QUANTUM FIELD THEORY PHYS40481 2012: Prof A Pilaftsis

EXAMPLES SHEET III: Quantum Electrodynamics

1 Weyl Spinors and Lorentz Symmetry

(i) Show that
$$\bar{\xi}\bar{\sigma}^\mu\eta \;\equiv\; \bar{\xi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\eta_\beta \;=\; -\eta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}} \;\equiv\; -\eta\sigma^\mu\bar{\xi}\;,$$
where $\sigma^\mu = (\mathbf{1}_2,\ \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (\mathbf{1}_2,\ -\boldsymbol{\sigma})$.

(ii) Use (i) to verify that up to a total derivative $\propto \partial_\mu(\bar{\eta}\bar{\sigma}^\mu\eta)$, we get
$$\begin{aligned}\mathcal{L}_{\text{Dirac}} &= \overline{\Psi}_D\, i\gamma^\mu\partial_\mu\Psi_D \;-\; m_D\overline{\Psi}_D\Psi_D\\ &= \bar{\xi}\, i\bar{\sigma}^\mu\partial_\mu\xi \;+\; \bar{\eta}\, i\bar{\sigma}^\mu\partial_\mu\eta \;-\; m_D\,(\xi\eta + \bar{\eta}\bar{\xi}\,)\,.\end{aligned}$$

(iii)* Show that
$$M\sigma_\mu M^\dagger = \Lambda^\nu{}_\mu\,\sigma_\nu \quad\text{and}\quad M^{\dagger -1}\bar{\sigma}_\mu M^{-1} = \Lambda^\nu{}_\mu\,\bar{\sigma}_\nu\,,$$
where $\Lambda^\mu{}_\nu \;\in\; \mathrm{SO}(1,3)^\uparrow$, satisfying $\eta^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\,\eta^{\alpha\beta}$ (with $\Lambda^0{}_0 > 0$), and $M \in \mathrm{SL}(2,\mathbf{C})$, with $\varepsilon_{\alpha\beta} \;=\; M_a{}^\gamma M_\beta{}^\delta\varepsilon_{\gamma\delta}$.

(iv) Use (iii) to show that $\mathcal{L}_{\text{Dirac}}$ is invariant under a Lorentz transformation:
$$x^\mu \;\to\; x'^\mu \;=\; \Lambda^\mu{}_\nu x^\nu.$$

(v) Show that
$$\begin{aligned}(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\beta} \;+\; (\sigma^\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} &= 2\eta^{\mu\nu}\delta_\alpha{}^\beta\,,\\ (\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\sigma^\nu)_{\beta\dot{\beta}} \;+\; (\bar{\sigma}^\nu)^{\dot{\alpha}\beta}(\sigma^\mu)_{\beta\dot{\beta}} &= 2\eta^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\,,\end{aligned}$$

implying that
$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2\,\eta^{\mu\nu}\,\mathbf{1}_4\,.$$

(vi) Use (iii) and (v) to prove that
$$\Lambda^\mu{}_\nu \;=\; \frac{1}{2}\,\mathrm{Tr}\left(M^\dagger\bar{\sigma}^\mu M\sigma_\nu\right)\,,$$

and hence the isomorphism: $\mathrm{SO}(1,3)^\uparrow \;\simeq\; \mathrm{SL}(2,\mathbf{C})/\mathbf{Z}_2$, with $\mathbf{Z}_2 = \{\mathbf{1}_2\,,\,-\mathbf{1}_2\}$.

2 Quantization of Dirac Fermion Fields

(i) If $u(\mathbf{0},s)$ and $v(\mathbf{0},s)$ are the positive and negative energy solutions of the Dirac equation at the rest frame $p^\mu_{\rm rest} = (m,\mathbf{0})$ in momentum representation, show that

$$u(\mathbf{p},s) \;=\; \frac{\not p + m}{\sqrt{2m(E_{\mathbf{p}}+m)}}\, u(\mathbf{0},s)\;, \qquad v(\mathbf{p},s) \;=\; \frac{-\not p + m}{\sqrt{2m(E_{\mathbf{p}}+m)}}\, v(\mathbf{0},s)$$

are the general solutions in an arbitrary Lorentz frame, with $p^\mu = \Lambda^\mu{}_\nu p^\nu_{\rm rest}$.

[*Hint:* More details may be found in Appendix A of the textbook by S. Pokorski (see page 3 of lecture notes).]

(ii) Given that $\Pi_\Psi(x) = i\bar\Psi(x)\gamma^0$ is the momentum operator conjugate to the Dirac field operator $\Psi(x)$, prove the equal-time anti-commutation relation:

$$\Big\{\Psi_\alpha(t,\mathbf{x})\,,\,[i\bar\Psi(t,\mathbf{y})\gamma_0]_\beta\Big\} \;=\; i\delta_{\alpha\beta}\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\;,$$

where $\alpha,\beta = 1,2,3,4$ are Dirac spinor indices.

[*Hint:* You may find useful to know that

$$\{b(\mathbf{k},s)\,,\,b^\dagger(\mathbf{k}',s')\} \;=\; \{d(\mathbf{k},s)\,,\,d^\dagger(\mathbf{k}',s')\} \;=\; \delta_{ss'}\,(2\pi^3)\,2E_{\mathbf{k}}\delta^{(3)}(\mathbf{k}-\mathbf{k}')\;.$$
]

(iii) Show that the Feynman propagator for a Dirac fermion field is given by

$$i\,S_F(x-y) \;\equiv\; \langle 0|T\{\Psi(x)\bar\Psi(y)\}|0\rangle \;=\; \int\frac{d^4k}{(2\pi)^4}\,\frac{i(\not k+m)\,e^{-ik\cdot(x-y)}}{k^2-m^2+i\varepsilon}\;.$$

(iv)* Why does the quantization of fermions in (ii) require equal-time anti-commutators, rather than commutators? What would go wrong in the quantization of the theory?

3 Gauge Symmetry and Photons

As discussed in the lectures, the Lagrange density of Quantum Electrodynamics includes the interaction of the photon with the electron and is given by:

$$\mathcal{L}_{\rm QED} \;=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \;+\; \bar\psi\,(i\,\not\partial \;-\; m \;-\; e\,\not A)\,\psi\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and we used the convention: $\not a \equiv \gamma_\mu a^\mu$.

(i) Derive the equation of motions with respect to the photon and electron fields.

(ii) Derive the conserved current and charge from $\mathcal{L}_{\text{QED}}$.

(iii) How should the Lagrangian describing a complex scalar field $\phi(x)$,

$$\mathcal{L} \;=\; (\partial^\mu \phi)^* \, (\partial_\mu \phi) \;-\; m^2 \, \phi^* \phi \, ,$$

be extended so as to become gauge symmetric under a U(1) local transformations? Derive the Feynman rules for this extended theory of Scalar Quantum Electrodynamics.

(iv)* *The Stueckelberg Model:* A Lorentz-invariant photon mass term is described by the Lagrangian $\mathcal{L}_{\text{mass}} \;=\; \frac{1}{2}\, m_A^2 \, A^\mu A_\mu$. Find a renormalizable gauge-symmetric extension of $\mathcal{L}_{\text{mass}}$.

(v)* Like in (iv), find a gauge-symmetric (non-renormalizable) extension of $\mathcal{L}_{\text{D}}$ without the need of introducing a vector field A^μ. What are the physical consequences of such an extension?

4 The Photon Propagator and Gauge Fixing

As discussed in the lectures, the gauge-fixing part of the QED Lagrange density reads:

$$\mathcal{L}_{\text{GF}} \;=\; -\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 \, ,$$

where ξ is the so-called gauge-fixing parameter.

(i) Derive the Euler-Lagrange equation of the photon in the presence of $\mathcal{L}_{\text{GF}}$.

(ii) Show that the photon propagator is given by the Green function:

$$\Delta_{\mu\nu}(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4} \left(-\,\eta_{\mu\nu} \;+\; (1-\xi)\,\frac{k_\mu k_\nu}{k^2} \right) \frac{e^{-ik\cdot(x-y)}}{k^2 + i\varepsilon} \, .$$

(iii) Use the equal-time commutators to show that

$$\langle 0 | T[A_\mu(x) A_\nu(y)] | 0 \rangle \;=\; i\Delta_{\mu\nu}(x-y)$$

in the Feynman gauge $\xi = 1$.

5 COURSEWORK III: Scattering Processes in Quantum Electrodynamics

(i) Show that

(a) $\mathrm{Tr}(\gamma_\mu \gamma_\nu) \; = \; 4\,\eta_{\mu\nu}$,

(b) $\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) \; = \; 4\,(\eta_{\mu\nu}\eta_{\rho\sigma} \; + \; \eta_{\mu\sigma}\eta_{\nu\rho} \; - \; \eta_{\mu\rho}\eta_{\nu\sigma})$,

(c) $\mathrm{Tr}(\gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_{2n+1}}) \; = \; 0$
(Hint: you may use the properties: $\{\gamma_5\,,\,\gamma_\mu\} = 0$ and $\gamma_5^2 = \mathbf{1}_4$, where $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$.),

(d) $\sum_{s=\pm 1/2} \bar{u}(p,s) M u(p,s) \; = \; \mathrm{Tr}\,[M\,(\not{p} + m)]$, where M is any arbitrary 4×4 matrix.

(ii) Use the Feynman rules for QED to write down the matrix element $\mathcal{M}_{fi}$ for the reaction $e^-(p_1)e^+(p_2) \to \mu^-(k_1)\mu^+(k_2)$.

(iii) With the aid of trace techniques given in (i), calculate $\overline{|\mathcal{M}_{fi}|}^2$, where the long bar indicates averaging over the spins of the electrons in the initial state.

(iv) Calculate analytically the differential cross section $d\sigma/d\Omega$ for $e^-e^+ \to \mu^-\mu^+$ which was taking place at the CERN LEP collider at CMS energies $\sqrt{s} = M_Z = 90$ GeV. Draw an accurate graph of $d\sigma/d\Omega$ as a function of $\cos\theta$.

(v) Supersymmetry predicts that in addition to muons $\mu^\pm$ there should be scalar muons $\tilde{\mu}^\pm$. Calculate $d\sigma/d\Omega$ for the process $e^-e^+ \to \tilde{\mu}^-\tilde{\mu}^+$. Plot $d\sigma/d\Omega$ as a function of $\cos\theta$ and comment on your results.