

Gaussian Prime Spiral

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We all know about complex numbers right?, They are of the form $z = a + bi$, where $i = \sqrt{-1}$. They are nothing fancy. They simply represent points in our good old coordinate plane. Like $z = a + bi$ represent the point (a, b) .

In **Julia programming Language**, we represent $i = \sqrt{-1}$ as "**im**". So, $z = a + bi$ is written as $a + b * im$. As an example,

```
z = 1 + 2im
• z = 1+2im
```

Real or Imaginary components of any *complex number* can be reached by using the command **real()** or **imag()** respectively. As an example,

```
1
• real(z)
```

```
2
• imag(z)
```

The absolute value can be found using the command **abs()** and the square of the absolute value can also be found by **abs2()**. As an example,

```
2.23606797749979
• abs(z)
```

```
5
• abs2(z)
```

To plot **Gaussian prime spiral**, we have to first understand what is a **gaussian prime** and how to find one. They are actually a special type of **gaussian integers**(complex numbers with *integer real and imaginary component*). The gaussian integers which doesn't have any factor in the complex field are called **gaussian prime**.

Like $5 + 3i$ can be factorised as $(1 + i)(3 - 2i)$. So, It is not a **gaussian prime**. But $3 - 2i$ is a **gaussian prime** as it cannot be further factorised. Likewise, 5 can be factorised as $(2 + i)(2 - i)$ so it is not a gaussian prime. But 3 is gaussian prime as it cannot be further factorised. Hence, **3+0i** is a **gaussian prime**.

To find if a number is gaussian prime or not, we will use 2 rules:

1. If the real or imaginary component of the given number is zero, then if the other one is a prime of form $4n + 3$, then it will be a gaussian prime. As an example, $5 + 0i$ is a gaussian prime as although its real component is zero, it is not of the form $4n + 3$. But $3i$ is a **gaussian prime**, as its real component is zero and $\text{im}(3i) = 3$ is a prime of form $4n + 3$.
2. If both real and imaginary components of a number $z = a + bi$ are non-zero, then we calculate $\text{abs}^2 = a^2 + b^2$. If abs is a prime, then it is a gaussian prime. This prime will be of form $4n + 1$ (from fermat's two square theorem).

Using this 2 points, we will write a function to check if any given number is gaussian prime or not. To check if any number is normal real prime(eg: 2,3,1033,...), we will use the **isprime** function **Primes** package. This returns true or false based upon the number given. Eg: **isprime(3)** > true ; **isprime(10)** > false.

```
• begin
•   using Primes
•   using Plots#We are also calling this as we will plot the results.
•   using PlutoUI#To see some output beautifully
• end
```

gaussian (generic function with 1 method)

```
• function gaussian(z)#Function to check gaussian prime or not.
•   ima = abs(imag(z)); rea = abs(real(z))
•   if z == 0
•       return false
•   elseif rea != 0 && ima != 0
•       d = abs2(z)
•       return isprime(d)
•   elseif rea == 0 || ima == 0
•       mod_z = abs(z) |> Int
•       if mod(mod_z,4) == 3
•           return isprime(mod_z)
•       else
•           return false
•       end
•   end
• end
```

Now, Lets's see if our code is working well or not. To check output use wikipedia list or any other list as the correct value.

false

```
• gaussian(5)
```

true

```
• gaussian(4+5im)
```

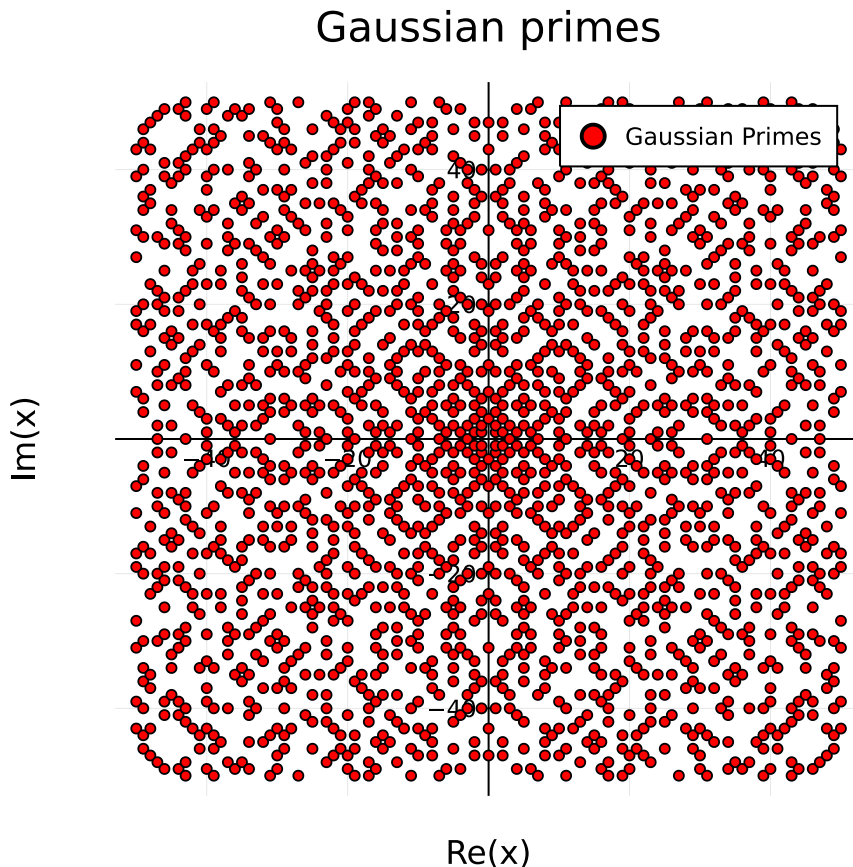
Now, let's plot all the gaussian primes in the range of 10 to -10 and $10i$ to $-10i$.

gaussian_prime_inrange (generic function with 1 method)

```
• function gaussian_prime_inrange(n::Int64)#Find gaussian prime in a circle of radius n
•   list = Complex{Int64}[]
•   for a = -n:n, b = -n:n
•       gaussian_int = a + b*im
•       push!(list,gaussian_int)
•   end
•   return filter(gaussian, list)
• end
```

n, -50+27im, -50+31im, -50+39im, -50+43im, -49-46im, -49-44im, -49-40im, -49-36im, ...

```
• gaussain_array = gaussian_prime_inrange(50)
```



```
• begin
•     scatter((gaussain_array), color=:red,markersize=3,size=(450,450),
•     title="Gaussian primes",framestyle=:origin,label="Gaussian Primes")
• end
```

Now, we are ready to solve a particular problem. Let's see what is it:(The problem is taken from **Learning Scientific Programming with Python**, 2nd edition, written by **Christian Hill**.)

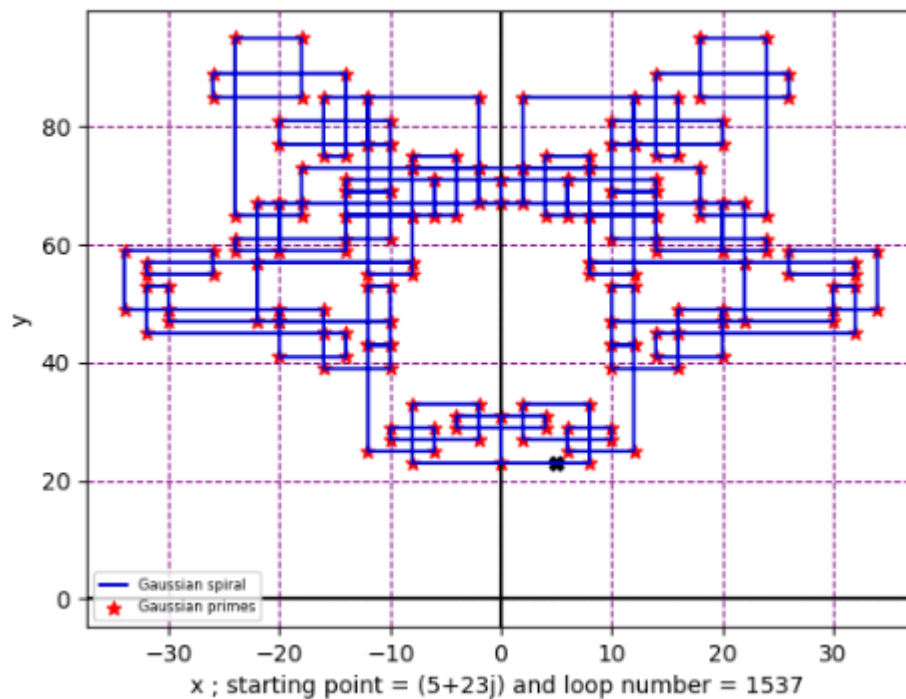
P3.2.2 A *Gaussian integer* is a complex number whose real and imaginary parts are both integers. A *Gaussian prime* is a Gaussian integer $x + iy$ such that either:

- one of x and y is zero and the other is a prime number of the form $4n + 3$ or $-(4n + 3)$ for some integer $n \geq 0$; or
- both x and y are nonzero and $x^2 + y^2$ is prime.

Consider the sequence of Gaussian integers traced out by an imaginary particle, initially at c_0 , moving in the complex plane according to the following rule: it takes integer steps in its current direction (± 1 in either the real or imaginary direction), but turns *left* if it encounters a Gaussian prime. Its initial direction is in the positive real direction ($\Delta c = 1 + 0i \Rightarrow \Delta x = 1, \Delta y = 0$). The path traced out by the particle is called a *Gaussian prime spiral*.

Write a program to plot the Gaussian prime spiral starting at $c_0 = 5 + 23i$.

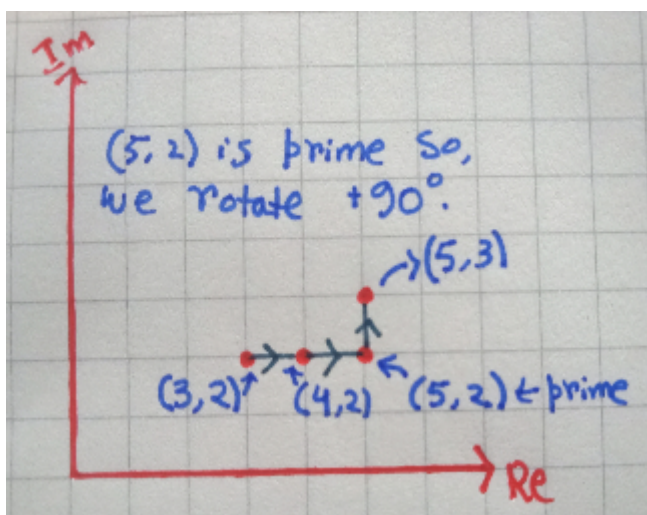
I have already solved it using python. The output should be something like this:



Let's see how to draw the spiral for a initial point $c_0 = 3 + 2i$ and $\Delta c = 1 + 0i = 1$.

1. For the first step, we don't care if c_0 is **gaussian prime** or not. We just add the step with it, i.e., we add Δc with c_0 . For our case it will give us $c_1 = (3 + 2i) + 1 = 4 + 2i$.
2. Then, we check if c_1 is a **gaussian prime** or not. In our case, $c_1 = 4 + 2i$ is not a gaussian prime. So, we repeat step 1(i.e., add Δc with it). This gives us $c_2 = 5 + 2i$. Again we check if c_2 is gaussian prime or not. In this case, c_2 **is a gaussian prime**. So, now we have to **rotate the direction 90° towards the left**, i.e., anti-clockwise. In complex plane, it is very easy. Just multiply the Δc by $i = \sqrt{-1}$ and that will be our new Δc . For our example, $c_3 = c_2 + \Delta c = 5 + 2i + (1 + 0i) \cdot i = 5 + 3i$.
3. From here, again we follow step-2, until we get the point from where we started with the same Δc or you can do it for your required step.

Here is a hand drawn **gaussian spiral** for 4 steps.



Now, let's write a program to draw these spirals.

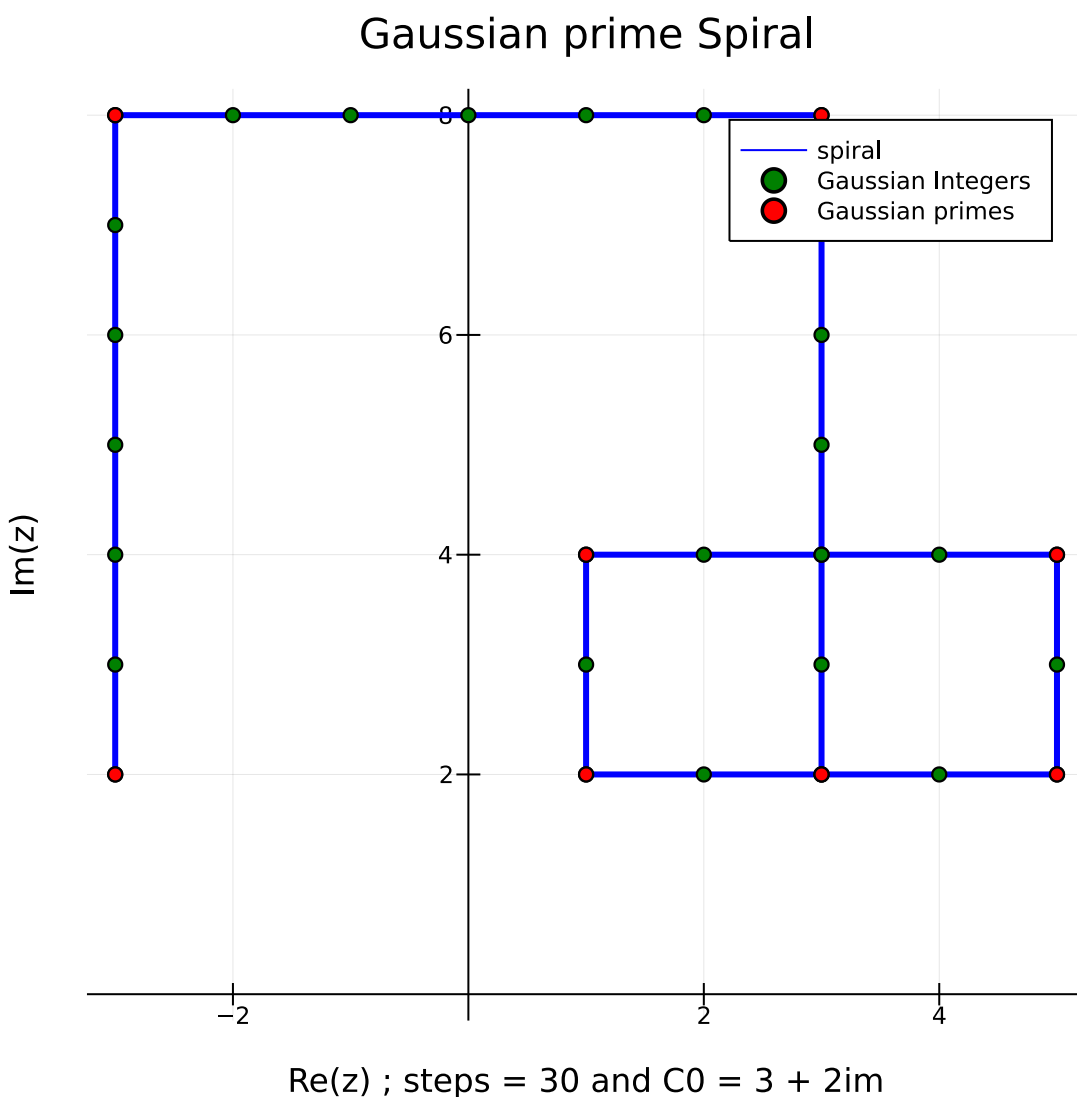
```
• begin
•   seed = 3+2*im; Δc = 1; step = 0; d = seed
•   prime_list = ComplexF64[]; points = ComplexF64[seed]
•   for i in 1:30
•       seed = seed + Δc; step += 1
•       push!(points, seed)
•       if gaussian(seed)
•           Δc = Δc*im
•           push!(prime_list, seed)
•       end
•   end
• end
```

```
ComplexF64[3.0 + 2.0im, 4.0 + 2.0im, 5.0 + 2.0im, 5.0 + 3.0im, 5.0 + 4.0im, 4.0 + 4.0im, 3.0 + 4.0im, 2.0 + 4.0im, 1.0 + 4.0im, 1.0 + 3.0im, 1.0 + 2.0im, 2.0 + 2.0im, 3.0 + 2.0im, 3.0 + 3.0im, 3.0 + 4.0im, 3.0 + 5.0im, 3.0 + 6.0im, 3.0 + 7.0im, 3.0 + 8.0im, 2.0 + 8.0im, 1.0 + 8.0im, 0.0 + 8.0im, -1.0 + 8.0im, -2.0 + 8.0im, -3.0 + 8.0im, -3.0 + 7.0im, -3.0 + 6.0im, -3.0 + 5.0im, -3.0 + 4.0im, -3.0 + 3.0im, -3.0 + 2.0im]
```

```
• with_terminal() do#I have given this just to show you a method to use terminal
•   println(points)
• end
```

This is the program to plot spiral. Here I have used **for loop** to just calculate for 30 steps. Here i have printed all the values, we get by following the step1 and step2 30 times.

Let's plot this. Remember the initial point is $3 + 2i$.



```

• begin
•   plot((points),color=:blue,width=3,title="Gaussian prime
Spiral",label="spiral",framestyle=:origin,size=(560,560))
•   scatter!((points),color=:green,label="Gaussian Integers");scatter!
((prime_list),label="Gaussian primes",color=:Red)
•   xlabel!("Re(z) ; steps = $step and C0 = $d"); ylabel!("Im(z)")
•
• end

```

Now, Let's define the plotting block of code as a function. That function can plot number of steps according to your wish or it can plot until it returns to it's starting state (i.e., $\Delta c = 1$ and starting value of c_0).

gaussian_spiral (generic function with 1 method)

```

• function gaussian_spiral(seed;loop_num=1,Δc=1,initial_con = true)
•   d = seed; points = Complex{Int64}[d]
•   if initial_con
•       while true
•           seed += Δc
•           push!(points,seed)
•           if seed == d
•               break
•           end
•           if gaussian(seed)
•               Δc = Δc*im
•           end
•       end
•   else
•       for i in 1:loop_num
•           seed += Δc
•           push!(points,seed)
•           if gaussian(seed)
•               Δc = Δc*im
•           end
•       end
•   end
•   prime_list = filter(gaussian,points)
•   return points, prime_list
• end

```

5 + 23im

```

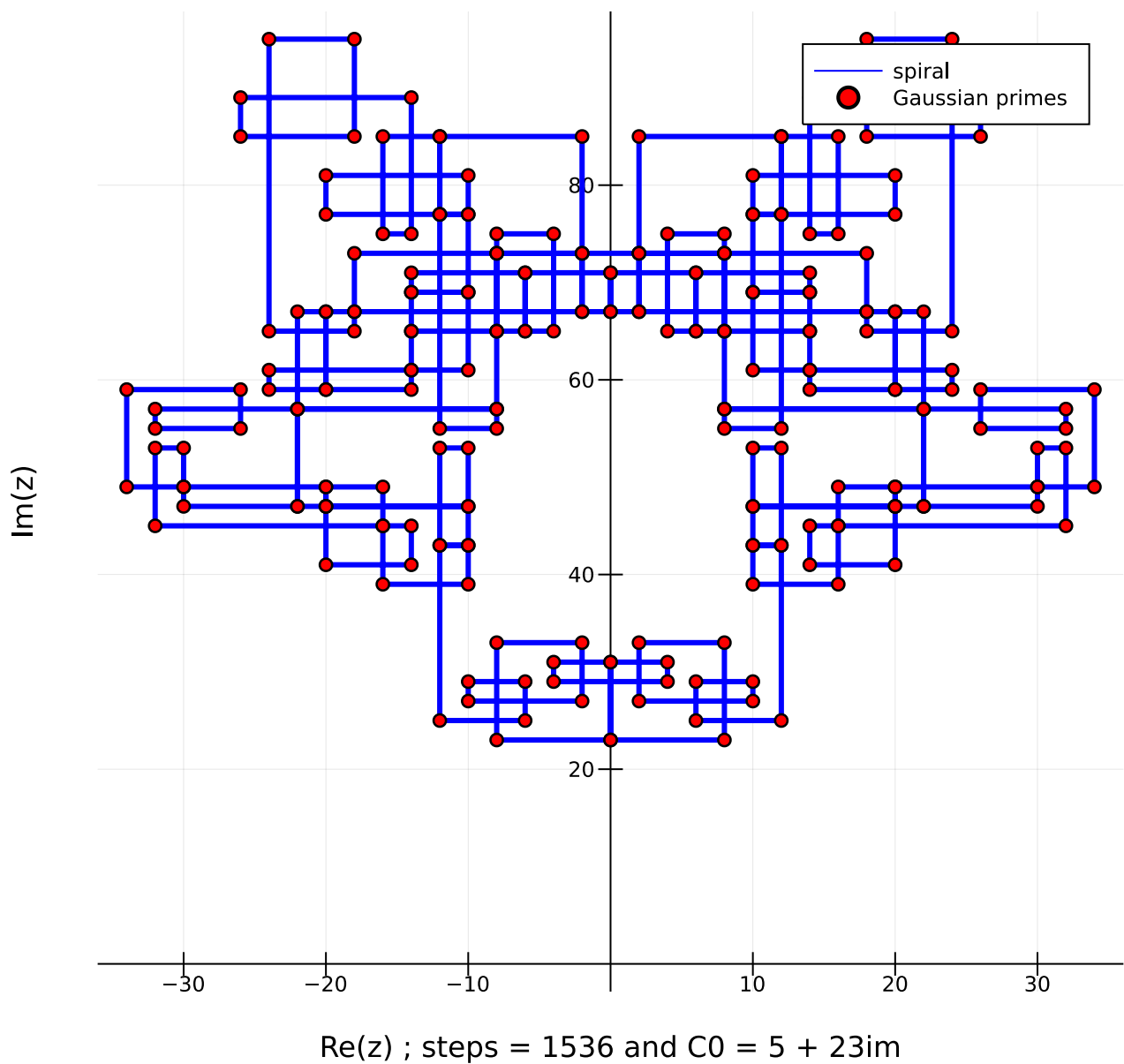
• begin
•   point, primes = gaussian_spiral(5+23*im)
•   step_count = size(point)[1] - 1
•   start = point[1]
• end

```

So, We have all the data in the 3 variables.

1. points = Contains all **Gaussian integers**, which will create our spiral.
2. primes = Contains all **Gaussian primes**, which are generated in the process.
3. step_count = which contains the number of steps needed for a particular spiral.

Gaussian prime Spiral



```

• begin
•   plot((point),color=:blue,width=3,title="Gaussian prime
Spiral",label="spiral",framestyle=:origin,size=(650,650))
•   scatter!((primes),label="Gaussian primes",color=:Red)
•   xlabel!("Re(z) ; steps = $step_count and C0 = $start"); ylabel!("Im(z)")
•
• end

```

Too beautiful!, Let's plot few more.

60 + 1im

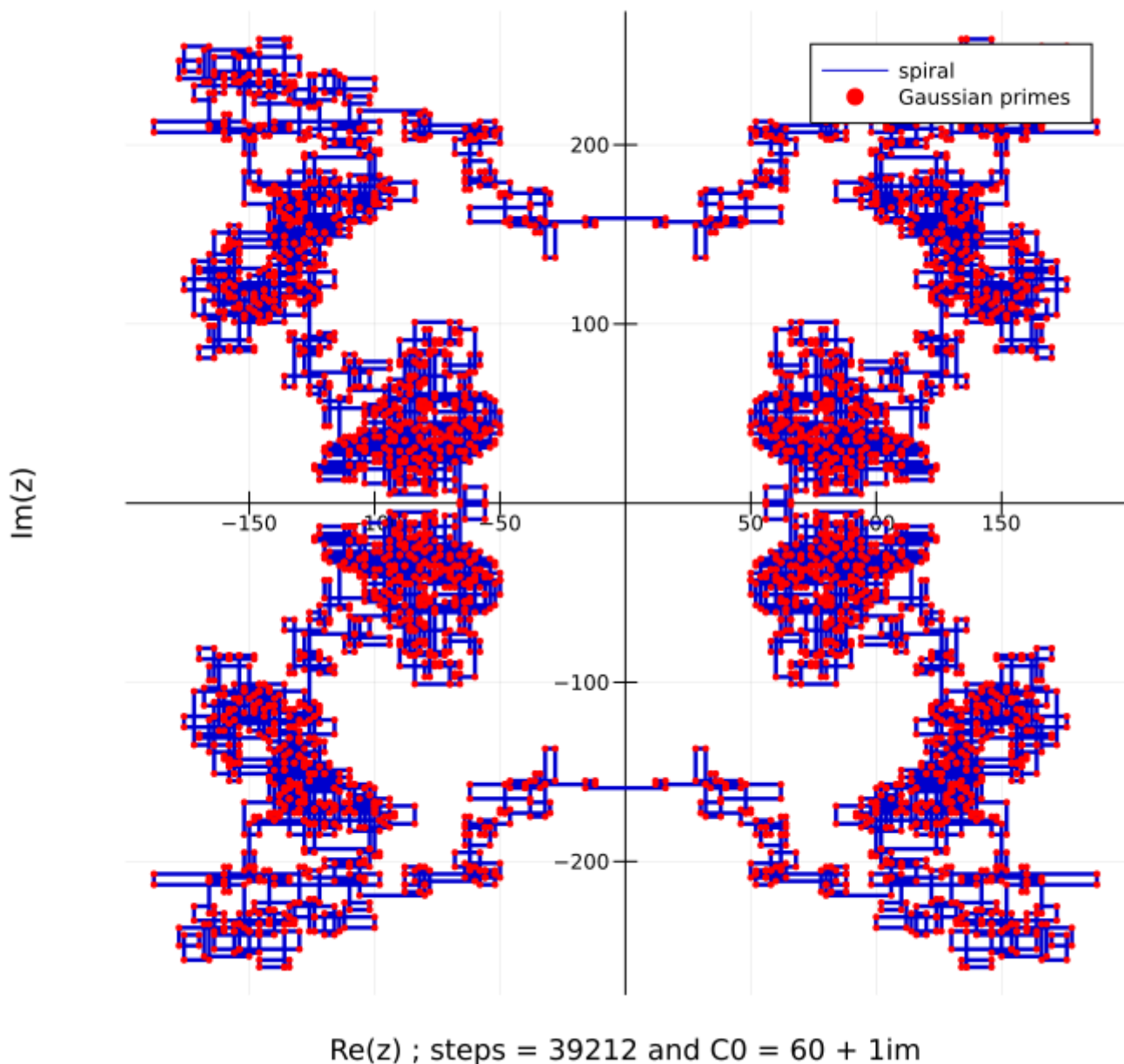
```

• begin
•   point1, prime1 = gaussian_spiral(60+im)
•   step_count1 = size(point1)[1] - 1
•   start1 = point1[1]
• end

```

Now, Let's plot it.

Gaussian prime Spiral



```

• begin
•   plot((point1),color=:blue3,width=2.4,title="Gaussian prime
Spiral",label="spiral",framestyle=:origin,size=(660,660))
•   scatter!((prime1),label="Gaussian
primes",color=:Red,markersize=3.32,markerstrokewidth = 0)
•   xlabel!("Re(z) ; steps = $step_count1 and C0 = $start1"); ylabel!("Im(z)")
•
• end

```

" If you don't like change the colour or maybe remove the gaussian primes.

Last one

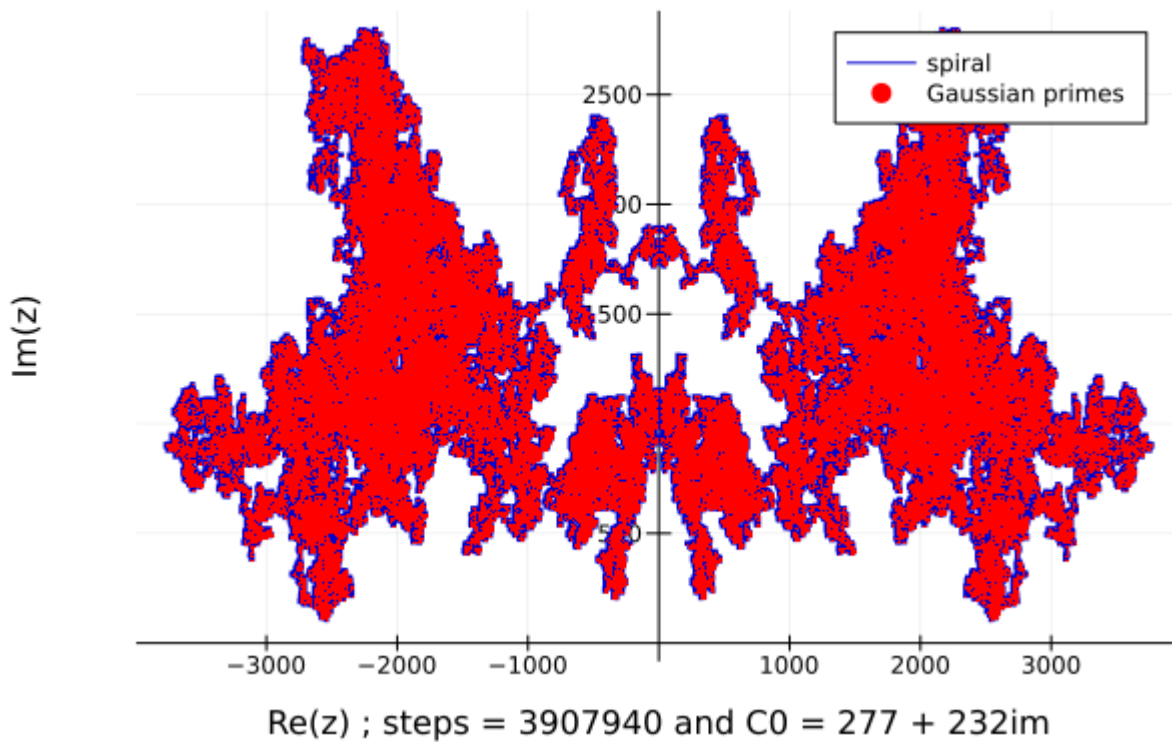
277 + 232im

```

• begin
•   point2, prime2 = gaussian_spiral(277+232*im)
•   step_count2 = size(point2)[1]-1
•   start2 = point2[1]
• end

```


Gaussian prime Spiral

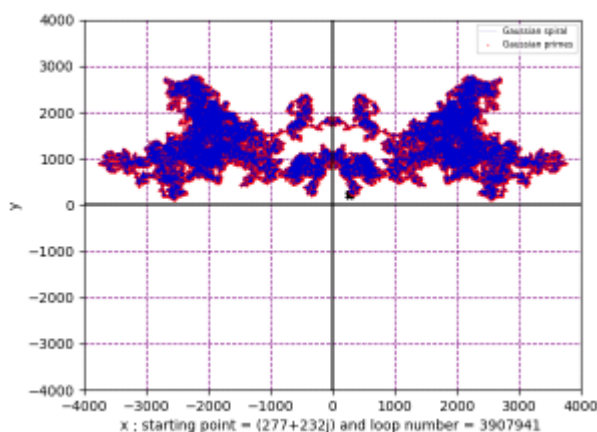


```

• begin
•   plot((point2),color=:blue3,width=2.4,title="Gaussian prime
Spiral",label="spiral",framestyle=:origin)
•   scatter!((prime2),label="Gaussian
primes",color=:Red,markersize=0.1,markerstrokewidth = 0)
•   xlabel!("Re(z) ; steps = $step_count2 and C0 = $start2"); ylabel!("Im(z)")
•
• end

```

Damn!!...It's too beautiful. I am just a beginner in **julia** so, the graphs are still not that good. Here is the same plot in **python**.



🤖 Am I seeing Batman doing back-flip?