Lecture 7

Non-Abelian Gauge Theories (or Yang-Mills theories)

These are gauge theories based on non-Abelian groups, which are compact, e.g. $SU(N)$, $SO(N>2)$ etc.

$\therefore$ $g_{ab} \mapsto \hat{g}_{ab} = \delta_{ab} = \hat{g}_{ab}$, $T_a = T^a$ and $f_{abc} = f^{abc}$.

Remember the gauge transf. in QED:

$U(1)$: $\quad A_\mu \mapsto A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta = U_1 A_\mu U_1^* + \frac{1}{ie} U_1 \partial_\mu U_1^*$, with $\underset{\underset{U(1)}{\cap}}{U_1 = e^{+i\theta}}$

Then, for $SU(N)$ gauge transfs.

$SU(N)$: $\quad \underline{A}_\mu \,\hat{=}\, A^a_\mu T^a \mapsto \underline{A}'_\mu = U \underline{A}_\mu U^\dagger + \frac{1}{ig} U \partial_\mu U^\dagger$, with $\underset{\underset{SU(N)}{\cap}}{U = e^{+i\theta^a T^a}}$;

N^2-1 gauge fields A^a_μ

gauge coupling of YM theory

$\theta^a = \theta n^a \in \mathbb{R}$

For global $SU(N)$ ($\theta^a = \text{const}$), we have

$$\underline{A}_\mu \mapsto \underline{A}'_\mu = U \underline{A}_\mu U^\dagger \quad \text{or} \quad A'^a_\mu (T^a)_i{}^j = U_i{}^k U^j{}_l A^a_\mu (T^a)_k{}^l = U_i{}^k U^{+j}_l A^a_\mu (T^a)_k{}^l$$

with $U_i{}^j \,\hat{=}\, U_{ij}$, $U^i{}_j \,\hat{=}\, U^*_{ij}$ and $U_k{}^i U^k{}_j = U^i{}_k U_j{}^k = \delta^i_j$

$\therefore$ Under global $SU(N)$ transfs, the YM field multiplet: $\underline{A}_\mu = A^a_\mu T^a$ transforms as a rank-2 $\binom{1}{1}$ $SU(N)$ tensor.

YM field strength tensor(s):

$$\underline{F}_{\mu\nu} \,\hat{=}\, F^a_{\mu\nu} T^a = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + ig\,[\underline{A}_\mu, \underline{A}_\nu]$$

$$= \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a + ig \underbrace{[A^b_\mu T^b, A^c_\nu T^c]}_{= A^b_\mu A^c_\nu [T^b, T^c]}$$

$$[T^b, T^c] = i f^{bca} T^a, \quad f^{bca} = f^{abc}$$

$$= \underbrace{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\right)}_{= F^a_{\mu\nu}} T^a \; ; \; a = 1, 2, \ldots, N^2-1$$

Under a SU(N) gauge transf., the YM multiplet strength tensor $\underline{F}_{\mu\nu} = F^a_{\mu\nu} T^a$ transforms as

$$SU(N): \quad \underline{F}_{\mu\nu} \mapsto \underline{F}'_{\mu\nu} = U \underline{F}_{\mu\nu} U^+ \quad , \text{with} \quad U \in SU(N)$$

Proof:

$$SU(N): \quad \underline{A}_\mu \mapsto \underline{A}'_\mu = U \underline{A}_\mu U^+ + \frac{1}{ig} U \partial_\mu U^+$$

Note that $\partial_\mu U^+ = - U^+(\partial_\mu U) U^+$, resulting from $\partial_\mu (U^+ U) = \partial_\mu \mathbf{1}_N = 0$

$$\underline{F}_{\mu\nu} \mapsto \underline{F}'_{\mu\nu} = \partial_\mu \underline{A}'_\nu - \partial_\nu \underline{A}'_\mu + ig [\underline{A}'_\mu, \underline{A}'_\nu]$$

$$= U(\partial_\mu \underline{A}_\nu) U^+ - U(\partial_\nu \underline{A}_\mu) U^+ + ig \underbrace{[U \underline{A}_\mu U^+, U \underline{A}_\nu U^+]}_{= U [\underline{A}_\mu, \underline{A}_\nu] U^+}$$

global SU(N) terms

$$+\delta \underline{F}^{(1)}_{\mu\nu} = \left\{ + \frac{1}{ig} \underbrace{\left[\partial_\mu (U \partial_\nu U^+) - \partial_\nu (U \partial_\mu U^+)\right]}_{= (\partial_\mu U)\cancel{(\partial_\nu U^+)} - \cancel{(\partial_\nu U)}(\partial_\mu U^+)} + \underbrace{\frac{ig}{(-g^2)}}_{= +1/ig} \underbrace{[U \partial_\mu U^+, U \partial_\nu U^+]}_{\substack{= U(\partial_\mu U^+) U (\partial_\nu U^+) - U(\partial_\nu U^+) U (\partial_\mu U^+) \\ = -(\partial_\mu U)\cancel{(\partial_\nu U^+)} + \cancel{(\partial_\nu U)}(\partial_\mu U^+)}} \right.$$

$$+\delta \underline{F}^{(2)}_{\mu\nu} = \left\{ \begin{array}{l} + \cancel{(\partial_\mu U) \underline{A}_\nu U^+} + \cancel{U \underline{A}_\nu (\partial_\mu U^+)} - (\mu \leftrightarrow \nu) \\ + \underbrace{[U \partial_\mu U^+, U \underline{A}_\nu U^+]}_{= \underbrace{U(\partial_\mu U^+) U}_{= -\partial_\mu U} \underline{A}_\nu U^+ - U \underline{A}_\nu \underbrace{U^+ U}_{= \mathbf{1}_N} (\partial_\mu U^+)} - (\mu \leftrightarrow \nu) \end{array} \right.$$

$$\rightsquigarrow \underline{F}'_{\mu\nu} = U \underline{F}_{\mu\nu} U^+ + \delta \underline{F}^{(1)}_{\mu\nu} + \delta \underline{F}^{(2)}_{\mu\nu} \quad , \text{with } \delta \underline{F}^{(1)}_{\mu\nu} = \delta \underline{F}^{(2)}_{\mu\nu} = 0$$

$$\therefore \quad \underline{F}'_{\mu\nu} = U \underline{F}_{\mu\nu} U^+ \qquad \text{q.e.d.}$$

The YM Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{2}\,\mathrm{Tr}\left(\underline{F}_{\mu\nu}\underline{F}^{\mu\nu}\right) = -\frac{1}{2}\,F^a_{\mu\nu}F^{b,\mu\nu}\underbrace{\mathrm{Tr}\left(T^aT^b\right)}_{=+\frac{1}{2}\delta^{ab}}$$

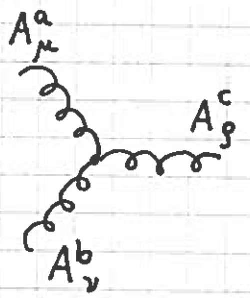
$$= -\frac{1}{4}\,F^a_{\mu\nu}F^{a,\mu\nu}$$

$\mathcal{L}_{YM}$ is invariant under SU(N) gauge (local) transfs. (Why?)

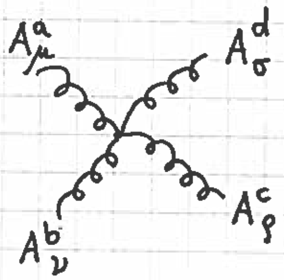
↔

REMARKS

- SU(N) YM theories predict N^2-1 gauge bosons (Why?)
- Consequently, the $SU(2)_L$ group of the SM predicts 3 weak bosons W^i_μ ($i=1,2,3$) which are responsible for the electroweak force.
- Quantum Chromodynamics (QCD) based on the $SU(3)_c$ group predicts 8 gluons $A^a_\mu \mathrel{\widehat{=}} G^a_\mu$ ($a=1,2,\dots,8$) mediating the strong force between quarks (and gluons).
- YM gauge fields self-interact!



$\propto \; g\, f^{abc}$



$\propto \; g^2 f^{xab} f^{xcd}$, (+cyclic perms of b,c,d)

Lecture 8

Interaction between Quarks q_i and Gluons G^a_μ in $SU(3)_c$

$SU(3)_c$: the colour group of Quantum $\underbrace{\text{Chromo}}$dynamics (QCD)

(Chroma = Χρῶμα = Colour

$q_i = (q_{red}, q_{green}, q_{blue})$: $\underbrace{\text{quarks}}$ in the fundamental rep. of $SU(3)_c$.

termed by M. Gell-Mann in '64.

Dirac fermions

G^a_μ $(a = 1, 2, \cdots, 8)$: Gluons

Like in QED, the interaction $\bar{q}_i - q_j - G^a_\mu$ is described by the Lagrangian

Dirac matrices

$$\mathcal{L}_q = \bar{q}_i \left[i \not{\partial} \delta_{ij} - m_q \delta_{ij} - g_s \not{G}^a_\mu (T^a)_{ij} \right] q_j \quad ; \quad \not{a} \mathrel{\hat{=}} \gamma^\mu a_\mu$$

$$\bar{q} \mathrel{\hat{=}} q^\dagger \gamma^0$$

$$= \bar{q}\, i\gamma^\mu \underbrace{\left[\mathbf{1}_3 \partial_\mu + i g_s \underline{G}_\mu \right]}_{\hat{=}\, D_\mu} q - m_q \bar{q} q$$

SU(3)$_c$ covariant derivative.

T^a : generators of $SU(3)_c$ in the fundamental rep,

i.e. $T^a = \frac{\lambda^a}{2}$ where λ^a are the so-called Gell-Mann matrices

see lecture notes for their explicit form, p.29

To prove $\mathcal{L}_q$ is invariant under $SU(3)_c$ gauge transfs, we first show that

$$SU(3)_c : \quad q \mapsto q' = \mathcal{U} q \quad , \text{ with } \mathcal{U} = e^{i\theta^a T^a} \in SU(3)$$

$$\underline{\text{and}} \quad D_\mu q \mapsto D'_\mu q' = \mathcal{U} D_\mu q$$

Proof: The transf. for $q = \{q_i\}$ in the fundamental rep. is self-evident.

Given that $\underline{A}'_\mu = \mathcal{U} \underline{A}_\mu \mathcal{U}^\dagger + \frac{1}{ig} \mathcal{U} \partial_\mu \mathcal{U}^\dagger$ for an SU(N) gauge multiplet,

$$D'_\mu q' = \left(\mathbf{1}_3 \partial_\mu + i g_s \mathcal{U} \underline{G}_\mu \mathcal{U}^\dagger + \mathcal{U} \underbrace{\partial_\mu \mathcal{U}^\dagger}_{= -\mathcal{U}^\dagger (\partial_\mu \mathcal{U}) \mathcal{U}^\dagger} \right) \mathcal{U} q$$

$$= \mathcal{U} \left(\mathbf{1}_3 \partial_\mu + i g_s \underline{G}_\mu \right) q + \underbrace{\left(\partial_\mu \mathcal{U} - \cancel{\mathcal{U}\mathcal{U}^\dagger} (\partial_\mu \mathcal{U}) \mathcal{U}^\dagger \mathcal{U} \right)}_{0} q = \mathcal{U} D_\mu q$$

q.e.d.

It is now not difficult to show that $\mathcal{L}_q$ is invariant under $SU(3)_c$ gauge transfs. Indeed, under $SU(3)_c$ local transfs, we have

$$SU(3)_c: \quad \mathcal{L}_q \mapsto \mathcal{L}'_q = \bar{q}' i\gamma^\mu D'_\mu q' - m_q \bar{q}' q' ,$$

with

$$q \mapsto q' = U q$$

$$\bar{q} \mapsto \bar{q}' = q'^{+}\gamma^0 = q^{+}U^{+}\gamma^0 = q^{+}\gamma^0 U^{+} = \bar{q}\, U^{+}$$

$$D_\mu q \mapsto D'_\mu q' = U D_\mu q$$

γ^μ and U^+ commute, as they act on two different group spaces. ⊛

$$\text{Hence, } \mathcal{L}'_q = \bar{q}\, U^{+} \overbrace{i\gamma^\mu U}^{= U\gamma^\mu} D_\mu q - m_q \bar{q}\, U^{+} U q$$

$$= \bar{q}\, i\gamma^\mu D_\mu q - m_q \bar{q} q = \mathcal{L}_q \qquad \text{q.e.d.}$$

⊛ **Mathematical supplement: Direct product of Groups**

A justification of the fact that $[\gamma^\mu, U] = 0$, with $\gamma^\mu \in GL(4,\mathbb{C})$ and $U \in SU(3)_c$ may be obtained by the notion of direct product or tensor product of groups.

$SU(3)_c \cap GL(4,\mathbb{C}) = \emptyset$

Loosely speaking, if A and B are two matrices belonging to two different group spaces, e.g. $A \in GL(n,\mathbb{C})$ and $B \in GL(m,\mathbb{C})$, then one can define a new matrix $M = A \otimes B \in GL(n{\cdot}x\, m, \mathbb{C})$, with the property: $M_1 \cdot M_2 = (A_1 \otimes B_1) \cdot (A_2 \otimes B_2) = (A_1 \cdot A_2) \otimes (B_1 \cdot B_2) \in GL(n x m, \mathbb{C})$

$\forall A_1, A_2 \in GL(n,\mathbb{C})$ and $B_1, B_2 \in GL(m,\mathbb{C})$.

E.g., $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

tensor product: $A \otimes B \mathrel{\hat{=}} \begin{bmatrix} a_1 B & a_2 B \\ a_3 B & a_4 B \end{bmatrix}$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_2 b_1 & a_2 b_2 \\ a_1 b_3 & a_1 b_4 & a_2 b_3 & a_2 b_4 \\ a_3 b_1 & a_3 b_2 & a_4 b_1 & a_4 b_2 \\ a_3 b_3 & a_3 b_4 & a_4 b_3 & a_4 b_4 \end{bmatrix} \neq B \otimes A .$$

Why?

Mathematical supplement (continued)

Hence, making use of tensor products, one may write γ^μ as $\gamma^\mu \otimes \mathbf{1}_3$, with $\mathbf{1}_3 \in SU(3)_c$, and U as $\mathbf{1}_4 \otimes U$, with $\mathbf{1}_4 \in GL(4,\mathbb{C})$ related to Dirac spinor space.

Evidently, we have

$$[\gamma^\mu, U] \mapsto [\gamma^\mu \otimes \mathbf{1}_3, \mathbf{1}_4 \otimes U] = (\gamma^\mu \mathbf{1}_4) \otimes (\mathbf{1}_3 U) - (\mathbf{1}_4 \gamma^\mu) \otimes (U \mathbf{1}_3) = 0$$

The notion of <u>tensor product</u> may generalize to <u>more than two</u> groups or matrices as follows:

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n) \cdot (B_1 \otimes B_2 \otimes \cdots \otimes B_n)$$

$$= (A_1 \cdot B_1) \otimes (A_2 \cdot B_2) \otimes \cdots \otimes (A_n \cdot B_n)$$

where pairwise $(A_1, B_1), (A_2, B_2) \ldots (A_n, B_n)$ belong to the same group spaces.

Note that a rep of the generators of the tensor group $G = G_A \otimes G_B$ and the corresponding Lie algebra $L = L_A \otimes L_B$ is given by

$$G(\theta^A, \theta^B) = e^{i\theta^A T^A \otimes \mathbf{1}_B + i\theta^B \mathbf{1}_A \otimes T_B} \quad ; \quad T^A \subset L_A, \; T^B \subset L_B$$

The generators of $G = G_A \otimes G_B$ are $T^A \otimes \mathbf{1}_B$, $\mathbf{1}_A \otimes T^B$, where $\mathbf{1}_A$ and $\mathbf{1}_B$ are the identity matrices of G_A and G_B, respectively.

<u>Exercise</u>: Show that $G(\theta^A, \theta^B)$ can also be written as

$$G(\theta^A, \theta^B) = e^{i\theta^A T^A} \otimes e^{i\theta^B T^B}$$

Lecture 9

Gauge Fixing in Yang-Mills (YM) Theories

To obtain a non-singular gauge-field propagator $\Delta^{ab}_{\mu\nu}(x-y)$ in YM theories, we proceed as in QED and add to $\mathcal{L}_{YM} = -\frac{1}{4} F^{a}_{\mu\nu} F^{a,\mu\nu}$ the gauge-fixing term $\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A^{a,\mu})(\partial_\nu A^{a,\nu})$

$\mathcal{L}_{GF}$ breaks explicity the SU(N) gauge symmetry, but removes the unphysical degrees of freedom related to longitudinal and time-like components of the YM field A^a_μ, i.e A^a_0 and A^a_3.

E-L equation of motion for A^a_μ derived from $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{GF}$ in the limit $g \to 0$:

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} \right] - \frac{\partial \mathcal{L}}{\partial A^a_\nu} = 0 \rightsquigarrow \left[\eta_{\mu\nu} \partial_k \partial^k - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] A^{a,\nu} = 0$$

The YM field propagator is given by the Green's function:

$$\left[\eta^{\mu\nu} \partial_k \partial^k - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] \Delta^{ab}_{\nu\lambda}(x-y) = \delta^{ab} \delta^\mu_\lambda \, \delta^{(4)}(x-y)$$

Set $\Delta^{ab}_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \tilde{\Delta}^{ab}_{\mu\nu}(k) \, e^{-ik\cdot(x-y)}$; $\tilde{\Delta}^{ab}_{\mu\nu}(k) = A^{ab}(k)\eta_{\mu\nu} + B^{ab}(k) k_\mu k_\nu$

In k^μ-space:

$$\left[-\eta^{\mu\nu} k^2 + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \right] \left(A^{ab}(k) \eta_{\nu\lambda} + B^{ab}(k) k_\nu k_\lambda \right) = \delta^{ab} \delta^\mu_\lambda$$

$$\rightsquigarrow \; -A^{ab} k^2 \delta^\mu_\lambda + k^\mu k_\lambda \overbrace{\left[\left(1 - \frac{1}{\xi}\right)\left(A^{ab} + k^2 B^{ab}\right) - k^2 B^{ab} \right]}^{=0} = \delta^{ab} \delta^\mu_\lambda$$

$$\rightsquigarrow \; A^{ab} = -\frac{1}{k^2} \delta^{ab} \quad \text{and} \quad B^{ab} = (\xi - 1) \frac{A^{ab}}{k^2} = \frac{1-\xi}{k^4} \delta^{ab}.$$

$$\therefore \quad \Delta^{ab}_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \left(-\eta_{\mu\nu} + (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \frac{\delta^{ab} e^{-ik\cdot(x-y)}}{k^2 + i\varepsilon} \; ; \; \varepsilon = 0^+.$$

Note that $\Delta^{ab}_{\mu\nu} \mapsto \infty$, as $\xi \mapsto \infty$ (unitary gauge)

Note that $\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A^{a,\mu})(\partial_\nu A^{a,\nu}) = -\frac{1}{\xi}\,\mathrm{Tr}\left[(\partial_\mu \underline{A}^\mu)(\partial_\nu \underline{A}^\nu)\right]$

Hence, $\mathcal{L}_{GF}$ is invariant only under global SU(N) transfs:

$$\underline{A}_\mu \mapsto \underline{A}'_\mu = \mathcal{U}\,\underline{A}_\mu\,\mathcal{U}^\dagger \quad , \text{ with } \partial_\mu \mathcal{U} = 0 \text{ and } \mathcal{U} \in SU(N) \quad (\text{Why?})$$

Q: Is there any way to restore local SU(N) symmetry at least at the infinitesimal level? (i.e. only to leading order in θ^a)

To do so, we introduce in the theory new Grassman-valued complex fields c^a and $\bar{c}^a$, the so-called Fadeev-Popov (FP) ghosts. The fields $c^a(x)$ and $\bar{c}^a(x)$ are bosons, but they satisfy anticommutation relations: $c^a(x_1)\,\overset{(-)}{c}{}^b(x_2) = -\overset{(-)}{c}{}^b(x_2)\,c^a(x_1)$, $c^a(x)\,\overset{(-)}{c}{}^b(x) = -\overset{(-)}{c}{}^b(x)\,c^a(x)$, and $\overset{(-)}{c}{}^a(x)\,\overset{(-)}{c}{}^b(x) = 0$ (as classical fields, not field operators).

This results in a new Lagrangian term for the FP ghosts:

$$\mathcal{L}_{FP} = -\bar{c}^a\,\partial^\mu\left[\delta^{ab}\partial_\mu + g f^{abc} A^c_\mu\right] c^b \;.$$

Then, the full YM Lagrangian: $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{GF} + \mathcal{L}_{FP}$ is invariant under the Becchi-Rouet-Stora (BRS) transfs:

$$\boxed{\begin{aligned} \delta A^a_\mu &\mathrel{\hat{=}} \omega\, s A^a_\mu = \omega\left[\delta^{ab}\partial_\mu + g f^{abc} A^c_\mu\right] c^b \\ \delta c^a &\mathrel{\hat{=}} \omega\, s c^a = \omega\,\frac{1}{2}\, g f^{abc} c^b c^c \\ \delta \bar{c}^a &\mathrel{\hat{=}} \omega\, s\bar{c}^a = -\omega\,\frac{1}{\xi}\,\partial^\mu A^a_\mu \quad , \text{ with } \omega^2 = 0 \text{ and } \partial_\mu \omega = 0. \end{aligned}}$$

See Ex III.3 (iv)

REMARK:

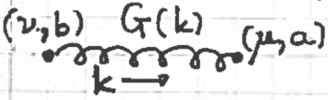
The BRS symmetry ensures unitarity and renormalizability of YM gauge theories, including spontaneously broken gauge theories, such as the Standard Model.

QCD Feynman rules

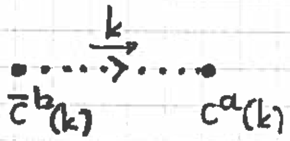
The complete QCD Lagrangian reads:

$$\boxed{T^a = \frac{\lambda^a}{2}}$$

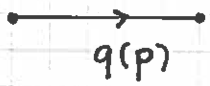
$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + \bar{q}_i \left[i \not\partial \delta_{ij} - m_q \delta_{ij} - g_s \not G^a (T^a)_{ij} \right] q_j$$
$$- \frac{1}{2\xi} (\partial_\mu G^{a,\mu})(\partial_\nu G^{a,\nu}) - \bar{c}^a \partial^\mu \left[\delta^{ab} \partial_\mu + g_s f^{abc} G^c_\mu \right] c^b$$



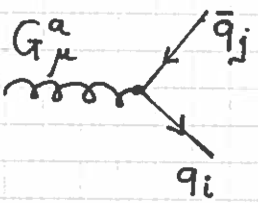
$$: \quad \frac{i\delta^{ab}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)}{k^2 + i\varepsilon}$$



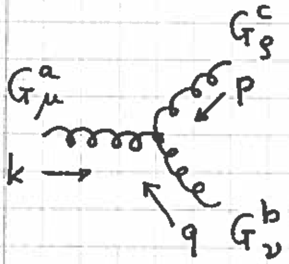
$$: \quad \frac{i\,\delta^{ab}}{k^2 + i\varepsilon}$$



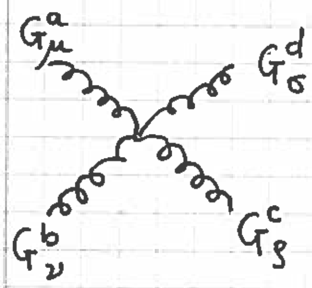
$$: \quad \frac{i}{\not p - m_q + i\varepsilon}$$



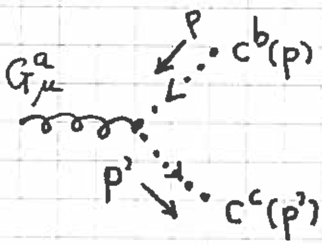
$$: \quad -i g_s \gamma_\mu \frac{(\lambda^a)_{ij}}{2}$$



$$: \quad -g_s f^{abc} \left[\eta^{\mu\nu}(k-q)^\rho + \eta^{\nu\rho}(q-p)^\mu + \eta^{\rho\mu}(p-k)^\nu \right]$$



$$: \quad -i g_s^2 \Big[f^{xab} f^{xcd} \left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} \right)$$
$$+ f^{xac} f^{xdb} \left(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma} \right)$$
$$+ f^{xad} f^{xbc} \left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} \right) \Big]$$



$$: \quad -g_s f^{abc} p'_\mu$$

Lecture 10

Revision of renormalization of ϕ^4-theory:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!}\lambda_0 \phi_0^4 + \Lambda_c^0$$

we ignore ren. of the cosmological constant

where $\phi_0 = Z_\phi^{1/2} \phi = \left(1 + \frac{1}{2}\delta Z_\phi\right)\phi = \phi + \delta\phi$

$$x_0 = Z_x x = (1 + \delta Z_x) = x + \delta x \quad ; \quad x \in \{m^2, \lambda, \Lambda_c\}$$

ϕ_0 : bare (unrenormalized) field

x_0 : bare (unrenormalized) parameter

$Z_\phi^{1/2}$: Wave-function renormalization of ϕ

ϕ: renormalized field

x: renormalized parameter

$\left.\begin{matrix}\delta\phi \\ \delta x\end{matrix}\right\}$: Counter-term (CT) of renormalization for : $\left\{\begin{matrix}\phi \\ x\end{matrix}\right.$

Renormalization programme:

(i) Calculate One-Particle-Irreducible (1PI) loop graphs $\Gamma^{(n)}$ using $\mathcal{L}_0$ only.

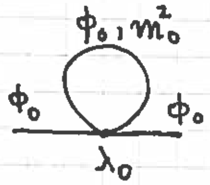
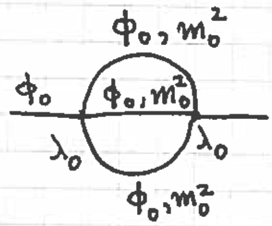
(ii) Define renormalization conditions to determine the CTs $\delta\phi = \delta\phi^{(1)} + \delta\phi^{(2)} + \dots$ and $\delta x = \delta x^{(1)} + \delta x^{(2)} + \dots$ through the desirable loop order. Note that for a renormalizable theory, the number of ren. conditions is finite.

(iii) Calculate physical observables, such as S-matrix elements, through loop order (l), using $\mathcal{L}_0$ only. Eliminate the Ultra-Violet (UV) infinities of the loop graphs against the UV infinities of $\delta\phi$ and δx contained in ϕ_0 and x_0, when the latter expanded to the appropriate loop order ($\leq l$).

(iv) Do not include loop corrections to the asymptotic (in and out) states of the S-matrix element to avoid double counting from $Z_\phi^{1/2}$.

• Mass and wave-function renormalization ($\delta Z_\phi, \delta m^2$)

1PI self-energy:

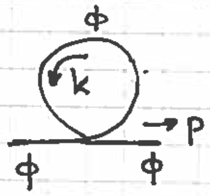
$$\Gamma^{(2)}(p^2) = i Z_\phi (p^2 - m^2 - \delta m^2) + $$  $$+$$ 

(1-loop) (2-loop)

On-mass shell (OS) Renormalization conditions:

$$\Gamma^{(2)}(p^2 = m^2) \overset{!}{=} 0$$

$$\frac{1}{i}\frac{d\Gamma^{(2)}}{dp^2}(p^2=m^2) \overset{!}{=} 1$$

$\Gamma^{(2)}_{(1\text{-loop})}(p^2)$:  $$= -\frac{i\lambda}{2}\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}$$

$\boxed{\text{Wick rotation to Euclidean space: } k^0 \mapsto ik_E^0}$

$$= \frac{\lambda}{2}\int\frac{i\,d^4k_E}{(2\pi)^4}\frac{1}{-k_E^2-m^2} = -\frac{i\lambda}{2}\int_0^{\Lambda^2}\frac{\pi^2 k_E^2\,dk_E^2}{(2\pi)^4}\frac{1}{k_E^2+m^2}$$

$\boxed{d^4k_E = 2\pi^2 k_E^3\,dk_E = \pi^2 k_E^2\,dk_E^2}$

$$= -\frac{i\lambda}{32\pi^2}\left[\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\right]$$

where Λ is the UV cut-off.

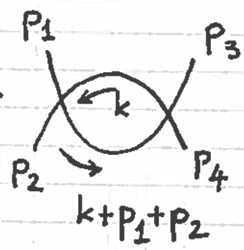
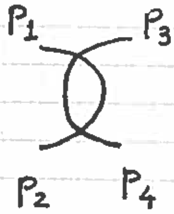
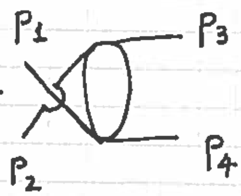
$\boxed{\text{independent of } p^2}$

$$\therefore \quad Z_\phi = 1, \text{ or } \delta Z_\phi^{(1)} = 0 \quad \text{and} \quad \delta m^{2(1)} = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\right]$$

Beyond 1-loop, $\delta Z_\phi^{(2)} \neq 0$.

$\longleftrightarrow$

• Quartic coupling renormalization ($\delta\lambda$)

$$\Gamma^{(4)}(p_i) = \underbrace{-iZ_\phi^2\lambda_0}_{=\lambda+\delta\lambda^{(1)}} + $$  $$+$$  $$+$$ 

$$\underbrace{}_{\hat{=}\,\tilde{\Gamma}[\underbrace{(p_1+p_2)^2}_{\hat{=}\,s}]} \qquad \underbrace{}_{=\tilde{\Gamma}(t)} \qquad \underbrace{}_{=\tilde{\Gamma}(u)}$$

Mandelstam variables:

$s = (p_1+p_2)^2$

$t = (p_1-p_3)^2$

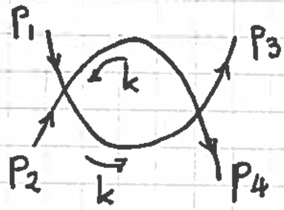
$u = (p_1-p_4)^2$

Renormalization conditions for $\delta\lambda$

There are several **renormalization** (ren) **schemes**:

- (i) IR ren. : $\Gamma^{(4)}(p_i=0) = -i\lambda$
- (ii) Symmetric ren. : $\Gamma^{(4)}\left(s=t=u=\frac{4m^2}{3}\right) = -i\lambda$
- (iii) Minimal Subtraction (MS) ren. : $\Gamma^{(4)}(p_i)\Big|_{\text{UV-part}} = 0$

We use (i) the IR scheme ($: p_{1,2,3,4} \mapsto 0$):



$$= \tilde{\Gamma}(0) = \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i^2}{(k^2-m^2+i\varepsilon)^2}$$

$$= \frac{i\lambda^2}{2}\int\frac{d^4k_E}{(2\pi)^4}\,\frac{1}{(k_E^2+m^2)^2} = \frac{i\lambda^2}{32\pi^2}\underbrace{\int\frac{k_E^2\,dk_E^2}{(k_E^2+m^2)^2}}$$

$$\boxed{= \ln\frac{\Lambda^2}{m^2} + m^2\left(\frac{1}{\Lambda^2}-\frac{1}{m^2}\right) \simeq \ln\frac{\Lambda^2}{m^2} - 1}$$

Hence, $\Gamma^{(4)}(0) = -i\lambda - i\delta\lambda^{(1)} + 3\tilde{\Gamma}(0) \overset{!}{=} -i\lambda \rightsquigarrow \delta\lambda^{(1)} = -3i\tilde{\Gamma}(0)$

$$\rightsquigarrow \delta\lambda^{(1)} = \frac{3\lambda^2}{32\pi^2}\left[\ln\frac{\Lambda^2}{m^2} - 1\right]$$

SUMMARY OF 1-LOOP RENORMALIZATION CONSTANTS (CTs):

$\delta\Lambda_c^{(1)} = -\frac{\Lambda^4}{64\pi^2}$ (not explicitly discussed in the lectures)

$$\left.\begin{aligned} &Z_\phi^{(1)} = 1, \ \text{or} \ \ \delta Z_\phi^{(1)} = 0 \\ &\delta m^{2\,(1)} = -\frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\right) \end{aligned}\right\} : \text{ OS scheme}$$

$\delta\lambda^{(1)} = \frac{3\lambda^2}{32\pi^2}\left[\ln\frac{\Lambda^2}{m^2} - 1\right] \longleftarrow$ IR scheme

Lecture 11

Dimensional Regularization (DR) is a regularization scheme based on the analytical continuation from 4 dimensions to $4-2\varepsilon$, after Wick's rotation in Euclidean space, where $\varepsilon \in \mathbb{C}$ and $|\varepsilon| \ll 1$. (see Mathematical Supplement for details.)

DR scheme is related to the Λ cut-off scheme as follows:

$$\Lambda^2 \longmapsto 0$$ (: no hard mass terms are introduced in DR)

$$\ln\frac{\Lambda^2}{m^2} \longleftrightarrow \frac{1}{\varepsilon} + 1 - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m^2}$$

where $\gamma_E \triangleq \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \approx 0.5772\ldots$ is the Euler-Mascheroni constant and μ is an arbitrary mass scale introduced by 't Hooft (who also introduced DR scheme, along with the MS scheme).

In the DR scheme, the one-loop CTs read:

$$Z_\phi^{1/2(0)} = \mu^{-\varepsilon} = e^{-\varepsilon\ln\mu} = 1 - \varepsilon\ln\mu + \mathcal{O}(\varepsilon^2),$$

$$m_0^2 = Z_{m^2}\, m^2 = m^2 + \delta m^{2(1)} + \delta m^{2(2)} + \ldots,$$

$$\text{with } \delta m^{2(1)} = +\frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\varepsilon} + 1 - \gamma_E + \ln\frac{4\pi\mu^2}{m^2}\right)$$

Note that in $4-2\varepsilon$ dims $[\phi_0] = 1-\varepsilon$, but $[\phi] = 1$. Also, $[m_0^2] = [m^2] = 2$ and $[\lambda_0] = 2\varepsilon$, with $[\lambda] = 0$. (Why?)

$$\lambda_0 = Z_\lambda \lambda = \mu^{2\varepsilon}\lambda + \delta\lambda^{(1)} + \delta\lambda^{(2)} + \ldots,$$

$$\text{with } \delta\lambda^{(1)} = (\mu^{2\varepsilon}\lambda)\frac{3\lambda}{32\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2}\right]$$

Theorem (without proof)

To <u>all orders</u> in perturbation theory, the renormalized 1PI effective action Γ does not depend on the UV cut-off scale Λ or the 't Hooft mass scale μ in the MS scheme.

For our ϕ^4-theory, the theorem implies

$$\phi^n(\mu)\,\Gamma^{(n)}\left[\lambda(\mu), m(\mu), \mu\right] = \phi^n(\mu_0)\,\Gamma^{(n)}\left[\lambda(\mu_0), m(\mu_0), \mu_0\right],$$

where $\Gamma^{(n)}$ is the <u>n-point</u> (or ϕ^n) 1PI <u>Green's function</u> (or otherwise called the <u>corellation function</u>), and μ and μ_0 are two arbitrary renormalization scales, with μ_0 being some reference scale.

Evidently, the renormalized field ϕ and the parameters λ and m^2 depend on μ through (DR scheme):

$$\phi_0(\varepsilon) = Z_\phi^{1/2}(\mu,\varepsilon)\,\phi(\mu)\,,\quad \lambda_0(\varepsilon) = Z_\lambda(\mu,\varepsilon)\,\lambda(\mu) \quad\text{and}\quad m_0^2(\varepsilon) = Z_{m^2}(\mu,\varepsilon)\,m^2(\mu)\,.$$

To describe this μ-dependence of ϕ, λ and m^2, we define the <u>dimensionless</u> parameters:

$$\gamma_\phi \mathrel{\hat=} \mu\frac{d\ln\phi(\mu)}{d\mu} = -\frac{1}{2}\mu\frac{d\ln Z_\phi}{d\mu}$$

$$\beta_\lambda \mathrel{\hat=} \mu\frac{d\lambda(\mu)}{d\mu} = -\mu\frac{d\ln Z_\lambda}{d\mu}\lambda$$

$$\gamma_{m^2} \mathrel{\hat=} \mu\frac{d\ln m^2(\mu)}{d\mu} = -\mu\frac{d\ln Z_{m^2}}{d\mu}$$

<u>See also Ex III.5(1).</u>

Knowing $\gamma_\phi(\mu)$, e.g. from perturbation theory, we can solve the 1st order differential equation,

$$\mu \frac{d\ln\phi}{d\mu} = \gamma_\phi \rightsquigarrow \ln\left(\frac{\phi(\mu)}{\phi(\mu_0)}\right) = \int_{\mu_0}^{\mu} d\ln\mu' \, \gamma_\phi(\mu')$$

$$\rightsquigarrow R(\mu;\mu_0) \,\hat{=}\, \frac{\phi(\mu)}{\phi(\mu_0)} = e^{\int_{\mu_0}^{\mu} d\ln\mu' \, \gamma_\phi(\mu')} \leftarrow \boxed{\text{it depends on } \mu}$$

$\gamma_\phi(\mu)$ is called the <u>anomalous dimension</u> of ϕ that goes beyond the naive <u>classical scaling</u> at the tree level, i.e. $[\phi] = 1$.

Now, the relation between two Green's functions renormalized at two different scales μ and μ_0 is given by

$$\Gamma^{(n)}[\mu] = \frac{\phi^n(\mu_0)}{\phi^n(\mu)} \Gamma^{(n)}[\mu_0] = R^{-n}(\mu;\mu_0)\, \Gamma^{(n)}[\mu_0] .$$

In principle, one could define the composition law of successive renormalizations (or R-operations):

$$R^{-n}(\mu;\mu_0) = R^{-n}(\mu;\mu_I)\, R^{-n}(\mu_I;\mu_0) ,$$

where μ_I is an arbitraty intermediate scale. By noticing that $R(\mu;\mu) = R(\mu_0;\mu_0) = 1$ (identity element) and $R(\mu;\mu') = R^{-1}(\mu';\mu)$ (the inverse element), the set of all R operations form a group which is called the <u>Renormalization Group</u> (RG).

In addition, the 1st order differential equations related to the running (or μ-dependence) of ϕ, λ and m^2 are called the <u>Renormalization Group Equations</u> (RGEs). ← See previous page.

The <u>'t Hooft mass scale</u> μ is also called the <u>RG scale</u>.

Lecture 12

We learnt that couplings and masses change as functions of the RG scale μ and the energy, if μ is identified with the typical energy of a scattering process, e.g. the centre-of-mass (CM) energy, or the transverse momentum p_T of a hard jet etc.

The strength of a quartic coupling λ or a gauge coupling g in the IR limit $\frac{\mu}{\mu_0} \mapsto 0$, or in the UV limit $\frac{\mu}{\mu_0} \mapsto \infty$, entirely depends on the roots of the β-functions $\beta_\lambda(\lambda)$ or $\beta_g(g)$, and the signature (positive or negative) of $\beta'_\lambda \hat{=} \frac{d\beta_\lambda}{d\lambda}$ or $\beta'_g \hat{=} \frac{d\beta_g}{dg}$ evaluated at these roots, i.e. at the points for which $\beta_\lambda(\lambda) = 0$ or $\beta_g(g) = 0$.

To illustrate this, let us introduce the parameter $t \hat{=} \ln\frac{\mu}{\mu_0}$, where μ_0 is a reference scale at which we know the value of the gauge coupling g: $g(\mu_0) = g(t=0) = g_*$.

e.g., measured by experiment at energies μ_0

Moreover, let us assume that the equation $\beta_g(g) = 0$ has a simple zero at $g = g_1$.

For values of $g \simeq g_1$, $\beta_g(g)$ may be approximated as

$$\beta_g(g) \simeq \beta_1\left[g(t) - g_1\right] \quad ; \quad \beta_1 = \beta'_g[g_1]$$

From the RGE for g: $\mu\frac{dg(\mu)}{d\mu} = \frac{dg(t)}{dt} = \beta_g[g(t)] \simeq \beta_1\left[g(t) - g_1\right]$,

we have $g(t) = g_1 + (g_* - g_1)\,e^{\beta_1 t}$.

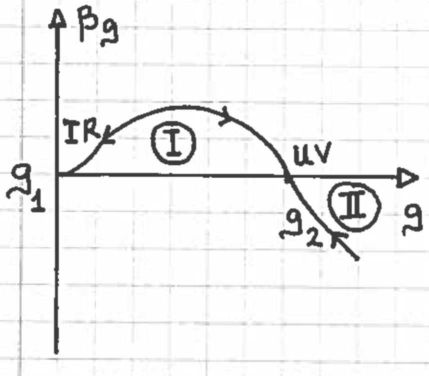
(a) $\beta_1 > 0$: $\quad g(t \mapsto -\infty) = g_1 \leftarrow$ IR stable fixed point.

(b) $\beta_1 < 0$: $\quad g(t \mapsto +\infty) = g_1 \leftarrow$ UV stable fixed point.

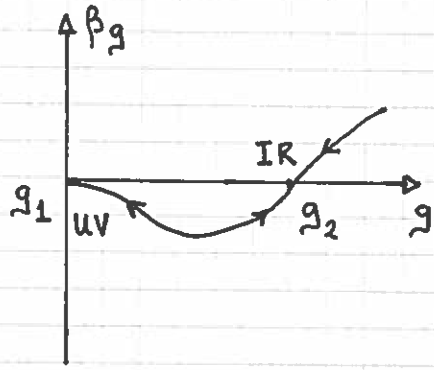
Examples:

Two (possibly gauge) theories exhibit the following β-functions

Theory A



Theory B



Theory A

$\beta'_g(g_1) > 0$ and $\beta'_g(g_2) < 0$

Note that $\beta'_g(g)$ is the tangent at the point g.

If g_* is found to be in the interval $g_1 < g_* < g_2$, i.e. Region I, then g_1 is an IR stable fixed point and g_2 is an UV stable fixed point

If $g_* > g_2$, the coupling g runs into the UV fixed point g_2.

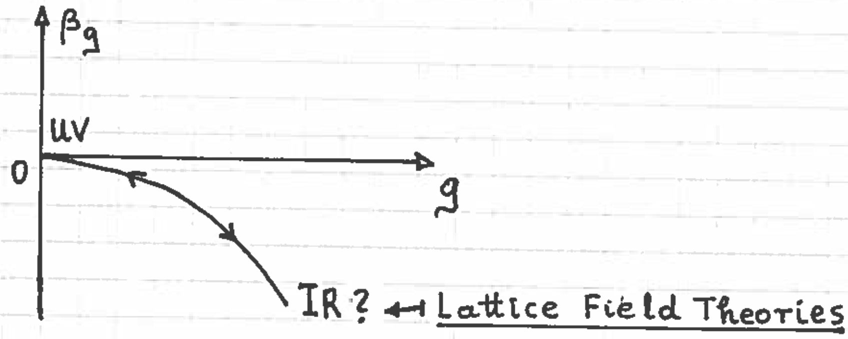
If $g_1 = 0$, the theory resembles QED with a possibly UV fixed point at g_2.

Theory B

$\beta'_g(g_1) < 0$ and $\beta'_g(g_2) > 0$

The theory B has an UV fixed point at g_1 and an IR one at g_2. If $g_1 = 0$, it is then said that the theory is asymptotically free, or it possesses asymptotic freedom.

Pure YM theories are asymptotically free.



Pure YM theories (e.g. QCD without quarks) are asymptotically free at infinite energies. Their one-loop β-function is given by (Politzer '73; Gross & Wilczek '73): See Ex III.5 (iv)

$$\beta_g = -\frac{11}{3} C_A \frac{g^3}{16\pi^2} < 0 \quad , \text{ where } C_A = N \text{ for } SU(N).$$

Hence, the YM coupling g becomes perturbative at sufficiently high energies, for which $\mu \gg \Lambda_{YM}$, and ordinary perturbation theory applies. However, for $\mu < \Lambda_{YM}$, the coupling $g(\mu) \gg 1$, and the theory becomes non-perturbative.

The scale Λ_{YM} is called the confinement scale, below which new phenomena due to quark and gluon bound states take place, as we know from QCD, for $\Lambda_{YM} = \Lambda_{QCD} \sim 300$ MeV.
This non-perturbative phase of the theory is called confinement.
Also, the RG scale μ, at which $g^2(\mu) \mapsto +\infty$ in perturbation theory, is sometimes called the Landau pole.

Remark. QCD is still the fundamental theory of strong interactions even for energies beyond the confinement scale.
Lattice field theory techniques incorporate the QCD Lagrangian and give remarkable predictions for the hadron and meson mass spectrum consistent with the experiment, within the level of the achieved theoretical accuracy.

FURTHER READING

- BRS symmetry plays a prominent role in path integral quantization of YM theories, which removes an infinite volume factor in gauge-field space.

- The gauge field A^a_μ in the covariant derivative $D_\mu = \partial_\mu + ig A^a_\mu (T^a)_{ij}$ may be interpreted as an affine connection in an abstract gauge-field space, i.e. $\Gamma^\mu_{ij} = A^{a,\mu}(T^a)_{ij}$. The analogues between General Relativity and YM theories are studied in Ex III.2 (iv)**.

- The BRS symmetry may be extended by promoting the gauge-fixing parameter ξ to an auxiliary field. This extended BRS invariance of the effective action Γ enables us to understand the ξ-independence of physical observables, such as the vacuum energy of a system or S-matrix elements.
  [e.g. see Papavassiliou, A.P., Binosi, PRD71(2005) 085007.]

- Lattice field theory techniques are based on discretization methods of the path integral, aiming at carrying out numerically all computations without perturbative expansions. Nevertheless, the finite lattice spacing (: the UV cut-off) and the overall size of the lattice (: the IR cut-off) determine both the CPU time and the theoretical accuracy achieved by these methods.