# GAUGE THEORIES

by Apostolos Pilaftsis

- Preliminaries
- Group Theory
- Quantum Chromodynamics
- The Standard Model of Electroweak Interactions
- Beyond the Standard Model

# Lecture 1

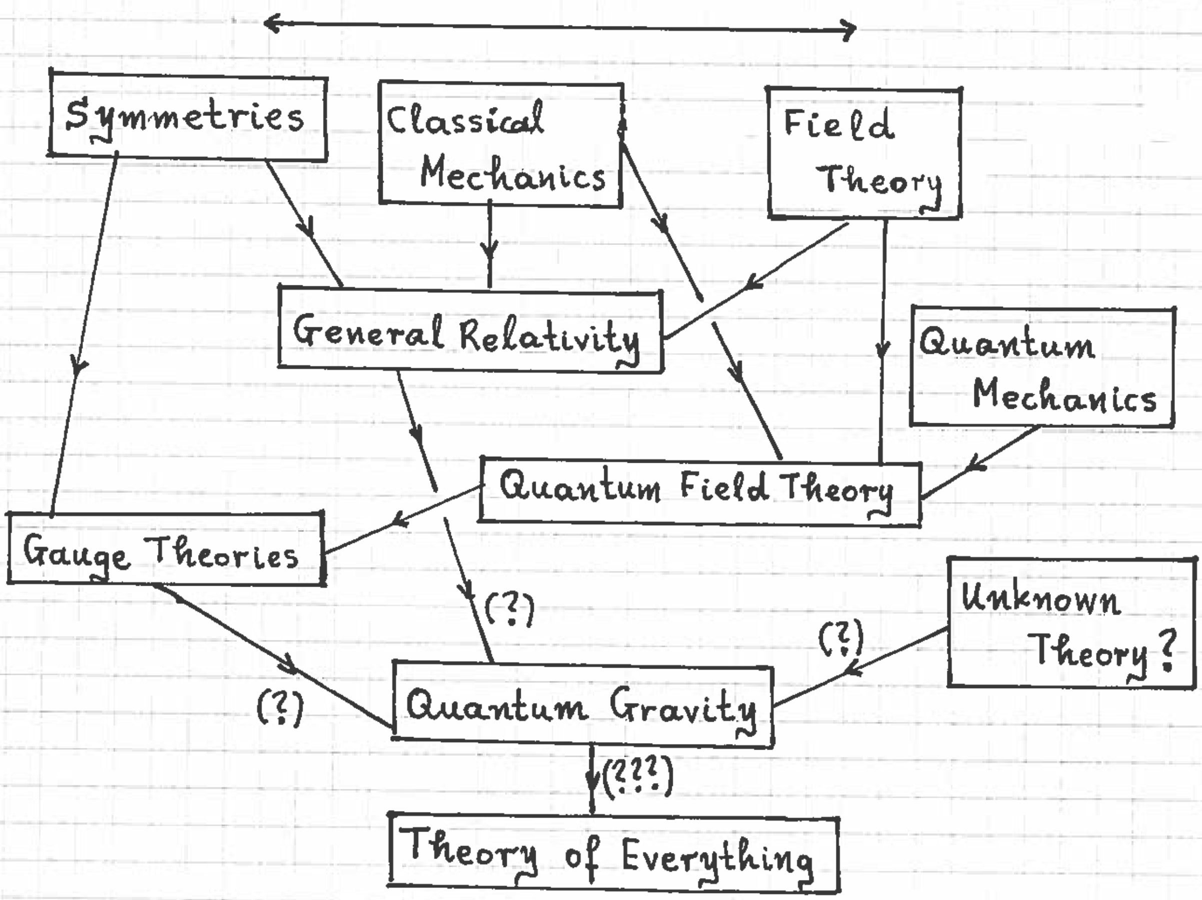
Introductory remarks:

- Electronic access of lecture notes (blackboard + personal website)
- Structure of the course

  Pre-requisites: Advanced Dynamics, Lagrangian Dynamics, Electrodynamics, Advanced Quantum Mechanics, Quantum Field Theory (QFT).

  Desirable: General Relativity (Gravitation), Mathematical Methods in Physics

- Literature (Revision: lecture notes on QFT by A.P.)
- Coursework for PhD, MSc students & UG students



# SUMMARY (Revision highlights)

## Classical Mechanics:

$$\dot{q}_i = \{q_i, H\}_{(q_j,p_j)} = \sum_{j=1}^{N} \underbrace{\frac{\partial q_i}{\partial q_j}}_{=\delta_{ij}} \frac{\partial H}{\partial p_j} - \cancelto{0}{\frac{\partial q_i}{\partial p_j}} \frac{\partial H}{\partial q_j} = \frac{\partial H}{\partial p_i}$$

Poisson brackets

$$\dot{p}_i = \{p_i, H\}_{(q_j,p_j)} = \sum_{j=1}^{N} \cancelto{0}{\frac{\partial p_i}{\partial q_j}} \frac{\partial H}{\partial p_j} - \underbrace{\frac{\partial p_i}{\partial p_j}}_{=\delta_{ij}} \frac{\partial H}{\partial q_j} = -\frac{\partial H}{\partial q_i}$$

generalized coordinates

conjugate momenta

and $\{q_i, p_j\} = \delta_{ij}$ ,

degrees of freedom

where $H(q_i, p_i) = \sum_{i=1}^{N} p_i \dot{q}_i - L(q_i, \dot{q}_i)$ and $p_i = \dfrac{\partial L(q_j, \dot{q}_j)}{\partial \dot{q}_i}$

Hamiltonian

Lagrangian

## Relativistic Quantum Mechanics:

Operators

Replace $\{q, p\} = 1$ with $-i[\hat{q}, \hat{p}] = 1$ (in units of $\hbar$)

or $[\hat{q}, \hat{p}] = i$

Relativistic formulation: $[\hat{x}^\mu(\tau), \hat{p}^\nu(\tau)] = -i\eta^{\mu\nu}$

$\tau$: proper time

$\eta^{\mu\nu} \equiv \mathrm{diag}(1,-1,-1,-1)$

$p_\mu = \dfrac{\partial L}{\partial \dot{x}^\mu} \mapsto \hat{p}_\mu(\tau)$: promoted to an operator in the Heisenberg picture

Equivalent to 1+0 dim QFT

## Quantum Field Theory:

Promote fields to field operators: $\Phi(t, \underline{x}) \mapsto \hat{\Phi}(t, \underline{x})$

and $\Pi(t, \underline{x}) = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} \mapsto \hat{\Pi}(t, \underline{x})$

Conjugate momentum operator

$\mathcal{L}$: Lagrange density.

Canonical quantization:

$$[\hat{\Phi}(t, \underline{x}), \hat{\Pi}(t, \underline{y})] = i\,\delta^{(3)}(\underline{x} - \underline{y})$$

1+3 dim QFT

# Lecture 2

## Symmetries of the Lagrangian

Consider $\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2$

$\mathcal{L}$ is invariant under : (i) Global $U(1)$ — $e^{i\theta} \in U(1)$

$\phi(x) \mapsto \phi'(x) = e^{i\theta} \phi(x)$ , with $\theta = \text{const.}$

(ii) Poincaré Group describing rotations, Lorentz boosts and spacetime translations (Why?)

$\mathcal{L}$ is not invariant under Local $U(1)$ transformations (Why?)

## Other possible symmetries (Global)

**SO(2)** $\mathcal{L}[\phi_i] = \frac{1}{2}(\partial^\mu \phi_i)(\partial_\mu \phi_i) - \frac{m^2}{2}\phi_i \phi_i - \frac{\lambda}{4}(\phi_i \phi_i)(\phi_j \phi_j)$ ; $i,j = 1,2$

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \Phi' = \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \Rightarrow \mathcal{L}[\phi'_i] = \mathcal{L}[\phi_i]$$

Set of $2\times 2$ real orthogonal matrices $O$ ($O^T O = O O^T = \mathbb{1}_2$) with $\det O = 1$
or $SO(2) = \{ O \in GL(2,\mathbb{R}) / O^T = O^{-1} \wedge \det O = 1 \}$

$SO(2) \ni O = e^{i\theta\sigma_2}$ ; $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Lie Generator of $SO(2)$

Property: $\phi_1'^2 + \phi_2'^2 = \phi_1^2 + \phi_2^2$

**SU(2)** $\mathcal{L}[\phi_i, \phi_i^*] = (\partial^\mu \phi_i^*)(\partial_\mu \phi_i) - m^2 \phi_i^* \phi_i - \lambda(\phi_i^* \phi_i)^2$

2x2 complex unitary matrices:
$U^\dagger U = U U^\dagger = \mathbb{1}_2$ with $\det U = 1$
or $SU(2) = \{ U \in GL(2,\mathbb{C}) / U^\dagger = U^{-1} \wedge \det U = 1 \}$
Property : $|\phi'_1|^2 + |\phi'_2|^2 = |\phi_1|^2 + |\phi_2|^2$

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \xrightarrow{SU(2)} \Phi' = \overbrace{e^{i\theta^a T^a}}^{\in SU(2)} \Phi \, ; \quad a = 1,2,3$$

$\Rightarrow \mathcal{L}[\phi'_i, \phi'^*_i] = \mathcal{L}[\phi_i, \phi^*_i]$

Lie Generators of $SU(2)$ : $T^{1,2,3} = \frac{\sigma_{1,2,3}}{2}$ (Pauli matrices) , $\theta^{1,2,3} \in \mathbb{R}$: group parameter

## Noether's theorem

If the action $S[\phi_i] = \int d^4x \, \mathcal{L}[\phi_i]$ is invariant under a given transf. of spacetime and/or fields, then there exists:

(i) Conserved current $J^\mu$ : $\partial_\mu J^\mu = 0$

(ii) Conserved charge $Q \mathrel{\hat=} \int_{V\mapsto\infty} d^3\underline{x}\, J^0(x)$ : $\frac{dQ}{dt} = 0$

~~its~~ existence depends on the vanishing of surface terms

Proof for global symmetries:

Consider $\phi_i \mapsto \phi_i' = \left(e^{i\theta^a T^a}\right)_i{}^j \phi_j = \phi_i + \underbrace{i\theta^a (T^a)_i{}^j \phi_j}_{\mathrel{\hat=}\, \delta\phi_i} + \mathcal{O}(\theta^{a2})$

possible total derivative

such that $\mathcal{L}[\phi_i, \partial_\mu\phi_i] = \mathcal{L}[\phi_i', \partial_\mu\phi_i'] + \partial_\mu \delta Q^\mu(\phi_i')$

$$= \mathcal{L}[\phi_i, \partial_\mu\phi_i] + \delta\mathcal{L}$$

where $\delta\mathcal{L} = \delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + (\partial_\mu\delta\phi_i)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} + \partial_\mu \delta Q^\mu = 0$

$$\rightsquigarrow \delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \delta Q^\mu\right] + \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right]}_{=\frac{\delta S}{\delta\phi_i} = 0}\delta\phi_i = 0$$

Euler-Lagrange equations of motion for $\phi_i$

$$\Rightarrow \quad \partial_\mu \underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \delta Q^\mu\right]}_{\propto J^\mu} = 0$$

Conserved current(s):

$$J^{a,\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\frac{\partial\delta\phi_i}{\partial\theta^a} + \frac{\partial\delta Q^\mu}{\partial\theta^a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} i(T^a)_i{}^j\phi_j + \frac{\partial\delta Q^\mu}{\partial\theta^a}$$

Conserved charge(s): $Q^a \mathrel{\hat=} \int_{V\mapsto\infty} d^3\underline{x}\, J^{a,0}(x)$

because $\frac{dQ^a}{dt} = \int_V d^3\underline{x}\, \partial_t J^{a,0} = -\int_V d^3\underline{x}\, \underline{\nabla}\cdot\underline{J}^a \overset{\text{Gauss}}{=} -\oint_{S(V)} d\underline{s}\cdot\underline{J}^a \underset{V\to\infty}{\longmapsto} 0$

assuming that surface terms = currents vanish at spatial infinity

# Revision material from lectures notes (pages 4-14)

- Lagrangian for a free Klein-Gordon field $\varphi$:

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

- Lagrangian for an electromagnetic field $A^\mu$:

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \quad ; \quad \underbrace{F_{\mu\nu}}_{\text{field strength tensor}} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Lagrangian for a Dirac fermion $\psi$:

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \, ; \quad \psi(x) = \begin{pmatrix} \xi_\beta(x) \\ \bar{\eta}^{\dot\beta}(x) \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix},$$

$$\bar\psi(x) \mathrel{\hat=} \left(\eta^\alpha(x), \bar\xi_{\dot\alpha}(x)\right),$$

and $\sigma^\mu \mathrel{\hat=} (\mathbb{1}_2, \underbrace{\sigma_1, \sigma_2, \sigma_3}_{\hat= \underline{\sigma}})$, $\bar\sigma^\mu \mathrel{\hat=} (\mathbb{1}_2, -\underline{\sigma})$

- Quantum Electrodynamics (QED)

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar\psi (i\partial\!\!\!/ - m - eA\!\!\!/)\psi$$

Gauge or local $U(1)$ invariance of $\mathcal{L}_{QED}$

under: $A_\mu \mapsto A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \theta$ , $\psi \mapsto \psi' = e^{i\theta}\psi$

- Photon propagator and Gauge fixing

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \frac{e^{-ik\cdot(x-y)}}{k^2 + i\varepsilon}$$

- QED Feynman Rules.

- Renormalization of scalar field theories

e.g. see lecture notes on QFT by A.P., pages 38-43.