Lecture 3

Definition of a Group $(G, \cdot)$. A set of elements $\{a, b, c, \dots\} \hat{=} G$ endowed with a composition law $\cdot$ that has the following properties:

(i) Closure. $\forall a, b \in G$, the element $c = a \cdot b \in G$.

(ii) Associativity. $\forall a, b, c \in G \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(iii) The identity element e. $\exists e \in G : e \cdot a = a \cdot e = a, \forall a \in G$

(iv) The inverse element. $\forall a \in G, \exists a^{-1} \in G : a \cdot a^{-1} = a^{-1} \cdot a = e$

Remarks

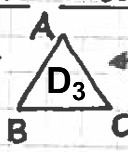
- The identity element $e \in G$ is unique (Why?)
- The inverse element a^{-1} of a is unique (Why?)
- If $a \cdot b = b \cdot a$, $\forall a, b \in G$, the group G is called Abelian

EXAMPLES:

- $(\mathbb{R}, +)$ is a group, but not $(\mathbb{R}, \cdot)$ (as there is no inverse for 0).

  In fact, $(\mathbb{R}, +, \cdot)$ is a field(*).

- Discrete Groups : S_n, Z_n and C_n

  E.g. S_3 : permutations of 3 objects which is equivalent to all possible symmetries of an equilateral triangle under rotations and reflections: $S_3 \simeq$ 

  Non-Abelian

- All $n \times n$ real matrices M, with $\det M \neq 0$, form the group $GL(n, \mathbb{R})$ under multiplication of matrices.

  General Linear

(*) There are $\underline{4}$ infinite fields: (i) $(\mathbb{Q}, +, \cdot)$ rational numbers; (ii) $(\mathbb{R}, +, \cdot)$ real numbers; (iii) $(\mathbb{C}, +, \cdot)$ complex numbers; (iv) $(\mathbb{H}, +, \cdot)$ quartenions introduced by Hamilton (non-Abelian).

<u>Cosets</u>. Let $H=\{h_1,h_2,\dots,h_r\}$ be a <u>proper subgroup</u> of group G i.e. $H\subset G$, with $H\neq I=\{e\}$ and $H\neq G$.

For a given $g\in G$, the set $gH=\{g\cdot h_1, g\cdot h_2,\dots,g\cdot h_r\}$ is called the <u>left coset</u> of H, and the set $Hg=\{h_1\cdot g, h_2\cdot g,\dots,h_r\cdot g\}$ is called the <u>right coset</u> of H.

<u>Example</u>: $C_6=\{e,a,\dots,a^5\}$, with $a=e^{2\pi i/6}=e^{i\pi/3}$ and $a^6=1=e$.

Proper subgroups: $C_2=\{e,a^3\}$ and $C_3=\{e,a^2,a^4\}$.

$\longleftrightarrow$

<u>Lagrange's Theorem</u>. For any two (left) cosets g_1H and g_2H, it holds: $g_1H=g_2H \;\vee$ (:<u>or</u>) $g_1H\cap g_2H=\varnothing$ (: the empty set).

<u>Proof</u>:

<u>Assume</u> $g_1H\neq g_2H \;\wedge\; g_1H\cap g_2H\neq\varnothing$ (*) is <u>true</u>.

$$\rightarrow \begin{cases} g_1\neq g_2, \text{ otherwise } g_1H=g_2H & \text{(a)} \\ \wedge\ \exists g_3\in g_1H\cap g_2H:\ g_3=g_1\cdot h_1=g_2\cdot h_2, \text{ with } h_1,h_2\in H & \text{(b)}\end{cases}$$

$$\text{(b)} \rightarrow g_2=g_1\cdot \underbrace{h_1\cdot h_2^{-1}}_{=h_3\in H}=g_1\cdot h_3 \rightarrow g_2\in g_1H$$

$$\rightarrow g_2H=\{g_1\cdot h_3\cdot h_\ell \,/\, \forall h_\ell\in H\}=\{g_1\cdot h_m \,/\, \forall h_m\in H\}=g_1H$$

$$\rightarrow g_2H=g_1H$$

in contradiction with (*)

$\therefore$ <u>Lagrange's theorem holds</u>

: <u>proof by contradiction</u>
$\updownarrow$
<u>reductio ad absurdum</u>
$\updownarrow$
<u>εἰς ἄτοπον απαγωγή</u>

$\longleftrightarrow$

<u>Coset decomposition</u>. If H is a proper subgroup of G, then $G=H\cup g_1H\cup g_2H\cdots\cup g_{\nu-1}H$; $g_{1,2,\dots\nu-1}\in G$, $g_1\notin H$, $g_2\notin H$, $g_2\notin g_1H$ etc.

ν: the <u>index</u> of H in G

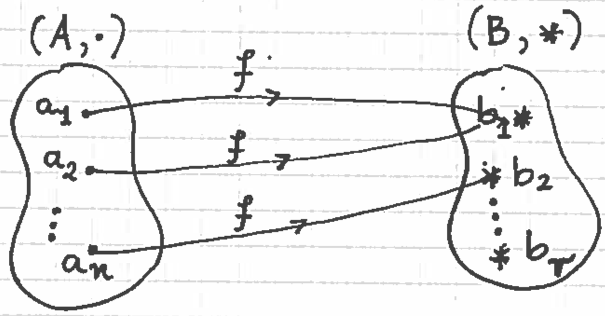
<u>Coset space</u>. $G/H=\{H,g_1H,\dots,g_{\nu-1}H\}$

<u>Examples</u>: $C_6/C_2=\{C_2,aC_2,a^2C_2\}$; $C_6/C_3=\{C_3,aC_3\}$.

Morphisms between Groups: $(A, \cdot)$ and $(B, *)$

Group Homomorphism

Functional mapping f :

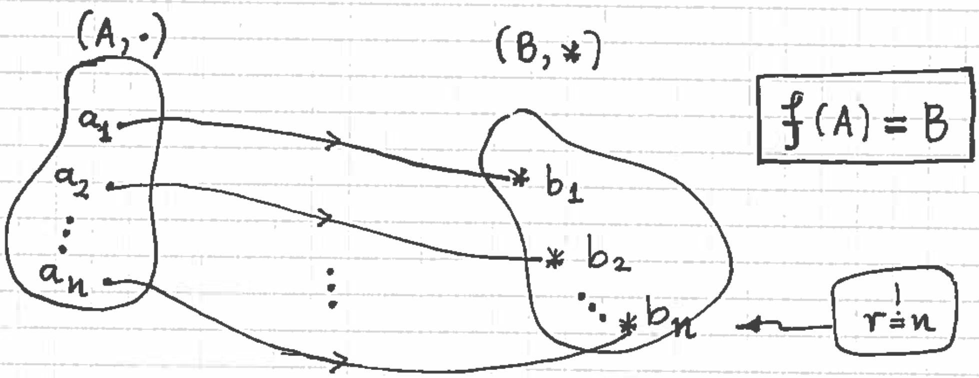


All elements $a \in A$ mapped to a single element $b = f(a) \in B$, such that $\boxed{f(a_1 \cdot a_2) = f(a_1) * f(a_2)}$.

Note that $f(A) \subset B$ and $f(A) \neq B$ in general, e.g. $r \neq n$

Group Isomorphism

1:1 mapping (: bijective mapping) of $(A, \cdot)$ onto $(B, *)$:



Composition law as above: $f(a_1 \cdot a_2) = f(a_1) * f(a_2)$.

Groups A and B are then said to be isomorphic: $A \cong B$

Group homomorphism $A \mapsto A$ is called endomorphism.

Group isomorphism $A \leftrightarrow A$ is called automorphism.

Examples of isomorphism: $C_n \cong Z_n$, $S_2 \cong C_2 \cong Z_2$

Lecture 4

Continuous Groups $GL(N,\mathbb{C})$, $SL(N,\mathbb{C})$, $O(N,\mathbb{R})$, $SO(N,\mathbb{R})$, $SU(N,\mathbb{C}) \hat{=} SU(N)$, $SO(N,M)$, E_6, E_7 etc

Glossary:

- G : General (with $\det M \neq 0$)
- L : Linear
- S : Special $\mapsto$ with $\det M = 1$
- O : Orthogonal
- U : Unitary
- E*: Exceptional Groups* ← relevant to Grand Unified Theories and String Theories

Group	no. of independent real parameters	Remarks
$GL(N,\mathbb{C})$	$2N^2$	$\det M \neq 0$
$SL(N,\mathbb{C})$	$2N^2 - 2$	$\det M = 1$
$SO(N,\mathbb{R})$	$\frac{1}{2}N(N-1)$	$\det O = 1$
$SU(N)$	$N^2 - 1$	$\det U = 1$
$SO(N,M)$	?	$\det \Lambda = 1$
$\vdots$		

E.g., $O(N,\mathbb{R}) \subset GL(N,\mathbb{R})$ has N^2 parameters, but not all are free.

Constraints : $O^T O = \mathbb{1}_N \rightsquigarrow O^T_{ab} O_{bc} = \delta_{ac}$

$\rightsquigarrow O_{ba} O_{bc} = \delta_{ac}$. It should be $c \geq a = 1, 2, \ldots, N$ to singly count the constraints.

No. of constraints: $R = \sum_{a=1}^{N} \underbrace{N - a + 1}_{C_{max}(a)} = N(N+1) - \frac{N(N+1)}{2} = \frac{N(N+1)}{2}$

Indep. real parameters: $N^2 - R = N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}$.

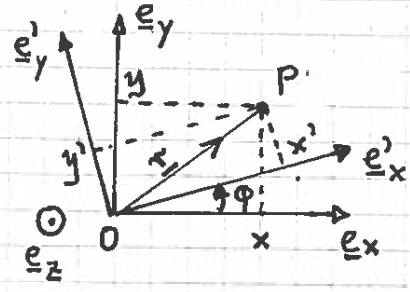
SO(2):

$$\begin{pmatrix} \underline{e}'_x \\ \underline{e}'_y \end{pmatrix} = \begin{pmatrix} c_\phi & s_\phi \\ -s_\phi & c_\phi \end{pmatrix} \begin{pmatrix} \underline{e}_x \\ \underline{e}_y \end{pmatrix} \quad ; \quad \begin{matrix} c_\phi \mathrel{\hat{=}} \cos\phi \\ s_\phi \mathrel{\hat{=}} \sin\phi \end{matrix}$$

$$\underline{r} = \vec{OP} = x\,\underline{e}_x + y\,\underline{e}_y = x'\underline{e}'_x + y'\underline{e}'_y$$

$$\leadsto \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}}_{\hat{=}\, O(\phi)\, \in\, SO(2)} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\hat{=}\, O(\phi) \in SO(2)$, since $\det O = 1$



Passive SO(2) rotation through φ about $\underline{e}_z$.

Basic property of SO(2):

$$O^T(\phi)\mathbb{1}_2 O(\phi) = \mathbb{1}_2 \quad \longleftrightarrow \quad x'^2 + y'^2 = x^2 + y^2$$

SO(2) is Abelian: $\quad O(\phi)\,O(\phi') = O(\phi+\phi') = O(\phi')\,O(\phi)$

Taylor expansion of $O(\phi)$ about $\phi = 0$:

$$O(\phi) = O(0) + \delta\phi\, O_\phi(0) + \mathcal{O}(\delta\phi^2) \quad ; \quad O_\phi(\phi) \mathrel{\hat{=}} \frac{d}{d\phi} O(\phi) = \begin{pmatrix} -s_\phi & -c_\phi \\ c_\phi & -s_\phi \end{pmatrix}$$

$$= \mathbb{1}_2 + \delta\phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \mathcal{O}(\delta\phi^2)$$

$$= \mathbb{1}_2 - i\,\delta\phi \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{=\sigma_2 = i\,O_\phi(0)} + \mathcal{O}(\delta\phi^2)$$

2nd Pauli matrix

Exponential representation:

$$O(\phi) = \lim_{N\to\infty} \Big[O\big(\underbrace{\tfrac{\phi}{N}}_{=\delta\phi}\big) \Big]^N = e^{-i\phi\sigma_2} \in SO(2)$$

σ_2: Generator of SO(2)

ϕ: Group parameter, $0 \le \phi < 2\pi$.

$U(1)$:

$SO(2)$ in $(V,\mathbb{R})$ is reducible in $(V,\mathbb{C})$ by means of a similarity transf:

$$M^{-1}\, O(\phi)\, M = \hat{O}(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = D^{(1)}(\phi) \oplus D^{(-1)}(\phi)$$

direct sum

with $M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, $M^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ independent of ϕ.

$D^{(\pm 1)}(\phi) = e^{\pm i\phi}$ are <u>faithful</u> <u>irreducible representations</u> (irreps) of $U(1)$

In $(V,\mathbb{C})$, we have: $SO(2) \cong U(1) \oplus \overline{U(1)}$

<u>bar</u> indicates complex conjugation

<u>General irrep of $U(1)$:</u>

$$D^{(m)}(\phi) = e^{-im\phi}$$

where $m \in \mathbb{Z}$, and $0 \leq \phi < 2\pi$.

The generators for these irreps of $U(1)$ are:

$$G^{(m)} \hat{=} i \frac{d}{d\phi} D^{(m)}(\phi)\Big|_{\phi=0} = m \in \mathbb{Z} \text{ : the integers}$$

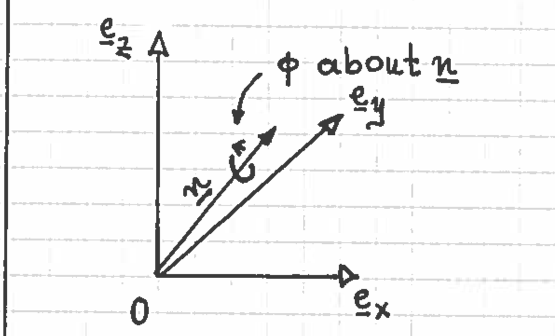
SO(3):

Proper rotations in 3D through ϕ about a given unit vector
$\underline{n} = n_x \underline{e}_x + n_y \underline{e}_y + n_z \underline{e}_z \mathrel{\hat=} (n_x, n_y, n_z) \mathrel{\hat=} (n_1, n_2, n_3)$ (with $\underline{n}^2 = 1$)

Proper (passive) rotations about $\underline{e}_x, \underline{e}_y, \underline{e}_z$ separately:

(i) $\underline{n} = \underline{e}_x = (1,0,0)$ counter-clockwise

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{pmatrix}$$



Geometric intepretation of SO(3) rotation.

(ii) $\underline{n} = \underline{e}_y = (0,1,0)$ counter-clockwise

$$R_2(\phi) = \begin{pmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{pmatrix}$$

(iii) $\underline{n} = \underline{e}_z = (0,0,1)$ counter-clockwise

$$R_3(\phi) = \begin{pmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Generators of SO(3): $\quad X_i = i \frac{d}{d\phi} R_i(\phi) \Big|_{\phi=0}$, with $i = 1,2,3$.

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(X_k)_{ij} = -i\,\varepsilon_{ijk} \;; \quad \varepsilon_{ijk} = \begin{cases} 1 & \text{for } (i,j,k) = (1,2,3) \text{ and even perm} \\ -1 & \text{for odd perms} \\ 0 & \text{otherwise} \end{cases}$$

Levi-Civita tensor

General representation of a group element of SO(3):

$$R(\phi, \underline{n}) = e^{-i\phi\,\underline{n}\cdot\underline{X}}$$

with $\underline{X} \doteq (X_1, X_2, X_3)$ and $0 \leq \phi < 2\pi$

Other reps possible, e.g. Euler rep and Euler angles α, β, γ.

SU(2):

Abstract passive rotation of a complex 2-dim vector $\underline{v} \hat{=} (v_1, v_2)^T$; $v_{1,2} \in \mathbb{C}$, through θ about $\underline{n}$:

$$SU(2): \ \underline{v} \mapsto \underline{v}' = \mathcal{U}(\theta, \underline{n})\, \underline{v} \quad ; \quad \underline{v}'^{*} \cdot \underline{v}' = \underline{v}^{*} \cdot \underline{v} \ ,$$

with $\det \mathcal{U} = 1$ and $\mathcal{U}(\theta, \underline{n}) = e^{-i\theta\, \underline{n} \cdot \frac{\underline{\sigma}}{2}}$

$$= \mathbb{1}_2 \cos\frac{\theta}{2} - i\, \underline{n} \cdot \underline{\sigma} \sin\frac{\theta}{2} \ ; \ 0 \le \theta < 4\pi$$

Note that $\underline{n} \in (V, \mathbb{R})$, with $\underline{n}^2 = 1$.

Generators of SU(2): $T_a = \frac{\sigma_a}{2} = i \frac{d}{d\theta} \mathcal{U}(\theta, \underline{e}_a)\Big|_{\theta=0}$; $\underline{e}_a = (\underset{1}{\underline{e}_x}, \underset{2}{\underline{e}_y}, \underset{3}{\underline{e}_z})$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices

Properties of σ_i (with $i = 1, 2, 3$):

(i) $\mathrm{Tr}\, \sigma_i = 0$

(ii) $\sigma_i = \sigma_i^{\dagger}$

(iii) $\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_2 + i \varepsilon_{ijk} \sigma_k$, or $\sigma_i^2 = \mathbb{1}_2$; $\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k$, for $i \neq j$

(iv) $\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i \varepsilon_{ijk} \frac{\sigma_k}{2}$, or $\left[T_a, T_b\right] = i \varepsilon_{abc} T_c$.

Lecture 5

Important remark:

$$\boxed{\begin{array}{ccccc} \text{No. of independent real parameters, or matrix elements in } \mathbb{R}. & = & \text{No. of Group generators} & = & \text{No. of real Group parameters or angles} \end{array}}$$

Note that $SO(3)$ and $SU(2)$ have the same number $\underline{\underline{3}}$ of generators (= group parameters):

$$SO(3) = \left\{ R(\phi, \underline{n}) = e^{-i\phi\, \underline{n}\cdot \underline{X}} \;\middle|\; 0 \leq \phi < 2\pi \right\}^{\circledast}$$

$$SU(2) = \left\{ U(\theta, \underline{n}) = e^{-i\theta\, \underline{n}\cdot \frac{\underline{\sigma}}{2}} \;\middle|\; 0 \leq \theta < 4\pi \right\}$$

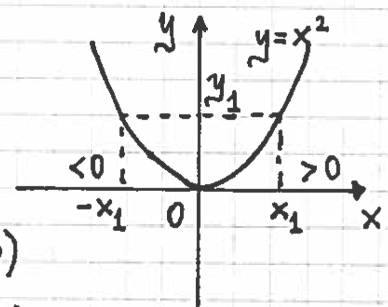
Also, $R(0, \underline{n}) = R(2\pi, \underline{n}) = \mathbb{1}_3$, whereas $U(0, \underline{n}) = \mathbb{1}_2$ but $U(2\pi, \underline{n}) = -\mathbb{1}_2$.

$\therefore$ A faithful 1:1 isomorphy is given by

$$SO(3) \cong SU(2)/\mathbb{Z}_2 = \left\{ U(\theta, \underline{n}) \cdot \mathbb{Z}_2 \;\middle|\; 0 \leq \theta < 2\pi \right\}$$

since $\mathbb{Z}_2 = \{\mathbb{1}_2, -\mathbb{1}_2\}$ is a proper subgroup of $SU(2)$.

Compare with $y = x^2$, where $y \in [0, +\infty)$ and $x \in (-\infty, +\infty)$



1:1 mapping $y \leftrightarrow x$, only for $(-\infty, +\infty)/_{\{1,-1\}}$
$= [0, +\infty) \cdot \{1, -1\}$
$\cong [0, +\infty)$

$\circledast$ Note that the generators X_i of $SO(3)$ satisfy the same commutation relation as the one by $\frac{\sigma_i}{2} = T_i$ of $SU(2)$:

$$[X_i, X_j] = i\varepsilon_{ijk} X_k$$

Lie Algebra and Lie Groups

A Lie algebra L over a number $d(G)$ of generators T_a is defined by the following two properties:

(i) $[T_a, T_b] = T_a \cdot T_b - T_b \cdot T_a = i f_{ab}^{\ c} T_c$; $f_{ab}^{\ c} \in \mathbb{C}$ are the structure constants of L

(ii) Jacobi identity:

$$[T_a, [T_b, T_c]] + [T_c, [T_a, T_b]] + [T_b, [T_c, T_a]] = 0 .$$

$\longleftrightarrow$

The set T_a of generators defines a $d(G)$-dim vector space $(V, \mathbb{C})$.

$\longleftrightarrow$

Fundamental rep of T_a are $d(F) \times d(F)$ matrices, where $d(F)$ is the least number of dimensions needed for T_a to realize the Lie algebra L and the respective continuous group G.

Examples: (i) $SO(3)$: $T_a = X_a$; (ii) $SU(2)$: $T_a = \frac{1}{2}\sigma_a$; (iii) $U(1)$: $m \in \mathbb{Z}$

Group elements of G :

$$G(\theta, \underline{n}) = e^{-i\theta\, \underline{n} \cdot \underline{T}} ,$$

where $\underline{n}^2 = 1$ and $\underline{T} = (T_1, T_2, \dots, T_{d_G})$, with $d_G \mathrel{\hat{=}} d(G)$.

Group Representations (reps)

The rep. of G depends on the rep. of T_a's, which need to satisfy the same Lie algebra L.

Example: The 3×3 matrices $X_{i=1,2,3}$ of $SO(3)$ form a higher rep of $SU(2)$, as both $SU(2)$ and $SO(3)$ groups satisfy the same Lie algebra.

The Lie-algebra commutator $[T_c, \cdot\,]$ (for fixed T_c) is a linear operator:

$$[T_c\,,\,\lambda_1 T_a + \lambda_2 T_b] = \lambda_1 [T_c, T_a] + \lambda_2 [T_c, T_b]\,, \forall T_a, T_b \subset L$$

The above linear homomorphic mapping from $L \mapsto L$ over $\mathbb{C}$ may be represented by the structure constants themselves:

$$[T_a\,, T_b] = i\, f^{c}_{ab}\, T_c \quad \rightarrow \quad [\mathcal{D}_{\mathcal{A}}(T_a)]^{c}{}_{b} = i\, f^{c}_{ab} \quad (= -i\, f^{c}_{ba})$$

linear operator

basis vectors in L

This rep of T_a is called the adjoint rep of L, denoted by $\mathcal{A}$.

The adjoint rep $\mathcal{D}_{\mathcal{A}}(T_a)$ satisfies the same Lie algebra L as the fundamental rep $T_a \hat{=} \mathcal{D}_F(T_a)$:

See Ex II.2(e)

(i) $$[\mathcal{D}_{\mathcal{A}}(T_a), \mathcal{D}_{\mathcal{A}}(T_b)] = i\, f^{c}_{ab}\, \mathcal{D}_{\mathcal{A}}(T_c)$$

(ii) $$[\mathcal{D}_{\mathcal{A}}(T_a), [\mathcal{D}_{\mathcal{A}}(T_b), \mathcal{D}_{\mathcal{A}}(T_c)]] + [\mathcal{D}_{\mathcal{A}}(T_c), [\mathcal{D}_{\mathcal{A}}(T_a), \mathcal{D}_{\mathcal{A}}(T_b)]]$$
$$+ [\mathcal{D}_{\mathcal{A}}(T_b), [\mathcal{D}_{\mathcal{A}}(T_c), \mathcal{D}_{\mathcal{A}}(T_a)]] = 0$$

Jacobi identity

Lecture 6

The generators $T^{\mathcal{A}}_a \hat{=} D_{\mathcal{A}}(T_a)$ define a metric vector space, i.e. a manifold, with the metric of the space defined by the Killing product form (: inner multiplication for matrices):

$$g_{ab} \hat{=} (T_a, T_b)_{\mathcal{A}} \hat{=} Tr\left[T^{\mathcal{A}}_a T^{\mathcal{A}}_b\right] \hat{=} Tr_{\mathcal{A}}(T_a T_b) = -f^d_{ac} f^c_{bd},$$

where g_{ab} is called the Cartan metric.

The Cartan metric g_{ab} can be used to lower the index of f^c_{ab}:

$$f_{abc} = f^d_{ab}\, g_{dc}$$

f^{abc} is fully antisymmetric in a, b, c.

Alternatively, one can show that $f_{abc} = -i Tr\left([T^{\mathcal{A}}_a, T^{\mathcal{A}}_b] T^{\mathcal{A}}_c\right)$. Indeed, starting from the RHS, we have

$$-i Tr\left([T^{\mathcal{A}}_a, T^{\mathcal{A}}_b] T^{\mathcal{A}}_c\right) = -i\, Tr\left(i f^x_{ab} T^{\mathcal{A}}_x T^{\mathcal{A}}_c\right) = f^x_{ab} \underbrace{Tr\left(T^{\mathcal{A}}_x T^{\mathcal{A}}_c\right)}_{= g_{xc}} = f_{abc}.$$

General Remarks:

- If all $f^c_{ab} \in \mathbb{R}$ for a Lie algebra L, then L is said to be a real Lie algebra

- If the Cartan metric g_{ab} is positive definite for a real L, then L is an algebra for a compact group, e.g. $SU(N)$ and $SO(N)$. In this case, g_{ab} can be diagonalized and rescaled to unit matrix, i.e. $g_{ab} \mapsto \hat{g}_{ab} = \mathbb{1}_{ab} = \delta_{ab}$ $(: T_a = T^a)$

- There is no adjoint rep for an Abelian group, such as $U(1)$ and $SO(2)$, since the stucture constants f^c_{ab} vanish.

Normalization of Generators

Fundamental rep: $\mathrm{Tr}[T_a \cdot T_b] = T_F \delta_{ab}$; $T_F = \frac{1}{2}$ ← for SU(N)

Adjoint rep: $\mathrm{Tr}[T_a^A \cdot T_b^A] = T_A \delta_{ab}$; $T_A = N$ ←
$= g_{ab}$

Casimir operators $\mathbf{T}_R^2$ of a Lie algebra L of a rep R are matrix reps that commute with all generators T_a^R of L in rep. R: $[\mathbf{T}_R^2, T_a^R] = 0$

Explicit construction of $\mathbf{T}_R^2$:

"$g_{ab}^{-1} = T_A^{-1} \delta^{ab}$" ← for compact groups

$$(\mathbf{T}_R^2)_{ij} = T_A \sum_{a,b=1}^{d_G} \sum_{k=1}^{d_R} \underbrace{[D_R(T_a)]_{ik}}_{\hat{=}(T_a^R)_{ik}} g^{ab} \underbrace{[D_R(T_b)]_{kj}}_{\hat{=}(T_b^R)_{kj}}$$

$$= (T_a^R T_a^R)_{ij} = \delta_{ij} C_R \rightsquigarrow \boxed{\mathbf{T}_R^2 = T_a^R T_a^R = \mathbf{1}_{d_R} C_R}$$

Note that $\mathrm{Tr}(\mathbf{T}_R^2) = \mathrm{Tr}(T_a^R T_a^R) = T_R d_G$

Also, $\mathrm{Tr}(T_a^R T_a^R) = C_R \,\mathrm{Tr}\, \mathbf{1}_{d_R} = C_R d_R$

$\Big\} \to$ $T_R d_G = C_R d_R$

$d_G \hat{=} d(G)$
$d_R \hat{=} d(R)$

Employing the last relation, we can derive for SU(N) theories:

$$C_F = \frac{N^2-1}{2N} \quad \text{and} \quad C_A = N$$

FURTHER READING

- Our lectures have only covered basic elements of Group Theory. Further discussion is given by former lecture notes by A.P. on "Symmetries in Physis" and literature provided there.

- Other topics of interest are the concepts: (i) Normal subgroups; (ii) Simple and semi-simple Lie algebras and ideals; (iii) Tensors in SU(N) and their decomposition with the use of Young Tableaux.

- Non-compact groups, such as the Lorentz and Poincaré groups. See exercise in Ex II.5 on the SL(2, $\mathbb{C}$) and SO(1,3) Groups.

- The Cartan programme of construction and classification of infinite groups (covered in advanced texts, e.g. H.F. Jones, "Groups, Reps. and Physics")

- Graded Lie-algebras and Supersymmetry. (lecture notes on "Supersymmetry" by A.P. also J. Wess and J. Bagger, "Supersymmetry and Supergravity")