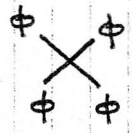
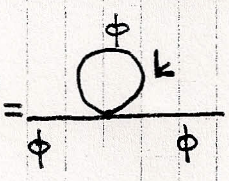
Lecture 18

Renormalizability

Consider first a ϕ^4 theory, i.e. $\mathcal{L}_{int} = -\frac{\lambda}{4!}\,\phi^4(x)$

Tree-level graph: $\Gamma^{(0)}_{\phi^4} =$  $: -i\lambda$

Loop graphs ($\underbrace{\text{1-particle irreducible}}_{\equiv\, 1PI}$)

$\Gamma^{(1)}_{\phi^2} =$ 

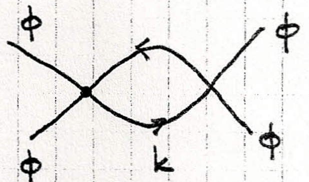
$$= -\frac{i\lambda}{2}\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2} \xrightarrow{|k|\gg m} \propto \int_0^{\Lambda}\frac{|k|^3 d|k|}{|k|} = \Lambda^2$$

it diverges for large momenta

cut-off regularization

$\rightsquigarrow$ superficial degree of divergence: $\underline{D=2}$

from $\sim\Lambda^D$: quadratic UV divergence

$\Gamma^{(1)}_{\phi^4} =$ 

$$= \frac{(-i\lambda)^2}{2}\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\,\frac{(i)^2}{(k^2-m^2)^2}$$

$$\xrightarrow[|k|\gg m]{\text{UV-limit}} \propto \int_m^{\Lambda}\frac{|k|^3 d|k|}{|k|^4} = \int_m^{\Lambda}\frac{d|k|}{|k|} = \ln\left(\frac{\Lambda}{m}\right)$$

superficial degree of divergence: $\underline{D=0}$

logarithmic UV divergence

Superficial degree of divergence of a $\overset{1PI}{\text{graph}}$ Γ is defined as

$D_\Gamma =$ number of loop momenta in the numerator

$-$ numbel of loop momenta in the denominator

$\Downarrow$ ϕ^4-theory

In d spacetime dimensions: $D_\Gamma = dL - 2P$,

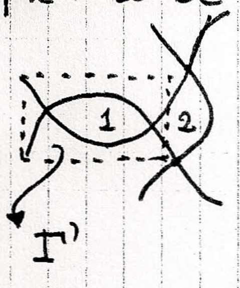
where L: # loops and P: # propagators of boson fields

Weinberg's theorem on renormalizability (without proof)

An 1PI loop graph Γ is UV finite, if D_Γ is negative ($D_\Gamma < 0$) and the superficial degree of all possible subgraphs is negative as well.

Observe that $D_\Gamma < 0$ is not sufficient for an 1PI graph to be UV convergent.

E.g.

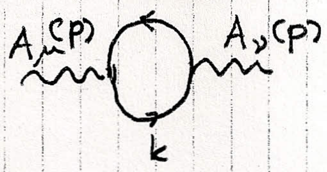


$$D_\Gamma = 4\times 2 - 2\times 5 = -2$$

but $D_{\Gamma'} = 0$

★ Possible caveats in Weinberg's theorem may occur if there are additional symmetries acting on the theory, such as gauge symmetry, chiral symmetry, supersymmetry etc.

E.g. $\Gamma^{(1)}_{A_\mu A_\nu}(p) =$



2 fermion propagators

$$D = 4 - 2\times 1 = +2$$ quadratically divergent

However, the gauge symmetry requires that $\Gamma_{A_\mu A_\nu}(p) = \{-g_{\mu\nu}p^2 + p_\mu p_\nu\}\,\Pi(p)$,

so that $p^\mu \Gamma_{A_\mu A_\nu} = 0 = p^\nu \Gamma_{A_\mu A_\nu}$

Hence, $D_\Gamma = 0 \mapsto \Gamma_{A_\mu A_\nu}$ is logarithmically divergent.

More detailed discussion is given in advanced texts.

General remarks on renormalizability of $\frac{1}{n!}\phi^n$ theories in d dimensions:

4-dimensions:

- ϕ^3- theory: Only a finite number of diagram are UV infinite.

  Such a theory is called super-renormalizable.

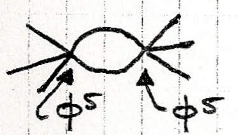
- ϕ^4- theory: New divergences occur at each loop order, However, all these UV divergences are related to 1PI amplitudes with a finite number of external legs.

$\boxed{\phi\text{- fields}}$

$\boxed{\text{Detailed discussion of renormalization will be given in the next lecture}}$

- $\phi^{n\geq 5}$- theories: all $\Gamma_{\phi^{n>4}}$ amplitudes are UV divergent for a high enough order in perturbation theory.

An example in ϕ^5- theory:



$\mapsto \Gamma_{\phi^6}$ has $D=0$

d-dimensions:

- $d=6$: $\phi^3 \mapsto$ renormalizable , $\phi^{n>3} \mapsto$ non-renormalizable
- $d=4$: $\begin{cases} \phi^3 \mapsto \text{super-renormalizable} \\ \phi^4 \mapsto \text{renormalizable} \\ \phi^{n>4} \mapsto \text{non-renormalizable} \end{cases}$

$\boxed{\text{e.g. quantum gravity}}$

$\boxed{S=\int d^4x\sqrt{-g}\,(m_{Pl}^2 R+\Lambda_c)}$

- $d=3$: $\begin{cases} \phi^4 \mapsto \text{super-renormalizable} \\ \phi^6 \mapsto \text{renormalizable} \end{cases}$

  $\phi^{n>6} \mapsto$ non-renormalizable
- $d=2$: $\phi^n \mapsto$ renormalizable for any power of n.

$\boxed{\text{string theory attempts, where } x^\mu = x^\mu(\tau,\sigma)}$

Lecture 19

Renormalization of ϕ^4-theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \Lambda_c$$

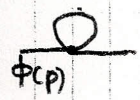
cosmological constant

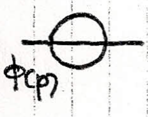
All UV divergences can be absorbed into a redefinition of m^2, λ, Λ_c and ϕ $\mapsto$ renormalizable theory.

More explicitly,

$N_\phi = 0$: $\Gamma^{(1)} =$ 

$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \sim \Lambda^4 \rightarrow$ UV infinity absorbed by Λ_c

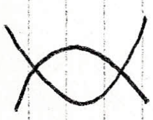
$N_\phi = 2$: $\Gamma^{(1)}_{\phi^2}(p) =$ 

$\Gamma^{(2)}_{\phi^2}(p) =$ 

$$\sim c_1\Lambda^2 + c_2 p^2 \ln\Lambda + \text{finite}$$

∞ to be absorbed into m^2

∞ absorbed by ϕ

$N_\phi = 4$: $\Gamma^{(1)}_{\phi^4} =$  $\sim \ln\Lambda + \text{finite}$

∞ to be absorbed into λ

Introduce UV finite (or renormalized) and bare (unrenormalized) quantities:

wave-function renormalization

parameter renormalization

$$\phi_0 = Z_\phi^{1/2}\phi_R \ , \quad m_0^2 = Z_{m^2} m_R^2 \ , \quad \lambda_0 = Z_\lambda \lambda_R \ , \quad \Lambda_c^0 = Z_{\Lambda_c}\Lambda_c^R$$

$$Z_x = 1 + \delta Z_x \ , \quad x \in \{m^2, \lambda, \Lambda_c\}$$

$$Z_\phi^{1/2} = 1 + \frac{1}{2}\delta Z_\phi$$

they diverge as $\Lambda \to +\infty$

where $\delta Z_x, \delta Z_\phi$ are UV infinite constants that are loopwise expanded : $\delta Z_x = \delta Z_x^{(1)} + \delta Z_x^{(2)} + \cdots$, $\delta Z_\phi = \delta Z_\phi^{(1)} + \delta Z_\phi^{(2)} + \cdots$

Alternative decomposition of $x_0 \in \{m_0^2, \lambda_0, \Lambda_c^0\}$ and ϕ_0:

$$\phi_0 = \phi_R + \delta\phi \, , \qquad x_0 = x_R + \delta x$$

where $\delta\phi = (Z_\phi^{1/2} - 1)\phi_R = \frac{1}{2}\delta Z_\phi \phi_R$, $\delta x = (Z_x - 1)x_R = \delta Z_x x_R$.

Renormalization programme to cancel the UV infinities of the 1PI loop graphs by $\delta\phi$ and δx (or δZ_ϕ and δZ_x).

- Calculate all Feynman graphs using the bare Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!}\lambda_0 \phi_0^4 + \Lambda_c^0$$

- Define renormalization conditions to determine $\delta\phi$ and δx:

Λ_c-renormalization:
For instance, $\Lambda_c^R = 0$ to all orders in perturbation theory.

$\Gamma_c = i\Lambda_c^0 +$  $+ \lambda_0$  $\lambda_0 + \cdots = i\Lambda_c^R = 0$

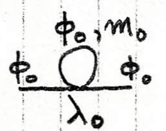
1-loop: $i\Lambda_c^0 + \frac{1}{2}\int_{-\Lambda}^{+\Lambda} \frac{d^4k}{(2\pi)^4} = i\Lambda_c^R + i\delta\Lambda_c + i\int_0^{\Lambda^2} \frac{|k|^2 d|k|^2}{32\pi^2} = i\Lambda_c^R$

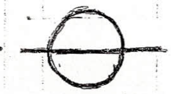
after Wick's rotation to Euclidean space

$$\rightsquigarrow \delta\Lambda_c = -\frac{\Lambda^4}{64\pi^2}$$

Mass and wave-function renormalization ($\delta Z_\phi, \delta m^2$)

$\Gamma_{\phi^2}(p^2) = \underbrace{iZ_\phi(p^2 - m_R^2 - \delta m^2)}_{\equiv \Gamma^{(0)}_{\phi^2}(p^2)} + \underbrace{\text{[tadpole]}}_{\equiv \Gamma^{(1)}_{\phi^2}(p^2)} + \underbrace{\text{[sunset]}}_{\equiv \Gamma^{(2)}_{\phi^2}(p^2)} + \cdots$





$$\left.\begin{aligned} &\Gamma_{\phi^2}(p^2 = m_R^2) = 0 \\ &\frac{1}{i}\frac{d\Gamma_{\phi^2}(p^2)}{dp^2}\Big|_{p^2 = m_R^2} = 1 \end{aligned}\right\}$$

On-shell renormalization condition.

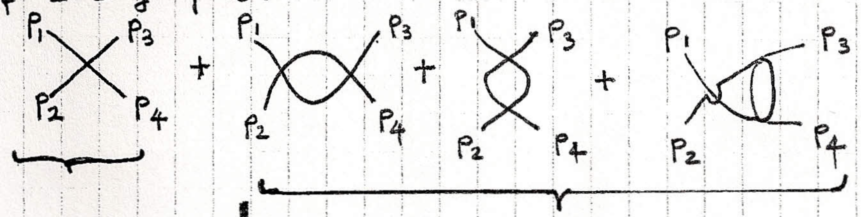
1-loop: $\Gamma^{(1)}_{\phi^2} = \frac{-i\lambda}{32\pi^2}\left[\Lambda^2 - m_R^2 \ln\frac{\Lambda^2}{m_R^2}\right]$

$\rightsquigarrow Z_\phi = 1$ (or $\delta Z_\phi = 0$) and $\delta m^2 = -i\Gamma^{(1)}_{\phi^2}$

Coupling-constant renormalization ($\delta\lambda$)

Sum of 1PI graphs:

$\Gamma_{\phi^4} =$



+ higher orders

$$\Gamma^{(0)}_{\phi^4} = -i\lambda_0 Z_\phi^2 \quad \Big| \quad \Gamma^{(1)}_{\phi^4}(p_1,p_2,p_3,p_4) = \Gamma^{(1)}_s + \Gamma^{(1)}_t + \Gamma^{(1)}_u$$

Renormalization conditions

$$\Gamma_{\phi^4}\Big|_{p_i=0} = -i\lambda_R \quad \text{or} \quad \Gamma_{\phi^4}\Big|_{s=t=u=\frac{4m_R^2}{3}} = -i\lambda_R$$

symmetric renormalization, since $s+t+u = 4m_R^2$.

or the minimal subtaction (MS) scheme

$$\Gamma_{\phi^4}\Big|_{\text{UV-part}} = 0 \quad , \quad \Gamma^{(1)}_s\Big|_{UV} = \Gamma^{(1)}_t\Big|_{UV} = \Gamma^{(1)}_u\Big|_{UV} = \frac{i\lambda_R^2}{32\pi^2}\ln\frac{\Lambda^2}{m_R^2}$$

In the MS scheme, we find

$$-i\,\delta\lambda^{(1)} + \frac{3i\lambda_R^2}{32\pi^2}\ln\frac{\Lambda^2}{m_R^2} = 0 \;\rightsquigarrow\; \delta\lambda^{(1)} = \frac{3\lambda_R^2}{32\pi^2}\ln\frac{\Lambda^2}{m_R^2}$$

Summary of 1-loop renormalization constants:

$$\delta\Lambda_c^{(1)} = -\frac{\Lambda^4}{64\pi^2}$$

$$Z_\phi^{(1)} = 1 \;,\; \delta Z_\phi^{(1)} = 0$$

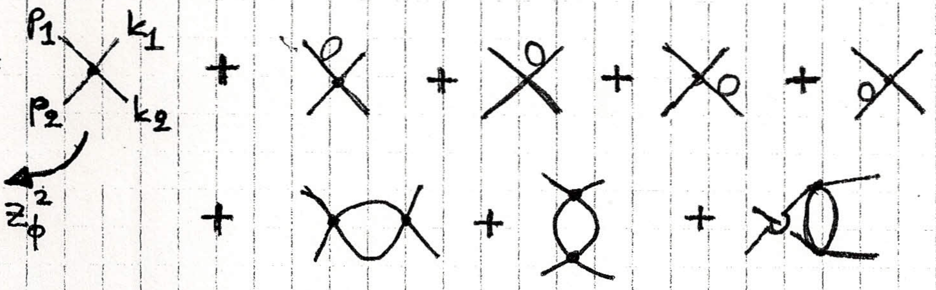
$$\delta m^{2(1)} = -\frac{\lambda_R}{32\pi^2}\left[\Lambda^2 - m_R^2\ln\frac{\Lambda^2}{m_R^2}\right]$$

$$\delta\lambda^{(1)} = \frac{3\lambda_R^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{m_R^2}\right)$$

Lecture 20

The $\overset{\text{1-loop}}{}$ transition amplitude $\phi(p_1)\phi(p_2) \mapsto \phi(k_1)\phi(k_2)$

$i\mathcal{T} =$



$-i\lambda_0 Z_\phi^2$ $\quad + \mathcal{O}(\hbar^2\lambda_0^3)$

$$= -i\left[\lambda_R + \delta\lambda^{(1)} + 4\times\left(\frac{\delta Z_\phi}{2}\right)\right] + 4\times$$

Remember $\delta Z_\phi = 0$

contribute only to mass ren. of the ϕ-field

see also LSZ formalism

$$+ I_s^{(1)} + I_t^{(1)} + I_u^{(1)}$$

where $I_s^{(1)} = \tilde{I}[(p_1+p_2)^2]$, $I_t^{(1)} = \tilde{I}[(p_1-k_1)^2]$

and $I_u^{(1)} = \tilde{I}[(p_1-k_2)^2]$.

$$\rightsquigarrow \; i\mathcal{T} = -i\lambda_R + i\delta\lambda^{(1)} + \tilde{I}(s) + \tilde{I}(t) + \tilde{I}(u)$$

where
$$\tilde{I}(p^2) = i\frac{\lambda_R^2}{32\pi^2}\ln\frac{\Lambda^2}{m_R^2} + \underbrace{\frac{i\lambda_R^2}{32\pi^2}\int_0^1 \frac{dx\,(1-x)(1-2x)\,p^2}{[m_R^2 - x(1-x)p^2 - i\varepsilon]}}_{\tilde{I}^{fin}(p^2) \leftarrow \text{UV-finite}}$$

Hence, the $\overset{\text{1-loop}}{}$ renormalized transition amplitude becomes

$$i\mathcal{T} = -i\lambda_R + \tilde{I}^{fin}(s) + \tilde{I}^{fin}(t) + \tilde{I}^{fin}(u)$$

Notice that all UV infinities were removed by $\delta\lambda^{(1)}$ and $\delta m^{(1)}$.
The transition amplitude $i\mathcal{T}$ only depends on λ_R and m_R, and it is thefore UV finite, i.e. the cut-off Λ-singularity is removed.

General remarks

- The UV finite part is independent of the regularization method. The same result would have been obtained for $\tilde{I}^{fin}(p^2)$, if we had used dimensional regularization, where the 4 dimensions are analytically extended to $4-2\varepsilon$. In this case, we get

$$\tilde{I}(p^2) = \frac{i\lambda_R^2}{32\pi^2}\left[\frac{1}{\varepsilon} - \overset{\text{Euler constant}}{\gamma_E} + \ln 4\pi - \ln\left(\frac{m_R^2}{\mu^2}\right)\right]$$

$$\underbrace{- \frac{i\lambda_R^2}{32\pi^2}\int_0^1 dx\left(\ln\frac{m_R^2 - x(1-x)p^2}{m_R^2}\right)}_{= \frac{i\lambda_R^2}{32\pi^2}\int_0^1 dx\,\frac{(1-x)(1-2x)p^2}{m_R^2 - x(1-x)p^2} \;=\; \tilde{I}^{fin}(p^2)}$$

after integrating by parts

- The correspondance between covariant cut-off regularization and dimensional regularization is as follows:

$$\frac{\lambda_R}{32\pi^2}\Lambda^2 \longleftrightarrow 0$$

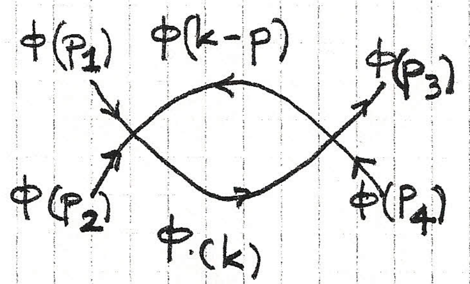
$$\frac{\lambda_R^2}{32\pi^2}\ln\frac{\Lambda^2}{m_R^2} \longleftrightarrow \frac{\lambda_R^2}{32\pi^2}\left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi\mu^2 + 1\right)$$

- The renormalized quantities λ_R, m_R^2 depend on the conditions of renormalization. However, physical observables, such as cross sections, are renormalization scheme independent.

- Finally, even a UV finite theory would have needed renormalization! (Why?)

MATHEMATICAL SUPPLEMENT

Dimensional regularization



; with $p = p_1 + p_2 = p_3 + p_4$; $p^2 = s$

$$I_s^{(1)} = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{(k^2 - m^2)[(k-p)^2 - m^2]}$$

$$= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)[(k-p)^2 - m^2]}$$

Use Feynman parameterization:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{[Ax + B(1-x)]^2}$$

to write

$$I_s^{(1)} = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[x(k^2 + p^2 - 2k\cdot p - m^2) + (1-x)(k^2 - m^2)]^2}$$

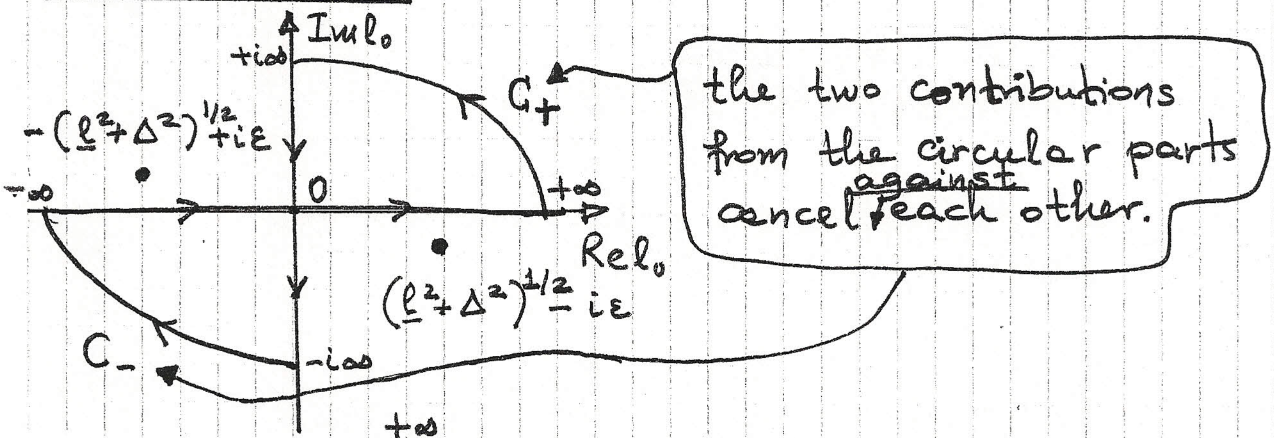
$$= \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{[\underbrace{k^2 - 2x k\cdot p + x p^2 - m^2 + i\varepsilon}_{(\underbrace{k - xp}_{\equiv \ell})^2 + x(1-x)p^2 - m^2 + i\varepsilon}]^2}$$

$$I_s^{(1)} = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - m^2 + x(1-x)(p^2 + i\varepsilon)]^2}$$

Integral to be calculated:

$$I(\Delta) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(\underbrace{l^2 - \Delta^2 + i\varepsilon}_{l_0^2 - [(\underline{l}^2+\Delta^2)^{1/2} - i\varepsilon]^2})^2} \quad ; \quad \Delta^2 = m^2 - x(1-x)p^2$$

Wick rotation:



$$I(\Delta) = \int \frac{d^3 \underline{l}}{(2\pi)^3} \underbrace{\int_{-\infty}^{+\infty} \frac{dl_0}{2\pi} \frac{1}{\{ l_0^2 - [(\underline{l}^2+\Delta^2)^{1/2} - i\varepsilon]\}^2}}$$

$$= \int_{-i\infty}^{i\infty} \frac{dl_0}{(2\pi)} \frac{1}{[l_0^2 - (\underline{l}^2+\Delta^2) + i\varepsilon]^2} \quad ; \quad l_0 = i l_4$$

$$= i \int_{-\infty}^{+\infty} \frac{dl_4}{(2\pi)} \frac{1}{[\underline{l}^2 + l_4^2 + \Delta^2 - i\varepsilon]^2}$$

$$\rightsquigarrow \quad I(\Delta) = i \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(|l_E|^2 + \Delta^2 - i\varepsilon)^2} \quad ; \quad l_E = (l_1, l_2, l_3, l_4)$$

The integral needs be calculated in the 4-dim Euclidean space.

The <u>Dimensional regularization</u> method consists in calculating $I(\Delta)$ in $\underline{n = 4 - 2\epsilon}$ dimensions, instead in 4, where ϵ is vanishingly small

This method was introduced by **G. 't Hooft**

The line element of <u>n-dim.</u> Euclidean space is

$$ds_n^2 = dr^2 + r^2 \underbrace{\left(d\theta_1^2 + \sin^2\theta_1 \, d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 \, d\theta_3^2 + \dots + \sin^2\theta_1 \sin^2\theta_2 \dots \sin^2\theta_{n-2} \, d\theta_{n-1}^2\right)}_{\equiv d\sigma_{n-1}^2}$$

$\leftarrow$ $\boxed{\text{line element of } S^{n-1}\text{; i.e. } (n-1)\text{-sphere.}}$

Hence,
$$\int d^n \ell_E = \int_0^{+\infty} |\ell_E|^{n-1} d|\ell_E| \int_0^{\pi} d\theta_1 \int_0^{\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_{n-1} \times (\sin\theta_1)^{n-2} (\sin\theta_2)^{n-3} \dots \sin\theta_1$$

$$= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{+\infty} |\ell_E|^{n-1} d|\ell_E|$$

<u>Gamma-function</u> (analytically continued)

$$\Gamma(z) \equiv \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,(n+z)} + \int_1^{+\infty} dt \; t^{z-1} e^{-t}$$

Properties of $\Gamma(z)$: (i) It has poles at $z = 0, -1, -2, \dots$

(ii) $\Gamma(z+1) = z\,\Gamma(z)$; $\Gamma(n+1) = n!$

(iii) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

(iv) $\Gamma(z) = \dfrac{1}{z} - \gamma_E + O(z)$

$\hookrightarrow$ <u>Euler constant</u>

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{1}{n} - \ln N\right) \equiv \gamma_E$$

We evaluate $I(\Delta)$ in $n=4-2\varepsilon$ Euclidean space:

$$I_{(4-2\varepsilon)}(\Delta) = i\,\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{+\infty} \frac{|l_E|^{n-1}d|l_E|}{(2\pi)^n}\,\frac{1}{(|l_E|^2+\Delta^2)^2}$$

$$= \frac{i}{\Gamma\left(\frac{n}{2}\right)2^n\pi^{n-n/2}} \underbrace{\int_0^{+\infty} \frac{(|l_E|^2)^{\frac{n}{2}-\frac{1}{2}}\,d|l_E|^2}{(|l_E|^2+\Delta^2)^2}}$$

see, e.g. Cheng + Li Eq. (2.84)

$$= \frac{1}{(\Delta^2)^{2-n/2}}\,\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(2-\frac{n}{2}\right)}{\underbrace{\Gamma(2)}_{\to 1}}$$

$$I_{(4-2\varepsilon)}(\Delta) = \frac{i\,(\Delta^2)^{-\varepsilon}}{16\pi^2(4\pi)^{\varepsilon}}\,\Gamma(\varepsilon)$$

$$= \frac{i}{16\pi^2}\,\underbrace{e^{+\varepsilon\ln 4\pi}}_{1+\varepsilon\ln 4\pi}\,\underbrace{e^{-\varepsilon\ln\Delta^2}}_{1-\varepsilon\ln\Delta^2}\left(\frac{1}{\varepsilon}-\gamma_E+\mathcal{O}(\varepsilon)\right)$$

$$\underline{I_{(4-2\varepsilon)}(\Delta) = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon}+\ln 4\pi-\gamma_E-\ln\Delta^2\right)}$$

Back to the original integral in $n=4-2\varepsilon$ dims:

$$I_s^{(1)}(s)\Big|_{n=4-2\varepsilon} = \frac{i\lambda^2\mu^{4\varepsilon}}{32\pi^2}\left(\frac{1}{\varepsilon}+\ln 4\pi-\gamma_E-\int_0^1 dx\,\ln\left(m^2-s\,x(1-x)\right)\right)$$

$\boxed{\mu\text{: t Hooft measure}}$

This is introduced, since λ is not dimensionless in $4-2\varepsilon$ dimensions,

i.e. $\lambda\phi^4 \to \lambda\mu^{2\varepsilon}\phi^4$; $[\phi]=1-\varepsilon$

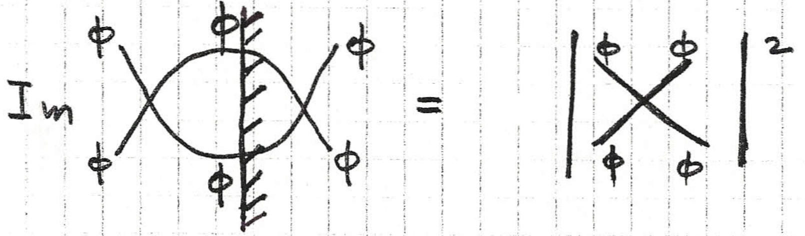
$\hookrightarrow \lambda=\lambda(\mu)$

$\boxed{\text{evolution governed by renormalization group eqs}}$

Final result :

$$I_s^{(1)}(s)\Big|_{n=4-2\varepsilon} = i(\lambda\mu^{2\varepsilon})\frac{\lambda}{32\pi^2}\left[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E - \int_0^1 dx\, \ln\left(\frac{m^2 - s\,x(1-x) - i\varepsilon}{\mu^2}\right)\right]$$

If $s > 4m^2$, the log-function has an imaginary part related to the on-shell production of $\phi\phi$ particles, i.e.



The above is also consistent with the <u>optical theorem</u>.
