PHYS40682: GAUGE THEORIES Prof A Pilaftsis

EXAMPLES SHEET III: Quantum Chromodynamics

1 Gauge Invariance in Yang–Mills Theories

The Lagrangian of an SU(N) Yang–Mills (YM) theory is given by

$$\mathcal{L}_{\rm YM} \;=\; -\,\frac{1}{4}\,F^a_{\mu\nu}\,F^{a,\mu\nu}\;,$$

where

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g\,f^{abc}\,A^b_\mu\,A^c_\nu\,,$$

is the field-strength tensor and $f^{abc} \equiv f_{abc}$ are the structure constants of the SU(N) Lie algebra. In this examples sheet, we assume that the Cartan metric g_{ab} of the SU(N) Lie algebra has been rescaled so as to become Euclidean, i.e. $g_{ab} \to \hat{g}_{ab} = \delta_{ab}$.

(i) Show that $\mathcal{L}_{\rm YM}$ is invariant under the *infinitesimal* SU(N) local transformations: $A^a_\mu \to A^a_\mu \,+\, \delta A^a_\mu$, with

$$\delta A^a_\mu \;=\; -\,\frac{1}{g}\,\partial_\mu\theta^a \;-\; f^{abc}\,\theta^b\,A^c_\mu\,.$$

(ii) The interaction of a Dirac fermion f in the fundamental representation of SU(N) with the respective YM gauge fields A^a_μ is governed by the Lagrangian

$$\mathcal{L}_f \;=\; \bar{f}_i\,\Big[\,i\,\partial\!\!\!/\,\delta_{ij} \;-\; m_f\,\delta_{ij} \;-\; g\,A\!\!\!/^a(T^a)_{ij}\Big]\,f_j\,,$$

where T^a are the generators of the SU(N) Lie algebra. Show that $\mathcal{L}_f$ is also invariant under *infinitesimal* SU(N) local transformations, provided that the fermion multiplet f_i transforms as

$$\delta f_i \;=\; i\theta^a\,(T^a)_{ij}\,f_j\;.$$

(iii) From part (ii), we see that the covariant derivative D_μ acting on an SU(N)-charged Dirac fermion f is

$$D_\mu f \;=\; \Big(\mathbf{1}_N\partial_\mu \;+\; ig\,T^a A^a_\mu\Big)\,f\;.$$

Given that $f \to f' = Uf$ (with $U \in$ SU(N)) under a finite SU(N) rotation, show that $D_\mu f$ transforms as follows:

$$D_\mu f \;\to\; D'_\mu f' \;=\; U D_\mu f\;,$$

thereby leaving $\mathcal{L}_f$ invariant under *finite* SU(N) local rotations.

2 Geometric Properties of YM Theories

This exercise will help us to understand some basic differential-geometric and topological properties of YM theories.

(i) Show that

$$-\frac{i}{g}\big[D_\mu\,,\ D_\nu\big]\ =\ \mathbf{F}_{\mu\nu}\ ,$$

where $\mathbf{F}_{\mu\nu} \equiv F^a_{\mu\nu}T^a = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig\big[\mathbf{A}_\mu\,, \mathbf{A}_\nu\big]$, with $\mathbf{A}_\mu \equiv A^a_\mu T^a$, is the SU($N$) field-strength tensor.

(ii) **The $\boldsymbol{\theta}$ term in YM theories.** Show that the term,

$$\mathcal{L}_\theta\ =\ -\,\frac{\theta}{4}\,F^a_{\mu\nu}\,\widetilde{F}^{a,\mu\nu}\ ,$$

is invariant under finite SU(N) gauge transformations, where $\widetilde{F}^{a,\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}F^a_{\rho\sigma}$ and $\varepsilon^{\mu\nu\rho\sigma}$ is the 4D Levi–Civita tensor (with the convention $\varepsilon^{0123} = +1$). Prove that $\mathcal{L}_\theta$ is a total derivative given by $\mathcal{L}_\theta = -\,\theta\,\partial_\mu K^\mu$, where

$$K^\mu\ =\ \varepsilon^{\mu\nu\rho\sigma}\ \mathrm{Tr}\Big(\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma\ +\ \frac{2}{3}\,ig\,\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma\Big)\ ,$$

is the so-called **Chern–Simons current**.

(iii)* Show that the θ-term is *odd* under **CP transformations**. The θ-term is related to the so-called strong CP problem of the QCD interactions. It also plays a profound role in understanding certain aspects of non-perturbative dynamics in QCD, through topological solutions, such as **instantons**.

(iv)** Prove the **Bianchi identity**:

$$\sum_{\substack{\rho,\mu,\nu\\ \text{cyclic}}} D_\rho\,\mathbf{F}_{\mu\nu}\ =\ 0\ ,$$

Compare this last equation with the corresponding Bianchi identity that you met in General Relativity (GR). Draw possible analogies of concepts, such as gauge transformations, the gauge field $A^a_\mu(T^a)_{ij}$, and the field strength tensor $\mathbf{F}_{\mu\nu}$, from related concepts in GR. Hence, guided by these findings, unravel the entire underlying differential-geometric structure of gauge theories.

3 Gauge Fixing and Becchi–Rouet–Stora (BRS) Transformations

To obtain a *non*-singular gauge-field propagator $\Delta^{ab}_{\mu\nu}(x-y)$ in YM theories, we must add to $\mathcal{L}_{\rm YM}$ a **covariant gauge-fixing term**:

$$\mathcal{L}_{\rm GF} \;=\; -\frac{1}{2\xi}\,(\partial_\mu A^{a,\mu})\,(\partial_\nu A^{a,\nu})\;.$$

(i) Derive the Euler–Lagrange equation of motion for the *free* YM field A^a_μ, by adding $\mathcal{L}_{\rm GF}$ to $\mathcal{L}_{\rm YM}$ (with $g=0$):

$$\Big[\eta_{\mu\nu}\,\partial_\kappa\partial^\kappa \;-\; \Big(1-\frac{1}{\xi}\Big)\,\partial_\mu\partial_\nu\Big]\,A^{a,\nu} \;=\; 0\;.$$

(ii) Show that the Green's function of the linear differential operator given in part (i) is given by the gauge-field Feynman propagator

$$\Delta^{ab}_{\mu\nu}(x-y) \;=\; \int\frac{d^4k}{(2\pi)^4}\,\Big(-\,\eta_{\mu\nu}\;+\;(1-\xi)\,\frac{k_\mu k_\nu}{k^2}\Big)\,\frac{\delta^{ab}\,e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}\;.$$

(iii) Show that $\mathcal{L}_{\rm GF}$ is invariant under an *infinitesimal* SU(N) transformation of the gauge-field A^a_μ: $\delta A^a_\mu \;=\; f^{abc}\,\theta^b\,A^c_\mu$, for which $\partial_\mu\theta^a=0$. What happens if $\partial_\mu\theta^a\neq 0$?

(iv) To partially restore gauge invariance at the level of *infinitesimal* local SU(N) transformations, we first introduce in the theory new Grassman-valued complex fields c^a and $\bar c^a$, the so-called **Fadeev–Popov** (FP) **ghosts**. This induces a Lagrangian term for the FP ghosts:

$$\mathcal{L}_{\rm FP} \;=\; -\,\bar c^a\,\partial^\mu\,\Big[\delta^{ab}\partial_\mu \;+\; g f^{abc}\,A^c_\mu\Big]\,c^b\;.$$

Show that the full Lagrangian $\mathcal{L}=\mathcal{L}_{\rm YM}+\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm FP}$ is invariant under the BRS transformations,

$$\begin{aligned}
\delta A^a_\mu \;&\equiv\; \omega\,sA^a_\mu \;=\; \omega\,\Big[\delta^{ab}\partial_\mu \;+\; g f^{abc}\,A^c_\mu\Big]\,c^b\;,\\
\delta c^a \;&\equiv\; \omega\,sc^a \;=\; \omega\,\frac{1}{2}\,g f^{abc}\,c^b\,c^c\;,\\
\delta \bar c^a \;&\equiv\; \omega\,s\bar c^a \;=\; -\,\omega\,\frac{1}{\xi}\,\partial^\mu A^a_\mu\;,
\end{aligned}$$

with $\omega^2=0$. You may assume that $s^2A^a_\mu=0$ (a proof is given in part (vi) below).

(v) Show that the quark–gauge field Lagrangian $\mathcal{L}_f$ given in Exercise 1(ii) is also invariant under BRS transformations, provided the fermion multiplets f_i and $\bar f_i$ transform as follows:

$$\delta f_i \;\equiv\; \omega\,sf_i \;=\; -\,\omega\,ig\,(T^a)_{ij}\,c^a\,f_j\;,\qquad \delta\bar f_i \;\equiv\; \omega\,s\bar f_i \;=\; \omega\,ig\,(T^a)_{ji}\,c^a\,\bar f_j\;.$$

(vi) Show that all BRS transformations are nilpotent, i.e.

$$s^2 f_i \;=\; s^2 A^a_\mu \;=\; s^2 c^a \;=\; 0\,,$$

except of

$$s^2 \bar{c}^a = -\frac{1}{\xi}\,\partial^\mu\left[\delta^{ab}\partial_\mu \;+\; g f^{abc}\,A^c_\mu\right] c^b\,.$$

What should one impose upon the ghost fields to also get $s^2\bar{c}^a = 0$?

Note that $s(AB) = (sA)\,B \pm A\,(sB)$, where the minus sign occurs if there is an odd number of ghosts or anti-ghosts in A.

4 The Gluon Self-energy

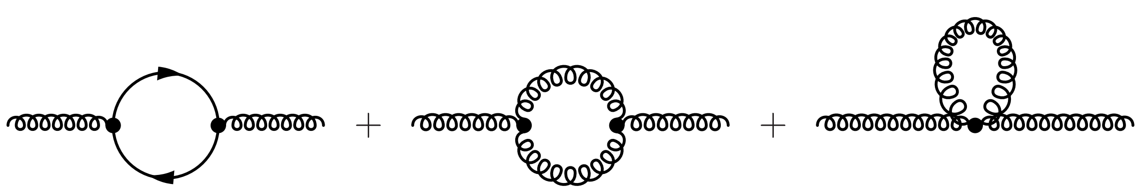
As shown in lectures, the *complete* Lagrangian of Quantum Chromodynamics (QCD) reads

$$\begin{aligned}\mathcal{L}_{\rm QCD} = &-\frac{1}{4}\,G^a_{\mu\nu}\,G^{a,\mu\nu} + \bar{q}_i\left[\,i\,\not{\partial}\,\delta_{ij} - m\,\delta_{ij} - g_s \not{G}^a (T^a)_{ij}\,\right] q_j \\ &-\frac{1}{2\xi}\,(\partial_\mu G^{a,\mu})\,(\partial_\nu G^{a,\nu}) - \bar{c}^a\,\partial^\mu\left[\delta^{ab}\partial_\mu + g_s f^{abc}\,G^c_\mu\right] c^b\,.\end{aligned}$$

(i) Use $\mathcal{L}_{\rm QCD}$ to derive the Feynman rules for the couplings: $G^a_\mu q_j \bar{q}_i$ and $G^a_\mu c^b \bar{c}^c$, as well as for the self-interactions:* $G^a_\mu G^b_\nu G^c_\rho$ and $G^a_\mu G^b_\nu G^c_\rho G^d_\sigma$. Consider the convention that all particle momenta flow into the vertex.

(ii) Use the Feynman rules deduced in part (i) to draw all graphs contributing to the gluon self-energy $\Pi^{ab}_{\mu\nu}(p)$, where p is the four-momentum of the gluon.

(iii) Write down the amplitudes for the following three graphs in the R_ξ gauge:



(iv) Calculate the colour factors for the three self-energy graphs shown in part (iii) in the Feynman–'t Hooft gauge $\xi = 1$.

5 The Renormalization Group

This exercise will help us to understand better the key fundamental properties of the Renormalization Group (RG). To start with, let us remind ourselves of the Lagrangian of an interacting scalar field theory:

$$\mathcal{L} \;=\; \frac{1}{2}\,(\partial_\mu \phi)\,(\partial^\mu \phi) \;-\; \frac{1}{2}\,m^2\,\phi^2 \;-\; \frac{1}{4!}\lambda\phi^4 \;.$$

(i) According to the renormalization programme outlined in the lectures of Quantum Field Theory (QFT), the relations between *bare* and *renormalized* quantities are given by

$$\phi_0 = Z_\phi^{1/2}\,\phi\,, \qquad m_0^2 = Z_{m^2}\,m^2\,, \qquad \lambda_0 = Z_\lambda\,\lambda\;,$$

where $Z_\phi^{1/2}$, Z_{m^2} and Z_λ are the renormalizations for the wavefunction of the field ϕ, its mass m and its self-coupling λ, respectively. By imposing the μ-*independence* of all bare parameters, derive the following differential RG equations:

$$\begin{aligned}
\gamma_\phi \;&\equiv\; \mu\,\frac{d\ln\phi(\mu)}{d\mu} \;=\; -\,\frac{1}{2}\,\mu\frac{d\ln Z_\phi}{d\mu}\;,\\
\beta_\lambda \;&\equiv\; \mu\,\frac{d\lambda(\mu)}{d\mu} \;=\; -\,\mu\frac{d\ln Z_\lambda}{d\mu}\,\lambda\;,\\
\gamma_{m^2} \;&\equiv\; \mu\,\frac{d\ln m^2(\mu)}{d\mu} \;=\; -\,\mu\frac{d\ln Z_{m^2}}{d\mu}\;.
\end{aligned}$$

(ii) Use the RG equation for the field ϕ obtained in part (i) to show that

$$\phi(\mu) \;=\; \phi(\mu_0)\;\exp\left[-\int_{\mu_0}^{\mu}\gamma_\phi(\mu')\;d\ln\mu'\right]\;.$$

(iii) Knowing that the one-loop beta function β_λ is

$$\beta_\lambda \;=\; \frac{3\,\lambda^2}{16\pi^2}\;,$$

calculate the RG energy scale $\mu = \Lambda_L$, at which the coupling $\lambda(\mu)$ diverges, i.e. when $\lambda(\Lambda_L) \to \infty$. The RG scale Λ_L is called the **Landau pole**.

(iv) **Asymptotic freedom**. The one-loop beta function of a gauge coupling β_g in a Yang–Mills theory SU(N) with n_F fermions is calculated to be

$$\beta_g \;=\; -\,g\,\frac{\alpha}{4\,\pi}\left(\frac{11}{3}\,C_{\mathcal{A}} \;-\; \frac{4}{3}\,T_F\,n_F\right)\;,$$

with $\alpha = g^2/(4\pi)$. Show that $g(\mu) \to 0$, as $\mu \to \infty$, provided $n_F < \frac{11}{2}\,N$.

(v) **The confinement scale** $\Lambda_{\rm QCD}$. In QCD, the strong fine structure constant α_s at energy scales $\mu = M_Z \simeq 91$ GeV is $\alpha_s(M_Z) \simeq 0.12$. Assuming that all quarks with masses smaller than M_Z are massless, determine the confinement scale $\Lambda_{\rm QCD}$.