QUANTUM FIELD THEORY PHYS40481 2012: Prof A Pilaftsis

EXAMPLES SHEET II: Canonical Quantization and the S-Matrix

1 Creation and Annihilation Operators

Show that

(i) $[a(\mathbf{p}),\, a^\dagger(\mathbf{q})] \;=\; (2\pi)^3\, 2E_p\, \delta^{(3)}(\mathbf{p}-\mathbf{q})$ is Lorentz invariant.

(ii) the Hermitian adjoint $a(\mathbf{p})$ is the **annihilation operator** removing a particle from a Fock state as follows:

$$a(\mathbf{p})\,|\mathbf{q}_1, \mathbf{q_2}, \ldots, \mathbf{q}_N\rangle \;=\; \sum_{i=1}^{N} (2\pi)^3 2E_p\, \delta^{(3)}(\mathbf{p}-\mathbf{q}_i)\,|\mathbf{q}_1, \ldots, \mathbf{q}_{i-1}, \mathbf{q}_{i+1}, \ldots, \mathbf{q}_N\rangle\ .$$

(iii) the integral measure

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \;=\; \int \frac{d^4k}{(2\pi)^4}\, 2\pi\, \theta(k_0)\, \delta(k^2 - m^2)$$

is Lorentz invariant, where $\theta(z)$ is the step function.

2 Quantization of Complex Fields

A theory for a complex field is described by the Lagrangian: $\mathcal{L} \;=\; (\partial^\mu \Phi^\dagger)\,(\partial_\mu \Phi) - m^2\, \Phi^\dagger \Phi$, where the complex field operator Φ is given by

$$\Phi(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_k} \Big(a(\mathbf{k})\, e^{-ik\cdot x} \;+\; b^\dagger(\mathbf{k})\, e^{ik\cdot x} \Big)\ .$$

Given that $[a(\mathbf{k}),\, a^\dagger(\mathbf{k}')] = [b(\mathbf{k}),\, b^\dagger(\mathbf{k}')] = (2\pi)^3 2E_k\, \delta^{(3)}(\mathbf{k}-\mathbf{k}')$ are the only non-zero commutators, prove the equal-time commutation relations:

$$[\Phi(t,\mathbf{x}),\, \Pi(t,\mathbf{y})] \;=\; [\Phi^\dagger(t,\mathbf{x}),\, \Pi^\dagger(t,\mathbf{y})] \;=\; i\delta^{(3)}(\mathbf{x}-\mathbf{y})\ ,$$

whereas all other commutators vanish, including $[\Phi(t,\mathbf{x}),\, \Pi^\dagger(t,\mathbf{y})] = 0$. Then, use $\mathcal{L}$ to derive the Hamilton density operator $\mathcal{H}$ in terms of the complex field operators $\Phi(x)$, $\Phi^\dagger(x)$ and their conjugate momentum operators $\Pi(x)$ and $\Pi^\dagger(x)$. Verify that $\mathcal{H}[\Phi, \Pi]$ is positive definite for $m^2 > 0$.

3 The Time Evolution Operator

(i) Prove the relation: $|A(t);t_0\rangle_I \;=\; U(t,t_0)\,|A;t_0\,\rangle_H$, where $U(t,t_0) = e^{iH_0^S(t-t_0)}\,e^{-iH^S(t-t_0)}$ is the time evolution operator.

(ii) Show that

$$i\,\frac{\partial}{\partial t}\,U(t,t_0) \;=\; H^I_{\rm int}(t)\,U(t,t_0)\;.$$

(iii) In the lectures, we argued that $U(t,t_0)$ can be written down as

$$\begin{aligned} U(t,t_0) \;&=\; \hat{\mathbf{1}} \,+\, (-i)\int_{t_0}^{t} dt_1\,H^I_{\rm int}(t_1) \,+\, \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\,T\{H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\} \,+\,\dots \\ &=\; T\Big\{\exp\Big[-i\int_{t_0}^{t} dt'\,H^I_{\rm int}(t')\Big]\Big\}\;. \end{aligned}$$

In order for this argument to go through, you need to show that

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\,T\{H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\} \;=\; \frac{1}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\,T\{H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\}\;.$$

4** Quantum Field Theory Techniques in Quantum Mechanics

The Hamilton operator describing an electron within a homogeneous magnetic field B_z in the z direction is

$$H_0 \;=\; E\,\mathbf{1}_2 \,+\, \mu B_z\,\sigma_3\;,$$

where σ_3 is the third Pauli matrix and μ is the magnetic moment of the electron. In this system, a small alternating magnetic field $b_x(t) = b_x \sin\Omega t$ (with $b_x \ll B_z$) is applied to the x direcion given by the interaction Hamiltonian

$$H_{\rm int}(t) \;=\; \mu b_x(t)\,\sigma_1\;,$$

where σ_1 is the first Pauli matrix. Introduce a time evolution operator $U(t;t_0)$ analogous to Example 3 to calculate the probability transition amplitude for an electron with spin $s_z = 1/2$ at time $t_0 = 0$ to flip spin to $s_z = -1/2$ at time t. Compare your result with the one obtained by ordinary Quantum Mechanics.

5 Feynman, Retarded and Advanced Propagators

(i) Show that $\Delta_F(x-y)$ is the Green function of the Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\, \Delta_F(x-y) = -\delta^{(4)}(x-y)\ .$$

(ii) The retarded and advanced propagator are defined as

$$i\Delta_{\rm R}(x-y) \ \equiv\ \theta(x^0-y^0)\, \langle 0|[\Phi(x)\,, \Phi(y)]|0\rangle\ , \qquad i\Delta_{\rm A}(x-y) \ \equiv\ -\theta(y^0-x^0)\, \langle 0|[\Phi(x)\,, \Phi(y)]|0\rangle\ .$$

Use complex integration techniques to show that these propagators can be written down as

$$i\Delta_{\rm R(A)}(x-y) \ =\ \int \frac{d^4k}{(2\pi)^4}\, \frac{i\, e^{-ik\cdot(x-y)}}{k^2\ -\ m^2\ +(-)\, ik^0\varepsilon}\ ,$$

with $\varepsilon = 0^+$.

(iii) Show that $i\Delta_{\rm R,A}(x-y)$ are also Green functions of the Klein–Gordon equation. What are their physical properties and their difference from the Feynman propagator?

6 Hamilton's Equations in Quantum Field Theory

The Hamilton operator of a quantum system is given by

$$H \ =\ \int d^3\mathbf{y}\ \mathcal{H}(t,\mathbf{y})\ ,$$

where $\mathcal{H}(t,\mathbf{y})$ is the Hamilton density operator derived in Example 2. Prove the quantum field-theoretic Hamilton equations:

$$\text{(a)} \qquad [\, H,\ \Phi(t,\mathbf{x})\,] \ =\ -\,i\, \frac{\partial \Phi(t,\mathbf{x})}{\partial t}\ ,$$

$$\text{(b)} \qquad [\, H,\ \Pi(t,\mathbf{x})\,] \ =\ -\,i\, \frac{\partial \Pi(t,\mathbf{x})}{\partial t}\ .$$

[*Hint:* Use the Euler–Lagrange equations of motion for the field operators Φ and $\Phi^\dagger$ to prove equation (b).]

7 COURSEWORK II: Scattering Processes and Cross Sections

(i) Show that

$$\frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\left(\frac{1}{2E_{p_1}}\right)\left(\frac{1}{2E_{p_2}}\right) \; = \; \frac{1}{2\,\lambda^{1/2}(s, m_{a_1}^2, m_{a_2}^2)}\;,$$

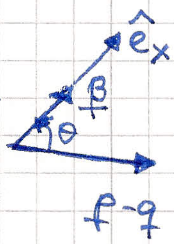
where $\lambda(x,y,z) = (x-y-z)^2 - 4yz$ and $s = (p_1+p_2)^2$, and so that σ is Lorentz invariant.

(ii) Calculate analytically the cross section σ for the scattering processes: (a) $\chi^+\chi^- \to \chi^+\chi^-$ and (b) $\chi^+\chi^- \to \Phi\Phi$.

(iii) Assuming that the scattering processes in (ii) take place at centre-of-mass energies $\sqrt{s} = 300$ GeV, estimate the value of σ in both GeV^{-2} and nb, for $M = 120$ GeV, $m = 50$ GeV and $\lambda = g/M = 0.5$.

(iv) How many scatterings do you expect to detect per year at a collider such as the TEVATRON with $L = 10^{31}\,\mathrm{cm}^{-2}s^{-1}$?

SOLUTIONS TO EXAMPLES SHEET II



1 (i) A general Lorentz boost of $p-q$ to a given direction β can be thought of as a rotation of the vector $p-q$ to become parallel to β and then a boost along β.

Under an orthogonal transformation of $p-q$:

$$p'^i - q'^i = O^{ij}(\theta)\,(p-q)^j\,,$$

we have $$\delta^{(3)}(p'-q') = \underbrace{\left|\det \frac{\partial(p'^i - q'^i)}{\partial(p-q)^j}\right|^{-1}}_{\underbrace{|\det O(\theta)|^{-1}}_{=1}} \delta^{(3)}(p-q)$$

Determinant of the Jacobian

$= 1$, since O is orthogonal matrix.

In addition, $E_{p'} = \sqrt{|p'|^2 + m^2} = \sqrt{p^2 + m^2} = E_p$

Hence, $E_{p'}\,\delta^{(3)}(p'-q') = E_p\,\delta^{(3)}(p-q)$ is invariant under rotations.

Assume $\beta \parallel \hat{e}_x \parallel p'$ & boost $\overset{(p'-q')}{\downarrow}$ along $\hat{e}_x$:

$$(p'' - q'')_x = \gamma\beta\,(E_{p'} - E_{q'}) + \gamma\,(p' - q')_x$$

$$p''_{y,z} = p'_{y,z}$$

$$E_{p''} = \gamma E_p + \gamma\beta p'_x = E_p\left(\gamma + \gamma\beta\frac{p'_x}{E_{p'}}\right)$$

Also, $$\frac{d(p''-q'')_x}{d p'_x} = \gamma\beta\frac{p'_x}{E_{p'}} + \gamma = \gamma\beta\frac{p'_x}{E_p} + \gamma = \frac{E_{p''}}{E_p}$$

Consequently, $$E_{p''}\,\delta^{(3)}(p''-q'') = E_{p''}\left(\frac{d(p''-q'')_x}{dp'_x}\right)^{-1}\delta^{(3)}(p'-q')$$

$$= E_{p''}\,\frac{E_p}{E_{p''}}\,\delta^{(3)}(p-q) = E_p\,\delta^{(3)}(p-q) \quad \underline{\text{q.e.d.}}$$

(ii)

$$a(p)\,|q_1, q_2, \dots, q_N\rangle = a_p\, a^\dagger_{q_1} a^\dagger_{q_2} \dots a^\dagger_{q_N} |0\rangle$$

$$= (2\pi)^3\, 2E_p\, \delta^{(3)}(p - q_1)\, a^\dagger_{q_2} \dots a^\dagger_{q_N} |0\rangle$$

$$+ \; a^\dagger_{q_1}\, a_p\, a^\dagger_{q_2} \dots a^\dagger_{q_N} |0\rangle$$

$$= (2\pi)^3\, 2E_p\, \delta^{(3)}(p - q_1)\, |q_2 \dots q_N\rangle$$

$$+ (2\pi)^3\, 2E_p\, \delta^{(3)}(p - q_2)\, a^\dagger_{q_1} a^\dagger_{q_3} \dots a^\dagger_{q_N} |0\rangle$$

$$+ \; a^\dagger_{q_1} a^\dagger_{q_2}\, a_p\, a^\dagger_{q_3} \dots a^\dagger_{q_N} |0\rangle = \dots$$

$$= (2\pi)^3\, 2E_p \sum_{i=1}^{N} \delta^{(3)}(p - q_i)\, |q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_N\rangle$$

$$+ \; \underbrace{a^\dagger_{q_1} a^\dagger_{q_2} \dots a^\dagger_{q_N}\, a_p |0\rangle}_{\to 0}$$

q.e.d.

(iii) $$\int \frac{d^4k}{(2\pi)^4}\, 2\pi\, \theta(k_0)\, \delta(k^2 - m^2)$$ ⟵ manifestly Lorentz invariant

$$= \int \frac{d^4k}{(2\pi)^4}\, 2\pi\, \theta(k_0)\, \frac{1}{2E_k} \left[\delta(k^0 - E_{\underline{k}}) + \delta(k^0 + E_{\underline{k}}) \right]$$

$$\boxed{\text{Only } k^0 = E_{\underline{k}} > 0 \text{ contributes}}$$

$$E_{\underline{k}} = \sqrt{\underline{k}^2 + m^2}$$

$$= \int \frac{d^3\underline{k}}{(2\pi)^3}\, \frac{1}{2E_k}$$

q.e.d.

2

$$\Phi(x) = \int_{\underline{k}} a_{\underline{k}}\, e^{-ik\cdot x} + b^{+}_{\underline{k}}\, e^{ik\cdot x} \quad ; \quad k^0 \equiv E_k \,,\ \int_{\underline{k}} \equiv \int \frac{d^3\underline{k}}{(2\pi)^3\, 2E_k}$$

$$\Pi(x) = \partial_t \Phi^{+}(x) = \int_{\underline{k}} +iE_k \left(a^{+}_{\underline{k}}\, e^{+ik\cdot x} - b_{\underline{k}}\, e^{-ik\cdot x} \right)$$

$$[\Phi(t,\underline{x}), \Pi(t,\underline{y})] = \int_{\underline{p}} \int_{\underline{q}} \left[a_{\underline{p}}\, e^{-ip\cdot x} + b^{+}_{\underline{p}}\, e^{ip\cdot x},\ +iE_q \left(a^{+}_{\underline{q}}\, e^{+iq\cdot y} - b_{\underline{q}}\, e^{-iq\cdot y} \right) \right]$$

with $x^0 = y^0 = t$

$$= \int_{\underline{p}} \int_{\underline{q}} +iE_q \left\{ \underbrace{[a_{\underline{p}}, a^{+}_{\underline{q}}]}_{(2\pi)^3\, 2E_p\, \delta^{(3)}(\underline{p}-\underline{q})} e^{-ip\cdot x + iq\cdot y} - \underbrace{[b^{+}_{\underline{p}}, b_{\underline{q}}]}_{-(2\pi)^3\, 2E_p\, \delta^{(3)}(\underline{p}-\underline{q})} e^{ip\cdot x - iq\cdot y} \right\}$$

$$= \int \frac{d^3\underline{p}}{(2\pi)^3} \frac{i}{2} \left[e^{-i\underline{p}\cdot(\underline{x}-\underline{y})} + e^{-i\underline{p}\cdot(\underline{x}-\underline{y})} \right] = i \int \frac{d^3\underline{p}}{(2\pi)^3}\, e^{i\underline{p}\cdot(\underline{x}-\underline{y})}$$

$$= i\, \delta^{(3)}(\underline{x}-\underline{y}) \quad \underline{\text{q.e.d.}}$$

integral $\int d^3\underline{p}$ sym: $\underline{p} \leftrightarrow -\underline{p}$

All other commutators vanish, e.g.

$$[\Phi(t,\underline{x}), \Pi^{+}(t,\underline{y})] = \int_{\underline{p}} \int_{\underline{q}} \left[a_{\underline{p}}\, e^{-ip\cdot x} + b^{+}_{\underline{p}}\, e^{ip\cdot x},\ -iE_q \left(a_{\underline{q}}\, e^{-iq\cdot y} - b^{+}_{\underline{q}}\, e^{iq\cdot y} \right) \right]$$

$$= 0 \ , \quad \text{since } [a_{\underline{p}}, a_{\underline{q}}] = [a_{\underline{p}}, b^{+}_{\underline{q}}] = 0$$

$$\mathcal{H}[\Phi, \Pi] = \Pi^{+} \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^{+}} + \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \Pi - \mathcal{L}[\Phi, \Phi^{+}] \quad ; \quad \mathcal{L} = (\partial_\mu \Phi)^{+} (\partial^\mu \Phi) - m^2 \Phi^{+} \Phi$$

$$= 2\Pi^{+}\Pi - \Pi^{+}\Pi + \underline{\nabla}\Phi^{+} \cdot \underline{\nabla}\Phi + m^2 \Phi^{+}\Phi$$

$$\rightsquigarrow \mathcal{H}[\Phi, \Pi] = \Pi^{+}\Pi + \underline{\nabla}\Phi^{+} \cdot \underline{\nabla}\Phi + m^2 \Phi^{+}\Phi$$

with $m^2 > 0$

Sum of positive definite operators

3

(i) Proceed as in lectures and require that

$$ {}_I\langle B(t);t_0|\underbrace{\hat{O}^I(t)}_{\equiv e^{iH_0^S(t-t_0)}\hat{O}^S e^{-iH_0^S(t-t_0)}}|A(t);t_0\rangle_I \overset{!}{=} {}_S\langle B(t);t_0|\hat{O}^S|A(t);t_0\rangle_S $$

$$ \rightsquigarrow e^{-iH_0^S(t-t_0)}|A(t);t_0\rangle_I = |A(t);t_0\rangle_S = e^{-iH^S(t-t_0)}\underbrace{|A(t_0);t_0\rangle_S}_{\overset{!}{=}|A;t_0\rangle_H} $$

$$ \rightsquigarrow |A(t);t_0\rangle_I = \underbrace{e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)}}_{\equiv \mathcal{U}(t,t_0)}|A;t_0\rangle_H $$

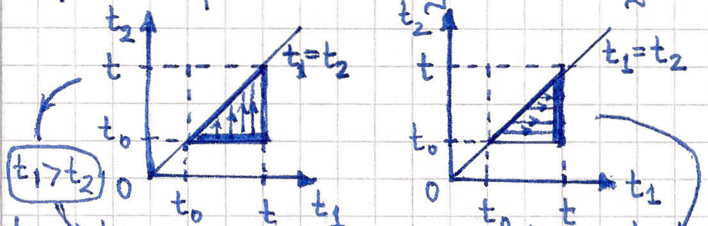
(ii)

$$ i\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = e^{iH_0^S(t-t_0)}\underbrace{\left(-H_0^S+H^S\right)}_{H^S_{int}}e^{-iH^S(t-t_0)} $$

insert $\mathbb{1} = e^{-iH_0^S(t-t_0)}e^{iH_0^S(t-t_0)}$

$$ = \underbrace{e^{iH_0^S(t-t_0)}H^S_{int}e^{-iH_0^S(t-t_0)}}_{\equiv H^I_{int}}\underbrace{e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)}}_{\equiv \mathcal{U}(t,t_0)} \quad \underline{\text{q.e.d.}} $$

(iii) Two possible ways to integrate:



; with $H^I_{(t)} \equiv H^I_{int}(t)$

$t_2 > t_1$

$$ \int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\,\underbrace{H^I(t_1)H^I(t_2)}_{=T\{H^I(t_1)H^I(t_2)\}} = \int_{t_0}^{t}dt_2\int_{t_2}^{t}dt_1\,H^I(t_1)H^I(t_2) \overset{\text{swap } t_1\leftrightarrow t_2}{=} \int_{t_0}^{t}dt_1\int_{t_1}^{t}dt_2\,\underbrace{H^I(t_2)H^I(t_1)}_{T\{H^I(t_1)H^I(t_2)\}} $$

$$ = \frac{1}{2}\int_{t_0}^{t}dt_1\underbrace{\left(\int_{t_0}^{t_1}dt_2+\int_{t_1}^{t}dt_2\right)}_{=\int_{t_0}^{t}dt_2}T\{H^I(t_1)H^I(t_2)\} \qquad \underline{\text{q.e.d.}} $$

4**

This is left as a topic of research.

5 (i) $$(\Box_x + m^2)\underbrace{\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-m^2+i\varepsilon}}_{=\Delta_F(x-y)} = \int \frac{d^4k}{(2\pi)^4}\frac{(\Box_x+m^2)\,e^{-ik\cdot(x-y)}}{k^2-m^2+i\varepsilon}$$

$$\Box_x = \boxed{\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}}$$

$$= \int \frac{d^4k}{(2\pi)^4}\underbrace{\frac{-k^2+m^2}{k^2-m^2}}_{=-1} e^{-ik\cdot(x-y)} = -\,\delta^{(4)}(x-y) \qquad \underline{\text{q.e.d.}}$$

(ii) From the lectures, we know

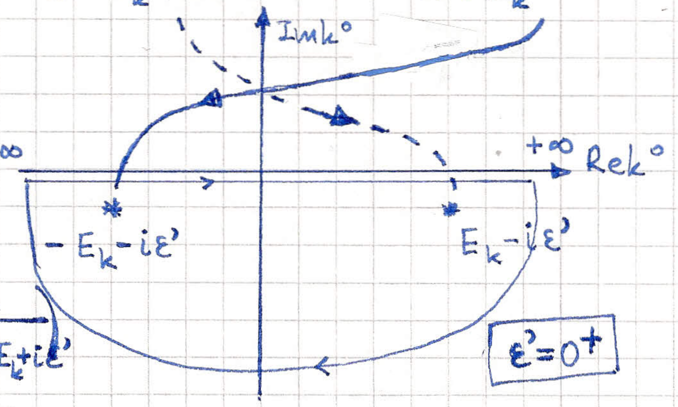
$$[\phi(x),\phi(y)] \equiv \Delta(x-y) = \int \frac{d^3k}{(2\pi)^3\,2E_k}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right) \qquad \boxed{\text{with } k^0 \equiv E_k}$$

Proceed as in lectures for $\Delta_F(x-y)$:

$$i\Delta_R(x-y) = \theta(x^0-y^0)\,\langle 0|\Delta(x-y)|0\rangle = \theta(x^0-y^0)\,\Delta(x-y)$$

$$= \int \frac{d^3k}{(2\pi)^3\,2E_k}\, e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\left[e^{-iE_k(x^0-y^0)}\theta(x^0-y^0) - e^{iE_k(x^0-y^0)}\theta(x^0-y^0)\right]$$

$$= \int \frac{d^3k}{(2\pi)^3\,2E_k}\, e^{i\underline{k}\cdot(\underline{x}-\underline{y})}\left[-\int_{-\infty}^{+\infty}\frac{dk^0}{2\pi i}\frac{e^{-ik_0(x^0-y^0)}}{k^0-E_k+i\varepsilon'} + \int \frac{dk^0}{2\pi i}\frac{e^{-ik_0(x^0-y^0)}}{k^0+E_k+i\varepsilon'}\right]$$



$$= \int \frac{d^4k}{(2\pi)^4}\frac{i\,e^{-ik\cdot(x-y)}}{2E_k}\left(\frac{1}{k^0-E_k+i\varepsilon'} - \frac{1}{k^0+E_k+i\varepsilon'}\right)$$

$$i\Delta_R(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i\,e^{-ik\cdot(x-y)}}{k^2-m^2+\underbrace{2i\varepsilon' k^0}_{i\varepsilon k^0}} \quad ; \quad \text{with } \varepsilon = 2\varepsilon' = 0^+$$

Proof for $i\Delta_A(x-y)$ goes analogously.

(iii) $i\Delta_{R(A)}(x-y) = 0$, for $(x-y)^2 < 0$, since $\Delta(x-y) = 0$, for $(x-y)^2 < 0$.

Hence, the retarded/advanced propagators are <u>causal</u>, unlike $i\Delta_F(x-y)$ which is <u>not</u>!

6 $\quad H = \int d^3\underline{y}\; \mathcal{H}(t,\underline{y}) \quad ; \quad \mathcal{H} = \Pi^\dagger\Pi + \underline{\nabla}\Phi^\dagger\underline{\nabla}\Phi + m^2\Phi^\dagger\Phi$

(a) $$[H, \Phi(t,\underline{x})] = \int d^3\underline{y}\; [\Pi^\dagger(t,\underline{y})\Pi(t,\underline{y}), \Phi(t,\underline{x})] + \boxed{\text{vanishing commutators}}$$

$$= \int d^3\underline{y}\Big\{\Pi^\dagger(t,\underline{y})\underbrace{[\Pi(t,\underline{y}), \Phi(t,\underline{x})]}_{-i\delta^{(3)}(\underline{x}-\underline{y})} + [\Pi^\dagger(t,\underline{y}), \Phi(t,\underline{x})]\Pi(t,\underline{y})\Big\}$$

0 ; see Example 2

$$= -i\,\Pi^\dagger(t,\underline{x}) = -i\frac{\partial}{\partial t}\Phi(t,\underline{x}) \qquad \underline{\text{q.e.d.}}$$

(b) $$[H, \Pi(t,\underline{x})] = \int d^3\underline{y}\; [\underline{\nabla}_{\underline{y}}\Phi^\dagger(t,\underline{y})\cdot\underline{\nabla}_{\underline{y}}\Phi(t,\underline{y}) + m^2\Phi^\dagger(t,\underline{y})\Phi(t,\underline{y}), \Pi(t,\underline{x})]$$

$$= \int d^3\underline{y}\Big\{\underline{\nabla}_{\underline{y}}\Phi^\dagger(t,\underline{y})\cdot\underline{\nabla}_{\underline{y}}\underbrace{[\Phi(t,\underline{y}), \Pi(t,\underline{x})]}_{i\delta^{(3)}(\underline{x}-\underline{y})} + m^2\Phi^\dagger(t,\underline{y})\underbrace{[\Phi(t,\underline{y}), \Pi(t,\underline{x})]}_{i\delta^{(3)}(\underline{x}-\underline{y})}\Big\}$$

integrate by parts

$$= -i\nabla^2_{\underline{x}}\Phi^\dagger(t,\underline{x}) + im^2\Phi^\dagger(t,\underline{x}) = i\underbrace{(\Box_x + m^2)\Phi^\dagger(t,\underline{x})}_{0,\ \text{due to KG eqn}} - i\frac{\partial^2}{\partial t^2}\Phi^\dagger(t,\underline{x})$$

$\boxed{\frac{\partial^2}{\partial t^2} - \nabla^2_{\underline{x}}}$

$$= -i\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial}{\partial t}\Phi^\dagger(t,\underline{x})\right)}_{=\Pi(t,\underline{x})} = -i\frac{\partial}{\partial t}\Pi(t,\underline{x}) \qquad \underline{\text{q.e.d.}}$$

7 $\quad$ COURSEWORK II is given on a separate file.