QUANTUM FIELD THEORY PHYS40481 2012: Prof A Pilaftsis

EXAMPLES SHEET I: PRELIMINARIES

1 Hamilton's Equations

Show Hamilton's equations:

$$\dot{q}_i \;=\; \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i \;=\; -\frac{\partial H}{\partial q_i}\,,$$

where $q_i = q_{1,2,\dots,n}$ are the generalized coordinates describing a dynamical system and $p_i = p_{1,2,\dots,n}$ their conjugate momenta.

[*Hint:* Use the Euler–Lagrange equations of motion to prove the second of Hamilton's equations.]

2 Hamilton's Principle

Hamilton's principle states that the actual motion of the system is determined by the stationary behaviour of S under small variations $\delta q_i(t)$ of the ith particle's generalized coordinate $q_i(t)$, with $\delta q_i(t_1) = \delta q_i(t_2) = 0$. In particular, show *rigorously* that the Euler–Lagrange equation of motion for the ith particle is given by

$$-\frac{\delta S[q_i]}{\delta q_i(t)} \;=\; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \;-\; \frac{\partial L}{\partial q_i} \;=\; 0\,,$$

where $L = L[q_i(t), \dot{q}_i(t)]$ is the Lagrangian of the system. Moreover, show that two Lagrangians L and L' that differ by an overall time derivative, i.e. $L' = L + \partial f(t)/\partial t$, where $f(t)$ is an arbitrary function of time, satisfy the same Euler–Lagrange equation of motion.

[*Hint:* Use the defining property of functional differentiation: $\delta q_j(t')/\delta q_i(t) = \delta_{ij}\delta(t-t')$.]

3 Equations of Motion for Acceleration-Dependent Lagrangians

Show that the Euler–Lagrange equations of motion for a particle system described by a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i)$ are

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_i} \;-\; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \;+\; \frac{\partial L}{\partial q_i} \;=\; 0\,,$$

with $\ddot{q}_i \equiv d^2 q_i/dt^2$.

[*Hint:* Consider only variations with $\delta q_i(t_{1,2}) = \delta \dot{q}_i(t_{1,2}) = 0$.]

4 Euler–Lagrange Equations of Motions for Electrodynamics

The Lagrangian (density) of Electrodynamics reads:

$$\mathcal{L}_{\rm ED} \;=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \;-\; J_\mu A^\mu\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and $J^\mu = (\rho\,,\,\mathbf{J})$ is the 4-vector current satisfying charge conservation: $\partial_\mu J^\mu = 0$. Use the Euler–Lagrange equations of motion for A_μ to obtain the first and the fourth Maxwell equations:

$$\partial_\mu F^{\mu\nu} \;=\; J^\nu\,,$$

with $\mu_0 = \varepsilon_0 = c = 1$. How could one obtain the remaining second and third Maxwell equations?

5* The SU(N) Group

The SU(N) group is the set of all $N \times N$ complex unitary matrices U, with $\det U = 1$. Prove that U may be expressed by $N^2 - 1$ real parameters. Moreover, show that *any* SU(N) matrix U may be represented, in terms of the $N^2 - 1$ Lie generators T^a (with $a = 1, 2, \ldots, N^2 - 1$), as follows:

$$U(\theta^a) \;=\; \exp\left[\, i \sum_{a=1}^{N^2-1} \theta^a T^a \,\right]\,,$$

where T^a are $N \times N$ Hermitean traceless matrices, i.e. satisfying the properties:

$$T^a \;=\; (T^a)^\dagger\,, \qquad {\rm Tr}\, T^a \;=\; 0\,.$$

[*Hint:* You may find useful the property of matrices M: $\ln(\det M) \;=\; {\rm Tr}\,(\ln M)$.]

6 COURSEWORK I: Noether's Theorem

Derive the conserved currents and charges for a scalar theory, whose action $S = \int d^4x\, \mathcal{L}(\phi_i, \partial_\mu\phi_i)$ is invariant under:

(i) the infinitesimal spacetime translations

$$x^\mu \;\to\; x'^\mu \;=\; x^\mu \,+\, \epsilon^\mu\,.$$

(ii)* the infinitesimal Lorentz transformations (LTs)

$$x^\mu \;\to\; x'^\mu \;\equiv\; \Lambda^\mu_{\ \nu}(\omega) x^\nu \;\approx\; x^\mu \,+\, \omega^\mu_{\ \nu}\, x^\nu\,,$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu} \ll 1$.

SOLUTIONS TO EXAMPLES SHEET I

1. $$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i)$$

with $$p_i = \frac{\partial L(q_j, \dot{q}_j)}{\partial \dot{q}_i} \xrightarrow{\text{invert}} \dot{q}_j = \dot{q}_j(q_i, p_i)$$

Variables q_i, p_i are independent, but not q_i & $\dot{q}_i$ or p_i & $\dot{q}_i$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \cancel{\left(\sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i}\right)} - \underbrace{\frac{\partial L(q_j, \dot{q}_j)}{\partial p_i}}_{\sum_j \frac{\partial \dot{q}_j}{\partial p_i} \frac{\partial L}{\partial \dot{q}_j} = \cancel{\sum_j \frac{\partial \dot{q}_j}{\partial p_i} p_j}}$$

$$= \dot{q}_i \qquad \underline{\text{q.e.d.}}$$

$$-\frac{\partial H}{\partial q_i} = -\cancel{\left(\sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i}\right)} + \frac{\partial L(q_j, \dot{q}_j)}{\partial q_i} + \cancel{\left(\sum_j \frac{\partial \dot{q}_j}{\partial q_i} \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{= p_j}\right)}$$

$$= \frac{\partial L(q_j, \dot{q}_j)}{\partial q_i} = \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{= p_i} = \dot{p}_i \qquad \underline{\text{q.e.d.}}$$

By assuming E-L eqns of motion

2. $$S[q_i+\delta q_i] = S[q_i] + \underbrace{\int_{t_1}^{t_2} dt'\, \delta q_j(t')\, \frac{\delta S}{\delta q_j(t')}}_{\equiv\, \delta S}$$

with $$S[q_i(t)] = \int_{t_1}^{t_2} dt\, L[q_i(t), \dot{q}_i(t)]$$

$$\delta S = \int_{t_1}^{t_2} dt'\, \delta q_j(t') \int_{t_1}^{t_2} dt\, \frac{\delta L[q_i(t), \dot{q}_i(t)]}{\delta q_j(t')}$$

index i is summed over

$$= \int_{t_1}^{t_2} dt'\, \delta q_j(t') \int_{t_1}^{t_2} dt \left[\underbrace{\frac{\delta q_i(t)}{\delta q_j(t')}}_{=\,\delta_{ij}\,\delta(t-t')} \frac{\partial L}{\partial q_i(t)} + \underbrace{\frac{\delta \dot{q}_i(t)}{\delta q_j(t')}}_{=\,\delta_{ij}\,\frac{d}{dt}\delta(t-t')} \frac{\partial L}{\partial \dot{q}_i(t)} \right]$$

$$= \int_{t_1}^{t_2} dt\, \delta q_i(t) \frac{\partial L}{\partial q_i(t)} + \int_{t_1}^{t_2} dt'\, \delta q_i(t') \underbrace{\int_{t_1}^{t_2} dt \left(\frac{d}{dt}\delta(t-t')\right) \frac{\partial L}{\partial \dot{q}_i(t)}}$$

integrate by parts $\longrightarrow$ $\delta(t-t')\dfrac{\partial L}{\partial \dot{q}_i(t)}\Big|_{t=t_1}^{t=t_2} - \delta(t-t')\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i(t)}$

$$= \int_{t_1}^{t_2} dt\, \delta q_i(t) \left[\frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)} \right] + \underbrace{\delta q_i(t) \frac{\partial L}{\partial \dot{q}_i(t)}\Big|_{t=t_1}^{t=t_2}}_{=\,0,\ \text{since}\ \delta q_i(t_1) = \delta q_i(t_2) = 0}$$

Hence, we get

$$\frac{\delta S}{\delta q_i(t)} = \frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)} \overset{!}{=} 0$$

it needs to be set to zero to extremize the action

3. Proceeding as in lectures, we have

$$\delta S = \int_{t_1}^{t_2} dt \left[\delta q_i(t) \frac{\partial L}{\partial q_i(t)} + \delta \dot{q}_i(t) \frac{\partial L}{\partial \dot{q}_i(t)} + \delta \ddot{q}_i(t) \frac{\partial L}{\partial \ddot{q}_i(t)} \right]$$

$$= \int_{t_1}^{t_2} dt \; \delta q_i(t) \left[\frac{\partial L}{\partial q_i(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i(t)} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i(t)} \right]$$

$$+ \underbrace{\delta q_i(t) \frac{\partial L}{\partial \dot{q}_i(t)} \Big]_{t=t_1}^{t=t_2}}_{=0:\ \text{since}\ \delta q_i(t_1) = \delta q_i(t_2) = 0} + \underbrace{\delta \dot{q}_i(t) \frac{\partial L}{\partial \ddot{q}_i(t)} \Big]_{t=t_1}^{t=t_2}}_{=0,\ \text{since}\ \delta \dot{q}_i(t_1) = \delta \dot{q}_i(t_2) = 0}$$

Requiring that $\frac{\delta S}{\delta q_i(t)} = 0$, we get

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i(t)} + \frac{\partial L}{\partial q_i(t)} = 0 \qquad \text{q.e.d.}$$

4. $\mathcal{L}_{ED} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - J_\alpha A^\alpha$; $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

E-L eqn of motion for A_ν :

$$\partial_\mu \frac{\partial \mathcal{L}_{ED}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_{ED}}{\partial A_\nu} = 0$$

$$\frac{\partial \mathcal{L}_{ED}}{\partial A_\nu} = -J_\alpha \frac{\partial A^\alpha}{\partial A_\nu} = -J_\alpha \eta^{\alpha\nu} = -J^\nu ; \quad J^\nu = (\rho, \underline{J})$$

Given that $\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta$, we may calculate

$$\frac{\partial}{\partial(\partial_\mu A_\nu)} \underbrace{\left[-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\right]}_{-\frac{1}{2}\left((\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - (\partial_\alpha A_\beta)(\partial^\beta A^\alpha)\right)} = -\delta^\mu_\alpha \delta^\nu_\beta (\partial^\alpha A^\beta) + \delta^\mu_\alpha \delta^\nu_\beta (\partial^\beta A^\alpha)$$

$$= -\partial^\mu A^\nu + \partial^\nu A^\mu$$

$$= -F^{\mu\nu}$$

$$\partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} \left[-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\right] = -\partial_\mu F^{\mu\nu}$$

Hence, we get $-\partial_\mu F^{\mu\nu} + J^\mu = 0 \rightsquigarrow \partial_\mu F^{\mu\nu} = J^\nu$

$\nu = 0$: $\partial_i F^{i0} = J^0 = \rho \rightsquigarrow \underline{\nabla} \cdot \underline{E} = \rho$ (M1) $(\varepsilon_0 = \mu_0 = c = 1)$

$\nu = i$: $\underbrace{\partial_t F^{0i}}_{-\partial_t E_i} + \partial_j \underbrace{F^{ji}}_{\parallel} = J^i$

$$-\partial_t E_i + \underbrace{\partial_j (\varepsilon^{jik} B_k)}_{(\underline{\nabla} \times \underline{B})_i} \quad \rightsquigarrow \underline{\nabla} \times \underline{B} = \underline{J} + \partial_t \underline{E} \quad \text{(M4)}$$

(M2) & (M3) are obtained from the Bianchi identity:

$$\partial_{[\mu} F_{\nu\lambda]} = \partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0$$

or $\varepsilon^{\kappa\mu\nu\lambda} \partial_{[\mu} F_{\nu\lambda]} = 0$

For $\kappa = 0$, we get $\varepsilon^{0ijk} \partial_i F_{jk} = \partial_i (\underbrace{\varepsilon^{ijk} F_{jk}}_{-B_i}) = 0 \rightsquigarrow \underline{\nabla} \cdot \underline{B} = 0$ (M2)

For $\kappa = i$, we get (M3): $\underline{\nabla} \times \underline{E} = -\partial_t \underline{B}$ (just check by yourself!)

5*. A $N \times N$ complex matrix $U \in GL(N, \mathbb{C})$ can be expressed by $2 \times N^2$ real parameters

Since U has to be unitary as well ($U \in SU(N)$), U has to satisfy the constraints

$$U^{+}U = \mathbb{1}_N \rightsquigarrow \sum_j U^{+}_{ij} U_{jk} = \delta_{ik} : \begin{cases} N \text{ constraints}; \; i=k \text{ real parameter} \\ 2 \times \frac{N(N-1)}{2} \text{ real par.}; \; i \neq k \text{ constraints} \end{cases}$$

Hence, a general unitary matrix U has

$$2N^2 - N(N-1) - N = N^2 \text{ real parameters}$$

Given that $\det(U^{+}U) = |\det U|^2 = 1 \rightsquigarrow \det U = e^{i\varphi}$,

the constraint $\det U = 1 \rightsquigarrow \varphi = 0$ (i.e. it removes one real parameter)

$\therefore$ $U \in SU(N)$ has $N^2 - 1$ real independent params.

If

$U(\theta^a) = \exp[i\theta^a T^a]$, then unitarity requires that

Generators of SU(N)

$$U^{+}(\theta^a) = U^{-1}(\theta^a) \rightsquigarrow \exp[-i\theta^a T^{a+}] = \exp[-i\theta^a T^a]$$

$$\rightsquigarrow T^{a+} = T^a$$

Also, it should be: $\det U = 1$ or $\ln(\det U) = 0$

$$\rightsquigarrow \operatorname{Tr}[\ln U] = 0 \rightsquigarrow i\theta^a \operatorname{Tr} T^a = 0$$

or $\operatorname{Tr} T^a = 0$, since θ^a can be arbitrary

q.e.d.

6 COURSEWORK I is given on a separate file.