$

$$= \int_{t_1}^{t_2} dt \; \delta q_i \underbrace{\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]}_{= \delta S / \delta q_i(t)} + \underbrace{\cancel{\delta q_i \frac{\partial L}{\partial \dot{q}_i}\Big|_{t_1}^{t_2}}}_{0} \overset{!}{=} 0$$

⇝ Euler-Lagrange eqn of motion:

$$\frac{\delta S}{\delta q_i(t)} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Example: (spring): $m\ddot{x} + kx = 0 \;⇝\; m\ddot{x} = -kx$

Proof of $\dot{p}_i = -\frac{\partial H}{\partial q_i}$:

$$-\frac{\partial H}{\partial q_i} = \cancel{-p_j \frac{\partial \dot{q}_j}{\partial q_i}} + \cancel{\frac{\partial \dot{q}_j}{\partial q_i} \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{= p_j}} + \frac{\partial L}{\partial q_i} \overset{\text{EOM}}{=} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i$$

Comment on differentiation: $\frac{\delta q_i(t)}{\delta q(t')} = \delta(t-t')$

This is an extension of the discrete differentiation:

$$\frac{dx_i}{dx_j} = \delta_{ij} \quad \text{or} \quad \boxed{\sum_j \frac{d}{dx_j} x_i = 1}$$

If i,j are continuous variables, we then have

$$\underbrace{\int}_{\Sigma} dt' \, \frac{\delta x(\overset{i}{t})}{\delta x(t')} = 1 \quad (1)$$

j

Obviously, $\frac{\delta x(t)}{\delta x(t')} = 0$, for $t \neq t'$ (2)

and $\frac{\delta x(t)}{\delta x(t')} \neq 0$, for $t = t'$ (3)

The only function with above (1), (2), (3) properties is

$$\frac{\delta x(t)}{\delta x(t')} = \delta(t-t')$$

Lagrangian Field Theory

Consider the Lagrangian of a real scalarfield:

$$L = \int d^3\underline{x} \; \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) \qquad ; \; \partial_\mu \mathrel{\widehat{=}} \frac{\partial}{\partial x^\mu}$$

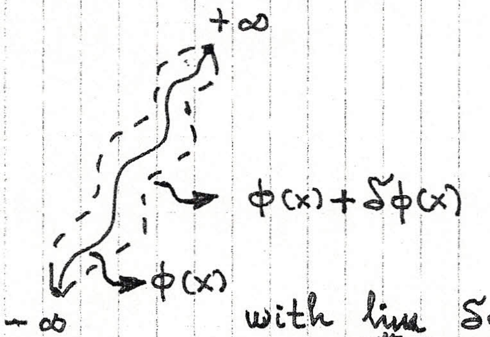
$\hookrightarrow$ $\boxed{\text{Lagrangian density or Lagrangian in FT.}}$

Action S

$$S[\phi(x)] = \int_{-\infty}^{+\infty} d^4x \; \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) \, .$$

Find the extrema of the action:

$$S[\phi(x) + \delta\phi(x)] = S[\phi(x)] + \underbrace{\int d^4y \, \delta\phi(y) \frac{\delta S}{\delta \phi(y)}}_{\delta S \, \stackrel{!}{=} \, 0}$$



with $\lim\limits_{x \to \pm\infty} \delta\phi(x) = 0$ and $\dfrac{\delta\phi(x)}{\delta\phi(y)} \stackrel{!}{=} \delta^{(4)}(x-y)$, we get

$$\delta S = \int_{-\infty}^{+\infty} d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \underbrace{\delta \partial_\mu \phi}_{= \partial_\mu(\delta\phi)} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right]$$

$$= \int_{-\infty}^{+\infty} d^4x \; \delta\phi \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] + \underbrace{\cancel{\int_{-\infty}^{+\infty} d^4x \, \partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}}_{0}$$

0, since $\delta\phi(x^\mu \to \pm\infty) = 0$

Euler–Lagrange eqn:

$$\frac{\delta S}{\delta \phi(x)} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 .$$

Lagrangian for a free scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

E-L eqn:

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \quad ; \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial}{\partial(\partial_\mu\phi)}\left[\frac{1}{2}(\partial_\nu\phi)(\partial^\nu\phi)\right]$$

$$= \delta^\mu_{\ \nu}\,\partial^\nu\phi = \partial^\mu\phi$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \rightsquigarrow (\partial_\mu\partial^\mu + m^2)\phi(x) = 0$$

This is the Klein-Gordon equation

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Lagrangian for the EM field A^μ

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \leftarrow$ EM field stregth tensor

N.B. $$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

E-L eqn: $$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \ ? = \partial_\mu F^{\mu\nu}$$

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = -J^\nu$$

$$\rightsquigarrow \partial_\mu F^{\mu\nu} = J^\nu \longleftarrow$$ ② of Maxwell's equs: $\begin{cases} \underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon_0} \\ \underline{\nabla}\times\underline{B} = \mu_0\underline{J} + \underbrace{\mu_0\varepsilon_0}_{1/c^2}\frac{\partial\underline{E}}{\partial t} \end{cases}$

$\hookrightarrow (\rho, \underline{J})$

Lecture 3:

Symmetries of the Lagrangian

Consider $\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^*\phi + \lambda(\phi^*\phi)^2$

$\mathcal{L}$ is invariant under a $U(1)$ rotation:

$$\phi(x) \mapsto \phi'(x) = e^{i\theta}\phi(x)$$

where θ does not depend on $x \equiv x^\mu$ (global transf.)

If $\theta = \theta(x)$ was depending on x, such a transf. is called local transformation.

It is left as an exercise to check that $\mathcal{L}$ is not invariant under a local transf.

Other symmetries ($SO(2)$: 2×2 real orthogonal matrices: $O^T O = O O^T = \mathbb{1}_2$, with $\det O = 1$)

E.g. $\underline{\Phi} = \begin{pmatrix} \mathrm{Re}\,\phi \\ \mathrm{Im}\,\phi \end{pmatrix}$ or $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$SO(2): \underline{\Phi} \mapsto \underline{\Phi}' = \begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{= e^{i\theta\sigma_2}} \underbrace{\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}}_{\underline{\Phi}}$$

Property:
$\phi_1^2 + \phi_2^2 = \phi_1'^2 + \phi_2'^2$

$= e^{i\theta\sigma_2}$; $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Lie Generator of $SO(2)$

$SU(2)$: 2×2 complex unitary matrices:

$$U^\dagger U = U U^\dagger = \mathbb{1}_2, \text{ with } \det U = 1$$

$$SU(2): \underline{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \mapsto \underline{\Phi}' = e^{i\theta^a T^a}\underline{\Phi} \quad ; \quad \theta^{1,2,3} \in \mathbb{R}$$

Lie generators: $T^{1,2,3} = \frac{\sigma_{1,2,3}}{2}$

Pauli matrices

Property:
$|\Phi_1|^2 + |\Phi_2|^2 = |\Phi_1'|^2 + |\Phi_2'|^2$

Noether's theorem

If a Lagrangian $\mathcal{L}$ is (up to a total derivative $\partial_\mu Q^\mu$) invariant under a given transf. of fields and spacetime, then there is a conserved current $J^\mu(x)$ and a conserved charge $Q = \int d^3\underline{x}\; J^0(x)$, such that

$$\partial_\mu J^\mu = 0 \quad \text{and} \quad \frac{dQ}{dt} = 0$$

Proof for a global symmetry

Consider : $\phi_i \mapsto \phi_i' = \phi_i + \delta\phi_i$, $\quad \delta\phi_i = i\theta^a (T^a)_i{}^j \phi_j$

such that $\mathcal{L}(\phi_i, \partial_\mu\phi_i) = \mathcal{L}(\phi_i', \partial_\mu\phi_i')$ $\boxed{+\, \partial_\mu \delta Q^\mu(\Phi_i, \partial_\mu \Phi_i)}$

$$= \mathcal{L}(\phi_i, \partial_\mu\phi_i) + \delta\mathcal{L},$$

where $\delta\mathcal{L} = \delta\phi_i \dfrac{\partial\mathcal{L}}{\partial\phi_i} + \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(\partial_\mu\delta\phi_i) = 0$

$$\rightsquigarrow \quad \delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right] + \left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right]\delta\phi_i = 0$$

$$\rightsquigarrow \quad \underbrace{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right]}_{\propto J^\mu} = \underbrace{\left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} - \frac{\partial\mathcal{L}}{\partial\phi_i}\right]}_{=0\text{, for } \phi_i \text{ satisfying E-L equs.}}\delta\phi_i = 0$$

Conserved current(s):

$$J^{a,\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\frac{\partial\delta\phi_i}{\partial\theta^a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\, i(T^a)_i{}^j\phi_j \boxed{+\, \frac{\partial\delta Q^\mu}{\partial\theta^a}}$$

Conserved charge: $Q^a(t) = \int d^3\underline{x}\; J^{a,0}(x)$,

because $\dfrac{dQ^a}{dt} = \int d^3\underline{x}\; \partial_t J^{a,0} = -\int d^3\underline{x}\; \underline{\nabla}\cdot\underline{J}^a \overset{\text{Gauss law}}{=} -\int d\underline{s}\cdot\underline{J}^a \mapsto 0$

surface terms vanish at infinity.

Lecture 4:

Canonical Quantization

Classical $\mapsto$ Quantum Mechanics

- Physical states are **ket** vectors $|A\rangle$ in the Hilbert space $\mathcal{H}$ (an infinite-dim vector space).
- If a system is in state $|A\rangle$, the probability to observe it in another state $|B\rangle$ is $P_{A\mapsto B} = |\langle B|A\rangle|^2$.

  E.g. $\langle x|A\rangle = \psi_A(x)$, $\langle x|B\rangle = \psi_B(x)$ (with $\hat{x}|x\rangle = x|x\rangle$)

  $$\rightsquigarrow \quad \langle B|A\rangle = \int dx\, \langle B|x\rangle\langle x|A\rangle = \int dx\, \psi_B^*(x)\,\psi_A(x)$$

  $\langle B|$ is Dirac's bra vector which is dual to $|B\rangle$, i.e.

  $$(\langle B|)^\dagger = |B\rangle, \quad \text{or} \quad \langle x|B\rangle^* = \langle B|x\rangle$$
- Observables are mapped into Hermitean operators $\hat{O}$
- Free particles are described by plane waves:

  $$\phi_{\underline{p}}(t,\underline{x}) = \langle \underline{x};t|\underline{p}\rangle = \sqrt{2E_p}\; e^{-ip_\mu x^\mu}\;;\quad E_p = \sqrt{\underline{p}^2+m^2}$$
- Completeness and orthogonality of $\mathcal{H}$:

  $$\sum_{\underline{p}} |\underline{p}\rangle\langle\underline{p}| = \int \frac{d^3\underline{p}}{(2\pi)^3(2E_p)}\, |\underline{p}\rangle\langle\underline{p}| = \mathbb{1}$$

  $$\langle \underline{k}|\underline{p}\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\underline{k}-\underline{p})$$

Example:

$$E^2 = \underline{p}^2 + m^2 \overset{QM}{\longmapsto} \left(i\hbar\frac{\partial}{\partial t}\right)^2 |\psi\rangle = \left[\left(\frac{\hbar}{i}\underline{\nabla}\right)^2 + m^2\right]|\psi\rangle$$

$\hbar = 1$

$$\rightsquigarrow \left(\frac{\partial^2}{\partial t^2} - \underline{\nabla}^2 + m^2\right)|\psi\rangle = 0 \rightsquigarrow \underline{(\Box + m^2)\,|\psi\rangle = 0}$$

Klein-Gordon eqn.

Particle States are vectors in the Fock space

$$\mathcal{F} \sim \sum_{i_n=1}^{+\infty} \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_{i_n}$$

$|0; t_0\rangle$: vacuum state with zero particles at t_0

$|\underline{p}; t_0\rangle$: state with one particle of momentum $\underline{p}$ at t_0

$|\underline{p}, \underline{q}; t_0\rangle$: state with two particles of $\underline{p}, \underline{q}$ at t_0

Creation operator $a^+(\underline{p})$ adds a particle of momentum $\underline{p}$ to the Fock state:

$$a^+(\underline{p})\, |\underline{q}_1, \underline{q}_2, \dots, \underline{q}_N\rangle \equiv |\underline{p}, \underline{q}_1, \underline{q}_2, \dots \underline{q}_N\rangle$$

Its adjoint $a(\underline{p})$ removes a particle of momentum $\underline{p}$ from the state on which it acts:

$$a(\underline{p})|0\rangle = 0 \quad , \quad a(\underline{p})|\underline{p}'\rangle = \,?\, = (2\pi)^3\, 2E_{\underline{p}}\, \delta^{(3)}(\underline{p}-\underline{p}')|0\rangle$$

$$\langle 0|\, a(\underline{p})\, |\underline{p}'\rangle = \langle \underline{p}'|\, a^+(\underline{p})\, |0\rangle^* = \langle \underline{p}'|\underline{p}\rangle$$

$$= (2\pi)^3\, 2E_{\underline{p}}\, \delta^{(3)}(\underline{p}-\underline{p}')$$

Vacuum normalization: $\langle 0|0\rangle = 1$.

Field operator for a free scalar theory:

$$\Phi(x) = \int \frac{d^3\underline{k}}{(2\pi)^3\, 2E_k} \left(a(\underline{k})\, e^{-ikx} + a^+(\underline{k})\, e^{ikx} \right)$$

with $\quad [a(\underline{p}), a^+(\underline{q})] = (2\pi)^3\, 2E_p\, \delta^{(3)}(\underline{p}-\underline{q})$

Indeed, $[a(\underline{p}), a^+(\underline{q})]|0\rangle = a_{\underline{p}} \underbrace{a^+_{\underline{q}} |0\rangle}_{|\underline{q}\rangle} - a^+_{\underline{q}} \overbrace{\cancel{a_{\underline{p}}} |0\rangle}^{0}$

$$= (2\pi)^3\, 2E_p\, \delta^{(3)}(\underline{p}-\underline{q})\, |0\rangle$$

Lecture 5:

Causality of the field operator $\Phi(x)$

$$[\Phi(x), \Phi(y)] = 0 \quad ; \text{ for } (x-y)^2 < 0$$

This means that field strength ϕ can be simultaneously measured at points with space-like separations

Proof:

$$[\Phi(x), \Phi(y)] = \int_{\underline{p}} \int_{\underline{q}} \left[\left(a_{\underline{p}}\, e^{-ipx} + a^+_{\underline{p}} e^{ipx}\right), \left(a_{\underline{q}} e^{-iqy} + a^+_{\underline{q}} e^{iqy}\right)\right]$$

with $\int_{\underline{p}} \equiv \int \frac{d^3\underline{p}}{(2\pi)^3\, 2E_p}$; $a_{\underline{p}} = a(\underline{p})$

$$[\Phi(x), \Phi(y)] = \int_{\underline{p}} \int_{\underline{q}} \Big\{ \underbrace{[a_{\underline{p}}, a^+_{\underline{q}}]}_{+(2\pi)^3 2E_p \delta^{(3)}(\underline{p}-\underline{q})} e^{-ipx+iqy} + \underbrace{[a^+_{\underline{p}}, a_{\underline{q}}]}_{-(2\pi)^3\, 2E_p\, \delta^{(3)}(\underline{p}-\underline{q})} e^{ipx-iqy} \Big\}$$

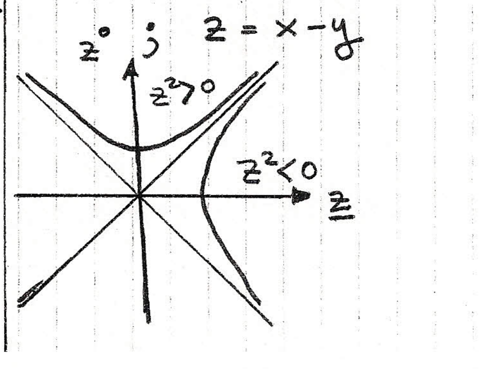
$$= \int \frac{d^3\underline{p}}{(2\pi)^3\, 2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) \equiv \Delta(x-y)$$

Properties: $\Delta(x-y) = -\Delta(y-x)$ and Lorentz invariant

Consider $x^0 = y^0$, but $\underline{x} \neq \underline{y}$, so that $\overbrace{(x-y)^2}^{-(\underline{x}-\underline{y})^2} < 0$,

then $\int_{-\infty}^{+\infty} \frac{d^3\underline{p}}{(2\pi)^3\, 2E_p}\, e^{-i\underline{p}\cdot\underline{z}} = -\int_{+\infty}^{-\infty} \frac{d^3\overbrace{(-\underline{p})}^{\tilde{\underline{p}}}}{(2\pi)^3\, 2E_p}\, e^{i\tilde{\underline{p}}\cdot\underline{z}}$

$$= \int_{-\infty}^{+\infty} \frac{d^3\tilde{\underline{p}}}{(2\pi)^3\, 2E_p}\, e^{i\underline{p}\cdot\underline{z}}$$



$\rightsquigarrow \Delta(x-y) = 0$, for $(x-y)^2 < 0$

You cannot go from $z^2>0$ to $z^2<0$ via a Lorentz transformation

Canonical Quantization of Scalar Field Theory

1-dim QFT

RQM $\qquad\qquad\qquad\qquad$ **QFT**

$$[\hat{x}^\mu(\tau), \hat{p}^\nu(\tau)] = -i\eta^{\mu\nu} \quad \mapsto \quad [\Phi(t,\underline{x}), \Pi(t,\underline{y})] = i\,\delta^{(3)}(\underline{x}-\underline{y})$$

τ is the proper time

Conjugate momentum operator

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \qquad \longrightarrow \qquad \Pi(t,x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} = \partial_0 \Phi(t,\underline{x})$$

$$[\Phi(t,\underline{x}), \Pi(t,\underline{y})] =$$

with $x^0 = y^0 = t$

$$= \int_p \int_q \left[a_p e^{-ipx} + a_p^\dagger e^{ipx},\; -iE_q\left(a_q e^{-iqy} - a_q^\dagger e^{iqy}\right)\right]$$

$$= \int_p \int_q -iE_q \Big\{ \underbrace{[a_p^\dagger, a_q]}_{-(2\pi)^3\, 2E_p\, \delta^{(3)}(p-q)} e^{it(E_p-E_q)}\, e^{-ip\cdot x + iq\cdot y} - \underbrace{[a_p, a_q^\dagger]}_{(2\pi)^3\, 2E_p\, \delta^{(3)}(p-q)} e^{-it(E_p-E_q)}\, e^{ip\cdot x - iq\cdot y} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{+i}{2}\right) \left\{ e^{+ip\cdot(\underline{x}-\underline{y})} + e^{-ip\cdot(\underline{x}-\underline{y})} \right\}$$

$$= i \int \frac{d^3p}{(2\pi)^3}\, e^{ip\cdot(\underline{x}-\underline{y})} = i\,\delta^{(3)}(\underline{x}-\underline{y}) \qquad \underline{\text{q.e.d.}}$$

Concluding remarks

To quantize a theory (for bosons), we proceed as follows:

- Write down the Lagrangian describing the system; e.g.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} m^2 \phi^2$$

- We promote the ~~classical~~ fields into operators, which satisfy the equal-time commutators:

(A) $[\Phi(t, \underline{x}), \Pi(t, \underline{y})] = i\,\delta^{(3)}(\underline{x} - \underline{y})$

(B) $[\Phi(t, \underline{x}), \Phi(t, \underline{y})] = [\Pi(t, \underline{x}), \Pi(t, \underline{y})] = 0$

where $\Pi(x)$ is the conjugate momentum operator.

The commutators (B) ensure the causality of the theory.

- The existence of a Hamilton (density) operator

$$\mathcal{H}(\Phi, \Pi) = \Pi\, \partial_0 \phi - \mathcal{L} \quad ; \quad \Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi$$

$$= \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\underline{\nabla} \Phi)\cdot(\underline{\nabla} \Phi) + \frac{1}{2} m \Phi^2$$

Lecture 6

Complex Fields and Anti-Particles

Consider the Lagrangian:

$$\mathcal{L} = (\partial^\mu \Phi^+)(\partial_\mu \Phi) - m^2 \Phi^+ \Phi$$

where ϕ and ϕ^+ are complex field operators:

$$\Phi(x) = \int \frac{d^3\underline{k}}{(2\pi)^3 2E_k} \left(a(\underline{k})\, e^{-ik\cdot x} + b^+(\underline{k})\, e^{ikx} \right) ,$$

$$\Phi^+(x) = \int \frac{d^3\underline{k}}{(2\pi)^3 2E_k} \left(a^+(\underline{k})\, e^{ikx} + b(\underline{k})\, e^{-ikx} \right) .$$

Conjugate momenta: $\Pi(x) = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} = \partial^0 \phi^+(x) = \partial_t \phi^+(x)$

and $\Pi^+(x) = \partial_t \phi(x)$.

One may use the commutation relations:

$$[a(\underline{k}), a^+(\underline{k}')] = [b(\underline{k}), b^+(\underline{k}')] = (2\pi)^3\, 2E_k\, \delta^{(3)}(\underline{k}-\underline{k}')$$

to show that (see exercise on p. 17):

$$[\phi(t,\underline{x}), \Pi(t,\underline{y})] = [\Phi^+(t,\underline{x}), \Pi^+(t,\underline{y})] = i\,\delta^{(3)}(\underline{x}-\underline{y})$$

whereas

$$[\Phi(t,\underline{x}), \Pi^+(t,\underline{y})] = 0$$ and all other

<u>equal-time</u> commutators vanish (why?)

Conserved current

The Lagrangian is invariant under $U(1)$ transf:

$$\Phi = e^{-i\theta}\Phi \quad ; \quad \Phi^{+} = e^{+i\theta}\Phi^{+} \quad (\delta\Phi = -i\theta\Phi, \delta\Phi^{+} = +i\theta\Phi^{+})$$

Noether's theorem implies

the conserved current: $J^{\mu} = \Phi^{+}(i\partial^{\mu}\Phi) - \Phi(i\partial^{\mu}\Phi^{+})$

and the conserved charge: $Q = \int d^3\underline{x} :J^0:$

this will be explained later

Derivation of J^{μ}:

$$J^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\underbrace{\frac{\partial\delta\phi}{\partial\theta}}_{(-i\phi)} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{+})}\underbrace{\frac{\partial\delta\Phi^{+}}{\partial\theta}}_{(+i\phi^{+})} = -(\partial^{\mu}\phi^{+})(i\phi) + (\partial^{\mu}\phi)(+i\phi^{+})$$

$$= \phi^{+}(i\partial^{\mu}\phi) - \phi(i\partial^{\mu}\phi^{+})$$

$Q = \int d^3\underline{x}\, J^0(x)$ ⟵ conserved charge operator

$$= \int d^3\underline{x}\left[\phi^{+}(i\partial_t\phi) - \phi(i\partial_t\phi^{+})\right] = \int d^3\underline{x}\left[\phi^{+}(i\partial_t\phi) + \text{H.c.}\right]$$

$$= \int d^3\underline{x}\left\{\int_{\underline{k}}\int_{\underline{p}}\left(a^{+}_{\underline{k}}e^{+ik\cdot x} + b_{\underline{k}}e^{-ik\cdot x}\right)E_p\left(a_{\underline{p}}e^{-ip\cdot x} - b^{+}_{\underline{p}}e^{ip\cdot x}\right) + \text{H.c.}\right\}$$

$$= \int_{\underline{k}}\int_{\underline{p}} E_p \int d^3\underline{x}\left[a^{+}_{\underline{k}}a_{\underline{p}}e^{i(k-p)\cdot x} - a^{+}_{\underline{k}}b^{+}_{\underline{p}}e^{i(k+p)\cdot x} + b_{\underline{k}}a_{\underline{p}}e^{-i(k+p)x} - b_{\underline{k}}b^{+}_{\underline{p}}e^{i(p-k)x} + \text{H.c.}\right]$$

$$= \int\frac{d^3\underline{k}}{(2\pi)^3 2E_k}\int\frac{d^3\underline{p}}{2}\left[\left(a^{+}_{\underline{k}}a_{\underline{p}} - b_{\underline{k}}b^{+}_{\underline{p}}\right)\delta^{(3)}(\underline{k}-\underline{p}) + \text{H.c.}\right.$$

$$\left. + \left(b_{\underline{k}}a_{\underline{p}}e^{-2iE_kt} - a^{+}_{\underline{k}}b^{+}_{\underline{p}}e^{2iE_kt}\right)\delta^{(3)}(\underline{k}+\underline{p}) + \text{H.c.}\right]$$

$$b^{+}_{\underline{p}}b^{+}_{\underline{k}}e^{2iE_kt} - b_{\underline{p}}a_{\underline{k}}e^{-2iE_kt}$$

factor 2

$$\rightsquigarrow \quad Q = \int \frac{d^3\underline{k}}{(2\pi)^3 2E_{\underline{k}}} \Big\{ a^\dagger_{\underline{k}} a_{\underline{k}} - \overbrace{b_{\underline{k}} b^\dagger_{\underline{k}}}^{b^\dagger_{\underline{k}} b_{\underline{k}} + (2\pi)^3 2E_{\underline{k}}\,\delta(0)}$$

$$+ \frac{1}{2}\Big[\underbrace{\big(b_{\underline{k}} a_{-\underline{k}} - \underbrace{b_{-\underline{k}} a_{\underline{k}}}_{b_{\underline{k}} a_{-\underline{k}}} \big) e^{-2iE_{\underline{k}}t}}_{\emptyset} + \text{H.c.} \Big] \Big\}$$

$\boxed{\text{after } \underline{k} \mapsto -\underline{k} \text{ change of integration}}$

$$Q = \int \frac{d^3\underline{k}}{(2\pi)^3 2E_{\underline{k}}} \Big[\big(a^\dagger_{\underline{k}} a_{\underline{k}} - b^\dagger_{\underline{k}} b_{\underline{k}} \big) + \underbrace{(2\pi)^3 2E_{\underline{k}}\,\delta(0)}_{\infty\,:\,\text{vacuum effect}} \Big]$$

<u>Wick's</u> or <u>normal</u> ordering removes vacuum effects; according to the prescription:

$$:\frac{1}{2}\big(a^\dagger_{\underline{k}} a_{\underline{k}} + a_{\underline{k}} a^\dagger_{\underline{k}}\big): \; = a^\dagger_{\underline{k}} a_{\underline{k}}$$

Hence, $\quad Q = \int d^3\underline{x} \, :J^0: \; = \int \frac{d^3\underline{k}}{(2\pi)^3 2E_{\underline{k}}} \big(a^\dagger_{\underline{k}} a_{\underline{k}} - b^\dagger_{\underline{k}} b_{\underline{k}} \big)$

Obviously, $\dfrac{dQ}{dt} = 0$

$$Q\, a^\dagger_{\underline{k}} |0\rangle = \int_{\underline{p}} a^\dagger_{\underline{p}} \underbrace{a_{\underline{p}} a^\dagger_{\underline{k}}}_{a^\dagger_{\underline{k}} a_{\underline{p}} + (2\pi)^3 2E_p \,\delta^{(3)}(\underline{k}-\underline{p})} |0\rangle = + a^\dagger_{\underline{k}} |0\rangle$$

Likewise, $\; Q\, b^\dagger_{\underline{k}} |0\rangle = - b^\dagger_{\underline{k}} |0\rangle$

Consequently, $a^\dagger_{\underline{k}}$ creates <u>particles</u> of charge $+1$ and momentum, whereas $b^\dagger_{\underline{k}}$ creates <u>anti-particles</u> of charge -1 and momentum $\underline{k}$.