Infinitesimal analysis of singular stochastic partial differential equations

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A papà

Abstract

The last decade has seen a considerable rise in the study of singular stochastic partial differential equations (SPDEs) which turned into the birth of many celebrated techniques for the development of a solution theory for such kind of equations. The present thesis is devoted to the study of some problems involving singular SPDEs with approaches based on the study of the infinitesimal generator of the semigroup related to the solution to the equation under investigation.

In the first part of the work, we study a probabilistic approach to singular SPDEs. More precisely, we deal with a martingale problem associated to the infinitesimal generator of the equation involved. Because of the irregular behaviour of the terms appearing in the equation, the first task is to give a meaning to the martingale problem itself, and only in a second moment one can proceed with studying existence and uniqueness for the martingale problem. In order to do so, we exploit stochastic calculus in infinite dimensions and the analysis of the infinitesimal generator corresponding to the solution of the equation, defining a suitable domain where we are able to solve the related Kolmogorov backward equation. As an application of the technique under consideration, we focus on (quasi-)stationary solutions to hyperviscous stochastic Navier–Stokes equation in two dimensions (both on the torus and on the plane), but such an approach was first developed for singular SPDEs by Gubinelli and Perkowski for the stochastic Burgers equation on the one-dimensional torus and on the real line.

The second part of the thesis is concerned with Euclidean quantum field theory. We approach the problem of stochastic quantization by providing a differential characterization of quantum field theories through the study of a singular integration by parts formula. In particular, we focus on the case of exponential interactions (alias Høegh–Krohn model) on the whole plane and show existence and uniqueness of a measure solving the associated renormalized integration by parts, that is a suitable Euclidean Dyson–Schwinger equation. This is achieved requiring that the measure can be compared with a Gaussian free field (meaning that it has a finite Wasserstein-type distance from it) and studying the corresponding symmetric Fokker–Planck–Kolmogorov equation. More precisely, we get existence of solutions exploiting Lyapunov functions, and uniqueness by analyzing the resolvent equation associated to the infinitesimal generator. This allows us to characterize the invariant measure of the stochastic quantization equation as the only measure satisfying the integration by parts formula.

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"Leaves are falling all around, it's time I was on my way.
Thanks to you, I'm much obliged for such a pleasant stay."
– Ramble On, **Led Zeppelin**

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Chapter 1
Introduction and preliminaries

Stochastic partial differential equations (SPDEs) arises in mathematical and physical modelling as a combination of partial differential equations (PDEs) and some random forcing term, two typical objects used to describe a vast variety of phenomena. Apart from the interest given by the possible applications, SPDEs are also widely studied in mathematics for the challenging analytical and probabilistic difficulties they bring. Solution theories for SPDEs have been constructed since the 1970s and, in particular, three main approaches were followed: the martingale approach (see [182]), the semigroup (or mild solution) approach (see the monographs [58, 59]), and the variational approach due to Pardoux [155] and Krylov and Rozovskii [125] (see also the books [132, 158]). The literature about such approaches is vast, we refer the interested reader to some other classical surveys on the topic [29, 49, 82, 109, 123].

Nevertheless, many of the non-linear SPDE models remain uncovered by such results because of their singular behaviour. More precisely, in some kind of equations, which the recent literature refers to as *singular SPDEs*, non-linear operations involving the solution of the equation appear. Such operations are often ill-defined since the solution (if it exists) is a priori a distribution due to the presence of a highly irregular noise – usually some sort of white noise. In order to deal with such non-linear operations, the idea of renormalization was introduced to SPDE theory (see [19, 54, 115]), which brought many results first on some models of singular SPDEs such as stochastic Navier–Stokes equation and Φ^4_2 equation (both on the two-dimensional torus) and then on more singular cases, like Φ^4_3 equation. The necessity of renormalization was pointed out in [9, 106]. Many important models were still out of reach but, in the last decade, this mathematical subject saw a great rise in the interest of researchers and consequently in the development of the field – documented by the introduction of a dedicated Mathematics Subject Classification (40H17) by the American Mathematical Society in 2020 – with mainly four approaches based on pathwise arguments: regularity structures, paracontrolled distributions, the Otto–Weber rough path approach, and the renormalization group theory technique. With the exception of the latter approach, they are all taking inspiration from the theory of rough paths introduced by Lyons [134, 136, 135] and from the closely related controlled rough paths by Gubinelli [87, 88].

The present thesis is concerned with problems related to singular SPDEs. In particular, it will deal with a probabilistic approach to such kind of equations and to problems of differential characterization of Euclidean quantum field theories through integration by parts formulae. Before proceeding with the main topics, let us give a brief overview of recent results on singular SPDEs obtained via pathwise techniques (Section 1.1). After that, the remaining sections of this introductory chapter are devoted to the presentation of preliminary results that are used later on in the present work. More precisely, Section 1.2 is concerned with semigroup theory and a brief introduction to Kolmogorov and Fokker–Planck–Kolmogorov equations, Section 1.3 introduces weighted Besov spaces and their properties, and Section 1.4 is devoted to Gaussian measures in infinite dimensional spaces. The thesis is then divided into two parts: Part I is related to the probabilistic approach to singular SPDEs, while Part II concerns stochastic quantization. Both of them contain an introduction to the subject with connection to the existing literature (Chapters 2 and 4, respectively) as well as the summaries of the two co-authored publications [61, 100] (Chapters 3 and 5) on which the present work relies and that are annexed in full in the appendices.

1.1 Singular SPDEs via pathwise techniques

It is worth to make some examples of singular SPDEs with the aim of clarifying a bit the situation presented above. First of all, let ξ be a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$, that is a centred Gaussian process with covariance $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$. It is worth to mention that the process ξ is almost surely a tempered distribution having negative Besov regularity given by $-(d+1)/2 - \varepsilon$, for any $\varepsilon > 0$ (see Example 1.4.18). We highlight here only some classical examples of singular SPDEs, collecting them into four categories. Let us stress that this is just an outline of the many models appearing in the literature. Let us also refer the reader the survey [51].

- **Hydrodynamics and KPZ.** In this class of singular SPDEs, we consider the well-known *KPZ equation* in one dimension, that is, $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ solves

  $$\partial_t h(t,x) = \partial_{xx}^2 h(t,x) + (\partial_x h(t,x))^2 + \sqrt{2}\,\xi(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

  as well as models appearing in hydrodynamics.

  KPZ equation arises when modelling large scale fluctuations of a growing interface whose height is described by h, and was first introduced by Kardar, Parisi, and Zhang [117]. The heuristic space derivative of h, $u = \partial_x h$, solves then the *stochastic Burgers equation*

  $$\partial_t u(t,x) = \partial_{xx}^2 u(t,x) + \partial_x(u(t,x)^2) + \sqrt{2}\,\partial_x \xi(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

  Another equation linked to KPZ is the stochastic heat equation with multiplicative noise

  $$\partial_t v(t,x) = \partial_{xx}^2 v(t,x) + \sqrt{2}\, v(t,x)\,\xi(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

  which can be heuristically obtained from h via the *Cole–Hopf transform* $v = e^h$.

  As mentioned above, due to the presence of terms involving the white noise ξ, all such equations present non-linear operations on terms with negative regularity, and therefore that are a priori ill-defined. The solution to KPZ equation on the torus was the first result achieved by Hairer's theory of regularity structures [102]. Similar results with the paracontrolled distributions approach on the equation on the torus can be found in the works by Gubinelli and Perkowski [96], which were then extended to the whole real line $\mathbb{R}$ by Perkowski and Rosati [157].

  Singular SPDEs are also important in the description of the dynamics of fluids. One of the more famous models is the *stochastic* (homogeneous, incompressible) *Navier–Stokes equation*, that is the velocity $u\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ solves

  $$\begin{aligned} \partial_t u(t,x) &= \Delta_x u(t,x) - (u(t,x)\cdot\nabla)u(t,x) - \nabla p(t,x) + \sqrt{2}\,\xi(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u(t,x) &= 0, \end{aligned}$$

  where $d = 2, 3$, and p is the pressure. Stochastic Navier–Stokes equation has been studied by Da Prato and Debussche [53] on the two-dimensional torus, and by Zhu and Zhu [184] on the three-dimensional torus both via regularity structures and paracontrolled distributions approaches.

- **Stochastic quantization equations.** The models of stochastic quantization are connected to Euclidean quantum field theories, we will discuss this relation further in Chapter 4. Given a potential $V \in C^1(\mathbb{R})$, namely in the space of continuous functions from $\mathbb{R}$ into itself with continuous first derivative V', we consider the equation for $\varphi: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ given by

$$\partial_t \varphi(t,x) = (\Delta_x - m^2)\varphi(t,x) - \lambda V'(\varphi(t,x)) + \xi(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where $m \in \mathbb{R}$, $\lambda > 0$. Heuristically, the invariant measure of the equation is the following:

$$\mu(\mathrm{d}\phi) = Z^{-1} e^{-\int_{\mathbb{R}^d} V(\phi(x))\mathrm{d}x} \nu(\mathrm{d}\phi),$$

where ν is the Gaussian free field measure with mass m (see Example 1.4.19).

The choice $V(\varphi) = \frac{1}{4}\varphi^4$ corresponds to the Φ^4_d*-model*. The case $d = 2$, which can also be extended to the case where V is a polynomial of even order (which is referred to as the $P(\varphi)_2$*-model*), was studied on the torus $\mathbb{T}^2$ by Da Prato and Debussche [54] and by Tsatsoulis and Weber [178], and on the whole $\mathbb{R}^2$ by Mourrat and Weber [141], by Albeverio and Röckner [19], and by Röckner, Zhu, and Zhu [163]. Let us also mention the results in the elliptic setting by Albeverio, De Vecchi, and Gubinelli [4], and in the hyperbolic setting by Gubinelli, Koch, Oh, and Tolomeo [94]. The case $d = 3$ was famously solved on the torus by Hairer [103] via regularity structures generalizing his own work on KPZ equation, by Kupiainen [129] via renormalization groups, and by Catellier and Chouk [44] via paracontrolled distributions. For more recent results on Φ^4_3 on the torus, see e.g. [15, 105, 107, 142]. For results on the whole space $\mathbb{R}^3$ we refer the interested reader to the works by Gubinelli and Hofmanová [90, 91], by Albeverio and Kusuoka [16], and by Moinat and Weber [140]. The case of dimension $d = 4 - \varepsilon$ was studied in the works by Chandra, Moinat, and Weber [48] via regularity structures and by Duch [64] with renormalization group techniques.

Taking $V(\varphi) = \beta^{-1}\cos(\beta\varphi)$, for some parameter β, gives the *sine-Gordon model*. Local solutions for such a model have been constructed by Hairer and Shen [106] in the case $\beta^2 < 16\pi/3$ and by the same two authors together with Chandra [47] for case $\beta^2 < 8\pi$.

The $\exp(\Phi)_2$*-model*, obtained when putting $V(\varphi) = \exp(\alpha\varphi)$, where α is a real parameter, was introduced by Høegh–Krohn [108] and further studied in [10, 13]. Other recent results are available for this model: let us mention the works by Garban [81] on the two-dimensional torus $\mathbb{T}^2$ and on the two-dimensional sphere $\mathbb{S}^2$ for $\alpha^2 < 4\pi(6 - 4\sqrt{2})$, by Hoshino, Kawabi, and Kusuoka [110, 111] on $\mathbb{T}^2$ for $\alpha^2 < 4\pi$ and $\alpha^2 < 8\pi$, respectively, by Oh, Robert, and Wang [150] and by the same authors together with Tzvetkov [149] for the case $\alpha^2 < 4\pi$ on $\mathbb{T}^2$ and on any connected, compact, orientable, two-dimensional manifold, respectively, and by Albeverio, De Vecchi, and Gubinelli [4] on the whole $\mathbb{R}^2$ for $\alpha < 4\pi + \varepsilon$ in the elliptic case.

It is worth to mention the recent results by Barashkov and De Vecchi [25] concerning the (elliptic) *sinh-Gordon model*, namely the case $V(\varphi) = \cosh(\beta\varphi)$, on the whole space $\mathbb{R}^2$ when taking $\beta^2 < 4\pi$.

- **Parabolic Anderson model.** The *generalized parabolic Anderson model* in dimensions two and three corresponds to the equation

$$\partial_t u(t,x) = \Delta u(t,x) + F(u(t,x))\,\zeta(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where $F\colon \mathbb{R}\to\mathbb{R}$ is a continuous function and ζ is a spatial white noise (that is constant in time). Well-posedness for such an equation on $\mathbb{T}^2$ was the first achievement of the paracontrolled distributions approach and it was obtained by Gubinelli, Imkeller and Perkowski [92]. The same result has been reached via the theory of regularity structure by Hairer [103]. Such an equation on a three-dimensional closed manifold was the object of study of the higher-order paracontrolled calculus work by Bailleul and Bernicot [23].

- **Quasilinear equations.** Quasilinear SPDEs are equations, for $u\colon \mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$, of the following form:

$$\partial_t u(t,x) = a(u(t,x))\,\Delta u(t,x) + b(u(t,x))\,\xi(t,x), \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d,$$

  where a and b are some smooth functions. Such a kind of equations was studied in the approach by Otto and Weber [154] on the one-dimensional torus $\mathbb{T}^1$, via paracontrolled distributions techniques on the two-dimensional torus $\mathbb{T}^2$ by Furlan and Gubinelli [80] and by Bailleul, Debussche, and Hofmanová [24], and with regularity structures by Gerencsér and Hairer [83].

Some of the most groundbreaking advances in singular SPDEs are due to Hairer, who exploited controlled rough paths to give a rigorous meaning to one-dimensional non-linearities [101]. Such an approach allowed him to solve the KPZ equation on the torus for the first time [102], and to develop then the theory of regularity structures [103], yielding to the solution of the Φ^4_3-model of stochastic quantization on the torus and of the (parabolic) generalized Anderson model on the torus. (See also [78] for an overview on rough paths and regularity structures).

At the same time, an alternative technique by Gubinelli, Imkeller, and Perkowski [92] introduced the theory of paracontrolled distributions, an extension of controlled rough paths to a multi-dimensional setting based on tools from harmonic analysis, which was then further extended to higher order by Bailleul and Bernicot [23].

In the most recent approach by Otto and Weber [154], the authors give an high dimensional generalization of controlled path allowing to work to quasi-linear equations.

Finally, another approach is given by renormalization group techniques adopted by Kupiainen [129], where the author decomposes random fields into various scales and relates them via recursive equations. It is worth to mention that this technique was extended further by Duch [64].

1.2 Semigroups and infinitesimal generators

We present here some basic semigroup theory and relate it to invariant measures. In particular, we will introduce the notion of semigroup, infinitesimal generators, with the associated Kolmogorov (backward ad forward) and Fokker–Planck–Kolmogorov equations, and resolvent operator and equation. These topics will be important in the upcoming parts of the thesis concerning martingale problems and stochastic quantization, where the notion of infinitesimal generator, resolvent operator and their properties will be exploited – or at the very least taken as point of inspiration – to study results about probability solutions for singular SPDEs, and about characterization results for quantum field theories.

As far as classic semigroup theory is concerned (Section 1.2.1) we will follow closely the references [68, 156], while for the part regarding invariant measures and the equations associated with infinitesimal generators (Section 1.2.2) the main references will be [37, 131, 175].

1.2.1 Semigroup theory

Let us start with some generalities on operator semigroups and their infinitesimal generators. Consider a Banach space E with norm $\|\cdot\|_E$. We also let E^* be the topological dual of E, equipped with norm $\|\cdot\|_{E^*}$, while the duality between the two spaces E^* and E will be denoted by $\langle\cdot,\cdot\rangle_{E^*,E}$.

Definition 1.2.1. (Semigroup) *A family of linear and bounded operators $T=(T_t)_{t\geqslant 0}$ on E is called a* semigroup *if*

$$T_0=\mathrm{id}_E, \qquad \text{and} \qquad T_{t+s}=T_tT_s, \quad t,s\geqslant 0,$$

where id_E *is the identity operator on E. A semigroup T is said to be* strongly continuous *(or C_0) if*

$$\lim_{t\to 0}\|T_tu-u\|_E=0, \qquad \text{for all } u\in E.$$

Moreover, a semigroup T is called contractive *(or* contraction*) if*

$$\|T_t\|\leqslant 1, \qquad \text{for all } t\geqslant 0.$$

A linear operator $\mathscr{L}$ on E is a linear mapping whose domain $D(\mathscr{L})$ is a subspace of E and whose *range* $\mathrm{ran}(\mathscr{L})$ lies in E, its *graph* is given by $\mathrm{graph}(\mathscr{L})=\{(u,\mathscr{L}u)\colon u\in D(\mathscr{L})\}\subset E\times E$. The operator $\mathscr{L}$ is said to be *closed* if $\mathrm{graph}(\mathscr{L})$ is a closed subspace of $E\times E$.

Definition 1.2.2. (Infinitesimal generator) *The* infinitesimal generator *$\mathscr{L}$ of a semigroup T on E is the linear operator defined by*

$$\mathscr{L}u=\lim_{t\to 0}\frac{1}{t}(T_tu-u), \qquad u\in D(\mathscr{L}),$$

where $D(\mathscr{L})$ is the domain of *$\mathscr{L}$, i.e. the linear subspace of E where the previous limit exists in E.*

We have the following relation between semigroups and their infinitesimal generators.

Proposition 1.2.3. *Let T be a strongly continuous semigroup on E with infinitesimal generator $\mathscr{L}$.*

i. If $u\in E$ and $t\geqslant 0$, then $\int_0^t T_su\,\mathrm{d}s\in D(\mathscr{L})$ and

$$T_tu-u=\mathscr{L}\int_0^t T_su\,\mathrm{d}s.$$

ii. If $u\in D(\mathscr{L})$ and $t\geqslant 0$, then $T_tu\in D(\mathscr{L})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}T_tu=\mathscr{L}T_tu=T_t\mathscr{L}u.$$

iii. If $u\in D(\mathscr{L})$ and $t\geqslant 0$, then

$$T_tu-u=\int_0^t\mathscr{L}T_su\,\mathrm{d}s=\int_0^t T_t\mathscr{L}u\,\mathrm{d}s.$$

Proof. See Proposition 1.1.5 in [68]. □

Corollary 1.2.4. *If $\mathscr{L}$ is the generator of a strongly continuous semigroup T on E, then $D(\mathscr{L})$ is dense in E and $\mathscr{L}$ is closed.*

Let us consider the case $T_t=e^{tA}=\sum_{k=1}^{\infty}\frac{1}{k!}t^kA^k$, where A is a bounded linear operator on E, then $D(\mathscr{L})=E$ and T has infinitesimal generator $\mathscr{L}=A$. Moreover, $(e^{t\mathscr{L}})_{t\geqslant 0}$ is strongly continuous. In most applications however, the domain of the generator is smaller than E and the generator $\mathscr{L}$ is not continuous on it. Usually, it is not easy or not even possible to explicitly find the domain of the generator.

Example 1.2.5. (Ornstein–Uhlenbeck semigroup) Let E be a separable Hilbert space equipped with inner product $(\cdot,\cdot)_E$ and with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and consider

$$T_t u = \sum_{n=1}^{\infty} e^{-nt}(u,e_n)_E\, e_n\,.$$

Then

$$D(\mathscr{L}) = \left\{ u\in E\colon \sum_{n=1}^{\infty} n^2|(u,e_n)_E|^2 < +\infty \right\}, \qquad \mathscr{L}u = -\sum_{n=1}^{\infty} n\,(u,e_n)_E\, e_n\,.$$

A particular case is given by the *Ornstein–Uhlenbeck semigroup* on $E = L^2(\mu)$, where μ is the standard Gaussian measure on $\mathbb{R}^d$, given by

$$T_t u(x) = \int_{\mathbb{R}^d} u\Big(e^{-t}x - \sqrt{1-e^{2t}}\, y\Big)\, \mu(\mathrm{d}y),$$

and with (e_n) being the basis formed by the Hermite polynomials H_n (see e.g. Example 2.9 in [112] or Section 1.1.1 in [148]). It is possible to show that $T_t H_n = e^{-nt}H_n$, and $T_t H_{n_1,\ldots,n_d} = e^{-(n_1+\cdots+n_d)t}(H_{n_1}\cdots H_{n_d})$. Here, the generator $\mathscr{L}$ of T coincides with the Ornstein-Uhlenbeck operator on $C_0^\infty(\mathbb{R}^d)$, that is

$$\mathscr{L}u(x) = \frac{1}{2}\,\Delta u(x) - \frac{1}{2}\,\langle x, \nabla u(x)\rangle.$$

Example 1.2.6. (Heat semigroup) The *heat semigroup* P on $L^p(\mathbb{R}^d)$, $p\in[1,+\infty]$, is defined by

$$P_t u(x) = \int_{\mathbb{R}^d} u(x - \sqrt{t}\, y)\ \mu(\mathrm{d}y).$$

Its infinitesimal generator is given by $\mathscr{L} = \Delta/2$ on the Sobolev space $W^{2,2}(\mathbb{R}^d)$.

A linear operator $\mathscr{L}$ on E is said to be *closable* if it has a closed linear extension. If $\mathscr{L}$ is closable then its minimal closed extension $\bar{\mathscr{L}}$ is called the *closure* of $\mathscr{L}$.

Definition 1.2.7. (Dissipative) *A linear operator $\mathscr{L}$ on E is called* dissipative *if, for every $u\in D(\mathscr{L})$, there exists $u^*\in E^*$ such that*

$$\|u^*\|_{E^*} = \|u\|_E, \qquad \langle u^*, u\rangle_{E^*,E} = \|u\|_E^2, \qquad \langle u^*, \mathscr{L}u\rangle_{E^*,E} \leqslant 0.$$

We have the following characterization of dissipative operators.

Proposition 1.2.8. *A linear operator $\mathscr{L}$ is dissipative if and only if*

$$\|(\lambda - \mathscr{L})u\| \geqslant \lambda\|u\|, \qquad \textit{for all } u\in D(\mathscr{L}) \textit{ and } \lambda>0.$$

Proof. See Theorem 4.2 in Section 1.4 of [156]. □

Let us state here some properties of dissipative operators.

Proposition 1.2.9. *Let $\mathscr{L}$ be a dissipative operator on E.*

i. If, for some $\lambda_0>0$, $\mathrm{ran}(\lambda_0-\mathscr{L}) = E$, then $\mathrm{ran}(\lambda-\mathscr{L}) = E$ for all $\lambda>0$.

ii. If $\mathscr{L}$ is closable, then $\bar{\mathscr{L}}$ is also dissipative.

iii. If $D(\mathscr{L})$ is dense in E, then $\mathscr{L}$ is closable and $\overline{\mathrm{ran}(\lambda-\mathscr{L})} = \mathrm{ran}(\lambda-\bar{\mathscr{L}})$.

Proof. See Theorem 4.5, Chapter 1 in [156] and Lemma 2.11 in Section 1.2 of [68]. □

A dissipative operator $\mathscr{L}$ is called *essentially m-dissipative* if

$$\operatorname{ran}(\lambda-\mathscr{L})=E, \qquad \text{for every } \lambda>0.$$

Notice that, by Proposition 1.2.9, it is sufficient that $\operatorname{ran}(\lambda I-\mathscr{L})=E$ holds for *some* $\lambda>0$.

Theorem 1.2.10. (Hille–Yosida, Lumer–Phillips) *A densely defined linear operator on a Banach space is essentially m-dissipative if and only if its closure is the generator of a strongly continuous contractive semigroup.*

Proof. See Theorem 2.12, Chapter 1 of [68] as well as Theorem 4.3, Section 1.4 of [156]. □

Let us introduce the notion of resolvent set and resolvent operator.

Definition 1.2.11. (Resolvent) *Given a densely defined operator $\mathscr{L}$ on a Banach space E, the* resolvent set $\rho(\mathscr{L})$ *of $\mathscr{L}$ consists of all $\lambda\in\mathbb{R}$ such that $\lambda-\mathscr{L}$ is one-to-one,* $\operatorname{ran}(\lambda-\mathscr{L})=E$, *and $R_\lambda=(\lambda-\mathscr{L})^{-1}$ is a bounded linear operator on E called the* resolvent operator *at λ of $\mathscr{L}$.*

It is follows from the previous definition that

$$R_\lambda-R_\theta=(\lambda-\theta)R_\lambda R_\theta, \qquad \lambda,\theta\in\rho(\mathscr{L}).$$

Proposition 1.2.12. *Let T be a strongly continuous contractive semigroup on E with infinitesimal generator $\mathscr{L}$. Then $(0,+\infty)\subset\rho(\mathscr{L})$ and*

$$R_\lambda u=\int_0^\infty e^{-\lambda t}T_t u\,\mathrm{d}t, \qquad \textit{for all } u\in E, \lambda>0.$$

Proof. See Proposition 2.1 in Chapter 1 of [68]. □

Let $\mathscr{L}$ be a closed linear operator on E. A subspace D of $D(\mathscr{L})$ is said to be a *core* for $\mathscr{L}$ if the closure of the restriction of $\mathscr{L}$ to D is equal to $\mathscr{L}$, i.e. if $\overline{\mathscr{L}|_D}=\mathscr{L}$.

Proposition 1.2.13. *Let $\mathscr{L}$ be the generator of a strongly continuous contractive semigroup on E. Then a subspace D of $D(\mathscr{L})$ is a core for $\mathscr{L}$ if and only if D is dense in E and* $\operatorname{ran}(\lambda-\mathscr{L}|_D)$ *is dense in E, for some $\lambda>0$.*

Proof. See Proposition 3.1 in Chapter 1 of [68]. □

Given a densely defined dissipative linear operator $\mathscr{L}$ on E, it is often useful to show that its closure $\bar{\mathscr{L}}$ generates a strongly continuous contraction semigroup on E. We already saw that an equivalent condition is that $\mathscr{L}$ is essentially m-dissipative on E (see Theorem 1.2.10), but we can also tackle this problem as a characterization problem for a core of the infinitesimal generator.

Proposition 1.2.14. *Let $\mathscr{L}$ be a dissipative linear operator on E and D_0 be a subspace of $D(\mathscr{L})$ that is dense in E. Suppose that, for every $u\in D_0$, there exists a continuous function $\varphi^u\colon[0,+\infty)\to E$ such that $\varphi^u(0)=u$, $\varphi^u(t)\in D(\mathscr{L})$ for all $t>0$, $\mathscr{L}\varphi^u\colon[0,+\infty)\to E$ is continuous, and*

$$\partial_t\varphi^u(t)=\mathscr{L}\varphi^u(t), \qquad \textit{for all } t>0. \tag{1.2.1}$$

Then $\mathscr{L}$ is closable, the closure of $\mathscr{L}$ generates a strongly continuous contractive semigroup T on E. Moreover, we have that $T_t u=\varphi^u(t)$, for all $u\in D_0$ and $t\geqslant 0$.

Proof. See Proposition 3.4, Chapter 1 in [68]. Point *iii.* in Proposition 1.2.9 yields that $\mathscr{L}$ is a closable operator. Let $u \in D_0$ and denote φ^u in the statement of the present Proposition by φ. Fix $t > t_0 > 0$, and notice that $\int_{t_0}^t e^{-s}\varphi(s)\,\mathrm{d}s \in D(\bar{\mathscr{L}})$ and

$$\bar{\mathscr{L}} \int_{t_0}^t e^{-s}\varphi(s)\,\mathrm{d}s = \int_{t_0}^t e^{-s}\mathscr{L}\varphi(s)\,\mathrm{d}s.$$

As a consequence

$$\begin{aligned}
\int_{t_0}^t e^{-s}\varphi(s)\,\mathrm{d}s &= (e^{-t} - e^{-t_0})\varphi(t_0) + \int_{t_0}^t e^{-s}\int_{t_0}^s \mathscr{L}\varphi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= (e^{-t} - e^{-t_0})\varphi(t_0) + \int_{t_0}^t (e^{-\tau} - e^{-t})\mathscr{L}\varphi(\tau)\,\mathrm{d}\tau \\
&= \bar{\mathscr{L}} \int_{t_0}^t e^{-s}\varphi(s)\,\mathrm{d}s + e^{-t_0}\varphi(t_0) - e^{-t}\varphi(t).
\end{aligned}$$

Since $\|\varphi(t)\| \leqslant \|u\|$ for all $t \geqslant 0$ by dissipativity of $\mathscr{L}$, we can let $t_0 \to 0$ and $t \to +\infty$ to get that $\int_0^{+\infty} e^{-s}\varphi(s)\,\mathrm{d}s \in D(\bar{\mathscr{L}})$ and

$$(1 - \bar{\mathscr{L}}) \int_0^{+\infty} e^{-s}\varphi(s)\,\mathrm{d}s = u.$$

We conclude that $\operatorname{ran}(1 - \bar{\mathscr{L}}) \supset D_0$, which, by Theorem 1.2.10, proves that $\bar{\mathscr{L}}$ generates a strongly continuous contraction semigroup T on E. For each $u \in D_0$, we have then

$$T_t u - T_{t_0} u = \int_{t_0}^t \bar{\mathscr{L}} T_s u\,\mathrm{d}s, \qquad \text{for all } t > t_0 > 0.$$

Subtracting equation (1.2.1) in the integral form $\varphi(t) - \varphi(t_0) = \int_{t_0}^t \mathscr{L}\varphi(s)\,\mathrm{d}s$ and exploiting once again the fact that $\|\varphi(t)\| \leqslant \|u\|$ for all $t \geqslant 0$, we obtain the second assertion of the statement. $\square$

1.2.2 Kolmogorov equations and invariant measures

When studying stochastic processes, and in particular solutions to S(P)DEs, it can be useful to consider some related problems which involve the infinitesimal generator $\mathscr{L}$ associated to the process itself (provided it is possible to define it). For instance, two equations that will be exploited and studied in the present thesis are the *Kolmogorov equations* (see [71, 72, 122]). As we will see, in the case of stochastic equations such problems become very useful tools when dealing with martingale problems or with invariant measures. Let $T > 0$, the *Kolmogorov backward equation* reads as

$$\partial_t \varphi(t) = \mathscr{L}\varphi(t), \qquad \varphi(T) = \varphi_T, \qquad t \in [0, T],$$

for some final condition φ_T, and it is linked to the problem of uniqueness (in law) of solutions to martingale problems (see e.g. [175] for the finite-dimensional case and [98] as well as Chapter 2 of the present thesis for an overview of the situation in the infinite-dimensional setting). On the other hand, the *Kolmogorov forward equation* is given by

$$\partial_t \mu(t) = \mathscr{L}^* \mu(t),$$

where $\mathscr{L}^*$ is the adjoint of $\mathscr{L}$, and it is often exploited to study invariant measures of the related S(P)DE (see [37, 131]).

Consider a measurable space $(X, \mathscr{B})$ and denote by $B_b(X)$ the space of bounded $\mathscr{B}$-measurable functions on X equipped with the sup-norm.

Definition 1.2.15. *If T is a semigroup of linear bounded operators on the space $B_b(X)$, then a bounded measure μ on $\mathscr{B}$ is called* invariant *for T if*

$$\int_X T_t u \,\mathrm{d}\mu = \int_X u \,\mathrm{d}\mu, \qquad \textit{for all } u \in B_b(X). \tag{1.2.2}$$

If μ is non-negative, then it is possible to similarly define the notion of invariant measure for semigroups on $L^1(\mu)$ or on $L^\infty(\mu)$, the semigroup T extends then from $B_b(X)$ to $L^1(\mu)$, and the measure μ will be invariant also for the extension. From an S(P)DEs viewpoint, an invariant measure is a measure μ such that, if the solution is started with initial distribution given by μ, then at every time the solution has the same distribution as the initial one.

Definition 1.2.16. *We say that the semigroup T is* symmetric *if*

$$\int_X T_t u \, v \,\mathrm{d}\mu = \int_X u \, T_t v \,\mathrm{d}\mu, \qquad \textit{for all } u, v \in B_b(X),$$

and the measure μ is called symmetric invariant *in this case.*

This case is characterized by the property that the generator $\mathscr{L}$ is symmetric in $L^2(\mu)$, that is

$$\int_X \mathscr{L}u \, v \,\mathrm{d}\mu = \int_X u \, \mathscr{L}v \,\mathrm{d}\mu, \qquad \text{for all } u, v \in B_b(X).$$

Semigroups on $B_b(X)$ with non-negative invariant measures often turn out to be strongly continuous on $L^1(\mu)$ and not on $B_b(X)$, e.g. when they are defined by transition probabilities of stochastic processes, and in that case we are able to define the corresponding infinitesimal generator $\mathscr{L}$ with domain $D(\mathscr{L}) \subset L^1(\mu)$, then equation (1.2.2) is equivalent to

$$\int_X \mathscr{L}u \,\mathrm{d}\mu = 0, \qquad \text{for all } u \in D(\mathscr{L}). \tag{1.2.3}$$

Notice that for the equivalence it is not sufficient to have this equality for all functions in a dense set in $L^1(\mu)$, it is important to have the identity above on all of $D(\mathscr{L})$. Let us remark that equation (1.2.3) is the integral version of the stationary Kolmogorov forward equation on $D(\mathscr{L})$.

It is often the case that one is able to have an explicit representation of the infinitesimal generator $\mathscr{L}$ on some class D_0 strictly contained in the domain $D(\mathscr{L})$. In this case, it can be useful to consider equation (1.2.3) on the space D_0.

Definition 1.2.17. *A measure μ on $\mathscr{B}$ is called* infinitesimally invariant *on D_0 if*

$$\int_X \mathscr{L}u \,\mathrm{d}\mu = 0, \qquad \textit{for all } u \in D_0. \tag{1.2.4}$$

We refer to equation (1.2.4) also as (stationary) *Fokker–Planck–Kolmogorov equation* on D_0. In general, equations (1.2.2) and (1.2.4) are not equivalent. It is possible to show that that any invariant measure is infinitesimally invariant but the vice versa is not clear a priori. An interesting task consists in studying whether the explicit representation of $\mathscr{L}$ on D_0 is enough to give results concerning the whole generator of the semigroup and on invariant measures. In the finite-dimensional case, many results on the link between invariant and infinitesimal invariant measures as well as on existence and uniqueness for the Fokker–Planck–Kolmogorov equation can be found in the books [37, 131].

Let us mention that, in a similar way, it is possible to define invariance in the case where X is a topological space, μ is a Borel measure on X, and T is a semigroup of bounded linear operators on the space $C_b(X)$ of bounded continuous functions.

Let us conclude the present subsection with an example of the link between SDEs and Kolmogorov equations in finite dimension. The following result can be found with more precise statements and details in the monographs [116, 151]. We consider the following SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = x_0, \tag{1.2.5}$$

where W is a Brownian motion, and $b, \sigma: \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous functions. We denote by X a solution to equation (1.2.5) (or sometimes X^{x_0} to specify the dependence on the initial data). For $\varphi \in C_b^2(\mathbb{R})$, we can define the semigroup T as follows

$$T_t\varphi(x) = \mathbb{E}_{X_0 = x_0}[\varphi(X_t)], \qquad t \geqslant 0, x \in \mathbb{R}.$$

It can be shown that T is a strongly continuous semigroup on $C_b^2(\mathbb{R})$. Moreover, applying Itô's formula, we have the following explicit representation of the generator for $\varphi \in C_b^2(\mathbb{R})$

$$\mathscr{L}\varphi(x) = b(x)\,\partial_x\varphi(x) + \frac{1}{2}\sigma^2(x)\,\partial_{xx}^2\varphi(x), \qquad x \in \mathbb{R},$$

and that $u(t,x) \coloneqq T_t\varphi(x)$ is a solution to the Kolmogorov backward equation

$$\partial_t u(t,x) = \mathscr{L}u(t,x), \qquad u(0,x) = \varphi(x).$$

Conversely, if $v \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ solves the Kolmogorov backward equation, then $v = u$.

Now, assume that X is a solution to the SDE starting at x_0 with $b \in C^1(\mathbb{R})$ and $\sigma^2 \in C^2(\mathbb{R})$, and that it has a density which we denote by p^{x_0}, so that

$$\mathbb{E}_{X_0 = x_0}[\varphi(X_t)] = \int_{\mathbb{R}} f(y) p^{x_0}(t,y)\,\mathrm{d}y, \qquad \varphi \in C_b^2(\mathbb{R}).$$

Assume further that $y \mapsto p^{x_0}(t,y)$ is smooth with respect to time and to the initial data of the SDE. Then p^{x_0} satisfies the Kolmogorov forward equation

$$\partial_t p(t,y) = \mathscr{L}^* p(t,y). \qquad p(0,y) = p^{x_0}(0,y),$$

where now

$$\mathscr{L}^* p(y) = -\partial_y(b(y)p(y)) + \frac{1}{2}\partial_{yy}^2(\sigma^2(y)\,p(y)).$$

1.3 Weighted Besov spaces

In this section, we introduce weighted Besov spaces $B_{p,q,\ell}^s$ and present some of their properties, characterizations, and how they relate with heat kernels. We consider the approach based on the Littlewood-Paley theory presented, for instance, by Bahouri, Chemin, and Danchin [22], but also refer to results presented by Triebel in [176, 177]. Besov spaces will be considered either on the whole $\mathbb{R}^n$ or on the n-dimensional torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, the presented results hold in both cases with minor modifications. Moreover, also space-time weighted Besov spaces will be treated.

1.3.1 Definition and embedding results

Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of *Schwartz functions* and $\mathcal{S}'(\mathbb{R}^n)$ be its topological dual, that is the space of *tempered distributions*. We also use the notation $\langle\cdot,\cdot\rangle_{\mathcal{S}',\mathcal{S}}$ for the duality between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. Hereafter, i denotes the imaginary unit. In what follows, we consider the *Fourier transform* $\mathcal{F}: L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ with the following definition

$$\hat{f}(\xi) \equiv \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}(\xi,x)} f(x)\,\mathrm{d}x,$$

together with its inverse

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\mathrm{i}(\xi,x)} f(\xi)\,\mathrm{d}\xi.$$

Such objects can be extended by duality to the space $\mathcal{S}'(\mathbb{R}^n)$ (see Section 1.2 in [22]).

Let us introduce a dyadic partition of unity and the corresponding Littlewood-Paley blocks. Hereafter, $B_r(x)$ denotes the ball centred at $x \in \mathbb{R}^n$ with radius $r > 0$, more precisely we have $B_r(x) = \{y \in \mathbb{R}^n : |y - x| \leqslant r\}$.

Definition 1.3.1. *We say that* (χ, φ) *is a* dyadic partition of unity *if* $\chi, \varphi: \mathbb{R}^n \to [0, 1]$ *are two smooth, compactly supported functions satisfying the following properties:*

i. $\operatorname{supp}(\chi) \subset B_{4/3}(0)$ *and* $\operatorname{supp}(\varphi) \subset B_{8/3}(0) \setminus B_{3/4}(0)$,

ii. $\chi(y) + \sum_{j\geqslant 0} \varphi(2^{-j}y) = 1$, *for any* $y \in \mathbb{R}^n$,

iii. $\operatorname{supp}(\chi) \cap \operatorname{supp}(\varphi(2^{-j}\cdot)) = \varnothing$, *for* $j \geqslant 1$,

iv. $\operatorname{supp}(\varphi(2^{-j}\cdot)) \cap \operatorname{supp}(\varphi(2^{-i}\cdot)) = \varnothing$, *for* $|i - j| > 1$.

It is possible to show that such a dyadic partition of unity exists (see [22], Section 2.2), and from now on we fix a dyadic partition of unity (χ, φ).

Definition 1.3.2. *Let* $\sigma \in C^\infty(\mathbb{R}^n)$ *growing at most polynomially at infinity. We define the operator* $\sigma(\mathrm{D}): \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *as*

$$\sigma(\mathrm{D})f = \mathcal{F}^{-1}(\sigma(\xi)\cdot\mathcal{F}(f)(\xi)), \qquad f \in \mathcal{S}(\mathbb{R}^n).$$

We now give the definition of Littlewood-Paley blocks.

Definition 1.3.3. *Let* $\varphi_{-1} = \chi$ *and* $\varphi_j(\cdot) = \varphi(2^{-j}\cdot)$, *we define the* Littlewood–Paley blocks

$$\Delta_j = \varphi_j(\mathrm{D}), \qquad \text{for } j \geqslant -1.$$

Let $K_{-1} = \mathcal{F}^{-1}(\chi)$ and $K_j = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot))$, for $j \geqslant 0$. If $u \in \mathcal{S}'(\mathbb{R}^n)$, then we have, for all $j \geqslant -1$,

$$\Delta_j u = K_j * u.$$

We also have, for $j \geqslant 0$,

$$K_j = 2^{jn} K_0(2^j\cdot).$$

It is possible to show that, if $u \in \mathcal{S}'(\mathbb{R}^n)$, then

$$\lim_{k\to+\infty} \sum_{j=-1}^{k} \Delta_j u = u.$$

Notice that, while $u \in \mathcal{S}'(\mathbb{R}^n)$ is a distribution, $\Delta_j u$ is a function, since, by definition of Littlewood–Paley blocks, $\Delta_j u$ has a compactly supported Fourier transform.

We are now in position to introduce weighted Besov spaces. We consider polynomial weights given by, for any $\ell > 0$,

$$\rho_\ell(y) = (1 + |y|^2)^{-\ell/2}, \qquad y \in \mathbb{R}^n,$$

and, for $p \in [1, +\infty]$, let $L^p_\ell(\mathbb{R}^n)$ denote the L^p-space with respect to the norm

$$\|u\|_{L^p_\ell} = \|\rho_\ell u\|_{L^p} = \left(\int_{\mathbb{R}^n} (\rho_\ell(y)\, u(y))^p \,\mathrm{d}y \right)^{1/p}.$$

Definition 1.3.4. (Besov space $B^s_{p,q,\ell}$) *Let $s \in \mathbb{R}$, $p, q, \in [1, +\infty]$, and $\ell \in \mathbb{R}$. For $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the norm*

$$\|u\|_{B^s_{p,q,\ell}} = \left(\sum_{j \geqslant -1} 2^{sqj} \|\Delta_j u\|^q_{L^p_\ell} \right)^{1/q}.$$

The weighted Besov space $B^s_{p,q,\ell}(\mathbb{R}^n)$ *is then defined as follows:*

$$B^s_{p,q,\ell}(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) \colon \|u\|_{B^s_{p,q,\ell}} < +\infty \}.$$

In the case $p = +\infty$ and $q = +\infty$, the norm reads as

$$\|u\|_{B^s_{\infty,\infty,\ell}} = \sup_{j \geqslant -1} 2^{sj} \|\Delta_j u\|_{L^\infty_\ell}.$$

We also write $B^s_{p,q}(\mathbb{R}^n) = B^s_{p,q,0}(\mathbb{R}^n)$ for the un-weighted Besov spaces.

Notice that in the definition of weighted Besov space $B^s_{p,q,\ell}(\mathbb{R}^n)$, the parameter s describes the regularity, namely the decay of the Littlewood–Paley blocks, while p describes the integrability. The parameter q is an additional refinement of the regularity scale, indeed we have

$$B^s_{p,q_1,\ell} \subset B^s_{p,q_2,\ell} \subset B^{s'}_{p,q_1,\ell}, \qquad q_1 \leqslant q_2, \quad s' < s.$$

It is possible to show that weighted Besov spaces are Banach spaces, Moreover, we have the equivalence $\|u\|_{B^s_{p,q,\ell}} \sim \|\rho_\ell u\|_{B^s_{p,q}}$, in fact

$$u \in B^s_{p,q,\ell} \quad \text{if and only if} \quad \rho_\ell u \in B^s_{p,q} \tag{1.3.1}$$

(see Theorem 6.5 in [177] and Section 4.2.2 of [67]).

Proposition 1.3.5. (Besov embedding) *Let $p_1, p_2, q_1, q_2 \in [1, +\infty]$, $\ell_1, \ell_2 \in \mathbb{R}$, and $s_1, s_2 \in \mathbb{R}$ be such that $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$ and $\ell_1 > \ell_2$. Then, we have the compact immersion*

$$B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n) \subset B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n).$$

Proof. The result can be found in Theorem 6.7 in [177], see also Section 4.2.3 of [67] for the full proof of the statement. □

We also have relations between Besov spaces and *Sobolev spaces* $W^{s,p}_\ell(\mathbb{R}^n)$, i.e. the spaces of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$\|f\|_{W^{s,p}_\ell} = \|(1 + |\cdot|^2)^{s/2}(\mathrm{D}) f\|_{L^p_\ell}$$

is finite. We also write $H^s_\ell(\mathbb{R}^n) = W^{s,2}_\ell(\mathbb{R}^n)$.

Proposition 1.3.6. (Besov–Sobolev embedding) *Let $s \in \mathbb{R}$, $\ell \in \mathbb{R}$, and $1 \leqslant p \leqslant 2$, then we have the continuous immersion*

$$B^s_{p,p,\ell}(\mathbb{R}^n) \subset W^{s,p}_\ell(\mathbb{R}^n) \subset B^s_{p,2,\ell}(\mathbb{R}^n).$$

For $2 \leqslant p < +\infty$, we have the continuous immersion

$$B^s_{p,2,\ell}(\mathbb{R}^n) \subset W^{s,p}_\ell(\mathbb{R}^n) \subset B^s_{p,p,\ell}(\mathbb{R}^n).$$

For $p = +\infty$, we have the continuous immersion

$$W^{s,\infty}_\ell(\mathbb{R}^n) \subset B^s_{\infty,\infty}(\mathbb{R}^n).$$

Proof. The proposition is proved in Theorem 6.4.4 and Theorem 6.2.4 of [32] for the case of unweighted spaces. The result for weighted spaces follows from (1.3.1) and from the fact that $g \in W^{s,p}_\ell(\mathbb{R}^n)$ if and only if $g \cdot \rho_\ell \in W^{s,p}(\mathbb{R}^n)$ (see Theorem 6.5 in [177] and Section 4.2.2 of [67]). $\square$

It follows from the previous proposition that $B^s_{2,2,\ell}(\mathbb{R}^n) = H^s_\ell(\mathbb{R}^n)$. The following proposition gives an interpolation result between weighted Besov spaces.

Proposition 1.3.7. *Consider $p_1, p_2, q_1, q_2 \in [1, +\infty]$, $\ell_1, \ell_2 \in \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$, and write, for any $\theta \in [0, 1]$,*

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \qquad \ell_\theta = \theta \ell_1 + (1-\theta)\ell_2, \qquad s_\theta = \theta s_1 + (1-\theta) s_2.$$

If $f \in B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n) \cap B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$, then $f \in B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}(\mathbb{R}^n)$, and furthermore

$$\|f\|_{B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}} \leqslant \|f\|^\theta_{B^{s_1}_{p_1,q_1,\ell_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

Proof. The proof is based on the fact that the complex interpolation of the two spaces $B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)$ and $B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$ is given by $B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}(\mathbb{R}^n)$. Such an interpolation is shown in Theorem 6.4.5 in [32] for unweighted Besov spaces. The proof for weighted spaces follows from relation (1.3.1). $\square$

We now want to give a characterization of weighted Besov spaces by means of differences norms. Let us introduce the notation $\tau_y u(\cdot) = u(\cdot + y)$, for $u: \mathbb{R}^n \to \mathbb{R}$ and $y \in \mathbb{R}^n$. Moreover, if $s \in (0, 1)$ and $\ell \in \mathbb{R}$, we let $C^s_\ell(\mathbb{R}^n)$ denote the space of *weighted Hölder continuous functions* on $\mathbb{R}^n$, i.e. of functions with norm, for $f \in C^s_\ell(\mathbb{R}^n)$, given by

$$\|f\|_{C^s_\ell} = \|f\|_{L^\infty_\ell} + \sup_{\substack{x,y \in \mathbb{R}^n \\ |y| < 1}} \rho_\ell(x) \frac{|f(x-y) - f(x)|}{|y|^s}.$$

Proposition 1.3.8. *Let $p, q \in [1, +\infty]$, $\ell \in \mathbb{R}$, and $0 < s < 1$. If $u \in B^s_{p,q,\ell}(\mathbb{R}^n)$, then*

$$\|u\|^q_{B^s_{p,q,\ell}} \sim \|u\|^q_{L^p_\ell} + \int_{|y| \leqslant 1} \frac{\|\tau_y u - u\|^q_{L^p_\ell}}{|y|^{n+qs}} \, \mathrm{d}y.$$

In particular,

$$B^s_{\infty,\infty,\ell}(\mathbb{R}^n) = C^s_\ell(\mathbb{R}^n),$$

in the sense that

$$\|f\|_{B^s_{\infty,\infty,\ell}} \sim \|f\|_{C^s_\ell}.$$

Proof. See e.g. Theorem 2.36 in [22] or Chapter 2.6.1 in [176] for the case of unweighted Besov spaces, and Theorem 6.9 in [177] or Section 5.1.4 in [164] for the generalization to weighted Besov spaces. $\square$

In the light of the second part of the previous result, we often write $C^s_\ell(\mathbb{R}^n)$ instead of $B^s_{\infty,\infty,\ell}(\mathbb{R}^n)$ whenever $s \in \mathbb{R} \setminus \mathbb{Z}$.

1.3.2 Bony's paraproducts

Let us discuss multiplication between elements of Besov spaces. Consider $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, then $fu \in \mathcal{S}'(\mathbb{R}^n)$, since

$$\langle uf, g\rangle_{\mathcal{S}',\mathcal{S}} = \langle u, fg\rangle_{\mathcal{S}',\mathcal{S}}, \qquad g \in \mathcal{S}(\mathbb{R}^n).$$

If instead $f \notin \mathcal{S}(\mathbb{R}^n)$ but, for instance, if it is only measurable, then the product fu is a priori not well-defined. In order to deal with this non-regular case, we introduce the notion of Bony's paraproducts (see [41]).

Suppose that $f \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$u = \sum_{j \geqslant -1} \Delta_j u, \qquad f = \sum_{i \geqslant -1} \Delta_i f,$$

and therefore

$$fu = \sum_{i,j \geqslant -1} \Delta_i f \, \Delta_j u.$$

Notice that the product $\Delta_i f \, \Delta_j u$ is always well-defined. The idea is then to use the previous representation of the factors to make sense of the product for more general terms. It will be important to split the sum as follows

$$\begin{aligned} fu &= \sum_{|j-i| \leqslant 1} \Delta_i f \, \Delta_j u + \sum_{|j-i| \geqslant 2} \Delta_i f \, \Delta_j u \\ &= \sum_{|j-i| \leqslant 1} \Delta_i f \, \Delta_j u + \sum_{j \geqslant -1} \sum_{i < j-1} (\Delta_j f \, \Delta_i u + \Delta_i f \, \Delta_j u) \\ &= \sum_{|j-i| \leqslant 1} \Delta_i f \, \Delta_j u + \sum_{j \geqslant -1} (S_{j-1} f \, \Delta_j u + S_{j-1} u \, \Delta_j f), \end{aligned} \tag{1.3.2}$$

where

$$S_j g = \sum_{-1 < i < j} \Delta_i g,$$

which is always a well-defined function by the property of the dyadic partition of unity since

$$S_j g = 2^{jn} K_{-1}(2^j \cdot) * g.$$

We call the terms on the right-hand side of equation (1.3.2) in the following way: $f \circ u$ is the *resonant product*

$$f \circ u = \sum_{|j-i| \leqslant 1} \Delta_i f \, \Delta_j u,$$

while $u \prec f$ and $u \succ f$ are the *paraproducts* of u by f, and of f by u, respectively, that is

$$u \prec f = f \succ u = \sum_{j \geqslant -1} S_{j-1} f \, \Delta_j u, \qquad u \succ f = f \prec u = \sum_{j \geqslant -1} S_{j-1} u \, \Delta_j f,$$

so that we have the *Bony decomposition*

$$fu = u \prec f + u \succ f + f \circ u.$$

Now suppose that u and f live in two Besov spaces. It is possible to show that $u \prec f$ and $u \succ f$ are always well-defined with Besov regularity not worse than the worst one between f and u. As far as the resonant product is concerned, it can be shown that it is well-defined provided that the sum of the Besov regularities of the two factors f and u is positive. More precisely, we have the following result.

Theorem 1.3.9. (Paraproduct) *Let $p_1, p_2, p, q_1, q_2, q \in [1, +\infty]$, $\ell_1, \ell_2, \ell, s_1, s_2, s \in \mathbb{R}$, be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \qquad \ell = \ell_1 + \ell_2, \qquad s_1 + s_2 > 0, \quad s = s_1 \wedge s_2. \tag{1.3.3}$$

Consider the bilinear map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, mapping $(f, g) \mapsto fg$. Then, there exists a unique continuous extension of the aforementioned map as a map

$$B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n) \times B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n) \to B^s_{p,q,\ell}(\mathbb{R}^n),$$

and we have, for any $f \in B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)$, $g \in B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$,

$$\|fg\|_{B^s_{p,q,\ell}} \lesssim \|f\|_{B^{s_1}_{p_1,q_1,\ell_1}} \|g\|_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

Proof. See Section 3.3 in [141] for Besov spaces with exponential weights. The proof for polynomial weights follows in a similar way. □

1.3.3 Relation with positive measures

In this section, we deal with the case of products in which one of the factors has positive regularity and is in L^∞ while the other factor is a positive measure with negative regularity. We have the following result.

Proposition 1.3.10. *Consider the same parameters as in Theorem 1.3.9 satisfying (1.3.3) and assume $s_1 > 0$, $s_2 \leqslant 0$. Suppose that $f \in B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and that $\mu \in B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$ is a positive measure, then we have*

$$\|f\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}} \lesssim \|f\|_{L^\infty} \|\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

We give a proof of the previous result following the one of Lemma 28 in [4]. We first need to introduce another representation of the weighted Besov norm. Let us denote by $\mathfrak{F}_r$, for $r \in \mathbb{N}$, the space of continuous functions $C^r(\mathbb{R}^n)$ with support in $B_1(0)$ and such that $\|f\|_{C^r} \leqslant 1$. If $f \in \mathfrak{F}_r$, then we write, for $y \in \mathbb{R}^n$ and $\lambda > 0$,

$$f_{y,\lambda}(\cdot) = \frac{1}{\lambda^n} f\left(\frac{\cdot - y}{\lambda}\right).$$

Proposition 1.3.11. *Let $s < 0$, $p, q \in [1, +\infty]$, and $\ell \in \mathbb{R}$. Then an equivalent norm in the space $B^s_{p,q,\ell}(\mathbb{R}^n)$ is given by, for $f \in B^s_{p,q,\ell}(\mathbb{R}^n)$,*

$$\|f\|_{B^s_{p,q,\ell}} \sim \left(\int_0^1 \frac{\| \sup_{g \in \mathfrak{F}_r} |\langle f, g_{\cdot,\lambda} \rangle| \|^q_{L^p_\ell}}{\lambda^{sq}} \frac{\mathrm{d}\lambda}{\lambda} \right)^{1/q}, \tag{1.3.4}$$

where $r \in \mathbb{N}$ is the first integer such that $r > -s$.

Proof. Theorem 6.15 in [177] proves the equivalence between the norm $\|\cdot\|_{B^s_{p,q,\ell}}$ and the norm of $B^s_{p,q,\ell}$ built using wavelets, while Proposition 2.4 in [104] shows the equivalence between the norm introduced in (1.3.4) and the norm of $B^s_{p,q,\ell}$ built exploiting wavelets. The combination of such results gives the claimed equivalence. □

Proof of Proposition 1.3.10. Notice first that by Theorem 1.3.9 the product $f\mu$ is well-defined. Exploiting the equivalent norm introduced in Proposition 1.3.11, we have

$$\|\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}} \sim \left(\int_0^1 \int_{\mathbb{R}^2} \left(\frac{\mu(B_\lambda(z))}{\lambda^{s_2}} \right)^{q_2} (\rho_{\ell_2}(z))^{q_2} \, \mathrm{d}z \, \frac{\mathrm{d}\lambda}{\lambda} \right)^{1/q_2}.$$

Proposition 1.3.11 also yields, provided f is a continuous bounded function,

$$\|f\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}} \lesssim \left(\int_0^1 \int_{\mathbb{R}^2} \left(\frac{(f\mu)(B_\lambda(z))}{\lambda^{s_2}}\right)^{q_2} (\rho_{\ell_2}(z))^{q_2}\,\mathrm{d}z\,\frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q_2} \lesssim \|f\|_{L^\infty}\|\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

If $f\in B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)$, then there exists a sequence $(f_k)_{k\in\mathbb{N}}$ of smooth functions such that, as $k\to+\infty$, $f_k\to f$ in $B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)$ and $\|f_k\|_{L^\infty}\leqslant\|f\|_{L^\infty}$. We then have, by the previous estimates, that $f_k\mu$ converges weakly in $\mathcal{S}'(\mathbb{R}^n)$ to $f\mu$, which gives

$$\|f\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}} \leqslant \liminf_{k\to+\infty}\|f_k\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}} \lesssim \|\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}} \liminf_{k\to+\infty}\|f_k\|_{L^\infty} \lesssim \|f\|_{L^\infty}\|\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

This concludes the proof. □

1.3.4 Space-time weighted Besov spaces

As in the study of evolution PDEs it is often useful to consider spaces such as $L^p([0,T];W^{s,p}(\mathbb{R}^n))$, $C^\alpha([0,T];W^{s,p}(\mathbb{R}^n))$, etc., in the study of SPDEs it can be convenient to introduce sets of functions from an interval into a Besov space. To this aim, let us consider the space

$$L^r_{\ell_1}(I;B^s_{p,q,\ell_2}(\mathbb{R}^n)),$$

where $I\subset\mathbb{R}$ is an interval, $r,p,q\in[1,+\infty]$, and $s,\ell_1,\ell_2\in\mathbb{R}$, with norm, for any $f:I\to B^s_{p,q,\ell_2}(\mathbb{R}^n)$, given by

$$\|f\|_{L^r_{\ell_1}(I;B^s_{p,q,\ell_2}(\mathbb{R}^n))} = \left(\int_I \|\rho_{\ell_1}(t)f(t)\|^r_{B^s_{p,q,\ell_2}}\,\mathrm{d}t\right)^{1/r}.$$

If $B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)\hookrightarrow B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$, then also $L^r(I;B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n))\hookrightarrow L^r(I;B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n))$, but this latter embedding is not compact. In order to get a compact embedding it is useful to introduce the following space

$$B^{\bar{s}}_{r,r,\bar{\ell}}(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n)),$$

for $\bar{s},\bar{\ell},\ell\in\mathbb{R}$, with norm given by

$$\|f\|_{B^{\bar{s}}_{r,r,\bar{\ell}}(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n))} = \sum_{j\geqslant -1}\int_{\mathbb{R}} \|\Delta^t_j f(t)\|^r_{B^s_{p,q,\ell}(\mathbb{R}^n)}(1+|t|^2)^{-\bar{\ell}r/2}\,\mathrm{d}t,$$

where Δ^t_j denotes the Littlewood–Paley block Δ_j taken only with respect to the time variable t.

By the characterization of Besov spaces with the (L^r) difference (see Theorem 1.3.8), we get that an equivalent norm of the previous spaces is given by

$$\|f\|^r_{B^{\bar{s}}_{r,r,\bar{\ell}}(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n))} \sim \|f\|^r_{L^r_{\bar{\ell}}(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n))} + \int_{\mathbb{R}}\int_{|k|<1} \frac{(\rho_{\bar{\ell}}(t))^r\|\tau_k f(t)-f(t)\|^r_{B^s_{p,q,\ell}(\mathbb{R}^n)}}{|k|^{1+rs}}\,\mathrm{d}k\,\mathrm{d}t. \quad (1.3.5)$$

Since $B^{\bar{s}}_{r,r,\bar{\ell}}(\mathbb{R}^n)\hookrightarrow B^{\bar{s}'}_{r',r',\bar{\ell}'}(\mathbb{R}^n)$ when $\bar{s}>\bar{s}'$, $\bar{\ell}<\bar{\ell}'$, $\bar{s}-1/r>\bar{s}'-1/r'$ in a compact way, then, if the immersion $B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)\hookrightarrow B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$ is compact, also the immersion

$$B^{\bar{s}}_{r,r,\bar{\ell}}(\mathbb{R},B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)) \hookrightarrow B^{\bar{s}'}_{r',r',\bar{\ell}'}(\mathbb{R},B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n))$$

is compact. A particular case is the one of Hölder space, i.e. when $r=\infty$, which gives

$$B^{\bar{s}}_{\infty,\infty,\bar{\ell}}(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n)) = C^{\bar{s}}_{\bar{\ell}}(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n)),$$

with norm

$$\|f\|_{C^{\bar{s}}_{\bar{\ell}}(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n))} = \|f\|_{L^\infty(\mathbb{R},B^s_{p,q,\ell}(\mathbb{R}^n))} + \sup_{\substack{t,k\in\mathbb{R}\\|k|<1}} \frac{\rho_{\bar{\ell}}(t)\,\|f(t+k)-f(t)\|_{B^s_{p,q,\ell}(\mathbb{R}^n)}}{|k|^{\bar{s}}}.$$

1.3.5 Heat flow and Besov regularity

Let Δ denote the Laplace operator on $\mathbb{R}^n$ and consider a mass $m>0$. Take $f\in B^s_{p,q}(\mathbb{R}^n)$ and define the *heat flow* on f as

$$e^{-(-\Delta+m^2)t}f(x)=\int_{\mathbb{R}^n}\mathfrak{K}^m(t,x-y)f(y)\,\mathrm{d}y,$$

where $\mathfrak{K}^m$ is the *heat kernel* with mass m:

$$\mathfrak{K}^m(t,x)=\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}+m^2t}.$$

If $f\in L^r(\mathbb{R},B^s_{p,q}(\mathbb{R}^n))$, then we also write

$$e^{-(-\Delta+m^2)}f(t,x)=\int_{-\infty}^t e^{-(-\Delta+m^2)(t-\tau)}f(\tau,x)\,\mathrm{d}\tau=\int_{-\infty}^t\int_{\mathbb{R}^n}\frac{1}{(4\pi(t-\tau))^{n/2}}e^{-\frac{|x-y|^2}{4(t-\tau)}+m^2(t-\tau)}f(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau.$$

Notice that $e^{-(-\Delta+m^2)}f(t,\cdot)$ is a distribution.

In this section, we restrict for simplicity to the case $n=2$. We have the following regularization property.

Theorem 1.3.12. *Consider $r\in[1,+\infty]$, $p,q\in[1,+\infty]$, $s,\ell_1,\ell_2\in\mathbb{R}$, and let $f\in L^r_{\ell_1}(\mathbb{R},B^s_{p,q,\ell_2}(\mathbb{R}^2))$. Then, for any $\beta_1,\beta_2>0$ such that $\beta_1+\beta_2<1$, we have*

$$e^{-(-\Delta+m^2)}f\in B^{\beta_2}_{r,r,\ell_1}\big(\mathbb{R},B^{s+2\beta_1}_{p,q,\ell_2}(\mathbb{R}^2)\big). \tag{1.3.6}$$

Notice that equation (1.3.6) states that we are gaining regularity β_2 in time and $2\beta_1$ in space. In order to prove Theorem 1.3.12, we need the following result saying that, when we apply the heat kernel at time t, we gain $2\beta_1$ in space-regularity, but we have to pay with a multiplicative factor of $t^{-\beta_1}$.

Lemma 1.3.13. *Let $m>0$ and consider $f\in B^s_{p,q,\ell}(\mathbb{R}^2)$, with $p,q\in[1,+\infty]$, $s,\ell\in\mathbb{R}$. Then, we have, for every $t>0$,*

$$\big\|e^{-(-\Delta+m^2)t}f\big\|_{B^{s+2\beta_1}_{p,q,\ell}}\lesssim t^{-\beta_1}e^{-m^2t}\|f\|_{B^s_{p,q,\ell}}.$$

Proof. See Proposition 5 in [141]. □

We will also need the following lemma saying that giving up some space-regularity we can gain a factor t^β on the right-hand side.

Lemma 1.3.14. *Consider $0<\beta<1$ and $f\in B^s_{p,q,\ell}(\mathbb{R}^2)$, with $p,q\in[1,+\infty]$, $s,\ell\in\mathbb{R}$. Then, we have, for any $t>0$,*

$$\big\|\big(1-e^{-(-\Delta+m^2)t}\big)f\big\|_{B^{s-2\beta}_{p,q,\ell}}\lesssim t^\beta\|f\|_{B^s_{p,q,\ell}}.$$

Proof. See Proposition 6 in [141]. □

Proof of Theorem 1.3.12. We give here a proof for unweighted Besov spaces, the general case follows the same lines. We exploit the difference characterization of space-time Besov spaces (1.3.5), which yields

$$\begin{aligned}\big\|e^{-(-\Delta-m^2)}f\big\|^r_{B^{\beta_2}_{r,r}(\mathbb{R},B^{s+2\beta_1}(\mathbb{R}^2))}\sim{}&\big\|e^{-(-\Delta-m^2)}f\big\|^r_{L^r(\mathbb{R},B^{s+2\beta_1}(\mathbb{R}^2))}\\&+\int_{\mathbb{R}}\int_{|\Delta t|\leqslant 1}\frac{\big\|e^{-(-\Delta-m^2)}f(t+\Delta t)-e^{-(-\Delta-m^2)}f(t)\big\|^r_{B^s_{p,q}}}{|\Delta t|^{1+r\beta_2}}\,\mathrm{d}(\Delta t)\,\mathrm{d}t.\end{aligned}\tag{1.3.7}$$

First, we prove that the first term on the right-hand side of equation (1.3.7) is finite. Write $\tilde{f} = e^{-(-\Delta+m^2)}f$, we have

$$\begin{aligned}
\|\tilde{f}\|_{L^r}^r &= \int_{\mathbb{R}} \|\tilde{f}(t)\|_{B_{p,q}^{s+2\beta_1}}^r \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \left\| \int_{-\infty}^t e^{-(-\Delta+m^2)(t-k)} f(k)\,\mathrm{d}k \right\|_{B_{p,q}^{s+2\beta_1}}^r \mathrm{d}t \\
&\lesssim \int_{\mathbb{R}} \left(\int_{-\infty}^t e^{-m^2(t-k)} \|e^{\Delta(t-k)} f(k)\|_{B_{p,q}^{s+2\beta_1}} \mathrm{d}k \right)^r \mathrm{d}t
\end{aligned}$$

Lemma 1.3.13 and Young inequality yield

$$\begin{aligned}
\|\tilde{f}\|_{L^r}^r &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\mathbb{I}_{[0,+\infty]}(t-k)}{(t-k)^{\beta_1}} e^{-m^2(t-k)} \|f(k)\|_{B_{p,q}^s} \mathrm{d}k \right)^r \mathrm{d}t \\
&\lesssim \left(\int_{\mathbb{R}} \frac{\mathbb{I}_{[0,+\infty]}(t-k)}{(t-k)^{\beta_1}} e^{-m^2(t-k)}\,\mathrm{d}k \right)^r + \|f\|_{L^r(\mathbb{R},B_{p,q}^s(\mathbb{R}^n))}^r,
\end{aligned}$$

where the first integral on the last step is finite if and only if $\beta_1 < 1$.

Consider now the difference term on the right-hand side of equation (1.3.7). We have

$$\begin{aligned}
\tilde{f}(t+\Delta t) - \tilde{f}(t) &= \int_t^{t+\Delta t} e^{-(-\Delta+m^2)(t+\Delta t-k)} f(k)\,\mathrm{d}k + \left(1 - e^{-(-\Delta+m^2)\Delta t}\right) \int_{-\infty}^t e^{-(-\Delta+m^2)(t-k)} f(k)\,\mathrm{d}k \\
&=: I_1 + I_2.
\end{aligned}$$

Now, by Lemma 1.3.13 and Young inequality,

$$\begin{aligned}
\|I_1\|_{L^r(\mathbb{R},B_{p,q}^{s+2\beta_1}(\mathbb{R}^2))}^r &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\mathbb{I}_{[-\Delta t,0]}(t-k) e^{-m^2(t+\Delta t-k)}}{(t+\Delta t-k)^{\beta_1}} \|f(k)\|_{B_{p,q}^s}\,\mathrm{d}k \right)^r \mathrm{d}t \\
&\lesssim (\Delta t)^{1-\beta_1} \|f\|_{L^r(\mathbb{R},B_{p,q}^s(\mathbb{R}^n))}
\end{aligned}$$

Consider $\beta_2 < \tilde{\beta} < 1 - \beta_1$, then, by Lemma 1.3.14,

$$\begin{aligned}
\|I_2\|_{L^r(\mathbb{R},B_{p,q}^{s+2\beta_1}(\mathbb{R}^2))}^r &\lesssim (\Delta t)^{\tilde{\beta}} \left\| e^{-(-\Delta+m^2)} f \right\|_{L^r(\mathbb{R},B_{p,q}^{s+2\beta_1+2\tilde{\beta}}(\mathbb{R}^n))} \\
&\lesssim (\Delta t)^{\tilde{\beta}} \|f\|_{L^r(\mathbb{R},B_{p,q}^s(\mathbb{R}^n))},
\end{aligned}$$

where we used the first part of the proof in the last step and the fact that $\beta_1 + \tilde{\beta} < 1$. Putting everything together, we get

$$\begin{aligned}
\int_{\mathbb{R}} \int_{|\Delta t| \leqslant 1} \frac{\|\tilde{f}(t+\Delta t) - \tilde{f}(t)\|_{B_{p,q}^s}^r}{|\Delta t|^{1+r\beta_2}} \mathrm{d}(\Delta t)\,\mathrm{d}t &\lesssim \|f\|_{L^1(\mathbb{R},B_{p,q}^s(\mathbb{R}^n))} \int_{|\Delta t| \leqslant 1} \frac{(\Delta t)^{(1-\beta_1)r} + (\Delta t)^{\tilde{\beta} r}}{|\Delta t|^{1+r\beta_2}} \mathrm{d}(\Delta t) \\
&\lesssim \|f\|_{L^1(\mathbb{R},B_{p,q}^s(\mathbb{R}^n))} \int_{|\Delta t| \leqslant 1} \frac{1}{|\Delta t|^{1-(\tilde{\beta}-\beta_2)r}} \mathrm{d}(\Delta t) \\
&\lesssim \|f\|_{L^r(\mathbb{R},B_{p,q}^s(\mathbb{R}^n))},
\end{aligned}$$

which gives the result. □

Let us conclude this section on weighted Besov spaces with the next result, which gives an equivalent norm by means of thermic expansions, i.e. involving the heat kernel $e^{-(-\Delta+m^2)t}$.

Proposition 1.3.15. *Let $s \in \mathbb{R}$, $p, q \in (0, +\infty]$, and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be such that*

$$k > \frac{s}{2}.$$

Consider a smooth and compactly supported function φ_0. Then, we have the following equivalence between norms

$$\|f\|_{B^s_{p,q,\ell}} \sim \|\varphi_0(\mathrm{D})f\|_{L^p_\ell} + \left(\int_0^{+\infty} t^{\left(k-\frac{s}{2}\right)q} \left\|\partial_{t^k}\left(e^{-(-\Delta+m^2)t}f\right)\right\|^q_{L^p_\ell} \frac{\mathrm{d}t}{t}\right)^{1/q}. \tag{1.3.8}$$

Proof. See Section 2.6.4 in [176] for a version of the theorem with a mass-less heat kernel. In particular, such a result differs from the one presented above since the integral with respect to t in equation (1.3.8) goes from 0 to 1 instead of going from zero to $+\infty$. This extension is possible thanks to the regularization properties of the heat kernel and the exponential decay thereof, due to the presence of the positive mass $m>0$. □

Remark 1.3.16. Under the same hypotheses as in Proposition 1.3.15, if we assume further that $s>0$, then the norm $\|\varphi_0(\mathrm{D})f\|_{L^p_\ell}$ appearing in equation (1.3.8) can be substituted by the weighted L^p-norm of f, since we have

$$\|\varphi_0(\mathrm{D})f\|_{L^p_\ell} \lesssim \|f\|_{L^p_\ell} \lesssim \|f\|_{B^s_{p,q,\ell}}.$$

Namely, for $s>0$, equivalence (1.3.8) becomes

$$\|f\|_{B^s_{p,q,\ell}} \sim \|f\|_{L^p_\ell} + \left(\int_0^{1} t^{\left(k-\frac{s}{2}\right)q} \left\|\partial_{t^k}\left(e^{-(-\Delta+m^2)t}f\right)\right\|^q_{L^p_\ell} \frac{\mathrm{d}t}{t}\right)^{1/q}.$$

1.4 Gaussian measures in infinite dimensions

This section is devoted to the study of Gaussian measures on infinite dimensional spaces, their properties and applications. In particular, after having introduced Gaussian measures on Banach and Hilbert spaces, we present Wick product and Wick exponential, Fock space and chaos decomposition, and hypercontractivity. We also give some examples that will be useful later on in the thesis, for example we introduce white noise and Gaussian free field. For a more detailed discussion of the aforementioned topics, we refer the reader to the classical monographs [33, 112, 126, 148, 179], on which the present section is based, as well as the work by Gross [86].

1.4.1 Definition and characterization

Let us consider a (in general, not separable) Banach space W, the example to keep in mind being $W=B^s_{p,q,\ell}(\mathbb{R}^n)$ with $s\in\mathbb{R}$, $p,q\in[1,+\infty]$, $\ell\in\mathbb{R}$, which is separable if $p,q\neq+\infty$ (for instance, if $W=H^s_\ell(\mathbb{R}^n)$), then W is separable), while it is not if $W=C^s_\ell$. Since W has a topology, we can consider the Borel σ-algebra

$$\mathscr{B}=\sigma(A\colon A \text{ open set in } W).$$

The element $x^*\in W^*$, where W^* is the strong dual of W, is a linear and bounded functional $x^*\colon W\to\mathbb{R}$, and therefore, we can also consider the σ-algebra generated by cylindrical sets

$$\mathscr{C}=\sigma(x^*(\cdot);x^*\in W^*).$$

Remark 1.4.1. While in general $\mathscr{C}\subset\mathscr{B}$, in the case where W is separable we have $\mathscr{C}=\mathscr{B}$.

Definition 1.4.2. *A probability measure μ on $(W,\mathscr{B})$ is called* Radon measure *if, given an open set A, for any $\varepsilon>0$, there exist a compact set $K\subset A$ such that*

$$\mu(A\smallsetminus K)<\varepsilon.$$

Remark 1.4.3. If W is separable, every Borel measure (i.e. a measure on $(W,\mathscr{B})$) is also a Radon measure.

Theorem 1.4.4. *Let μ be a Radon probability measure on $(W,\mathscr{B})$. Then, for any Borel set $B\in\mathscr{B}$, there is a cylindrical set $C\in\mathscr{C}$ such that*

$$\mu(B\triangle C)=0,$$

where $B\triangle C=(B\setminus C)\cup(C\setminus B)$ is the symmetric union of B and C.

Proof. See Proposition A.3.12 in [33]. □

A consequence of the previous theorem is the following result.

Theorem 1.4.5. *Suppose that μ and μ' are two Radon measures on $(W,\mathscr{B})$ such that, for any $x_1^*,\dots,x_k^*\in W^*$, writing as $\mu_{(x_1^*,\dots,x_k^*)}$ (respectively, $\mu'_{(x_1^*,\dots,x_k^*)}$) the probability law on $\mathbb{R}^k$ induced by the random variables $(x_1^*,\dots,x_k^*)\colon W\to\mathbb{R}^k$ and by the measure μ (respectively, μ'), we have that*

$$\mu_{(x_1^*,\dots,x_k^*)}=\mu'_{(x_1^*,\dots,x_k^*)}.$$

Then $\mu=\mu'$.

Proof. See Corollary A.3.13 in [33]. □

We want to work with Borel σ-algebras in order to have that any continuous function is also measurable, on the other hand we want to work with Radon measures in order to have the property stated by the previous result.

Remark 1.4.6. Theorem 1.4.5 holds also if we have equality of finite laws taking $x_1^*,\dots,x_k^*\in\tilde{W}^*\subsetneq W^*$, where $\tilde{W}^*$ separates the points of W, i.e., for any $w_1,w_2\in W$, there is $x^*\in\tilde{W}^*$ such that $x^*(w_1)\neq x^*(w_2)$.

Let us define what a Gaussian measure on the Banach space W is.

Definition 1.4.7. *Consider a Banach space W and a Radon measure μ on it. We say that μ is* Gaussian *if, for any $x_1^*,\dots,x_k^*\in W^*$, we have that $\mu_{(x_1^*,\dots,x_k^*)}$ is a Gaussian measure on $\mathbb{R}^k$.*

Definition 1.4.8. *If μ is a Radon measure, we define its* Fourier transform *(or* characteristic function*) as the continuous map $\mathscr{F}(\mu)\equiv\hat{\mu}\colon W^*\to\mathbb{C}$ given by*

$$\mathscr{F}(\mu)(x^*)\equiv\hat{\mu}(x^*)=\int_W e^{\mathrm{i}x^*(w)}\,\mu(\mathrm{d}w).$$

As a consequence of Theorem 1.4.5, if two Radon measures μ and μ' share the same Fourier transform, then $\mu=\mu'$. We have the following characterization for Gaussian measures.

Theorem 1.4.9. *A probability Radon measure μ on W is Gaussian if and only if there exist a symmetric, bilinear and non-negative definite functional $\Sigma\colon W^*\times W^*\to\mathbb{R}$ and $m\colon W^*\to\mathbb{R}$ such that*

$$\mathscr{F}(\mu)(x^*)=\exp\left(\mathrm{i}m(x^*)-\frac{1}{2}\Sigma(x^*,x^*)\right). \tag{1.4.1}$$

Proof. See Theorem 2.2.4 in [33] or Lemma 2.1 and Theorem 2.3 in [128]. □

Remark 1.4.10. If we choose Σ and m as above, it is in general *not* true that there exists a Radon Gaussian measure μ such that (1.4.1) holds.

It is possible to choose m and Σ as follows:

$$\begin{aligned} m(x^*) &= \int_W x^*(w)\,\mu(\mathrm{d}w), \qquad x^*\in W^*, \\ \Sigma(x^*,y^*) &= \int_W (x^*(w)-m(x^*))(y^*(w)-m(y^*))\,\mu(\mathrm{d}w), \qquad x^*,y^*\in W^*. \end{aligned}$$

Example 1.4.11. (Classical Wiener space) Let $W=C^0([0,T])$, with dual $W^*=\mathscr{P}([0,T])$ being the space of bounded, signed, Borel measures on $[0,T]$. Define a Gaussian measure μ on W having mean zero and covariance operator given by $\Sigma(\eta,\gamma)=\int_{[0,T]^2}\min(s,t)\,\eta(\mathrm{d}s)\,\gamma(\mathrm{d}t)$. Then μ is the measure induced by Brownian motion in the space of continuous functions on $[0,T]$. For instance, in the case $\eta=\delta_{s_0}$ and $\gamma=\delta_{t_0}$ for some $s_0,t_0\in[0,T]$, then $\Sigma(\delta_{s_0},\delta_{t_0})=\min(s_0,t_0)=\mathbb{E}[B_{s_0}B_{t_0}]$ where B is a Brownian motion on $[0,T]$. See Section I.3 in [128] for more details.

From now on, we restrict to the case $W=H$, where H is a separable Hilbert space equipped with an inner product $(\cdot,\cdot)_H$. As a first consequence of this choice, we have the identification $H^*\cong H$ by Riesz theorem, that is we can identify any element of the dual with an element of the Hilbert space itself, so for any $x^*\in H^*$ there is $x\in H$ such that $x^*(\cdot)=(x,\cdot)_H$.

Furthermore, if $\Sigma\colon H^*\times H^*\to\mathbb{R}$ is symmetric, bilinear, positive definite and continuous, then we can identify it with $\Sigma\colon H\times H\to\mathbb{R}$, and therefore there exists a unique linear and bounded operator $S\colon H\to H$ such that, for any $x,y\in H$,

$$(Sx,y)_H=\Sigma(x,y).$$

Moreover, S is also positive (namely $(Sx,x)\geqslant 0$, for any $x\in H$) and self-adjoint (that is $(Sx,y)_H=(x,Sy)_H$, for any $x,y\in H$). We call S the *covariance* operator of μ.

As far as the operator m is concerned, it is a map from $H^*\to\mathbb{R}$. If m is continuous, then $m\in(H^*)^*=H$, with no need for Riesz theorem, and therefore we have $(m,x)_H=m(x^*)$. Such an $m\in H$ is called the *mean* of the Gaussian measure μ.

Definition 1.4.12. *Let $O\colon H\to R$ be a positive operator. Then the* trace *of O, is given by*

$$\mathrm{tr}(O)=\sum_{n\in\mathbb{N}}(Oe_n,e_n)_H\in[0,+\infty],$$

where $(e_n)_{n\in\mathbb{N}}$ is any basis of H. We say that the operator O is trace-class *if* $\mathrm{tr}(O)<+\infty$. *We also write* tr_H *to specify that the trace is taken with the inner product with which H is equipped.*

It is possible to prove that the trace does not depend the basis.

Theorem 1.4.13. *Let μ be a Gaussian measure on a separable Hilbert space H. Then,*

$$\mathscr{F}(\mu)(x)=\exp\left(\mathrm{i}\,(m,x)_H-\frac{1}{2}(Sx,x)_H\right), \qquad x\in H,$$

where $m\in H$ and S is a bounded, positive, self-adjoint, trace-class operator.

Conversely, if $m\in H$ and S is a positive, self-adjoint, trace-class operator, then there exists a unique Gaussian measure $\mu_{m,S}$ such that

$$\mathscr{F}(\mu_{m,S})(x)=\exp\left(\mathrm{i}\,(m,x)_H-\frac{1}{2}(Sx,x)_H\right), \qquad x\in H.$$

Proof. See Theorem 2.3.1 in [33] or Theorem 5.1 in [126]. □

1.4.2 Some examples

Let us consider the case $H = L^2(M, \mathrm{d}\nu)$, where M is a topological measure space and $\mathrm{d}\nu$ is a positive Borel measure (e.g. $M = \mathbb{R}^n, \mathbb{T}^n, \mathbb{R}\times\mathbb{T}^n, \ldots$, and $\mathrm{d}\nu = \mathrm{d}x, \rho_\ell^2(x)\,\mathrm{d}x, \ldots$). It is often the case that the covariance operator S is given by an integral operator, that is, for any $f \in L^2(M, \mathrm{d}\nu)$,

$$Sf(a) = \int_M K(a,b)\, f(b)\, \mathrm{d}\nu(b), \tag{1.4.2}$$

where $K: M\times M \to \mathbb{R}$ is a measurable function. If S is of the form (1.4.2) and it is also positive and such that K is continuous, then its trace is given by

$$\mathrm{tr}(S) = \int_M K(b,b)\, \mathrm{d}\nu(b). \tag{1.4.3}$$

(See page 65 in Section XI.4 in [171].)

Example 1.4.14. (Brownian motion on an interval) Let $T > 0$ and let $B = (B_t)_{t\in[0,T]}$ be a Brownian motion. We want to look at such a stochastic process as a Gaussian measure on $L^2([0,T])$. Consider $f \in L^2([0,T])$, then we have

$$(B,f)_{L^2} = \int_0^T B_s f(s)\, \mathrm{d}s,$$

and, for $f, g \in L^2([0,T])$,

$$\begin{aligned} \mathbb{E}[(B,f)_{L^2}(B,g)_{L^2}] &= \int_{[0,T]^2} f(t)\, g(s)\, \mathbb{E}[B_s B_t]\, \mathrm{d}t\, \mathrm{d}s \\ &= \int_0^T \left(\int_0^T \min(s,t)\, f(t)\, \mathrm{d}t \right) g(s)\, \mathrm{d}s \\ &= (A_K(f), g)_{L^2}, \end{aligned}$$

where

$$A_K(f)(s) = \int_0^T K(s,t)\, f(t)\, \mathrm{d}t, \qquad K(s,t) = \min(s,t).$$

On the other hand, Itô's formula gives

$$\int_0^T \left(\int_s^T f(\tau)\, \mathrm{d}\tau \right) \mathrm{d}B_s - \int_0^T f(s) B_s\, \mathrm{d}s = B_T \int_T^T f(\tau)\, \mathrm{d}\tau - B_0 \int_0^T f(\tau)\, \mathrm{d}\tau = 0,$$

and thus

$$(B,f)_{L^2} = \int_0^T B_s f(s)\, \mathrm{d}s = \int_0^T \left(\int_s^T f(\tau)\, \mathrm{d}\tau \right) \mathrm{d}B_s.$$

Therefore, exploiting Itô isometry, we have

$$\begin{aligned} \mathbb{E}[(B,f)_{L^2}(B,g)_{L^2}] &= \mathbb{E}\left[\left(\int_0^T \left(\int_s^T f(\tau)\, \mathrm{d}\tau \right) \mathrm{d}B_s \right) \left(\int_0^T \left(\int_s^T g(\tau)\, \mathrm{d}\tau \right) \mathrm{d}B_s \right) \right] \\ &= \int_0^T \left(\int_s^T f(\tau)\, \mathrm{d}\tau \right) \left(\int_s^T g(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s. \end{aligned}$$

It is then possible to show that

$$A_K(f)(t) = \int_0^t \left(\int_s^T f(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s.$$

Therefore, A_K is positive and self-adjoint. Moreover, it is trace class, since by equation (1.4.3) we have

$$\mathrm{tr}(A_K) = \int_0^T K(t,t)\, \mathrm{d}t = \int_0^T t\, \mathrm{d}t = T^2/2 < +\infty.$$

This gives the existence of a unique Gaussian measure μ on $H = L^2([0,T])$ with mean $m = 0$ and covariance operator $S = A_K$.

Let us consider now the Hilbert space $H = B^s_{2,2,\ell}(\mathbb{R}^n) = H^s_\ell(\mathbb{R}^n)$, $s, \ell \in \mathbb{R}$. Recall that we have the inclusions $\mathcal{S}(\mathbb{R}^n) \subset B^s_{2,2,\ell}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, with $\mathcal{S}(\mathbb{R}^n)$ dense in $B^s_{2,2,\ell}(\mathbb{R}^n)$. If $x, y \in B^s_{2,2,\ell}(\mathbb{R}^n)$ and $S: B^s_{2,2,\ell}(\mathbb{R}^n) \to B^s_{2,2,\ell}(\mathbb{R}^n)$, then

$$(Sx, y)_{B^s_{2,2,\ell}} = (\rho_\ell(-\Delta+1)^{s/2}(Sx), \rho_\ell(-\Delta+1)^{s/2}(y))_{L^2}.$$

Recall that, if μ is a Gaussian measure on $H = B^s_{2,2,\ell}(\mathbb{R}^n)$ with zero mean, then the covariance operator has the form

$$(Sx, y)_H = \int_H (x,h)_H\,(y,h)_H\,\mathrm{d}\mu(h).$$

If $x, y \in \mathcal{S}(\mathbb{R}^n)$, we can then define the operator

$$\langle S'x, y\rangle_{\mathcal{S}',\mathcal{S}} := \int_{H \subset \mathcal{S}'} \langle h, x\rangle_{\mathcal{S}',\mathcal{S}}\,\langle h, y\rangle_{\mathcal{S}',\mathcal{S}}\,\mathrm{d}\mu(h).$$

It is possible to show that $S'x \in \mathcal{S}'(\mathbb{R}^n)$, and hence $S': \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.4.15. *Let $H = B^s_{2,2,\ell}(\mathbb{R}^n)$, for some $s \in \mathbb{R}$ and $\ell \in \mathbb{R}$, then we have*

$$\mathrm{tr}_H(S) = \mathrm{tr}_{L^2(\mathbb{R}^d)}(\rho_\ell(-\Delta+1)^{s/2} S'\,(-\Delta+1)^{s/2}\rho_\ell).$$

The previous result also holds true with minor modifications if we substitute the space $H = B^s_{2,2,\ell}(\mathbb{R}^n)$ with $B^s_{2,2}(\mathbb{T}^n)$, $B^m_{2,2,\ell}([0,T], B^{m'}_{2,2,\ell}(\mathbb{R}^n))$, etc. In order to prove Theorem 1.4.15 we need the following result.

Lemma 1.4.16. *Under the previous hypotheses we have that*

$$S(x) = S'\{(-\Delta+1)^{s/2}[\rho_{2\ell}(-\Delta+1)^{s/2}(x)]\}, \qquad x \in H.$$

Proof. Consider $x \in \mathcal{S}(\mathbb{R}^n)$ and $h \in B^s_{2,2,\ell}(\mathbb{R}^n)$ then

$$(h, x)_H = \langle \rho_\ell(-\Delta+1)^{s/2}(h), \rho_{\ell/2}(-\Delta+1)^{s/2}(x)\rangle_{\mathcal{S}',\mathcal{S}}.$$

Thus, taking $\tilde{x} \in \mathcal{S}(\mathbb{R}^n)$ we get

$$\begin{aligned}
\langle h, \tilde{x}\rangle_{\mathcal{S}',\mathcal{S}} &= \langle (-\Delta+1)^{s/2}(h), (-\Delta+1)^{-s/2}(\tilde{x})\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \langle \rho_{2\ell}(-\Delta+1)^{s/2}(h), \rho_{-2\ell}(-\Delta+1)^{-s/2}(\tilde{x})\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \langle \rho_{2\ell}(-\Delta+1)^{s/2}(h), (-\Delta+1)^{s/2}\{(-\Delta+1)^{-s/2}[\rho_{-2\ell}(-\Delta+1)^{-s/2}(\tilde{x})]\}\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \langle \rho_\ell(-\Delta+1)^{s/2}(h), \rho_\ell(-\Delta+1)^{s/2}\{(-\Delta+1)^{-s/2}[\rho_{-2\ell}(-\Delta+1)^{-s/2}(\tilde{x})]\}\rangle_{\mathcal{S}',\mathcal{S}} \\
&= (h, (-\Delta+1)^{-s/2}[\rho_{-2\ell}(-\Delta+1)^{-s/2}(\tilde{x})])_H.
\end{aligned}$$

We then have, for any $\tilde{y}, \tilde{x} \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
&\langle S'\tilde{x}, \tilde{y}\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \int_{H \subset \mathcal{S}'} \langle h, \tilde{x}\rangle_{\mathcal{S}',\mathcal{S}}\,\langle h, \tilde{y}\rangle_{\mathcal{S}',\mathcal{S}}\,\mathrm{d}\mu(h) = \\
&= \int_H (h, (-\Delta+1)^{-s'}[\rho_{-2\ell}(-\Delta+1)^{-s'}(\tilde{x})])_H (h, (-\Delta+1)^{-s'}[\rho_{-2\ell}(-\Delta+1)^{-s'}(\tilde{y})])_H\,\mathrm{d}\mu(h) \\
&= \big(S\{(-\Delta+1)^{-s'}[\rho_{-2\ell}(-\Delta+1)^{-s'}(\tilde{x})]\}, (-\Delta+1)^{-s'}[\rho_{-2\ell}(-\Delta+1)^{-s'}(\tilde{y})]\big)_H \\
&= \langle \rho_\ell(-\Delta+1)^{s'}\{S\{(-\Delta+1)^{-s'}[\rho_{-2\ell}(-\Delta+1)^{-s'}(\tilde{x})]\}\}, \rho_{-\ell}(-\Delta+1)^{-s'}(\tilde{y})\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \langle S\{(-\Delta+1)^{-s'}[\rho_{-2\ell}(-\Delta+1)^{-s'}(\tilde{x})]\}, \tilde{y}\rangle_{\mathcal{S}',\mathcal{S}},
\end{aligned}$$

where $s' = s/2$, from which we get, by density of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{2,2,\ell}(\mathbb{R}^n)$,

$$S'(\tilde{x}) = S\{(-\Delta+1)^{-s/2}[\rho_{-2\ell}(-\Delta+1)^{-s/2}(\tilde{x})]\}.$$

choosing $\tilde{x} = (-\Delta+1)^{s/2}[\rho_{2\ell}(-\Delta+1)^{s/2}(\tilde{x})]$ yields the result. □

Proof of Theorem 1.4.15. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ be an orthonormal basis of H (such a basis exists since $\mathcal{S}(\mathbb{R}^n)$ is dense in H), then, by definition of the norm of H,

$$(\tilde{f}_n)_{n\in\mathbb{N}} := (\rho_\ell(-\Delta+1)^{s/2}(f_n))_{n\in\mathbb{N}}$$

is an orthonormal basis of L^2, since

$$(\rho_\ell(-\Delta+1)^s(f_n), \rho_\ell(-\Delta+1)^s(f_m))_{L^2} = (f_n, f_m)_H = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases}$$

and since $(-\Delta+1)$ and the multiplication by ρ_ℓ are bijective maps in $\mathcal{S}(\mathbb{R}^n)$ (which is dense in $L^2(\mathbb{R}^n)$). By Lemma 1.4.16, we have

$$\begin{aligned}
\mathrm{tr}_H(S) &= \sum_{n\in\mathbb{N}} (S(f_n), f_n)_H \\
&= \sum_{n\in\mathbb{N}} (\rho_\ell(-\Delta+1)^{s/2} S f_n, \rho_\ell(-\Delta+1)^{s/2}(f_n))_{L^2(\mathbb{R}^d)} \\
&= \sum_{n\in\mathbb{N}} (\rho_\ell(-\Delta+1)^{s/2} S'\{(-\Delta+1)^{s/2}[\rho_{2\ell}\cdot(-\Delta+1)^{s/2}(f_n)]\}, \rho_\ell(-\Delta+1)^{s/2}(f_n))_{L^2(\mathbb{R}^d)} \\
&= \sum_{n\in\mathbb{N}} (\rho_\ell(-\Delta+1)^{s/2} S'\{(-\Delta+1)^{s/2}(\rho_\ell \tilde{f}_n)\}, \tilde{f}_n)_{L^2(\mathbb{R}^d)} \\
&= \mathrm{tr}_{L^2}(\rho_\ell(-\Delta+1)^{s/2} S' (-\Delta+1)^{s/2}\rho_\ell),
\end{aligned}$$

which gives the result. □

Example 1.4.17. (Brownian motion on $\mathbb{R}_+$) Recalling the scenario presented in Example 1.4.14, let B be a Brownian motion on $\mathbb{R}_+$ and let $\ell > 1$. Then B induces a Gaussian measure on $L^2_\ell(\mathbb{R}_+)$. Let

$$S'f(t) = \int_{\mathbb{R}_+} K(t,s) f(s)\, \mathrm{d}s, \qquad f \in \mathcal{S}(\mathbb{R}_+), t \in \mathbb{R}_+,$$

where $K(t,s) = \min(s,t)$, for $s, t \in \mathbb{R}_+$. Then we have

$$\mathbb{E}[\langle B, f\rangle_{\mathcal{S}',\mathcal{S}} \langle B, g\rangle_{\mathcal{S}',\mathcal{S}}] = \langle S'f, g\rangle_{\mathcal{S}',\mathcal{S}}.$$

By Theorem 1.4.15 and Lemma 1.4.16, letting

$$Sf = S'(\rho_{2\ell} f), \qquad f \in L^2_\ell(\mathbb{R}_+),$$

we get that

$$\mathrm{tr}_{L^2_\ell}(S) = \mathrm{tr}_{L^2}(\rho_\ell S' \rho_\ell) = \int_{\mathbb{R}_+} \rho_{2\ell}(t) K(t,t)\, \mathrm{d}t = \int_{\mathbb{R}_+} \frac{t}{(t+1)^{\ell/2}} \mathrm{d}t,$$

which is finite if and only if $\ell > 1$.

Example 1.4.18. (White noise) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *white noise* is a Gaussian random field $\xi: \Omega \to \mathcal{S}'(\mathbb{R}^n)$ such that

$$\mathbb{E}[\langle \xi, f\rangle_{\mathcal{S}',\mathcal{S}} \langle \xi, g\rangle_{\mathcal{S}',\mathcal{S}}] = (f, g)_{L^2}.$$

Let $S' = \delta_\cdot$, i.e. the Dirac delta, in such a way that

$$S'f(x) = \int_{\mathbb{R}^d} \delta_x(\mathrm{d}y)\, f(y) = f(x), \qquad x \in \mathbb{R}^n,$$

which gives

$$\langle S'f, g\rangle_{\mathcal{S}',\mathcal{S}} = (f,g)_{L^2}.$$

We can then realize ξ as a Gaussian measure on $H_\ell^{-\alpha}(\mathbb{R}^n) = B_{2,2,\ell}^{-\alpha}(\mathbb{R}^n)$, where $\ell > d/2$ and $\alpha > n/2$. We have in fact, by Theorem 1.4.15,

$$\begin{aligned}\mathrm{tr}_{H_\ell^{-\alpha}}(S) &= \mathrm{tr}_{L^2}(\rho_\ell(-\Delta+1)^{-\alpha/2} S'(-\Delta+1)^{-\alpha/2}\rho_\ell)\\ &= \int_{\mathbb{R}^n} \rho_\ell^2(x)\, K(x,x)\, \mathrm{d}x,\end{aligned}$$

where $K(x,y)$ is the integral kernel of the operator

$$Sf(x) = (-\Delta+1)^{-\alpha/2}(S'(-\Delta+1)^{-\alpha/2}(f))(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, \mathrm{d}y.$$

Now,

$$\begin{aligned}(-\Delta+1)^{-\alpha/2} S'(-\Delta+1)^{-\alpha/2}(f) &= \mathcal{F}^{-1}((|k|^2+1)^{-\alpha}\mathcal{F}(f))\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\mathrm{i}(x\cdot k)}\frac{1}{(|k|^2+1)^\alpha} e^{\mathrm{i}(x'\cdot k)} f(x')\, \mathrm{d}x'\, \mathrm{d}k\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{e^{\mathrm{i}((x'-x)\cdot k)}}{(|k|^2+1)^\alpha}\mathrm{d}k\right) f(x')\, \mathrm{d}x'.\end{aligned}$$

Therefore

$$K(x,x') = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{\mathrm{i}((x'-x)\cdot k)}}{(|k|^2+1)^\alpha}\mathrm{d}k,$$

and

$$K(x,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{1}{(|k|^2+1)^\alpha}\mathrm{d}k,$$

which is a positive constant that is finite if and only if $\alpha > n/2$. Hence, we have

$$\mathrm{tr}_{H_\ell^{-\alpha}}(S) = \left(\int_{\mathbb{R}^n}\rho_\ell^2(x)\, \mathrm{d}x\right) K(x,x) < +\infty$$

if and only if $\ell > n/2$ and $\alpha > n/2$.

Example 1.4.19. (Gaussian free field) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Fix $m > 0$ and let ψ_m: $\Omega \to \mathcal{S}'(\mathbb{R}^d)$ be a Gaussian random field with

$$\mathbb{E}[\langle \psi_m, f\rangle_{\mathcal{S}',\mathcal{S}}\langle \psi_m, g\rangle_{\mathcal{S}',\mathcal{S}}] = \int_{\mathbb{R}^d}\frac{\mathcal{F}(f)(k)\,\overline{\mathcal{F}(g)}(k)}{|k|^2+m^2}\mathrm{d}k.$$

We can then define

$$S'f = (-\Delta+m^2)^{-1} f.$$

The random field ψ_m can be seen as a Gaussian probability law on $B_{2,2,\ell}^s(\mathbb{R}^d)$ when $\ell > d/2$ and $s < (2-d)/2$. We then get

$$\begin{aligned}(-\Delta+1)^s S'(-\Delta+1)^s(f)(x) &= \mathcal{F}^{-1}((|k|^2+1)^s(|k|^2+m^2)\mathcal{F}(f))\\ &= \int_{\mathbb{R}^d} K(x,x')\, f(x')\, \mathrm{d}x',\end{aligned}$$

with

$$K(x,x')=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{(|k|^2+1)^s}{|k|^2+m^2}e^{\mathrm{i}(x'-x)\cdot k}\mathrm{d}k.$$

We have that

$$K(x,x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{(|k|^2+1)^s}{|k|^2+m^2}\mathrm{d}k$$

is a finite constant if and only if $s<(2-d)/2$. We have then

$$\begin{aligned}\mathrm{tr}_{H^s_\ell}(S) &= \mathrm{tr}_{L^2_\ell}((-\Delta+1)^{s/2}S'(-\Delta+1)^{s/2})\\ &= \left(\int_{\mathbb{R}^d}\rho^2_\ell(x)\,\mathrm{d}x\right)K(x,x)<+\infty\end{aligned}$$

if and only if $\ell>d/2$. It is worth to notice that in this case the parameter s can be positive if $d<2$. Hence in dimension $d=1$, ψ_m is a random field, since $\psi_m(x)$ is well-defined. While for $d\geqslant 2$, ψ_m is only a random distribution.

1.4.3 The Cameron–Martin space

Hereafter, we denote by H a separable Hilbert space equipped with inner product $(\cdot,\cdot)_H$. Moreover, from now on we focus only on Gaussian measures with zero mean and covariance operator S with trivial kernel, $\ker(S)=\{0\}$. In fact, this is always possible since, if μ is a Gaussian measure on H with mean m and covariance S, then the measure μ_0, given by

$$\mu_0(B)=\mu(B+m),\qquad B\in\mathscr{B}(H),$$

is a Gaussian measure on H with mean zero and with the same covariance S as μ. Moreover,

$$\mathrm{supp}(\mu_0)\subset(\ker(S))^\perp,$$

where $\mathrm{supp}(\mu)$ is the support of μ, $\ker(S)\coloneqq\{x\in H\colon Sx=0\}$ denotes the kernel of S, and $A^\perp\coloneqq\{h\in H\colon (x,h)_H=0,\ \text{for any } x\in A\}$, for $A\subset H$.

Let us recall that the range of an operator S is given by $\mathrm{ran}(S)=\{Sx\colon x\in H\}$. If S is a self-adjoint operator, then $\overline{\mathrm{ran}(S)}=(\ker(S))^\perp$. Indeed, let $x\in\mathrm{ran}(S)$ with $y\in H$ such that $x=Sy$ and consider $z\in\ker(S)$, then $(x,z)_H=(Sy,z)_H=(y,Sz)_H=0$. Moreover,

$$\ker(S|_{\ker(S)^\perp})=\ker(S)\cap\ker(S)^\perp=\{0\}.$$

Since S is self-adjoint, trace-class and positive, then there exist a basis $(e_n)_{n\in\mathbb{N}}$ of H and some non-zero elements $(\lambda_n)_{n\in\mathbb{N}}\subset\mathbb{R}_+$ converging to zero as $n\to\infty$, such that

$$S(h)=\sum_{n\in\mathbb{N}}\lambda_n(e_n,h)_He_n.$$

So, for any $\alpha\in\mathbb{R}_+$, we can define

$$S^\alpha(h)=\sum_{n\in\mathbb{N}}\lambda_n^\alpha(e_n,h)_He_n,$$

which is bounded, self-adjoint and positive. Moreover, if $\alpha>1$, it is trace-class, and if $\alpha=1/2$ then $S^{1/2}$ is Hilbert-Schmidt (i.e. the sum $\sum_{n\in\mathbb{N}}\|S^{1/2}e_n\|_H^2$ is finite), since $\mathrm{tr}(S^{1/2}\cdot(S^{1/2})^*)=\mathrm{tr}(S)<\infty$.

For $\alpha\in\mathbb{R}_+$, we have

$$\overline{\mathrm{ran}(S^\alpha)}=(\ker(S^\alpha))^\perp=(\{0\})^\perp=H,$$

since since $\ker(S^\alpha)=\ker(S)=\{0\}$.

Definition 1.4.20. *Let μ be a Gaussian measure with mean $m=0$ and covariance S with trivial kernel* $\ker(S)=\{0\}$. *We call the subspace*

$$H_{\mathrm{CM}}=\operatorname{ran}(S^{1/2})$$

the Cameron–Martin space *of the measure μ and we define on H_{CM} the scalar product*

$$(x,y)_{H_{\mathrm{CM}}}=(S^{-1/2}x,S^{-1/2}y)_H.$$

It is possible to show that the space $(H_{\mathrm{CM}},(\cdot,\cdot)_{H_{\mathrm{CM}}})$ is an Hilbert space.

Let us point out that $S^{1/2}$ is a square root of the covariance operator S in the following sense: A linear and bounded operator S_1 on H is said to be a *square root* of S if

$$S=S_1^*S_1.$$

It is possible to show that, for any square root S_1 of S, there is an unitary operator $U\colon H\to\overline{\operatorname{ran}(S_1)}$ (where $\overline{\operatorname{ran}(S_1)}\subset H$ is equipped with the induced scalar product) such that

$$S_1=US^{1/2}.$$

For a square root S_1 we have $\ker(S_1)\subseteq\ker(S)=\{0\}$, and hence $\ker(S_1)=\{0\}$. On the other hand,

$$\ker(S_1^*)=(\operatorname{ran}(S_1))^{\perp}$$

and

$$\ker(S_1^*)\cap\overline{\operatorname{ran}(S_1)}=\{0\}.$$

This means that there exists a unique operator (in general, unbounded)

$$S_1^{*,-1}\colon\operatorname{ran}(S_1^*)\to\overline{\operatorname{ran}(S_1)},$$

which is the inverse of the operator S_1^*. If $x,y\in\overline{\operatorname{ran}(S_1)}$ are such that $S_1^*x=S_1^*y$, then $S_1^*(x-y)=0$, which gives $(x-y)\in\ker(S_1^*)$ and then $(x-y)\in\ker(S_1)\cap\overline{\operatorname{ran}(S_1)}=\{0\}$. Thus, $S_1^*(S_1^{*,-1}(x))=x$.

Theorem 1.4.21. *Let S be a positive, self-adjoint, trace-class and with* $\ker(S)=0$. *Then we have that, for any square root S_1 of S,*

$$\operatorname{ran}(S^{1/2})=\operatorname{ran}(S_1^*).$$

Furthermore, for any $x\in\operatorname{ran}(S_1^*)$, we have

$$(S_1^{*,-1}x,S_1^{*,-1}x)_H=(S^{-1/2}x,S^{-1/2}x)_H.$$

Example 1.4.22. (Brownian motion, continued) Recall that a Brownian motion B on $[0,T]$ induces a Gaussian measure on $L^2([0,T])$ with covariance operator

$$Sf(t)=\int_0^T\min(s,t)\,f(s)\,\mathrm{d}s=\int_0^t\left(\int_s^T f(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

with $S\colon L^2([0,T])\to L^2([0,T])$ (see Example 1.4.14). A square root of S is given by

$$S_1f(t)=\int_t^T f(\tau)\,\mathrm{d}\tau,$$

with

$$S_1^* f(t) = \int_0^t f(\tau)\,\mathrm{d}\tau.$$

Moreover, we have

$$\begin{aligned} H_{\mathrm{CM}} &= \left\{ g \colon \exists f \in L^2([0,T]), g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau \right\} \\ &= H^1([0,T]) \cap \{g \in C^0([0,T]); g(0) = 0\} \\ &= H_0^1([0,T]) \subset C^0([0,T]). \end{aligned}$$

Furthermore,

$$(S_1^{*,-1} g, S_1^{*,-1} g)_H = \int_0^T (S_1^{*,-1} g)^2(t)\,\mathrm{d}t = \int_0^T (g'(t))^2\,\mathrm{d}t,$$

and hence the Cameron–Martin space of μ is given by

$$(H_{\mathrm{CM}}, (\cdot,\cdot)_{H_{\mathrm{CM}}}) = \left(H_0^1([0,T]), \int_0^T (g'(t))^2\,\mathrm{d}t \right).$$

The following result introduces the notion of Wiener–Itô integral $\delta(h)$, for $h \in H_{\mathrm{CM}}$, also denoted as $I(h)$, $W(h)$, etc., in the literature.

Theorem 1.4.23. *There exists a unique isometry, called* Wiener–Itô integral*,*

$$\delta \colon (H_{\mathrm{CM}}, (\cdot,\cdot)_{H_{\mathrm{CM}}}) \to L^2(H, \mu),$$

such that, for any $x \in \operatorname{ran}(S) \subset \operatorname{ran}(S^{1/2})$*, we have*

$$\delta(x) = (S^{-1} x, \cdot)_H.$$

For $x, y \in H_{\mathrm{CM}}$, we have

$$\mathbb{E}[\delta(x)\delta(y)] = (x,y)_{H_{\mathrm{CM}}} = (S^{-1/2} x, S^{-1/2} y)_H.$$

Example 1.4.24. (Brownian motion, continued) With the same notation as in Example 1.4.14 and 1.4.22, we have

$$\operatorname{ran}(S) = H_0^2([0,T]) = \{g \in H^2([0,T]); g(0) = 0, g'(T) = 0\},$$

and $S^{-1} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}$, so that, for $g \in H^2([0,T])$,

$$\delta(g) = -\int_0^T g''(t) B_t\,\mathrm{d}t = \int_0^T g'(t)\,\mathrm{d}B_t$$

where we used Itô formula. Notice that the last integral is defined also when $g \in H^1$. Moreover,

$$\mathbb{E}[\delta(x)\delta(y)] = \mathbb{E}\left[\int_0^T g'(t)\,\mathrm{d}B_t \int_0^T f'(t)\,\mathrm{d}B_t \right] = \int_0^T g'(t) f'(t)\,\mathrm{d}t,$$

by Itô isometry.

In order to prove Theorem 1.4.23, we need first the following result.

Lemma 1.4.25. *We have that* $\operatorname{ran}(S) \subset H_{\mathrm{CM}}$ *and* $\overline{\operatorname{ran}(S)}^{H_{\mathrm{CM}}} = H_{\mathrm{CM}}$.

Proof. Since $(S^{1/2})^2 = S$, then $\mathrm{ran}(S) = S^{1/2}(S^{1/2}(H)) \subset S^{1/2}(H) = \mathrm{ran}(S^{1/2}) = H_{\mathrm{CM}}$. Moreover, we have $S^{1/2}\colon H_{\mathrm{CM}} \to \mathrm{ran}(S) \subset H_{\mathrm{CM}}$. Introduce then the operator

$$T = S^{1/2}|_{H_{\mathrm{CM}}},$$

which is bounded with respect to the metric $(\cdot,\cdot)_{H_{\mathrm{CM}}}$ since, for $x \in H_{\mathrm{CM}}$,

$$\|Tx\|^2_{H_{\mathrm{CM}}} = (S^{-1/2}S^{1/2}x, S^{-1/2}S^{1/2}x)_H = \|x\|^2_H \leqslant \|S\|\,\|x\|^2_{H_{\mathrm{CM}}}.$$

T is also self-adjoint with respect to $(\cdot,\cdot)_{H_{\mathrm{CM}}}$, since, for $x, y \in H_{\mathrm{CM}}$,

$$(Tx, y)_{H_{\mathrm{CM}}} = (S^{-1/2}S^{1/2}x, S^{-1/2}y)_H = (x, S^{-1/2}y)_H = (S^{1/2}S^{-1/2}x, S^{-1/2}y)_H,$$

where we used the fact that S is injective, and, since S is self-adjoint,

$$(Tx, y)_{H_{\mathrm{CM}}} = (S^{-1/2}x, S^{1/2}S^{-1/2}y)_H = (S^{-1/2}x, y)_H = (S^{-1/2}x, S^{-1/2}S^{1/2}y)_H = (x, Ty)_{H_{\mathrm{CM}}}.$$

Therefore, $\ker(T)^\perp = \ker(T) \subset \ker(S) = \{0\}$, and thus $\mathrm{ran}(S|_H) = \mathrm{ran}(T|_{H_{\mathrm{CM}}})$ is dense in H_{CM} with respect to $(\cdot,\cdot)_{H_{\mathrm{CM}}}$. □

Proof of Theorem 1.4.23. Since $\mathrm{ran}(S)$ is dense with respect to the norm $\|\cdot\|_{H_{\mathrm{CM}}}$, the only thing to prove is that $\delta(x)$ is an isometry on H, for $x \in H$. We have, for $x, y \in \mathrm{ran}(S^{-1})$,

$$\begin{aligned}
\int_H \delta(x)(h)\delta(y)(h)\mu(\mathrm{d}h) &= \int_H (S^{-1}x, h)_H (S^{-1}y, h)_H\, \mu(\mathrm{d}h) \\
&= (SS^{-1}x, S^{-1}y)_H \\
&= (x, S^{-1}y)_H \\
&= (S^{1/2}S^{-1/2}x, S^{-1}y)_H \\
&= (S^{-1/2}x, S^{1/2}S^{-1}y)_H \\
&= (S^{-1/2}x, S^{-1/2}y)_H \\
&= (x, y)_{H_{\mathrm{CM}}}.
\end{aligned}$$

This implies that δ can be extended to an isometry from $\overline{\mathrm{ran}(S)}^{H_{\mathrm{CM}}} = H_{\mathrm{CM}}$ into $L^2(H, \mu)$. □

So far, we constructed the Cameron–Martin space starting from a Gaussian measure μ on an Hilbert space H. One may wonder whether it is possible to go the other way around, i.e. construct a Gaussian measure whose Cameron–Martin space is given by an Hilbert space we start from. Suppose that we have a separable Hilbert space H and a subspace $\tilde{H}_{\mathrm{CM}} \subset H$ with the following properties:

1. $\overline{\tilde{H}_{\mathrm{CM}}}^H = H$.
2. There is a scalar product $(\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}}$ such that $(\tilde{H}_{\mathrm{CM}}, (\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}})$ is an Hilbert space.
3. $(\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}}$ is stronger than $(\cdot,\cdot)_H$, i.e. there exists a constant $C > 0$ such that, for every $x \in \tilde{H}_{\mathrm{CM}}$, we have $(x, x)_H \leqslant C(x, x)_{\tilde{H}_{\mathrm{CM}}}$.

Remark 1.4.26. Since $\tilde{H}_{\mathrm{CM}}$ is dense in H, we have that $\dim(H) = \dim(\tilde{H}_{\mathrm{CM}}) \leqslant \aleph_0$ (where $\aleph_0$ is the countable cardinal number). In particular, there exists $C_1\colon H \to \tilde{H}_{\mathrm{CM}}$ that is an isomorphism, namely, C_1 is injective, surjective, and an isometry, namely, for any $x, y \in H$, we have $(C_1x, C_1y)_{\tilde{H}_{\mathrm{CM}}} = (x, y)_H$.

Lemma 1.4.27. *Take H and $(\tilde{H}_{\mathrm{CM}}, (\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}})$ as above with the properties 1.–3. holding true. Then we have the following statements.*

i. *The isomorphism $C_1: H \to H_{\mathrm{CM}} \hookrightarrow H$ defined in Remark 1.4.26 is continuous in H.*

ii. *We can define $S = C_1 C_1^*: H \to H$. Then, S does independent on the choice of C_1.*

iii. *Suppose further that $(e_j^{\tilde{H}_{\mathrm{CM}}})_{j\in\mathbb{N}}$ is any basis of $(\tilde{H}_{\mathrm{CM}}, (\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}})$, then*

$$\mathrm{tr}(S) = \sum_{j\in\mathbb{N}} \left\| e_j^{\tilde{H}_{\mathrm{CM}}} \right\|_H^2,$$

(Notice the presence of the H-norm on the right-hand side).

Proof.

i. The first point follows from the fact that $(\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}}$ is stronger than $(\cdot,\cdot)_H$, since

$$\|C_1 x\|_H^2 = (C_1 x, C_1 x)_H \leqslant C\,(C_1 x, C_1 x)_{\tilde{H}_{\mathrm{CM}}} = C(x,x)_H.$$

ii. Consider two isomorphisms $C_1, C_2: H \to \tilde{H}_{\mathrm{CM}}$, then there is an isometry $U: H \to H$ such that $C_1 = C_2 U$. Recall that $C_2^{-1}: \tilde{H}_{\mathrm{CM}} \to H$ is an isomorphism, and thus $U = C_2^{-1} C_1: H \to H$ is well-defined. Moreover, for $x, y \in H$,

$$(Ux, Uy)_H = (C_2^{-1} C_1 x, C_2^{-1} C_1 y)_H = (C_1 x, C_1 y)_{\tilde{H}_{\mathrm{CM}}} = (x,y)_H,$$

and therefore $UU^* = \mathrm{id}_H$ and $S' = C_2 C_2^* = C_1 U U^* C_1^* = C_1 C_1^* = S = S^{1/2} S^{1/2}$, and in particular $S^{1/2}$ is one of these isomorphisms, $(S^{-1/2} x, S^{-1/2} y)_H = (x,y)_{\tilde{H}_{\mathrm{CM}}}$.

iii. Take $(e_j^{\tilde{H}_{\mathrm{CM}}})_{j\in\mathbb{N}}$ to be a basis of $(\tilde{H}_{\mathrm{CM}}, (\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}})$. Since $S^{-1/2}$ is an isomorphism between $\tilde{H}_{\mathrm{CM}}$ and H, then $(S^{-1/2} e_j^{\tilde{H}_{\mathrm{CM}}})_{j\in\mathbb{N}}$ is a basis of H. Moreover ,

$$\mathrm{tr}(S) = \sum_{j\in\mathbb{N}} (S S^{-1/2} e_j^{\tilde{H}_{\mathrm{CM}}}, S^{-1/2} e_j^{\tilde{H}_{\mathrm{CM}}})_H = \sum_{j\in\mathbb{N}} (e_j^{\tilde{H}_{\mathrm{CM}}}, S^{1/2} S^{-1/2} e_j^{\tilde{H}_{\mathrm{CM}}})_{\tilde{H}_{\mathrm{CM}}} = \sum_{j\in\mathbb{N}} \left\| e_j^{\tilde{H}_{\mathrm{CM}}} \right\|_H^2,$$

which concludes the proof. □

Theorem 1.4.28. *Take H and $(\tilde{H}_{\mathrm{CM}}, (\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}})$ as above with the properties 1.–3. holding true. Then there exists a unique Gaussian measure μ with mean zero such that $(H_{\mathrm{CM}}, (\cdot,\cdot)_{H_{\mathrm{CM}}}) = (\tilde{H}_{\mathrm{CM}}, (\cdot,\cdot)_{\tilde{H}_{\mathrm{CM}}})$ is the Cameron–Martin space of μ if and only if there exists (at least) one basis $(e_j^{\tilde{H}_{\mathrm{CM}}})_{j\in\mathbb{N}}$ of $\tilde{H}_{\mathrm{CM}}$ such that*

$$\sum_{j\in\mathbb{N}} \|e_j^{\tilde{H}_{\mathrm{CM}}}\|_H^2 < +\infty.$$

Proof. With the same notation as in Lemma 1.4.27, if such a measure μ does exist, then it must have covariance operator $S = C_1 C_1^*$. Such an operator S is a covariance of some Gaussian measure if and only if $\mathrm{tr}(S) < +\infty$, and by Lemma 1.4.27 the statement follows. □

Let us conclude state the following result.

Theorem 1.4.29. *Let H be an Hilbert space with Cameron-Martin space given by $(H_{\mathrm{CM}}, (\cdot,\cdot)_{H_{\mathrm{CM}}})$ with basis $(e_j^{H_{\mathrm{CM}}})_{j\in\mathbb{N}}$. Then the series $(J_n)_{n\in\mathbb{N}}$ given by*

$$J_n(h) = \sum_{j=1}^{n} \delta(e_j^{H_{\mathrm{CM}}})(h)\, e_j^{H_{\mathrm{CM}}} \in H, \qquad h\in H,$$

is convergent in $L^2(H,\mu)$. Moreover, $J(h) = \mathrm{id}_H(h)$, μ-a.s.

Proof. See Lemma B.1.1 in [180]. □

1.4.4 Wick product and Wick exponential

We define here the Wick product and show some of its properties. Let us start with the definition of Feynman diagrams.

Definition 1.4.30. *A* Feynman diagram γ *of* order $n = n(\gamma) \geqslant 0$ *and* rank $r = r(\gamma) \geqslant 0$ *is a graph consisting of a set of n vertices and a set of r edges without common endpoints.*

Notice that, with the same notation as in the previous definition, a Feynman diagram γ has $r(\gamma)$ disjoint pairs of vertices, each joined by an edge, and $n(\gamma) - 2r(\gamma)$ unpaired vertices. Moreover, a Feynman diagram γ is *complete* if $r(\gamma) = n(\gamma)/2$, i.e. if all vertices are paired off, and *incomplete* if $r(\gamma) < n(\gamma)/2$, i.e. if some vertices are unpaired.

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with expectation $\mathbb{E}$.

Definition 1.4.31. *A* Feynman diagram γ labelled by n random variables $F_1,\dots,F_n$ on $(\Omega,\mathscr{F},\mathbb{P})$ *is a Feynman diagram of order n with vertices $1,\dots,n$, where we think of F_i as attached to the vertex i, for every $i = 1,\dots,n$.*

We introduce now Wick polynomials and Wick products. Hereafter, we use a bold notation to denote vectors, e.g. $\boldsymbol{k} = (k_1,\dots,k_n)$, without specifying the number of elements whenever it is clear by the context. We also use such a notation for some operations, e.g. $\partial_{\boldsymbol{t}}^{\boldsymbol{k}} = \partial_{t_1}^{k_1}\cdots\partial_{t_n}^{k_n}$ or $\boldsymbol{t}^{\boldsymbol{k}} = t_1^{k_1}\cdots t_n^{k_n}$. Let us also denote by 1_i, for $i = 1,\dots,n$, the vector in $\mathbb{R}^n$ with all zero elements but having value 1 in the i-th position.

Definition 1.4.32. *Let $F_1,\dots,F_n$ be (real) random variables on $(\Omega,\mathscr{F},\mathbb{P})$. Then, if $\boldsymbol{k}\in\mathbb{N}_0^n$, we call* Wick polynomial *related to $\boldsymbol{F} = (F_1,\dots,F_n)$ any element $W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F})$ of the space $\mathbb{R}[\boldsymbol{f}]$ of polynomials with coefficient in $\mathbb{R}$ and with variables $\boldsymbol{f} = (f_1,\dots,f_n)$, defined in the following way:*

i. If $\boldsymbol{k} = 0$, then

$$W_0(\boldsymbol{f}|\boldsymbol{F}) = 1,$$

ii. If $\boldsymbol{k} \neq 0$ and $k_i \geqslant 1$ for some $i = 1,\dots,n$, then

$$\partial_{f_i} W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F}) = k_i W_{\boldsymbol{k}-1_i}(\boldsymbol{f}|\boldsymbol{F}),$$

and

$$\mathbb{E}[W_{\boldsymbol{k}}(\boldsymbol{F}|\boldsymbol{F})] = 0.$$

We call ($\boldsymbol{k}$-th) Wick product *of $F_1,\ldots,F_n$ the particular case of the random variable $W_{\boldsymbol{k}}(\boldsymbol{F}|\boldsymbol{F})$, which is also denoted as*

$$:F_1^{k_1}\cdots F_n^{k_n}: = W_{\boldsymbol{k}}(\boldsymbol{F}|\boldsymbol{F}).$$

If $n=1$, then $W_0(f|F)=1$ and $\frac{\partial}{\partial f}W_1(f|F)=1$. Therefore $W_1(f|F)=f+c$, where the real constant c can be determined by

$$0=\mathbb{E}[W_1(F|F)]=\mathbb{E}[F+c]=\mathbb{E}[F]+c,$$

yielding

$$W_1(f|F)=f-\mathbb{E}[F].$$

In the same way it goes for W_2, giving

$$W_2(f|F)=f^2-2\mathbb{E}[F]f+2\mathbb{E}[F]^2-\mathbb{E}[F^2].$$

Let $\boldsymbol{t}=(t_1,\ldots,t_n)\in\mathbb{R}_+^n$, we introduce the generating power series related to Wick polynomials:

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^N}\frac{\boldsymbol{t}^{\boldsymbol{k}}}{\boldsymbol{k}!}W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F}) := \sum_{\boldsymbol{k}\in\mathbb{N}_0^n}\frac{t_1^{k_1}\cdots t_n^{k_n}}{k_1!\cdots k_n!}W_{\boldsymbol{k}}((f_1,\ldots,f_n)|(F_1,\ldots,F_n)). \tag{1.4.4}$$

Theorem 1.4.33. *Suppose that $\mathbb{E}[\exp(t_1F_1+\cdots+t_nF_n)]$ is an entire function in $\boldsymbol{t}\in\mathbb{C}^n$ (i.e. it is holomorphic in all its variables on the whole $\mathbb{C}$). Then*

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^n}\frac{\boldsymbol{t}^{\boldsymbol{k}}}{\boldsymbol{k}!}W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F})=\frac{\exp(t_1f_1+\cdots+t_nf_n)}{\mathbb{E}[\exp(t_1F_1+\cdots+t_nF_n)]}=:\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|\boldsymbol{t}).$$

Proof. The statement is equivalent to the following: for any $\boldsymbol{k}\in\mathbb{N}_0^n$,

$$\partial_{\boldsymbol{t}}^{\boldsymbol{k}}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|0)=\partial_{t_1}^{k_1}\cdots\partial_{t_n}^{k_n}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|0)=W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F}). \tag{1.4.5}$$

We prove (1.4.5) by induction over $|\boldsymbol{k}|=k_1+\cdots+k_n$. For $k=0$, we have $W_0(\boldsymbol{f}|\boldsymbol{F})=0$, and

$$\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|0)=\frac{\exp(0)}{\mathbb{E}[\exp(0)]}=1.$$

Suppose that equation (1.4.5) is true for $|\boldsymbol{k}|<n$ and take $|\boldsymbol{k}'|=n$ with $\boldsymbol{k}'=\boldsymbol{k}+1_\ell$, for some $\ell\in\{1,\ldots,n\}$. By induction hypothesis, we have

$$\partial_{\boldsymbol{t}}^{\boldsymbol{k}}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|0)=W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F}).$$

Consider then

$$\tilde{W}_{\boldsymbol{k}'}(\boldsymbol{f}|\boldsymbol{F})=\partial_{\boldsymbol{t}}^{\boldsymbol{k}'}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|0)$$

and take the partial derivative with respect to f_ℓ, to get, if $k_\ell\geqslant 1$,

$$\begin{aligned}
\partial_{f_\ell}\tilde{W}_{\boldsymbol{k}'}(\boldsymbol{f}|\boldsymbol{F}) &= \partial_{f_\ell}\partial_{\boldsymbol{t}}^{\boldsymbol{k}'}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|0)\\
&= [\partial_{\boldsymbol{t}}^{\boldsymbol{k}'}\partial_{f_\ell}\mathrm{EXP}](\boldsymbol{f}|\boldsymbol{F}|0)\\
&= \partial_{\boldsymbol{t}}^{\boldsymbol{k}'}(t_\ell\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|\boldsymbol{t}))|_{\boldsymbol{t}=0}\\
&= \left[t_\ell\partial_{\boldsymbol{t}}^{\boldsymbol{k}'}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|\boldsymbol{t})+k_\ell\,\partial_{\boldsymbol{t}}^{\boldsymbol{k}'-1_\ell}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|\boldsymbol{t})\right]\Big|_{\boldsymbol{t}=0}\\
&= k_\ell\,\partial_{\boldsymbol{t}}^{\boldsymbol{k}'-1_\ell}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|0),
\end{aligned}$$

and, since $|\boldsymbol{k}'-1_\ell|\leqslant n-1$, we get $\partial_{f_\ell}\tilde{W}_{\boldsymbol{k}'}(\boldsymbol{f}|\boldsymbol{F})=k_\ell W_{\boldsymbol{k}'-1_\ell}(\boldsymbol{f}|\boldsymbol{F})$. Moreover, $\mathbb{E}[\tilde{W}_{\boldsymbol{k}}(\boldsymbol{F}|\boldsymbol{F})]=0$:

$$\begin{aligned}\mathbb{E}[\tilde{W}_{\boldsymbol{k}}(\boldsymbol{F}|\boldsymbol{F})] &= \partial_{\boldsymbol{t}}^{\boldsymbol{k}'}\mathbb{E}[\mathrm{EXP}(\boldsymbol{F}|\boldsymbol{F}|\boldsymbol{t})]|_{\boldsymbol{t}=0}\\ &= \partial_{\boldsymbol{t}}^{\boldsymbol{k}'}\mathbb{E}\left[\frac{\exp(\boldsymbol{F}\times\boldsymbol{t})}{\mathbb{E}[\boldsymbol{F}\times\boldsymbol{t}]}\right]\Bigg|_{\boldsymbol{t}=0}\\ &= [\partial_{\boldsymbol{t}}^{\boldsymbol{k}'}(1)]=0.\end{aligned}$$

Therefore $\tilde{W}_{\boldsymbol{k}'}(\boldsymbol{f}|\boldsymbol{F})=W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F})$. □

Corollary 1.4.34. *We have*

$$W_{\boldsymbol{k}}(\boldsymbol{f}|\boldsymbol{F})=[\partial_{\boldsymbol{t}}^{\boldsymbol{k}}\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|\boldsymbol{t})]|_{\boldsymbol{t}=0}.$$

Also in this case, $\mathrm{EXP}(\boldsymbol{F}|\boldsymbol{F}|\boldsymbol{t})$ is a particular case, we encode it in the following definition.

Definition 1.4.35. *The* Wick exponential *of F is defined as, for $\boldsymbol{t}\in\mathbb{R}^n$,*

$$:\!\exp(\boldsymbol{F}\times\boldsymbol{t})\!:=\mathrm{EXP}(\boldsymbol{F}|\boldsymbol{F}|\boldsymbol{t})=\sum_{\boldsymbol{k}\in\mathbb{N}_0^n}\frac{\boldsymbol{t}^{\boldsymbol{k}}}{\boldsymbol{k}!}W_{\boldsymbol{k}}(\boldsymbol{F}|\boldsymbol{F})=\sum\frac{\boldsymbol{t}^{\boldsymbol{k}}}{\boldsymbol{k}!}:\!F_1^{k_1}\cdots F_n^{k_n}\!:=\frac{\exp(\boldsymbol{F}\times\boldsymbol{t})}{\mathbb{E}[\exp(\boldsymbol{F}\times\boldsymbol{t})]}.$$

Remark 1.4.36. If we define $:\!F_1\cdots F_n\!:$ for generic random variables, then we have defined also the more generic Wick product with powers, $:\!G_1^{k_1}\cdots G_m^{k_m}\!:$, by simply taking $n=k_1+k_2+\cdots+k_m$ and $F_1=G_1$, ...,$F_{k_1}=G_1$, $F_{k_1+1}=G_2$, ..., $F_{k_1+k_2}=G_2$, ..., $F_{k_1+\cdots+k_{m-1}}=G_{M-1}$, $F_{k_1+\cdots+k_{m-1}+1}=G_m$, $\ldots$, and $F_{k_1+\cdots+k_m}=G_m$. Let us prove this. We have that

$$\mathrm{EXP}(\boldsymbol{f}|\boldsymbol{F}|\boldsymbol{t})=\frac{\exp(\boldsymbol{f}\times\boldsymbol{t})}{\mathbb{E}[\exp(\boldsymbol{F}\times\boldsymbol{t})]},\quad\text{ and }\quad\mathrm{EXP}(\boldsymbol{g}|\boldsymbol{G}|\tilde{\boldsymbol{t}})=\frac{\exp(\boldsymbol{g}\times\tilde{\boldsymbol{t}})}{\mathbb{E}[\exp(\boldsymbol{G}\times\tilde{\boldsymbol{t}})]}.$$

If we choose F_i and G_i satisfying the previous relations we have that

$$\begin{aligned}&\mathrm{EXP}((g_1,\cdots,g_1,g_2,\cdots,g_2,\cdots,g_m,\cdots,g_m)|\boldsymbol{F}|\boldsymbol{t})=\\ &\qquad=\mathrm{EXP}(\boldsymbol{g}|\boldsymbol{G}|(t_1+\cdots t_{k_1},t_{k_1+1}+\cdots t_{k_1+k_2},\cdots,t_{k_1+\cdots k_{m-1}+1}+\cdots+t_{k_1+\cdots+k_m}))\end{aligned}$$

where the variables g_i in the previous equality are repeated exactly k_i times, for each $i=1,\ldots,m$. This implies that

$$\begin{aligned}:\!F_1\cdots F_n\!: &= :\!G_1\cdots G_1G_2\cdots G_2\cdots G_m\cdots G_m\!:\\ &= W_1(\boldsymbol{F}|\boldsymbol{F})\\ &= \partial_{t_1}\cdots\partial_{t_n}\mathrm{EXP}(\boldsymbol{F}|\boldsymbol{F}|\boldsymbol{t})|_{\boldsymbol{t}=0}\\ &= \partial_{t_1}\cdots\partial_{t_n}\mathrm{EXP}(\boldsymbol{G}|\boldsymbol{G}|(t_1+\cdots+t_{k_1},\cdots,t_{k_1+\cdots k_{m-1}+1}+\cdots+t_{k_1+\cdots+k_m}))|_{\boldsymbol{t}=0}\\ &= \partial_{\tilde{t}_1}^{k_1}\cdots\partial_{\tilde{t}_m}^{k_m}\mathrm{EXP}(\boldsymbol{G}|\boldsymbol{G}|\tilde{\boldsymbol{t}})|_{\tilde{\boldsymbol{t}}=0}\\ &= W_k(\boldsymbol{G}|\boldsymbol{G})\\ &= :\!G_1^{k_1}\cdots G_m^{k_m}\!:,\end{aligned}$$

where we used the general fact that

$$\partial_{\ell_1}\cdots\partial_{\ell_p} f(\ell_1+\cdots+\ell_p)=(\partial_x^p f(x))|_{x=\ell_1+\cdots+\ell_p}$$

for any smooth function f.

Suppose that $\boldsymbol{F}$ is an $\mathbb{R}^n$-Gaussian random variable with zero mean. Then, we have the explicit expression

$$\mathbb{E}[\exp(\boldsymbol{F}\times\boldsymbol{t})]=\exp\left(\frac{1}{2}\sum_{i,j=1}^{n} t_i t_j\,\mathbb{E}[F_iF_j]\right).$$

Theorem 1.4.37. (Wick) *Let $F_1,\ldots,F_n$ be n Gaussian random variables of mean zero. Then,*

$$\mathbb{E}[F_1\cdots F_n]=\sum_{\gamma}\prod_{k=1}^{r(\gamma)}\mathbb{E}[F_{i_k}F_{j_k}],$$

where the sum runs over all complete Feynman diagrams γ labelled by $F_1,\ldots,F_n$ with edges (i_k,j_k), $k=1,\ldots,r(\gamma)$. Notice that, if n is odd, then $\mathbb{E}[F_1\cdots F_n]=0$.

Proof. The proof can be found also in Theorem 1.28 (see also Theorem 1.36) in [112]. We have

$$\begin{aligned}\mathbb{E}[\exp(t_1F_1+\cdots+t_nF_n)] &= \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right)\\ &= \exp\left(\sum_{\substack{i,j=1\\ i<j}}^{n} t_it_j\mathbb{E}[F_iF_j]+\frac{1}{2}\sum_{i=1}^{n} t_i^2\mathbb{E}[F_i^2]\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[F_1\cdots F_n] &= \partial_{\boldsymbol{t}}\,\mathbb{E}[\exp(\boldsymbol{t}\times\boldsymbol{F})]|_{\boldsymbol{t}=0}\\ &= \partial_{t_1}\cdots\partial_{t_n}\exp\left(\sum_{\substack{i,j=1\\ i<j}}^{n} t_it_j\mathbb{E}[F_iF_j]+\frac{1}{2}\sum_{i=1}^{n} t_i^2\mathbb{E}[F_i^2]\right)\Bigg|_{\boldsymbol{t}=0}\\ &= \partial_{t_1}\cdots\partial_{t_{n-1}}\left[\sum_{i=1}^{n-1} t_i\mathbb{E}[F_iF_n]\exp\left(\sum_{\substack{i,j=1\\ i<j}}^{n} t_it_j\mathbb{E}[F_iF_j]+\frac{1}{2}\sum_{i=1}^{n} t_i^2\mathbb{E}[F_i^2]\right)\right]\Bigg|_{\boldsymbol{t}=0}\\ &= \partial_{t_1}\cdots\partial_{t_{n-2}}\Bigg[\left(\mathbb{E}[F_{n-1}F_n]+\sum_{i=1}^{n-1} t_i\mathbb{E}[F_iF_n]+\sum_{i=1}^{n-2} t_i\mathbb{E}[F_iF_{n-1}]+t_n\mathbb{E}[F_{n-1}F_n]\right)\\ &\qquad\cdot\exp\left(\sum_{\substack{i,j=1\\ i<j}}^{n} t_it_j\mathbb{E}[F_iF_j]+\frac{1}{2}\sum_{i=1}^{n} t_i^2\mathbb{E}[F_i^2]\right)\Bigg]\Bigg|_{\boldsymbol{t}=0}\\ &\;\vdots\\ &= \sum_{\gamma}\prod_{k=1}^{r(\gamma)}\mathbb{E}[F_{i_k}F_{j_k}],\end{aligned}$$

giving the result. □

Proposition 1.4.38. *Suppose that* $\boldsymbol{F}$ *and* $\boldsymbol{G}$ *are two* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, Gaussian random variables with zero mean. Then*

$$\mathbb{E}[:F_1\cdots F_n::G_1\cdots G_m:]=\begin{cases}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n\mathbb{E}[G_iF_{\sigma(i)}], & n=m,\\ 0, & n\neq m,\end{cases}$$

where $\mathfrak{S}_n$ *is the set of permutations of* $\{1,\ldots,n\}$.

Proof. A proof of a more general result can be found in Theorem 3.9 and Theorem 3.12 in [112]. We have

$$:G_1\cdots G_m:=\partial_s\frac{\exp(\boldsymbol{G}\times\boldsymbol{s})}{\mathbb{E}[\exp(\boldsymbol{G}\times\boldsymbol{s})]}\bigg|_{\boldsymbol{s}=0}$$

and analogously for $\boldsymbol{F}$, then

$$\begin{aligned}&\mathbb{E}[:G_1\cdots G_m::F_1\cdots F_n:]\\ =\ &\partial_s\partial_t\mathbb{E}\left[\frac{\exp(\boldsymbol{G}\times\boldsymbol{s})\exp(\boldsymbol{F}\times\boldsymbol{t})}{\mathbb{E}[\exp(\boldsymbol{G}\times\boldsymbol{s})]\,\mathbb{E}[\exp(\boldsymbol{F}\times\boldsymbol{t})]}\right]\bigg|_{\boldsymbol{t},\boldsymbol{s}=0}\\ =\ &\partial_s\partial_t\frac{\exp\left(\frac{1}{2}\sum_{i,j=1}^m s_is_j\mathbb{E}[G_iG_j]\right)\exp\left(\frac{1}{2}\sum_{i,j=1}^n t_it_j\mathbb{E}[F_iF_j]\right)\exp(\sum_{i,j=1}^{n,m}t_is_j\mathbb{E}[F_iG_j])}{\exp\left(\frac{1}{2}\sum_{i,j=1}^m s_is_j\mathbb{E}[G_iG_j]\right)\exp\left(\frac{1}{2}\sum_{i,j=1}^n t_it_j\mathbb{E}[F_iF_j]\right)}\Bigg|_{\boldsymbol{t},\boldsymbol{s}=0}\\ =\ &\partial_s\partial_t\exp\left(\sum_{i,j=1}^{n,m}t_is_j\mathbb{E}[F_iG_j]\right)\bigg|_{\boldsymbol{t},\boldsymbol{s}=0}\\ =\ &\partial_s\left(\prod_{i=1}^n\left(\sum_{j=1}^m s_j\mathbb{E}[F_iG_j]\right)\right)\bigg|_{\boldsymbol{s}=0}.\end{aligned}$$

The quantity $\prod_{i=1}^n(\sum_{j=1}^m s_j\,\mathbb{E}[F_iG_j])$ is a homogeneous polynomial of degree n in $\mathrm{s}_1,\ldots,s_m$. Therefore, when $m>n$ we are differentiating more times than the degree of the polynomial, while if $m<n$ we obtain a polynomial of degree $n-m>0$ evaluated at $\boldsymbol{t}=0$; in both cases we get then 0. The only case left is the one where $m=n$, which gives

$$\mathbb{E}[:F_1\cdots F_n::G_1\cdots G_m:]=\begin{cases}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n\mathbb{E}[G_iF_{\sigma(i)}], & \text{if } n=m,\\ 0, & \text{if } n\neq m,\end{cases}$$

concluding the proof. □

Proposition 1.4.39. *Let* $F_1,\ldots,F_n$ *be n Gaussian random variables of mean zero. Then,*

$$:F_1\cdots F_n:=\sum_\gamma(-1)^{r(\gamma)}\prod_{k=1}^{r(\gamma)}\mathbb{E}[F_{i_k}F_{j_k}]\prod_{i\in A}F_i,$$

where the sum runs over all Feynman diagrams γ *labelled by* $F_1,\ldots,F_n$ *with edges* (i_k,j_k)*, for* $k=1,\ldots,r(\gamma)$*, and where A is the set of* $n-2r(\gamma)$ *unpaired vertices.*

Proof. A proof of this result can be found in Theorem 3.2 in [112]. We have

$$
\begin{aligned}
:F_1\cdots F_n: &= \partial_t \left.\frac{\exp(\boldsymbol{t}\times\boldsymbol{F})}{\mathbb{E}[\exp(\boldsymbol{t}\times\boldsymbol{F})]}\right|_{t=0} \\
&= \partial_{t_1}\cdots\partial_{t_n}\left.\left(\exp(\boldsymbol{t}\times\boldsymbol{F})\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right)\right)\right|_{t=0} \\
&= \partial_{t_1}\cdots\partial_{t_{n-1}}\Bigg[F_n\exp(\boldsymbol{t}\times\boldsymbol{F})\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right) \\
&\quad \left.+\exp(\boldsymbol{t}\times\boldsymbol{F})\,\partial_{t_n}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right)\Bigg]\right|_{t=0} \\
&= \partial_{t_1}\cdots\partial_{t_{n-2}}\Bigg[F_nF_{n-1}\exp(\boldsymbol{t}\times\boldsymbol{F})\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right) \\
&\quad +F_n\exp(\boldsymbol{t}\times\boldsymbol{F})\,\partial_{t_{n-1}}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right) \\
&\quad +F_{n-1}\exp(\boldsymbol{t}\times\boldsymbol{F})\,\partial_{t_n}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right) \\
&\quad \left.+\exp(\boldsymbol{t}\times\boldsymbol{F})\,\partial_{t_{n-1}}\partial_{t_n}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right)\Bigg]\right|_{t=0} \\
&\;\;\vdots \\
&= \sum_{\gamma}\left(\prod_{i\in A}F_i\right)\left.\left[\left(\prod_{j\notin A}\partial_{t_j}\right)\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} t_it_j\mathbb{E}[F_iF_j]\right)\right]\right|_{t=0},
\end{aligned}
$$

which gives the result. □

From the previous result, it is possible to get the following result.

Proposition 1.4.40. *Suppose that $\boldsymbol{F}_1=(F_{11},\dots,F_{1n_1})$ and $\boldsymbol{F}_2=(F_{21},\dots,F_{2n_2})$ are two $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, Gaussian random variables with zero mean. Then*

$$
:F_{11}\cdots F_{1n_1}::F_{21}\cdots F_{2n_2}:=\sum_{\gamma}\prod_{k=1}^{r(\gamma)}\mathbb{E}[F_{1i_k}F_{2j_k}]:\prod_{i\in A}F_i:,
$$

where the sum runs over all Feynman diagrams γ labelled by $\{F_{11},\dots,F_{1n_1},F_{21},\dots,F_{2n_2}\}$ with edges $(1i_k,2j_k)$, for $k=1,\dots,r(\gamma)$, and where A is the set of $n_1+n_2-2r(\gamma)$ unpaired vertices.

Proof. The results follows in the same way as in the proof Proposition 1.4.39, considering the formula

$$
:F_{11}\cdots F_{1n_1}::F_{21}\cdots F_{2n_2}:=\partial_{t_1}\partial_{t_2}\left.\frac{\exp(\boldsymbol{F}_1\times\boldsymbol{t}_1)\exp(\boldsymbol{F}_2\times\boldsymbol{t}_2)}{\mathbb{E}[\exp(\boldsymbol{F}_1\times\boldsymbol{t}_1)]\,\mathbb{E}[\exp(\boldsymbol{F}_2\times\boldsymbol{t}_2)]}\right|_{t_1,t_2=0}.
$$

For a different proof of the same result, see also Theorem 3.15 in [112]. □

1.4.5 Fock spaces and chaos decomposition

Let us now recall the notions of tensor product and symmetric tensor product on Hilbert spaces. Take an Hilbert space H and consider its n-times tensor product with itself:

$$H^{\otimes n} = H \otimes \cdots \otimes H,$$

that is, if $h_1, \ldots, h_n \in H$, then $h_1 \otimes \cdots \otimes h_n \in H^{\otimes n}$, but also linear combinations of elements of this form live in $H^{\otimes n}$. There is a natural scalar product on $H^{\otimes n}$, given by, for $h_1, \ldots, h_n, h'_1, \ldots, h'_n \in H$,

$$\langle h_1 \otimes \cdots \otimes h_n, h'_1 \otimes \cdots \otimes h'_n \rangle_{H^{\otimes n}} = \prod_{i=1}^{n} (h_i, h'_i)_H.$$

However, $H^{\otimes n}$ is not a complete space with this scalar product, so we can consider its completion $H^{\hat{\otimes} n}$ with respect to the scalar product itself, to get an Hilbert space.

Example 1.4.41. Let M be a topological measure space equipped with a measure η and consider $H = L^2(M, \mathrm{d}\eta)$, then $H^{\hat{\otimes} n}$ is isomorphic to the space

$$L^2(M^n, \mathrm{d}(\eta \otimes \cdots \otimes \eta)).$$

That is, for $h_1, \ldots, h_n \in H$ and $x_1, \ldots, x_n \in M$, we have

$$(h_1 \otimes \cdots \otimes h_n)(x_1, \ldots, x_n) = h_1(x_1) \cdots h_n(x_n) \in L^2(M \times \cdots \times M, \eta(\mathrm{d}x_1) \cdots \eta(\mathrm{d}x_n)).$$

If $H = H_0^1([0,1])$, then

$$H^{\otimes n} = \{ f \in L^2([0,1]^n), f(0) = 0, \partial_{t_1} \cdots \partial_{t_n} f \in L^2([0,1]^n) \},$$

that is, to $f_1 \otimes f_2$ corresponds to $f_1(t_1) f_2(t_2) \in H^{\otimes 2}$, and the scalar product on $H^{\otimes 2}$ is given by

$$(f_1 \otimes f_2, g_1 \otimes g_2)_{H^{\otimes 2}} = \int_{[0,1]^2} \partial_{t_1} f_1(t_1) \partial_{t_2} f_2(t_2) \partial_{t_1} g_1(t_1) \partial_{t_2} g_2(t_2) \, \mathrm{d}t_1 \, \mathrm{d}t_2,$$

where $g_1, g_2 \in H_0^1([0,1])$.

Now, we consider the Hilbert space $H^{\hat{\odot} n} \subset H^{\otimes n}$ generated by elements of the form

$$f_1 \odot \cdots \odot f_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}, \qquad h_i \in H.$$

Notice that such a product is symmetric. We can construct a scalar product in the following way:

$$(h_1 \odot \cdots \odot h_n, h'_1 \odot \cdots \odot h'_n)_{H^{\hat{\odot} n}} = \frac{1}{(n!)^2} \sum_{\sigma, \sigma' \in \mathfrak{S}_n} \prod_{i=1}^{n} (h_{\sigma(i)}, h'_{\sigma'(i)})_H = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} (h_i, h'_{\sigma(i)})_H.$$

Example 1.4.42. In the case of $H = L^2(M, \mathrm{d}\eta)$, we have that $H^{\hat{\odot} n}$ is the set of symmetric functions with respect to the permutation of $x_1, \ldots, x_n$, with product

$$h_1 \odot \cdots \odot h_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} h_{\sigma(i)}(x_i),$$

and

$$h \odot \cdots \odot h = h(x_1) \cdots h(x_n).$$

The next result introduces the n-th order Wiener–Itô integral δ^n, which is often denoted also as I_n or W_n in the literature.

Theorem 1.4.43. *There exists a (quasi-)isometry δ^n, called n-th order Wiener–Itô integral, between $H_{\mathrm{CM}}^{\hat{\odot} n}$ into $L^2(H,\mu)$ such that, if $h_1,\dots,h_n \in H$, we have*

$$\delta^n(h_1 \odot \cdots \odot h_n) = :\delta(h_1)\cdots\delta(h_n):,$$

where δ is the Wiener–Itô integral introduced in Theorem 1.4.23. Moreover, for $f,g \in H_{\mathrm{CM}}^{\hat{\odot} n}$,

$$\mathbb{E}[\delta^n(f)\,\delta^n(g)] = n!(f,g)_{H_{\mathrm{CM}}^{\hat{\odot} n}}. \tag{1.4.6}$$

Proof. Since the linear combinations of elements of the form $h_1 \odot \cdots \odot h_n$ are dense in $H_{\mathrm{CM}}^{\hat{\odot} n}$, the theorem follows by proving equation (1.4.6) for elements of that form. We have

$$\begin{aligned}
\mathbb{E}[\delta^n(h_1 \odot \cdots \odot h_n)\,\delta^n(h'_1 \odot \cdots \odot h'_n)] &= \mathbb{E}[:\delta(h_1)\cdots\delta(h_n)::\delta(h'_1)\cdots\delta(h'_n):] \\
&= \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \mathbb{E}[\delta(h_i)\,\delta(h'_{\sigma(i)})] \\
&= \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n (h_i, h'_{\sigma(i)})_{H_{\mathrm{CM}}} \\
&= n!\,(h_1 \odot \cdots \odot h_n, h'_1 \odot \cdots \odot h'_n)_{H_{\mathrm{CM}}^{\hat{\odot} n}},
\end{aligned}$$

which yields the proof. □

Before proceeding further with the definition of Fock space generated by H and the chaos decomposition, let us state here an iterative formula for products of the form integrals of different order $\delta^n(h_1 \odot \cdots \odot h_n)\,\delta^m(h'_1 \odot \cdots \odot h'_m)$.

Proposition 1.4.44. *Let $h_1,\dots,h_m,h'_1,\dots,h'_m \in H$. Then*

$$\begin{aligned}
&\delta^n(h_1 \odot \cdots \odot h_n)\,\delta^m(h'_1 \odot \cdots \odot h'_m) \\
&= \sum_{\gamma} \left(\prod_{k=1}^{r(\gamma)} \mathbb{E}[\delta(h_{i_k})\delta(h'_{j_k})]\right) \delta^{n+m-2r(\gamma)}\Big(\bigodot_{i \in A_1(\gamma)} h_i \odot \bigodot_{j \in A_2(\gamma)} h'_j\Big),
\end{aligned}$$

where the sum runs over all Feynman diagrams γ labelled by $\{h_1,\dots,h_n,h'_1,\dots,h'_m\}$ with edges (i_k,j_k), for $k=1,\dots,r(\gamma)$, and where $A = A_1 \cup A_2$ is the set of $n+m-2r(\gamma)$ unpaired vertices.

Proof. The result follows by Proposition 1.4.40, see also Proposition 1.1.3 in [148]. □

Example 1.4.45. (Brownian motion on $[0,1]$) Let us consider a Brownian motion B on $[0,1]$ and recall by Example 1.4.14, 1.4.22 and 1.4.24 that it can be seen as a Gaussian measure on $L^2([0,1])$ with Cameron–Martin space given by $H_0^1([0,1])$. Let $n \in \mathbb{N}$ and consider $K \in (H_0^1([0,1]))^{\odot n}$, then it is possible to show that

$$\delta_B^N(K) = N! \int_0^1 \int_{t_1 < \cdots < t_N} \partial_{t_1}\cdots\partial_{t_N} K(t_1,\dots,t_N)\,\mathrm{d}B_{t_1}\cdots\mathrm{d}B_{t_N}.$$

Definition 1.4.46. *Let H be an Hilbert space. We call ΓH the* Fock space generated by *H, with*

$$\Gamma H = \bigoplus_{n=0}^{\infty} H^{\hat{\odot} n},$$

(here $H^{\hat{\odot} 0} = \mathbb{R}$) with scalar product, for $f = (f_0,\dots,f_n,\dots) \in \Gamma H$ and $g = (g_0,\dots,g_n,\dots) \in \Gamma H$,

$$(f,g)_{\Gamma H} = \sum_{n=0}^{\infty} n!(f_n,g_n)_{H^{\hat{\odot} n}}.$$

Remark 1.4.47. ΓH is an Hilbert space.

The next result gives an isomorphism between the space $L^2(H,\mu)$ and the Fock space ΓH_{CM}. It was first proved by Segal in [168], see also the work by Friedrichs [77].

Theorem 1.4.48. *The maps defined in Theorem 1.4.43 give rise to an isometry $\delta^{(\infty)}$ between the Fock space ΓH_{CM} and the space $L^2(H,\mu)$ defined as*

$$\delta^{(\infty)} = \bigoplus_{n=0}^{\infty} \delta^{(n)}.$$

Furthermore, $\delta^{(\infty)}$ is an isomorphism between ΓH_{CM} and $L^2(H,\mu)$.

Proof. We know that $\delta^{(n)}\colon H^{\odot n} \to L^2(H,\mu)$. Moreover,

$$\mathbb{E}[\delta^{(n)}(f)\delta^{(m)}(g)] = \int \delta^{(n)}(f)(h)\,\delta^{(m)}(g)(h)\,\mu(\mathrm{d}h) = 0, \qquad n \neq m,$$

and $\delta^{(\infty)}$ is an isometry, since, in the natural scalar product of ΓH_{CM}, $H_{\mathrm{CM}}^{\hat{\odot} n}$ is orthogonal to $H_{\mathrm{CM}}^{\hat{\odot} m}$, for $m \neq n$, and therefore $\delta^{(\infty)}$ is also injective.

We are left to prove that $\delta^{(\infty)}$ is surjective. Since it is an isometry, we have that its range, $\mathrm{ran}(\delta^{(\infty)})$, is closed. We have to show that $\mathrm{ran}(\delta^{(\infty)}) = L^2(H,\mu)$. Let us denote by $\mathscr{P}_n(H_{\mathrm{CM}})$ the set of polynomials of degree n with respect to elements of the form $\delta(f)$, $f \in H_{\mathrm{CM}}$. Then, we have $\mathscr{P}_n(H_{\mathrm{CM}}) \subset L^2(H,\mu)$, and moreover

$$\overline{\bigcup_{n\in\mathbb{N}} \mathscr{P}_n(H_{\mathrm{CM}})} \subset \mathrm{ran}(\delta^{(\infty)}).$$

We show that

$$\overline{\mathscr{P}} := \overline{\bigcup_{n\in\mathbb{N}} \mathscr{P}_n(H_{\mathrm{CM}})} = L^2(H,\mu).$$

Let us start proving that, if $f_1,\ldots,f_n \in H_{\mathrm{CM}}$, then, for any $\alpha_1,\ldots,\alpha_n \in \mathbb{R}$,

$$e^{\mathrm{i}(\alpha_1\delta(f_1)+\cdots+\alpha_n\delta(f_n))} \in \overline{\mathscr{P}}.$$

We have, for $f \in H_{\mathrm{CM}}$ and $\alpha \in \mathbb{R}$,

$$e^{\mathrm{i}\alpha\delta(f)} = \sum_{k=0}^{\infty} \frac{(\mathrm{i}\alpha\delta(f))^k}{k!},$$

and, if we consider the truncated sum, for some $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} \frac{(i\alpha\delta(f))^k}{k!} \in \mathscr{P}_N(H_{\mathrm{CM}}),$$

then the following bound holds: for any $p \geqslant 1$,

$$\begin{aligned} \left|\sum_{k=0}^{N} \frac{(\mathrm{i}\alpha\delta(f))^k}{k!}\right| &\leqslant \sum_{k=0}^{N} \frac{|\alpha|^k}{k!}|\delta(f)|^k \\ &\leqslant e^{\alpha|\delta(f)|} \in L^p(H,\mu), \end{aligned}$$

where we used the fact that $\delta(f)$ is Gaussian. Therefore, we have

$$\int_H \left| e^{\mathrm{i}\alpha\delta(f)} - \sum_{k=0}^{N} \frac{(\mathrm{i}\alpha\delta(f))^k}{k!} \right|^2 \to 0, \qquad \text{as } N \to +\infty,$$

by the Lebesgue dominated convergence theorem, since

$$\left| e^{\mathrm{i}\alpha\delta(f)} - \sum_{k=0}^{N} \frac{(\mathrm{i}\alpha\delta(f))^k}{k!} \right|^2 \leqslant 1 + e^{\alpha|\delta(f)|}.$$

This gives $e^{\mathrm{i}(\alpha_1\delta(f_1)+\cdots+\alpha_n\delta(f_n))}\in L^2(H,\mu)$. Now, if $F\in\overline{\mathscr{P}}^{\perp}$, then

$$\mathbb{E}_\mu[Fe^{\mathrm{i}(\alpha_1\delta(f_1)+\cdots+\alpha_n\delta(f_n))}]=0,$$

for $\alpha_i\in\mathbb{R}$ and $f_i\in H_{\mathrm{CM}}$, $i=1,\ldots,n$. Take an orthonormal basis $(e_j^{H_{\mathrm{CM}}})_{j\in\mathbb{N}}$ of H_{CM} such that $e_j^{H_{\mathrm{CM}}}\in\mathrm{ran}(S)$, that is $\delta(e_j^{H_{\mathrm{CM}}})=(S^{-1}e_j^{H_{\mathrm{CM}}},\cdot)$, and consider, for $R>0$, the filtration given by

$$\mathscr{F}_R=\sigma\big(\big\{\delta(e_j^{H_{\mathrm{CM}}}): j\leqslant R\big\}\big).$$

Notice that

$$\mathscr{F}_R\to\mathscr{F}_\infty=\mathscr{C},\qquad \text{as } R\to+\infty.$$

If we consider $F\in L^2(H,\mu)$, then

$$F_R:=\mathbb{E}_\mu[F|\mathscr{F}_R]\to\mathbb{E}_\mu[F|\mathscr{F}_\infty]=:F,\qquad \text{a.s.},$$

by the martingale convergence theorem.

On the other hand, we have

$$\begin{aligned}\mathbb{E}_\mu\Big[F_R e^{\mathrm{i}\left(\alpha_1\delta(e_1^{H_{\mathrm{CM}}})+\cdots+\alpha_n\delta(e_n^{H_{\mathrm{CM}}})\right)}\Big] &= \mathbb{E}_\mu\Big[\mathbb{E}_\mu[F|\mathscr{F}_R]e^{\mathrm{i}\left(\alpha_1\delta(e_1^{H_{\mathrm{CM}}})+\cdots+\alpha_n\delta(e_n^{H_{\mathrm{CM}}})\right)}\Big]\\ &= \mathbb{E}_\mu\Big[Fe^{\mathrm{i}\left(\alpha_1\delta(e_1^{H_{\mathrm{CM}}})+\cdots+\alpha_n\delta(e_n^{H_{\mathrm{CM}}})\right)}\Big]=0,\end{aligned}$$

since F_R is a measurable function of $\delta(e_j^{H_{\mathrm{CM}}})$, $j\leqslant R$, by the uniqueness of Fourier transforms in finite dimension, we have that $F_R=0$. Therefore, $F=\lim_{R\to\infty}F_R=0$, a.s. This implies that $\overline{\mathscr{P}}^{\perp}=\{0\}$, and so $\overline{\mathscr{P}}=H$, since $\overline{\mathscr{P}}$ is closed. □

In the light of Theorem 1.4.48, if $F\in L^2(H,\mu)$, then we have the following representation, called *Wiener chaos decomposition*,

$$F=\sum_{n=0}^{\infty}\delta^{(n)}F_n,$$

for some $F_n\in H_{\mathrm{CM}}^{\odot n}$, $n\in\mathbb{N}$, with the sum on the right-hand side converging in $L^2(H,\mu)$.

With an abuse of notation, we can write

$$\Gamma^n H_{\mathrm{CM}}=\mathrm{ran}(\delta^{(n)})=\mathrm{ran}(\delta^{(\infty)}|_{H_{\mathrm{CM}}^{\odot n}}),$$

and $\Gamma^0 H_{\mathrm{CM}}=\mathbb{R}$. In this way, we also have the following representation for $F\in L^2(H,\mu)$, which is referred to as *Wiener chaos decomposition* as well,

$$F=\sum_{n=0}^{\infty}\tilde{F}_n,$$

where $\tilde{F}_n\in\Gamma^n H_{\mathrm{CM}}$, i.e. $\tilde{F}_n$ is a Wick polynomial of degree n, for every $n\in\mathbb{N}$. Moreover, if $P(h)\in L^2(H,\mu)$ is a Wick polynomial of degree n, then

$$P(h)\in\bigcup_{i=1}^{n}\Gamma^i H_{\mathrm{CM}}.$$

In particular,

$$\|h\|_H^2\in\bigcup_{i=1}^{2}\Gamma^i H_{\mathrm{CM}}.$$

We now want to introduce the notion of Malliavin derivative. Let H be a separable Hilbert space and let μ be a Gaussian measure on H with Cameron–Martin space H_{CM}.

Definition 1.4.49. *We say that* $F: H \to \mathbb{R}$ *is a* cylinder function , *and we write* $F \in \mathrm{Cyl}_H$, *if it has the form*

$$F(x) = f(\delta(h_1)(x), \ldots, \delta(h_n)(x)), \qquad x \in H, \tag{1.4.7}$$

for some $n \in \mathbb{N}$, $h_1, \ldots, h_n \in H_{\mathrm{CM}}$, $f \in \mathcal{S}(\mathbb{R}^n)$.

It is then possible to define a notion of derivative for cylinder functions.

Definition 1.4.50. *Let* $F \in \mathrm{Cyl}_H$ *be a cylinder function of the form (1.4.7) an consider* $h \in H_{\mathrm{CM}}$. *The* derivative of F in direction h *is defined as*

$$\mathrm{D}_h F(x) = \frac{\mathrm{d}}{\mathrm{d}\lambda} F(x + \lambda h)\big|_{\lambda=0} = \sum_{i=1}^{n} \partial_{x_i} f(\delta(h_1)(x), \ldots, \delta(h_n)(x))\, h_i, \qquad x \in H.$$

Notice that, for $x \in H$ fixed, $h \mapsto \mathrm{D}_h F(x)$ is linear and bounded on H, and it then determines an element of $H_{\mathrm{CM}}^* = H_{\mathrm{CM}}$, which we denote by $\mathrm{D}F(x)$. Therefore, $F \mapsto \mathrm{D}F$ is a linear operator from the real-valued cylindrical functions into the space of H_{CM}-valued Wiener functionals $L^p(H, \mu; H_{\mathrm{CM}})$, for any $p > 1$. It is possible to show that the operator D is closable from $L^p(H, \mu)$ into $L^p(H, \mu; H_{\mathrm{CM}})$, for any $p > 1$ (see Proposition B.3.1 in [180] or Proposition 1.2.1 in [148]).

Definition 1.4.51. *We denote by* $\mathbb{D}^{1,p} = \mathbb{D}^{1,p}(\mathbb{R})$ *the set of (equivalence classes of) functionals* F *on* H *such that there exists a sequence of cylindric random variables* $(F_n)_{n \in \mathbb{N}}$ *converging to* F *in* $L^p(H, \mu)$ *such that* $(\mathrm{D}F_n)_{n \in \mathbb{N}}$ *is a Cauchy sequence in* $L^p(H, \mu; H_{\mathrm{CM}})$. *We denote the limit* $\lim_{n \to +\infty} \mathrm{D}F_n$ *by* $\mathrm{D}F$.

It is possible to show that the limit $\mathrm{D}F$ defined in the previous proposition is independent of the choice of the approximating sequence of F. Moreover, $\mathbb{D}^{1,p}$ is a Banach space with the norm

$$\|F\|_{\mathbb{D}^{1,p}} = \|F\|_{L^p(H,\mu)} + \|\mathrm{D}F\|_{L^p(H,\mu;H_{\mathrm{CM}})}$$

(see Appendix B.3 in [180] or Section 1.2 in [148]).

We then get the following iterative construction.

Definition 1.4.52. *Let* $p > 1$ *and* $k \geqslant 1$. *We define the space* $\mathbb{D}^{k,p}$ *as follows:*

i. $F \in \mathbb{D}^{2,p}$ *if* $\mathrm{D}F \in \mathbb{D}^{1,p}(\mathbb{R} \odot H_{\mathrm{CM}})$, *we also write* $\mathrm{D}^2 F = \mathrm{D}(\mathrm{D}F)$.

ii. $F \in \mathbb{D}^{k,p}$ *if* $\mathrm{D}^{k-1} F \in \mathbb{D}^{1,p}(\mathbb{R} \odot H_{\mathrm{CM}}^{\odot(k-1)})$.

Proposition 1.4.53. *Suppose that* H_{CM} *is a subspace of measurable functions on a set* $\mathcal{X}$. *Let* $F \in \mathbb{D}^{1,2}$ *be a square integrable random variable with chaos decomposition* $F = \sum_{n=0}^{\infty} \delta^n F_n$, *for* $F_n \in H_{\mathrm{CM}}^{\odot n}$, $n \in \mathbb{N}$. *Then, we have*

$$\mathrm{D}F(x) = \sum_{n=1}^{\infty} n \delta^{n-1} F_n(x, \cdot), \qquad x \in \mathcal{X}.$$

Proof. See Proposition 1.2.7 in [148]. □

Remark 1.4.54. A consequence of the previous result is that, if F is as in Proposition 1.4.53 with chaos decomposition $F = \sum_{n=0}^{\infty} \pi_n F$, where π_n denotes the projection on the n-th Wiener-Itô chaos $\Gamma^n H_{\mathrm{CM}}$, we then have $\mathrm{D}\pi_n F = \pi_{n-1} \mathrm{D}F$.

1.4.6 Hypercontractivity

Let us introduce the following operator between Fock spaces.

Definition 1.4.55. *Let A be a linear and bounded operator from H_{CM} into itself, then we write*

$$\Gamma A: L^2(H,\mu) \to L^2(H,\mu),$$

associating each $f_1,\ldots,f_n \in H_{\mathrm{CM}}$ to

$$\Gamma A(:\delta(f_1)\cdots\delta(f_n):) = :\delta(Af_1)\cdots\delta(Af_n):.$$

In the present section, we want to show that, if A is a strict contraction and $1<p<+\infty$, then the hypercontractivity property holds, that is ΓA is a contraction of L^p into L^q for some $q>p$. This subsection follows closely Chapter III inside the reference [179].

Take $r\in[0,1]$, and define the linear and bounded operator T_r on $L^2(H,\mu)$ given by

$$T_r = \Gamma(r\,\mathrm{id}_{H_{\mathrm{CM}}}),$$

meaning that, if $f_1,\ldots,f_n\in H_{\mathrm{CM}}$, then

$$T_r(:\delta(f_1)\cdots\delta(f_n):) = :\delta(rf_1)\cdots\delta(rf_n): = r^n :\delta(f_1)\cdots\delta(f_n):.$$

It is also possible to express this operator as $T_r = \Gamma(r\,\mathrm{id}_{H_{\mathrm{CM}}}) = r^{\mathcal{N}}$, where $\mathcal{N}$ is the (unbounded) *number operator* on $L^2(H,\mu)$ defined on the set of $L^2(H,\mu)$ functionals with finite chaos expansion by extending via linearity the following definition:

$$\mathcal{N}(:\delta(f_1)\cdots\delta(f_n):) = n:\delta(f_1)\cdots\delta(f_n):.$$

The operator $\mathcal{N}$ is known as the *number operator* in quantum field theory and as *Ornstein–Uhlenbeck operator* in stochastic analysis. Notice that the operators $\Gamma(e^{-t}\,\mathrm{id}_{H_{\mathrm{CM}}}) = e^{-t\mathcal{N}}$, $t\geqslant 0$, form an operator semigroup called *Ornstein–Uhlenbeck semigroup* (see Example 1.2.5).

Lemma 1.4.56. *T_r is a contraction, that is,*

$$\|T_r(F)\|_{L^2(H,\mu)} \leqslant \|F\|_{L^2(H,\mu)},$$

and, moreover, it is a self-adjoint operator.

Proof. Let $F\in L^2(H,\mu)$ and consider its chaos decomposition for $\tilde{F}_n\in\Gamma^n H_{\mathrm{CM}}$, $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$, then

$$T_r\tilde{F}_n = r^n\tilde{F}_n,$$

and then, since $r\leqslant 1$,

$$\|T_r\tilde{F}_n\|_{L^2} = \|r^n\tilde{F}_n\|_{L^2} = r^n\|\tilde{F}_n\|_{L^2} \leqslant \|\tilde{F}_n\|_{L^2}.$$

Therefore

$$\left\|T_r\left(\sum_{n=0}^{\infty}\tilde{F}_n\right)\right\|_{L^2}^2 = \sum_{n=0}^{\infty} r^{2n}\|\tilde{F}_n\|_{L^2}^2 \leqslant \sum_{n=0}^{\infty}\|\tilde{F}_n\|_{L^2}^2 = \|F\|_{L^2}^2,$$

by the properties of the chaos decomposition (cf. Theorem 1.4.43 and 1.4.48).

To prove that T_r is self-adjoint, take $F,G\in\Gamma^n H_{\mathrm{CM}}$, then

$$(G,T_rF)_{L^2} = (G,r^nF)_{L^2} = (r^nG,F)_{L^2} = (T_rG,F)_{L^2},$$

while, if $F\in\Gamma^n H_{\mathrm{CM}}$ and $G\in\Gamma^m H_{\mathrm{CM}}$, $n\neq m$, then, by orthogonality,

$$(G,T_rF)_{L^2} = (G,r^nF)_{L^2} = 0 = (r^nG,F)_{L^2} = (T_rG,F)_{L^2}.$$

which gives the statement by the boundedness of the operator T_r. □

We have the following characterization of T_r.

Theorem 1.4.57. *Let $F \in L^2(H,\mu)$. Then, for $h \in H$,*

$$T_rF(h) = \int_H F\Big(rh + \sqrt{1-r^2}\,\tilde{h}\Big)\mu(\mathrm{d}\tilde{h}). \tag{1.4.8}$$

Proof. See Section 4.2 in [112]. □

Theorem 1.4.58. *T_r is a continuous operator from L^p into itself, for any $p \in [1,+\infty]$.*

Proof. When $p = +\infty$, we have $\|T_rF\|_{L^\infty} \leqslant \int \|F\|_{L^\infty}\,\mathrm{d}\mu = \|F\|_{L^\infty}$, which gives the result. For $p = 1$, we have

$$\begin{aligned}
\|T_rF\|_{L^1} &= \int_H \left|\int_H F\Big(rh + \sqrt{1-r^2}\,\tilde{h}\Big)\mu(\mathrm{d}\tilde{h})\right|\mu(\mathrm{d}h)\\
&\leqslant \int_H \int_H \left|F\Big(rh + \sqrt{1-r^2}\,\tilde{h}\Big)\right|\mu(\mathrm{d}\tilde{h})\,\mu(\mathrm{d}h)\\
&= (1, T_r|F|)_{L^2}\\
&= (T_r1, |F|)_{L^2}\\
&= (1, |F|)_{L^2}\\
&= \|F\|_{L^1}.
\end{aligned}$$

Via interpolation we can get the result for any $p \in [1,+\infty]$. □

We are then in position to prove the hypercontractivity result.

Theorem 1.4.59. (Hypercontractivity) *Let $1 < p \leqslant q < +\infty$ and $r \in (0,1)$ be such that $r^2(q-1) \leqslant p-1$. Then T_r is a contraction from $L^p(H,\mu)$ into $L^q(H,\mu)$.*

It is worth to stress the fact that $p \leqslant q$, namely the operator T_r improves the integrability:

$$\|T_rF\|_{L^q} \leqslant \|F\|_{L^p}.$$

Let us see some consequences of this result.

Corollary 1.4.60. *If $F \in \Gamma^n H_{\mathrm{CM}}$, then we have, for any $q \in [2,+\infty)$,*

$$\|F\|_{L^q} \leqslant \big(\sqrt{q-1}\big)^n\,\|F\|_{L^2}.$$

Proof. Recall that $T_rF = r^nF$. We can apply Theorem 1.4.59 for $p = 2$, taking $r \leqslant \big(\sqrt{q-1}\big)^{-1}$, to get

$$\big(\sqrt{q-1}\big)^{-n}\|F\|_{L^q} = \|T_rF\|_{L^q} \leqslant \|F\|_{L^2},$$

which is what stated. □

Corollary 1.4.61. *Let F be a polynomial of degree at most n. Then, for any $q \in [2,+\infty)$,*

$$\|F\|_{L^q} \leqslant \big(\sqrt{q-1}\big)^n\|F\|_{L^2}.$$

Proof. Notice that $F\in\bigcup_{k=0}^{n}\Gamma^k H_{\mathrm{CM}}$. We can then use Corollary 1.4.60 to get

$$\|F_k\|_{L^q}\leqslant\left(\sqrt{q-1}\right)^k\|F_k\|_{L^2}\leqslant\left(\sqrt{q-1}\right)^n\|F_k\|_{L^2},$$

and then the result for F. □

Corollary 1.4.62. (Fernique Theorem for Hilbert spaces) *Let μ be a Gaussian measure (with mean zero) on a Hilbert space H. Then, there exists $\alpha>0$ such that*

$$\int_H e^{\alpha\|h\|_H^2}\mu(\mathrm{d}h)<+\infty.$$

Proof. We have

$$\int_H e^{\alpha\|h\|_H^2}\mu(\mathrm{d}h)=\sum_{n=0}^{+\infty}\int_H\frac{\alpha^n\|h\|_H^{2n}}{n!}\mu(\mathrm{d}h)=\sum_{n=0}^{+\infty}\frac{\alpha^n}{n!}\|\,\|h\|_H^2\|_{L^n}^n.$$

Since S is a trace class, positive self-adjoint operator, then there is a basis $\{e_j\}_{j\in\mathbb{N}}$ of H consisting of eigenvalues of S, i.e. $Se_j=\sigma_j^2 e_j$ for some $\sigma_j^2\geqslant 0$. In particular, this implies that $(Se_j,e_i)_H=\delta_{ij}\sigma_j^2$. Since $\delta(Se_j)=(S^{-1}(Se_j),\cdot)_H=(e_j,\cdot)_H$, then we get $\|h\|_H^2=\sum_{j\in\mathbb{N}}(\delta(Se_j))^2$. We have that

$$\begin{aligned}
\|\,\|h\|_H^2\|_{L^2}^2 &= \sum_{i,j\in\mathbb{N}}\int\delta(Se_j)^2\,\delta(Se_i)^2\,\mu(\mathrm{d}h)\\
&= \sum_{i,j\in\mathbb{N},i\neq j}(Se_j,e_j)_H\,(Se_i,e_i)_H+\sum_{j\in\mathbb{N}}\int\delta(Se_j)^4\mu(\mathrm{d}h)\\
&= \sum_{i,j\in\mathbb{N},i\neq j}(Se_j,e_j)_H\,(Se_i,e_i)_H+3\sum_{j\in\mathbb{N}}(Se_j,e_j)_H^2\\
&\leqslant \mathrm{tr}(S)^2+C(1+\mathrm{tr}(S))\\
&< +\infty.
\end{aligned}$$

where $C>0$ is a suitable constant and we used the fact that $\int\delta(Se_j)^4\mu(\mathrm{d}h)=3(Se_j,e_j)_H^2$ (being $\delta(Se_j)$ a Gaussian random variable with variance $(Se_j,e_j)_H$), and that $\ell^2(\mathbb{N})\subset\ell^1(\mathbb{N})$. Therefore, $\|h\|_H^2\in\bigcup_{n=0}^{2}\Gamma^n H_{\mathrm{CM}}$, and we can exploit hypercontractivity (Theorem 1.4.59) to get

$$\|\,\|h\|_H^2\|_{L^n}^n\leqslant\left(\sqrt{n-1}\right)^{2n}\|\,\|h\|_H^2\|_{L^2}^n.$$

Thus,

$$\int_H e^{\alpha\|h\|_H^2}\mu(\mathrm{d}h)\leqslant\sum_{n=0}^{+\infty}\frac{\alpha^n}{n!}(n-1)^n\|\,\|h\|_H^2\|_{L^2}^n,$$

and by Stirling formula, namely $(n-1)^n\leqslant C'^n n!$ for some positive constant C', we have

$$\int_H e^{\alpha\|h\|_H^2}\mu(\mathrm{d}h)\leqslant\sum_{n=0}^{+\infty}(\alpha C'\|\,\|h\|_H^2\|_{L^2})^n,$$

and taking $\alpha<(C'\|\,\|h\|_H^2\|_{L^2})^{-1}$ yields the statement of the corollary. □

The remainder of the present section is devoted to the proof of Theorem 1.4.59. We follow closely the proof presented in Chapter III in [179] (which in turn follows [147]), but let us mention that the hypercontractivity property was first proved by Nelson [146]. We also refer the reader to Chapter V in [112] and the references therein, as well as Example 7.15 from the same book.

Proof of Theorem 1.4.59

It is enough to consider H and H_{CM} as finite-dimensional spaces, since the statement of the theorem does not depend on the dimension of the space. Indeed, if H is infinite-dimensional, then we have $F_n \to F$ in L^q such that $F_n = \tilde{F}_n(\delta(e_1), \ldots, \delta(e_n))$, where $(e_i)_{i\in\mathbb{N}}$ is a basis of H_{CM}. Therefore, if the result holds in finite dimensions,

$$\|T_r\tilde{F}_n\|_{L^p(\mathbb{R}^n, N(0,I))} \leqslant \|\tilde{F}_n\|_{L^q},$$

and $\|T_rF_n\|_{L^p} = \|T_r\tilde{F}_n\|$, $\|F_n\|_{L^q} = \|\tilde{F}_n\|_{L^q}$, giving the result for the infinite-dimensional case. Here, we denoted by $N(0,I)$ the space of n-dimensional Gaussian random variables with zero mean and variance given by the identity I.

We let X and Z be two n-dimensional Gaussian random variables with variance given by the identity I, we write $X, Z \sim N(0,I)$, and consider, for $0 < r < 1$, $Y = rX + \sqrt{1-r^2}Z$. Then Y is another Gaussian random variable and $E[XY] = rI$. From now on, we denote by $\mathscr{F}^X$ the σ-algebra generated by the random variable X.

Lemma 1.4.63. *Consider $X, Y \sim N(0,I)$ such that $\mathbb{E}[XY] = rI$. If $F(X) \in L^2$, then*

$$\mathbb{E}[F(X)|\,\mathscr{F}^Y] = \int F\Big(rY + \sqrt{1-r^2}z\Big)\,\mu_Z(\mathrm{d}z) = T_rF(Y),$$

where $Z \sim N(0,I)$.

Proof. Let $G \in L^\infty(\mathscr{F}^Y)$. Since X and Y are Gaussian with correlation rI, then there exists a unique random measure $Z \sim N(0,I)$ independent of Y, such that

$$Y = rX + \sqrt{1-r^2}Z.$$

Now, we have

$$\begin{aligned}\mathbb{E}[F(X)G(Y)] &= \mathbb{E}\Big[F(X)G\Big(rX + \sqrt{1-r^2}Z\Big)\Big]\\ &= \mathbb{E}\Big[F\Big(rX + \sqrt{1-r^2}Z\Big)G(Z)\Big],\end{aligned}$$

since T_r is self-adjoint. Moreover, $Z \sim X$ and $rX + \sqrt{1-r^2}Z \sim Y$, and therefore, by Theorem 1.4.57,

$$\begin{aligned}\mathbb{E}[F(X)G(Y)] &= \mathbb{E}[T_rF(Z)G(Z)]\\ &= \mathbb{E}[T_rF(Y)G(Y)],\end{aligned}$$

since this is true for any $G \in L^\infty(\Omega, \mathscr{F}^Y, \mathbb{P})$, we have

$$\mathbb{E}[F(X)|\,\mathscr{F}^Y] = T_rF(Y), \qquad \text{a.s.} \qquad \square$$

Consider two n-dimensional Brownian motions $X_t = (X_{1,t}, \ldots, X_{n,t})$ and $Y_t = (Y_{1,t}, \ldots, Y_{n,t})$, for $t \in [0,1]$, with covariance rt for some $r \in (0,1)$, that is

$$\mathbb{E}[X_{i,t}Y_{j,t}] = \delta_{ij}rt.$$

Let $\mathscr{F}_t^X$ and $\mathscr{F}_t^Y$ be the natural filtrations of X_t and Y_t, respectively. By the previous lemma we have that, if $F \in L^p(\sigma(X_1))$, then

$$T_rF(Y_1) = \mathbb{E}[F(X_1)|Y_1] = \mathbb{E}[F(X_1)|\mathscr{F}^Y],$$

since the Brownian motion is Markovian.

If $G \in L^p(\mathscr{F}_1^X)$, we use the following notation

$$\tilde{T}G = \mathbb{E}[G|\mathscr{F}_1^Y].$$

Proposition 1.4.64. *Let $1<p\leqslant q<+\infty$ and $r\in(0,1)$ such that $p-1\geqslant r^2(q-1)$, then $\tilde{T}=\mathbb{E}[\cdot|\mathscr{F}_1^Y]$: $L^p(\mathscr{F}_1^X)\to L^q(\mathscr{F}_1^Y)$ and moreover, for any $G\in L^p(\mathscr{F}_1^X)$, we have*

$$\|\tilde{T}G\|_{L^q}\leqslant\|G\|_{L^p}. \tag{1.4.9}$$

Proof. Since $\tilde{T}$ is positive, namely

$$|\tilde{T}(G)|=|\mathbb{E}[G|\mathcal{F}_1^Y]|\leqslant\mathbb{E}[|G||\mathcal{F}_1^Y]=\tilde{T}(|G|),$$

it is enough to prove the theorem for $G\geqslant 0$. Furthermore, since $L^\infty(\mathcal{F}_1^Y)$ is dense in $L^p(\mathcal{F}_1^Y)$ for any $p\in[1,+\infty]$, we can restrict to consider the case $0\leqslant G\leqslant b$ for some $b>0$. Finally, since $G+\varepsilon\to G$ as $\varepsilon\to 0$ in $L^p(\mathcal{F}_1^Y)$ it is sufficient to prove inequality (1.4.9) for any $G\in L^\infty(\mathscr{F}_1^X)$ such that $0<a\leqslant G\leqslant b$, for some $a,b>0$.

Since linear combinations of element of the previous form are total in $L^p(\mathcal{F}_1^Y)$ for $p\in(1,+\infty)$ (i.e. they separate the point of the dual), inequality (1.4.9) is equivalent to prove that $F\in L^\infty(\mathscr{F}^Y)$ such that $0<a'\leqslant F\leqslant b'$ (for some $a',b'>0$) we have

$$\mathbb{E}[\tilde{T}(G)F]\leqslant\|G\|_{L^q}\|F\|_{L^s},$$

where $1/s+1/q=1$. This is because it would imply the same inequality for the moduli, namely

$$\mathbb{E}[|\tilde{T}(G)||F|]\leqslant\|G\|_{L^q}\|F\|_{L^s},$$

which gives

$$\|\tilde{T}G\|_{L^q}=\sup_{\substack{\|F\|_{L^s}=1\\ a'\leqslant F\leqslant b',\text{ for some }a',b'>0}}\mathbb{E}[|\tilde{T}(G)||F|]\leqslant\|G\|_{L^q}.$$

Now, we take $M_t=\mathbb{E}[G^p|\mathscr{F}_t^X]$, $N_t=\mathbb{E}[F^s|\mathscr{F}_t^Y]$. These are martingales, and by the Itô representation theorem we have

$$\begin{aligned} M_t &= M_0+\int_0^t\phi_\tau\cdot\mathrm{d}X_\tau,\\ N_t &= N_0+\int_0^t\psi_\tau\cdot\mathrm{d}Y_\tau,\end{aligned}$$

where $\phi,\psi\in L^2([0,1]\times\Omega,\mathbb{R}^n)$ are predictable. Then, taking $\alpha=1/p$ and $\beta=1/s$, we have by Itô formula

$$M_t^\alpha N_t^\beta=M_0^\alpha N_0^\beta+\int_0^t\alpha M_\tau^{\alpha-1}N_\tau^\beta\mathrm{d}M_\tau+\int_0^t\beta M_\tau^\alpha N_\tau^{\beta-1}\mathrm{d}N_\tau+\frac{1}{2}\int_0^1 M_t^\alpha N_t^\beta A_t\,\mathrm{d}t,$$

with

$$A_t=\alpha(\alpha-1)\left|\frac{\phi_t}{M_t}\right|^2+2r\alpha\beta\frac{\phi_t}{M_t}\cdot\frac{\psi_t}{N_t}+\beta(\beta-1)\left|\frac{\psi_t}{N_t}\right|^2,$$

and we could apply Itô formula because $M_t,N_t\geqslant a>0$.

Since M and N are martingales, we have

$$\mathbb{E}\left[M_1^\alpha N_1^\beta\right]=\mathbb{E}\left[M_0^\alpha N_0^\beta\right]+\frac{1}{2}\int_0^1\mathbb{E}\left[M_t^\alpha N_t^\beta A_t\right]\mathrm{d}t.$$

Now, $\mathbb{E}[M_1^\alpha]=G$ and $\mathbb{E}\left[N_1^\beta\right]=F$, moreover

$$\mathbb{E}[GF]=\mathbb{E}[\mathbb{E}[G^p|\mathscr{F}_0^X]^{1/p}\mathbb{E}[F^s|\mathscr{F}_0^Y]^{1/s}]=\|G\|_{L^p}\|F\|_{L^s}+\frac{1}{2}\int_0^1\mathbb{E}\left[M_t^\alpha N_t^\beta A_t\right]\mathrm{d}t.$$

Notice that $A_t \leqslant 0$. This is because it is an homogeneous polynomial of degree 2 in ϕ_t/M_t and ψ_t/N_t, and, since $\alpha = 1/p < 1$ and $\beta = 1/s < 1$,

$$\operatorname{tr}\begin{pmatrix} \alpha(\alpha-1) & r\alpha\beta \\ r\alpha\beta & \beta(\beta-1) \end{pmatrix} = \alpha(\alpha-1) + \beta(\beta-1) < 0.$$

Moreover,

$$\det\begin{pmatrix} \alpha(\alpha-1) & r\alpha\beta \\ r\alpha\beta & \beta(\beta-1) \end{pmatrix} = \alpha\beta(\alpha-1)(\beta-1) - r^2\alpha^2\beta^2 \geqslant 0,$$

which holds because it is equivalent to $p-1 \geqslant r^2(q-1)$.

Therefore,

$$\mathbb{E}[GF] \leqslant \|G\|_{L^p} \|F\|_{L^s}.$$

Now GF is measurable with respect to $\mathscr{F}^Y$ and we have

$$\mathbb{E}[\mathbb{E}[GF|\mathscr{F}^Y]] = \mathbb{E}[\mathbb{E}[G|\mathscr{F}^Y]F] = \mathbb{E}[\tilde{T}(G)\, F],$$

which gives $\mathbb{E}[\tilde{T}(G)F] \leqslant \|G\|_{L^p} \|F\|_{L^s}$ and thus $\tilde{T}$ is hypercontractive, which concludes the proof of the statement. □

We have then $\tilde{T}(G(X_1)) = T_r(G)(Y_1)$ and hence

$$\|T_r(G)(Y_1)\|_{L^q} \leqslant \|G(Y_1)\|_{L^p},$$

for $Y_1 \sim N(0, I)$. This concludes the proof of hypercontractivity, i.e. Theorem 1.4.59.

Part I

Probabilistic approach to singular SPDEs

Chapter 2
Introduction

The four methods mentioned in Section 1.1 for solving singular SPDEs present the common feature of bypassing probability theory and approaching the problem from an analytic point of view, focusing in particular on pathwise arguments. Nevertheless, some progress on a probabilistic perspective has been made in recent years. We focus here on the study of weak solutions to singular SPDEs via their martingale problem representation (we also refer the reader to [98] for a recent overview on the subject).

2.1 Martingale problems for SDEs with distributional drift

In order to showcase some of the difficulties one encounters when trying to give a probabilistic formulation to singular SPDEs, let us start by working on a simpler model. Indeed, since in most cases singular SPDEs are nothing but SPDEs with a distributional drift, we consider here a finite-dimensional example given by the following stochastic *ordinary* differential equation with distributional drift:

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t, \qquad t \in \mathbb{R}_+, \tag{2.1.1}$$

where X is real-valued, W is a one-dimensional Brownian motion, and $b \in \mathcal{S}'(\mathbb{R})$ is a Schwartz distribution on $\mathbb{R}$. In particular, $b(z)$ is not well-defined for $z \in \mathbb{R}$, and hence the term $b(X_t)$ in equation (2.1.1) is ill-defined since X_t has values in $\mathbb{R}$. A possible probabilistic approach is to consider the martingale problem associated with the infinitesimal generator $\mathscr{L}$ of the solution X to equation (2.1.1). Heuristically, if one considers a smooth test function φ and applies Itô's formula to $\varphi(X_t)$, one should get the relation

$$M_t^{\varphi} = \varphi(X_t) - \varphi(X_0) - \int_0^t \mathscr{L}\varphi(X_s)\,\mathrm{d}s, \tag{2.1.2}$$

where M_t^{φ} is a (local) martingale and where $\mathscr{L}\varphi(X_t) = \partial_{xx}^2\varphi(X_t) + b(X_t)\,\partial_x\varphi(X_t)$ is (an explicit representation of) the infinitesimal generator of the solution X for the test functions φ. The martingale problem consists then in finding a stochastic process X such that the right-hand side of equation (2.1.2) is a local martingale. Still, an a priori ill-defined term appears in $\mathscr{L}\varphi$, namely $b(X_t)\,\partial_x\varphi(X_t)$, which is a product between a distribution and a smooth function – notice that multiplying by a smooth function does not increase the regularity a priori (e.g. multiplication by a constant) –, and thus it does not make it possible to make sense of the term $\int_0^t \mathscr{L}\varphi(X_s)\,\mathrm{d}s$. The idea is then to consider test functions φ that are non-smooth but with a structure somehow adapted to the term b, with the aim of increasing the regularity when the multiplication occurs – for instance, think of multiplying a non-regular function f by $1/f$: such an operation increases the regularity. Solving the equation

$$\mathscr{L}\varphi = F,$$

for some given continuous function F, would allow us to make sense of the integral term $\int_0^t \mathscr{L}\varphi(X_s)\,\mathrm{d}s = \int_0^t F(X_s)\,\mathrm{d}s$ and to then study the martingale problem. In particular, solving $\mathscr{L}\varphi = F$, for all $F \in C_b$, would put us in position to identify explicitly the domain of the operator $\mathscr{L}$ as

$$D(\mathscr{L}) = \{\varphi : \mathscr{L}\varphi = F, \text{for } F \in C_b\},$$

determining uniquely the distribution of X via the martingale problem. The main problem is then to deal with the equation $\mathscr{L}\varphi = F$, which still involves the term $b(X_t)\,\partial_x\varphi(X_t)$. In the case where b is the derivative of a continuous function B, it is possible to show that the equation $\mathscr{L}\varphi = F$ is well-defined and that it admits a unique solution as it is established in the works by Flandoli, Russo, and Wolf [74, 75]. The same strategy of considering the problem $\mathscr{L}\varphi = F$ can be adopted in the multidimensional setting or when the distributional drift is time-inhomogeneous, i.e. when $b(t, X_t)$ depends on time t, but in order to deal with the well-posedness of the equation it is necessary to work with paracontrolled distributions techniques. We refer the reader to the work by Flandoli, Issoglio, and Russo [73] for the multidimensional case when b has Hölder regularity greater than $-1/2$, to the paper by Delarue and Diel [62] for the one-dimensional case with b having Hölder regularity grater than $-2/3$, and to the article by Cannizzaro and Chouk [42] for the multidimensional case when b is of Hölder regularity greater than $-2/3$.

It is worth to point out that the aforementioned works provide results for probabilistically weak solutions to the equation, while if b is a (possibly non-smooth) function instead of a distribution, it is then possible to get a unique probabilistically strong solution (see e.g. the paper by Krylov and Röckner [124]).

2.2 Energy solutions to singular SPDEs

We shall now turn our attention to the infinite-dimensional scenario. The first results on martingale solutions to singular SPDEs were obtained on KPZ equation on the one-dimensional torus, in particular passing through the stochastic Burgers equation

$$\partial_t u(t,x) = \Delta u(t,x) + \partial_x u^2(t,x) + \sqrt{2}\,\partial_x \xi(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{T}. \tag{2.2.1}$$

where ξ is a space-time white noise. The dynamics of u are known to be invariant under the law μ of the space-time white noise on $\mathbb{T}$, and in particular $u(t,\cdot)$ is a distribution of negative regularity $-1/2 - \varepsilon$ for every $t \in \mathbb{R}_+$ (see Example 1.4.18). Since products of distributions with such a regularity are not well-defined, the square of u does not make sense from an analytical point of view. But starting the solution u at stationarity (i.e. in the stationary measure μ), allows us to make sense of u^2 as a square of a white noise.

Let $\varphi \in C^\infty(\mathbb{T})$ be a test function. Following the formal argument of the singular drift case, a process $u \in C(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}))$ is called a *martingale solution* to the stochastic Burgers equation (2.2.1) if the following process is a continuous martingale

$$M_t^\varphi = \varphi(u_t) - \varphi(u_0) - \int_0^t u_s(\Delta\varphi)\,\mathrm{d}s - \int_0^t u_s^2(\partial_x\varphi)\,\mathrm{d}s,$$

with quadratic variation $2t\|\partial_x\varphi\|^2_{L^2}$. Here, the square of u appears, which, as mentioned before, is not well-defined. An idea is to consider a sequence of regularized solutions u^N (e.g. convoluting the original solution against a sequence of mollifiers ρ^N) and to proceed defining the integral $\int_0^t u_s^2(\partial_x\varphi)\,\mathrm{d}s$ as a limit (in probability) as $N\to+\infty$ of the integrals for the regularized solutions $\int_0^t (u_s^N)^2(\partial_x\varphi)\,\mathrm{d}s$. While it is possible to show existence of martingale solutions, the lack of control on the non-linearity does not make it possible to show uniqueness. Gonçalves and Jara [85] introduced the notion of *energy solution* to equation (2.2.1) by requiring further that u satisfies some inequality – called *energy estimate* – allowing them to conclude that $\int_0^\cdot \partial_x u_s^2\,\mathrm{d}s\in C(\mathbb{R}_+,\mathcal{S}'(\mathbb{T}))$ and that the process $\int_0^\cdot \partial_x u_s^2(\varphi)\,\mathrm{d}s$ has zero quadratic variation for every test function φ. There is still no way to compare two energy solutions, but a possible way to get uniqueness is to prove that every energy solution u to the stochastic Burgers equation satisfies the *Cole–Hopf transformation* $u=\partial_x\log v$, where v solves the stochastic heat equation with multiplicative noise

$$\partial_t v(t,x)=\Delta v(t,x)+v(t,x)\,\xi(t,x), \tag{2.2.2}$$

that is a linear SPDE that can be solved in a classical way. In particular, the assumption in the definition of energy solutions about the estimate of the non-linearity makes it possible to get an Itô formula to work with, which would heuristically allow to perform such a Cole-Hopf transformation. But in practice, one needs some extra control on additive functionals of the form $\int_0^t F(u_s)\,\mathrm{d}s$ to get the proper transformation.

Gubinelli and Jara [93] gave a slightly refined notion of energy solution – also called *forward-backward solution* (see e.g. [98]) – as an energy solution u to equation (2.2.1) with the extra requirement that the reversed process $\hat{u}_t=u_{T-t}$ is an energy solution to the equation

$$\partial_t\hat{u}(t,x)=\Delta\hat{u}(t,x)-\partial_x\hat{u}^2(t,x)+\sqrt{2}\partial_x\xi(t,x),$$

with respect to the backward filtration. The authors proved existence of such solutions exploiting an inequality called *Itô trick* (or sometimes *martingale trick*) satisfied by the forward-backward solutions, which allows them to control the mean of the time-integral of the symmetric part of the infinitesimal generator of the process u by means of the initial condition. We will dig deeper on such an inequality in Chapter 3 since, as we will see, it will be crucial for our results on the probabilistic approach to singular SPDEs. Let us mention that the authors also provide existence for a generalized stochastic Burgers equation (and uniqueness for some cases not including the "classical" one given by equation (2.2.1)), and for the hyperviscous stochastic Navier–Stokes equation, that will be one of the subjects of the present thesis.

In particular, the Itô trick allows to control the aforementioned additive functionals $\int_0^t F(u_s)\,\mathrm{d}s$ and therefore to implement the Cole–Hopf transformation for the forward-backward solutions. This also yields an Itô's formula allowing them to map solutions to the stochastic Burgers equation (2.2.1) into solutions to the stochastic heat equation (2.2.2) with multiplicative noise. In this way, uniqueness of forward-backward solutions for the stochastic Burgers equation on the real line and on the torus was proved by Gubinelli and Perkowski in [97].

A further development of this approach to stochastic Burgers equation was given by Gubinelli and Perkowski in [99]. The authors introduced a more generic notion of martingale problem, involving explicitly the requirement that the process solves the Itô trick mentioned above, and based the approach on constructing a suitable domain for the infinitesimal generator $\mathscr{L}$ consisting of distributions and solving the associated Kolmogorov backward equation (see Section 1.2.2), borrowing some ideas from paracontrolled calculus. Notice that this is the same idea exploited above for distributional drift (see Section 2.1). In particular, the authors got existence and uniqueness results for the stochastic Burgers equation on the torus $\mathbb{T}$ and on the whole real line $\mathbb{R}$ without relying on whether the solution satisfies the Cole–Hopf transformation or not. The authors also adapted their approach to solve multi-component stochastic Burgers equation and fractional stochastic Burgers equation. Let us mention that one of the peculiar properties that is exploited by the authors, is that the invariant measure is a Gaussian measure, hinting to the fact that such an approach can be applied to any singular SPDEs with this property. The first main result of the present thesis (see the summary in Chapter 3 and the full article in Chapter A), published in a joint work with Gubinelli [99], is that it is in fact possible to define such a (quasi-)stationary martingale problem (and get existence and uniqueness results) also for hyperviscous stochastic Navier–Stokes equations. Let us also mention the works by Luo and Zhu [133], where the authors partially apply the same approach to study the stochastic modified surface quasi-geostrophic equations.

Chapter 3

Hyperviscous stochastic Navier–Stokes equations with white noise invariant measure

This chapter summarizes the results obtained in the published paper

[100] Massimiliano Gubinelli and Mattia Turra. Hyperviscous stochastic Navier-Stokes equations with white noise invariant measure. *Stoch. Dyn.*, 20(6):2040005–39, 2020.

The full version of the paper can be found in Appendix A.

The research undertaken in the article in question is a collaboration with M. Gubinelli. All the authors have contributed significant parts to each section of the work.

3.1 Introduction

As mentioned in the introduction, the paper [100] is a continuation of the probabilistic approach to singular SPDEs. In particular, it consists in an extension of the results obtained by Gubinelli and Perkowski in [99] to the model of stochastic Navier–Stokes equations in two dimensions. More precisely, the equation under consideration is the following singular SPDE for the velocity field $u = u(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^2$:

$$\begin{aligned} \partial_t u &= -(-\Delta)^\theta u - \lambda u \cdot \nabla u - \nabla p - \sqrt{2}\nabla^\perp(-\Delta)^{\frac{\theta-1}{2}}\xi, \\ \operatorname{div} u &= 0. \end{aligned} \tag{3.1.1}$$

Here, ξ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$, λ is a coupling constant, p is the pressure, $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ and $\theta > 0$ is a positive parameter. Let us remark that the choice of a forcing term given by $\sqrt{2}\nabla^\perp(-\Delta)^{(\theta-1)/2}\xi$ formally guarantees that the spatial white noise μ on $\mathbb{T}^2$, which we can formally write as the *energy measure* $\mu(\mathrm{d}u) = Z^{-1}\exp(-\|u\|_{L^2}^2/2)\,\mathrm{d}u$, where Z is a renormalization constant, is an invariant for the dynamics. In the present work, we stick to the case $\theta > 1$, that is the *hyperviscous* case, but it is worth to mention the results by Cannizzaro and Kiedrowski [43] for the case $\theta \in (0,1]$, where the authors prove that, if $\theta = 1$ and with an appropriate regularization and choice of the coupling constant, the sequence of regularized solutions is tight and the nonlinearity does not vanish in the limit, while for $\theta \in (0,1)$, the large scale behaviour of the velocity is trivial, and the non-linear term vanishes with solution converging to the equation with $\lambda = 0$. However, as far as we understand, the authors do not give any result concerning the analysis of the generator of the limiting process, which instead is the core of our result on uniqueness for the case $\theta > 1$.

Hereafter, we consider the case $\theta > 1$ and $\lambda = 1$, and take the initial condition to be distributed as the energy measure or an L^2-perturbation thereof, namely $u_0 \sim \eta\ \mathrm{d}\mu$ with $\eta \in L^2(\mu)$. Let us point out that the existence of an energy solution to the stationary hyperviscous stochastic Navier-Stokes equation was obtained by Gubinelli and Jara in [93]. Let us also remark that, while here, for the sake of simplicity, the results are presented on the two-dimensional torus $\mathbb{T}^2$ setting, all of them hold true also on the whole $\mathbb{R}^2$ space scenario with some adaptations (see Section A.7 for more details).

3.2 Main results

Let us rewrite equation (3.1.1) in the vorticity form. Let $\omega = \nabla^{\perp} \cdot u$, then ω solves the equation

$$\partial_t \omega = -(-\Delta)^{\theta} \omega - (K * \omega) \cdot \nabla \omega + \sqrt{2} (-\Delta)^{\frac{\theta+1}{2}} \xi, \tag{3.2.1}$$

where K is the *Biot–Savart kernel* on $\mathbb{T}^2$, namely $K(x) = -\sum_{k \in \mathbb{Z}_0^2} 2\pi \mathrm{i} k^{\perp} |2\pi k|^{-2} e^{2\pi \mathrm{i} k \cdot x}$, with $\mathbb{Z}_0^2 = \mathbb{Z}^2 \backslash \{0\}$. Notice that $u = K * \omega$ and therefore the incompressibility condition (i.e. $\operatorname{div} u = 0$) is already included in the new formulation (3.2.1). We also use the notation $B(\omega) = (K * \omega) \cdot \nabla \omega$ for the non-linearity. With such a formulation for the equation, we have that the energy measure

$$\mu(\mathrm{d}\omega) = Z^{-1} e^{-\frac{1}{2} \|(-\Delta)^{-1/2} \omega\|_{L^2}^2} \mathrm{d}\omega,$$

where Z is a renormalization constant, is an invariant measure for the equation, and it characterizes the law of the solution ω as a centred Gaussian process indexed by $H_0^1(\mathbb{T}^2) = \{\psi \in H^1(\mathbb{T}^2) \colon \hat{\psi}(0) = 0\}$, i.e. if we start the equation with distribution μ, then ω is a centred Gaussian process (see Section 1.4) with covariance

$$\mathbb{E}[\omega(f)\, \omega(g)] = \langle f, g \rangle_{H_0^1}, \qquad f, g \in H_0^1(\mathbb{T}^2).$$

The first problem to be tackled is the definition of solution to the martingale problem associated with equation (3.2.1), indeed, as we mentioned in the introduction of the present thesis for the case of stochastic Burgers equations, it would involve the requirement that the process

$$M_t^{\varphi} = \varphi(\omega_t) - \varphi(\omega_0) - \int_0^t \mathscr{L} \varphi(\omega_s)\, \mathrm{d}s, \qquad t \geqslant 0,$$

is a continuous martingale with quadratic variation $\int_0^t \|(-\Delta)^{(\theta+1)/2} D\varphi(\omega_s)\|_{L^2(\mathbb{T}^2)}^2 \mathrm{d}s$. Here, $\mathscr{L}$ should correspond to the infinitesimal generator of equation (3.2.1). Let us consider a cylinder function φ of the form $\varphi(\omega) = \Phi(\omega(f_1), \ldots, \omega(f_n))$, where Φ is a C_b^2-function and $f_1, \ldots, f_n$ are smooth (see Definition 1.4.49). Then, if ω is a solution to equation (3.2.1), we have by Itô formula

$$\mathrm{d}\varphi(\omega_t) = \mathscr{L} \varphi(\omega_s)\, \mathrm{d}s + \mathrm{d}M_t^{\varphi},$$

where $\mathscr{L} = \mathscr{L}_{\theta} + \mathscr{G}$ is given by

$$\begin{aligned} \mathscr{L}_{\theta} \varphi(\omega) &= -\sum_{i=1}^n \partial_i \Phi(\omega(f_1), \ldots, \omega(f_n))\, \omega((-\Delta)^{\theta} f_i) + \sum_{i,j=1}^n \partial_{ij}^2 \Phi(\omega(f_1), \ldots, \omega(f_n))\, \langle (-\Delta)^{\theta+1} f_i, f_j \rangle, \\ \mathscr{G} \varphi(\omega) &= -\sum_{i=1}^n \partial_i \Phi(\omega(f_1), \ldots, \omega(f_n))\, \langle (K * \omega) \cdot \nabla \omega, f_i \rangle. \end{aligned} \tag{3.2.2}$$

Notice that the non-linear operation appearing in (3.2.2) yields problems of ill-definition for the term $\int_0^t \mathscr{L} \varphi(\omega_s)\, \mathrm{d}s$.

Introducing a Galerkin approximation $B^m(\omega)$ for the non-linear term, which essentially projects on a finite number of order m of Fourier modes the non-linearity, it is possible to define the integral $\int_0^t \mathscr{L} \varphi(\omega_s)\, \mathrm{d}s$ for cylinder functions φ as a limit as $m \to +\infty$, provided that we assume that the process ω starting at $\omega_0 \sim \eta\, \mathrm{d}\mu$, for $\eta \in L^2(\mu)$, is *incompressible*, i.e., for any cylinder function φ, we have

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}|\varphi(\omega_t)| \lesssim_T \|\varphi\|_{L^2(\mu)},$$

and that it satisfies the *Itô trick*, namely that, for any cylinder function φ and $p\geqslant 1$, we have

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \varphi(\omega_s)\,\mathrm{d}s\right|^p \lesssim (T^{p/2}\vee T^p)\|c_{2p}^{\mathcal{N}}(1-\mathcal{L}_\theta)^{-1/2}\varphi\|_{L^2(\mu)}^p, \tag{3.2.3}$$

where $\mathcal{L}_\theta$ is the infinitesimal generator corresponding to the linearized equation and $\mathcal{N}$ is the *number* (or *Ornstein–Uhlenbeck*) *operator* (see Example 1.2.5 and Section 1.4.6). The aforementioned assumptions become then part of the problem, but it is worth to notice that they are quite natural, in the sense that they are satisfied by the solution ω^m to the approximating problem.

Introducing the Fock space representation $\mathcal{H}=\Gamma H_0^1(\mathbb{T}^2)$ of the space $L^2(\mu)$ (see Lemma A.2.5 and also Section 1.4.5), we can get some a priori estimates for the approximating operator $\mathcal{L}^m$:

Lemma 3.2.1. (see Lemma A.2.7) *Let $w:\mathbb{N}_0\to\mathbb{R}_+$ and $\varphi\in\mathcal{H}$. The following m-dependent bound holds:*

$$\|w(\mathcal{N})\mathcal{G}^m\varphi\|_{\mathcal{H}}\lesssim m\|(w(\mathcal{N}+1)+w(\mathcal{N}-1))(1+\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi\|_{\mathcal{H}}.$$

Moreover, uniformly in m, we have

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}_+^m\varphi\|_{\mathcal{H}}\lesssim\|w(\mathcal{N}+1)(1+\mathcal{N})(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma}\varphi\|_{\mathcal{H}},\quad \textit{for all } \gamma>\frac{1}{2\theta},$$

and

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}_-^m\varphi\|_{\mathcal{H}}\lesssim\|w(\mathcal{N}-1)\mathcal{N}^{3/2}(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma}\varphi\|_{\mathcal{H}},\quad \textit{for all } \gamma<\frac{1}{2}.$$

By the previous lemma, it is possible to show that, while we have that $\mathcal{L}^m\varphi\notin\mathcal{H}$, it does hold that $(1-\mathcal{L}_\theta)^{1/2}\mathcal{L}^m\varphi\in\mathcal{H}$, and we are therefore able to exploit the incompressibility of ω and the Itô trick to bound $\int_0^t\mathcal{L}^m\varphi(\omega_s)\,\mathrm{d}s$ and pass to the limit as $m\to+\infty$. Such a reasoning brings us in position to give a meaningful definition of the martingale problem.

Definition 3.2.2. (see Definition A.3.2) *A process $(\omega_t)_{t\geqslant 0}$ with trajectories in $C(\mathbb{R}_+;\mathcal{S}')$ solves the* cylinder martingale problem for $\mathcal{L}$ with initial distribution ν *if $\omega_0\sim\nu$ and if the following conditions are satisfied:*

i. $(\omega_t)_t$ is incompressible,

ii. the Itô trick (3.2.3) works,

iii. for any cylinder function φ, the process

$$M_t^\varphi=\varphi(\omega_t)-\varphi(\omega_0)-\int_0^t\mathcal{L}\varphi(\omega_s)\,\mathrm{d}s,\qquad t\geqslant 0,$$

is a continuous martingale with quadratic variation $\langle M^\varphi\rangle_t=\int_0^t\mathcal{E}(\varphi)(\omega_s)\,\mathrm{d}s$, where $\mathcal{E}(\varphi)=2\int_{\mathbb{T}^2}|(-\Delta)_x^{(\theta+1)/2}\mathrm{D}_x\varphi|^2\,\mathrm{d}x$ and D denotes the Malliavin derivative (see Definition 1.4.50).

The proof of existence of a solution to the cylinder martingale problem with initial condition $\omega_0\sim\eta\,\mathrm{d}\mu$, for $\eta\in L^2(\mu)$, follows a tightness argument on the sequence $(\omega^m)_{m\in\mathbb{N}}$ of solutions to the approximating problems (see the proof of Theorem A.3.3).

As far as uniqueness is concerned, we exploit a duality argument with the Kolmogorov backward equation (recall Section 1.2.2). Suppose that there is a domain $D(\mathcal{L})\subset\mathcal{H}$ for the operator $\mathcal{L}$ such that $(\mathcal{L},D(\mathcal{L}))$ is a densely defined dissipative operator on $\mathcal{H}$ and that there exists a unique solution $\varphi\in C(\mathbb{R}_+,D(\mathcal{L}))\cap C^1(\mathbb{R}_+,\mathcal{H})$ to the Kolmogorov backward equation

$$\partial_t\varphi(t)=\mathcal{L}\varphi(t), \tag{3.2.4}$$

for any initial condition φ in a dense subset $\mathscr{U}\subseteq\mathscr{H}$. Then $(\mathscr{L},D(\mathscr{L}))$ is closable and its closure is the unique extension of $\mathscr{L}$ being the generator of a strongly continuous semigroup of contractions $(T_t)_{t\geqslant 0}$ and moreover $\varphi(t)=T_t\varphi$, for any $\varphi\in\mathscr{U}$ (see Proposition 1.2.14).

Moreover, if $\varphi\in C(\mathbb{R}_+,D(\mathscr{L}))\cap C^1(\mathbb{R}_+,\mathscr{H})$ and ω solves the cylinder martingale problem for $\mathscr{L}$, then also the process

$$\varphi(t,\omega_t)-\varphi(0,\omega_0)-\int_0^t(\partial_s+\mathscr{L})\varphi(s,\omega_s)\,\mathrm{d}s,\qquad t\geqslant 0,$$

is a martingale (see Lemma A.4.2). The proof of uniqueness of solutions to the cylinder martingale problem consists then in determining uniquely the n-point marginals via the initial condition, for instance, for the case $n=2$ we have

$$\mathbb{E}[\varphi(\omega_t)\,\psi(\omega_s)]=\mathbb{E}\left[\left(T_{t-s}\varphi(\omega_s)+\int_0^t(\partial_r+\mathscr{L})T_{t-r}\varphi(\omega_r)\,\mathrm{d}r\right)\psi(\omega_s)\right],$$

and exploiting the incompressibility of ω and the fact that $T_t\varphi$ solves the Kolmogorov backward equation (3.2.4) with initial condition φ, we have that $\int_0^t(\partial_r+\mathscr{L})T_{t-r}\varphi(\omega_r)\,\mathrm{d}r=0$, and hence

$$\mathbb{E}[\varphi(\omega_t)\psi(\omega_s)]=\mathbb{E}[T_{t-s}\varphi(\omega_s)\psi(\omega_s)].$$

By induction on n we can then determine uniquely the law of ω.

Theorem 3.2.3. (see Theorems A.3.3 and A.4.3) *There exists a unique solution ω to the cylinder martingale problem for $\mathscr{L}$ with initial distribution $\omega_0\sim\eta\,\mathrm{d}\mu$ with $\eta\in L^2(\mu)$. Moreover, ω is a homogeneous Markov process with transition kernel $(T_t)_{t\geqslant 0}$ and with invariant measure μ.*

In order to complete the (heuristic) proof of the theorem, we should then determine the domain $D(\mathscr{L})$ of $\mathscr{L}$ and prove existence and uniqueness for the Kolmogorov backward equation (3.2.4). Let us start with the former problem. We have to determine a dense domain $D(\mathscr{L})$ for $\mathscr{L}$ such that $\mathscr{L}\varphi\in\mathscr{H}$, whenever $\varphi\in D(\mathscr{L})$. The method follows the idea of the finite-dimensional scenario (see Section 2.1), namely it is based on the idea of solving the resolvent equation (cf. Section 1.2.1)

$$(\lambda-\mathscr{L})\varphi=F,$$

for some regular F. So far, we only dealt with the regularity of the non-linear term $\mathscr{G}$ with respect to the linear part $\mathscr{L}_\theta$, but we also need to take care of the behaviour of $\mathscr{G}$ with respect to the number operator $\mathscr{N}$. In order to deal with this problem, we exploit the decomposition

$$\mathscr{G}=\mathbf{1}_{|\mathscr{L}_\theta|\gtrsim\mathscr{N}^{\alpha(\theta)}}\mathscr{G}+\mathbf{1}_{|\mathscr{L}_\theta|\lesssim\mathscr{N}^{\alpha(\theta)}}\mathscr{G}=:\mathscr{G}^{\succ}+\mathscr{G}^{\prec},$$

where $\alpha(\theta)$ is some parameter depending on $\theta>1$ (for more details, see Section A.5). Notice that the term $\mathscr{G}^{\succ}$ contains all the non-regular part with respect to $\mathscr{N}$.

Let $w(\mathscr{N})$, for $w\colon\mathbb{N}_0\to\mathbb{R}_+$ be some positive weight. We then proceed introducing an auxiliary problem allowing us to treat $\mathscr{G}^{\succ}$ and $\mathscr{G}^{\prec}$ separately. First of all, it is possible to show the following bound for $\mathscr{G}^{\succ}$ (see Lemma A.5.4): For $L\geqslant 1$, $\bar{\varepsilon}\in(0,c(\theta))$, where $c(\theta)$ is some positive constant depending on $\theta>1$, we have

$$\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-1/2}\mathscr{G}^{\succ}\psi\|_{\mathscr{H}}\lesssim|w|L^{-c(\theta)+\bar{\varepsilon}}\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{1/2}\psi\|_{\mathscr{H}}.$$

Therefore, if we consider $\varphi^\sharp\in w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H}$ (and take $L>0$ large enough), we can exploit (cf. Lemma A.5.4) a fixed point theorem argument to conclude that there exists a unique solution $\varphi=\mathscr{K}\varphi^\sharp\in w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H}$ to the following equation:

$$\varphi=\varphi^\sharp+(1-\mathscr{L}_\theta)^{-1}\mathscr{G}^{\succ}\varphi. \tag{3.2.5}$$

Moreover (see Proposition A.5.6), if w is a polynomial weight, we have the estimate

$$\|w(\mathscr{N})(1-\mathscr{L}_\theta)^\gamma \mathscr{G}^\prec \varphi\|_{\mathscr{H}} \lesssim \|w(\mathscr{N})(1+\mathscr{N})^{\beta(\theta)}(1-\mathscr{L}_\theta)^{1/2}\varphi^\sharp\|_{\mathscr{H}},$$

where $\beta(\theta)=(5\theta-2)/(2\theta-2)$, and hence, if $\varphi=\mathscr{K}\varphi^\sharp$, for

$$\varphi^\sharp \in w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1}\mathscr{H} \cap w(\mathscr{N})^{-1}(1+\mathscr{N})^{-\beta(\theta)}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H},$$

we then have $\mathscr{L}\varphi \in w(\mathscr{N})^{-1}\mathscr{H}$, where we exploit the relation $(1-\mathscr{L})\varphi=(1-\mathscr{L}_\theta)\varphi^\sharp+\mathscr{G}^\prec\varphi$.

Such results give us a definition (see Lemma A.5.7) of a dense domain for the operator $\mathscr{L}$ as

$$D(\mathscr{L}) := \{\mathscr{K}\varphi^\sharp : \varphi^\sharp \in (1-\mathscr{L}_\theta)^{-1}\mathscr{H} \cap (\mathscr{N}+1)^{-\beta(\theta)}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H}\}.$$

Moreover, it is easy to show, exploiting an adjointness relation (modulo a sign) of $\mathscr{G}_+$ and $\mathscr{G}_-$ (see equation (A.2.6)), that $(D(\mathscr{L}),\mathscr{L})$ is a dissipative operator (see Lemma A.5.8).

We can then consider the Kolmogorov backward equation (3.2.4). In view of the representation of $\varphi=\mathscr{K}\varphi^\sharp$ via $\varphi^\sharp$ and the decomposition of $\mathscr{G}$, we can rewrite equation (3.2.4) as follows:

$$\begin{aligned}\partial_t\varphi^\sharp+(1-\mathscr{L}_\theta)\varphi^\sharp &= \mathscr{L}\varphi+(1-\mathscr{L}_\theta)\varphi^\sharp-(1-\mathscr{L}_\theta)^{-1}\mathscr{G}^\succ\partial_t\varphi\\ &= \varphi+\mathscr{G}^\prec\varphi-(1-\mathscr{L}_\theta)^{-1}\mathscr{G}^\succ(\varphi+\mathscr{G}^\prec\varphi-(1-\mathscr{L}_\theta)\varphi^\sharp).\end{aligned}$$

Working with the aforementioned Galerkin approximation, it is possible to exploit some Schauder's estimates and the estimates on $\mathscr{G}^\prec$ and $\mathscr{G}^\succ$ to get some bounds on the right-hand side of the rewriting by means of $\varphi_0^{m,\sharp}$ corresponding to the initial condition $\varphi_0^m=\mathscr{K}^m\varphi_0^{m,\sharp}$ (see Section A.5.4). By doing so, we can then determine a dense set $\mathscr{U}\subseteq\mathscr{H}$ for φ_0, allowing us to show compactness of the sequence $(\varphi^{m,\sharp})_m$ of solutions to the Kolmogorov backward equation (3.2.4) with initial condition $\varphi_0^m=\mathscr{K}^m\varphi_0^\sharp$, where $\varphi_0^\sharp=\mathscr{K}^{-1}\varphi_0$, and whose limit $\varphi^\sharp$ (along a converging subsequence) solves the Kolmogorov backward equation (3.2.4) with initial condition $\varphi_0^\sharp$. Uniqueness of solutions to equation (3.2.4) follows then by the dissipativity of the operator $(D(\mathscr{L}),\mathscr{L})$. (See Theorem A.5.10).

3.3 Further developments

In this section, we want to give some possible developments of the martingale approach adopted in this part of the thesis to analyze singular SPDEs. In particular, as we mentioned before, one of the peculiarities exploited in the paper [100] is that the considered model has Gaussian invariant measure, which is crucial e.g. in order to exploit the chaos decomposition (cf. Section 1.4.5). Taking inspiration by the works by Jara and Menezes [113, 114] on non-equilibrium fluctuations of interacting particle systems, we try to deal with KPZ equation in dimension three and higher by considering an approximation of the invariant measure that allows us to "control" the law of the stationary process with the new approximation by means of some relative entropy estimate. Let us point out that most of the content of this section is to be considered at a heuristic level.

Consider the KPZ equation in dimensions $d\geqslant 3$, namely the following equation on $\mathbb{R}_+\times\mathbb{T}_L^d$, where $\mathbb{T}_L^d=\mathbb{R}^d/(2\pi L\mathbb{Z})^d$ and $L\in\mathbb{N}$,

$$\partial_t h=\nu\Delta h+\lambda|\nabla h|^2+\sqrt{\sigma}\xi, \tag{3.3.1}$$

where ξ is a space-time white noise, while ν, λ and $\sigma>0$ are constants called *viscosity*, *deposition rate*, and *noise strength*, respectively. Recall that, in the case $d\geqslant 3$, the invariant measure of the equation is not known a priori. Introduce the rescaling

$$h^\varepsilon(t,x)=\varepsilon^{-\frac{d-2}{2}}h(\varepsilon^{-2}t,\varepsilon^{-1}x),$$

then h^ε solves the equation

$$\partial_t h^\varepsilon = \nu\Delta h^\varepsilon + \varepsilon^{\frac{d-2}{2}}\lambda|\nabla h^\varepsilon|^2 + \sqrt{\sigma}\,\xi^\varepsilon, \tag{3.3.2}$$

where $\xi^\varepsilon(t,x) = \varepsilon^{-\frac{d+2}{2}}\xi(\varepsilon^{-2}t,\varepsilon^{-1}x)$ is a space-time white noise itself. With the same computations as in Chapters 2 and 3, we can write the generator of the rescaled equation (3.3.2) for some cylinder function $\varphi(h) = \Phi(h(f_1),\ldots,h(f_n))$ as

$$\mathscr{L}^\varepsilon\varphi(h) = \mathscr{L}_0\varphi(h) + \mathscr{G}^\varepsilon\varphi(h),$$

where now

$$\mathscr{L}_0\varphi(h) = \nu\sum_{i=1}^n \partial_i\Phi(h(f_1),\ldots,h(f_n))\,h(\Delta f_i) + \frac{\sigma}{2}\sum_{i,j=1}^n \partial_{ij}^2\Phi(h(f_1),\ldots,h(f_n))\,\langle f_i,f_j\rangle,$$

and

$$\mathscr{G}^\varepsilon\varphi(h) = \varepsilon^{\frac{d-2}{2}}\lambda\sum_{i=1}^n \partial_i\Phi(h(f_1),\ldots,h(f_n))\langle f_i,|\nabla h|^2\rangle.$$

The martingale problem associated to $\mathscr{L}^\varepsilon$ then would involve the requirement that the process

$$M_t^\varepsilon(\varphi) = \varphi(h_t^\varepsilon) - \varphi(h_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon\varphi(h_s^\varepsilon)\,\mathrm{d}s$$

is a martingale with quadratic variation $\int_0^t \|\mathrm{D}\varphi(h_s^\varepsilon)\|_{L^2(\mathbb{T}^d)}\,\mathrm{d}s$.

It is well-known – at least in the case of a white noise regularized in the space variable – (see e.g. [50, 65, 137]) that the solution to equation (3.3.2) converges in distribution, as $\varepsilon\to 0$, to the one of the *Edwards–Wilkinson model*, that is the linear SPDE given by

$$\partial_t X = \nu_{\mathrm{eff}}\,\Delta X + \sqrt{\sigma_{\mathrm{eff}}}\,\xi, \tag{3.3.3}$$

where ν_{eff} and σ_{eff} are some renormalized coefficients. It is worth to notice that equation (3.3.3) has an invariant measure given by a Gaussian measure μ_{eff} on $\mathcal{S}'(\mathbb{T}^d)$ with covariance $\alpha_{\mathrm{eff}}(-\Delta)^{-1}$ with $\alpha_{\mathrm{eff}} = \sigma_{\mathrm{eff}}\,(2\nu_{\mathrm{eff}})^{-1}$. Moreover, let us denote by μ_{lin} the invariant measure of the linear part of equation (3.3.2), which is a Gaussian measure with covariance $\alpha(-\Delta)^{-1}$ with $\alpha = \sigma\,(2\nu)^{-1}$. An idea is then to consider a Gaussian measure of the form

$$\mu_t(\mathrm{d}h) = Z^{-1}\exp\left(-\frac{1}{2}\|\Sigma(t,\mathrm{D})h\|_{L^2}^2\right)\mathrm{d}h,$$

where Σ is to be determined in such a way that μ interpolates μ_{eff} and μ_{lin} in some sense, and D denotes the derivative operator. The measure μ satisfies the integration by parts formula

$$\int \mathrm{D}F(h)\,\mu_t(\mathrm{d}h) = \int \Sigma(t,\mathrm{D})^2\,hF(h)\,\mu_t(\mathrm{d}h).$$

In order to prove tightness of the solutions to the martingale problem, we need to obtain a bound on the term $\mathbb{E}_{\mu^\varepsilon}[|\varphi(h_t^\varepsilon) - \varphi(h_s^\varepsilon)|^p]$, and in particular on the term involving the non-linear part $\mathscr{G}^\varepsilon$ of the infinitesimal generator, that is

$$\mathbb{E}\left[\left|\int_s^t \mathscr{G}^\varepsilon\varphi(h_\tau^\varepsilon)\,\mathrm{d}\tau\right|^p\right].$$

Let us first introduce the notion of relative entropy, that is, if μ is a measure and f is a density with respect to that measure, we let

$$H(f;\mu) := \int f\log(f)\,\mathrm{d}\mu.$$

In particular, we have the following estimate

$$\mathbb{P}\left(\int_s^t \mathscr{G}^\varepsilon\varphi(h_\tau^\varepsilon)\,\mathrm{d}\tau > l\right) \leqslant \frac{H(f_s^\varepsilon;\mu_s^\varepsilon)+C}{\log \mathbb{P}^{\mu_\varepsilon^s}(\int_0^{t-s}\mathscr{G}^\varepsilon\varphi(h_\tau^\varepsilon)\,\mathrm{d}\tau > l)^{-1}},$$

where f_s^ε is the density of the law of h_s^ε with respect to μ_s^ε.

Lemma 3.3.1. *Let $V\colon[0,T]\times\Omega\to\mathbb{R}$ be a bounded function and assume $h_0^\varepsilon\sim\mu_0^\varepsilon$. Then*

$$\begin{aligned}&\log \mathbb{E}^{\mu_0^\varepsilon}\left[\exp\left\{\int_0^T V_t(h_t^\varepsilon)\,\mathrm{d}t\right\}\right]\\ &\leqslant \int_0^T \sup_f\left\{-\int \Gamma^\varepsilon\sqrt{f}\,\mathrm{d}\mu_t^\varepsilon + \int\left(V_t+\frac{1}{2}(\mathscr{L}^\varepsilon)_t^*\mathbb{1}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log\psi_t^\varepsilon\right)f\,\mathrm{d}\mu_t^\varepsilon\right\}\mathrm{d}t,\end{aligned}\tag{3.3.4}$$

where the supremum is taken over all densities f with respect to μ_t^ε.

Let us write then $\mathscr{A}\varphi(h)=\varepsilon^{-\frac{d-2}{2}}\lambda^{-1}\mathscr{G}^\varepsilon\varphi(h)$. Then, Chebyshev inequality (namely the following relation: $\mathbb{P}(w\geqslant\varepsilon)=\mathbb{P}(e^{wv}\geqslant e^{\varepsilon v})\leqslant\mathbb{E}[e^{v(w-\varepsilon)}]$) and the variational inequality (3.3.4) with μ_{s+t}^ε instead of μ_t^ε (notice that the initial law in the present setting is μ_s^ε), we have

$$\begin{aligned}&\log \mathbb{P}^{\mu_s^\varepsilon}\left(\int_0^{t-s}\mathscr{A}(h_\tau^\varepsilon)\,\mathrm{d}\tau > l\lambda^{-1}\varepsilon^{-\frac{d-2}{2}}\right)\\ &\leqslant \log\mathbb{E}^{\mu_s^\varepsilon}\left[\exp\left\{\int_0^{t-s}\mathscr{A}(h_\tau^\varepsilon)\,\mathrm{d}\tau - l\lambda^{-1}\varepsilon^{-\frac{d-2}{2}}\right\}\right]\\ &\leqslant \int_0^{t-s}\sup_f\left\{-\int\left\|\mathrm{D}\sqrt{f}\right\|^2\mathrm{d}\mu_{s+\tau}^\varepsilon+\int\mathscr{A}(h_{s+\tau}^\varepsilon)f\,\mathrm{d}\mu_{s+\tau}^\varepsilon+\right.\\ &\qquad\left.+\frac{1}{2}\int\mathscr{J}_{s+\tau}^\varepsilon f\,\mathrm{d}\mu_{s+\tau}^\varepsilon\right\}\mathrm{d}\tau - l\lambda^{-1}\varepsilon^{-\frac{d-2}{2}}\end{aligned}\tag{3.3.5}$$

where the supremum is taken over all densities f with respect to $\mu_{s+\tau}^\varepsilon$, and

$$\mathscr{J}_t^\varepsilon := (\mathscr{L}^\varepsilon)_t^*\mathbb{1}-\frac{\mathrm{d}}{\mathrm{d}t}\log\psi_t^\varepsilon,\tag{3.3.6}$$

where $(\mathscr{L}^\varepsilon)_t^*$, the adjoint of $\mathscr{L}^\varepsilon$, is obtained from the relation

$$\int(\mathscr{L}^\varepsilon)_t^*\mathbb{1}f\,\mathrm{d}\mu_t^\varepsilon=\int\mathscr{L}^\varepsilon f\,\mathrm{d}\mu_t^\varepsilon,$$

by integration by parts, and ψ_t^ε is the Radon–Nikodym derivative (see e.g. Section 14.13 in [183]) of μ_t^ε with respect to the reference measure ν such that μ_t^ε is absolutely continuous with respect to ν for every $t\geqslant 0$.

Let us mention that, as far as we understand, the irregularity of the non-linear term does not allow to get a meaningful estimate in the present scenario, since, in particular, it makes a third order term appear in the adjoint of $\mathscr{L}^\varepsilon$. A solution could be to consider a non-linearity given by a bounded function of ∇h that rescales in the same way as described above and that allows us to get some useful estimates.

Part II

Differential characterization of quantum field theories

Chapter 4
Introduction

Singular SPDEs find applications also in constructive quantum field theory via the stochastic quantization procedure of studying the measure related to a quantum field as the invariant measure of a SPDE. In this chapter, we give a brief, heuristic background on quantum field theory, introducing the reader to the approach adopted in the paper [61] (see also its summary in Chapter 5). We refer the reader to the references [11, 21, 84, 170, 173, 174] as well as to the works mentioned throughout the chapter for a more detailed and complete vision of the subjects presented here.

4.1 Constructive and Euclidean quantum field theory

Building a model of quantum field theory (QFT) is equivalent to building some Hilbert space H, equipped with an inner product $(\cdot,\cdot)_H$, and a positive, self-adjoint, densely defined operator $\mathscr{H}$ with unique ground state $\Omega\in H$, namely Ω is the unique (up to multiplication by a scalar) eigenvector of the lowest eigenvalue of $\mathscr{H}$, invariant with respect to the Poincaré group. Restricting to the case of scalar bosonic fields and adopting an heuristic viewpoint, a quantum field is a map φ from $\mathbb{R}^{1+d}$ to the space of self-adjoint operators on H (more precisely, φ is a map from the space $\mathcal{S}(\mathbb{R}^{1+d})$ of Schwartz functions on $\mathbb{R}^{1+d}$ to the space of self-adjoint operator on H) such that, for $(t_1,x_1),(t_2,x_2)\in\mathbb{R}^{1+d}$, $\varphi(t_1,x_1)$ commutes with $\varphi(t_2,x_2)$ whenever (t_1-t_2,x_1-x_2) is a space-type vector in the Minkowski metric. Furthermore, the field φ is related to the Hamiltonian $\mathscr{H}$ in the following way:

$$\varphi(t+s,x)=e^{-i\mathscr{H}s}\,\varphi(t,x)\,e^{i\mathscr{H}s},$$

where $e^{i\mathscr{H}s}$ is the unitary semigroup on H generated by $\mathscr{H}$.

The most famous formulation of rigorous constructive QFT is due to Wightman, who describes the theory not in terms of the aforementioned objects but exploiting *Wightman functions* (more precisely, Wightman distributions), which, sticking to the previous notation, are defined as

$$\begin{aligned} W_n((t_1,x_1),\ldots,(t_n,x_n)) &= (\varphi(t_n,x_n)\cdots\varphi(t_1,x_1)\Omega,\Omega)_H \\ &= (e^{-i\mathscr{H}t_n}\varphi(0,x)\,e^{i\mathscr{H}(t_n-t_{n-1})}\varphi(0,x)\cdots\varphi(0,x)\,e^{i\mathscr{H}t_1}\Omega,\Omega)_H, \end{aligned}$$

for $n\in\mathbb{N}$ and $(t_1,x_1),\ldots,(t_n,x_n)\in\mathbb{R}^{1+d}$. A QFT is then built from a set of Wightman functions W_n satisfying some conditions: the *Wightman axioms* (see [173]). Let us mention that the Wightman axioms do not identify uniquely a particular QFT. Instead, different QFTs are expected to satisfy different *Dyson–Schwinger equations*, namely a system of partial differential equations relating Wightman functions and encoding the local and hyperbolic equations of motions for the quantum fields [60, 167].

A groundbreaking progress in the understanding of constructive Wightman QFT was to switch the focus on the analytic continuations on imaginary time of Wightman functions, namely the *Schwinger functions*. These are given by

$$\begin{aligned} S_n((s_1,x_1),\ldots,(s_n,x_n)) &= W_n((is_1,x_1),\ldots,(is_n,x_n)) \\ &= (e^{\mathscr{H}s_n}\varphi(0,x)\,e^{-\mathscr{H}(s_n-s_{n-1})}\varphi(0,x)\cdots\varphi(0,x)\,e^{-\mathscr{H}s_1}\Omega,\Omega)_H, \end{aligned}$$

which is well-defined for $s_1 \leqslant \cdots \leqslant s_n$ since $\mathscr{H}$ is positive and self-adjoint, and Ω is an invariant vector for the related semigroup. Such objects have then to satisfy analogous conditions to the Wightman axioms, namely the *Osterwalder–Schrader axioms* (see [152, 153]).

It was Nelson [144, 145, 146] who observed and exploited for the first time the fact that, in many cases (among which the scalar bosonic field mentioned above), the Schwinger functions can be built as moments of a (classical) random field defined on $\mathbb{R}^{1+d}$, namely there is a random field ϕ having measure μ such that

$$S_n((s_1,x_1),\ldots,(s_n,x_n)) = \mathbb{E}_\mu[\phi(s_1,x_1),\ldots,\phi(s_n,x_n)].$$

The measure μ is related to the physical structure of the problem in the following heuristic way: the measure μ is the *Gibbs measure* associated with the classical action of the quantum field φ, that in the scalar bosonic case is given by

$$\mu(\mathrm{d}\phi) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}^{1+d}} (|\nabla\phi|^2 + m^2\phi^2 + 2V(\phi))\,\mathrm{d}y\right)\mathscr{D}\phi, \tag{4.1.1}$$

where $\mathscr{D}\phi$ is a (non-existing) Lebesgue measure on the space of tempered distributions $\mathscr{S}'(\mathbb{R}^{1+d})$ and V is a regular function called *self-interactive potential*. The functional

$$S(\phi) = \frac{1}{2}\int_{\mathbb{R}^{1+d}} (|\nabla\phi|^2 + m^2\phi^2 + 2V(\phi))\,\mathrm{d}y$$

is the *action functional* describing the classical limit of the quantum field φ.

The problem now becomes to give a rigorous meaning to the expression (4.1.1). In the non-interacting case, i.e. when $V \equiv 0$, the measure $\mu = \mu^{\mathrm{free}}$ is the Gaussian measure on $\mathscr{S}'(\mathbb{R}^{1+d})$ with Cameron-Martin space $H^1(\mathbb{R}^{1+d})$ (cf. Section 1.4) equipped with the norm

$$\frac{1}{2}\int_{\mathbb{R}^{1+d}} (|\nabla\phi|^2 + m^2\phi^2)\,\mathrm{d}y,$$

The random field ϕ associated with the measure $\mu = \mu^{\mathrm{free}}$ is the so-called Gaussian free field (compare with Example 1.4.19).

In the interactive case, the measure μ can then be written in terms of μ^{free} as follows:

$$\mu(\mathrm{d}\phi) = Z^{-1}\exp\left(-\int_{\mathbb{R}^{1+d}} V(\phi)\,\mathrm{d}y\right)\mu^{\mathrm{free}}(\mathrm{d}\phi),$$

where Z is a renormalization constant.

For $d \geqslant 1$, the measure μ^{free} is supported on a space of proper distributions growing at infinity, making the integral in the exponential of (4.1.1) meaningless. The main idea is then to consider a sequence of mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$ such that $\rho_\varepsilon \to \delta$ as $\varepsilon \to 0$, where δ is the Dirac delta, a sequence of compactly supported cut-off function $(f_\varepsilon)_{\varepsilon>0}$ such that $f_\varepsilon \to 1$ as $\varepsilon \to 0$, and define the measure μ as the limit as $\varepsilon \to 0$ of the measures

$$\mu_\varepsilon(\mathrm{d}\phi) = Z_\varepsilon^{-1}\exp\left(-\int_{\mathbb{R}^{1+d}} f_\varepsilon V_\varepsilon(\rho_\varepsilon * \phi)\,\mathrm{d}y\right)\mu^{\mathrm{free}}(\mathrm{d}\phi), \tag{4.1.2}$$

where V_ε is a (diverging) modification of the self-interactive potential V for which it holds that the limit $\lim_{\varepsilon\to 0} V_\varepsilon(\rho_\varepsilon * \phi)$ converges to some distribution μ^{free}-a.s. It is worth to mention that such a modification V_ε of V is usually called *renormalized potential*, and the previous approximation procedure is referred to as *renormalization procedure* (see [31]). With this method, it is possible to define the measure μ through the limit as $\varepsilon \to 0$.

One of the methods adopted to implement the technique just described is called *stochastic quantization* and relies on studying the measure μ_ε defined in equation (4.1.2) as an invariant measure of the following SPDE, called *stochastic quantization equations* (cf. Section 1.1):

$$\partial_t \phi_\varepsilon(t,x) = -\frac{\delta S_\varepsilon}{\delta\phi}(\phi_\varepsilon(t,x)) + \xi(t,x), \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \tag{4.1.3}$$

with ξ denoting a space-time white noise (see Example 1.4.18) and S_ε being given by

$$S_\varepsilon(\phi) = \frac{1}{2}\int_{\mathbb{R}^{1+d}} (|\nabla\phi|^2 + m^2\phi^2 + 2f_\varepsilon V_\varepsilon(\rho_\varepsilon * \phi))\,\mathrm{d}y, \tag{4.1.4}$$

where $\delta S_\varepsilon/\delta\phi$ denotes the functional derivative of the approximating action S_ε with respect to ϕ. The aim of this approach is then to show the existence of a limit $\phi = \lim_{\varepsilon\to 0}\phi_\varepsilon$ solving some SPDE

$$\partial_t \phi(t,x) = -\frac{\delta S}{\delta\phi}(\phi(t,x)) + \xi(t,x), \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d,$$

and such that its invariant measure satisfies the physical properties of quantum field measures (e.g. the Osterwalder–Schrader axioms). We refer the reader to the references mentioned in Section 1.1 on stochastic quantization or more details and recent results on this very interesting subject.

4.2 Differential non-perturbative approach to EQFT

Let us mention that the procedure described in Section 4.1 is not the only possible way of characterizing a measure. Indeed, an alternative approach comes from the study of *integration by parts formulae*. More precisely, consider a cylinder function $F(\phi)$ of the form $F(\phi) = \tilde{F}(\langle f_1, \phi\rangle, \ldots, \langle f_n, \phi\rangle)$ for some smooth and bounded function $\tilde{F}$ and $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^{1+d})$ (cf. Definition 1.4.49), and consider its Malliavin derivative $\mathrm{D}F(\phi)$ (see Definition 1.4.50), namely

$$\mathrm{D}F(\phi) = \sum_{i=1}^{n} \partial_i \tilde{F}(\langle f_1, \phi\rangle, \ldots, \langle f_n, \phi\rangle) f_i \in \mathcal{S}(\mathbb{R}^{1+d}).$$

Let $g \in \mathcal{S}(\mathbb{R}^{1+d})$, we then say that a measure μ satisfies the *integration by parts formula* related to the action $S(\phi)$ provided that the following holds: for any cylinder function $F(\phi)$, we have

$$\mathbb{E}_\mu[\langle g, \mathrm{D}F(\phi)\rangle] = -\mathbb{E}_\mu\Big[\langle g, \frac{\delta S}{\delta\phi}(\phi)\rangle F(\phi)\Big], \tag{4.2.1}$$

where $\delta S/\delta\phi$ denotes the functional derivative of the action S with respect to ϕ, while $\mathbb{E}_\mu$ denotes the integration with respect to the measure μ.

The expression (4.2.1) is well-defined when $V \equiv 0$, and the unique measure satisfying such a condition is the aforementioned Gaussian free field μ^{free}. In the interactive case, we can rephrase the previous integration by parts formula as follows:

$$\mathbb{E}_\mu[\langle g, \nabla F(\phi)\rangle] = -\lim_{\varepsilon\to 0}\mathbb{E}_\mu\Big[\langle g, \frac{\delta S_\varepsilon}{\delta\phi}(\phi)\rangle F(\phi)\Big], \tag{4.2.2}$$

where, sticking to the notation adopted in Section 4.1, S_ε is given by

$$S_\varepsilon(\phi) = \frac{1}{2}\int_{\mathbb{R}^{1+d}} (|\nabla\phi|^2 + m^2\phi^2 + 2f_\varepsilon V_\varepsilon(\rho_\varepsilon * \phi))\,\mathrm{d}y.$$

The problem now is to prove that the equation (4.2.2) has some solution μ and that such a solution coincides with the limit of the approximation procedure described in Section 4.1.

As we will see in Chapter 5 (see also Appendix B), the integration by parts problem (4.2.2) is heavily connected to the Fokker–Planck–Kolmogorov (FPK) equation (cf. Section 1.2.2) for some Borel measure μ

$$\lim_{\varepsilon\to 0}\int \mathscr{L}^{\varepsilon}(F)(\varphi)\,\mu(\mathrm{d}\varphi)=0,$$

where $\mathscr{L}^{\varepsilon}$ is the infinitesimal generator (cf. Section 1.2) of a solution ϕ_ε to equation (4.1.3) that is given by

$$\begin{aligned}\mathscr{L}^{\varepsilon}(F)(\varphi) &= \frac{1}{2}\mathrm{tr}_{L^2}\bigg(\frac{\delta^2 F}{\delta\varphi^2}(\varphi)\bigg)-\langle\frac{\delta S_\varepsilon}{\delta\varphi},\frac{\delta F}{\delta\varphi}\rangle\\ &= \frac{1}{2}\mathrm{tr}_{L^2}\bigg(\frac{\delta^2 F}{\delta\varphi^2}(\varphi)\bigg)-\int(-\Delta\varphi+m^2\varphi+g_\varepsilon*(f_\varepsilon V'(g_\varepsilon*\varphi)))(x)\,\frac{\delta F}{\delta\varphi}(x)\,\mathrm{d}x.\end{aligned}$$

It is worth to remark that measures characterization through integration by parts formulae is a widely studied problem in the stochastic analysis community and it often appears under different formulations, some examples being the problem of showing existence and uniqueness of a measure with a given logarithmic gradient (see the monograph [34]) or the unique closability of a minimally defined pre-Dirichlet form (see [2, 20, 38] and the references therein). Kirillov applied such problems to QFT in the case of the sine-Gordon model (cf. Section 1.1) without renormalization (see the works [118, 119, 120, 121]); the technique exploited by the author in order to show existence of solutions to the integration by parts formula relies on Lyapunov functions and it has been generalized to non-singular (namely with no need for renormalization) FPK equations by Bogachev and Röckner [39, 40].

Up to our knowledge, only a few results about uniqueness of solutions to the integration by parts formula (or to an infinite-dimensional FPK equation) are available in the literature; for instance, Bogachev, Da Prato and Röckner [35, 36] and Röckner, Zhu and Zhu [161] consider the case of a dissipative non-regular drift (i.e. without renormalization). Let us refer the reader also to the books [34, 37] for more details on the subject. The case of the $P(\varphi)_2$ stochastic quantization equation on the two-dimensional torus $\mathbb{T}^2$ (cf. Section 1.1) has been widely addressed as far as uniqueness of solutions to FPK equations or of invariant measures is concerned (see e.g. the works [55, 127, 162, 163, 178]), but, as far as we are aware, it is still not clear whether the techniques adopted in the aforementioned papers can be extended to models on the non-compact space $\mathbb{R}^2$ or in dimension larger than two. It is worth to mention that the study of uniqueness in the framework of Dirichlet forms for the exponential model has been discussed by Albeverio, Kawabi and Röckner in [14] for the one-dimensional non-singular case, and by the same authors together with Mihalache in [13] for the two-dimensional setting on the torus. See also [18] for a review of the existing literature on the Dirichlet approach to the problem.

In the present thesis (see Chapter 5 and Appendix B), we give a suitable formulation of the integration by parts formula when it does involve a renormalization procedure in its definition, which is the usual situation in constructive Euclidean QFT, testing our approach on the case with exponential interaction and positive mass $m>0$, which is also referred to as $\exp(\Phi)_2$-model or *Høegh–Krohn model* [10] (cf. Secion 1.1), on the whole space $\mathbb{R}^2$. The exponential interaction in the case of mass $m=0$ [8, 12, 160] is a classical example of conformal field theory [143, 165] and it finds important applications in Liouville quantum gravity [66, 130].

Chapter 5

A singular integration by parts formula for the exponential Euclidean QFT on the plane

This chapter summarizes the results obtained in the paper

[61] Francesco C. De Vecchi, Massimiliano Gubinelli, and Mattia Turra. A singular integration by parts formula for the exponential Euclidean QFT on the plane. *ArXiv*, arXiv:2212.05584, 2022.

The full version of the paper can be found in Appendix B.

The research undertaken in the article in question is a collaboration with F. C. De Vecchi and M. Gubinelli. All the authors have contributed significant parts to each section of the work.

5.1 Introduction

As we briefly mentioned in Chapters 1 and 4, the aim of the paper [61] is to characterize Euclidean quantum field theories via integration by parts (IbP) problems with renormalization, i.e. through the study of Euclidean Dyson–Schwinger equations. Let us be more precise and introduce the problem under investigation. Let $E \subset \mathcal{S}'(\mathbb{R}^2)$ be a Banach space with norm $\|\cdot\|_E$ and denote by $\mathscr{M}$ a subset of the space $\mathscr{P}(E)$ of (Radon) probability measures on E. A general IbP problem for a measure $\nu \in \mathscr{M}$ has the form

$$\int_E \langle \nabla_\varphi F - FB, f \rangle \, \mathrm{d}\nu = 0, \qquad \text{for any } F \in \mathrm{Cyl}_E^b,$$

where Cyl_E^b is the set of smooth and bounded cylinder functions (cf. also Definition 1.4.49), and $B: E \to \mathcal{S}'(\mathbb{R}^2)$ is a local functional of the form $B(\varphi)(x) = p(\varphi(x))$, for $x \in \mathbb{R}^2$, for some smooth function $p: \mathbb{R} \to \mathbb{R}$. Let us mention that such kind of functions B are rarely well-defined on the set E on which we could hope that any solution ν would be supported. Typically, this support looks very much like the support of the GFF and therefore non-linear local functionals are not automatically well-defined and need to be approached via an ultraviolet regularization and subsequent renormalization.

We consider then a sequence of maps $(B_\varepsilon)_{\varepsilon > 0}$ such that, for every $\varepsilon > 0$, we have $B_\varepsilon: E \to \mathcal{S}'(\mathbb{R}^2)$ and for which we recover locality in the limit as $\varepsilon \to 0$. They are typically of the form

$$B_\varepsilon(\varphi)(x) = p_\varepsilon((g_\varepsilon * \varphi)(x)), \qquad x \in \mathbb{R}^2,$$

where $(g_\varepsilon)_{\varepsilon \geqslant 0}$ is some sequence of local smoothing kernels for which $(g_\varepsilon * \varphi) \to \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$ and $(p_\varepsilon: \mathbb{R} \to \mathbb{R})_\varepsilon$ is a sequence of smooth function chosen to deliver the expected renormalization, typical of EQFT in two and three dimensions.

Problem A. *We say that a measure $\nu \in \mathscr{M}$ satisfies the* integration by parts formula with respect to $(B_\varepsilon)_{\varepsilon>0}$ and $\mathscr{M}$ *if, for any $f \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\int_E \langle \nabla_\varphi F(\varphi), f\rangle \, \nu(\mathrm{d}\varphi) = \lim_{\varepsilon\to 0} \int_E F(\varphi)\langle B_\varepsilon(\varphi), f\rangle \, \nu(\mathrm{d}\varphi), \qquad \text{for any } F \in \mathrm{Cyl}_E^b. \tag{5.1.1}$$

Let us remark that Problem A strongly depends on the choice of the subset $\mathscr{M}$, which can be neither too large nor too small. For instance, choosing $\mathscr{M} = \mathscr{P}(E)$ leads to problems on the existence of a limit for the sequence $(B_\varepsilon)_\varepsilon$ as $\varepsilon \to 0$. The choice we made is to consider a set of measures $\mathscr{M}_B$ which are close to the Gaussian free field with mass $m > 0$ with respect to some Wasserstein distance depending on a convex cone B in E with stronger norm than the one of E, but without requiring absolute continuity of the measures with respect to the GFF (i.e. the case $B = H^1(\mathbb{R}^2)$, the Cameron–Martin space of the GFF). The class of measures $\mathscr{M}_B$ encodes the existence of sufficiently regular couplings between our target measures and the GFF.

5.2 Main results

We would like to provide complete well-posedness results within this framework of singular IbP formulas and, for this reason, we focus on the specific case of the exponential interaction (cf. Section 1.1). More precisely, we take $E = B_X + B_Y$, where $B_X = C_\ell^{-\delta}(\mathbb{R}^2)$, i.e. the (weighted) Besov–Hölder space with negative regularity $-\delta$ (cf. Section 1.3), and B_Y is such that

$$B_Y \subset B_{p,p,\ell}^{s(\alpha)-\delta}(\mathbb{R}^2),$$

where $s(\alpha) > 0$ satisfies some conditions depending on the parameter α (see Definition B.2.6) and $B_{p,p,\ell}^{s(\alpha)-\delta}$ is a weighted Besov space (cf. Section 1.3). Moreover, we set

$$B_\varepsilon(\varphi) := (-\Delta + m^2)\varphi + \alpha f_\varepsilon \, e^{\alpha(g_\varepsilon * \varphi) - \frac{\alpha^2}{2} c_\varepsilon}, \tag{5.2.1}$$

where $\alpha, m \in \mathbb{R}_+$, f_ε is a smooth, spatial cut-off function such that $f_\varepsilon \to 1$, $g_\varepsilon = \varepsilon^{-2} g(\varepsilon^{-1} \cdot)$ is a regular mollifier, and

$$c_\varepsilon := \int_{\mathbb{R}^2} g_\varepsilon(z)(-\Delta + m^2)^{-1} g_\varepsilon(z) \, \mathrm{d}z \tag{5.2.2}$$

is a renormalization constant diverging logarithmically to $+\infty$ as $\varepsilon \to 0$. Finally, we consider the space of measures $\mathscr{M}$ in Problem A to be $\mathscr{M}_{B_Y}$, that is an intermediate regime between the case $\mathscr{M}_{H^1}$ (i.e. the space of measures that are absolutely continuous with respect to the GFF) and $\mathscr{M}_E$ (which coincides with the Wasserstein space $\mathscr{W}_E^1$, see Chapter 6 in [181]).

Let $\tilde{\gamma}_{\max} := 3 - 2\sqrt{2} \approx 0.172$. In the present setting, it is possible to obtain the following result.

Theorem 5.2.1. *Suppose that $\alpha^2 < 4\pi\tilde{\gamma}_{\max}$. Consider $\mathscr{M}_{B_Y}$ with $E = B_X + B_Y$, where*

$$B_X = C_\ell^{-\delta}(\mathbb{R}^2) \quad \textit{and} \quad B_Y = B_Y^{\leqslant} := B_{p,p,\ell}^{s(\alpha)-\delta}(\mathbb{R}^2) \cap \{f : \mathbb{R}^2 \to \mathbb{R}, f \leqslant 0\}.$$

Then there exists a unique solution to Problem A with respect to $(B_\varepsilon)_\varepsilon$ (given by equation (5.2.1)) and the space of measures $\mathscr{M}_{B_Y}$.

It is also possible to obtain an existence result for the whole regime $\alpha^2 < 8\pi$.

Theorem 5.2.2. *Suppose that $\alpha^2 < 8\pi$ and assume that the same hypotheses on the spaces B_X, B_Y, E, $\mathscr{M}_{B_Y}$ and the drift $(B_\varepsilon)_\varepsilon$ as in Theorem B.1.2 hold. Then there exists a solution to Problem A with respect to $(B_\varepsilon)_\varepsilon$ and $\mathscr{M}_{B_Y}$.*

An important consequence of Theorem 5.2.1 and Theorem 5.2.2 is a differential characterization of the exponential measure (see Theorem B.2.13 for details).

Let us mention that in the present thesis we also get a uniqueness result for a slightly more restrictive formulation of Problem A (cf. Problem B in Section B.2.1) in the regime $\alpha^2 < 4\pi\gamma_{\max}$, where $\gamma_{\max} \approx 0.55$ is defined as in Remark B.1.1 (notice $\tilde{\gamma}_{\max} < \gamma_{\max}$). The possible measures solving this latter formulation of the IbP problem contain the invariant measure of the stochastic quantization equation with exponential interaction.

We want to illustrate briefly how the proof of uniqueness of solutions to Problem A was carried out in our work. Let us first go back to the case of a general $(B_\varepsilon)_{\varepsilon>0}$: It is worth to mention that we can give an equivalent formulation of Problem A, consisting in a symmetric generalization of the Fokker–Planck–Kolmogorov (FPK) equation (see also Section 1.2.2). More precisely, we consider $E = B_X + B_Y$, where B_X is some space supporting the measure of the Gaussian free field with mass $m > 0$, and $B_Y = B$. Let us also introduce the natural projection $P^X: B_X \times B_Y \to B_X$ and the map $P^{X+Y}: B_X \times B_Y \to E$ such that $(X, Y) \mapsto X + Y$. Suppose that B_ε is regular enough and define the following second order operator

$$\mathfrak{L}_\varepsilon F := \frac{1}{2} \operatorname{tr}_{L^2(\mathbb{R}^2)}(\nabla_\varphi^2 F) - \langle B_\varepsilon, \nabla_\varphi F \rangle, \qquad F \in \mathrm{Cyl}_E.$$

It is possible to show (cf. Problem A' in Appendix B as well as Theorem B.2.5) that, provided that $\sup_{\varepsilon>0} \int |\langle B_\varepsilon(\varphi), \nabla_\varphi F \rangle| \, \nu(\mathrm{d}\varphi) < +\infty$, for any $F \in \mathrm{Cyl}_E$ and $\nu \in \mathscr{M}$, Problem A is equivalent to asking that the measure $\nu \in \mathscr{M}$ satisfies

$$\lim_{\varepsilon \to 0} \int \left[(\mathfrak{L}_\varepsilon F)\, G - F\, (\mathfrak{L}_\varepsilon G) \right] \mathrm{d}\nu = 0, \qquad \text{for any } F \in \mathrm{Cyl}_E^b, G \in \mathrm{Cyl}_E.$$

We can moreover lift the problem from the space E to the space $B_X \times B_Y$. We start with the notion of coupling.

Definition 5.2.3. *The subset of measures $\mathscr{M}$ satisfies the* coupling hypotheses *if, for any $\nu \in \mathscr{M}$, there exists a Radon measure μ on $B_X \times B_Y$ with the following properties:*

i. $P_*^X \mu = \nu^{\text{free}}$*, where ν^{free} is the law of the Gaussian free field on B_X,*

ii. $P_*^{X+Y} \mu = \nu$*,*

iii. $\int \|Y\|_{B_Y} \mu(\mathrm{d}X, \mathrm{d}Y) < +\infty$*,*

we call μ a coupling of ν with the free field. *We denote by $\mathscr{M}_{B_X \times B_Y}$ the set of Radon measures on $B_X \times B_Y$ satisfying condition i. and iii.*

We can consider then an operator $\mathscr{L}_\varepsilon$ on the space of regular functions on $B_X \times B_Y$ of the form

$$\mathscr{L}_\varepsilon \Phi(X, Y) := \frac{1}{2} \operatorname{tr}(\nabla_X^2 \Phi) - \langle (-\Delta + m^2) X, \nabla_X \Phi \rangle - \langle B_\varepsilon(X + Y) - (-\Delta + m^2) X, \nabla_Y \Phi \rangle.$$

It can be shown (see Theorem B.2.5) that, if we assume further that $\mathscr{M} = \mathscr{M}_{B_Y}$, then Problem A is equivalent to the following formulation (which we call Problem A" to be consistent with the notation adopted in Appendix B):

Problem A". *We say that a measure* $\nu \in \mathscr{M}_{B_Y}$ *satisfies the* symmetric Fokker–Planck–Kolmogorov equation related to B_ε *if*

$$\lim_{\varepsilon\to 0}\int \left[\mathscr{L}_\varepsilon(F\circ P^{X+Y})\,G\circ P^{X+Y} - F\circ P^{X+Y}\,\mathscr{L}_\varepsilon(G\circ P^{X+Y})\right]\,\mu(\mathrm{d}X,\mathrm{d}Y)=0, \quad \text{for } F\in \mathrm{Cyl}_E^b, G\in \mathrm{Cyl}_E, \tag{5.2.3}$$

where μ *is a coupling of* ν *with the free field.*

The proof of uniqueness of solutions for the case of exponential interaction was obtained working with the formulation given in Problem A". Let us give here an heuristic presentation of the adopted method and refer the interested reader to Appendix B.3 for more details. Indeed, assume that B_ε is of the form given in equation (5.2.1) and take B_X and B_Y as in Theorem 5.2.1. We study then the resolvent equation for the operator $\mathscr{L}_\varepsilon$, namely, for $F\in \mathrm{Cyl}_{B_X\times B_Y}$, we consider

$$(\lambda-\mathscr{L}_\varepsilon)G_\varepsilon^\lambda = F, \qquad \lambda\in\mathbb{R}_+. \tag{5.2.4}$$

Equation (5.2.4) admits a classical solution G_ε^λ such that $\mathscr{L}_\varepsilon G_\varepsilon^\lambda$ is integrable with respect to any measure in $\mathscr{M}_{B_X\times B_Y}$ (see Proposition B.3.5).

Let μ_1 and μ_2 be two solutions of Problem A". We want to show that, for any $F\in \mathrm{Cyl}_{B_X\times B_Y}$ with compact support in Fourier variables, $\int F\mathrm{d}\mu_1 = \int F\mathrm{d}\mu_2$. This implies $\mu_1=\mu_2$ since $F\in \mathrm{Cyl}_{B_X\times B_Y}$ with compact support in Fourier variables separates points of the space of Radon measures on $B_X\times B_Y$.

Let $i=1,2$, since μ_i is a solution of Problem A", then $\int \mathscr{L}G_\varepsilon^\lambda\,\mathrm{d}\mu_i := \lim_{\bar{\varepsilon}\to 0}\int \mathscr{L}_{\bar{\varepsilon}}G_\varepsilon^\lambda\,\mathrm{d}\mu_i = 0$, and by equation (5.2.4) we have

$$\int F\mathrm{d}\mu_i = \lambda\int G_\varepsilon^\lambda\mathrm{d}\mu_i - \int \nabla_{Y_0}G_\varepsilon^\lambda(\mathscr{G}-\mathscr{G}_\varepsilon)\,\mathrm{d}\mu_i, \qquad i=1,2,$$

where $\mathscr{G}(X,Y)=\alpha{:}e^{\alpha X}{:}\,e^{\alpha Y}$ and $\mathscr{G}_\varepsilon(X,Y)=\alpha f_\varepsilon{:}e^{\alpha(g_\varepsilon * X)}{:}\,e^{\alpha(g_\varepsilon*Y)}$. Taking the difference of such a relation for $j=1,2$ yields

$$\int F\mathrm{d}\mu_1 - \int F\mathrm{d}\mu_2 = \lambda\int G_\varepsilon^\lambda\mathrm{d}\mu_1 - \lambda\int G_\varepsilon^\lambda\mathrm{d}\mu_2 - \int \nabla_{Y_0}G_\varepsilon^\lambda(\mathscr{G}-\mathscr{G}_\varepsilon)\,\mathrm{d}\mu_1 + \int \nabla_{Y_0}G_\varepsilon^\lambda(\mathscr{G}-\mathscr{G}_\varepsilon)\,\mathrm{d}\mu_2.$$

It is then possible to show that (by Proposition B.3.5 whose proof relies on the dissipativity of the drift B_ε)

$$\lambda\left|\int G_\varepsilon^\lambda\,(\mathrm{d}\mu_1-\mathrm{d}\mu_2)\right|\to 0, \qquad \text{as } \lambda\to 0,$$

uniformly in $\varepsilon>0$. Moreover, we have that

$$\left|\int \nabla_{Y_0}G_\varepsilon^\lambda(\mathscr{G}-\mathscr{G}_\varepsilon)\,\mathrm{d}\mu_i\right|\to 0, \qquad \text{as } \varepsilon\to 0. \tag{5.2.5}$$

Therefore, taking $\lambda\to 0$ and then $\varepsilon\to 0$, we have that

$$\int F\mathrm{d}\mu_1 = \int F\mathrm{d}\mu_2.$$

Let us remark that the condition $\alpha^2 < 4\pi\tilde{\gamma}_{\max}$ comes from the proof of the limit in equation (5.2.5) and depends on the chosen approximation.

The proof of existence (cf. Appendix B.4) is based on the existence of Lyapunov functions for the sequence of operators $(\mathscr{L}_\varepsilon)_{\varepsilon>0}$ uniform in ε. Let us mention that such an approach was already proposed by Kirillov for the case non-renormalized equations (see [118, 119, 120, 121]). More precisely, we introduce a further approximation $\mathscr{L}_{M,N,\varepsilon}$ of the operator $\mathscr{L}_\varepsilon$, depending on two additional parameters M and N in $\mathbb{N}$, chosen in such a way that the corresponding FPK equation admits a solution $\mu_{N,M,\varepsilon}\in\mathscr{M}_{B_X\times B_Y}$ that is a finite-dimensional approximation of the measure of the Gaussian free field. The existence (and uniqueness) of such a measure is based on standard results for finite-dimensional FPK equations (see [37]). We then prove that there are some regular functions $V_1, V_2: B_X\times B_Y\to\mathbb{R}$, and $V_3: B_X\to\mathbb{R}$, such that

i. V_2 and V_3 are positive,

ii. The inequality

$$\mathscr{L}_{M,N,\varepsilon}V_1(X,Y)\leqslant -V_2(X,Y)+V_3(X)$$

holds true.

See Appendix B.4.1 for the precise choice of V_1, V_2, and V_3. Exploiting the fact that $\mu_{M,N,\varepsilon}$ is a solution to the FPK equation, and therefore $\int\mathscr{L}_{M,N,\varepsilon}V_1(X,Y)\,\mu_{M,N,\varepsilon}(\mathrm{d}X,\mathrm{d}Y)=0$, we get

$$\int V_2(X,Y)\,\mu_{M,N,\varepsilon}(\mathrm{d}X,\mathrm{d}Y)\leqslant\int V_3(X)\,\mu_{M,N,\varepsilon}(\mathrm{d}X,\mathrm{d}Y)=\int V_3(X)\,\nu_M^{\mathrm{free}}(\mathrm{d}X),$$

where ν_M^{free} is the Gaussian free field measure on the two-dimensional torus of size M. Since $\sup_{M\in\mathbb{N}}\int V_3(X)\,\mu_M^{\mathrm{free}}(\mathrm{d}X)<+\infty$, and taking V_2 to have compact sub-levels, we get tightness. The existence of a solution to the original problem then follows after showing that

$$\int\mathscr{L}_{N,M,\varepsilon}(\Phi(X,Y))\,\mu_{N,M,\varepsilon}(\mathrm{d}X,\mathrm{d}Y)\to\int\mathscr{L}(\Phi(X,Y))\,\mu(\mathrm{d}X,\mathrm{d}Y), \tag{5.2.6}$$

for any Φ in a suitable class of regular functions, see Definition B.3.1 for a precise definition of such a class of functions and Appendix B.4.4 for the proof of the limit in (5.2.6).

5.3 Further developments

To the best of our knowledge, what we presented in this chapter and is published in our paper [61] are the first results on singular integration by parts problems in the whole space concerning any kind of interactions. Up to now, the only known achievements were results about the $P(\varphi)_2$-type interaction on the two-dimensional torus (see [55, 127, 162, 163, 178]). Therefore, many possible developments are still available topics of future research.

The most direct generalization is to consider the exponential (or alternatively the sinh-Gordon) model on $\mathbb{R}^2$ for a charge parameter α in the full L^1-regime (or sub-critical regime), namely the case $\alpha^2<8\pi$. Although there are results on such a model in the whole subcritical regime exploiting the stochastic quantization approach (see [111]), our uniqueness result holds only for stricter conditions on α (cf. Theorem 5.2.1). As mentioned in the previous section, a possibility is that this is due to some technical issues involving the chosen approximation in our work, and that it can be solved exploiting a different choice (for instance, a lattice approximation as it has been done in [25, 91]).

Another generalization concerning the exponential interactions is the choice of a wider class of measures as possible solutions, with the aim to reach the whole space of Borel measures on $C_\ell^{-\delta}(\mathbb{R}^2)$, which in the discussion above means $B_Y = B_X = C_\ell^{-\delta}(\mathbb{R}^2)$.

A more ambitious project is to address the IbP problem for singular non-convex potentials, which simplest example is given by the polynomial $P(\varphi)_2$-model on $\mathbb{R}^2$ (but other important examples in QFT are e.g. the sine-Gordon model, the Φ^4_3-model, Yang-Mills model, and so on). In this case, the uniqueness problem is more complex since the proof proposed in this thesis relied heavily on the convexity of the renormalized exponential. Uniqueness of solutions to the IbP formula is closely related to uniqueness of invariant measures of the associated stochastic quantization equation, which, to the best of our knowledge, remains completely an open problem for the models mentioned above (on the whole space).

A final interesting direction of investigation would be getting a better general understanding of the relation between the uniqueness of solutions to the IbP formula presented in the thesis with the unique closure of the related Dirichlet form as well as with the uniqueness of the invariant measure of the associated SPDE dynamics. The relation between these three problems – IbP, Dirichlet forms, and SPDE – is very well understood in finite-dimensional settings (and the notions of uniqueness for the three problems coincide under mild hypotheses), see [37] and the references therein, but very few results in this direction exist in infinite dimensions. In particular, there are no generic hypotheses on the coefficients of an SPDE (and the related FPK equation) for which the infinitesimal invariance of a measure is equivalent to the notion of stationarity with respect to the stochastic dynamics.

Appendices

Appendix A

Hyperviscous stochastic Navier–Stokes equations with white noise invariant measure

Abstract We prove existence and uniqueness of martingale solutions to a (slightly) hyper-viscous stochastic Navier–Stokes equation in 2d with initial conditions absolutely continuous with respect to the Gibbs measure associated to the energy, getting the results both in the torus and in the whole space setting.

A.1 Introduction

Consider the following stochastic hyper-viscous Navier–Stokes equation on $\mathbb{R}_+ \times \mathbb{T}^2$

$$\begin{aligned} \partial_t u &= -A^\theta u - u \cdot \nabla u - \nabla p - \sqrt{2} \nabla^\perp A^{(\theta-1)/2} \xi, \\ \operatorname{div} u &= 0, \end{aligned} \tag{A.1.1}$$

where $\mathbb{T}^2$ is the two dimensional torus, $A = -\Delta$ on $\mathbb{T}^2$, $\nabla^\perp := (\partial_2, -\partial_1)$, $\theta > 1$, and ξ denotes a space-time white noise. The initial condition for u will be taken distributed according to the white noise on $\mathbb{T}^2$ or an absolute continuous perturbation thereof with density in L^2. The white noise on $\mathbb{T}^2$ is formally invariant for the dynamics described by (A.1.1) and the existence theory for the corresponding stationary process has been addressed by Gubinelli and Jara in [93] using the concept of *energy solutions* for any $\theta > 1$. Uniqueness was left open in the aforementioned paper, and the main aim of the present work, which can be thought of as a continuation of [99], is to introduce a martingale problem formulation (A.1.1) for which we can prove uniqueness.

In order to properly formulate the martingale problem, we need to investigate the infinitesimal generator for eq. (A.1.1) and uniqueness will result from suitable solutions of the associated Kolmogorov backward equation.

The variable u appearing in eq. (A.1.1) represents physically the *velocity* of a fluid. Rewriting the equation for the *vorticity* $\omega := \nabla^\perp \cdot u$ yields

$$\partial_t \omega = -A^\theta \omega - u \cdot \nabla \omega + \sqrt{2} A^{(\theta+1)/2} \xi. \tag{A.1.2}$$

We also have the relation $u = K * \omega$, for the *Biot-Savart kernel* K on $\mathbb{T}^2$ given by

$$K(x) = \frac{1}{2\pi\imath} \sum_{k \in \mathbb{Z}_0^2} \frac{k^\perp}{|k|^2} e^{2\pi\imath k \cdot x} = -\sum_{k \in \mathbb{Z}_0^2} \frac{2\pi\imath k^\perp}{|2\pi k|^2} e^{2\pi\imath k \cdot x},$$

where $k^\perp = (k_2, -k_1)$ and $\mathbb{Z}_0^2 = \mathbb{Z}^2 \backslash \{0\}$. It is more convenient to work with the scalar quantity ω and with eq. (A.1.2).

The standard stochastic Navier-Stokes equation corresponds to the case $\theta=1$. However, this regime is quite singular for the white noise initial condition and no results are known, not even existence of a stationary solution, e.g. from limit of Galerkin approximations. While a bit unphysical, we will stick here to the *hyper-viscous regime*, namely $\theta>1$. Note that the noise has to be coloured accordingly in order to preserve the white noise as invariant measure. Moreover, we call *energy measure* the law under which the velocity field is a (vector-valued, incompressible) white noise. In terms of vorticity ω, the kinetic energy of the fluid configuration u is

$$\|u\|_{L^2}^2=\int_{\mathbb{T}^2}|K*\omega|^2(x)\,\mathrm{d}x=\sum_{k\in\mathbb{Z}_0^2}|\hat{K}(k)|^2|\hat{\omega}(k)|^2=\sum_{k\in\mathbb{Z}_0^2}\left|\frac{2\pi\imath k^\perp}{|2\pi k|^2}\right|^2|\hat{\omega}(k)|^2=\|(-\Delta)^{-1/2}\omega\|_{L^2}^2,$$

where $\hat{f}\colon\mathbb{Z}^2\to\mathbb{C}$ denotes the Fourier transform of $f\colon\mathbb{T}^2\to\mathbb{R}$ defined as to have $f(x)=\sum_{k\in\mathbb{Z}^2}e^{2\pi\imath k\cdot x}\hat{f}(k)$. The energy measure is thus formally given by

$$\mu(\mathrm{d}\omega)=\frac{1}{C}e^{-\frac{1}{2}\|A^{-1/2}\omega\|_{L^2}^2}\mathrm{d}\omega, \tag{A.1.3}$$

where $\mathrm{d}\omega$ denotes the "Lebesgue measure" on functions on $\mathbb{T}^2$. Rigorously, this of course means the product Gaussian measure

$$\mu(\mathrm{d}\omega)=\prod_{k\in\mathbb{Z}_0^2}\frac{1}{C_k}\exp\left(-\frac{|\hat{\omega}(k)|^2}{2|2\pi k|^2}\right)\mathrm{d}\hat{\omega}(k),$$

with the restriction that $\hat{\omega}(-k)=\overline{\hat{\omega}(k)}$. For $f,g\in C^\infty(\mathbb{T}^2)$, we have

$$\int\omega(f)\omega(g)\,\mu(\mathrm{d}\omega)=\sum_{k\in\mathbb{Z}_0^2}|2\pi k|^2\,\overline{\hat{f}(k)}\,\hat{g}(k)=\langle A^{1/2}f,A^{1/2}g\rangle_{L^2(\mathbb{T}^2)}=\langle f,g\rangle_{H^1(\mathbb{T}^2)}.$$

We can use the right-hand side as the definition of the covariance of $(\omega(f))_{f\in C^\infty(\mathbb{T}^2)}$, which determines the law of ω as a centred Gaussian process indexed by $H^1(\mathbb{T}^2)$. If η is a white noise on $L^2(\mathbb{T}^2)$, then μ has the same distribution as $A^{1/2}\eta$ and it is only supported on $H^{-2-}(\mathbb{T}^2)$.

A different situation occurs if we consider initial conditions distributed according to the *enstrophy measure*, namely the Gaussian measure for which the initial vorticity is a white noise. This measure is more regular than the energy measure and more results are known, both for the Euler dynamics (i.e., without dissipation and noise) and for the stochastic Navier-Stokes dynamics, see e.g. [3, 6, 7].

As we already remarked, we use here the technique introduced in [99] and strongly rooted in the notion of energy solution of Gonçalves and Jara [85], extended in [93]. With respect to [99] we give a slightly different formulation which simplifies certain technical estimates. The core of the argument however remains the same. The main point is to consider the well-posedness problem for (A.1.1) as a problem of *singular diffusion*, i.e. diffusions with distributional drift. The papers [74, 75, 62, 42] all follow a similar strategy in order to identify a domain for the formal infinitesimal generator $\mathscr{L}=\frac{1}{2}\Delta+b\cdot\nabla$ of a finite dimensional diffusion. Then they show existence and uniqueness of solutions for the corresponding martingale problem. The key difficulty is that for distributional b the domain does not contain any smooth functions and instead one has to identify a class of non-smooth test functions with a special structure, adapted to b. Roughly speaking they must be local perturbations of a linear functional constructed from b. Recently other results of regularisation by noise for SPDEs [56, 57] have been obtained. An important difference is that our drift is unbounded and not even a function. The connection between energy solutions and regularisation by noise was first observed in [93].

Plan of the paper In Section A.2 we introduce a Galerkin approximation for the nonlinearity $u\cdot\nabla\omega$ and study the infinitesimal generator of the approximating equation. The martingale problem for cylinder function related to eq. (A.1.2) is introduced in Section A.3. In Section A.4 we prove uniqueness for the martingale problem via existence of classical solutions to the backward Kolmogorov equation for the operator $\mathscr{L}$ involved in the martingale problem. The construction of a domain to such an operator is the core of the work and it will be tackled in Section A.5, where we provide also existence and uniqueness for the associated Kolmogorov equation. In Section A.6 we show some crucial bounds on the drift. Finally, Section A.7 extends the results obtained in the previous part of the paper to the whole space case, that is, to the hyper-viscous stochastic Navier–Stokes equation on $\mathbb{R}^2$. Appendix A.8 contains some auxiliary results.

Notation Let us fix here some notation that will be adopted throughout the paper. The Schwartz space on $\mathbb{T}^2$ is denoted by $\mathcal{S}(\mathbb{T}^2)$ and its dual $\mathcal{S}'(\mathbb{T}^2)$ is the space of tempered distributions. We denote by $H^s(\mathbb{T}^2)$, $s\in\mathbb{R}$, the completion of the space of functions $f\in\mathcal{S}(\mathbb{T}^2)$ such that

$$\|f\|^2_{H^s(\mathbb{T}^2)} := \int_{\mathbb{T}^2} |z|^{2s}|\hat{f}(z)|^s\,\mathrm{d}z < +\infty,$$

identifying f and g whenever $\|f-g\|_{H^s(\mathbb{T}^2)}=0$. From now on, we write $C(X,Y)$ to indicate the space of continuous functions from X to Y. We also write $a\lesssim b$ or $b\gtrsim a$ if there exists a constant $C>0$, independent of the variables under consideration, such that $a\leqslant C\cdot b$, and $a\simeq b$ if both $a\lesssim b$ and $b\lesssim a$. If the aforementioned constant C depends on a variable, say $C=C(x)$, then we use the notation $a\lesssim_x b$, and similarly for $\gtrsim$. For the sake of brevity, we will also use the notation $k_{1:n}=(k_1,\ldots,k_n)$.

A.2 Galerkin approximations

In order to rigorously study the eq. (A.1.2), consider the solution $(\omega_t^m)_{t\geqslant 0}$ to its Galerkin approximation:

$$\partial_t\omega^m = -A^\theta\omega^m - B_m(\omega^m) + \sqrt{2}A^{(1+\theta)/2}\xi, \tag{A.2.1}$$

where

$$B_m(\omega) := \operatorname{div}\Pi_m((K*\Pi_m\omega)\Pi_m\omega),$$

and Π_m denotes the projection onto Fourier modes of size less than m, namely $\Pi_m f(x) = \sum_{|k|\leqslant m} e^{2\pi\iota k\cdot x}\hat{f}(k)$.

Proposition A.2.1. *Eq. (A.2.1) has a unique strong solution $\omega^m\in C(\mathbb{R}_+,H^{-2-}(\mathbb{T}^2))$ for every deterministic initial condition in $H^{-2-}(\mathbb{T}^2)$. The solution is a strong Markov process and it is invariant under μ.*

Proof. We can rewrite ω^m in Fourier variables as $\omega^m = w^m_{\text{fin}} + w^m_{\text{lin}} := \Pi_m\omega^m + (1-\Pi_m)\omega^m$, in such a way that w^m_{fin} and w^m_{lin} solve a finite-dimensional SDE with locally Lipschitz continuous coefficients and an infinite-dimensional linear SDE, respectively. Global existence and invariance of μ follow by Section 7 in [93]. Now, w^m_{fin} has compact spectral support and therefore $w^m_{\text{fin}}\in C(\mathbb{R}_+,C^\infty(\mathbb{T}))$, while it can be proved that w^m_{lin} has trajectories in $C(\mathbb{R}_+,H^{-2-}(\mathbb{T}^2))$. Thus, ω^m has trajectories in $C(\mathbb{R}_+,H^{-2-}(\mathbb{T}^2))$. □

We define the semigroup of ω^m for all bounded and measurable functions φ as $T_t^m\varphi(\omega_0) := \mathbb{E}_{\omega_0}[\varphi(\omega_t^m)]$, where, under $\mathbb{P}_{\omega_0}$, the process ω^m solves (A.2.1) with initial condition $\omega_0 \in H^{-2-}(\mathbb{T}^2)$.

Lemma A.2.2. *For all $p \in [1, \infty]$, the family of operators $(T_t^m)_{t \geqslant 0}$ can be uniquely extended to a contraction semigroup on $L^p(\mu)$ which is continuous for $p \in [1, \infty[$.*

Definition A.2.3. *Let $\mathscr{C} = \mathrm{Cyl}_{\mathbb{T}^2}$ denote the set of cylinder functions on $H^{-2-}(\mathbb{T}^2)$, namely those functions $\varphi: H^{-2-}(\mathbb{T}^2) \to \mathbb{R}$ of the form $\varphi(\omega) = \Phi(\omega(f_1), \ldots, \omega(f_n))$ for some $n \geqslant 1$ where $\Phi: \mathbb{R}^n \to \mathbb{R}$ is smooth and $f_1, \ldots, f_n \in C^\infty(\mathbb{T}^2)$, here and in the rest of the paper we will denote by $\nu(g)$ the duality between a distribution ν and a function g.*

On such cylinder functions the generator of the semigroup T^m has an explicit representation: Itô's formula gives, for $\varphi \in \mathrm{Cyl}_{\mathbb{T}^2}$ as in Definition A.2.3,

$$\mathrm{d}\varphi(\omega_t^m) = \mathscr{L}^m \varphi(\omega_t^m)\mathrm{d}t + \sum_{i=1}^n \partial_i \Phi(\omega_t^m(f_1), \ldots, \omega_t^m(f_n))\mathrm{d}M_t(f_i), \tag{A.2.2}$$

where $\mathscr{L}^m := \mathscr{L}_\theta + \mathscr{G}^m$ with

$$\mathscr{L}_\theta \varphi(\omega) = \sum_{i=1}^n \partial_i \Phi(\omega(f_1), \ldots, \omega(f_n))\omega(-A^\theta f_i) + \frac{1}{2}\sum_{i=1}^n \partial_{i,j}^2 \Phi(\omega(f_1), \ldots, \omega(f_n))\langle A^{\theta+1} f_i, f_j \rangle,$$

and

$$\mathscr{G}^m \varphi(\omega) = -\sum_{i=1}^n \partial_i \Phi(\omega(f_1), \ldots, \omega(f_n))\langle B_m(\omega), f_i \rangle.$$

Here, $(M_t(f_i))_{t \geqslant 0}$ is a continuous martingale with quadratic variation

$$\langle M(f_i) \rangle_t = 2t \| A^{(\theta+1)/2} f_i \|_{L^2(\mathbb{T}^2)}^2,$$

and therefore $\int_0^\cdot \sum_{i=1}^n \partial_i \Phi(\omega_t^m(f_1), \ldots, \omega_t^m(f_n))\mathrm{d}M_t(f_i)$ is a martingale. Consequently, we have

$$T_t^m \varphi(\omega) - \varphi(\omega) = \int_0^t T_s^m(\mathscr{L}^m \varphi)(\omega)\,\mathrm{d}s, \qquad \text{for all } \omega \in H^{-2-}.$$

To extend this to more general functions φ, we work via Fock space techniques. The Hilbert space $L^2(\mu)$ can be identified with the Fock space $\mathscr{H} = \Gamma H_0^1(\mathbb{T}^2) := \bigoplus_{n=0}^\infty (H_0^1(\mathbb{T}^2))^{\otimes n}$ with $H_0^1(\mathbb{T}^2) := \{\psi \in H^1(\mathbb{T}^2): \hat{\psi}(0) = 0\}$ and norm

$$\|\varphi\|^2 = \sum_{n=0}^\infty n!\, \|\varphi_n\|_{(H_0^1(\mathbb{T}^2))^{\otimes n}}^2 = \sum_{n=0}^\infty n! \sum_{k_{1:n} \in (\mathbb{Z}_0^2)^n} \left(\prod_{i=1}^n |2\pi k_i|^2 \right) |\hat{\varphi}_n(k_{1:n})|^2,$$

by noting that any $\varphi \in L^2(\mu)$ can be written in chaos expansion $\varphi = \sum_{n \geq 0} W_n(\varphi_n)$, where W_n is the n-th order Wiener-Itô integral and $\varphi_n \in H_0^1(\mathbb{T}^2)^{\otimes n}$ for every $n \in \mathbb{N}$, see e.g. [112, 148] for details. We will use the convention that φ_n is symmetric in its n arguments, that is, we identify it with its symmetrisation. Note that cylinder functions are dense in $\mathscr{H}$. We denote by $\mathscr{N}$ the *number operator*, i.e. the self–adjoint operator on $\mathscr{H}$ such that $(\mathscr{N}\varphi)_n := n\varphi_n$. It is well known that the semigroup generated by the number operator satisfies an hypercontractivity estimate, see Theorem 1.4.1 in [148]. We record it in the next lemma.

Lemma A.2.4. *For $p \geqslant 2$, let $c_p = \sqrt{p-1}$. Then*

$$\| |\varphi|^{p/2} \|^2 \leqslant \| c_p^{\mathscr{N}} \varphi \|^p, \qquad \textit{for every } \varphi \in \mathscr{H}.$$

With these preparations we are ready to give expressions for the operators $\mathscr{L}_\theta$ and $\mathscr{G}^m$ in terms of the Fock space representation of $\mathscr{H}$.

Lemma A.2.5. *For sufficiently nice $\varphi \in \mathscr{H}$, the operator $\mathscr{L}_\theta$ is given by*

$$\mathscr{F}(\mathscr{L}_\theta \varphi)_n(k_{1:n}) = -(2\pi)^{2\theta} L_\theta(k_{1:n}) \hat{\varphi}_n(k_{1:n}) \tag{A.2.3}$$

where $L_\theta(k_{1:n}) := |k_1|^{2\theta} + \cdots + |k_n|^{2\theta}$. Moreover, writing $\mathscr{G}^m = \mathscr{G}^m_+ + \mathscr{G}^m_-$ we have

$$\mathscr{F}(\mathscr{G}^m_+ \varphi)_n(k_{1:n}) = (n-1) 1_{|k_1|, |k_2|, |k_1+k_2| \leqslant m} \frac{(k_1^\perp \cdot (k_1+k_2))((k_1+k_2)\cdot k_2)}{|k_1|^2 |k_2|^2} \hat{\varphi}_{n-1}(k_1+k_2, k_{3:n}), \tag{A.2.4}$$

$$\mathscr{F}(\mathscr{G}^m_- \varphi)_n(k_{1:n}) = (2\pi)^2 (n+1) n \sum_{p+q=k_1} 1_{|k_1|, |p|, |q| \leqslant m} \frac{(k_1^\perp \cdot p)(k_1 \cdot q)}{|k_1|^2} \hat{\varphi}_{n+1}(p, q, k_{2:n}). \tag{A.2.5}$$

For all $\varphi_{n+1} \in (H_0^1(\mathbb{T}^2))^{\otimes(n+1)}$ and for all $\varphi_n \in (H_0^1(\mathbb{T}^2))^{\otimes n}$, we have

$$\langle \varphi_{n+1}, \mathscr{G}^m_+ \varphi_n \rangle = -\langle \mathscr{G}^m_- \varphi_{n+1}, \varphi_n \rangle. \tag{A.2.6}$$

Proof. The computations are analogous to those of Lemma 3.7 of [97] for $\mathscr{L}_\theta$ and of Lemma 2.4 and Lemma 2.7 in [99]. □

Remark A.2.6. $\mathscr{G}^m_+$ and $\mathscr{G}^m_-$ are (unbounded) operators which increase and decrease, respectively, the "number of particles" by one. Moreover, we know from (A.2.6) that they are formally the adjoint of the other (modulo a sign change).

A key result is given by the following bounds for $\mathscr{G}^m_\pm$ acting on weighted subspaces of $\mathscr{H}$.

Lemma A.2.7. *Let $w: \mathbb{N}_0 \to \mathbb{R}_+$ and $\varphi \in \mathscr{H}$. The following m-dependent bound holds:*

$$\|w(\mathscr{N}) \mathscr{G}^m \varphi\| \lesssim m \|(w(\mathscr{N}+1) + w(\mathscr{N}-1))(1+\mathscr{N})(1-\mathscr{L}_\theta)^{1/2} \varphi\|. \tag{A.2.7}$$

Moreover, uniformly in m, we have

$$\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-\gamma} \mathscr{G}^m_+ \varphi\| \lesssim \|w(\mathscr{N}+1)(1+\mathscr{N})(1-\mathscr{L}_\theta)^{(1+1/\theta)/2-\gamma} \varphi\|, \qquad \textit{for all } \gamma > \frac{1}{2\theta}, \tag{A.2.8}$$

and

$$\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-\gamma} \mathscr{G}^m_- \varphi\| \lesssim \|w(\mathscr{N}-1)\mathscr{N}^{3/2}(1-\mathscr{L}_\theta)^{(1+1/\theta)/2-\gamma} \varphi\|, \qquad \textit{for all } \gamma < \frac{1}{2}. \tag{A.2.9}$$

These bounds will be proven later on in Section A.6. In view of eq. (A.2.7), it is natural to identify a dense domain $\mathscr{D}(\mathscr{L}^m)$ for $\mathscr{L}^m$ as

$$\mathscr{D}(\mathscr{L}^m) := \{\varphi \in \mathscr{H} : \|(1+\mathscr{N})(1-\mathscr{L}_\theta)\varphi\| < \infty\} = (1+\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1}\mathscr{H}.$$

Note that $\langle \psi, (\mathscr{L}_\theta + \mathscr{G}^m)\varphi \rangle = \langle (\mathscr{L}_\theta - \mathscr{G}^m)\psi, \varphi \rangle$ for $\psi, \varphi \in \mathscr{D}(\mathscr{L}^m)$ and in particular that $\mathscr{L}_\theta$ is dissipative since for all $\varphi \in \mathscr{D}(\mathscr{L}^m)$ we have

$$\langle \varphi, (\mathscr{L}_\theta + \mathscr{G}^m)\varphi \rangle = \langle \mathscr{L}_\theta \varphi, \varphi \rangle = -\|(-\mathscr{L}_\theta)^{1/2}\varphi\|^2 \leqslant 0.$$

A priori $\mathscr{L}^m$ is only the restriction to $\mathscr{D}(\mathscr{L}^m)$ of the generator $\hat{\mathscr{L}}^m$ of the semigroup $(T_t^m)_t$. However, we will also prove in Lemma A.5.2 below that the operator $\mathscr{L}^m$ is closable and that its closure is indeed the generator $\hat{\mathscr{L}}^m$.

In order to exploit these pieces of information, we have to work with solutions of Galerkin approximations having "near-stationary" fixed-time marginal.

Definition A.2.8. *We say that a stochastic process $(\omega_t)_{t\geqslant 0}$ with values in $\mathcal{S}'(\mathbb{T}^2)$ is (L^2)-incompressible if, for all $T>0$, there exists a constant $C(T)$ such that we have*

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}|\varphi(\omega_t)| \leqslant C(T)\|\varphi\|, \qquad \varphi\in\mathscr{C}.$$

For an incompressible process $(\omega_t)_{t\geqslant 0}$ it makes sense, using a density argument involving cylinder functions, to define $s\mapsto\varphi(\omega_s)$ for all $\varphi\in\mathscr{H}$ as a stochastic process continuous in L^1.

Lemma A.2.9. *Let $\mathbb{E}_{\eta \mathrm{d}\mu}$ be the law of the solution ω^m to the Galerkin approximation (A.2.1) starting from an initial condition $\omega_0^m\sim\eta\mathrm{d}\mu$ with $\eta\in L^2(\mu)$. Then, for any $\Psi\colon C(\mathbb{R}_+;\mathcal{S}')\to\mathbb{R}$,*

$$\mathbb{E}_{\eta\mathrm{d}\mu}|\Psi(\omega^m)| \leqslant \|\eta\|\, \mathbb{E}_\mu(\Psi(\omega^m)^2)^{1/2}.$$

In particular, any such process is incompressible uniformly in m.

Proof. We get

$$\mathbb{E}_{\eta\mathrm{d}\mu}|\Psi(\omega^m)| = \mathbb{E}_\mu[\eta(w_0)|\Psi(\omega^m)|] \leqslant \|\eta\|(\mathbb{E}_\mu\Psi(\omega^m)^2)^{1/2}.$$

Incompressibility easily follows from the fact that μ is an invariant measure for the Galerkin approximations independently of m. □

Definition A.2.10. *A* weight *is a measurable increasing map $w\colon\mathbb{R}_+\to(0,\infty)$ such that there exists $C>0$ with $w(x)\leqslant Cw(x+y)$, for all $x\geqslant 1$ and for $|y|\leqslant 1$. We write as $|w|$ the smallest such constant C. We denote $w(\mathcal{N})$ the self-adjoint operator on $\mathscr{H}$ defined as spectral multiplier.*

We will use the notation D_x to indicate the Malliavin derivative, see e.g. [99], which acts on cylinder functions φ as in Definition A.2.3 as follows,

$$D_x\varphi = \sum_{k=1}^n \partial_{x_k}\Phi(\omega(f_1),\ldots,\omega(f_n))\, f_k(x).$$

Lemma A.2.11. *Let $\eta\in L^2(\mu)$ and let ω^m be a solution to (A.2.1) with* $\mathrm{Law}(\omega_0^m)\sim\eta\mathrm{d}\mu$. *Then this solution is incompressible and, for any $\varphi\in\mathscr{D}(\mathscr{L}^m)$, the process*

$$M_t^{m,\varphi} = \varphi(\omega_t^m) - \varphi(\omega_0^m) - \int_0^t \mathscr{L}^m\varphi(\omega_s^m)\,\mathrm{d}s, \qquad t\geqslant 0,$$

is a continuous martingale with quadratic variation

$$\langle M^{m,\varphi}\rangle_t = \int_0^t \mathscr{E}(\varphi)(\omega_s^m)\,\mathrm{d}s, \qquad \textit{with} \qquad \mathscr{E}(\varphi) = 2\int_{\mathbb{T}^2} |A_x^{\frac{\theta+1}{2}} D_x\varphi|^2\,\mathrm{d}x. \tag{A.2.10}$$

For any weight w, we have

$$\|w(\mathcal{N})(\mathscr{E}(\varphi))^{1/2}\| \lesssim \|w(\mathcal{N}-1)(1-\mathscr{L}_\theta)^{1/2}\varphi\|. \tag{A.2.11}$$

Moreover, for all $p\geqslant 1$, it holds

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \varphi(\omega_s^m)\,\mathrm{d}s\right|^p \lesssim (T^{p/2}\vee T^p)\|c_{2p}^{\mathcal{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|^p, \tag{A.2.12}$$

uniformly in m.

Proof. If φ is a cylinder function, then we have eq. (A.2.2) and in that case Doob's inequality and Lemma A.2.9 yield, for all $T>0$,

$$\mathbb{E}\sup_{t\in[0,T]}|M_t^{m,\varphi}|\lesssim\mathbb{E}(\langle M^{m,\varphi}\rangle_T^{1/2})\leqslant\|\eta\|\mathbb{E}_\mu(\langle M^{m,\varphi}\rangle_T)^{1/2}\lesssim\|\eta\|T^{1/2}\|(\mathscr{E}(\varphi))^{1/2}\|_{L^2(\mu)}.$$

The norm appearing on the right-hand side can be estimated as follows:

$$\begin{aligned}
\|w(\mathscr{N})(\mathscr{E}(\varphi))^{1/2}\|^2 &= 2\int_x\left\|w(\mathscr{N})A_x^{\frac{\theta+1}{2}}D_x\varphi\right\|^2\mathrm{d}x\\
&= 2\int_x\left(\sum_{n\geq 0}(n-1)!w(n-1)^2n^2\left\|A_x^{\frac{\theta+1}{2}}\varphi_n(x,\cdot)\right\|^2_{H_0^1(\mathbb{T}^2)^{\otimes(n-1)}}\right)\\
&\simeq 2\sum_{n\geq 1}n!w(n-1)^2n\sum_{k_{1:n}}\left(\prod_{i=2}^n|2\pi k_i|^2\right)|k_1|^{2(\theta+1)}|\hat{\varphi}_n(k_{1:n})|^2\\
&= 2\sum_{n\geq 1}n!w(n-1)^2n\sum_{k_{1:n}}\left(\prod_{i=1}^n|2\pi k_i|^2\right)|k_1|^{2\theta}|\hat{\varphi}_n(k_{1:n})|^2\\
&= 2\sum_{n\geq 1}n!w(n-1)^2\sum_{k_{1:n}}\left(\prod_{i=1}^n|2\pi k_i|^2\right)L_\theta(k_{1:n})|\hat{\varphi}_n(k_{1:n})|^2\\
&\lesssim 2\|w(\mathscr{N}-1)(1-\mathscr{L}_\theta)^{1/2}\varphi\|^2,
\end{aligned}$$

where we used a symmetrisation in the arguments of $\hat{\varphi}_n$ in the 5th line. Using the bounds (A.2.7) and (A.2.11), one can extend formula (A.2.11) to all functions in $\mathscr{D}(\mathscr{L}^m)$ by a density argument.

As far as (A.2.12) is concerned, let us remark that, provided the process ω^m is started from its stationary measure μ, then the *reversed process* $(\tilde{\omega}_t=\omega_{T-t})_{t\geqslant 0}$ is also stationary and with (martingale) generator $\tilde{\mathscr{L}}^m=\mathscr{L}_\theta-\mathscr{G}^m$. The forward–backward Itô trick used in [93] allows us to represent additive functionals of the form $\int_0^t\mathscr{L}_\theta\psi(\omega_s^m)\,\mathrm{d}s$ as a sum of forward and backward martingales whose quadratic variations satisfy (A.2.10). Therefore,

$$\begin{aligned}
&\mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t\mathscr{L}_\theta\varphi(\omega_s^m)\,\mathrm{d}s\right|^p\right]\lesssim T^{p/2}\|(\mathscr{E}(\varphi))^{p/4}\|^2\\
&\qquad\lesssim T^{p/2}\|c_p^{\mathscr{N}}\mathscr{E}(\varphi)^{1/2}\|^p\lesssim T^{p/2}\|c_p^{\mathscr{N}}(1-\mathscr{L}_\theta)^{1/2}\varphi\|^p.
\end{aligned}\tag{A.2.13}$$

Let $\psi=(1-\mathscr{L}_\theta)^{-1}\varphi$ and exploit (A.2.13) to compute

$$\begin{aligned}
\mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t\varphi(\omega_s^m)\,\mathrm{d}s\right|^p\right] &= \mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t(1-\mathscr{L}_\theta)\psi(\omega_s^m)\,\mathrm{d}s\right|^p\right]\\
&\lesssim \mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t(-\mathscr{L}_\theta)\psi(\omega_s^m)\,\mathrm{d}s\right|^p\right]+\mathbb{E}_\mu\left[\sup_{t\in[0,T]}\left|\int_0^t\psi(\omega_s^m)\,\mathrm{d}s\right|^p\right]\\
&\lesssim T^{p/2}\|c_p^{\mathscr{N}}(1-\mathscr{L}_\theta)^{1/2}\psi\|^p+T^p\|c_p^{\mathscr{N}}\psi\|^p\\
&\lesssim (T^{p/2}\vee T^p)(\|c_p^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|^p+\|c_p^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1}\varphi\|^p)\\
&\lesssim (T^{p/2}\vee T^p)\|c_p^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|^p,
\end{aligned}$$

which is uniform in m. □

A.3 The cylinder martingale problem

We want now to take limits of Galerkin approximations and have a characterisation of the limiting dynamics. The main problem is that the formal limiting (martingale) generator $\mathscr{L}$ does not send cylinder functions to $\mathscr{H}$, therefore we cannot properly formulate a martingale problem for incompressible solutions. However, estimate (A.2.12) suggests that it is reasonable to ask that any limit process $(\omega_t)_{t\geqslant 0}$ satisfies

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \varphi(\omega_s)\,\mathrm{d}s\right|^p \lesssim (T^{p/2}\vee T^p)\|c_{2p}^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|^p, \tag{A.3.1}$$

for all $p\geqslant 1$ and all cylinder functions $\varphi\in\mathscr{C}$. The proof of the next lemma is almost immediate.

Lemma A.3.1. *Assume that a process $(\omega_t)_t$ satisfies (A.3.1) and let $I_t(\varphi)=\int_0^t\varphi(\omega_s)\,\mathrm{d}s$ for all $\varphi\in\mathscr{C}$. Then the map $\varphi\mapsto(I_t(\varphi))_{t\geqslant 0}$ can be extended to all $\varphi\in(1-\mathscr{L}_\theta)^{1/2}\mathscr{H}$. The process $(I_t(\varphi))_{t\geqslant 0}$ is almost surely continuous.*

Proof. Take $(\varphi_n)_n\subseteq\mathscr{C}$ such that $\sum_n\|(1-\mathscr{L}_\theta)^{1/2}\varphi_n-\varphi\|<\infty$, then it is easy to see that $(I(\varphi_n))_n$ is a Cauchy sequence in $C([0,T];\mathbb{R})$ a.s. with limit $I(\varphi)\in C([0,T];\mathbb{R})$. It satisfies (A.3.1) by Fatou's lemma and, therefore, depends only on φ and not on the particular approximating sequence. □

From this we deduce that for such processes we have

$$\lim_{m\to\infty}\int_0^t(\mathscr{L}^m\varphi)(\omega_s)\mathrm{d}s=\int_0^t(\mathscr{L}\varphi)(\omega_s)\mathrm{d}s,$$

in probability and in L^p for cylinder functions $\varphi\in\mathscr{C}$. Here, on the right-hand side the quantity $\mathscr{L}\varphi$ is defined as $\mathscr{L}\varphi=\mathscr{L}_\theta\varphi+\lim_{m\to\infty}\mathscr{G}^m\varphi$, that is an element of the space of distributions $(1-\mathscr{L}_\theta)^{1/2}\mathscr{H}$. The limit exists and is unique thanks to the uniform estimates on $\mathscr{G}^m$ in Lemma A.2.7. As a consequence, we have also a notion of martingale problem w.r.t. the operator $\mathscr{L}$ involving only cylinder functions.

Definition A.3.2. *A process $(\omega_t)_{t\geqslant 0}$ with trajectories in $C(\mathbb{R}_+;\mathscr{S}')$ solves the* cylinder martingale problem for $\mathscr{L}$ *with initial distribution ν if $\omega_0\sim\nu$ and if the following conditions are satisfied:*

i. $(\omega_t)_t$ is incompressible,

ii. the Itô trick works: for all cylinder functions φ and all $p\geqslant 1$, we have eq. (A.3.1).

iii. for any $\varphi\in\mathscr{C}$, the process

$$M_t^\varphi=\varphi(\omega_t)-\varphi(\omega_0)-\int_0^t\mathscr{L}\varphi(\omega_s)\,\mathrm{d}s,\qquad t\geqslant 0, \tag{A.3.2}$$

is a continuous martingale with quadratic variation $\langle M^\varphi\rangle_t=\int_0^t\mathscr{E}(\varphi)(\omega_s)\,\mathrm{d}s$. The integral on the right-hand side of eq. (A.3.2) is defined according to Lemma A.3.1.

Theorem A.3.3. *Let $\eta\in L^2(\mu)$ and, for each $m\geqslant 1$, let (ω^m) be the solution to (A.2.1) with $\omega_0^m\sim\eta\,\mathrm{d}\mu$. Then the family $(\omega^m)_{m\in\mathbb{N}}$ is tight in $C(\mathbb{R}_+;\mathscr{S}')$ and any weak limit ω solves the cylinder martingale problem for $\mathscr{L}$ with initial distribution $\eta\,\mathrm{d}\mu$ according to Definition A.3.2 and we have*

$$\mathbb{E}[|\varphi(\omega_t)-\varphi(\omega_s)|^p]\lesssim(|t-s|^{p/2}\vee|t-s|^p)\|c_{4p}^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|^p \tag{A.3.3}$$

for any $p\geqslant 2$ and $\varphi\in\mathscr{C}$.

Proof. The proof follows the one for Theorem 4.6 in [99].

Step 1. Consider $p\geqslant 2$ and $\varphi\in\mathscr{C}$. We want to derive an estimate for $\mathbb{E}[|\varphi(\omega_t^m)-\varphi(\omega_s^m)|^p]$. We write then $\varphi(\omega_t^m)-\varphi(\omega_s^m)=\int_s^t\mathscr{L}^m\varphi(\omega_r^m)\,\mathrm{d}r+M_t^{m,\varphi}-M_s^{m,\varphi}$, and get from Lemma A.2.9 and eq. (A.2.12) the following bound

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t\mathscr{L}^m\varphi(\omega_r^m)\mathrm{d}r\right|^p\right] &\lesssim \left[\mathbb{E}_\mu\left|\int_s^t\mathscr{L}^m\varphi(\omega_r^m)\mathrm{d}r\right|^{2p}\right]^{1/2}\\ &\lesssim (|t-s|^{p/2}\vee|t-s|^p)\|c_{4p}^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|^p.\end{aligned}$$

The martingale term can be bounded by means of the Burkholder-Davis-Gundy inequality and (A.2.11) as follows:

$$\begin{aligned}\mathbb{E}[|M_t^{m,\varphi}-M_s^{m,\varphi}|^p] &\lesssim \mathbb{E}\left[\left(\int_s^t\mathscr{E}(\varphi)(\omega_r^m)\,\mathrm{d}r\right)^{p/2}\right]\lesssim\left[\mathbb{E}_\mu\left(\int_s^t\mathscr{E}(\varphi)(\omega_r^m)\,\mathrm{d}r\right)^p\right]^{1/2}\\ &\lesssim |t-s|^{p/2}\|(\mathscr{E}(\varphi))^{p/2}\|\lesssim|t-s|^{p/2}\|c_{2p}^{\mathscr{N}}(\mathscr{E}(\varphi))^{1/2}\|^p\\ &\lesssim |t-s|^{p/2}\|c_{2p}^{\mathscr{N}}(1-\mathscr{L}_\theta)^{1/2}\varphi\|^p.\end{aligned}$$

Therefore,

$$\mathbb{E}[|\varphi(\omega_t^m)-\varphi(\omega_s^m)|^p]\lesssim(|t-s|^{p/2}\vee|t-s|^p)\|c_{4p}^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|^p. \tag{A.3.4}$$

The law of the initial condition $\varphi(\omega_0^m)$ is independent of m, and by Kolmogorov's continuity criterion the sequence of real-valued processes $(\varphi(\omega^m))_m$ is tight in $C(\mathbb{R}_+;\mathbb{R})$ whenever $p\geqslant 4$ and $\varphi\in\mathscr{C}$ is such that $\|c_{4p}^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\|<\infty$. Note that this space contains in particular all the functions of the form $\varphi(\omega)=\omega(f)$ with $f\in C^\infty(\mathbb{T}^2)$. Hence, we can apply Mitoma's criterion [139] to get the tightness of the sequence $(\omega^m)_m$ in $C(\mathbb{R}_+;\mathscr{S}')$.

Step 2. Since $\omega_0^m\sim\eta\,\mathrm{d}\mu$, any weak limit has initial distribution $\eta\,\mathrm{d}\mu$. Incompressibility is also clear since, for any $\varphi\in\mathscr{H}$, we have

$$\mathbb{E}[|\varphi(\omega_t)|]\leqslant\liminf_{m\to\infty}\mathbb{E}[|\varphi(\omega_t^m)|]\leqslant\|\eta\|\|\varphi\|.$$

Using cylinder functions, we can pass to the limit in eq. (A.2.12) and prove that any accumulation point $(\omega_t)_t$ satisfies eq. (A.3.1). It remains to check the martingale characterisation (A.3.2). Fix $\varphi\in\mathscr{C}$ and let $(\psi_n)_n\subseteq\mathscr{C}$ be such that $\psi_n\to\mathscr{L}\varphi$ in $(1+\mathscr{L}_\theta)^{1/2}\mathscr{H}$. By convergence in law, incompressibility, eq. (A.2.12) and eq. (A.3.1), we have that

$$\begin{aligned}&\mathbb{E}\left[\left(\varphi(\omega_t)-\varphi(\omega_s)-\int_s^t\mathscr{L}\varphi(\omega_r)\,\mathrm{d}r\right)G((\omega_r)_{r\in[0,s]})\right]\\ =&\lim_{n\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t)-\varphi(\omega_s)-\int_s^t\psi_n(\omega_r)\,\mathrm{d}r\right)G((\omega_r)_{r\in[0,s]})\right]\\ =&\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t^m)-\varphi(\omega_s^m)-\int_s^t\psi_n(\omega_r^m)\,\mathrm{d}r\right)G((\omega_r^m)_{r\in[0,s]})\right]\\ =&\lim_{m\to\infty}\mathbb{E}\left[\left(\varphi(\omega_t^m)-\varphi(\omega_s^m)-\int_s^t\mathscr{L}\varphi(\omega_r^m)\,\mathrm{d}r\right)G((\omega_r^m)_{r\in[0,s]})\right],\end{aligned}$$

where the exchange of limits in the last line is justified by the uniformity in m of the bound in eq. (A.2.12). By dominated convergence in the estimates leading to Lemma A.2.11 one has

$$\|(1-\mathscr{L}_\theta)^{-1/2}(\mathscr{L}\varphi-\mathscr{L}^m\varphi)\|=\|(1-\mathscr{L}_\theta)^{-1/2}(\mathscr{G}\varphi-\mathscr{G}^m\varphi)\|\lesssim o(1)\|(1+\mathscr{N})^{3/2}(1-\mathscr{L}_\theta)^{1/2}\varphi\|$$

as $m \to \infty$. This is enough to conclude (again using eq. (A.2.12)) that

$$\begin{aligned} & \lim_{m\to\infty} \mathbb{E}\left[\left(\varphi(\omega_t^m) - \varphi(\omega_s^m) - \int_s^t \mathscr{L}\varphi(\omega_r^m)\,\mathrm{d}r\right) G((\omega_r^m)_{r\in[0,s]})\right] \\ = & \lim_{m\to\infty} \mathbb{E}\left[\left(\varphi(\omega_t^m) - \varphi(\omega_s^m) - \int_s^t \mathscr{L}^m\varphi(\omega_r^m)\,\mathrm{d}r\right) G((\omega_r^m)_{r\in[0,s]})\right] = 0, \end{aligned} \tag{A.3.5}$$

since $(\omega_t^m)_t$ solves indeed the martingale problem for $\mathscr{L}^m$. This establishes that any accumulation point $(\omega_t)_t$ is a solution to the cylinder martingale problem for $\mathscr{L}$. Similarly, one can pass to the limit on the martingales $(M_t^{m,\varphi})_t$ to show that the limiting quadratic variation is as claimed. □

A.4 Uniqueness of solutions

Uniqueness of solutions to the cylinder martingale problem depends on the control of the associated Kolmogorov equation.

The following standard fact on generators of semigroups that will be useful in our further considerations. For the sake of the reader we provide also a proof to illustrate the relation between the Kolmogorov equation for a concrete operator and abstract semigroup theory.

Lemma A.4.1. *Let $\mathscr{A}$ be a densely defined, dissipative operator on $\mathscr{H}$ and assume that we can solve the Kolmogorov equation $\partial_t\varphi(t) = \mathscr{A}\varphi(t)$ in $C(\mathbb{R}_+;\mathscr{D}(\mathscr{A})) \cap C^1(\mathbb{R}_+;\mathscr{H})$ with initial condition $\varphi(0) = \varphi_0$ in a dense set $\mathscr{U}_{\mathscr{A}} \subseteq \mathscr{D}(\mathscr{A})$. Then $\mathscr{A}$ is closable and its closure $\mathscr{B}$ is the unique extension of $\mathscr{A}$ which generates a strongly continuous semigroup of contractions $(T_t)_{t\geqslant 0}$. Moreover, we have*

$$\mathscr{A}T_t\varphi_0 = T_t\mathscr{A}\varphi_0, \tag{A.4.1}$$

for all $\varphi_0 \in \mathscr{U}_{\mathscr{A}}$.

Proof. Since $\mathscr{A}$ is dissipative, the solution to the Kolmogorov equation is unique and $\|\varphi(t)\| \leqslant \|\varphi_0\|$. Then, if we let $T_t\varphi_0 = \varphi(t)$ for $\varphi_0 \in \mathscr{U}_{\mathscr{A}}$ we can extend T_t by continuity to the whole space $\mathscr{H}$ as a contraction. By uniqueness, we have then $T_{t+s}\varphi_0 = T_tT_s\varphi_0$, since $t \mapsto T_{t+s}\varphi_0$ solves the equation with initial condition $T_s\varphi_0$. Moreover, for $\varphi_0 \in \mathscr{U}_{\mathscr{A}}$, we have that

$$T_t\varphi_0 - \varphi_0 = \int_0^t \mathscr{A}T_s\varphi_0\,\mathrm{d}s, \tag{A.4.2}$$

which implies that $t \mapsto T_t\varphi_0$ is strongly continuous. Again by density, we deduce that $(T_t)_{t\geqslant 0}$ is a strongly continuous semigroup. Let now $\mathscr{B}$ be its Hille–Yosida generator. Then (A.4.2) implies that $\mathscr{B}\varphi_0 = \partial_tT_t\varphi_0|_{t=0} = \mathscr{A}\varphi_0$ for all $\varphi_0 \in \mathscr{U}_{\mathscr{A}}$, and therefore for all $\varphi_0 \in \mathscr{D}(\mathscr{A})$ since $\mathscr{B}$ is closed. So $\mathscr{B}$ is an extension of $\mathscr{A}$ and therefore $\mathscr{A}$ is closable. Assume now that there exists another extension $\tilde{\mathscr{B}}$ which is the generator of another strongly continuous semigroup $(S_t)_{t\geqslant 0}$ of contractions. Now, for all $\varphi_0 \in \mathscr{U}_{\mathscr{A}} \subseteq \mathscr{D}(\mathscr{A}) \subseteq \mathscr{D}(\tilde{\mathscr{B}})$ we have $\partial_tS_t\varphi_0 = \tilde{\mathscr{B}}S_t\varphi_0$, but also $\partial_tT_t\varphi_0 = \mathscr{A}T_t\varphi_0 = \tilde{\mathscr{B}}T_t\varphi_0$. Since $\tilde{\mathscr{B}}$ is dissipative (due to the fact that its semigroup is contractive), the associated Kolmogorov equation must have a unique solution and, as a consequence, $T_t\varphi_0 = S_t\varphi_0$, which by density implies that $T = S$ and that $\mathscr{B} = \tilde{\mathscr{B}}$. Now observe that, if $\varphi_0 \in \mathscr{U}_{\mathscr{A}}$, then $T_t\varphi_0 \in \mathscr{D}(\mathscr{A})$ and by standard results on contraction semigroups (see e.g. Proposition 1.1.5 in [68]) we have $\mathscr{A}T_t\varphi_0 = \mathscr{B}T_t\varphi_0 = T_t\mathscr{B}\varphi_0 = T_t\mathscr{A}\varphi_0$. □

Theorem A.5.10 below tells us that we can find a dense domain $\mathscr{D}(\mathscr{L})\subseteq\mathscr{H}$ for $\mathscr{L}$ such that the Kolmogorov equation

$$\partial_t\varphi(t)=\mathscr{L}\varphi(t),\qquad t\geqslant 0, \tag{A.4.3}$$

has a unique solution in $C(\mathbb{R}_+;\mathscr{D}(\mathscr{L}))\cap C^1(\mathbb{R}_+;\mathscr{H})$ for any initial condition in a dense set $\mathscr{U}\subseteq\mathscr{H}$. As a first consequence, Lemma A.4.1 tells us that $\mathscr{L}$ is closable and its closure $\mathscr{L}^\natural$ is the generator of a strongly continuous semigroup $(T_t)_{t\geqslant 0}$ and $\varphi(t)=T_t\varphi$ for all $\varphi\in\mathscr{U}$.

Lemma A.4.2. *Let $\varphi\in C(\mathbb{R}_+;\mathscr{D}(\mathscr{L}))\cap C^1(\mathbb{R}_+;\mathscr{H})$ and let ω be a solution to the cylinder martingale problem for $\mathscr{L}$. Then*

$$\varphi(t,\omega_t)-\varphi(0,\omega_0)-\int_0^t(\partial_s+\mathscr{L})\varphi(s,\omega_s)\,\mathrm{d}s,\qquad t\geqslant 0,$$

is a martingale.

Proof. By an approximation argument, it is easy to see that for any $\varphi\in\mathscr{D}(\mathscr{L})$ the process

$$\varphi(\omega_t)-\varphi(\omega_0)-\int_0^t\mathscr{L}\varphi(\omega_s)\,\mathrm{d}s,\qquad t\geqslant 0,$$

is a martingale, where the integral on the right-hand side is now understood as a standard Lebesgue integral of the continuous process $s\mapsto(\mathscr{L}\varphi)(\omega_s)$ (which is well defined a.s.). The proof of the extension to time-dependent functions follows the same lines as that of Lemma A.3 in [99]. □

For an incompressible process we have that, for all $t\geqslant 0$,

$$\int_0^s(\partial_r+\mathscr{L})T_{t-r}\varphi(\omega_r)\,\mathrm{d}r=0,\qquad s\in[0,t]$$

for all $\varphi\in\mathscr{D}(\mathscr{L}^\natural)$, and therefore also that $(T_{t-s}\varphi(\omega_s))_{s\in[0,t]}$ is a martingale for any solution of the cylinder martingale problem for $\mathscr{L}$. This easily implies the main result of the paper.

Theorem A.4.3. *There exists a unique solution ω to the cylinder martingale problem for $\mathscr{L}$ with initial distribution $\omega_0\sim\eta\,\mathrm{d}\mu$ with $\eta\in L^2(\mu)$. Moreover, ω is a homogeneous Markov process with transition kernel $(T_t)_{t\geqslant 0}$ and with invariant measure μ.*

Proof. Let us first prove that $(\omega_t)_{t\geqslant 0}$ is Markov. Let $0\leqslant t<s$, let X be an $\mathscr{F}_t$-measurable bounded random variable, where $\mathscr{F}_t=\sigma(\omega_r: r\in[0,t])$, and let $\varphi\in\mathscr{D}(\mathscr{L}^\natural)$, then $(T_{s-t}\varphi(\omega_t))_{t\in[0,s]}$ is a martingale and

$$\mathbb{E}[X\varphi(\omega_s)]\;=\;\mathbb{E}[XT_{s-s}\varphi(\omega_s)]=\mathbb{E}[XT_{s-t}\varphi(\omega_t)]$$

i.e., $\mathbb{E}[\varphi_0(\omega_s)|\mathscr{F}_t]=T_{s-t}\varphi(\omega_t)=\mathbb{E}[\varphi_0(\omega_s)|\omega_t]$, and the Markov property is a consequence of another density argument. Moreover, its transition kernel is given by the semigroup $(T_t)_{t\geqslant 0}$. By an induction argument, it is clear that any finite-dimensional marginal is determined by T and by the law of $\omega_0\sim\eta\,\mathrm{d}\mu$. As a consequence, the law of the process is unique. If $\omega_0\sim\mu$, then the process is stationary. □

Remark A.4.4. As a by-product note that the formula $(T_t\varphi)(\omega_0)=\mathbb{E}[\varphi(\omega_t)|\omega_0]$ allows to extend the semigroup T to a bounded semigroup in L^p for all $p\in[1,\infty]$ since

$$|\langle\psi,T_t\varphi\rangle|=|\mathbb{E}_\mu[\psi(\omega_0)(T_t\varphi)(\omega_0)]|=|\mathbb{E}_\mu[\psi(\omega_0)\varphi(\omega_t)]|\leqslant\|\psi\|_{L^p}\|\varphi\|_{L^q}$$

for all $\psi, \varphi \in L^\infty(\mu)$ and all $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Therefore $\|T_t \varphi\|_{L^q} \leqslant \|\varphi\|_{L^q}$. Moreover for all $\varphi \in \mathscr{C}$ such that $\|c_{4p}^{\mathscr{N}}(1-\mathscr{L}_\theta)^{-1/2}\varphi\| < \infty$ we have

$$\|T_t\varphi - \varphi\|_{L^p} = \sup_{\psi : \|\psi\|_{L^q} \leqslant 1} \mathbb{E}_\mu[\psi(\omega_0)(\varphi(\omega_t) - \varphi(\omega_0))] \leqslant (\mathbb{E}_\mu[|\varphi(\omega_t) - \varphi(\omega_0)|^p])^{1/p} \to 0$$

as $t \to 0$ by eq. (A.3.3). An approximation argument gives that $(T_t)_{t \geqslant 0}$ is strongly continuous in L^p for all $1 \leqslant p < \infty$.

A.5 The Kolmogorov equation

It remains to determine a suitable domain for $\mathscr{L}$ and solve the Kolmogorov backward equation

$$\partial_t \varphi(t) = \mathscr{L}\varphi(t),$$

for a sufficiently large class of initial data. In order to do so, we consider the backward equation for the Galerkin approximation with generator $\mathscr{L}^m$ and derive uniform estimates. By compactness, this yields the existence of strong solutions to the backward equation after removing the cutoff. Uniqueness follows by the dissipativity of $\mathscr{L}$.

A.5.1 A priori estimates

Lemma A.5.1. *For any $\varphi_0 \in \mathscr{V} := (1+\mathscr{N})^{-2}(1-\mathscr{L}_\theta)^{-1}\mathscr{H}$, there exists a solution*

$$\varphi^m \in C(\mathbb{R}_+, \mathscr{D}(\mathscr{L}^m)) \cap C^1(\mathbb{R}_+; \mathscr{H})$$

to the backward Kolmogorov equation

$$\partial_t \varphi^m(t) = \mathscr{L}^m \varphi^m(t)$$

with $\varphi^m(0) = \varphi_0$ and which satisfies the estimates

$$\|(1+\mathscr{N})^p \varphi^m(t)\|^2 + \int_0^t e^{-C(t-s)} \|(1+\mathscr{N})^p (1-\mathscr{L}_\theta)^{1/2} \varphi^m(s)\|^2 \mathrm{d}s \lesssim_p e^{Ct} \|(1+\mathscr{N})^p \varphi_0\|^2,$$

and

$$\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)\varphi^m(t)\| \lesssim_{t,m,p} \|(1+\mathscr{N})^{p+1}(1-\mathscr{L}_\theta)\varphi_0\|,$$

for all $t \geqslant 0$ and $p \geqslant 1$.

Proof. Take $h > 0$ and let $\mathscr{G}^{m,h} = J_h \mathscr{G}^m J_h$, where $J_h = e^{-h(\mathscr{N} - \mathscr{L}_\theta)}$. The operator $\mathscr{G}^{m,h}$ is bounded on $\mathscr{H}$ by the estimates in Lemma A.2.7. Consider $\varphi_0^m \in \mathscr{D}(\mathscr{L}^m)$. Using the fact that $\mathscr{L}_\theta$ is the generator of a contraction semigroup, we take $(\varphi^m(t))_{t \geqslant 0}$ to be the solution to the integral equation

$$\varphi^{m,h}(t) = e^{\mathscr{L}_\theta t}\varphi_0 + \int_0^t e^{\mathscr{L}_\theta(t-s)} \mathscr{G}^{m,h} \varphi^{m,h}(s) \mathrm{d}s \tag{A.5.1}$$

in $C(\mathbb{R}_+; (1-\mathscr{L}_\theta)\mathscr{H})$ and deduce easily that $\varphi^{m,h}$ solves the equation $\partial_t \varphi^{m,h}(t) = (\mathscr{L}_\theta + \mathscr{G}^{m,h})\varphi^{m,h}(t)$. Moreover

$$\|(1-\mathscr{L}_\theta)(1+\mathscr{N})^{2p}\varphi^{m,h}(t)\| \leqslant C_{t,h,m} \|(1-\mathscr{L}_\theta)(1+\mathscr{N})^{2p}\varphi_0\|,$$

for any finite $t \geqslant 0$ and $p > 0$ but not uniformly in h and m. Now

$$\begin{aligned}
& \langle (1+\mathscr{N})^{2p}\varphi^{m,h}(t), \mathscr{G}^{m,h}\varphi^{m,h}(t) \rangle \\
= \; & \langle (1+\mathscr{N})^{2p}\varphi^{m,h}(t), \mathscr{G}_+^{m,h}\varphi^{m,h}(t) \rangle - \langle \mathscr{G}_+^{m,h}(1+\mathscr{N})^{2p}\varphi^{m,h}(t), \varphi^{m,h}(t) \rangle \\
= \; & \langle (1+\mathscr{N})^{2p}\varphi^{m,h}(t), \mathscr{G}_+^{m,h}\varphi^{m,h}(t) \rangle - \langle \mathscr{N}^{2p}\mathscr{G}_+^{m,h}\varphi^{m,h}(t), \varphi^{m,h}(t) \rangle \\
= \; & \langle ((1+\mathscr{N})^{2p} - \mathscr{N}^{2p})\varphi^{m,h}(t), \mathscr{G}_+^{m,h}\varphi^{m,h}(t) \rangle.
\end{aligned}$$

Using $|(1+\mathscr{N})^{2p}-\mathscr{N}^{2p}|\lesssim(1+\mathscr{N})^{2p-1}$ and the uniform estimates in Lemma A.2.7 we have that, for some $\sigma\in(0,1/2)$,

$$\begin{aligned}
&|\langle((1+\mathscr{N})^{2p}-\mathscr{N}^{2p})\varphi^{m,h}(t),\mathscr{G}_+^{m,h}\varphi^{m,h}(t)\rangle|\\
\leqslant\ & \|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^\sigma\varphi^{m,h}(t)\|\|(1+\mathscr{N})^{p-1}(1-\mathscr{L}_\theta)^{-\sigma}\mathscr{G}_+^{m,h}\varphi^{m,h}(t)\|\\
\leqslant\ & \|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^\sigma\varphi^{m,h}(t)\|^2.
\end{aligned}$$

Therefore,

$$|\langle(1+\mathscr{N})^{2p}\varphi^{m,h}(t),\mathscr{G}^{m,h}\varphi^{m,h}(t)\rangle|\lesssim\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^\sigma\varphi^{m,h}(t)\|^2$$

and by interpolation we can bound this by

$$|\langle(1+\mathscr{N})^{2p}\varphi^{m,h}(t),\mathscr{G}^{m,h}\varphi^{m,h}(t)\rangle|\leqslant C_\delta\|(1+\mathscr{N})^p\varphi^{m,h}(t)\|^2+\delta\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1/2}\varphi^{m,h}(t)\|^2,$$

for some small $\delta>0$. Therefore, we have

$$\begin{aligned}
&\partial_t\frac{1}{2}\|(1+\mathscr{N})^p\varphi^{m,h}(t)\|^2=\langle(1+\mathscr{N})^{2p}\varphi^{m,h}(t),(\mathscr{L}_\theta+\mathscr{G}^{m,h})\varphi^{m,h}(t)\rangle\\
=\ & -\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1/2}\varphi^{m,h}(t)\|^2+\|(1+\mathscr{N})^p\varphi^{m,h}(t)\|^2+\langle(1+\mathscr{N})^{2p}\varphi^{m,h}(t),\mathscr{G}^{m,h}\varphi^{m,h}(t)\rangle\\
\leqslant\ & -(1-\delta)\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1/2}\varphi^{m,h}(t)\|^2+C_\delta'\|(1+\mathscr{N})^p\varphi^{m,h}(t)\|^2
\end{aligned}$$

uniformly in m and h. Integrating this inequality gives

$$\|(1+\mathscr{N})^p\varphi^{m,h}(t)\|^2+\int_0^t e^{-C(t-s)}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1/2}\varphi^{m,h}(s)\|^2\mathrm{d}s\lesssim e^{Ct}\|(1+\mathscr{N})^p\varphi_0\|^2$$

for all $p\geqslant1$ where the constants are uniform in m and h. Inserting this a priori bound in the mild formulation in eq. (A.5.1) we obtain

$$\begin{aligned}
\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)\varphi^{m,h}(t)\|&\leqslant\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)\varphi^{m,h}(0)\|+\int_0^t\|(1+\mathscr{N})^{p+1}(1-\mathscr{L}_\theta)^{1/2}\varphi^{m,h}(s)\|\mathrm{d}s\\
&\lesssim_t\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)\varphi_0\|+\|(1+\mathscr{N})^{p+1}\varphi_0\|,
\end{aligned}$$

where we also used that

$$\begin{aligned}
\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)\mathscr{G}^{m,h}\varphi^{m,h}(s)\| &\leqslant C(m)\|(1+\mathscr{N})^{p+1}\mathscr{G}^{m,h}\varphi^{m,h}(s)\|\\
&\lesssim_m \|(1+\mathscr{N})^{p+1}(1-\mathscr{L}_\theta)^{1/2}\varphi^{m,h}(s)\|
\end{aligned}$$

by the presence of the Galerkin projectors and our (non-uniform) bounds. Indeed, note that

$$(1-\mathscr{L}_\theta)\Pi_m\lesssim|m|^{2\theta}(1+\mathscr{N})\Pi_m.$$

We conclude that

$$\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)\varphi^{m,h}(t)\|\lesssim_{t,m}\|(1+\mathscr{N})^{p+1}(1-\mathscr{L}_\theta)\varphi_0\|,$$

uniformly in h. We can then pass to the limit (by subsequence) as $h\to0$ and obtain a function $\varphi^m\in C(\mathbb{R}_+,(1+\mathscr{N})^{-p}(1-\mathscr{L}_\theta)^{-1}\mathscr{H})$ satisfying the estimates

$$\|(1+\mathscr{N})^p\varphi^m(t)\|^2+\int_0^t e^{-C(t-s)}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1/2}\varphi^m(s)\|^2\mathrm{d}s\lesssim e^{Ct}\|(1+\mathscr{N})^p\varphi_0\|^2$$

and

$$\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)\varphi^m(t)\|\lesssim_{t,m}\|(1+\mathscr{N})^{p+1}(1-\mathscr{L}_\theta)\varphi_0\|,$$

for all $t\geqslant0$ and $p\geqslant1$. As a consequence, $\varphi^m\in C(\mathbb{R}_+,\mathscr{D}(\mathscr{L}^m))$ for all $t\geqslant0$ as soon as $\|(1+\mathscr{N})^2(1-\mathscr{L}_\theta)\varphi_0\|<\infty$. By passing to the limit in the equation, φ^m also satisfies

$$\partial_t\varphi^m(t)=(\mathscr{L}_\theta+\mathscr{G}^m)\varphi^m(t)=\mathscr{L}^m\varphi^m(t). \qquad \square$$

Recall that we write T^m to indicate the semigroup generated by the Galerkin approximation ω^m. Moreover, if we denote by $\hat{\mathscr{L}}^m$ its Hille–Yosida generator, we have the following result.

Lemma A.5.2. *$(\mathscr{L}^m, \mathscr{D}(\mathscr{L}^m))$ is closable and its closure is the generator $\hat{\mathscr{L}}^m$. In particular, if $\varphi \in \mathscr{V}$, then $\varphi^m(t) = T_t^m \varphi$ solves*

$$\partial_t \varphi^m(t) = \mathscr{L}^m \varphi^m(t),$$

and we have

$$\mathscr{L}^m T_t^m \varphi = T_t^m \mathscr{L}^m \varphi.$$

Proof. Let $(\omega_t^m)_{t \geqslant 0}$ be a solution to the Galerkin approximation (A.2.1) with initial condition ω_0. If $\varphi \in \mathscr{C}$ is a cylinder function, then we have

$$T_t^m \varphi(\omega_0) - \varphi(\omega_0) = \mathbb{E}_{\omega_0}\left[\int_0^t \mathscr{L}^m \varphi(\omega_s^m)\, \mathrm{d}s\right] = \int_0^t T_s^m(\mathscr{L}^m \varphi)(\omega_0)\, \mathrm{d}s.$$

By approximation (using a Bochner integral in $\mathscr{H}$ on the right-hand side), we can extend this point-wise formula to all $\varphi \in \mathscr{D}(\mathscr{L}^m)$ obtaining for them that $T_t^m \varphi - \varphi = \int_0^t T_s^m \mathscr{L}^m \varphi \mathrm{d}s$ in $\mathscr{H}$. For every $\varphi \in \mathscr{D}(\mathscr{L}^m)$, Lemma A.2.2 implies that the map $s \mapsto T_s^m \mathscr{L}^m \varphi \in \mathscr{H}$ is continuous, and therefore

$$\frac{T_t^m \varphi - \varphi}{t} \to \mathscr{L}^m \varphi, \qquad \text{as } t \to 0, \qquad \varphi \in \mathscr{D}(\mathscr{L}^m),$$

with convergence in $\mathscr{H}$. As a consequence, $\varphi \in \mathscr{D}(\hat{\mathscr{L}}^m)$ and we conclude that $\hat{\mathscr{L}}^m$ is an extension of $(\mathscr{L}^m, \mathscr{D}(\mathscr{L}^m))$. By Lemma A.4.1, we have that the closure of $\mathscr{L}^m$ is $\hat{\mathscr{L}}^m$ and that $\mathscr{L}^m T_t^m \varphi = T_t^m \mathscr{L}^m \varphi$ for all $\varphi \in \mathscr{V}$. □

Using the commutation $\mathscr{L}^m T_t^m \varphi = T_t^m \mathscr{L}^m \varphi$, we are able to get better estimates, uniform in m.

Corollary A.5.3. *For all $\varphi_0 \in \mathscr{V}$ and for all $\alpha \geqslant 1$, we have*

$$\|(1+\mathscr{N})^\alpha \partial_t \varphi^m(t)\|^2 = \|(1+\mathscr{N})^\alpha \mathscr{L}^m \varphi^m(t)\|^2 \lesssim e^{tC} \|(1+\mathscr{N})^\alpha \mathscr{L}^m \varphi_0\|^2, \tag{A.5.2}$$

and

$$\|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \varphi^m(t)\|^2 \lesssim t e^{tC} \|(1+\mathscr{N})^\alpha \mathscr{L}^m \varphi_0^m\|^2 + \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \varphi_0\|^2. \tag{A.5.3}$$

Proof. Recall $T_t^m \varphi_0^m = \varphi^m(t)$. We already know

$$e^{-tC} \|(1+\mathscr{N})^\alpha T_t^m \varphi_0^m\|^2 + \int_0^\infty e^{-sC} \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} T_s^m \varphi_0^m\|^2 \mathrm{d}s \lesssim \|(1+\mathscr{N})^\alpha \varphi_0\|^2,$$

which yields

$$\begin{aligned}
\int_0^\infty e^{-tC} \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \partial_t T_t^m \varphi_0^m\|^2 \mathrm{d}t &= \int_0^\infty e^{-tC} \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} T_t^m \mathscr{L}^m \varphi_0\|^2 \mathrm{d}t \\
&\lesssim \|(1+\mathscr{N})^\alpha \mathscr{L}^m \varphi_0\|^2,
\end{aligned}$$

and

$$\begin{aligned}
& \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} T_t^m \varphi_0\|^2 \\
\lesssim\ & \left\| \int_0^t (1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \partial_s T_s^m \varphi_0^m \, \mathrm{d}s \right\|^2 + \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \varphi_0\|^2 \\
\leqslant\ & t \int_0^t \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \partial_s T_s^m \varphi_0\|^2 \mathrm{d}s + \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \varphi_0\|^2 \\
\leqslant\ & t e^{tC} \int_0^t e^{-sC} \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \partial_s T_s^m \varphi_0\|^2 \mathrm{d}s + \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \varphi_0\|^2 \\
\lesssim\ & t e^{tC} \|(1+\mathscr{N})^\alpha \mathscr{L}^m \varphi_0\|^2 + \|(1+\mathscr{N})^\alpha (1-\mathscr{L}_\theta)^{1/2} \varphi_0\|^2,
\end{aligned}$$

which is what claimed. □

A.5.2 Controlled structures

The a priori bounds (A.5.2) and (A.5.3) bring us in position to control $\|\varphi^m(t)\|$, $\|\partial_t\varphi^m(t)\|$, and $\|\mathscr{L}^m\varphi^m(t)\|$ uniformly in m and locally uniformly in t, but in order to study the limiting Kolmogorov backward equation we have first to deal with the limiting operator $\mathscr{L}$ and to define a domain $\mathscr{D}(\mathscr{L})$.

To take care of the term $\mathscr{G}$ in the limiting operator $\mathscr{L}$, we decompose it by means of a cut-off function $\mathscr{M}=M(\mathscr{N})$ as follows

$$\mathscr{G}^m = 1_{|\mathscr{L}_\theta|\geqslant\mathscr{M}}\mathscr{G}^m + 1_{|\mathscr{L}_\theta|<\mathscr{M}}\mathscr{G}^m =: \mathscr{G}^{m,\succ} + \mathscr{G}^{m,\prec}.$$

We then set

$$\varphi^{m,\sharp} := \varphi^m - (1-\mathscr{L}_\theta)^{-1}\mathscr{G}^{m,\succ}\varphi^m, \tag{A.5.4}$$

so that

$$(1-\mathscr{L}^m)\varphi^m = (1-\mathscr{L}_\theta)\varphi^{m,\sharp} + \mathscr{G}^{m,\prec}\varphi^m. \tag{A.5.5}$$

Lemma A.5.4. *Let w be a weight, $L\geqslant 1$, $\bar{\varepsilon}\in]0,(\theta-1)/(2\theta)[$ and $M(n)=L\,(n+1)^{3\theta/(\theta-1-2\theta\varepsilon)}$. Then we have*

$$\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-1/2}\mathscr{G}^{m,\succ}\psi\| \lesssim |w| L^{-\frac{(\theta-1)}{2\theta}+\bar{\varepsilon}} \|w(\mathscr{N})(1-\mathscr{L}_\theta)^{1/2}\psi\|. \tag{A.5.6}$$

Consequently, there exists $L_0=L_0(|w|)$ such that, for all $L\geqslant L_0$ and all $\varphi^\sharp\in w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H}$, there is a unique $\varphi^m=\mathscr{K}^m\varphi^\sharp$ such that

$$\varphi^m = (1-\mathscr{L}_\theta)^{-1}\mathscr{G}^{m,\succ}\varphi^m + \varphi^\sharp \in w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H},$$

which satisfies the bound

$$\begin{aligned} &\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{1/2}\mathscr{K}^m\varphi^\sharp\| + |w|^{-1}L^{(\theta-1)/(2\theta)-\bar{\varepsilon}}\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{1/2}(\mathscr{K}^m\varphi^\sharp-\varphi^\sharp)\| \\ \lesssim\ &\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{1/2}\varphi^\sharp\|. \end{aligned} \tag{A.5.7}$$

All the estimates are uniform in m and true in the limit $m\to\infty$. We denote $\mathscr{K}=\mathscr{K}^\infty$.

Proof. We start with the estimate on $\mathscr{G}_+^{m,\succ}$. We have, for $\varepsilon\in]0,1/2-1/(2\theta)[$, using Lemma A.2.7,

$$\begin{aligned} \|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-1/2}\mathscr{G}_+^{m,\succ}\psi\| &\lesssim \|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-1/2}1_{|\mathscr{L}_\theta|\geqslant M(\mathscr{N})}\mathscr{G}_+^m\psi\| \\ &\lesssim \|w(\mathscr{N})M(\mathscr{N})^{-1/2+1/(2\theta)+\varepsilon}(1-\mathscr{L}_\theta)^{-1/(2\theta)-\varepsilon}\mathscr{G}_+^m\psi\| \\ &\lesssim \|w(\mathscr{N}+1)M(\mathscr{N}+1)^{-1/2+1/(2\theta)+\varepsilon}(\mathscr{N}+1)(1-\mathscr{L}_\theta)^{1/2-\varepsilon}\psi\|. \end{aligned}$$

The bound on $\mathscr{G}_-^{m,\succ}$ can be obtained using again Lemma A.2.7:

$$\begin{aligned} \|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-1/2}\mathscr{G}_-^{m,\succ}\psi\| &\lesssim \|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-1/2}1_{|\mathscr{L}_\theta|\geqslant M(\mathscr{N})}\mathscr{G}_-^m\psi\| \\ &\lesssim \|w(\mathscr{N})M(\mathscr{N})^{-1/2+1/(2\theta)}(1-\mathscr{L}_\theta)^{-1/(2\theta)}\mathscr{G}_-^m\psi\| \\ &\lesssim \|w(\mathscr{N}-1)M(\mathscr{N}-1)^{-1/2+1/(2\theta)}\mathscr{N}^{3/2}(1-\mathscr{L}_\theta)^{1/2}\psi\|. \end{aligned}$$

In conclusion, for $\varepsilon\in]0,(\theta-1)/(2\theta)[$, choosing $M(n)=L\,(n+1)^{3\theta/(\theta-1-2\theta\varepsilon)}$, for $L\geqslant 1$,

$$\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-1/2}\mathscr{G}^{m,\succ}\psi\| \lesssim L^{-1/2+1/(2\theta)+\varepsilon}|w|\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{1/2}\psi\|.$$

Now let $\varphi^\sharp\in w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H}$, the map

$$\begin{aligned} \Psi^m\colon w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H} &\to w(\mathscr{N})^{-1}(1-\mathscr{L}_\theta)^{-1/2}\mathscr{H}, \\ \psi &\mapsto \Psi^m(\psi) := (1-\mathscr{L}_\theta)^{-1}\mathscr{G}^{m,\succ}\psi + \varphi^\sharp, \end{aligned}$$

satisfies, for some positive constant C,

$$\begin{aligned}\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\Psi^m(\psi)\| &\leqslant \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}^{m,\succ}\psi\| + \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|\\ &\leqslant CL^{-1/2+1/(2\theta)+\varepsilon}|w|\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\| + \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|.\end{aligned}$$

Namely, Ψ^m is well-defined and, choosing L large enough, it is a contraction leaving the ball of radius $2\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|$ invariant. Therefore, it has a unique fixed point $\mathcal{K}^m\varphi^\sharp$ satisfying the claimed inequalities. □

Remark A.5.5. In the previous lemma, the cut-off $M(n)$ depends via $|w|$ on the weight w. In the following we will only use *polynomial weights* of the form $w(n) = (1+n)^\alpha$ with $|\alpha| \leqslant K$ for a fixed K. In this case $|w|$ is uniformly bounded and it is possible to select a cut-off which is adapted to all those weights. This will be fixed once and for all and not discussed further.

Proposition A.5.6. *Let w be a polynomial weight, $\gamma \geqslant 0$, $\bar{\varepsilon}$ as in Lemma A.5.4,*

$$\alpha(\gamma) = \frac{\theta(6\gamma+5)-2}{2(\theta-1)}.$$

Let

$$\varphi^\sharp \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(1+\mathcal{N})^{-\alpha(\gamma)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H},$$

and set $\varphi^m := \mathcal{K}^m\varphi^\sharp$. Then $\mathcal{L}^m\varphi^m$ is a well-defined operator and we have the bound

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma\mathcal{G}^{m,\prec}\varphi^m\| \lesssim \|w(\mathcal{N})(1+\mathcal{N})^{\alpha(\gamma)}(1-\mathcal{L}_\theta)^{1/2}\varphi^\sharp\|. \tag{A.5.8}$$

Proof. By eq. (A.5.5) we need only to estimate $\mathcal{G}^{m,\prec}\varphi^m$. We first deal with $\mathcal{G}_+^{m,\prec}$: we have by (A.2.8), for $\delta < 1/2 - 1/(2\theta)$,

$$\begin{aligned}\|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma\mathcal{G}_+^{m,\prec}\varphi^m\| &= \|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma 1_{|\mathcal{L}_\theta|<M(\mathcal{N})}\mathcal{G}_+^m\varphi^m\|\\ &\lesssim \|w(\mathcal{N})M(\mathcal{N})^{\gamma+1/2-\delta}(1-\mathcal{L}_\theta)^{-1/2+\delta}\mathcal{G}_+^m\varphi^m\|\\ &\lesssim \|w(\mathcal{N}+1)M(\mathcal{N}+1)^{\gamma+1/2}(\mathcal{N}+1)(1-\mathcal{L}_\theta)^{1/(2\theta)+\delta}\varphi^m\|.\end{aligned}$$

For $\mathcal{G}_-^{m,\prec}$, it follows in a similar way from estimate (A.2.9) that, for every $\delta \in]0, 1/(2\theta)]$,

$$\begin{aligned}\|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma\mathcal{G}_-^{m,\prec}\varphi^m\| &= \|w(\mathcal{N})(1-\mathcal{L}_\theta)^\gamma 1_{|\mathcal{L}_\theta|<M(\mathcal{N})}\mathcal{G}_-^m\varphi^m\|\\ &\lesssim \|w(\mathcal{N})M(\mathcal{N})^{\gamma+1/(2\theta)}(1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-^m\varphi^m\|\\ &\lesssim \|w(\mathcal{N}-1)M(\mathcal{N}-1)^{\gamma+1/(2\theta)}\mathcal{N}^{3/2}(1-\mathcal{L}_\theta)^{1/2}\varphi^m\|.\end{aligned}$$

These bounds and the definition of $M(n)$ give the claimed bound on $\mathcal{G}^{m,\prec}$. □

A.5.3 Limiting generator and its domain

Lemma A.5.7. *Let w be a weight and take a cut-off function as in Proposition A.5.6 with $\gamma = 0$. Set*

$$\mathcal{D}_w(\mathcal{L}) := \{\mathcal{K}\varphi^\sharp : \varphi^\sharp \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}\}.$$

Then $\mathcal{D}_w(\mathcal{L})$ is dense in $w(\mathcal{N})^{-1}\mathcal{H}$. If $w \equiv 1$ we simply write $\mathcal{D}(\mathcal{L})$.

Proof. Note that $w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$ is dense in $w(\mathcal{N})^{-1}\mathcal{H}$, therefore, in order to prove Lemma A.5.7, it suffices to show that, for any $\psi \in w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$ and for all $\nu \geqslant 1$, there exists $\varphi^\nu \in \mathcal{D}_w(\mathcal{L})$ such that

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}(\varphi^\nu - \psi)\| \lesssim \nu^{-(\theta-1)/(2\theta)+\bar{\varepsilon}}\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\|, \tag{A.5.9}$$

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi^\nu\| \lesssim \|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\psi\|, \tag{A.5.10}$$

$$\begin{aligned}\|w(\mathcal{N})(1-\mathcal{L})\varphi^\nu\| &\lesssim \nu^{1/(2\theta)+\delta}(\|w(\mathcal{N})(1-\mathcal{L}_\theta)\psi\|\\ &\quad + \|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi\|),\end{aligned} \tag{A.5.11}$$

for some $\delta > 0$. By Lemma A.5.4, there exists $\varphi^\nu \in w(\mathcal{N})^{-1}\mathcal{H}$ such that

$$\varphi^\nu = 1_{\nu M(\mathcal{N}) \leqslant |\mathcal{L}_\theta|}(1-\mathcal{L}_\theta)^{-1}\mathcal{G}\varphi^\nu + \psi$$

and satisfying estimates (A.5.9)–(A.5.10). We are left to show that $\varphi^\nu \in \mathcal{D}_w(\mathcal{L})$ and (A.5.11). Note that

$$\varphi^\nu = (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{\succ}\varphi^\nu + \varphi^{\nu,\sharp},$$

where

$$\varphi^{\nu,\sharp} = \psi - 1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}(1-\mathcal{L}_\theta)^{-1}\mathcal{G}\varphi^\nu.$$

In particular, we have $\mathcal{L}\varphi^\nu = \varphi^\nu + \mathcal{G}^{\prec}\varphi^\nu - (1-\mathcal{L}_\theta)\varphi^{\nu,\sharp}$, and, by Proposition A.5.6, it suffices to estimate $\varphi^{\nu,\sharp}$ in $w(\mathcal{N})^{-1}(1-\mathcal{L}_\theta)^{-1}\mathcal{H} \cap w(\mathcal{N})^{-1}(\mathcal{N}+1)^{-\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\mathcal{H}$. The first contribution, ψ, satisfies the required bounds by assumption, so it is enough to show that the second contribution, which we denote by ψ^ν, satisfies

$$\|w(\mathcal{N})(1-\mathcal{L}_\theta)\psi^\nu\| \lesssim \nu^{1/(2\theta)+\delta}\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi\|, \tag{A.5.12}$$
$$\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi^\nu\| \lesssim \|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi\|. \tag{A.5.13}$$

Notice that $(1-\mathcal{L}_\theta)\psi^\nu = -1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}\mathcal{G}\varphi^\nu$, hence estimate (A.5.12) can be obtained from the uniform bounds in Lemma A.2.7 as follows (note that those bounds are valid also when $m = +\infty$). We have, for $\mathcal{G}_+$,

$$\begin{aligned}
&\|w(\mathcal{N})\, 1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}\, \mathcal{G}_+\varphi\| \\
\lesssim\ &\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/(2\theta)+\delta}\, 1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}\, (1-\mathcal{L}_\theta)^{-1/(2\theta)-\delta}\mathcal{G}_+\varphi\| \\
\lesssim\ &\nu^{1/(2\theta)+\delta}\|w(\mathcal{N}+1)M(\mathcal{N}+1)^{1/(2\theta)+\delta}(\mathcal{N}+1)(1-\mathcal{L}_\theta)^{1/2-\delta}\varphi\|.
\end{aligned}$$

For $\mathcal{G}_-$ we have, instead

$$\begin{aligned}
&\|w(\mathcal{N})1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}\mathcal{G}_-\varphi\| \\
\lesssim\ &\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{1/(2\theta)}\, 1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}\, (1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-\varphi\| \\
\lesssim\ &\nu^{1/(2\theta)}\|w(\mathcal{N}-1)M(\mathcal{N}-1)^{1/(2\theta)}\mathcal{N}^{3/2}\, 1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}\, (1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-\varphi\|,
\end{aligned}$$

which gives estimate (A.5.12) if we choose $\bar{\varepsilon}$ small enough. In order to obtain estimate (A.5.13), note that, for $\kappa \in]0, (\theta-1)/(2\theta)[$,

$$\begin{aligned}
\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi^\nu\| &= \|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{-1/2}\, 1_{M(\mathcal{N}) \leqslant |\mathcal{L}_\theta| < \nu M(\mathcal{N})}\, \mathcal{G}\varphi^\nu\| \\
&\lesssim M(n)^{-(\theta-1)/\theta+2\kappa}\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{-1/(2\theta)-\kappa}\mathcal{G}_+\varphi^\nu\| \\
&\quad + M(n)^{-(\theta-1)/\theta+2\kappa}\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{-1/(2\theta)}\mathcal{G}_-\varphi^\nu\|
\end{aligned}$$

Now recall that $M(n) \simeq (n+1)^{3\theta/(\theta-1-2\theta\bar{\varepsilon})}$ and get by (A.2.8)–(A.2.9) the inequality

$$\|w(\mathcal{N})(\mathcal{N}+1)^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\psi^\nu\| \lesssim \|w(\mathcal{N})(1+\mathcal{N})^{\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi^\nu\|.$$

Applying (A.5.7) yields the result. □

Lemma A.5.8. *For any $\varphi \in \mathcal{D}(\mathcal{L})$, we have*

$$\langle \varphi, \mathcal{L}\varphi \rangle \leqslant 0.$$

In particular, the operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is dissipative.

Proof. Notice that $\varphi \in \mathcal{D}(\mathcal{L})$ implies $\mathcal{L}_\theta\varphi, \mathcal{G}\varphi \in (1-\mathcal{L}_\theta)^{1/2}\mathcal{H}$ and $\varphi \in (1-\mathcal{L}_\theta)^{-1/2}(1+\mathcal{N})^{-1}\mathcal{H}$. These regularities are enough to proceed by approximation and establish that

$$\langle \varphi, \mathcal{L}\varphi \rangle = -\langle \varphi, (-\mathcal{L}_\theta)\varphi \rangle + \langle \varphi, \mathcal{G}\varphi \rangle = -\langle \varphi, (-\mathcal{L}_\theta)\varphi \rangle = -\|(-\mathcal{L}_\theta)^{1/2}\varphi\|^2 \leqslant 0,$$

where we used the anti-symmetry of the form associated to $\mathcal{G}$, i.e. $\langle \varphi, \mathcal{G}\varphi\rangle = 0$. □

A.5.4 Existence and uniqueness for the Kolmogorov equation

Having defined a domain for $\mathcal{L}$ it remains to study the Kolmogorov equation $\partial_t \varphi = \mathcal{L}\varphi$. In particular, we consider the equation for $\varphi^{m,\sharp}$, which was defined in (A.5.4),

$$\begin{aligned}
\partial_t \varphi^{m,\sharp} + (1-\mathcal{L}_\theta)\varphi^{m,\sharp} &= \mathcal{L}^m \varphi^m + (1-\mathcal{L}_\theta)\varphi^{m,\sharp} - (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}\partial_t \varphi^m \\
&= \varphi^m + \mathcal{G}^{m,\prec}\varphi^m - (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}\partial_t\varphi^m \\
&= \varphi^m + \mathcal{G}^{m,\prec}\varphi^m - (1-\mathcal{L}_\theta)^{-1}\mathcal{G}^{m,\succ}(\varphi^m + \mathcal{G}^{m,\prec}\varphi^m - (1-\mathcal{L}_\theta)\varphi^{m,\sharp}) \\
&=: \Phi^{m,\sharp}.
\end{aligned}$$

We want to get a suitable bound in terms of $\varphi_0^{m,\sharp}$ for each term of $\Phi^{m,\sharp}$. The Schauder estimate in Lemma A.8.2 will be crucial. We will also need the following result.

Lemma A.5.9. *We have*

$$\begin{aligned}
&\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(t)\| \\
&\lesssim (te^{tC}+1)^{1/2}\big(\big\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\big\| + \big\|(1+\mathcal{N})^{p+\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\big\|\big).
\end{aligned} \tag{A.5.14}$$

Proof. By (A.5.3) and Lemma A.5.4 it follows that

$$\begin{aligned}
&\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(t)\| \\
&\lesssim \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi^m(t)\| + \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{-1/2}\mathcal{G}^{m,\succ}\varphi^m(t)\| \\
&\lesssim te^{tC}\|(1+\mathcal{N})^p\mathcal{L}^m\varphi_0^m\| + \|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{1/2}\varphi_0^m\| \\
&\lesssim (te^{tC}+1)^{1/2}\big(\big\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\big\| + \big\|(1+\mathcal{N})^{p+\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\big\|\big),
\end{aligned}$$

where in the last step we exploited Proposition A.5.6. □

For $\gamma \in]1/2, 1-1/(2\theta)[$, we have that, by the estimates (A.2.8) and (A.2.9),

$$\begin{aligned}
\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma-1}\mathcal{G}^{m,\succ}(1-\mathcal{L}_\theta)\varphi^{m,\sharp}(s)\| &\lesssim \|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma+1/2+1/(2\theta)}\varphi^{m,\sharp}(s)\| \\
&\lesssim \|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma+1/2+1/(2\theta)}\varphi^{m,\sharp}(s)\|.
\end{aligned}$$

By interpolation for products, there exists $q>0$ such that, for all $\varepsilon \in]0,1[$,

$$\begin{aligned}
\|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma+1/2+1/(2\theta)}\varphi^{m,\sharp}(s)\| \lesssim\ & C_\varepsilon\|(1+\mathcal{N})^q(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\| \\
&+\varepsilon\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma+1}\varphi^{m,\sharp}(s)\|,
\end{aligned}$$

where the first term on the right-hand side can be controlled via the a priori estimate (A.5.14), while the second term can be absorbed on the left-hand side. Moreover, we have by (A.5.8) and by estimate (A.5.14),

$$\begin{aligned}
\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^\gamma\mathcal{G}^{m,\prec}\varphi^m(s)\| &\lesssim \|(1+\mathcal{N})^{p+\alpha(\gamma)}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\| \\
&\lesssim \big\|(1+\mathcal{N})^{p+\alpha(\gamma)}(1-\mathcal{L}_\theta)\varphi_0^{m,\sharp}\big\| \\
&\quad+\big\|(1+\mathcal{N})^{p+\alpha(\gamma)+\alpha(0)}(1-\mathcal{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\big\|.
\end{aligned}$$

Recalling $\gamma \in]1/2, 1-1/(2\theta)[$ and exploiting estimates (A.2.8)–(A.2.9), we get

$$\begin{aligned}
&\|(1+\mathcal{N})^p(1-\mathcal{L}_\theta)^{\gamma-1}\mathcal{G}^{m,\succ}\mathcal{G}^{m,\prec}\varphi^m(s)\| \\
&\lesssim \|(1+\mathcal{N})^{p+3/2}(1-\mathcal{L}_\theta)^{\gamma-1/2+1/(2\theta)}\mathcal{G}^{m,\prec}\varphi^m(s)\| \\
&\lesssim \|(1+\mathcal{N})^{p+3/2+\alpha(\gamma-1/2+1/(2\theta))}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\| \\
&\lesssim \|(1+\mathcal{N})^{p+\alpha(\gamma)}(1-\mathcal{L}_\theta)^{1/2}\varphi^{m,\sharp}(s)\|,
\end{aligned}$$

where we used $3/2+\alpha(\gamma-1/2+1/(2\theta))<\alpha(\gamma)$ whenever $\bar{\varepsilon}<1/3-1/(3\theta)$. This bound can be controlled via (A.5.14) as above. As a consequence, we established that, after renaming $q=q(p,\gamma)>0$,

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^\gamma\Phi^{m,\sharp}(t)\| \lesssim_T\ & \left\|(1+\mathscr{N})^q(1-\mathscr{L}_\theta)\varphi_0^{m,\sharp}\right\| \\ & +\left\|(1+\mathscr{N})^{q+\alpha(0)}(1-\mathscr{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\right\| \\ & +\varepsilon\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{\gamma+1}\varphi^{m,\sharp}(t)\|,\end{aligned}$$

and hence, for $\gamma\in]1/2,1-1/(2\theta)[$,

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1+\gamma}\varphi^{m,\sharp}(t)\| \lesssim_T\ & \left\|(1+\mathscr{N})^q(1-\mathscr{L}_\theta)\varphi_0^{m,\sharp}\right\|+\left\|(1+\mathscr{N})^q(1-\mathscr{L}_\theta)^{1/2}\varphi_0^{m,\sharp}\right\| \\ & +\left\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\right\| \\ \lesssim_T\ & \left\|(1+\mathscr{N})^q(1-\mathscr{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\right\|.\end{aligned}$$

Recall $\partial_t\varphi^{m,\sharp}=(1-\mathscr{L}_\theta)\varphi^{m,\sharp}+\Phi^{m,\sharp}(t)$, so that

$$\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^\gamma\partial_t\varphi^{m,\sharp}(t)\|\lesssim\left\|(1+\mathscr{N})^q(1-\mathscr{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\right\|.$$

By interpolation, this gives

$$\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1+\gamma/2}(\varphi^{m,\sharp}(t)-\varphi^{m,\sharp}(s))\|\leqslant|t-s|^\kappa\left\|(1+\mathscr{N})^q(1-\mathscr{L}_\theta)^{1+\gamma}\varphi_0^{m,\sharp}\right\|.$$

Introduce now, for $p>0$, the sets

$$\mathscr{U}_p=\bigcup_{\gamma\in\left]\frac{1}{2},1-\frac{1}{2\theta}\right[}\mathscr{K}(1+\mathscr{N})^{q(p,\gamma)}(1-\mathscr{L}_\theta)^{-1-\gamma}\mathscr{H}\subset\mathscr{H},$$

and $\mathscr{U}=\bigcup_{p>\alpha(0)}\mathscr{U}_p$.

Theorem A.5.10. *Let $p>0$ and $\varphi_0\in\mathscr{U}_p$. Then there exists a solution*

$$\varphi\in\bigcup_{\delta>0}C(\mathbb{R}_+;(1+\mathscr{N})^{-p+\delta}(1-\mathscr{L}_\theta)^{-1}\mathscr{H})$$

to the Kolmogorov backward equation $\partial_t\varphi=\mathscr{L}\varphi$ with initial condition $\varphi(0)=\varphi_0$. For $p>\alpha(0)$, we have $\varphi\in C(\mathbb{R}_+,\mathscr{D}(\mathscr{L}))\cap C^1(\mathbb{R},\mathscr{H})$ and, by dissipativity of $\mathscr{L}$, this solution is unique.

Proof. Let $\varphi_0\in\mathscr{U}_p$ and set $\varphi_0^\sharp:=\mathscr{K}^{-1}\varphi_0\in(1+\mathscr{N})^{-q}(1-\mathscr{L}_\theta)^{-1-\gamma}\mathscr{H}$ for $\gamma\in]1/2,1-1/(2\theta)[$ and $p>0$. For $m\in\mathbb{N}$, let φ^m be the solution to $\partial_t\varphi^m=\mathscr{L}^m\varphi^m$ with initial condition $\varphi^m(0)=\mathscr{K}^m\varphi_0^\sharp$. A diagonal argument yields the relative compactness of bounded sets of $(1+\mathscr{N})^{-p}(1-\mathscr{L}_\theta)^{-1-\gamma/2}\mathscr{H}$ in the space $(1+\mathscr{N})^{-p+\delta}(1-\mathscr{L}_\theta)^{-1}\mathscr{H}$ for $\delta>0$, with the consequence that, by Ascoli-Arzelà the sequence $(\varphi^{m,\sharp})_m$ is relatively compact in $C(\mathbb{R}_+;(1+\mathscr{N})^{-p+\delta}(1-\mathscr{L}_\theta)^{-1}\mathscr{H})$ equipped with the topology of uniform convergence on compact sets. We denote $\varphi^\sharp$ a limit point of such a sequence and let $\varphi=\mathscr{K}\varphi^\sharp$. Then, along the convergent subsequence,

$$\begin{aligned}\varphi(t)-\varphi(0) &= \lim_{m\to\infty}(\varphi^m(t)-\varphi^m(0)) \\ &= \lim_{m\to\infty}\int_0^t\mathscr{L}^m\varphi^m(s)\,\mathrm{d}s \\ &= \lim_{m\to\infty}\int_0^t(\mathscr{L}_\theta\varphi^{m,\sharp}(s)+\mathscr{G}^{m,\prec}\mathscr{K}^m\varphi^{m,\sharp}(s))\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \lim_{m\to\infty}\int_0^t (\mathscr{L}_\theta\varphi^\sharp(s)+\mathscr{G}^{m,\prec}\mathscr{K}^m\varphi^{m,\sharp}(s))\,\mathrm{d}s\\
&= \int_0^t (\mathscr{L}_\theta\varphi^\sharp(s)+\mathscr{G}^{\prec}\mathscr{K}\varphi^\sharp(s))\,\mathrm{d}s,
\end{aligned}
$$

where we exploited our uniform bounds on $\mathscr{L}_\theta$, $\mathscr{G}^{m,\succ}$, $\mathscr{K}^m$ and the convergence of $\varphi^{m,\sharp}$ to $\varphi^\sharp$ as $m\to\infty$ to get the 4th equality, while the last step follows from our bounds for $\mathscr{G}^\prec$ and $\mathscr{K}$, together with the dominated convergence theorem.

If we take $p>\alpha(0)$, then by definition (cfr. Lemma A.5.7) $\varphi\in\mathscr{D}(\mathscr{L})$. Furthermore, $\mathscr{L}\varphi\in C(\mathbb{R}_+;\mathscr{H})$ and we have $\varphi\in C^1(\mathbb{R}_+;\mathscr{H})$ because of the relation $\varphi(t)-\varphi(s)=\int_s^t\mathscr{L}\varphi(\tau)\,\mathrm{d}\tau$. We can hence compute,

$$
\partial_t\|\varphi(t)\|^2=2\langle\varphi(t),\mathscr{L}\varphi(t)\rangle\leqslant 0,
$$

by the dissipativity of the operator $\mathscr{L}$ given by Lemma A.5.8. Therefore, for any solution we have $\|\varphi(t)\|\leqslant\|\varphi_0\|$, which together with the linearity of the equation yields the uniqueness. □

A.6 Bounds on the drift

We prove there the key bounds on the drift $\mathscr{G}^m$.

Proof of Lemma A.2.7. We start by estimating $\mathscr{G}_+^m$. We have, by Lemma A.8.1 and since $\gamma>1/(2\theta)$,

$$
\begin{aligned}
&\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-\gamma}\mathscr{G}_+^m\varphi\|^2=\sum_{n\geqslant 0}n!w(n)^2\sum_{k_{1:n}}\left(\prod_{i=1}^n|2\pi k_i|^2\right)|\mathscr{F}((1-\mathscr{L}_\theta)^{-\gamma}\mathscr{G}_+^m\varphi)_n(k_{1:n})|^2\\
\lesssim&\sum_{n\geqslant 2}n!n^2w(n)^2\sum_{k_{1:n}}\left(\prod_{i=1}^n|2\pi k_i|^2\right)\frac{1_{|k_1|,|k_2|,|k_1+k_2|\leqslant m}|k_1+k_2|^4}{(1+L_\theta(k_{1:n}))^{2\gamma}|k_1|^2|k_2|^2}|\hat{\varphi}_{n-1}(k_1+k_2,k_{3:n})|^2\\
\lesssim&\sum_{n\geqslant 2}n!n^2w(n)^2\sum_{\ell,k_{3:n}}\left(\prod_{i=3}^n|2\pi k_i|^2\right)|\ell|^4|\hat{\varphi}_{n-1}(\ell,k_{3:n})|^2\sum_{k_1+k_2=\ell}(1+L_\theta(k_{1:n}))^{-2\gamma}\\
\lesssim&\sum_{n\geqslant 2}n!n^2w(n)^2\sum_{\ell,k_{3:n}}\left(\prod_{i=3}^n|2\pi k_i|^2\right)|\ell|^4(1+L_\theta(\ell,k_{3:n}))^{-2\gamma+1/\theta}|\hat{\varphi}_{n-1}(\ell,k_{3:n})|^2.
\end{aligned}
\tag{A.6.1}
$$

Introducing the notation $\ell_1=\ell=k_1+k_2$ and $\ell_i=k_{i+1}$ for $i\geqslant 2$, we get

$$
\lesssim\sum_{n\geqslant 2}n!n^2w(n)^2\sum_{\ell_{1:n-1}}\left(\prod_{i=1}^{n-1}|2\pi\ell_i|^2\right)|\ell_1|^2(1+L_\theta(\ell_{1:n-1}))^{-2\gamma+1/\theta}|\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2.
$$

then using the symmetry of $\hat{\varphi}_{n-1}$ we reduce this to

$$
\lesssim\sum_{n\geqslant 2}n!nw(n)^2\sum_{\ell_{1:n-1}}\left(\prod_{i=1}^{n-1}|2\pi\ell_i|^2\right)\frac{|\ell_1|^2+\cdots+|\ell_n|^2}{1+L_\theta(\ell_{1:n-1})}(1+L_\theta(\ell_{1:n-1}))^{-2\gamma+1/\theta+1}|\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2.
$$

from which we obtain

$$\begin{aligned}
&\lesssim \sum_{n\geqslant 1} n!\,(n+1)^2\, w(n+1)^2 \sum_{\ell_{1:n}} \left(\prod_{i=1}^{n} |2\pi\ell_i|^2\right)(1+L_\theta(\ell_{1:n}))^{-2\gamma+1/\theta+1}|\hat{\varphi}_n(\ell_{1:n})|^2\\
&\lesssim \|w(\mathcal{N}+1)(1+\mathcal{N})(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma}\varphi\|^2.
\end{aligned}$$

For $\mathcal{G}^m_-$, note first that, by the Cauchy-Schwarz inequality and by Lemma A.8.1 (since $\gamma<1/2$),

$$\begin{aligned}
&\left|\sum_{p+q=k_1} (k_1^\perp\cdot p)\,(k_1\cdot q)\hat{\varphi}_{n+1}(p,q,k_{2:n})\right|^2\\
\lesssim\ & \sum_{p+q=k_1} (1+|p|^{2\theta}+|q|^{2\theta})^{2\gamma-1-1/\theta}\\
&\times \sum_{p+q=k_1} (1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|k_1^\perp\cdot p|^2|k_1\cdot q|^2|\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2\\
\lesssim\ & (1+|k_1|^{2\theta})^{2\gamma-1}\sum_{p+q=k_1} (1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|k_1^\perp\cdot p|^2|k_1\cdot q|^2|\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2,
\end{aligned}$$

therefore,

$$\begin{aligned}
&\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}^m_-\varphi\|^2 = \sum_{n\geqslant 0} n!\,w(n)^2\sum_{k_{1:n}}\left(\prod_{i=1}^{n}|2\pi k_i|^2\right)|\mathcal{F}((1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}^m_-\varphi)_n(k_{1:n})|^2\\
\lesssim\ & \sum_{n\geqslant 0} n!\,w(n)^2(n+1)^4\sum_{k_{1:n}}\left(\prod_{i=1}^{n}|2\pi k_i|^2\right)\frac{1}{|k_1|^4(1+L_\theta(k_{1:n}))^{2\gamma}}\\
&\times\left|\sum_{p+q=k_1}(k_1^\perp\cdot p)\,(k_1\cdot q)\hat{\varphi}_{n+1}(p,q,k_{2:n})\right|^2\\
\lesssim\ & \sum_{n\geqslant 0} n!\,w(n)^2(n+1)^4\sum_{k_{1:n}}\left(\prod_{i=1}^{n}|2\pi k_i|^2\right)\frac{1}{|k_1|^4(1+L_\theta(k_{1:n}))^{2\gamma}}(1+|k_1|^{2\theta})^{2\gamma-1}\\
&\times\sum_{p+q=k_1}(1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|k_1|^4|p|^2|q|^2|\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2\\
\lesssim\ & \sum_{n\geqslant 0} n!\,w(n)^2(n+1)^4\sum_{k_{1:n}}\sum_{p+q=k_1}\left(\prod_{i=2}^{n}|2\pi k_i|^2\right)|2\pi p|^2|2\pi q|^2\\
&\times(1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2,
\end{aligned}$$

we now let $\ell_1=p$, $\ell_2=q$, and $\ell_i=k_{i-1}$ for $3\leqslant i\leqslant n+1$, so that

$$\begin{aligned}
&\|w(\mathcal{N})(1-\mathcal{L}_\theta)^{-\gamma}\mathcal{G}^m_-\varphi\|^2\\
\lesssim\ & \sum_{n\geqslant 0} n!\,w(n)^2(n+1)^4\sum_{\ell_{1:n+1}}\left(\prod_{i=1}^{n+1}|2\pi\ell_i|^2\right)(1+|\ell_1|^{2\theta}+|\ell_2|^{2\theta})^{1+1/\theta-2\gamma}|\hat{\varphi}_{n+1}(\ell_{1:n+1})|^2\\
\lesssim\ & \sum_{n\geqslant 0} n!\,w(n)^2(n+1)^4\sum_{\ell_{1:n+1}}\left(\prod_{i=1}^{n+1}|2\pi\ell_i|^2\right)(1+|\ell_1|^{2\theta}+\cdots+|\ell_{n+1}|^{2\theta})^{1+1/\theta-2\gamma}|\hat{\varphi}_{n+1}(\ell_{1:n+1})|^2\\
\lesssim\ & \sum_{n\geqslant 1} n!\,w(n-1)^2n^3\sum_{\ell_{1:n}}\left(\prod_{i=1}^{n}|2\pi\ell_i|^2\right)(1+|\ell_1|^{2\theta}+\cdots+|\ell_n|^{2\theta})^{1+1/\theta-2\gamma}|\hat{\varphi}_n(\ell_{1:n})|^2\\
\lesssim\ & \|w(\mathcal{N}-1)\mathcal{N}^{3/2}(1-\mathcal{L}_\theta)^{(1+1/\theta)/2-\gamma}\varphi\|^2
\end{aligned}$$

which gives the uniform bound.

Let us now discuss the m-dependent estimates, we have for $\mathcal{G}_+^m$

$$
\begin{aligned}
\|w(\mathcal{N})\mathcal{G}_+^m\varphi\|^2 &= \sum_{n\geqslant 0} n!w(n)^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) |\mathcal{F}(\mathcal{G}_+^m\varphi)_n(k_{1:n})|^2 \\
&\lesssim \sum_{n\geqslant 2} n!w(n)^2 n^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) 1_{|k_1|,|k_2|,|k_1+k_2|\leqslant m} \frac{|k_1+k_2|^4}{|k_1|^2|k_2|^2} |\hat{\varphi}_{n-1}(k_1+k_2, k_{3:n})|^2 \\
&\lesssim \sum_{n\geqslant 2} n!w(n)^2 n^2 \\
&\quad \times \sum_{k_{1:n}} \left(\prod_{i=3}^n |2\pi k_i|^2\right) |2\pi(k_1+k_2)|^2 1_{|k_1|,|k_2|,|k_1+k_2|\leqslant m} |k_1+k_2|^{2\theta} |\hat{\varphi}_{n-1}(k_1+k_2, k_{3:n})|^2 \\
&\lesssim m^2 \sum_{n\geqslant 2} n!w(n)^2 n^2 \sum_{\ell_{1:n-1}} \left(\prod_{i=1}^{n-1} |2\pi \ell_i|^2\right) |\ell_1|^{2\theta} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\
&\lesssim m^2 \sum_{n\geqslant 2} n!w(n)^2 n \sum_{\ell_{1:n-1}} \left(\prod_{i=1}^{n-1} |2\pi \ell_i|^2\right) L_\theta(\ell_{1:n-1}) |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\
&\lesssim m^2 \sum_{n\geqslant 1} n!w(n+1)^2 (n+1)^2 \sum_{\ell_{1:n}} \left(\prod_{i=1}^{n} |2\pi \ell_i|^2\right) (1+L_\theta(\ell_{1:n})) |\hat{\varphi}_{n}(\ell_{1:n})|^2 \\
&\lesssim m^2 \|w(\mathcal{N}+1)(1+\mathcal{N})(1-\mathcal{L}_\theta)^{1/2}\varphi\|^2.
\end{aligned}
$$

Finally, for $\mathcal{G}_-^m$ we have,

$$
\begin{aligned}
\|w(\mathcal{N})\mathcal{G}_-^m\varphi\|^2 &= \sum_{n\geqslant 0} n!w(n)^2 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) |\mathcal{F}(\mathcal{G}_-^m\varphi)_n(k_{1:n})|^2 \\
&\lesssim \sum_{n\geqslant 0} n!w(n)^2 (n+1)^4 \sum_{k_{1:n}} \left(\prod_{i=1}^n |2\pi k_i|^2\right) \frac{1_{|k_1|,|p|,|q|\leqslant m}}{|k_1|^4} \\
&\quad \times \left| \sum_{p+q=k_1} (k_1^\perp\cdot p)\,(k_1\cdot q)\hat{\varphi}_{n+1}(p,q,k_{2:n}) \right|^2 \\
&\lesssim \sum_{n\geqslant 0} n!w(n)^2 (n+1)^4 \\
&\quad \times \sum_{k_{1:n}} \sum_{p+q=k_1} \left(\prod_{i=2}^n |2\pi k_i|^2\right) |2\pi p|^2 |2\pi q|^2 1_{|k_1|,|p|,|q|\leqslant m} |k_1|^2 |\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2 \\
&\lesssim m^2 \sum_{n\geqslant 0} n!w(n)^2 (n+1)^3 \sum_{p,q,k_{2:n}} \left(\prod_{i=2}^n |2\pi k_i|^2\right) \\
&\quad \times |2\pi p|^2 |2\pi q|^2 (|p|^{2\theta} + |q|^{2\theta} + |k_2|^{2\theta} + \cdots + |k_n|^{2\theta}) |\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2 \\
&\lesssim m^2 \sum_{n\geqslant 0} n!w(n)^2 (n+1)^3 \sum_{\ell_{1:n+1}} \left(\prod_{i=1}^{n+1} |2\pi \ell_i|^2\right) L_\theta(\ell_{1:n+1}) |\hat{\varphi}_{n+1}(\ell_{1:n+1})|^2 \\
&\lesssim m^2 \sum_{n\geqslant 1} n!w(n-1)^2 n^2 \sum_{\ell_{1:n}} \left(\prod_{i=1}^{n} |2\pi \ell_i|^2\right) L_\theta(\ell_{1:n}) |\hat{\varphi}_{n}(\ell_{1:n})|^2 \\
&\lesssim m^2 \|w(\mathcal{N}-1)\mathcal{N}(1-\mathcal{L}_\theta)^{1/2}\varphi\|^2.
\end{aligned}
$$

This concludes the proof. □

A.7 Stochastic Navier–Stokes on the plane

In this section we prove that the main results of the paper, namely existence and uniqueness of energy solution for the hyper-viscous Navier–Stokes eq. (A.1.1) extends very naturally to the setting of the whole plane $\mathbb{R}^2$. We will discuss first existence of martingale solutions via a limiting procedure involving finite volume approximations, then we will show that the Kolmogorov equation can be solved also in the full space, which implies, as in the periodic setting, uniqueness in law.

The invariant measure At the beginning of the paper, we introduced the invariant measure of the problem, i.e., the energy measure μ given by eq. (A.1.3). The computation of the covariance in the current case gives, for any $\varphi, \psi \in \mathscr{S}$,

$$\mathbb{E}[\omega(\varphi)\omega(\psi)] = \langle (-\Delta)^{1/2}\varphi, (-\Delta)^{1/2}\psi \rangle_{L^2(\mathbb{R}^2)} =: \langle \varphi, \psi \rangle_{\dot{H}^1(\mathbb{R}^2)},$$

where $\dot{H}^s(\mathbb{R}^d)$ denotes the so-called *homogeneous Sobolev space* of $L^2(\mathbb{R}^d)$ functions f having norm $\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi$ finite. We denote by $\mu_{\mathbb{R}^2}$ the energy measure in the case of the whole space.

A new approximating problem In order to approximate stochastic Navier-Stokes equations on the whole space, we study Galerkin approximation problems on scaled tori $\mathbb{T}^2_\lambda := \mathbb{R}^2 \setminus (2\pi\lambda\mathbb{Z}^2)$, $\lambda > 0$, with the goal to take first the limit as $\lambda \to \infty$, allowing us to pass to the case of $\mathbb{R}^2$. For f: $\mathbb{T}^2_\lambda \to \mathbb{R}$, we define the Fourier transform $\mathscr{F}_\lambda(f) = \hat{f} : \lambda^{-1}\mathbb{Z}^2 \to \mathbb{R}$ as

$$\mathscr{F}_\lambda(f)(k) = \hat{f}(k) = \int_{\mathbb{T}^2_\lambda} e^{-2\pi\imath k\cdot x} f(x)\,\mathrm{d}x, \qquad k \in \lambda^{-1}\mathbb{Z}^2,$$

while the inverse transform is given by

$$\mathscr{F}^{-1}_\lambda(f)(x) = (2\pi\lambda)^{-2} \sum_{k\in\lambda^{-1}\mathbb{Z}^2} e^{2\pi\imath k\cdot x} f(k), \qquad x \in \mathbb{T}^2_\lambda.$$

Plancherel theorem now reads as

$$(2\pi\lambda)^{-2} \sum_{k\in\lambda^{-1}\mathbb{Z}^2} \mathscr{F}_\lambda(f)(k)\,\overline{\mathscr{F}_\lambda(g)(k)} = \int_{\mathbb{T}^2_\lambda} f(x) g(x)\,\mathrm{d}x.$$

The $\mathscr{H}$-norm is now given by

$$\|\varphi\|_{\mathscr{H}_\lambda} = \sum_{n=0}^{\infty} n!\, \|\varphi_n\|^2_{(H^1_0(\mathbb{T}^2_\lambda))^{\otimes n}} \simeq \sum_{n=0}^{\infty} n!\,\lambda^{-2n} \sum_{k_{1:n}\in(\lambda^{-1}\mathbb{Z}^2_0)^n} \left(\prod_{i=1}^{n} |k_i|^2\right) |\hat{\varphi}_n(k_{1:n})|^2.$$

The Biot-Savart kernel is $K(x) = -(2\pi)^{-3}\lambda^{-2}\imath \sum_{k\in\lambda^{-1}\mathbb{Z}^2} k^\perp |k|^{-2} e^{2\pi\imath k\cdot x}$, for $x \in \mathbb{T}^2_\lambda$, since from the relation $\omega = \nabla^\perp \cdot u$ we get

$$\hat{\omega}(k) = 2\pi\imath k_2\,\hat{u}_1(k) - 2\pi\imath k_1\,\hat{u}_2(k) = 2\pi\imath k^\perp \cdot \hat{u}(k),$$

which gives $\hat{u}(k) = -2\pi\imath k^\top |2\pi k|^{-2} \cdot \hat{\omega}(k)$.

$\mathscr{L}^{\lambda,m}_\theta$ can again be represented in Fourier terms by (A.2.3) for $k_1, \ldots, k_n \in \lambda^{-1}\mathbb{Z}^2$. The ($\lambda$-)Fourier transform of $\mathscr{G}^{\lambda,m}_+$ is exactly the same as in (A.2.4), while the one of $\mathscr{G}^{\lambda,m}_-$ is as in (A.2.5) but multiplied by a factor λ^{-2} due to the convolution. Following the proof of Lemma A.2.7, we get some estimates of $\mathscr{G}^{\lambda,m}_\pm$ uniform both in m and in λ (up to the λ-dependence of the $\mathscr{H}_\lambda$-norm). After getting this estimate we obtain the same result as in Lemma A.2.11, for every $\lambda > 0$.

Proof of Lemma A.2.7. We show here the two bounds for $\mathscr{G}^{\lambda,m}_{\pm}$. For $\mathscr{G}^{\lambda,m}_{+}$, the main difference with respect to the proof presented in Section A.6 is the presence of the λ^{-2n} term in the definition of the norm (and, of course, of a different Fourier transform), the sum on $k_1+k_2=\ell$ in the third step of the inequality (A.6.1) eats also a term λ^{-2}, hence thereafter we will have $\lambda^{-2(n-1)}$, which is the correct term that will enter the norm at the end of the estimate.

For the term $\mathscr{G}^{\lambda,m}_{-}$ we will use the fact that, by Lemma A.8.1,

$$
\begin{aligned}
&\left|\lambda^{-2}\sum_{p+q=k_1}(k_1^{\perp}\cdot p)(k_1\cdot q)\hat{\varphi}_{n+1}(p,q,k_{2:n})\right|^2\\
\lesssim\ &\lambda^{-2}\sum_{p+q=k_1}(1+|p|^{2\theta}+|q|^{2\theta})^{2\gamma-1-1/\theta}\\
&\times\lambda^{-2}\sum_{p+q=k_1}(1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|k_1^{\perp}\cdot p|^2|k_1\cdot q|^2|\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2\\
\lesssim\ &(1+|k_1|^{2\theta})^{2\gamma-1}\lambda^{-2}\sum_{p+q=k_1}(1+|p|^{2\theta}+|q|^{2\theta})^{1+1/\theta-2\gamma}|k_1^{\perp}\cdot p|^2|k_1\cdot q|^2|\hat{\varphi}_{n+1}(p,q,k_{2:n})|^2,
\end{aligned}
$$

which implies that $\|w(\mathscr{N})(1-\mathscr{L}_\theta)^{-\gamma}\mathscr{G}^m_{-}\varphi\|^2$ can be bounded as in the proof in Section A.6 with the extra term $\lambda^{-2(n+1)}$, that is exactly the term that will enter the norm, yielding the claimed estimate. □

Existence for the cylinder martingale problem We now want to give a proper definition of infinitesimal generator and of martingale problem on the whole space $\mathbb{R}^2$ (cfr. Section A.3), and in particular we want to show the existence of solutions obtained in Theorem A.3.3 for the present scenario.

Let us therefore focus our attention on the proof of Theorem A.3.3. Following Step 1, we want to show tightness of the sequence $(\omega^{\lambda,m})_{\lambda,m}$. With Step 2, we are going to conclude the existence for the martingale problem as $\lambda,m\to\infty$. Let us assume for the moment that we can associate to each cylinder function, $\varphi\in\mathrm{Cyl}_{\mathbb{R}^2}$, of the form $\varphi(\omega)=\Phi(\omega(f_1),\dots,\omega(f_n))$, where $f_1,\dots,f_n\in\mathcal{S}(\mathbb{R}^2)$, to $\varphi^\lambda(\omega)=\Phi(\omega(f_1^\lambda),\dots,\omega(f_n^\lambda))$ with $f_1^\lambda,\dots,f_n^\lambda\in\mathcal{S}(\mathbb{T}^2_\lambda)$ in such a way that

$$\varphi^\lambda_\ell(k_{1:\ell})=\varphi_\ell(k_{1:\ell}),\qquad \text{for } k_1,\dots,k_\ell\in\lambda^{-1}\mathbb{Z}^2,$$

where we are exploiting the chaos decompositions of φ and φ^λ. It is then possible to recover the bound (A.3.4) for $\omega^{\lambda,m}$ and therefore tightness, since $\mathscr{L}^{\lambda,m}_\theta$ has $\mathscr{L}^\infty_\theta$ as a limit of a sum converging to an integral. As regards Step 2, the crucial part is to pass from $\mathscr{L}^{\lambda,m}$ to $\mathscr{L}^{\infty,m}$ in (A.3.5) when taking the limit as $\lambda\to\infty$. To do this, we have to show that $\|(1-\mathscr{L}_\theta)^{-1/2}(\mathscr{L}^{\lambda,m}\varphi-\mathscr{L}^{\infty,m}\varphi)\|_{\mathscr{H}_\lambda}$ tends to 0 when $\lambda\to\infty$, which reduces to prove that

$$\|(1-\mathscr{L}_\theta)^{-1/2}(\mathscr{G}^{\lambda,m}_{-}\varphi-\mathscr{G}^{\infty,m}_{-}\varphi)\|_{\mathscr{H}_\lambda}\to 0,\qquad \text{as } \lambda\to\infty.$$

Comparing the explicit formulas for $\mathscr{G}^{\lambda,m}_{-}$ and $\mathscr{G}^{\infty,m}_{-}$, we have that the only difference between the two is the fact that in $\mathscr{G}^{\lambda,m}_{-}$ the sum given by the convolution becomes an integral when taking the limit. In particular,

$$
\begin{aligned}
&\|(1-\mathscr{L}_\theta)^{-1/2}(\mathscr{G}^{\lambda,m}_{-}\varphi-\mathscr{G}^{\infty,m}_{-}\varphi)\|_{\mathscr{H}_\lambda}\\
\simeq\ &\sum_{n\geqslant 0}n!\,\lambda^{-2n}n(n+1)\sum_{k_{1:n}\in(\lambda^{-1}\mathbb{Z}^2)^n}\left(\prod_{i=1}^n|k_i|^2\right)\frac{1}{(1+L_\theta(k_{1:n}))^{1/2}}\times\\
&\times\left|\lambda^{-2}\sum_{p+q=k_1}1_{|k_1|,|p|,|q|\leqslant m}\frac{k_1^{\perp}\cdot p|q|^2}{|k_1|^2}\mathscr{F}_\lambda(\varphi_{n+1})(p,q,k_{3:n})-\right.
\end{aligned}
$$

$$- \int_{\mathbb{R}^2} 1_{|k_1|,|s|,|k_1-s|\leqslant m} \frac{k_1^{\perp}\cdot s|k_1-s|^2}{|k_1|^2} \mathscr{F}_\lambda(\varphi_{n+1})(s,k_1-s,k_{3:n})\,\mathrm{d}s \Bigg|^2$$

$$\lesssim \sum_{n\geqslant 0} n!\,\lambda^{-2n} n(n+1) \sum_{k_{1:n}\in(\lambda^{-1}\mathbb{Z}^2)^n} \left(\prod_{i=2}^n |k_i|^2\right) \frac{1_{|k_1|\leqslant m}}{(1+L_\theta(k_{1:n}))^{1/2}} \times$$

$$\times \Bigg| \lambda^{-2} \sum_p 1_{|p|,|k_1-p|\leqslant m} p|k_1-p|^2 \mathscr{F}_\lambda(\varphi_{n+1})(p,k_1-p,k_{3:n}),$$

$$- \int_{\mathbb{R}^2} 1_{|s|,|k_1-s|\leqslant m} s|k_1-s|^2 \mathscr{F}_\lambda(\varphi_{n+1})(s,k_1-s,k_{3:n})\,\mathrm{d}s \Bigg|^2,$$

and the right-hand side goes to zero as $\lambda\to\infty$.

In both cases it is important to understand the role played by the test functions with respect to the norm (which depends on λ) we are considering. We want in fact to take $\varphi\in\mathrm{Cyl}_{\mathbb{R}^2}$, but we will evaluate it on $\omega^{m,\lambda}$. This is a non-trivial step and it is worth to spend a few words on it adopting a chaos expansion point of view. To apply the Mitoma's criterion we only need to test on linear functions, see [139], while for the second step it suffices to consider functions φ of the form

$$\varphi(\omega) = {:}e^{\imath\langle\omega,f\rangle}{:} = e^{\imath\omega(f)+\frac{1}{2}\|f\|^2}, \qquad \text{for } f\in\mathscr{S}(\mathbb{R}^2),$$

where ${:}e^{\imath g}{:}$ indicates the Wick exponential of g, see also [89, 95]. Focusing on the latter case, we can identify φ with the sequence of chaoses $(\varphi_n(k_{1:n}))_n$ where $\varphi_n(k_{1:n})=\imath^n\hat{f}(k_1)\cdots\hat{f}(k_n)$ for $k_{1:n}\in(\mathbb{R}^2)^n$, so that, after noticing

$$\varphi^\lambda(\omega^{m,\lambda}) := \varphi(\omega^{m,\lambda}) = e^{\imath\omega^{m,\lambda}(f)+\frac{1}{2}\|f\|^2_{\mathscr{S}(\mathbb{R}^2)}} = C_\lambda(f)\, e^{\imath\omega^\lambda(f)+\frac{1}{2}\|f\|^2_{\mathscr{S}(\mathbb{T}^2_\lambda)}},$$

with $C_\lambda(f)\to 1$ as $\lambda\to\infty$, we have $\varphi_n^\lambda(k_{1:n})=\imath^n C_\lambda(f)\hat{f}(k_1)\cdots\hat{f}(k_n)$ for $k_{1:n}\in(\lambda^{-1}\mathbb{Z}^2)^n$.

The Kolmogorov backward equation in the plane Let us now turn to the study of the Kolmogorov backward equation for the whole space setting. In order to get the result we have obtained in Section A.5 in the case of periodic boundary conditions, we need first to give a proper description of the space we are working in, that is $\mathscr{H}_{\mathbb{R}^2}=L^2(\mu_{\mathbb{R}^2})$. In particular, we need that the operator $\mathscr{L}$ is well-defined on this space.

We start by describing the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^2)$. As remarked in Proposition 1.34 of [22] this space is not a space of functions. Indeed consider a smooth bump function $\theta:\mathbb{R}^2\to\mathbb{R}$ compactly supported and such that $\theta(0)=1$. Let $\theta_\varepsilon(x)=\theta(\varepsilon x)$, then as $\varepsilon\to 0$ we have $\|\theta_\varepsilon\|_{\dot{H}^1(\mathbb{R}^2)}\approx 1$, $\theta_\varepsilon\to 1$ pointwise and $\theta_\varepsilon\to 0$ weakly in $\dot{H}^1(\mathbb{R}^2)$. The elements of $\dot{H}^1(\mathbb{R}^2)$ consists of equivalence classes of functions modulo constants and we will have to take this into account in our analysis. We say that $\varphi\in\dot{H}^1(\mathbb{R}^2)$ if there exists a tempered distribution $\tilde{\varphi}\in\mathscr{S}'(\mathbb{R}^2)$ such that

$$\|\tilde{\varphi}\|_{\dot{H}^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |k|^2 |\hat{\tilde{\varphi}}(k)|^2 \mathrm{d}k < \infty,$$

and for which

$$\langle\varphi,\psi\rangle_{\dot{H}^1(\mathbb{R}^2)} = \langle\tilde{\varphi},\psi\rangle_{\dot{H}^1(\mathbb{R}^2)}, \qquad \text{for all } \psi\in\mathscr{S}(\mathbb{R}^2). \tag{A.7.1}$$

Note that the equality (A.7.1) implies an identification of elements whose difference is a constant. Indeed, for $C\in\mathbb{R}$,

$$\langle\varphi,\psi\rangle_{\dot{H}^1(\mathbb{R}^2)} = \langle\tilde{\varphi},\psi\rangle_{\dot{H}^1(\mathbb{R}^2)} = \tilde{\varphi}(|\mathrm{D}|^2\psi) + C\widehat{\delta_0}(|\mathrm{D}|^2\psi) = \hat{\tilde{\varphi}}(|\cdot|^2\hat{\psi}(\cdot)),$$

where D denotes the derivative operator. This means that we identify φ with $\tilde{\varphi}+C$ and write

$$\|\varphi\|_{\dot{H}^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |k|^2 |\hat{\varphi}(k)|^2 \mathrm{d}k.$$

As a consequence, the tensor product $(\dot{H}^1(\mathbb{R}^2))^{\otimes n}$, understood as a tensor product of Hilbert spaces, can be described in the following way: for every $\varphi \in (\dot{H}^1(\mathbb{R}^2))^{\otimes n}$, there exists $\tilde{\varphi} \in \mathcal{S}_s'((\mathbb{R}^2)^n)$, the space of symmetric tempered distributions on $(\mathbb{R}^2)^n$, such that

$$\|\varphi\|^2_{(\dot{H}^1(\mathbb{R}^2))^{\otimes n}} = \int_{(\mathbb{R}^2)^n} \left(\prod_{i=1}^n |k_i|^2\right) |\hat{\tilde{\varphi}}(k_{1:n})|^2 \mathrm{d}k_{1:n} < \infty,$$

and for which

$$\langle \varphi, \psi \rangle_{(\dot{H}^1(\mathbb{R}^2))^{\otimes n}} = \hat{\tilde{\varphi}}\left(\left(\prod_{i=1}^n |\cdot_i|^2\right)\hat{\psi}\right), \qquad \psi \in \mathcal{S}_s((\mathbb{R}^2)^n),$$

that leaves the freedom to change $\hat{\tilde{\varphi}}$ in $Z_n = \bigcup_{i=1}^n \{k_i = 0\} \subset (\mathbb{R}^2)^n$ and defines it modulo a symmetric distribution η whose Fourier transform is supported in Z_n. Therefore, we can identify φ with $\tilde{\varphi}+\eta$, where $(\prod_{i=1}^n |k_i|^2)\hat{\eta}(k_{1:n}) = 0$.

Let us now study the operators $\mathcal{G}_\pm$. With the identification above we can define

$$\begin{aligned}
\mathcal{F}(\mathcal{G}_+^m \varphi)_n(k_{1:n}) &= (n-1)\chi_m(k_1,k_2,k_1+k_2)\frac{(k_1^\perp \cdot (k_1+k_2))((k_1+k_2)\cdot k_2)}{|k_1|^2|k_2|^2}\hat{\varphi}_{n-1}(k_1+k_2,k_{3:n}),\\
\mathcal{F}(\mathcal{G}_-^m \varphi)_n(k_{1:n}) &= (2\pi)^2(n+1)n\int_{\mathbb{R}^2} \chi_m(p,k_1-p,k_1)\frac{(k_1^\perp \cdot p)(k_1\cdot q)}{|k_1|^2}\hat{\varphi}_{n+1}(p,k_1-p,k_{2:n}),
\end{aligned}$$

with χ_m a smooth function such that $\chi_m(k_1,k_2,k_3) \approx 1_{|k_1|,|k_2|,|k_3| \leqslant m}$. These formulas have to be understood via duality

$$\begin{aligned}
\langle \psi, (\mathcal{G}_+^m \varphi)_n \rangle_{(\dot{H}^1)^{\otimes n}} &= (n-1)\int_{(\mathbb{R}^2)^n} \left[\prod_{i=1}^n |k_i|^2\right] \chi_m(k_1,k_2,k_1+k_2)\,\frac{(k_1^\perp \cdot (k_1+k_2))((k_1+k_2)\cdot k_2)}{|k_1|^2|k_2|^2}\\
&\quad \times \hat{\varphi}_{n-1}(k_1+k_2,k_{3:n})\,\hat{\psi}_n(k_{1:n})\,\mathrm{d}k_1\cdots\mathrm{d}k_n\\
&= (n-1)\int_{(\mathbb{R}^2)^n} \left[\prod_{i=3}^n |k_i|^2\right] \chi_m(k_1,k_2,k_1+k_2)\,(k_1^\perp \cdot (k_1+k_2))((k_1+k_2)\cdot k_2)\\
&\quad \times \hat{\varphi}_{n-1}(k_1+k_2,k_{3:n})\,\hat{\psi}_n(k_{1:n})\,\mathrm{d}k_1\cdots\mathrm{d}k_n.
\end{aligned}$$

In order to check that this definition is correct, we need to make sure that whenever $\hat{\varphi}_{n-1}$ is supported in Z_{n-1} or when $\hat{\psi}$ is supported in Z_n the result is zero. This is obvious for $\hat{\psi}$, so let us check it for $\hat{\varphi}_{n-1}$. Assume $\hat{\varphi}_{n-1}$ is supported in Z_{n-1}, then either $k_1+k_2=0$ or $k_i=0$ for some $i=3,\ldots,n$. In the first case the result is zero due to the multiplicative factor $(k_1^\perp \cdot (k_1+k_2))((k_1+k_2)\cdot k_2)$, while in the second the result is again zero because of the factor $\prod_{i=3}^n |k_i|^2$. The same works for $\mathcal{G}_-$ since it is the adjoint of $\mathcal{G}_+$.

The expression of the norms, and therefore the results about the estimates on the operator and so on, are exactly the same as in the periodic setting modulo changing the sums into integrals. As already shown in Section A.4 for the torus case, the existence and uniqueness result for the Kolmogorov backward equation yields, via duality, uniqueness of solutions to the cylinder martingale problem also for the case of the whole space $\mathbb{R}^2$.

A.8 Appendix: Some auxiliary results

Lemma A.8.1. *Let $C,\beta\geqslant 0$, $\alpha>(d+\beta)/(2\theta)$. Then, for every λ large,*

$$\lambda^{-2}\sum_{p\in\lambda^{-1}\mathbb{Z}^d}\frac{|p|^\beta}{(|p|^{2\theta}+|k-p|^{2\theta}+C)^\alpha}\lesssim(|k|^{2\theta}+C)^{(\beta+d)/(2\theta)-\alpha},\qquad k\in\lambda^{-1}\mathbb{Z}^d,$$

uniformly in λ.

Proof. Since $|p|^{2\theta}+|k-q|^{2\theta}\gtrsim|p|^{2\theta}+|k|^{2\theta}$, we have

$$\lambda^{-2}\sum_p\frac{|p|^\beta}{(|p|^{2\theta}+|k-p|^{2\theta}+C)^\alpha}\ \lesssim\ \int_{\mathbb{R}^d}\frac{|y|^\beta}{(|y|^{2\theta}+|k|^{2\theta}+C)^\alpha}\mathrm{d}y$$

By scaling

$$\int_{\mathbb{R}^d}\frac{|y|^\beta}{(|y|^{2\theta}+|k|^{2\theta}+C)^\alpha}\mathrm{d}y=(|k|^{2\theta}+C)^{(\beta+d)/2\theta-\alpha}\int_{\mathbb{R}^d}\frac{|y|^\beta}{(|y|^{2\theta}+1)^\alpha}\mathrm{d}y$$

and the integral is finite if $\beta-2\theta\alpha<-d$. □

Lemma A.8.2. *We have, for any $T>0$, $\gamma>0$,*

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1+\gamma}\psi(t)\| &\leqslant \|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1+\gamma}\psi(0)\|\\ &\quad+\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^\gamma(\partial_t-(1-\mathscr{L}_\theta))\psi(t)\|.\end{aligned}$$

Proof. The proof is standard and proceeds by spectral calculus. Write $\Psi(t):=(\partial_t-(1-\mathscr{L}_\theta))\psi(t)$

$$\Psi_i(s)=1_{|1-\mathscr{L}_\theta|\sim 2^i}\Psi(s),$$

where $1_{|1-\mathscr{L}_\theta|\sim 2^i}$ denotes a dyadic partition of unity such that $\|\varphi\|^2\approx\sum_i\|1_{|1-\mathscr{L}_\theta|\sim 2^i}\varphi\|^2$ for any φ.

Let $S_t=e^{-t(1-\mathscr{L}_\theta)}$, so that

$$\psi(t)=S_t\psi(0)+\int_0^t S_{t-s}\Psi(s)\mathrm{d}s.$$

Then, using $\|(1-\mathscr{L}_\theta)^{1+\gamma}S_{t-s}\psi\|\lesssim((t-s)^{-1-\gamma}\vee 1)\|\psi\|$ and $\|(1-\mathscr{L}_\theta)^{1+\gamma}1_{|1-\mathscr{L}_\theta|\sim 2^i}\|\lesssim 2^{(1+\gamma)i}$, and letting $\delta=2^{-i}$, we have

$$\begin{aligned}&\left\|(1-\mathscr{L}_\theta)^{1+\gamma}\int_0^t S_{t-s}\Psi_i(s)\mathrm{d}s\right\|\\ \leqslant\ &\left\|(1-\mathscr{L}_\theta)^{1+\gamma}\int_0^{t-\delta}S_{t-s}\Psi_i(s)\mathrm{d}s\right\|+\left\|(1-\mathscr{L}_\theta)^{1+\gamma}\int_{t-\delta}^t S_{t-s}\Psi_i(s)\mathrm{d}s\right\|\\ \lesssim\ &\int_0^{t-\delta}((t-s)^{-1-\gamma}\vee 1)\|\Psi_i(s)\|\,\mathrm{d}s+2^{(1+\gamma)i}\int_{t-\delta}^t\|S_{t-s}\Psi_i(s)\|\,\mathrm{d}s\\ \lesssim\ &(\delta^{-\gamma}+2^{i(1+\gamma)}\delta)\sup_{0\leqslant s\leqslant T}\|\Psi_i(s)\|\\ \lesssim\ &2^{i\gamma}\sup_{0\leqslant s\leqslant T}\|\Psi_i(s)\|\\ \lesssim\ &\sup_{0\leqslant s\leqslant T}\|(1-\mathscr{L}_\theta)^\gamma\Psi_i(s)\|,\end{aligned}$$

and, as a consequence,

$$\begin{aligned}\left\|(1-\mathscr{L}_\theta)^{1+\gamma}\int_0^t S_{t-s}\Psi(s)\mathrm{d}s\right\|^2 &\lesssim \sum_i \left\|(1-\mathscr{L}_\theta)^{1+\gamma}\int_0^t S_{t-s}\Psi_i(s)\mathrm{d}s\right\|^2\\ &\lesssim \sup_{0\leqslant s\leqslant T}\sum_i \|(1-\mathscr{L}_\theta)^{\gamma}\Psi_i(s)\|^2\\ &\lesssim \sup_{0\leqslant s\leqslant T}\|(1-\mathscr{L}_\theta)^{\gamma}\Psi(s)\|^2.\end{aligned}$$

Therefore, since $\mathscr{N}$ commutes with $\mathscr{L}_\theta$, we also have

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1+\gamma}\psi(t)\| &\leqslant \|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{1+\gamma}\psi(0)\|\\ &\quad+\sup_{0\leqslant t\leqslant T}\|(1+\mathscr{N})^p(1-\mathscr{L}_\theta)^{\gamma}\Psi(s)\|,\end{aligned}$$

that is the claimed estimate. □

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Appendix B

A singular integration by parts formula for the exponential Euclidean QFT on the plane

Abstract. We give a novel characterization of the Euclidean quantum field theory with exponential interaction ν on $\mathbb{R}^2$ through a renormalized integration by parts (IbP) formula, or otherwise said via an Euclidean Dyson–Schwinger equation for expected values of observables. In order to obtain the well-posedness of the singular IbP problem, we import some ideas used to analyse singular SPDEs and we require the measure to "look like" the Gaussian free field (GFF) in the sense that a suitable Wasserstein distance from the GFF is finite. This guarantees the existence of a nice coupling with the GFF which allows to control the renormalized IbP formula.

B.1 Introduction

One of the first steps in constructive quantum field theory (QFT) is to build a family of distributions satisfying Wightman axioms [84, 173], which can thus be interpreted as the Wightman functions of a field theory with unique ground state, invariant with respect to the Poincaré group. Wightman axioms, however, do not identify uniquely a particular QFT. Instead different QFTs are expected to satisfy different *Dyson–Schwinger equations*: a systems of partial differential equations (PDEs) relating Wightman functions and encoding the local and hyperbolic equations of motions for the quantum fields [60, 167].

An important progress in the analysis of Wightman QFTs was the introduction of Schwinger functions, namely the analytic continuations on imaginary time of Wightman functions, which are described by a set of axioms introduced by Osterwalder and Schrader (see [84, 152, 153, 170]). As observed by Nelson (see [144, 145, 146]), in many cases (such as the scalar bosonic QFT) Schwinger functions are the moments of a probability measure ν on Schwarz distributions (see [84, 170] for systematic applications of this approach). In particular, the Dyson–Schwinger equations translate in an integration by parts (IbP) formula for the measure ν [17, 63, 70, 91] and it becomes natural to investigate the problem of existence and uniqueness of probability measures satisfying prescribed integration by parts formulas.

The characterization of a measure through some IbP formula is a classical subject in stochastic analysis which has different formulations, such as existence and uniqueness of a measure with given logarithmic gradient [34] or the unique closability of a minimally defined pre-Dirichlet form [2, 20, 38] (and the references therein). The application of logarithmic gradients and integration by parts formulas to quantum field theory was already proposed by Kirillov in the case of sine–Gordon models, see [118, 119, 120, 121] where the problem is considered without renormalization. In the aforementioned works by Kirillov, the author exploits also a Lyapunov functions technique to show existence of solutions to the integration by parts formulas; a generalization of this technique is applied to the related problem of non-singular (i.e. with no need for renormalization) Fokker–Planck–Kolmogorov equations by Bogachev and Röckner [39, 40]. The problem of uniqueness of solutions to integration by parts formula or, similarly, to an infinite-dimensional Fokker–Planck–Kolmogorov equation, is solved only in some particular cases, see for instance the books [34, 37], and the works by Bogachev, Da Prato and Röckner [35, 36] and by Röckner, Zhu and Zhu [161], where a dissipative non-regular drift (without renormalization) is considered. Let us also mention the studies about uniqueness of solutions to Fokker–Planck–Kolmogorov equations or of invariant measures of the $P(\varphi)_2$ stochastic quantization equation on the two-dimensional torus [55, 127, 162, 163, 178]. Although in this case the problem of existence and uniqueness is solved, it is not clear whether the techniques used in the aforementioned papers can be extended to the models on the non-compact space $\mathbb{R}^2$ or in dimension greater than two. It is worth to mention that the study of uniqueness in the framework of Dirichlet forms for the exponential model has been discussed by Albeverio, Kawabi and Röckner in [14] for the one-dimensional non-singular case, and by the same authors together with Mihalache in [13] for the two-dimensional setting on the torus. See also [18] for a review of the existing literature on the Dirichlet form approach to the dynamical problem. Let us mention that our concern here is different from that in Dirichlet form theory since here the measure is an unknown in the problem and not given a priori.

The key open problem that we address in this paper is to provide a suitable setting in which existence and uniqueness of measures satisfying given some *singular* IbP formula, that is one involving a renormalization procedure in its definition, which is the usual situation in constructive Euclidean QFT. Instead of attempting a general framework, we concentrate in a particular case where we can establish a reasonable well-posedness theory for the singular IbP problem: we test our ideas on the EQFT with exponential interaction and positive mass $m>0$, or Høegh–Krohn model [10] on the whole space $\mathbb{R}^2$. The exponential interaction in the case of mass $m=0$ [8, 12, 160] is a classical example of conformal field theory [143, 165] and it finds important applications in Liouville quantum gravity [66, 130]. As far as we know, this contribution of ours is the first which manages to address this question for an EQFT requiring renormalization and in the infinite volume limit.

Let us give a more detailed description of the problem that we consider here. Let $\mathcal{S}(\mathbb{R}^2)$ be the space of Schwartz functions and denote its dual, that is the space of tempered distributions, by $\mathcal{S}'(\mathbb{R}^2)$. We fix a Banach space (or a topological vector space) $E\subset\mathcal{S}'(\mathbb{R}^2)$ and we consider a family $\mathfrak{F}$ of functions $F\colon E\to\mathbb{R}$ which are Fréchet differentiable. In particular, for any $\varphi\in E$, we can consider the derivative $D_fF(\varphi)$ of F in the direction $f\in\mathcal{S}(\mathbb{R}^2)$; the map $f\mapsto D_fF(\varphi)$ is linear and bounded in E, and thus in $\mathcal{S}(\mathbb{R}^2)$. This means that, since $\mathcal{S}'(\mathbb{R}^2)$ is the topological dual of $\mathcal{S}(\mathbb{R}^2)$, there exists a unique $\nabla_\varphi F(\varphi)\in\mathcal{S}'(\mathbb{R}^2)$ such that $D_fF(\varphi)=\langle\nabla_\varphi F(\varphi),f\rangle$, where $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{\mathcal{S}',\mathcal{S}}$ is the duality between $\mathcal{S}'(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^2)$.

Let us denote by $\mathscr{M}$ a family of probability measures on E, and let $B\colon E\to\mathcal{S}'(\mathbb{R}^2)$ a given map. Then a generic IbP problem, or Euclidean Schwinger–Dyson equation for a measure $\nu\in\mathscr{M}$, has the general form

$$\int_E\langle\nabla_\varphi F-FB,f\rangle\,\mathrm{d}\nu=0,\qquad\text{for any }F\in\mathrm{Cyl}_E^b.$$

We are interested in *local* functionals B which have the form

$$B(\varphi)(x) = p(\varphi(x)), \qquad x \in \mathbb{R}^2,$$

for some smooth function $p: \mathbb{R} \to \mathbb{R}$. This locality of the IbP formula is peculiar of EQFT where locality (or reflection positivity, or domain Markov property) is structurally linked to the finite speed of propagation of signals in the Minkowski theory. Unfortunately, such kind of functions B are seldom well-defined on the set E on which we could hope that any solution ν would be supported. Typically, this support looks very much like the support of the GFF and therefore non-linear local functionals are not automatically well-defined and need to be approached via an ultraviolet regularization and subsequent renormalization. In this sense, we talk about a *singular* IbP formula adopting the term from the recent literature in singular stochastic PDEs.

Given this motivation we consider a sequence of maps $(B_\varepsilon)_{\varepsilon>0}$ such that, for every $\varepsilon > 0$, we have $B_\varepsilon: E \to \mathcal{S}'(\mathbb{R}^2)$ and for which we recover locality in the limit as $\varepsilon \to 0$. They are typically of the form

$$B(\varphi)(x) = p_\varepsilon((g_\varepsilon * \varphi)(x)), \qquad x \in \mathbb{R}^2,$$

where $(g_\varepsilon)_{\varepsilon \geqslant 0}$ is some sequence of local smoothing kernels for which $(g_\varepsilon * \varphi) \to \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$ and $(p_\varepsilon: \mathbb{R} \to \mathbb{R})_\varepsilon$ is a sequence of smooth function chosen to deliver the expected renormalization, typical of EQFT in two and three dimensions. The main problem we analyze in this paper can therefore be formulated abstractly as follows:

Problem A. *We say that a measure $\nu \in \mathcal{M}$ satisfies the* integration by parts formula with respect to $(B_\varepsilon)_{\varepsilon>0}$ and $\mathcal{M}$ *if, for any $f \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\int_E \langle \nabla_\varphi F(\varphi), f \rangle \, \nu(\mathrm{d}\varphi) = \lim_{\varepsilon \to 0} \int_E F(\varphi) \langle B_\varepsilon(\varphi), f \rangle \nu(\mathrm{d}\varphi), \qquad \text{for any } F \in \mathrm{Cyl}_E^b, \tag{B.1.1}$$

where Cyl_E^b is the set of smooth and bounded cylinder functions (cf. the Notation section at the end of the present introduction).

Let us remark that the problem of existence and uniqueness of solutions to Problem A strongly depends on the subset $\mathcal{M}$ of the space $\mathcal{P}(E)$ of (Radon) probability measures on E. One of the main problems is that, if we consider $\mathcal{M} = \mathcal{P}(E)$, i.e. we consider a generic Radon probability measure on E, then it is not clear if $(B_\varepsilon)_\varepsilon$ admits a limit in probability as $\varepsilon \to 0$ and whether the limit depends on the measure ν or not. For example, if we take B_ε to be the drift of Φ_2^4 measure in $\mathbb{R}^2$, namely

$$B_\varepsilon(\varphi) = (g_\varepsilon * \varphi)^4 - 6c_\varepsilon (g_\varepsilon * \varphi)^2 + 3c_\varepsilon^2$$

with $c_\varepsilon = \|(-\Delta + m^2)^{-1/2} g_\varepsilon\|_{L^2(\mathbb{R}^2)}^2$, then it is known that B_ε converges to the unique limit $:\!\varphi^4\!:$ (where $:\!\cdot\!:$ is the Wick product of Gaussian random fields, see Chapter 1 in [170]) when the measure ν is absolutely continuous with respect to the Gaussian free field with mass $m > 0$. On the other hand, if ν is supported on the space of smooth functions, such a limit does not exist. It is useful then to consider a class of measures ν for which it is possible to make sense (almost surely) of the limit $\lim_{\varepsilon \to 0} B_\varepsilon$.

The set $\mathscr{M}$ can therefore be neither too large nor too small and we find useful to focus on measures which "look like" the Gaussian free field with mass $m>0$. We formalize this idea introducing an appropriate Wasserstein metric which encodes this proximity without requiring absolute continuity, which in the full $\mathcal{S}'(\mathbb{R}^2)$ is anyway never an appropriate condition. Let B be a convex cone in E equipped with a norm $\|\cdot\|_B$ that is stronger than the one of E, then we define the function $d_B: \mathscr{P}(E)\times\mathscr{P}(E)\to[0,+\infty]$ as

$$d_B(\nu,\nu') = \inf_{\pi\in\Pi(\nu,\nu')} \int_{E\times E} \|x-y\|_B\, \chi_B(x-y)\, \pi(\mathrm{d}x,\mathrm{d}y), \qquad \nu,\nu'\in\mathcal{P}(E),$$

where $\Pi(\nu,\nu')\subset\mathscr{P}(E^2)$ is the set of couplings between ν and ν', and

$$\chi_B(x) := \begin{cases} 1, & \text{if } x\in B \\ +\infty & \text{if } x\notin B \end{cases}.$$

Let us suppose that E contains the support of the measure ν^{free} of the Gaussian free field with mass $m>0$ and define

$$\mathscr{M}_B := \{\nu\in\mathscr{P}(E) \,|\, d_B(\nu,\nu^{\text{free}}) < +\infty\}. \tag{B.1.2}$$

The space $\mathscr{M}_B$ strongly depends on the convex cone B together with its norm. For example, if we consider $B=H^1(\mathbb{R}^2)$ we have

$$\mathscr{M}_{H^1(\mathbb{R}^2)} = \{\nu\in\mathscr{P}(E) \,|\, \nu \text{ is absolutely continuous with respect to } \nu^{\text{free}}\}.$$

Another simple case is when $B=E$ equipped with its natural norm, this gives

$$\mathscr{M}_E = \mathscr{W}^1(E),$$

namely the Wasserstein space of measures on E (see Chapter 6 in [181]).

The class of measures $\mathscr{M}_B$ encodes the existence of sufficiently regular couplings between our target measures and the GFF. This mirrors the situation in singular SPDEs and in other recent development in EQFT where the existence of such couplings has been a key technical aspect to develop a suitable stochastic analysis of singular dynamics and EQFTs. In the fundamental work [54], Da Prato and Debussche indeed introduced a notion of solution to stochastic quantization equations in two dimensions as the sum of the Gaussian free field with mass m and a solution to a (random) non-linear PDE. In other words, they look at the solution as a perturbation of the solution to the linear stochastic heat equation. More recently, Barashkov and Gubinelli [26, 27, 28] developed a coupling approach based on an optimal control problem, Bauerschmidt and Hofstetter [30] used a similar coupling to study the pathwise properties of the two dimensional sine–Gordon EQFT on a torus and Shenfeld and Mikulincer [169, 138] linked these ideas with current developments in the theory of optimal transport and functional inequalities. Similar ideas are fundamental also in the context of singular stochastic PDEs, see regularity structures [45, 46, 103, 106], paracontrolled calculus [15, 16, 91, 92], and renormalization group theory [64, 129].

As we already mentioned, we would like to provide complete well-posedness results within this framework of singular IbP formulas and, for this reason, we focus on the specific case of the exponential interaction, namely we take $E=B_X+B_Y$, where $B_X=C_\ell^{-\delta}(\mathbb{R}^2)$, i.e. the (weighted) Besov–Hölder space with negative regularity $-\delta$ (see Appendix B.5), and B_Y is taken to be

$$B_Y\subset B_{p,p,\ell}^{s(\alpha)-\delta}(\mathbb{R}^2),$$

where $s(\alpha)>0$ satisfies some conditions depending on the parameter α (see Definition B.2.6 below) and $B^{s(\alpha)-\delta}_{p,p,\ell}$ is a weighted Besov space (see Appendix B.5). Moreover, we set

$$B_\varepsilon(\varphi) := (-\Delta+m^2)\varphi + \alpha f_\varepsilon e^{\alpha(g_\varepsilon*\varphi)-\frac{\alpha^2}{2}c_\varepsilon}, \tag{B.1.3}$$

where $\alpha, m \in \mathbb{R}_+$, f_ε is a smooth, spatial cut-off function such that $f_\varepsilon \to 1$, $g_\varepsilon = \varepsilon^{-2} g(\varepsilon^{-1}\cdot)$ is a regular mollifier, and

$$c_\varepsilon := \int_{\mathbb{R}^2} g_\varepsilon(z)(-\Delta+m^2)^{-1} g_\varepsilon(z)\,\mathrm{d}z \tag{B.1.4}$$

is a renormalization constant diverging logarithmically to $+\infty$ as $\varepsilon \to 0$. Finally, we consider the space of measures $\mathscr{M}$ in Problem A to be $\mathscr{M}_{B_Y}$ (see equation (B.1.2)), that is an intermediate regime between the case $\mathscr{M}_{H^1}$ (i.e. the space of measures that are absolutely continuous with respect to the Gaussian free field) and $\mathscr{M}_E$ (which coincides with the Wasserstein space $\mathscr{W}^1_E$).

Remark B.1.1. In the following, we shall consider the number $\gamma_{\max} \approx 0.55$ given by the maximum taken over all $r>1$, satisfying Definition B.2.6, of

$$\frac{2(r-1)^2}{r((r-1)^2+1)}. \tag{B.1.5}$$

Moreover, we let $\tilde{\gamma}_{\max} := 3-2\sqrt{2} \approx 0.172 < \gamma_{\max}$.

In the present setting, it is possible to obtain the following results.

Theorem B.1.2. *Suppose that $\alpha^2 < 4\pi\tilde{\gamma}_{\max}$. Consider $\mathscr{M}_{B_Y}$ (see Definition B.2.2) with $E = B_X + B_Y$, where*

$$B_X = C^{-\delta}_\ell(\mathbb{R}^2) \quad \text{and} \quad B_Y = B_Y^{\leqslant} := B^{s(\alpha)-\delta}_{p,p,\ell}(\mathbb{R}^2) \cap \{f : \mathbb{R}^2 \to \mathbb{R}, f \leqslant 0\}.$$

Then there exists a unique solution to Problem A with respect to $(B_\varepsilon)_\varepsilon$ (given by equation (B.1.3)) and the space of measures $\mathscr{M}_{B_Y}$.

It is also possible to obtain an existence result for the whole regime $\alpha^2 < 8\pi$.

Theorem B.1.3. *Suppose that $\alpha^2 < 8\pi$ and assume that the same hypotheses on the spaces B_X, B_Y, E, $\mathscr{M}_{B_Y}$ and the drift $(B_\varepsilon)_\varepsilon$ as in Theorem B.1.2 hold. Then there exists a solution to Problem A with respect to $(B_\varepsilon)_\varepsilon$ and $\mathscr{M}_{B_Y}$.*

Theorem B.1.3 is obtained by building suitable Lyapunov functions independent of $\varepsilon>0$, similarly to what was done by Kirillov for the not-renormalized equation (see [118, 119, 120, 121]), and reducing the infinite-dimensional problem to the existence of a symmetric invariant measure for a finite-dimensional differential operator.

An important consequence of Theorem B.1.2 and Theorem B.1.3 is a differential characterization of the exponential measure (see Theorem B.2.13 for details).

Let us mention that in the present paper we also get a uniqueness result for a slightly more restrictive formulation of Problem A (cf. Problem B in Section B.2.1) in the regime $\alpha^2 < 4\pi\gamma_{\max}$. The possible measures solving this latter formulation of the IbP problem contain the invariant measure of the stochastic quantization equation with exponential interaction.

Plan of the paper

Let us present here the structure of the paper. In Section B.2, we discuss the IbP Problem A in the general setting, establishing some equivalent formulations that will be useful to address Problem A itself. We also consider the more restrictive Problem B and Problem B-sym, their relation with Problem A and some properties of the solutions to such problems, such as the negativity of the coupling between the Gaussian free field and the quantum field measure with exponential interaction. Section B.3 is devoted to the study of uniqueness of solutions to Problem A showing the uniqueness result stated in Theorem B.1.2. Since the proof relies on some properties of the solution to the resolvent equation associated to the drift B_ε, most parts of the section is dedicated to the study of such an object. The existence of solutions to Problem A is proved in Section B.4. The proof is based on an approximation method and it involves Lyapunov functions. In Appendices B.5 and B.6 we recall the definitions and properties of weighted Besov spaces and Wick exponential, respectively, which are used throughout the paper. Appendices B.7, B.8, and B.9 are concerned with some technical, analytical results on (S)PDEs exploited in the paper.

Notation

We fix here some notation adopted throughout the paper.

We write $a \lesssim b$ or $b \gtrsim a$ if there exists a constant $C>0$, independent of the variables under consideration, such that $a \leqslant Cb$, and $a \simeq b$ if both $a \lesssim b$ and $b \lesssim a$. If the aforementioned constant C depends on a variable, say $C = C(x)$, then we use the notation $a \lesssim_x b$, and similarly for $\gtrsim$.

The space $L(A,B)$ is the space of linear and bounded functionals from the Banach space A into the Banach space B. $L(A,B)$ is equipped with its natural operator norm $\|\cdot\|_{L(A,B)}$. We also write $L(A) = L(A,A)$.

The space of Schwartz functions on $\mathbb{R}^d$ is denoted by $\mathcal{S}(\mathbb{R}^d)$ and its dual, that is the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^d)$. Moreover, we let $C_b^m(\mathbb{R}^d)$ be the space of m-times differentiable functions on $\mathbb{R}^d$ with continuous and bounded derivatives. We also use the notation $B^r_{p,q,\ell}(\mathbb{R}^d)$ to denote the weighted Besov space on $\mathbb{R}^d$ (see Appendix B.5 for more results on Besov spaces). For $k, \ell > 0$, we define the weight $\rho_\ell^k(x) = (1 + k|x|^2)^{-\ell/2}$, $x \in \mathbb{R}^d$, and $\rho_\ell := \rho_\ell^1$.

Let K be a convex subset of $\mathcal{S}'(\mathbb{R}^d)$, we define the set Cyl_K of *cylinder functions* on K to be the set of functions $F: K \to \mathbb{R}$ such that there exist a function $\tilde{F} \in C^2(\mathbb{R}^n)$, with at most linear growth at infinity and all bounded derivatives, and $u_1,...,u_n \in \mathcal{S}(\mathbb{R}^d)$, such that $F(\kappa) = \tilde{F}(\langle u_1, \kappa\rangle,..., \langle u_n, \kappa\rangle)$, for every $\kappa \in K$. We say that $F \in \mathrm{Cyl}_K$ has compact support in Fourier variables if the Fourier transforms $\hat{u}_j$ of u_j, $j = 1, \ldots, n$, have compact supports on $\mathbb{R}^d$. Moreover, we denote by Cyl_K^b the subset of Cyl_K such that, sticking with the previous notation, the function $\tilde{F}$ is also bounded.

We adopt the notation $P_t = e^{-(-\Delta + m^2)t}$ for the heat kernel with mass m (cf. Appendix B.5).

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B.2 Formulation of the problem

B.2.1 Reformulation of Problem A

In this section, we want to reformulate Problem A when the space of measures $\mathscr{M}$ is the space $\mathscr{M}_B$ introduced in (B.1.2). For this reason, we consider $E = B_X + B_Y$, where B_X is some space supporting the measure of the Gaussian free field with mass $m > 0$, and $B_Y = B$. We consider also the natural projection $P^X: B_X \times B_Y \to B_X$ and the map $P^{X+Y}: B_X \times B_Y \to E$ such that $(X, Y) \mapsto X + Y$.

Let us reformulate Problem A by means of a second order operator. Consider B_ε to be regular enough and let

$$\mathfrak{L}_\varepsilon F := \frac{1}{2} \mathrm{tr}_{L^2(\mathbb{R}^2)}(\nabla_\varphi^2 F) - \langle B_\varepsilon, \nabla_\varphi F \rangle, \qquad F \in \mathrm{Cyl}_E.$$

Problem A'. *We say that a measure $\nu \in \mathscr{M}$ satisfies the* symmetric Fokker–Planck–Kolmogorov equation related to B_ε *if*

$$\lim_{\varepsilon \to 0} \int \left[(\mathfrak{L}_\varepsilon F)\, G - F\, (\mathfrak{L}_\varepsilon G) \right] \mathrm{d}\nu = 0, \qquad \textit{for any } F \in \mathrm{Cyl}_E^b, G \in \mathrm{Cyl}_E. \tag{B.2.1}$$

Remark B.2.1. A consequence of equation (B.2.1) is that

$$\lim_{\varepsilon \to 0} \int \mathfrak{L}_\varepsilon F \mathrm{d}\nu = 0, \qquad \text{for any } F \in \mathrm{Cyl}_E. \tag{B.2.2}$$

That is what is usually called *Fokker-Planck-Kolmogorov equation*.

We now want to lift the problem from the space E to the space $B_X \times B_Y$. In order to do so, we introduce the following notion:

Definition B.2.2. *The subset of measures $\mathscr{M}$ satisfies the* coupling hypotheses *if, for any $\nu \in \mathscr{M}$, there exists a Radon measure μ on $B_X \times B_Y$ with the following properties:*

i. $P_*^X \mu = \mu^{\mathrm{free}}$, *where μ^{free} is the law of the Gaussian free field on B_X,*

ii. $P_*^{X+Y} \mu = \nu$,

iii. $\int \|Y\|_{B_Y} \mu(\mathrm{d}X, \mathrm{d}Y) < +\infty$,

we call μ a coupling of ν with the free field. *We denote by $\mathscr{M}_{B_X \times B_Y}$ the set of Radon measures on $B_X \times B_Y$ satisfying condition i. and iii.*

Remark B.2.3. With the same notation as in Definition B.2.2, it is clear that $\mathscr{M}_{B_Y}$ defined as in (B.1.2) where $B = B_Y$, coincides with the space $P_*^{X+Y} \mathscr{M}_{B_X \times B_Y}$. Indeed, $\nu \in \mathscr{M}_{B_Y}$ if and only if there exists a coupling $\pi(\mathrm{d}x, \mathrm{d}z)$ between $\nu(\mathrm{d}z)$ and the free field $\mu^{\mathrm{free}}(\mathrm{d}x)$ such that the difference $x - z$ is supported on B_Y. The coupling π is related with a measure μ in $\mathscr{M}_{B_X \times B_Y}$ via the transformation $y = x - z$.

We consider now an operator $\mathscr{L}_\varepsilon$ on the space of regular functions on $B_X \times B_Y$ of the form

$$\mathscr{L}_\varepsilon \Phi(X, Y) := \frac{1}{2} \mathrm{tr}(\nabla_X^2 \Phi) - \langle (-\Delta + m^2) X, \nabla_X \Phi \rangle - \langle B_\varepsilon(X + Y) - (-\Delta + m^2) Y, \nabla_Y \Phi \rangle.$$

Problem A''. *We say that a measure $\nu \in \mathscr{M}_{B_Y}$ satisfies the* symmetric Fokker–Planck–Kolmogorov equation related to B_ε *if*

$$\lim_{\varepsilon \to 0} \int \left[\mathscr{L}_\varepsilon (F \circ P^{X+Y})\, G \circ P^{X+Y} - F \circ P^{X+Y}\, \mathscr{L}_\varepsilon (G \circ P^{X+Y}) \right] \mu(\mathrm{d}X, \mathrm{d}Y) = 0, \quad \textit{for any } F \in \mathrm{Cyl}_E^b, G \in \mathrm{Cyl}_E, \tag{B.2.3}$$

where μ is a coupling of ν with the free field.

Remark B.2.4. As in the case of Problem A' (see also Remark B.2.1), equation (B.2.3) implies

$$\lim_{\varepsilon\to 0}\int \mathscr{L}_\varepsilon(F\circ P^{X+Y})\ \mu(\mathrm{d}X,\mathrm{d}Y)=0, \qquad \text{for any } F\in \mathrm{Cyl}_E, \tag{B.2.4}$$

We often say that μ solves Problem A'' if there is a measure ν solving the FPK equation with μ as coupling of ν with the free field.

Problem A'' is based on a formulation of integration by parts formula related to stationary solutions to the Fokker–Planck–Kolmogorov equation, making the operator $\mathscr{L}_\varepsilon$ symmetric on its domain (see Problem A' in Section B.2.1 or [1, 38] for more details on the relations between the formulations).

It is convenient to introduce new equations related to Problem A'' described above, where the argument of $\mathscr{L}_\varepsilon$ in equation (B.2.4) is not necessarily of the form $F\circ P^{X+Y}$.

Problem B. *We say that a Radon measure μ on $B_X\times B_Y$ satisfies the* Fokker–Planck–Kolmogorov equation related to B_ε, *if*

$$\lim_{\varepsilon\to 0}\int \mathscr{L}_\varepsilon\Phi\,\mathrm{d}\mu=0, \qquad \textit{for any } \Phi\in \mathrm{Cyl}_{B_X\times B_Y}. \tag{B.2.5}$$

Problem B-sym. *We say that μ satisfies the* symmetric Fokker–Planck–Kolmogorov equation related to B_ε *if, furthermore,*

$$\lim_{\varepsilon\to 0}\int \left[\mathscr{L}_\varepsilon(F\circ P^{X+Y})\,G\circ P^{X+Y}-F\circ P^{X+Y}\,\mathscr{L}_\varepsilon(G\circ P^{X+Y})\right]\,\mathrm{d}\mu=0, \tag{B.2.6}$$

for any $F,G\colon B_X+B_Y\to\mathbb{R}$ such that $F\circ P^{X+Y}\in \mathrm{Cyl}_E^b, G\circ P^{X+Y}\in \mathrm{Cyl}_E$.

We have then the following result.

Theorem B.2.5. *Suppose that $\sup_{\varepsilon>0}\int|\langle B_\varepsilon(\varphi),\nabla_\varphi F\rangle|\ \nu(\mathrm{d}\varphi)<+\infty$, for any $F\in\mathrm{Cyl}_E$ and $\nu\in\mathscr{M}$. Then the following statements hold:*

i. Problem A is equivalent to Problem A'.

ii. Assume further that $\mathscr{M}=\mathscr{M}_{B_Y}$, then Problem A, Problem A' and Problem A'' are equivalent.

iii. Assume the same hypothesis as in point ii.. Then, a solution to Problem B-sym is a solution to Problem A''.

Summarizing, if all the hypotheses of points i., ii., and iii., then Problem A is equivalent to Problem A'', and Problem B implies Problems A and A'', that is

$$\textit{Problem A} \iff \textit{Problem A'} \iff \textit{Problem A''} \impliedby \textit{Problem B-sym}$$

Proof. Let us give the proof point by point.

Proof of point *i*. Let us prove that a solution to Problem A' is also a solution to Problem A. Let $\nu\in\mathscr{M}$ solve Problem A', take $F\in\mathrm{Cyl}_E^b$, and consider $G(\cdot)=\langle\cdot,f\rangle$, for $f\in\mathcal{S}(\mathbb{R}^2)$. Then, by equation $(B.2.2)$, we have

$$\lim_{\varepsilon\to 0}\int \mathfrak{L}_\varepsilon(F\cdot G)\mathrm{d}\nu=0,$$

where

$$\begin{aligned}\mathfrak{L}_\varepsilon(F\cdot G) &= \mathfrak{L}_\varepsilon(F)\,G+F\,\mathfrak{L}_\varepsilon(G)+2\langle\nabla_\varphi F,\nabla_\varphi G\rangle\\ &= \mathfrak{L}_\varepsilon(F)\,G+F\,\mathfrak{L}_\varepsilon(G)+2\langle\nabla_\varphi F,f\rangle.\end{aligned}$$

Thus, we have

$$\begin{aligned}0 &= \lim_{\varepsilon\to0}\int\mathfrak{L}_\varepsilon(F\cdot G)\mathrm{d}\nu=\lim_{\varepsilon\to0}\int(\mathfrak{L}_\varepsilon(F)\,G+F\,\mathfrak{L}_\varepsilon(G)+2\langle\nabla_\varphi F,f\rangle)\mathrm{d}\nu\\ &= \lim_{\varepsilon\to0}\int(2F\,\mathfrak{L}_\varepsilon(G)+2\langle\nabla_\varphi F,f\rangle)\mathrm{d}\nu=\lim_{\varepsilon\to0}\int(2F\langle B_\varepsilon,f\rangle+2\langle\nabla_\varphi F,f\rangle)\mathrm{d}\nu.\end{aligned}$$

We now show that any solution to Problem A is a solution to Problem A'. Let $\mu\in\mathscr{M}$ solve Problem A. Consider an orthonormal basis $(e_n)_n$ of $L^2(\mu)$ and take $F=\langle\nabla G,e_n\rangle$, for some $G\in\mathrm{Cyl}_E$. We have then, by equation (B.1.1),

$$\int\nabla^2G(e_n,e_n)\mathrm{d}\nu=\int\langle\nabla F,e_n\rangle\mathrm{d}\nu=\int\langle\nabla G,e_n\rangle\langle B_\varepsilon,e_n\rangle\mathrm{d}\nu.$$

Taking the sum over $n\in\mathbb{N}$ on both sides, noticing that by the properties of cylinder functions we can exchange the integral with the sum, and exploiting Parseval's identity we have

$$\int\mathrm{tr}_{L^2(\mathbb{R}^2)}(\nabla^2G)\mathrm{d}\nu=\int\langle\nabla G,B_\varepsilon\rangle\mathrm{d}\nu,$$

that is $\int\mathfrak{L}_\varepsilon G\mathrm{d}\nu=0$. Now, consider again an orthonormal basis $(e_n)_n$ of $L^2(\mu)$, but take $F=G\langle\nabla H,e_n\rangle$, for some $G,H\in\mathrm{Cyl}_E$. Let us note that

$$\langle\nabla F,e_n\rangle=\langle\nabla(G\langle\nabla H,e_n\rangle),e_n\rangle=\langle\nabla G,e_n\rangle\langle\nabla H,e_n\rangle+G\nabla^2H(e_n,e_n).$$

On the other hand, equation (B.1.1) gives

$$\int\langle\nabla(G\langle\nabla H,e_n\rangle),e_n\rangle\mathrm{d}\nu=\int G\langle\nabla H,e_n\rangle\langle B_\varepsilon,e_n\rangle\mathrm{d}\nu.$$

This yields

$$\int G\langle\nabla H,e_n\rangle\langle B_\varepsilon,e_n\rangle\mathrm{d}\nu=\int\big(\langle\nabla G,e_n\rangle\langle\nabla H,e_n\rangle+G\nabla^2H(e_n,e_n)\big)\mathrm{d}\nu,$$

and taking the sum over $n\in\mathbb{N}$ and using again the properties of cylinder functions, we get

$$\int G\langle\nabla H,B_\varepsilon\rangle\mathrm{d}\nu=\int\big(\langle\nabla G,\nabla H\rangle+G\mathrm{tr}_{L^2(\mathbb{R}^2)}(\nabla^2H)\big)\mathrm{d}\nu,$$

which implies

$$0=\int\big(\langle\nabla G,\nabla H\rangle+G\mathrm{tr}(\nabla^2H)-G\langle\nabla H,B_\varepsilon\rangle\big)\mathrm{d}\nu=\int\big(\langle\nabla G,\nabla H\rangle+G\mathfrak{L}_\varepsilon H\big)\mathrm{d}\nu,$$

namely

$$\int G\,\mathfrak{L}_\varepsilon(H)\mathrm{d}\nu=-\int\langle\nabla G,\nabla H\rangle\mathrm{d}\nu.$$

Doing the same computation exchanging the roles of G and H we also get $\int H\,\mathfrak{L}_\varepsilon(G)\nu(\mathrm{d}\varphi)=-\int\langle\nabla G,\nabla H\rangle\nu(\mathrm{d}\varphi)$, that gives

$$\int G\,\mathfrak{L}_\varepsilon(H)\mathrm{d}\nu-\int H\,\mathfrak{L}_\varepsilon(G)\mathrm{d}\nu=0,$$

and concludes the proof of the first point of the theorem.

Proof of point *ii.* Assume further that $\mathscr{M}=\mathscr{M}_{B_Y}$. Notice that $\mathscr{L}_\varepsilon(F\circ P^{X+Y})=\mathfrak{L}_\varepsilon(F)\circ P^{X+Y}$, and since $P_*^{X+Y}\mu=\nu$ we have that

$$\int \mathscr{L}_\varepsilon(F\circ P^{X+Y})\mathrm{d}\mu=\int \mathfrak{L}_\varepsilon(F)\circ P_*^{X+Y}\mathrm{d}\mu=\int \mathfrak{L}_\varepsilon F\mathrm{d}\nu,$$

which gives the equivalence between equations (B.2.4) and (B.2.2) and in the same way we have the equivalence between equations (B.2.3) and (B.2.1).

Proof of point *iii.* Let μ be a solution to Problem B. We only have to show that $\mu\in\mathscr{M}_{B_X\times B_Y}$, that is, the only condition that we need to verify is that $P_*^X\mu=\mu^{\mathrm{free}}$.

Since the terms involving derivatives with respect to X of $\mathscr{L}_\varepsilon$ do not depend on ε, in this proof we write $\mathscr{L}$ for the operator $\mathscr{L}_\varepsilon$. Furthermore, for a solution μ to (B.2.5)–(B.2.6) we have that, if Φ does not depend on Y,

$$0=\lim_{\varepsilon\to 0}\int \mathscr{L}_\varepsilon\Phi(X)\,\mu(\mathrm{d}X,\mathrm{d}Y)=\int \mathscr{L}_\varepsilon\Phi(X)\,\mu(\mathrm{d}X,\mathrm{d}Y)=\int \mathscr{L}\Phi(X)\,\mu(\mathrm{d}X,\mathrm{d}Y).$$

Step 1. We start with the proof that the free field measure with mass $m>0$ solves equations (B.2.5)–(B.2.3).

Consider, for $X_0\in B_X$,

$$\mu_t\sim X_t=P_tX_0+\int_0^t P_{t-s}\xi_s\,\mathrm{d}s,$$

and let μ_∞ be the limit

$$\mu_\infty(\mathrm{d}X)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T \mu_t(\mathrm{d}X)\,\mathrm{d}t.$$

We want to show that $\int\mathscr{L}\Phi\mathrm{d}\mu_\infty=0$, for any Φ smooth cylindric function. By Itô's formula (see Theorem 4.32 in [59]), taking expectation and dividing by $T>0$, we have that

$$\begin{aligned}\frac{1}{T}\mathbb{E}[\Phi(X_T)-\Phi(X_0)] &= \frac{1}{T}\int_0^T \mathbb{E}[\mathscr{L}\Phi(X_s)]\,\mathrm{d}s\\ &= \frac{1}{T}\int_0^T\int \mathscr{L}\Phi(X)\,\mu_s(\mathrm{d}X)\,\mathrm{d}s\\ &= \int \mathscr{L}\Phi(X)\left(\frac{1}{T}\int_0^T \mu_s(\cdot)\,\mathrm{d}s\right)(\mathrm{d}X),\end{aligned}$$

and letting $T\to+\infty$, recalling that Φ is bounded, we get

$$0=\int \mathscr{L}\Phi(X)\left(\lim_{T\to+\infty}\frac{1}{T}\int_0^T \mu_s(\cdot)\,\mathrm{d}s\right)(\mathrm{d}X)=\int \mathscr{L}\Phi(X)\,\mu_\infty(\mathrm{d}X).$$

Step 2. We now show uniqueness of the measure μ. Consider

$$G^\lambda(X_0)=\mathbb{E}_\xi\int_0^{+\infty} e^{-\lambda s}F(X_s)\,\mathrm{d}s,$$

where $F\in\mathrm{Cyl}_{B_X}$ of the form

$$F(X)=\tilde{F}(\langle u_1,X\rangle,\ldots,\langle u_n,X\rangle).$$

Step 2.a. Let us prove that $\mathscr{L}_\varepsilon G^\lambda$ is well-defined, recall

$$\mathscr{L}_\varepsilon G^\lambda(X_0)=\mathscr{L}G^\lambda(X_0)=\frac{1}{2}\mathrm{tr}(\nabla_{X_0}^2G^\lambda)-\langle(-\Delta+m^2)X_0,\nabla_{X_0}G^\lambda\rangle,$$

where

$$\nabla_{X_0}G^\lambda=\mathbb{E}_\xi\int_0^{+\infty} e^{-\lambda s}\langle\nabla_{X_0}F(X_s),\nabla_{X_0}X_s\rangle\,\mathrm{d}s=\mathbb{E}_\xi\int_0^{+\infty} e^{-\lambda s}\sum_{k=1}^n\partial_k\tilde{F}(\langle u_1,X\rangle,\ldots,\langle u_n,X\rangle)\,P_tu_k\,\mathrm{d}s,$$

and

$$\begin{aligned}\nabla^2_{X_0} G^\lambda &= \mathbb{E}\int_0^{+\infty} e^{-\lambda s}\nabla_{X_0}[\langle \nabla_{X_0}F(X_s), \nabla_{X_0}X_s\rangle]\,\mathrm{d}s\\ &= \mathbb{E}_\xi\int_0^{+\infty} e^{-\lambda s}\sum_{k,j=1}^n \partial^2_{k,j}\tilde{F}(\langle u_1,X\rangle,\ldots,\langle u_n,X\rangle)\,\langle P_t u_k, P_t u_j\rangle\,\mathrm{d}s,\end{aligned}$$

since

$$\nabla_{X_0}X_t(h) = P_t h, \qquad \nabla^2_{X_0}X_t(h) = 0.$$

This gives

$$\begin{aligned}\mathscr{L}G^\lambda(X_0) &= \frac{1}{2}\mathrm{tr}_{L^2(\mathbb{R}^2)}\left(\mathbb{E}_\xi\int_0^{+\infty} e^{-\lambda s}\sum_{k,j=1}^n \partial^2_{k,j}\tilde{F}(\langle u_1,X\rangle,\ldots,\langle u_n,X\rangle)\,\langle P_t u_k, P_t u_j\rangle\,\mathrm{d}s\right)\\ &\quad -\mathbb{E}_\xi\int_0^{+\infty} e^{-\lambda s}\sum_{k=1}^n \partial_k\tilde{F}(\langle u_1,X\rangle,\ldots,\langle u_n,X\rangle)\langle(-\Delta+m^2)X_0, P_t u_k\rangle\,\mathrm{d}s,\end{aligned}$$

and thus $\mathscr{L}G^\lambda$ is well-defined.

Step 2.b. We now show that

$$\int (\lambda-\mathscr{L})G^\lambda\,\mathrm{d}\mu = \int F\,\mathrm{d}\mu. \tag{B.2.7}$$

By Itô formula (see Theorem 4.32 in [59]), we have

$$e^{-\lambda t}G^\lambda(X_t) - G^\lambda(X_0) = \int_0^t e^{-\lambda s}\mathscr{L}_\varepsilon G^\lambda(X_s)\,\mathrm{d}s + \int_0^t e^{-\lambda s}\nabla_{X_0}G^\lambda(X_s)\,\mathrm{d}X_s - \lambda\int_0^t e^{-\lambda s}G^\lambda(X_s)\,\mathrm{d}s,$$

and taking the expectation we get

$$e^{-\lambda t}\mathbb{E}[G^\lambda(X_t)] - \mathbb{E}[G^\lambda(X_0)] = \mathbb{E}\int_0^t e^{-\lambda s}\mathscr{L}_\varepsilon G^\lambda(X_s)\,\mathrm{d}s - \lambda\mathbb{E}\int_0^t e^{-\lambda s}G^\lambda(X_s)\,\mathrm{d}s. \tag{B.2.8}$$

On the other hand,

$$e^{-\lambda t}G^\lambda(X_t) - G^\lambda(X_0) = \int_0^{+\infty} e^{-\lambda(t+s)}\mathbb{E}_\xi[F(X_{t+s})]\,\mathrm{d}s - \int_0^{+\infty} e^{-\lambda s}\mathbb{E}_\xi[F(X_s)]\,\mathrm{d}s,$$

and taking the expectation

$$e^{-\lambda t}\mathbb{E}[G^\lambda(X_t)] - \mathbb{E}[G^\lambda(X_0)] = -\mathbb{E}\int_0^t e^{-\lambda s}\mathbb{E}_\xi[F(X_s)]\,\mathrm{d}s. \tag{B.2.9}$$

Comparing equations (B.2.8) and (B.2.9), we get

$$\lambda\mathbb{E}\int_0^t e^{-\lambda s}G^\lambda(X_s)\,\mathrm{d}s - \mathbb{E}\int_0^t e^{-\lambda s}\mathscr{L}_\varepsilon G^\lambda(X_s)\,\mathrm{d}s = \mathbb{E}\int_0^t e^{-\lambda s}\mathbb{E}_\xi[F(X_s)]\,\mathrm{d}s,$$

dividing by $t>0$ and letting $t\to 0$, we have the result.

Step 2.c. Take $Z\sim\mu$ and $Z'\sim\mu_\infty$, and call ψ^Z and $\psi^{Z'}$ the solutions to equation

$$(\partial_t - \Delta + m^2)\psi = \xi,$$

with initial conditions $\psi(0)=Z$ and $\psi(0)=Z'$, respectively. We show that $\mu\sim\mu_\infty$.

Consider the product measure $\mu\otimes\mu_\infty$ between μ and μ_∞. Since we are working in separable spaces, any Borel measure is a Radon measure (see, e.g. Theorem 9 in Chapter 2, Section 3 in [166]), and therefore we have that, for every $\varepsilon>0$, there exists $r>0$ such that

$$\mu\otimes\mu_\infty(\|Z\|>r, \|Z'\|>r) < \varepsilon. \tag{B.2.10}$$

Since

$$
\begin{aligned}
|\tilde{F}(\langle u_1,\psi^Z\rangle,\ldots,\langle u_n,\psi^Z\rangle)-\tilde{F}(\langle u_1,\psi^{Z'}\rangle,\ldots,\langle u_n,\psi^{Z'}\rangle)| &\leqslant \|\nabla\tilde{F}\|_{L^\infty}\sum_{k=1}^n |\langle u_k,\psi^Z-\psi^{Z'}\rangle| \\
&\leqslant \|\nabla\tilde{F}\|_{L^\infty}\sum_{k=1}^n \|u_k\|\,\|P_t(Z-Z')\| \\
&\leqslant C\lambda\int_0^{+\infty} e^{-\lambda t}e^{-m^2 t}(\|Z\|+\|Z'\|)\,\mathrm{d}t \\
&\leqslant C'\frac{\lambda}{\lambda+m^2}(\|Z\|+\|Z'\|),
\end{aligned}
$$

then we have, recalling $\int\lambda G_\lambda\,\mathrm{d}\mu_\infty=\int F\,\mathrm{d}\mu_\infty$,

$$
\int\lambda G_\lambda\,\mathrm{d}\mu-\int F\,\mathrm{d}\mu_\infty=\mathbb{E}[\lambda G_\lambda(Z)-\lambda G_\lambda(Z')]\leqslant C'\frac{\lambda}{\lambda+m^2}\mathbb{E}(\|Z\|+\|Z'\|).
$$

The right-hand side of the previous inequality converges to zero as $\lambda\to+\infty$ by property (B.2.10), and thus $\int F\,\mathrm{d}\mu_\infty=\int F\,\mathrm{d}\mu$ since, by equation (B.2.7) and being μ a solution to Problem B, we have $\lambda\int G_\lambda\,\mathrm{d}\mu=\int F\,\mathrm{d}\mu$. This yields $\mu\sim\mu_\infty$ and uniqueness of the measure. □

B.2.2 A priori deductions for exponential interaction

Let $\alpha,m\in\mathbb{R}_+$, and $\gamma:=\alpha^2/(4\pi)$.

Definition B.2.6. *We define*

$$B_X:=C_\ell^{-\delta}(\mathbb{R}^2),$$

for some $\delta>0$ small enough, and with $\ell>0$. Let us recall that ℓ denotes the presence of a weight, in fact $C_\ell^\sigma(\mathbb{R}^2)$ and $B_{p,q,\ell}^\sigma(\mathbb{R}^\sigma)$ are weighted Besov spaces (see Appendix B.5). Also, we let B_Y to be a space of Besov functions with positive regularity, in particular, we choose

$$B_Y\subset B_{p,p,\ell}^{s-\delta}(\mathbb{R}^2),$$

where $1<p<+\infty$ and $s>0$ satisfies the following condition depending on γ:

$$0<s<\gamma+2-\sqrt{8\gamma},$$

arising when dealing with the problem of existence of solutions to the Fokker–Planck–Kolmogorov equation (cf. Theorem B.4.4). We suppose further that there exists $r>1$ such that

$$\frac{1}{p}+\frac{1}{r}\leqslant 1,\qquad s-\gamma(r-1)-2\delta>0,\qquad \gamma r<2.$$

Notice that such a condition is always satisfied for some $s>0$, $p>1$ and $r>1$, whenever $\gamma<2$. Let us also recall the definition of the space

$$B_Y^{\leqslant}:=B_{p,p,\ell}^{s-\delta}(\mathbb{R}^2)\cap\{f:\mathbb{R}^2\to\mathbb{R}, f\leqslant 0\}\tag{B.2.11}$$

featured in Theorem B.1.2.

Let us introduce the functional $\mathcal{G}_\varepsilon: B_X\times B_Y\to\mathcal{S}'(\mathbb{R}^2)$

$$\mathcal{G}_\varepsilon(X,Y):=\alpha f_\varepsilon :e^{\alpha(g_\varepsilon*X)}: e^{\alpha(g_\varepsilon*Y)}.\tag{B.2.12}$$

where the term $:e^{\alpha(g_\varepsilon*X)}:$ is defined as

$$:e^{\alpha(g_\varepsilon*X)}:=e^{\alpha(g_\varepsilon*X)-\frac{\alpha^2}{2}c_\varepsilon},\tag{B.2.13}$$

with the constant c_ε introduced in equation (B.1.4). Here, f_ε is a smooth cut-off function with compact support and with derivatives uniformly bounded in ε such that $f_\varepsilon \to 1$ uniformly with respect to a polynomially weighted norm, and we let $g_\varepsilon = \varepsilon^{-2} g(\varepsilon^{-1} \cdot)$, where g is a positive, smooth, compactly supported function on $\mathbb{R}^2$ with Lebesgue integral equal to 1, and such that there exists its convolutional square root $\tilde{g}_\varepsilon$, i.e. $g_\varepsilon = \tilde{g}_\varepsilon * \tilde{g}_\varepsilon$, which is also positive, smooth, and compactly supported function with $\int \tilde{g}_\varepsilon(x)\,\mathrm{d}x = 1$. We also assume that $g_\varepsilon(x) = g_\varepsilon(-x)$, for all $x \in \mathbb{R}^2$. Notice that this last property implies that $\hat{g}_\varepsilon$ takes real values.

Consider an operator $\mathscr{L}_\varepsilon$ of the form

$$\mathscr{L}_\varepsilon \Phi(X,Y) = \frac{1}{2}\operatorname{tr}(\nabla_X^2 \Phi) - \langle(-\Delta + m^2)X, \nabla_X \Phi\rangle - \langle(-\Delta + m^2)Y + \mathscr{G}_\varepsilon(X,Y), \nabla_Y \Phi\rangle, \tag{B.2.14}$$

which is well-defined for $\Phi: B_X \times B_Y \to \mathbb{R}$ in a suitable class $\mathscr{F}$ of regular functions to be specified below (see Definition B.3.1). For the moment, we only require the set $\mathrm{Cyl}_{B_X \times B_Y}$ of smooth cylindrical functions to be contained in the family $\mathscr{F}$.

Remark B.2.7. Our choice of B_X and B_Y implies that the space B_X contain the Gaussian free field with mass m and that

$$\begin{aligned} (-\Delta + m^2)X &\in C_\ell^{-2-\delta}(\mathbb{R}^2), \\ (-\Delta + m^2)Y &\in B_{p,p,\ell}^{s-2-\delta}(\mathbb{R}^2), \\ :\!\exp(\alpha X)\!: &\in B_{r,r,\ell'}^{-\gamma(r-1)-\delta}(\mathbb{R}^2), \qquad r\ell' > 2, \\ \exp(\alpha Y) &\in L^\infty(\mathbb{R}^2) \cap B_Y, \qquad \text{if } Y \leqslant 0, \end{aligned}$$

where all the parameters are as in Definition B.2.6 (see [4, 110, 111] and Appendix B.6). For simplicity, we adopt the notation

$$B_{\exp}^{\tilde{s},\tilde{\ell}} = B_{\tilde{s},\tilde{s},\tilde{\ell}}^{-\gamma(\tilde{s}-1)-\delta}(\mathbb{R}^2),$$

and we omit the parameters whenever $\tilde{s} = r$ and $\tilde{\ell} = \ell'$, i.e. $B_{\exp} = B_{\exp}^{\tilde{s},\tilde{\ell}}$.

Let us introduce also the notation

$$\mathscr{G} = \mathscr{G}(X,Y) := \alpha :\!e^{\alpha X}\!: e^{\alpha Y},$$

where the measurable function $:\!\exp(\alpha X)\!:$ is defined on a subset of B_X of full measure (with respect to the free field measure with mass m) as the limit of $:\!\exp(\alpha g_\varepsilon * X)\!:$ as $\varepsilon \to 0$ (see Proposition B.6.1 for details). The operator $\mathscr{L}_\varepsilon$ defined in equation (B.2.14) is an approximation for the operator

$$\mathscr{L}\Phi := \frac{1}{2}\operatorname{tr}_{L^2(\mathbb{R}^2)}(\nabla_X^2 \Phi) - \langle(-\Delta + m^2)X, \nabla_X \Phi\rangle - \langle(-\Delta + m^2)Y + \mathscr{G}(X,Y), \nabla_Y \Phi\rangle, \tag{B.2.15}$$

for $\Phi \in \mathrm{Cyl}_{B_X \times B_Y}$.

Theorem B.2.8. *Let B_X and B_Y be as in Definition B.2.6 with the additional condition $B_Y = B_Y^{\leqslant}$ and consider a Radon measure $\mu \in \mathscr{M}_{B_X \times B_Y}$. Then, for any $\Phi, \Psi \in \mathrm{Cyl}_{B_X \times B_Y}$, we have*

$$\lim_{\varepsilon \to 0} \int (\mathscr{L}_\varepsilon \Phi)\Psi\,\mathrm{d}\mu = \int (\mathscr{L}\Phi)\Psi\,\mathrm{d}\mu. \tag{B.2.16}$$

Proof. We show the proof only for the case $\Psi = 1$, the general case can be deduced via Lebesgue's dominated convergence theorem. Let us prove that $\mathscr{L}\Phi \in L^1(\mu)$, for $\Phi \in \mathrm{Cyl}_{B_X \times B_Y}$. By definition, we have that $\operatorname{tr}_{L^2(\mathbb{R}^2)}(\nabla_X^2 \Phi)$ is bounded, and therefore in $L^1(\mu)$. Moreover,

$$|\langle(-\Delta + m^2)X, \nabla_X \Phi\rangle| = |\langle X, (-\Delta + m^2)\nabla_X \Phi\rangle| \leqslant \|X\|_{B_X} \|\nabla_X \Phi\|_{B_{1,1,-\ell}^{2-\delta}},$$

and since $\|X\|_{B_X}\in L^1(\mu)$, $P_*^X\mu$ being the Gaussian free field, and $\|\nabla_X\Phi\|_{B^{2-\delta}_{1,1,-\ell}}\in L^\infty(B_X\times B_Y,\mathbb{R})$, then also the term $\langle(-\Delta+m^2)X,\nabla_X\Phi\rangle$ is in $L^1(\mu)$. Now, the term $\langle(-\Delta+m^2)Y,\nabla_Y\Phi\rangle$ can be handled similarly by using the hypothesis $\|Y\|_{B_Y}\in L^1(\mu)$.

We are left to consider the term $\langle\mathscr{G},\nabla_Y\Phi\rangle=\langle\alpha: e^{\alpha X}\colon e^{\alpha Y},\nabla_Y\Phi\rangle$. First of all, we note that the product $:\exp(\alpha X)\colon\exp(\alpha Y)$ is well-defined. Indeed, since $P_*^X\mu$ is the Gaussian free field we can exploit Proposition B.6.1 to get $:\exp(\alpha X)\colon\in B^{r,\ell'}_{\exp}$ (cf. Remark B.2.7). Furthermore, since $Y\leqslant 0$ and $Y\in B^{s-\delta}_{p,p,\ell}(\mathbb{R}^2)$, by composition of a smooth bounded function with a Besov function we have $\exp(\alpha Y)\in B^{s-\delta}_{p,p,\ell}(\mathbb{R}^2)$. Thus, by Theorem B.5.3, provided

$$\frac{1}{r'}:=\frac{1}{r}+\frac{1}{p}<1,\qquad s-\frac{\alpha^2}{4\pi}(r-1)-2\delta>0,$$

we have that $:\exp(\alpha X)\colon\exp(\alpha Y)\in B^{r',\ell+\ell'}_{\exp}$, and the product is well-defined and continuous. On the other hand, since we have also $\exp(\alpha Y)\in L^\infty(\mathbb{R}^2)$, then Proposition B.5.4 implies that $:\exp(\alpha X)\colon\exp(\alpha Y)\in B^{r,\ell}_{\exp}$ and

$$\|:e^{\alpha X}\colon e^{\alpha Y}\|_{B^{r,\ell}_{\exp}}\lesssim\|:e^{\alpha X}\colon\|_{B^{r,\ell}_{\exp}}.$$

Thus,

$$|\langle\mathscr{G},\nabla_Y\Phi\rangle|\lesssim\|:e^{\alpha X}\colon\|_{B^{r,\ell}_{\exp}}\|\nabla_Y\Phi\|_{(B^{r,\ell}_{\exp})^*}.$$

By hypothesis we have $\|\nabla_Y\Phi\|_{(B^{r,\ell}_{\exp})^*}\in L^\infty(B_X\times B_Y,\mathbb{R})$ and, since $r\alpha^2<8\pi$ by our assumptions in Definition B.2.6, we also have that $\|:\exp(\alpha X)\colon\|_{B^{r,\ell}_{\exp}}\in L^r(\mu)$. Hence,

$$|\langle\mathscr{G},\nabla_Y\Phi\rangle|\in L^r(\mu).$$

We are left to show that equation (B.2.16) holds. This is equivalent to showing

$$\lim_{\varepsilon\to 0}\int(\mathscr{L}_\varepsilon\Phi-\mathscr{L}\Phi)\Psi\mathrm{d}\mu=\lim_{\varepsilon\to 0}\int\langle\mathscr{G}_\varepsilon-\mathscr{G},\nabla_Y\Phi\rangle\Psi\mathrm{d}\mu=0.$$

By the same reasoning as in the previous part of the proof and by Proposition B.6.1, we have that $\langle\mathscr{G}_\varepsilon-\mathscr{G},\nabla_Y\Phi\rangle\in L^r(\mu)$, uniformly with respect to $0<\varepsilon<1$. In order to show that the previous limit holds, it is sufficient to prove that

$$\langle\mathscr{G}_\varepsilon-\mathscr{G},\nabla_Y\Phi\rangle\to 0,$$

in probability as $\varepsilon\to 0$. On the other hand, we know that

$$\|\mathscr{G}_\varepsilon-\mathscr{G}\|_{B^{r,\ell'}_{\exp}}<+\infty,$$

uniformly with respect to ε, and by Proposition B.6.1 we have that $\mathscr{G}_\varepsilon\to\mathscr{G}$, weakly in $\mathscr{S}'(\mathbb{R}^2)$. Therefore, we have that $\mathscr{G}_\varepsilon\to\mathscr{G}$ weakly in $B^{r,\ell'}_{\exp}$, and in probability with respect to μ. By definition of weak convergence and since $\nabla_Y\Phi\in(B^{r,\ell'}_{\exp})^*$ by hypothesis (see Definition B.3.1), we have that $\langle\mathscr{G}_\varepsilon-\mathscr{G},\nabla_Y\Phi\rangle\to 0$, in probability. Finally, by property *iv.* in Definition B.3.1 and the fact that

$$\|\mathscr{G}_\varepsilon-\mathscr{G}\|_{B^{r,\ell'}_{\exp}}\in L^r(\mu)$$

uniformly in $\varepsilon>0$ by Proposition B.6.1, we have that $\langle\mathscr{G}_\varepsilon-\mathscr{G},\nabla_Y\Phi\rangle\in L^p(\mu)$ uniformly in $\varepsilon>0$, for some $p>1$, and thus the result follows from Lebesgue's dominated convergence theorem. $\square$

Remark B.2.9. By Theorem B.2.8 it is possible to take the limit $\varepsilon \to 0$ inside the integration with respect to μ in equations (B.2.4)–(B.2.3) and (B.2.5)–(B.2.6) so that the mentioned equations are equivalent to the one where the limit disappears and the operator $\mathscr{L}_\varepsilon$ is replaced by $\mathscr{L}$. Another consequence of the previous theorem is that, if we consider $\tilde{\mathscr{L}}_\varepsilon$ to be another approximation of $\mathscr{L}$ such that

$$\lim_{\varepsilon \to 0} \int (\tilde{\mathscr{L}}_\varepsilon \Phi)\Psi \mathrm{d}\mu = \int (\mathscr{L}\Phi)\Psi \mathrm{d}\mu, \quad \text{for any } \Phi, \Psi \in \mathrm{Cyl}_{B_X \times B_Y}, \tag{B.2.17}$$

then Problems A'' and B with the operator $\tilde{\mathscr{L}}_\varepsilon$ are equivalent to Problems A'' and B with the operator $\mathscr{L}_\varepsilon$. This means that the formulations given in Problems A'' and B do not depend on the precise form of the approximating operator $\mathscr{L}_\varepsilon$ but only on its limit $\mathscr{L}$.

We now want to prove a result that justifies our restriction in taking $B_Y = B_Y^{\leqslant}$ in the results above and in the rest of the paper. Indeed, if we focus on Problem B, the solutions are always supported on the space of negative functions on the Y component. In the light of point *iii.* in Theorem B.2.5, the next result implies that any solution μ to Problem B belongs to the spaces $\mathscr{M}_{B_X \times B_Y}$.

Theorem B.2.10. *Let μ be a solution to Problem B, then we have*

$$\operatorname{supp} P_*^Y(\mu) \subset \{Y \leqslant 0\}.$$

Proof. Suppose that $Y \in B_Y = B_{p,p,\ell}^{s-\delta}(\mathbb{R}^2)$ (see Definition B.2.6). Define, for $r, k > 0$,

$$\rho_r^k(x) = (1 + k|x|^2)^{-r/2}, \qquad x \in \mathbb{R}^2,$$

and

$$f(x) = x \vee 0, \qquad I(x) = \int_0^x f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

We consider, for $\eta > 0$, the convolution $g_\eta * Y$ which, by definition of B_Y, belongs to $B_{2,2,\ell'}^{s'}$ for any $\ell', s' \geqslant 1$ such that $p\ell' > 2$. Take

$$F(Y) = \arctan(\langle I(g_\eta * Y), \rho_r^k \rangle),$$

so that, by integration by parts,

$$\begin{aligned}
\mathscr{L}_\varepsilon F(Y) &= \frac{1}{1 + \langle I(g_\eta * Y), \rho_r^k \rangle^2} \langle f(g_\eta * Y), \rho_r^k g_\eta * (-(-\Delta + m^2)Y - \mathscr{G}_\varepsilon(X, Y)) \rangle \\
&= -\frac{1}{1 + \langle I(g_\eta * Y), \rho_r^k \rangle^2} \langle f(g_\eta * Y), \rho_r^k((-\Delta + m^2)(g_\eta * Y) + g_\eta * \mathscr{G}_\varepsilon(X, Y)) \rangle \\
&= -\frac{1}{1 + \langle I(g_\eta * Y), \rho_r^k \rangle^2} \big[\langle f(g_\eta * Y), \rho_r^k m^2 (g_\eta * Y) \rangle + \langle f(g_\eta * Y), \rho_r^k(-\Delta)(g_\eta * Y) \rangle \\
&\qquad + \langle f(g_\eta * Y), \rho_r^k (g_\eta * \mathscr{G}_\varepsilon(X, Y)) \rangle \big] \\
&= -\frac{1}{1 + \langle I(g_\eta * Y), \rho_r^k \rangle^2} \big[\langle f(g_\eta * Y), \rho_r^k m^2 (g_\eta * Y) \rangle + \langle f'(g_\eta * Y)(g_\eta * \nabla Y), \rho_r^k (g_\eta * \nabla Y) \rangle \\
&\qquad + \langle f(g_\eta * Y), \nabla \rho_r^k (g_\eta * \nabla Y) \rangle + \langle f(g_\eta * Y), \rho_r^k (g_\eta * \mathscr{G}_\varepsilon(X, Y)) \rangle \big].
\end{aligned}$$

Consider the term $\langle f(g_\eta * Y), \nabla \rho_r^k (g_\eta * \nabla Y) \rangle$, and multiply and divide by ρ_r^k to get

$$\langle f(g_\eta * Y), \nabla \rho_r^k (g_\eta * \nabla Y) \rangle = \langle \rho_r^k f(g_\eta * Y), \frac{\nabla \rho_r^k}{\rho_r^k} (g_\eta * \nabla Y) \rangle.$$

By Young inequality, we have

$$\begin{aligned}\mathscr{L}_\varepsilon F(Y) \leqslant & -\frac{1}{1+\langle I(g_\eta * Y), \rho_{\ell'}^k\rangle^2}\Big[\langle f(g_\eta * Y), \rho_r^k m^2(g_\eta * Y)\rangle + \langle f'(g_\eta * Y)(g_\eta * \nabla Y), \rho_r^k(g_\eta * \nabla Y)\rangle \\ & -\frac{1}{2}\langle \rho_r^k f^2(g_\eta * Y), \Big(\frac{\nabla \rho_r^k}{\rho_r^k}\Big)^2\rangle - \frac{1}{2}\langle \rho_r^k, |g_\eta * \nabla Y|^2 \mathbb{1}_{\{g_\eta * Y \geqslant 0\}}\rangle\Big],\end{aligned}$$

where we also used that the term $\langle f(g_\eta * Y), \rho_r^k(g_\eta * \mathscr{G}_\varepsilon(X,Y))\rangle$ is positive. We want to show that $\nabla\rho_r^k/\rho_r^k$ is bounded. Taking $y=\sqrt{k}x$, we have, for $k>0$ small enough,

$$\frac{\nabla \rho_r^k}{\rho_r^k} = \frac{2kx_1}{1+k|x|^2} = \frac{2\sqrt{k}\,y_1}{1+|y|^2} \leqslant 2\sqrt{k}\sup_{y\in\mathbb{R}^2}\frac{y_1}{1+|y|^2} \leqslant C_k,$$

which gives, if $C_k/2 - m^2 < -\zeta$,

$$\mathscr{L}_\varepsilon F(Y) < -\frac{1}{1+\langle I(g_\eta * Y), \rho_{\ell'}^k\rangle^2}\Big[\zeta\langle f^2(g_\eta * Y), \rho_r^k\rangle + \frac{1}{2}\langle \rho_r^k, |g_\eta * \nabla Y|^2 \mathbb{1}_{\{g_\eta * Y \geqslant 0\}}\rangle\Big],$$

since $f(g_\eta * Y) = g_\eta * Y$ on $\mathrm{supp}(f)$.

Since the right-hand side does not depend on ε, and the left-hand side converges to 0 when integrated with respect to μ, we have

$$\int \frac{\zeta}{1+\langle I(g_\eta * Y), \rho_{\ell'}^k\rangle^2}\Big[\langle f^2(g_\eta * Y), \rho_r^k\rangle + \frac{1}{2}\langle \rho_r^k, |g_\eta * \nabla Y|^2 \mathbb{1}_{\{g_\eta * Y \geqslant 0\}}\rangle\Big]\mathrm{d}\mu \leqslant 0,$$

and since the terms $2^{-1}\langle \rho_r^k, |g_\eta * \nabla Y|^2 \mathbb{1}_{\{g_\eta * Y \geqslant 0\}}\rangle$, ρ_r^k, $|g_\eta * \nabla Y|^2$, and $\zeta(1+\langle I(g_\eta * Y), \rho_{\ell'}^k\rangle^2)^{-1}$ are all positive, we have

$$\int \zeta\langle f^2(g_\eta * Y), \rho_r^k\rangle \,\mathrm{d}\mu \leqslant 0.$$

By Fatou's Lemma we can take the limit as $\eta \to 0$ and get that

$$\mathbb{1}_{\{Y \geqslant 0\}} = 0, \qquad \mu\text{-a.s.} \qquad \square$$

Remark B.2.11. A consequence of Theorem B.2.10 is that for the case of exponential interaction, a solution to Problem B with $B_Y = B_{p,p,\ell}^{s-\delta}(\mathbb{R}^2)$ is necessarily a solution to Problem A'' with $B_Y = B_Y^{\leqslant}$.

B.2.3 A description of the exponential measure

What has been described in the previous section has the main aim of giving a characterization to a unique measure which we call the exponential QFT. In this section, we want to connect our discussion of the IbP formula with the standard approach to construct EQFT via Gibbsian modifications of the GFF. Assume that μ_M^{free} is the measure of the free field with mass m on the torus $\mathbb{T}_M^2$ of size M. We define the interaction, for $\varepsilon > 0$,

$$V_{M,\varepsilon}^{\exp,\alpha}(\varphi) := \int_{\mathbb{T}_M^2} f_\varepsilon(x) e^{\alpha(g_\varepsilon * \varphi) - \frac{\alpha^2}{2}c_\varepsilon}\mathrm{d}x,$$

with f, g, c_ε and α are as in Section B.2.2. We also consider

$$Z_{M,\varepsilon} := \int e^{-V_{M,\varepsilon}^{\exp,\alpha}(\varphi)}\mu_M^{\text{free}}(\mathrm{d}\varphi).$$

Definition B.2.12. *We say that the measure* $\nu_m^{\exp,\alpha}$ *on* $\mathcal{S}'(\mathbb{R}^2)$ *is the measure related with the Euclidean QFT having action*

$$S(\varphi)=\frac{1}{2}\int_{\mathbb{R}^2}(|\nabla\varphi(x)|^2+m^2\varphi(x)^2)\,\mathrm{d}x+\int_{\mathbb{R}^2}:e^{\alpha\varphi(x)}:\mathrm{d}x,$$

if there are two sequences $\varepsilon_n\to 0$ *and* $M_{n'}\to+\infty$, *with* $\varepsilon_n>0$ *and* $N_n\in\mathbb{N}$ *such that*

$$\nu_m^{\exp,\alpha}(\mathrm{d}\varphi)=\lim_{n'\to+\infty}\lim_{n\to+\infty}Z_{M_{n'},\varepsilon_n}^{-1}e^{-V_{M_{n'},\varepsilon_n}^{\exp,\alpha}(\varphi)}\mu_{M_{n'}}^{\mathrm{free}}(\mathrm{d}\varphi), \tag{B.2.18}$$

where the limits are taken in weak sense in the space of probability measures.

Such a measure $\nu_m^{\exp,\alpha}$ was first built by Albeverio and Høegh-Krohn in [10] (see also [79]) using techniques from constructive QFT. More recently, this model was studied in the context of stochastic quantization on the torus or on a compact manifold (see e.g. [13, 27, 81, 110, 111, 149, 150]), and on $\mathbb{R}^2$ (see [4]). See also [25] for the related $\cosh(\Phi)_2$-model.

A consequence of the results we presented in the previous sections of the paper (see Theorem B.1.2, Theorem B.1.3) is the following differential characterization of the measure related to the exponential interaction.

Theorem B.2.13. *The following statements hold:*

i. *For any* $\alpha^2<8\pi$, *there exists a measure related to the exponential interaction defined by the limit (B.2.18).*
ii. *Let* $B_X=C_\ell^{-\delta}(\mathbb{R}^2)$ *and* $B_Y=B_{\tilde{Y}}^{\leqslant}$. *Then, for any* $\alpha^2<4\pi\tilde{\gamma}_{\max}$, *the measure* $\nu_m^{\exp,\alpha}$ *is the unique measure in the space* $\mathcal{M}_{B_X+B_Y}$ *such that, for any* $F\in\mathrm{Cyl}_E^b$ *and* $h\in\mathcal{S}(\mathbb{R}^2)$, *we have*

$$\int\langle\nabla_\varphi F,h\rangle\,\nu(\mathrm{d}\varphi)=\lim_{\varepsilon\to 0}\int F(\varphi)\,\langle(-\Delta+m^2)\varphi+\alpha f_\varepsilon e^{\alpha g_\varepsilon*\varphi-\frac{\alpha^2}{2}c_\varepsilon},h\rangle\,\nu(\mathrm{d}\varphi).$$

B.3 Uniqueness of solution

B.3.1 Proof of uniqueness of solutions to Problem A''

In this section, we discuss the uniqueness of solutions to Problems A, A'', and B in the case where B_X, B_Y, $\mathcal{M}=\mathcal{M}_{B_Y}$, and $E=B_X+B_Y$ are as in Section B.2.2 and B_ε is the drift of the exponential interaction (see equation (B.1.3)). The method that we adopt to prove uniqueness is the study of the resolvent equation for the operator $\mathcal{L}_\varepsilon$ defined in equation (B.2.14), namely

$$(\lambda-\mathcal{L}_\varepsilon)G_\varepsilon^\lambda=F,\qquad \lambda\in\mathbb{R}_+,. \tag{B.3.1}$$

We are interested in classical solutions to the previous equation, namely solutions G_ε^λ that are at least in $C^2(B_X\times B_Y)$ and to which the operator $\mathcal{L}_\varepsilon$ can be applied.

Let us first introduce here a class of functions where the solutions of the resolvent equation belongs.

Definition B.3.1. *Recall that* $\gamma=\alpha^2/(4\pi)$. *Let* p,s,ℓ,r *be the same parameters as in Definition B.2.6 and let us introduce the space*

$$\hat{B}_X=B_{\tilde{p},\tilde{p},\ell}^{-\delta}(\mathbb{R}^2),$$

for some $1<\tilde{p}<+\infty$ large enough. Note that $B_X\subset\hat{B}_X$. Denote by $\mathscr{F}$ the class of bounded, measurable functions $\Phi:\hat{B}_X\times B_Y\to\mathbb{R}$ such that

i. $\nabla_Y\Phi\in C^0(\hat{B}_X\times B_Y,B_{l,l,-\ell}^{(2-s)\wedge(\gamma(r-1))+\delta}(\mathbb{R}^2))$, *for any $\delta>0$, $1<l<+\infty$.*

ii. $\nabla_X\Phi\in C^0(\hat{B}_X\times B_Y,B_{1,1,-\ell}^{2-\delta}(\mathbb{R}^2))$, *so that $\langle\nabla_X\Phi,(-\Delta+m^2)X\rangle$ is well-defined.*

iii. *The operator $\nabla_X^2\Phi\in C^0(\hat{B}_X\times B_Y,L(\hat{B}_X,B_X))$ can be extended in a unique continuous way to an operator $\nabla_X^2\Phi\in C^0(\hat{B}_X\times B_Y,L(H_\ell^{-\kappa}(\mathbb{R}^2),H_{-\ell}^{\kappa}(\mathbb{R}^2)))$, where $\kappa>1$ and $\ell>1$.*

iv. *There exists some $f_\Phi(X)\in L^p(\mu^{\text{free}})$, $p\in[1,+\infty)$, such that*

$$\begin{aligned}\|\nabla_Y\Phi(X,Y)\|_{B_{l,l,-\ell}^{(2-s)\wedge(\gamma(r-1))+\delta}(\mathbb{R}^2)} &\leqslant f_\Phi(X),\\ \|\nabla_X\Phi(X,Y)\|_{B_{1,1,-\ell}^{2-\delta}(\mathbb{R}^2)} &\leqslant f_\Phi(X),\\ \|\nabla_X^2\Phi(X,Y)\|_{L(H_\ell^{-\kappa}(\mathbb{R}^2),H_{-\ell}^{\kappa}(\mathbb{R}^2))} &\leqslant f_\Phi(X).\end{aligned}$$

We define the space $\mathfrak{F}$ as the space of functions $F:\tilde{B}_X+B_Y\to\mathbb{R}$ such that

$$F\circ P^{X+Y}\in\mathscr{F}.$$

Remark B.3.2. Under condition *iii.* of the previous definition, since the immersion $H_\ell^{-\kappa}(\mathbb{R}^2)\hookrightarrow L^2(\mathbb{R}^2)$ is an Hilbert-Schmidt operator, then $\nabla_X^2\Phi,\rho_{-\ell}\nabla_X^2\Phi\in L_{\text{loc}}^\infty(\hat{B}_X\times B_Y,\text{TC}(L^2))$, where $\text{TC}(L^2)$ is the space of trace-class operators on $L^2(\mathbb{R}^2)$.

Remark B.3.3. The classes $\mathscr{F}$ and $\mathfrak{F}$ are chosen in such a way that they satisfy the following two important properties: (i) $\mathscr{F}$ and $\mathfrak{F}$ contain $\text{Cyl}_{B_X\times B_Y}$ and Cyl_E, respectively, and (ii) if $B_Y=B_Y^{\leqslant}$ then, for any Radon measure $\mu\in\mathscr{M}_{B_X\times B_Y}$, we have $\sup_{\varepsilon>0}\int|\mathscr{L}_\varepsilon\Phi|^\sigma\text{d}\mu<+\infty$, for any $\Phi\in\mathscr{F}$ and $\sigma\geqslant1$.

The classes $\mathscr{F}$ and $\mathfrak{F}$ satisfy the following important lemma.

Lemma B.3.4. *Suppose that $\mu\in\mathscr{M}_{B_X\times B_Y}$ and satisfies equation (B.2.5) for any $\Phi\in\text{Cyl}_{B_X\times B_Y}$, then it satisfies the same equation for every $\Phi\in\mathscr{F}$. Suppose that $\nu\in\mathscr{M}_{B_Y}$ and satisfies equation (B.2.1) for any $\Phi\in\text{Cyl}_E$, then it satisfies the same equation for every $\Phi\in\mathfrak{F}$.*

Proof. See Appendix B.8. □

Proposition B.3.5. *Let $F\in\text{Cyl}_{B_X\times B_Y}$, then there exists a classical solution $G_\varepsilon^\lambda\in C^2(B_X\times B_Y)$ to the resolvent equation (B.3.1) with the following properties:*

i. *$G_\varepsilon^\lambda\in\mathscr{F}$ and moreover, if $F=\bar{F}\circ P^{X+Y}$ for some $\bar{F}\in\text{Cyl}_E$, then there exists $\bar{G}_\varepsilon^\lambda\in\mathfrak{F}$ such that we have $G_\varepsilon^\lambda=\bar{G}_\varepsilon^\lambda\circ P^{X+Y}$.*

ii. *Suppose that F has compact support in Fourier variables, then there exists $\varepsilon_0>0$ such that, for every $\mu_1,\mu_2\in\mathscr{M}_{B_X\times B_Y}$ and for every $\varsigma\in(0,1)$, there are two constants $C_{\mu_1,\mu_2,\varsigma}>0$ and $K>0$ such that*

$$\lambda\left|\int G_\varepsilon^\lambda\,(\text{d}\mu_1-\text{d}\mu_2)\right|\lesssim\varsigma+\frac{\lambda}{\lambda+K}C_{\mu_1,\mu_2,\varsigma},$$

where the constant included in the symbol $\lesssim$ does not depend on λ, μ_1, μ_2, ε or ς.

iii. If $\alpha^2 < 4\pi\tilde{\gamma}_{\max}$, then there exists $q > 1$ such that, for every measurable $\mathscr{K} : B_X \times B_Y \to B_{\exp}^{q,\ell/2}$ and every $\mu \in \mathscr{M}_{B_X \times B_Y}$, we have

$$\int |\langle \nabla_{Y_0} G_\varepsilon^\lambda, \mathscr{K} \rangle| \,\mathrm{d}\mu \lesssim_\lambda \left(\int \|\mathscr{K}\|_{B_{\exp}^{q,\ell/2}}^q \,\mathrm{d}\mu \right)^{1/q},$$

uniformly in $\varepsilon > 0$.

Let $\tilde{\gamma}_{\max}$ be the parameter defined in Remark B.1.1. In the light of Proposition B.3.5, the proof of which we postpone to Section B.3.2, we can then proceed with the proof of uniqueness of solutions to Problem A''.

Theorem B.3.6. *Let $\alpha^2 < 2\pi\tilde{\gamma}_{\max}$ and take $B_X = C_\ell^{-\delta}(\mathbb{R}^2)$ and $B_Y = B_Y^{\leqslant}$, where the parameters are taken as in Definition B.2.6 and the space $B_Y^{\leqslant}$ is defined as in equation (B.2.11). Then the solution to FPK equation in the sense of Problem A'' is unique.*

Proof. Let μ_1 and μ_2 be two solutions of Problem A''. We want to show that, for any $F \in \mathrm{Cyl}_{B_X \times B_Y}$ with compact support in Fourier variables, $\int F \mathrm{d}\mu_1 = \int F \mathrm{d}\mu_2$. This implies $\mu_1 = \mu_2$ since $F \in \mathrm{Cyl}_{B_X \times B_Y}$ with compact support in Fourier variables separates points of the space of Radon measures on $B_X \times B_Y$. Consider the solution G_ε^λ of the resolvent equation (B.3.1) given by Proposition B.3.5 and recall that, by point *ii.* in Proposition B.3.5. Since μ_1 and μ_2 are solutions to the FPK equations we have in particular that equation (B.2.5) holds. Therefore, integrating with respect to μ_1 (respectively, with respect to μ_2) the resolvent equation equation (B.3.1) and subtracting the integral $\int \mathscr{L} G_\varepsilon^\lambda \,\mathrm{d}\mu_1 = 0$ (respectively, $\int \mathscr{L} G_\varepsilon^\lambda \,\mathrm{d}\mu_2 = 0$) which holds by Lemma B.3.4, we get the relation

$$\int F \mathrm{d}\mu_j = \lambda \int G_\varepsilon^\lambda \mathrm{d}\mu_j - \int \nabla_{Y_0} G_\varepsilon^\lambda (\mathscr{G} - \mathscr{G}_\varepsilon) \,\mathrm{d}\mu_j, \qquad j = 1, 2,$$

where we omitted the dependences on (X_0, Y_0) for the sake of brevity. Taking the difference of such a relation for $j = 1, 2$ yields

$$\int F \mathrm{d}\mu_1 - \int F \mathrm{d}\mu_2 = \lambda \int G_\varepsilon^\lambda \,\mathrm{d}\mu_1 - \lambda \int G_\varepsilon^\lambda \,\mathrm{d}\mu_2 - \int \nabla_{Y_0} G_\varepsilon^\lambda (\mathscr{G} - \mathscr{G}_\varepsilon) \,\mathrm{d}\mu_1 + \int \nabla_{Y_0} G_\varepsilon^\lambda (\mathscr{G} - \mathscr{G}_\varepsilon) \,\mathrm{d}\mu_2.$$

Now, point *ii.* in Proposition B.3.5 gives us, for any $\varsigma > 0$,

$$\left| \int F \mathrm{d}\mu_1 - \int F \mathrm{d}\mu_2 \right| \lesssim \varsigma + (1 + C_\varsigma) \frac{\lambda}{\lambda + K} + \left| \int \nabla_{Y_0} G_\varepsilon^\lambda (\mathscr{G} - \mathscr{G}_\varepsilon) \,\mathrm{d}\mu_1 \right| + \left| \int \nabla_{Y_0} G_\varepsilon^\lambda (\mathscr{G} - \mathscr{G}_\varepsilon) \,\mathrm{d}\mu_2 \right|$$

On the other hand, point *iii.* in Proposition B.3.5 implies that

$$\left| \int \nabla_{Y_0} G_\varepsilon^\lambda (\mathscr{G} - \mathscr{G}_\varepsilon) \,\mathrm{d}\mu_j \right| \lesssim_\lambda \left(\int \|\mathscr{G} - \mathscr{G}_\varepsilon\|_{B_{\exp}^{q,\ell/2}}^q \,\mathrm{d}\mu_j \right)^{1/q}.$$

By the proof of Theorem B.2.8, Proposition B.6.1, and using the fact that $P_*^X \mu_j = \mu^{\mathrm{free}}$, we get that

$$\left(\int \|\mathscr{G} - \mathscr{G}_\varepsilon\|_{B_{\exp}^{q,\ell/2}}^q \,\mathrm{d}\mu_j \right)^{1/q} \to 0 \qquad \text{as } \varepsilon \to 0.$$

Thus, if we take the limit $\varepsilon \to 0$ and then $\lambda \to 0$, we get $\left| \int F \mathrm{d}\mu_1 - \int F \mathrm{d}\mu_2 \right| \lesssim \varsigma$, and since $\varsigma > 0$ is arbitrary, we get $\mu_1 = \mu_2$. □

In the case of Problem B a better result, in the sense that $\gamma_{\max} > \tilde{\gamma}_{\max}$ (cf. Remark B.1.1), can be achieved exploiting the properties of the resolvent operator proved in Section B.3.3.

Theorem B.3.7. *Let $\alpha^2 < 2\pi\gamma_{\max}$, $B_X = C_\ell^{-\delta}(\mathbb{R}^2)$ and $B_Y = B_{p,p,\ell}^{s-\delta}(\mathbb{R}^2)$, where the parameters are taken as in Definition B.2.6. Then the solution to FPK equation in the sense of Problem B is unique.*

Proof. The proof is the same as in the case of Problem A", but substituting point *iv.* of Proposition B.3.5, with Proposition B.3.11 below. □

B.3.2 Analysis of the resolvent equation

To solve the resolvent equation as needed in Proposition B.3.5, we use a probabilistic representation and look for solutions in the form

$$G_\varepsilon^\lambda(X_0, Y_0) = \mathbb{E}_\xi\left[\int_0^{+\infty} e^{-\lambda t} F(X_t^\varepsilon, Y_t^\varepsilon)\,\mathrm{d}t\right], \tag{B.3.2}$$

where X_t^ε and Y_t^ε are solutions to the stochastic differential system

$$\partial_t X_t^\varepsilon = -(-\Delta + m^2) X_t^\varepsilon + \xi_t, \tag{B.3.3}$$
$$\partial_t Y_t^\varepsilon = -(-\Delta + m^2) Y_t^\varepsilon - \mathcal{G}_\varepsilon(X_t^\varepsilon, Y_t^\varepsilon), \tag{B.3.4}$$

where $\mathcal{G}_\varepsilon$ is defined in equation (B.2.12), ξ is a Gaussian space-time white noise, and with initial conditions $(X_0, Y_0) \in B_X \times B_Y$. This is a triangular system, where the first equation does not depend on the second variable Y. The first equation is a linear SDE with additive noise with stationary solution given by the Ornstein-Uhlenbeck process, whose invariant measure is the Gaussian free field with mass $m > 0$. To prove that G_ε^λ is a classical solution we need some properties on the flow induced by the SPDEs (B.3.3)–(B.3.4). We will only state such properties in the present section – postponing their proof to Appendix B.9 – and then proceed by showing that Proposition B.3.5 holds.

First, let us write down the equations for the derivatives of the flow. Let us denote by X and Y the solutions to equations (B.3.3)–(B.3.4), dropping the dependence on the parameter $\varepsilon > 0$ for simplicity of notation. The derivatives of X solve

$$(\partial_t - \Delta + m^2)\nabla_{X_0} X_t = 0, \qquad \nabla_{X_0} X(0) = \mathrm{id}, \tag{B.3.5}$$
$$(\partial_t - \Delta + m^2)\nabla_{X_0}^2 X_t = 0, \qquad \nabla_{X_0}^2 X(0) = 0, \tag{B.3.6}$$
$$(\partial_t - \Delta + m^2)\nabla_{Y_0} X_t = 0, \qquad \nabla_{Y_0} X(0) = 0. \tag{B.3.7}$$

Whenever $Y_0 \in B_Y^{\leqslant}$,

$$(\partial_t - \Delta + m^2)\nabla_{Y_0} Y_t(Y_0)[h] = -\mathrm{D}_Y \mathcal{G}_\varepsilon(X_t, Y_t)[\nabla_{Y_0} Y_t(Y_0)[h]], \qquad \nabla_{Y_0} Y_0 = h, \tag{B.3.8}$$

for $h \in B_Y \cup B_{\exp}^{r,\ell}$, and

$$\begin{aligned}(\partial_t - \Delta + m^2)\nabla_{X_0} Y_t(Y_0)[h] &= -\mathrm{D}_Y \mathcal{G}_\varepsilon(X_t, Y_t)[\nabla_{X_0} Y_t(Y_0)[h]] - \mathrm{D}_X \mathcal{G}_\varepsilon(X_t, Y_t)[\nabla_{X_0} X_t[h]], \\ \nabla_{X_0} Y_0 &= 0,\end{aligned} \tag{B.3.9}$$

for $h \in B_X$, and

$$\begin{aligned}(\partial_t - \Delta + m^2)\nabla_{X_0}^2 Y_t(Y_0)[h, h'] &= -\mathrm{D}_Y \mathcal{G}_\varepsilon(X_t, Y_t)[\nabla_{X_0}^2 Y_t(Y_0)[h, h']] \\ &\quad -\mathrm{D}_Y^2 \mathcal{G}_\varepsilon(X_t, Y_t)[\nabla_{X_0} Y_t(Y_0)[h], \nabla_{X_0} Y_t(Y_0)[h']] \\ &\quad -2\mathrm{D}_{X,Y}^2 \mathcal{G}_\varepsilon(X_t, Y_t)[\nabla_{X_0} X_t[h], \nabla_{X_0} Y_t(Y_0)[h']] \\ &\quad -\mathrm{D}_X^2 \mathcal{G}_\varepsilon(X_t, Y_t)[\nabla_{X_0} X_t(Y_0)[h], \nabla_{X_0} X_t(Y_0)[h']], \\ \nabla_{X_0}^2 Y_0 &= 0,\end{aligned} \tag{B.3.10}$$

for $h, h' \in L^2(\mathbb{R}^2)$. The derivatives $\mathrm{D}\mathscr{G}_{\varepsilon,\bar{\varepsilon}}$ have the following expressions

$$\mathrm{D}_X\mathscr{G}_\varepsilon(X_t, Y_t)[\varphi] = \mathrm{D}_Y\mathscr{G}_\varepsilon(X_t, Y_t)[\varphi] = \alpha^2 f_\varepsilon(:e^{\alpha(X_t * g_\varepsilon)}: e^{\alpha(Y_t * g_\varepsilon)})(g_\varepsilon * \varphi),$$

and

$$\begin{aligned}\mathrm{D}^2_X\mathscr{G}_\varepsilon(X_t, Y_t)[\varphi,\psi] = \mathrm{D}^2_Y\mathscr{G}_\varepsilon(X_t, Y_t)[\varphi,\psi] &= \mathrm{D}^2_{X,Y}\mathscr{G}_\varepsilon(X_t, Y_t)[\varphi,\psi]\\ &= \alpha^3 f_\varepsilon(:e^{\alpha(X_t * g_\varepsilon)}: e^{\alpha(Y_t * g_\varepsilon)})(g_\varepsilon * \varphi)(g_\varepsilon * \psi).\end{aligned}$$

Hereafter, we will use the notation $\gamma = \alpha^2/(4\pi)$, and

$$\tilde{B}_Y = B_Y \cup B^{-(2-s)\wedge(\gamma(r-1))-\delta}_{p\wedge r, p\wedge r, \ell}(\mathbb{R}^2), \qquad \tilde{B}_X = B^{-2+\delta}_{\infty,\infty,\ell}(\mathbb{R}^2), \tag{B.3.11}$$

where all the appearing parameters are defined as in Definition B.2.6.

Proposition B.3.8. *For any $\varepsilon > 0$, if $(X_0, Y_0) \in \hat{B}_X \times \{B_Y \cup B^{r,\ell}_{\exp}\}$, then there exists a unique solution (X, Y) to equations (B.3.3)–(B.3.4) such that*

$$(X_t, Y_t) \in \hat{B}_X \times \{B_Y \cup B^{r,\ell}_{\exp}\}, \qquad t \in \mathbb{R}_+.$$

Let X and Y be solutions to equations (B.3.3)–(B.3.4).

i. For every $\varepsilon > 0$, we have that that the derivatives $\nabla_{X_0}X_t$, $\nabla^2_{X_0}X_t$, $\nabla_{Y_0}X_t$, $\nabla_{X_0}Y$, $\nabla^2_{X_0}Y$, and $\nabla_{Y_0}Y$ of X and Y exist and satisfy equations (B.3.5), (B.3.6), (B.3.7), (B.3.8), (B.3.9), and (B.3.10), respectively. Furthermore, they are all continuous functions with respect to X_0 and Y_0.

ii. For every $\delta \in (0,1)$, $\theta \in (0, 1-\delta)$, $\ell, \ell' \geqslant 1$, we have the estimates

$$\begin{aligned}&\|\nabla_{Y_0}Y_t(Y_0)[h]\|_{C^\theta_{\ell'}(\mathbb{R}_+, C^{2-2\theta-2\delta}_{-\ell}(\mathbb{R}^2)) \oplus L^\infty(\mathbb{R}_+, \tilde{B}_Y)}\\ &\lesssim_{g_\varepsilon} \mathfrak{P}_2\big(\big\|f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)}: e^{\alpha P_t(g_\varepsilon * h)}\big\|_{L^\infty_{\ell'}(\mathbb{R}_+, L^\infty_{-\ell}(\mathbb{R}^2))}, \|h\|_{\tilde{B}_Y}\big), \quad h \in \tilde{B}_Y,\end{aligned} \tag{B.3.12}$$

and

$$\begin{aligned}&\|\nabla_{X_0}Y_t(Y_0)[h]\|_{C^\theta_{\ell'}(\mathbb{R}_+, C^{2-2\theta-2\delta}_{-\ell}(\mathbb{R}^2))}\\ &\lesssim_{g_\varepsilon} \tilde{\mathfrak{P}}_2(\|f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)}: e^{\alpha P_t(g_\varepsilon * h)}\|_{L^\infty_{\ell'}(\mathbb{R}_+, L^\infty_{-\ell}(\mathbb{R}^2))}), \quad h \in \tilde{B}_X,\end{aligned} \tag{B.3.13}$$

where $\mathfrak{P}_2$ and $\tilde{\mathfrak{P}}_2$ are two second degree polynomials.

iii. For every $\ell, \kappa \geqslant 0$, we have that there exist $\beta, \delta > 0$ such that

$$\|\nabla^2_{X_0}Y_t(Y_0)\|_{L(H^{-\kappa}_{\ell}, H^{\kappa}_{-\ell})} \lesssim \left(\int_{\mathbb{R}^2} \alpha^2 f_\varepsilon(z') \; :e^{\alpha(g_\varepsilon * X_t)(z')}: \; e^{\alpha(g_\varepsilon * Y_t)(z')} \; e^{(\delta - m^2)t}(1 + |z'|^\beta)\,\mathrm{d}z'\right)^2. \tag{B.3.14}$$

The remainder of the present subsection is devoted to the proof of Proposition B.3.5.

Proof of Proposition B.3.5. Let us start by proving that G^λ_ε is a classical solution of the resolvent equation. We exploit Itô's formula appearing in Theorem 4.32 in [59]. Notice that it can be applied to G^λ_ε since, by point *i.* in Proposition B.3.8, G^λ_ε is a C^2-function with trace-class second derivative (continuity of the second derivative can be deduced with similar techniques as the ones adopted in Proposition B.9.7 and point *i.* in Proposition B.3.5).

We use the notation $X_t^{X_0}$ to denote a process X at time t, with starting point X_0 at time $t=0$. Recalling the definition of G_ε^λ in equation (B.3.2), we have

$$G_\varepsilon^\lambda(X_t^{X_0}, Y_t^{Y_0}) = \int_0^{+\infty} e^{-\lambda s}\, \mathbb{E}_\xi[F(X_s^{X_t}, Y_s^{Y_t}) | \mathscr{H}_t]\, \mathrm{d}s,$$

where $(\mathscr{H}_t)_{t>0}$ is the filtration generated by the white noise at time t and the initial conditions X_0 and Y_0. By Markovianity of the process, we get

$$G_\varepsilon^\lambda(X_t^{X_0}, Y_t^{Y_0}) = \int_0^{+\infty} e^{-\lambda s}\, \mathbb{E}_\xi[F(X_{t+s}^{X_0}, Y_{t+s}^{Y_0})\, | \mathscr{H}_t]\, \mathrm{d}s.$$

Since from now on X and Y always start at X_0 and Y_0, respectively, when $t=0$, we drop the dependence on the initial conditions. We have, by Itô's formula,

$$\begin{aligned} & e^{-\lambda t} G_\varepsilon^\lambda(X_t, Y_t) - G_\varepsilon^\lambda(X_0, Y_0) \\ = & \int_0^t e^{-\lambda s} \mathscr{L}_\varepsilon(G_\varepsilon^\lambda(X_s, Y_s))\, \mathrm{d}s + \int_0^t e^{-\lambda s} \nabla_{X_0} G_\varepsilon^\lambda(X_s, Y_s) \cdot \mathrm{d}X_s - \lambda \int_0^t e^{-\lambda s} G_\varepsilon^\lambda(X_s, Y_s)\, \mathrm{d}s. \end{aligned}$$

On the other side of the equation, we have

$$\int_0^{+\infty} e^{-\lambda(t+s)} \mathbb{E}_\xi[F(X_{t+s}, Y_{t+s}) | \mathscr{H}_t]\, \mathrm{d}s - \int_0^{+\infty} e^{-\lambda s} \mathbb{E}_\xi[F(X_s, Y_s) | \mathscr{H}_t]\, \mathrm{d}s.$$

Notice that

$$\mathbb{E} \int_0^{+\infty} e^{-\lambda(t+s)} \mathbb{E}[F(X_{t+s}, Y_{t+s}) | \mathscr{H}_t]\, \mathrm{d}s = \mathbb{E} \int_t^{+\infty} e^{-\lambda s} \mathbb{E}[F(X_s, Y_s) | \mathscr{H}_t]\, \mathrm{d}s,$$

and therefore

$$\mathbb{E} \int_t^{+\infty} e^{-\lambda s} \mathbb{E}[F(X_s, Y_s) | \mathscr{H}_t]\, \mathrm{d}s - \mathbb{E} \int_0^{+\infty} e^{-\lambda s} \mathbb{E}[F(X_s, Y_s) | \mathscr{H}_t]\, \mathrm{d}s = -\mathbb{E} \int_0^t e^{-\lambda s} \mathbb{E}[F(X_s, Y_s) | \mathscr{H}_t]\, \mathrm{d}s.$$

Dividing by t and letting $t \to 0$, we get

$$-\frac{1}{t} \mathbb{E} \int_0^t e^{-\lambda s} \mathbb{E}[F(X_s, Y_s) | \mathscr{H}_t]\, \mathrm{d}s \to -F(X_0, Y_0).$$

Taking expectation also on the first side, we have the desired relation.

Proof of point *i*. When $F \in \mathrm{Cyl}^b_{B_X \times B_Y}$, the derivative of $G_\varepsilon^\lambda(X_0, Y_0)$ with respect to Y_0 is given by

$$\nabla_{Y_0} G_\varepsilon^\lambda(X_0, Y_0)(h) = \mathbb{E}_\xi \left[\int_0^{+\infty} e^{-\lambda t} \Big(\sum_{k=1}^N \partial_k \tilde{F}\, \langle \nabla_{Y_0} Y_t(h), v_k \rangle \Big)\, \mathrm{d}t \right].$$

Therefore, we have

$$|\nabla_{Y_0} G_\varepsilon^\lambda(X_0, Y_0)(h)| \lesssim \sup_{k=1,\ldots,N} \mathbb{E}_\xi \left[\int_0^{+\infty} e^{-\lambda t}\, \|\partial_k \tilde{F}\|_{L^\infty} |\langle \nabla_{Y_0} Y_t(h), v_k \rangle|\, \mathrm{d}t \right].$$

Applying the estimate (B.3.12), we get

$$\begin{aligned} |\langle \nabla_{Y_0} Y_t(h), v_k \rangle| & \lesssim \rho_{\ell'}(t)\, \|\nabla_{Y_0} Y_t(Y_0)(h)\|_{C^\theta_{\ell'}(\mathbb{R}_+, C^{2-2\theta-2\delta}_{-\ell}(\mathbb{R}^2)) \oplus L^\infty(\mathbb{R}_+, \tilde{B}_Y)} \|v_k\|_{\tilde{B}_Y^*} \\ & \lesssim \rho_{\ell'}(t)\, P_2\big(\|f_\varepsilon : e^{\alpha(g_\varepsilon * X_t)} \colon e^{\alpha P_t(g_\varepsilon * h)}\|_{L^\infty_{\ell'}(\mathbb{R}_+, L^\infty_{-\ell}(\mathbb{R}^2))}, \|h\|_{\tilde{B}_Y} \big)\, \|v_k\|_{\tilde{B}_Y^*}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
|\nabla_{Y_0} G_\varepsilon^\lambda(X_0,Y_0)(h)| \lesssim\ & \sup_k (\|\partial_k \tilde{F}\|_{L^\infty} \|v_k\|_{\tilde{B}_Y^*}) \int_0^{+\infty} e^{-\lambda t} \rho_{\ell'}(t) \\
& \times \mathbb{E}_\xi \Big[P_2\big(\|f_\varepsilon : e^{\alpha(g_\varepsilon * X_t)} \colon e^{\alpha P_t(g_\varepsilon * h)}\|_{L^\infty_{\ell'}(\mathbb{R}_+, L^\infty_{-\ell}(\mathbb{R}^2))}, \|h\|_{\tilde{B}_Y}\big)\Big] \,\mathrm{d}t.
\end{aligned}
$$

We have that

$$
\begin{aligned}
& \mathbb{E}_\xi \Big[P_2\big(\|f_\varepsilon : e^{\alpha(g_\varepsilon * X_t)} \colon e^{\alpha P_t(g_\varepsilon * h)}\|_{L^\infty_{\ell}(\mathbb{R}_+, L^\infty_{-\ell}(\mathbb{R}^2))}, \|h\|_{\tilde{B}_Y}\big)\Big] \\
\lesssim\ & R_{2,\varepsilon}\big(\|f_\varepsilon e^{\alpha|g_\varepsilon * X_0|}\|_{L^\infty}, e^{\alpha^2 \sup_{t\geqslant 0} \mathbb{E}[(g_\varepsilon * X_t)^2]}, e^{\alpha \|h\|_{\tilde{B}_Y}}\big),
\end{aligned}
$$

where $R_{2,\varepsilon}$ is a suitable second degree polynomial. Taking the supremum over $h \in \tilde{B}_Y$ with $|h| \leqslant 1$, recalling that $\tilde{B}_Y^* \subset B^{(2-s)\wedge(\gamma(r-1))+\delta}_{l,l,-\ell}(\mathbb{R}^2)$, for some $l \in (1,+\infty)$, and taking $f_{G_\varepsilon^\lambda}(X_0)$ as in Definition B.3.1 to be proportional to $\|f_\varepsilon \exp(\alpha|g_\varepsilon * X_0|)\|^2_{L^\infty} + 1$, we get that G_ε^λ satisfies the first condition of Definition B.3.1. For the remaining conditions of Definition B.3.1, similar arguments with the application of inequalities (B.3.13) and (B.3.14) allow us to conclude that $G_\varepsilon^\lambda \in \mathscr{F}$.

Since in our approximation we have that $\mathscr{G}_\varepsilon(X,Y) = \alpha f_\varepsilon e^{\alpha(g_\varepsilon * (X+Y)) - \frac{\alpha^2}{2} c_\varepsilon}$, then the sum $\varphi_t^\varepsilon = X_t^\varepsilon + Y_t^\varepsilon$ solves the SPDE

$$
(\partial_t - \Delta + m^2)\varphi_t^\varepsilon = -\alpha f_\varepsilon e^{\alpha g_\varepsilon * \varphi_t^\varepsilon - \frac{\alpha^2}{2} c_\varepsilon} + \xi_t.
$$

Therefore, if $F \in \mathrm{Cyl}^b_{B_X \times B_Y}$ is of the form $F = \bar{F} \circ P^{X+Y}$ for some $\bar{F} \in \mathrm{Cyl}^b_E$, we have

$$
G_\varepsilon^\lambda(X_0,Y_0) = \bar{G}_\varepsilon^\lambda(X_0+Y_0) = \mathbb{E}_\xi \left[\int_0^{+\infty} e^{-\lambda t} \bar{F}(\varphi_t^\varepsilon)\,\mathrm{d}t \,\middle|\, \varphi_0 = X_0 + Y_0\right].
$$

Proof of point *ii.* In the following, for $j=1,2$, we denote by $(X_t^{\varepsilon,j}, Y_t^{\varepsilon,j})$ the solution to the system of equations (B.3.3)–(B.3.4) with initial conditions $(X_0,Y_0) \sim \mu_j$. Sometimes we also write (X_0^j, Y_0^j) to indicate that $(X_0,Y_0) \sim \mu_j$, for $j=1,2$. By definition of G_ε^λ (i.e. by equation (B.3.2)), we have

$$
\begin{aligned}
& \lambda \int G_\varepsilon^\lambda(X_0,Y_0)\, \mu_1(\mathrm{d}X_0, \mathrm{d}Y_0) - \lambda \int G_\varepsilon^\lambda(X_0,Y_0)\, \mu_2(\mathrm{d}X_0,\mathrm{d}Y_0) \\
=\ & \lambda \int \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_\xi[F(X_t^{\varepsilon,1}, Y_t^{\varepsilon,1})]\,\mathrm{d}t\,\mathrm{d}\mu_1 - \lambda \int \int_0^{+\infty} e^{-\lambda t} \mathbb{E}_\xi[F(X_t^{\varepsilon,2}, Y_t^{\varepsilon,2})]\,\mathrm{d}t\,\mathrm{d}\mu_2 \\
=\ & \lambda \int_0^{+\infty} e^{-\lambda t} \mathbb{E}[F(X_t^{\varepsilon,1}, Y_t^{\varepsilon,1}) - F(X_t^{\varepsilon,2}, Y_t^{\varepsilon,2})]\,\mathrm{d}t,
\end{aligned}
$$

Notice that, for any $\varsigma > 0$, there exist two compact sets K_1, K_2, such that $\mu_1(K_1), \mu_2(K_2) > 1 - \varsigma$. Therefore, the following inequality holds

$$
\begin{aligned}
& \left|\lambda \int_0^{+\infty} e^{-\lambda t} \mathbb{E}[F(X_t^{\varepsilon,1}, Y_t^{\varepsilon,1}) - F(X_t^{\varepsilon,2}, Y_t^{\varepsilon,2})]\,\mathrm{d}t\right| \\
\leqslant\ & \left|\lambda \int_0^{+\infty} e^{-\lambda t} \mathbb{E}\big[|F(X_t^{\varepsilon,1}, Y_t^{\varepsilon,1}) - F(X_t^{\varepsilon,2}, Y_t^{\varepsilon,2})|\, \mathbb{I}_{(X_0^1,Y_0^1)\in K_1, (X_0^2,Y_0^2)\in K_2}\big]\,\mathrm{d}t\right| + 2\varsigma \|F\|_{L^\infty} \lambda \int_0^{+\infty} e^{-\lambda \tau} \mathrm{d}\tau.
\end{aligned}
$$

Let us then focus on the case where $(X_0^1, Y_0^1) \in K_1$ and $(X_0^2, Y_0^2) \in K_2$. By Lagrange's theorem, we have

$$
F(X_t^{\varepsilon,1}, Y_t^{\varepsilon,1}) - F(X_t^{\varepsilon,2}, Y_t^{\varepsilon,2}) = \int_0^1 \langle \mathrm{d}F_\beta, \mathfrak{D}_\beta \rangle\,\mathrm{d}\beta,
$$

where, for $\beta \in (0,1)$,

$$
\mathrm{d}F_\beta = \mathrm{d}F(X_t^\varepsilon((1-\beta)X_0^1 + \beta X_0^2), Y_t^\varepsilon((1-\beta)Y_0^1 + \beta Y_0^2)),
$$

with $\mathrm{d}F(X,Y)=\sum_{j=1}^{N}\partial_j\tilde{F}\langle u_j,X\rangle+\sum_{j=1}^{M}\partial_j\tilde{F}\langle v_j,Y\rangle$, for $F\in\mathrm{Cyl}^b_{B_X\times B_Y}$ having the following form $F(X,Y)=\tilde{F}(\langle u_1,X\rangle,\ldots,\langle u_N,X\rangle,\langle v_1,Y\rangle,\ldots,\langle v_M,Y\rangle)$, with $\tilde{F}\in C_b^2(\mathbb{R}^{N+M})$ and $u_i\in\mathcal{S}(\mathbb{R}^2)$, $i=1,\ldots,N$, and $v_i\in\mathcal{S}(\mathbb{R}^2)$, $i=1,\ldots,M$, and, for $\beta\in(0,1)$,

$$\mathfrak{D}_\beta=(\nabla_{X_0^1-X_0^2}X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2),0,\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2),\nabla_{Y_0^1-Y_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)).$$

We want to get some bounds for $\left|\int_0^1\langle\mathrm{d}F_\beta,\mathfrak{D}_\beta\rangle\mathrm{d}\beta\right|$.

For the term $|\langle\mathrm{d}F_\beta,\nabla_{X_0^1-X_0^2}X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2)\rangle|$ we have

$$\begin{aligned}\left|\langle\mathrm{d}F_\beta,\nabla_{X_0^1-X_0^2}X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2)\rangle\right| &= |\langle\mathrm{d}F_\beta,P_t((1-\beta)X_0^1+\beta X_0^2)\rangle|\\ &\leqslant e^{-m^2t}\|\mathrm{d}F_\beta\|\,\|(1-\beta)X_0^1+\beta X_0^2\|_{C_\ell^{-\delta}}.\end{aligned}$$

For the term $|\langle\mathrm{d}F_\beta,\nabla_{Y_0^1-Y_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\rangle|$, considering equation (B.3.8), and multiplying it by $\tilde{g}_\varepsilon*\nabla_{Y_0}Y_t$ and exploiting the negativity of the non-linearity, we get the following a priori estimate for some constant $k\in(0,m^2)$

$$\partial_t\|\tilde{g}_\varepsilon*\nabla_{Y_0^1-Y_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2+k\|\tilde{g}_\varepsilon*\nabla_{Y_0^1-Y_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2\leqslant 0,$$

and therefore, by Gronwall lemma,

$$\|\tilde{g}_\varepsilon*\nabla_{Y_0^1-Y_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2\lesssim e^{-kt},$$

and the estimate is independent of ε.

Consider now the operator $f\mapsto\tilde{g}_\varepsilon*f$, and recall that in Fourier representation convolution corresponds to multiplication, i.e. the previous operator can be viewed as $\hat{f}\mapsto\hat{\tilde{g}}_\varepsilon\cdot\hat{f}$. When $\hat{f}$ has compact support, such an operator is invertible if $0<\varepsilon\leqslant\varepsilon_0$ for some positive constant ε_0 depending on the size of the support of $\hat{f}$, with inverse given by

$$\hat{\mathcal{Q}}_\varepsilon:\hat{f}\mapsto\frac{1}{\hat{\tilde{g}}_\varepsilon}\cdot\hat{f},$$

which is a well-defined operation because $\hat{\tilde{g}}_\varepsilon(k)=\hat{\tilde{g}}_1(\varepsilon k)$, $\hat{\tilde{g}}_1(0)=1$, and $\hat{\tilde{g}}_1$ is smooth since $\tilde{g}_1$ is a Schwartz function. Using the previous notation, we have

$$\begin{aligned}|\langle\mathrm{d}F_\beta,\nabla_{Y_0^1-Y_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\rangle| &= |\langle\mathrm{d}F_\beta\circ\mathcal{Q}_\varepsilon,\tilde{g}_\varepsilon*\nabla_{Y_0^1-Y_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\rangle|\\ &\lesssim e^{-kt}\|\mathrm{d}F_\beta\circ\mathcal{Q}_\varepsilon\|\\ &\lesssim e^{-kt}\sup_{0<\varepsilon\leqslant\varepsilon_0}\|\mathrm{d}F_\beta\circ\mathcal{Q}_\varepsilon\|,\end{aligned}$$

where we used the fact that F has compact support in Fourier variables and so the norm $\|\mathrm{d}F_\beta\circ\mathcal{Q}_\varepsilon\|$ is bounded for any $0<\varepsilon\leqslant\varepsilon_0$, and also that

$$\sup_{\substack{0<\varepsilon<\varepsilon_0\\ k\in\mathrm{supp}(F)}}|\hat{\tilde{g}}_\varepsilon(k)|^{-1}<+\infty.$$

Now, focus on the term $|\langle\mathrm{d}F_\beta,\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\rangle|$. By equation (B.3.9), we have

$$\begin{aligned}&\partial_t\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2+\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{H^1}^2\\ &+m^2\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2\\ \lesssim&\int_{\mathbb{R}^2}\mathrm{D}_X\mathcal{G}_\varepsilon(X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2),Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2))\nabla_{X_0^1-X_0^2}X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2)\\ &\times\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\,\mathrm{d}z,\end{aligned}$$

and by Hölder and Young inequalities, and re-absorbing the terms properly, we have that, for any $\varsigma>0$, the following inequality holds

$$\begin{aligned}
&\partial_t\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2+k\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2\\
\lesssim\ &\frac{1}{\varsigma}\|\mathrm{D}_X\mathscr{G}_\varepsilon(X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2),Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2))\|_{B_{p,p}^{-s}}^2\|\nabla_{X_0^1-X_0^2}X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2)\|_{B_{q,q}^s}^2.
\end{aligned}$$

Now, we can bound the norm $\|\nabla_{X_0^1-X_0^2}X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2)\|_{B_{q,q}^s}^2$ by a constant since X_0^1 and X_0^2 belong to compact sets. Moreover, by Lemma B.5.7, we have

$$\begin{aligned}
&\partial_t\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2+k\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2\\
\lesssim\ &\frac{e^{-m^2t}}{t^{\delta+s}}\|\mathrm{D}_X\mathscr{G}_\varepsilon(X_t^\varepsilon((1-\beta)X_0^1+\beta X_0^2),0)\|_{B_{p,p}^{-s}}^2.
\end{aligned}$$

Applying Gronwall lemma yields the following inequality

$$\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2\lesssim e^{-kt}\int_0^t\frac{e^{(k-m^2)\tau}}{\tau^{\delta+s}}\|\mathrm{D}_X\mathscr{G}_\varepsilon(X_\tau^\varepsilon((1-\beta)X_0^1+\beta X_0^2),0)\|_{B_{p,p}^{-s}}^2\mathrm{d}\tau.$$

Thus, we get, for some constant $C_\varsigma>0$ depending on $\varsigma>0$,

$$\begin{aligned}
&\mathbb{E}\left[\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}\mathbb{1}_{(X_0^1,Y_0^1)\in K_1,(X_0^2,Y_0^2)\in K_2}\right]\\
\leqslant\ &\mathbb{E}\left[\|\nabla_{X_0^1-X_0^2}Y_t^\varepsilon((1-\beta)Y_0^1+\beta Y_0^2)\|_{L^2}^2\mathbb{1}_{(X_0^1,Y_0^1)\in K_1,(X_0^2,Y_0^2)\in K_2}\right]^{1/2}\\
\lesssim\ &C_\varsigma\left(e^{-kt}\int_0^t\frac{e^{(k-m^2)\tau}}{\tau^{\delta+s}}\mathbb{E}\left[\|\mathrm{D}_X\mathscr{G}_\varepsilon(X_\tau^\varepsilon((1-\beta)X_0^1+\beta X_0^2),0)\|_{B_{p,p}^{-s}}^2\right]\mathrm{d}\tau\right)^{1/2}\\
\lesssim\ &C_\varsigma\left(e^{-kt}\int_0^t\frac{e^{(k-m^2)\tau}}{\tau^{\delta+s}}\mathrm{d}\tau\right)^{1/2},
\end{aligned}$$

since the law of $X_\tau^\varepsilon((1-\beta)X_0^1+\beta X_0^2)$ is that of a Gaussian free field independent of $\beta\in(0,1)$ and of $\varepsilon>0$.

Putting everything together, we get

$$\begin{aligned}
&\mathbb{E}\left[(F(X_t^{\varepsilon,1},Y_t^{\varepsilon,1})-F(X_t^{\varepsilon,2},Y_t^{\varepsilon,2}))\mathbb{1}_{(X_0^1,Y_0^1)\in K_1,(X_0^2,Y_0^2)\in K_2}\right]\\
\lesssim\ &e^{-m^2t}\|\mathrm{d}F_\beta\|\mathbb{E}\left[\|(1-\beta)X_0^1+\beta X_0^2\|_{C_\ell^{-\delta}}\right]+e^{-kt}\|\mathrm{d}F_\beta\|+C_\varsigma\left(e^{-kt}\int_0^t\frac{e^{(k-m^2)\tau}}{\tau^{\delta+s}}\mathrm{d}\tau\right)^{1/2}\\
\lesssim\ &e^{-m^2t}\|\mathrm{d}F_\beta\|+e^{-kt}\|\mathrm{d}F_\beta\|+C_\varsigma\left(e^{-kt}\int_0^t\frac{e^{(k-m^2)\tau}}{\tau^{\delta+s}}\mathrm{d}\tau\right)^{1/2},
\end{aligned}$$

where we used that the law of $(1-\beta)X_0^1+\beta X_0^2$ does not depend on $\beta\in(0,1)$.

Therefore, we get

$$\begin{aligned}
&\left|\lambda\int G_\varepsilon^\lambda(X_0,Y_0)\,\mu_1(\mathrm{d}X_0,\mathrm{d}Y_0)-\lambda\int G_\varepsilon^\lambda(X_0,Y_0)\,\mu_2(\mathrm{d}X_0,\mathrm{d}Y_0)\right|\\
\lesssim\ &\left|\lambda\int_0^{+\infty}\left[e^{-\lambda t}e^{-m^2t}\|\mathrm{d}F_\beta\|+e^{-kt}\sup_{0<\varepsilon\leqslant\varepsilon_0}\|\mathrm{d}F_\beta\circ\mathcal{Q}_\varepsilon\|(1+C_\varsigma)\left(1+\int_0^t\frac{e^{(k-m^2)\tau}}{\tau^{\delta+s}}\mathrm{d}\tau\right)^{1/2}\right]\mathrm{d}t\right|\\
&+2\varsigma\|F\|_{L^\infty}\lambda\int_0^{+\infty}e^{-\lambda\tau}\mathrm{d}\tau.
\end{aligned}$$

For some constant $K>0$, we have

$$\left|\lambda\int_0^{+\infty}\left[e^{-\lambda t}e^{-m^2 t}\|\mathrm{d}F_\beta\|+e^{-kt}\sup_{0<\varepsilon\leqslant\varepsilon_0}\|\mathrm{d}F_\beta\circ\mathcal{Q}_\varepsilon\|(1+C_\varsigma)\left(1+\int_0^t\frac{e^{(k-m^2)\tau}}{\tau^{\delta+s}}\mathrm{d}\tau\right)^{1/2}\right]\mathrm{d}t\right|$$
$$\lesssim (1+C_\varsigma)\lambda\int_0^{+\infty}e^{-\lambda t-Kt}\mathrm{d}t$$
$$\lesssim (1+C_\varsigma)\frac{\lambda}{\lambda+K},$$

and the last term converges to zero as $\lambda\to 0$. Thus, we have

$$\left|\lambda\int G_\varepsilon^\lambda(X_0,Y_0)\ \mu_1(\mathrm{d}X_0,\mathrm{d}Y_0)-\lambda\int G_\varepsilon^\lambda(X_0,Y_0)\ \mu_2(\mathrm{d}X_0,\mathrm{d}Y_0)\right|\lesssim\varsigma+(1+C_\varsigma)\frac{\lambda}{\lambda+K}.$$

Proof of point *iii.* Let $\mathscr{K}\in B_{\exp}^{q,\ell/2}$ and note that $\nabla_{Y_0}Y_t^\varepsilon[\mathscr{K}]$ solves equation (B.3.8). We write

$$\nabla\mathscr{Y}_t^\varepsilon=\nabla_{Y_0}Y_t^\varepsilon[\mathscr{K}]-P_t\mathscr{K}. \tag{B.3.15}$$

Recall that in the proof of point *i.* in Proposition B.3.5 we saw that

$$\nabla_{Y_0}G_\varepsilon^\lambda(X_0,Y_0)=\mathbb{E}_\xi\left[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y_t^\varepsilon)\,\nabla_{Y_0}Y_t^\varepsilon\,\mathrm{d}t\right].$$

Now, recalling the representation (B.3.15) of $\nabla\mathscr{Y}_t^\varepsilon$, we have

$$\int\mathbb{E}_\xi\left[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y_t^\varepsilon)\,\nabla_{Y_0}Y_t^\varepsilon[\mathscr{K}]\,\mathrm{d}t\right]\mathrm{d}\mu$$
$$\lesssim \|\mathrm{d}F\|\int\mathbb{E}_\xi\left[\int_0^{+\infty}e^{-\lambda t}\|\nabla_{Y_0}Y_t^\varepsilon[\mathscr{K}]\|_{B_{\exp}^{q,\ell/2}}\mathrm{d}t\right]\mathrm{d}\mu$$
$$\lesssim \int\mathbb{E}_\xi\left[\int_0^{+\infty}e^{-\lambda t}\left(\|P_t\mathscr{K}\|_{B_{\exp}^{q,\ell/2}}+\|\nabla\mathscr{Y}_t^\varepsilon\|_{L_\ell^2(\mathbb{R}^2)}\right)\mathrm{d}t\right]\mathrm{d}\mu.$$

By Lemma B.3.9 below, we have

$$\int\mathbb{E}_\xi\left[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y_t^\varepsilon)\,\nabla_{Y_0}Y_t^\varepsilon[\mathscr{K}]\,\mathrm{d}t\right]\mathrm{d}\mu$$
$$\lesssim \int\mathbb{E}_\xi\left[\int_0^{+\infty}e^{-\lambda t}\left(1+(1+t)^\sigma\|f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:\|_{L_{\ell'''}^r(\mathbb{R}_+,B_{\exp}^{r,\ell/2})}\right)\|\mathscr{K}\|_{B_{\exp}^{q,\ell/2}}\mathrm{d}t\right]\mathrm{d}\mu.$$

Hölder inequality with respect to all measures, together with the fact that $\lambda>0$, yields

$$\int\mathbb{E}_\xi\left[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y_t^\varepsilon)\,\nabla_{Y_0}Y_t^\varepsilon[\mathscr{K}]\,\mathrm{d}t\right]\mathrm{d}\mu$$
$$\lesssim_\lambda \left(\int\|\mathscr{K}\|_{B_{\exp}^{q,\ell/2}}^q\mathrm{d}\mu\right)^{1/q}\left(\int\mathbb{E}_\xi\left[\|f_\varepsilon:e^{\alpha(g_\varepsilon*X_t)}:\|_{L_{\ell'''}^r(\mathbb{R}_+,B_{\exp}^{r,\ell/2})}^r\right]\mathrm{d}\mu\right)^{1/r}\int_0^{+\infty}e^{-\lambda t}(1+(1+t)^\sigma)\,\mathrm{d}t.$$

Here, since $\lambda>0$, the integral $\int_0^{+\infty}e^{-\lambda t}(1+(1+t)^\sigma)\,\mathrm{d}t$ is finite. Moreover, by Proposition B.6.3, we have that the term

$$\left(\int\mathbb{E}_\xi\left[\|f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:\|_{L_{\ell'''}^r(\mathbb{R}_+,B_{\exp}^{r,\ell/2})}^r\right]\mathrm{d}\mu\right)^{1/r}$$

is bounded with respect to $\varepsilon>0$. This concludes the proof of Proposition B.3.5. □

We close the section with an auxiliary lemma used in the previous proof.

Lemma B.3.9. *Let $\alpha^2 < 4\pi\tilde{\gamma}_{\max}$. Then, for every $\varepsilon > 0$, $Y_0 \in B_Y^{\leqslant}$, and X_0 in a set of full measure with respect to the free field measure μ^{free} with mass $m > 0$, we have that*

$$\|\nabla \mathscr{Y}_t^\varepsilon\|_{L_\ell^2(\mathbb{R}^2)} \lesssim (1+t)^\sigma \|f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)}:\|_{L_{\ell'''}^r(\mathbb{R}_+, B_{\exp}^{r,\ell/2})} \|\mathscr{K}\|_{B_{\exp}^{q,\ell/2}}.$$

Proof. Let $\gamma = \alpha^2/(4\pi) < \tilde{\gamma}_{\max}$. The proof is similar to the one of Theorem B.7.1. Indeed, by multiplying the equation for $\nabla \mathscr{Y}^\varepsilon$ by $g_\varepsilon * \nabla \mathscr{Y}^\varepsilon$, and integrating, we get

$$\begin{aligned}
&\rho_{\ell'}(t)\|\tilde{g}_\varepsilon * \nabla \mathscr{Y}_t^\varepsilon\|_{L_\ell^2}^2 + \int_0^t \rho_{\ell'}(s)\|\tilde{g}_\varepsilon * \nabla \mathscr{Y}_s^\varepsilon\|_{H_\ell^1}^2 \,\mathrm{d}s \\
&\quad + \int_0^t \rho_{2\ell}(z)\rho_{\ell'}(s)\alpha^2 f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)(z)}: e^{\alpha(g_\varepsilon * Y_s)(z)}(g_\varepsilon * \nabla \mathscr{Y}_s^\varepsilon(z))^2 \,\mathrm{d}s\,\mathrm{d}z \\
\lesssim &\int_0^t \rho_{\ell'}(s)\big(\rho_{2\ell}(z) f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)(z)}: e^{\alpha(g_\varepsilon * Y_s)(z)}\big(g_\varepsilon * e^{-(-\Delta+m^2)s}\mathscr{K}\big)\big) g_\varepsilon * \nabla \mathscr{Y}_s^\varepsilon(z)\,\mathrm{d}s\,\mathrm{d}z.
\end{aligned}$$

In the last line, we have

$$\int_0^t \rho_{\ell'}(s)\,\rho_{2\ell}(z)\, f_\varepsilon \underbrace{:e^{\alpha(g_\varepsilon * X_s)}:}_{L^p(\mathbb{R}, B_{\exp}^{r,\ell/2}(\mathbb{R}^2))} \underbrace{e^{\alpha(g_\varepsilon * Y_s)}}_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)} \underbrace{\big(g_\varepsilon * e^{-(-\Delta+m^2)s}\mathscr{K}\big)}_{L^\infty(t^{-\beta}\mathbb{R}_+, B_{q,q,\ell/2}^{-\gamma(q-1)-\delta+2\beta-2/q}(\mathbb{R}^2))} \underbrace{g_\varepsilon * \nabla \mathscr{Y}_s^\varepsilon}_{\substack{L_{\ell'/2}^\infty(\mathbb{R}_+, L_\ell^2(\mathbb{R}^2)) \cap \\ \cap L_{\ell'/2}^2(\mathbb{R}_+, H_\ell^1(\mathbb{R}^2))}} \,\mathrm{d}s.$$

Now, whenever $\gamma < \tilde{\gamma}_{\max}$, there exist $p, q \geqslant 1$, $0 \leqslant \theta \leqslant 1$, $0 \leqslant \beta \leqslant 1$, $\delta > 0$, such that the following system has a solution

$$\begin{cases}
-\gamma(q-1) + 2\beta - \frac{2}{q} > 0, \\
-\gamma(p-1) - \delta + \theta > 0, \\
\frac{1}{p} + \frac{1}{2} < 1, \\
\frac{1}{p} + \beta - \delta + \frac{\theta}{2} \leqslant 1.
\end{cases}$$

The proof is complete. □

B.3.3 Proof of uniqueness of solutions to Problem B

In the case of Problem B a better result can be obtained. It is useful to introduce a new approximating operator $\mathscr{L}_{\varepsilon,\bar{\varepsilon}}$ for $\mathscr{L}$ of the form

$$\mathscr{L}_{\varepsilon,\bar{\varepsilon}}(\Phi) := \frac{1}{2}\operatorname{tr}(\nabla_X^2 \Phi) - \langle(-\Delta + m^2)X, \nabla_X \Phi\rangle - \langle(-\Delta + m^2)Y + \mathscr{G}_{\varepsilon,\bar{\varepsilon}}(X, Y), \nabla_Y \Phi\rangle,$$

where

$$\mathscr{G}_{\varepsilon,\bar{\varepsilon}}(X, Y) := \alpha f_\varepsilon(:e^{\alpha(g_\varepsilon * X)}: e^{\alpha(g_{\bar{\varepsilon}} * Y)}).$$

Here f_ε and g_ε are defined as in Section B.2.2. Recall that by Remark B.2.9, since $\lim_{\bar{\varepsilon},\varepsilon \to 0} \mathscr{L}_{\varepsilon,\bar{\varepsilon}} = \mathscr{L}$, Problem B with the operator $\mathscr{L}_{\varepsilon,\bar{\varepsilon}}$ is equivalent to the one with operator $\mathscr{L}_\varepsilon$ defined in equation (B.2.14). We can then consider the resolvent equation associated to $\mathscr{L}_{\varepsilon,\bar{\varepsilon}}$, namely

$$(\lambda - \mathscr{L}_{\varepsilon,\bar{\varepsilon}}) G_{\varepsilon,\bar{\varepsilon}}^\lambda = F,$$

where $F \in \mathrm{Cyl}_{B_X \times B_Y}$ with compact support in Fourier variables. A solution to such equation is given by

$$G_{\varepsilon,\bar{\varepsilon}}^\lambda(X_0, Y_0) = \mathbb{E}_\xi\left[\int_0^{+\infty} e^{-\lambda t} F(X_t, Y_t^{\varepsilon,\bar{\varepsilon}})\,\mathrm{d}t\right], \tag{B.3.16}$$

where $X_t, Y_t^{\varepsilon,\bar{\varepsilon}}$ solves the following system of equations

$$\begin{aligned}(\partial_t - \Delta + m^2) X_t &= \xi_t, & X(0) &= X_0, &\quad \text{(B.3.17)}\\ (\partial_t - \Delta + m^2) Y_t^{\varepsilon,\bar{\varepsilon}} &= -\mathscr{G}_{\varepsilon,\bar{\varepsilon}}(X_t, Y_t^{\varepsilon,\bar{\varepsilon}}), & Y^{\varepsilon,\bar{\varepsilon}}(0) &= Y_0. &\quad \text{(B.3.18)}\end{aligned}$$

It is easy to show that all the results in Proposition B.3.8 hold also for equations (B.3.17)–(B.3.18), adapting the form of the equations for the derivatives. This implies that points *i.*, *ii.* stated in Proposition B.3.5 hold true also for $G_{\varepsilon,\bar{\varepsilon}}^{\lambda}$. As far as point *iii* in Proposition B.3.5 is concerned, we can get a slightly better result in the present scenario. Recall that $\gamma_{\max}$ is defined as in Remark B.1.1.

Notice that the operator (B.3.16) and equations (B.3.17)–(B.3.18) can be defined also for the case $\bar{\varepsilon} = 0$. Moreover, (a suitable adaptation of) Proposition B.3.8 holds also for the case $\bar{\varepsilon} = 0$, and point *iii.* of Proposition B.3.5 holds also for $G_{\varepsilon,0}^{\lambda}$.

Remark B.3.10. It is worth to note that the operator $\mathscr{L}_{\varepsilon,\bar{\varepsilon}}$ cannot be use directly to solve Problem A", since the solution to the resolvent equation depends in general on (X, Y) and not only on $X + Y$.

Proposition B.3.11. *Let $F \in \mathrm{Cyl}_{B_X \times B_Y}$ and consider $G_{\varepsilon,\bar{\varepsilon}}^{\lambda}$ given by (B.3.16). If $\alpha^2 < 4\pi\gamma_{\max}$, then there exists $q > 1$ such that, for every $\mu \in \mathscr{M}_{B_X \times B_Y}$, we have*

$$\lim_{\bar{\varepsilon} \to 0} \int |\langle \nabla_{Y_0} G_{\varepsilon,\bar{\varepsilon}}^{\lambda}, \mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}} \rangle| \,\mathrm{d}\mu \lesssim_{\lambda} \left(\int \|\mathscr{G} - \mathscr{G}_{\varepsilon,0}\|_{B_{\exp}^{q,\ell/2}}^q \,\mathrm{d}\mu \right)^{1/q},$$

uniformly in $\varepsilon > 0$.

In order to prove the previous result, we need some technical lemmas. First, we deal with the convergence as $\bar{\varepsilon} \to 0$. In particular, whenever an object has $\bar{\varepsilon}$ as one of its parameters (e.g. $\nabla_{Y_0} G_{\varepsilon,\bar{\varepsilon}}^{\lambda}$), the same notation with $\bar{\varepsilon} = 0$ indicates that it is the limiting object as $\bar{\varepsilon} \to 0$ (e.g. $\nabla_{Y_0} G_{\varepsilon,0}^{\lambda} = \lim_{\bar{\varepsilon} \to 0} \nabla_{Y_0} G_{\varepsilon,\bar{\varepsilon}}^{\lambda}$), whenever it exists.

Lemma B.3.12. *For every $\varepsilon > 0$, $Y_0 \in B_Y^{\leqslant}$, and X_0 in a set of full measure with respect to the free field measure μ^{free} with mass $m > 0$, we have that $\nabla_{Y_0} Y_t^{\varepsilon,\bar{\varepsilon}}(\mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}})$ converges to $\nabla_{Y_0} Y_t^{\varepsilon,0}(\mathscr{G} - \mathscr{G}_{\varepsilon,0})$ in $B_{\exp}$, as $\bar{\varepsilon} \to 0$.*

Proof. Recall that $\nabla_{Y_0} Y_t^{\varepsilon,\bar{\varepsilon}}(\mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}})$ solves equation (B.3.8). This means that

$$\nabla \mathscr{Y}_t^{\varepsilon,\bar{\varepsilon}} = \nabla_{Y_0} Y_t^{\varepsilon,\bar{\varepsilon}}(\mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}}) - P_t(\mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}})$$

solves the equation

$$\begin{aligned}(\partial_t - \Delta + m^2) \nabla \mathscr{Y}_t^{\varepsilon,\bar{\varepsilon}} &= -\mathrm{D}_Y \mathscr{G}_{\varepsilon,\bar{\varepsilon}}(X_t, Y_t^{\varepsilon,\bar{\varepsilon}})[\nabla \mathscr{Y}_t^{\varepsilon,\bar{\varepsilon}}] - \mathrm{D}_Y \mathscr{G}_{\varepsilon,\bar{\varepsilon}}(X_t, Y_t^{\varepsilon,\bar{\varepsilon}})[P_t(\mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}})], \quad \text{(B.3.19)}\\ \nabla \mathscr{Y}_0^{\varepsilon,\bar{\varepsilon}} &= 0.\end{aligned}$$

By Corollary B.7.3, we get the estimate

$$\|\nabla \mathscr{Y}_t^{\varepsilon,\bar{\varepsilon}}\| \lesssim \rho_{\ell''}(t) P_2(\|f_\varepsilon : e^{\alpha(g_\varepsilon * X_t)} : \|_{L^\infty}, \|\mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}}\|_{B_{\exp}}).$$

Such a bound is uniform with respect to $\bar{\varepsilon} > 0$ and therefore we get a converging subsequence whose limit $\nabla \mathscr{Y}_t^{\varepsilon,0}$ solves equation (B.3.19) with $\bar{\varepsilon} = 0$. Since the term $P_t(\mathscr{G} - \mathscr{G}_{\varepsilon,\bar{\varepsilon}})$ converges point-wise to $P_t(\mathscr{G} - \mathscr{G}_{\varepsilon,0})$, the result is proved. □

Let us prove that it makes sense to consider some parameters satisfying certain conditions that will be useful in upcoming results.

Lemma B.3.13. *Suppose that $\gamma=\alpha^2/(4\pi)<\gamma_{\max}$. Then there exist $q>1$ such that $\gamma q<2$, $\kappa>0$, and r and δ as in Definition B.2.6, such that the following inequalities are satisfied*

$$-\gamma(r-1)-\gamma(q-1)-2\delta+\kappa>0,\qquad \frac{1}{q}+\frac{1}{r}<1,\qquad \frac{\kappa}{2}q<1. \tag{B.3.20}$$

Proof. In order to prove the result, it is enough to show that there exists some solution with the previous properties to the system of equations

$$-\gamma(r-1)-\gamma\frac{1}{r-1}+\frac{2(r-1)}{r}=0,\qquad \frac{1}{q}+\frac{1}{r}=1,\qquad \frac{\kappa}{2}q=1.$$

From such relations we get the equality

$$\gamma=\frac{2(r-1)^2}{r((r-1)^2+1)}. \tag{B.3.21}$$

We get the maximum value $\gamma_{\max}$ of γ for some $r=\bar{r}\approx 2.52$. Since, with this choices of parameters, we have $q\approx 1.21$, the result is proved. □

In the proof of Lemma B.3.12, we introduced the object

$$\nabla\mathscr{Y}_t^{\varepsilon,\bar{\varepsilon}}=\nabla_{Y_0}Y_t^{\varepsilon,\bar{\varepsilon}}(\mathscr{G}-\mathscr{G}_{\varepsilon,\bar{\varepsilon}})-P_t(\mathscr{G}-\mathscr{G}_{\varepsilon,\bar{\varepsilon}}), \tag{B.3.22}$$

satisfying equation (B.3.19) and admitting a limit $\nabla\mathscr{Y}_t^{\varepsilon,0}$ as $\bar{\varepsilon}\to 0$.

Lemma B.3.14. *Suppose that $\alpha^2<4\pi\gamma_{\max}$, consider the parameters q and r defined as in Lemma B.3.13, and ℓ,ℓ''' such that $\ell q/2>2$, $\ell r/2>2$ and $\ell''' r>1$. Then, for every $\varepsilon>0$, $Y_0\in B_Y^{\leqslant}$, and X_0 in a set of full measure with respect to the free field measure μ^{free} with mass $m>0$, we have that*

$$\|\nabla\mathscr{Y}_t^{\varepsilon,0}\|_{L^1_\ell(\mathbb{R}^2)}\lesssim(1+t)^\sigma\|f_\varepsilon :e^{\alpha(g_\varepsilon*X_s)}:\|_{L^r_{\ell'''}(\mathbb{R}_+,B^{r,\ell/2}_{\exp}(\mathbb{R}^2))}\|\mathscr{G}-\mathscr{G}_{\varepsilon,0}\|_{B^{q,\ell/2}_{\exp}(\mathbb{R}^2)}.$$

Proof. Let $\gamma=\alpha^2/(4\pi)$. With the same notation as in Lemma B.3.12, we consider the limit $\nabla\mathscr{Y}_t^{\varepsilon,0}$ and show that it converges to zero. Indeed, it satisfies

$$(\partial_t-\Delta_z+m^2+\alpha^2 f_\varepsilon :e^{\alpha(g_\varepsilon*X_t)(z)}: e^{\alpha Y_t(z)})\nabla\mathscr{Y}_t^{\varepsilon,0}=-\alpha^2 f_\varepsilon :e^{\alpha(g_\varepsilon*X_t)(z)}: e^{\alpha Y_t(z)}P_t(\mathscr{G}-\mathscr{G}_{\varepsilon,0}),$$

with $\nabla\mathscr{Y}_0^{\varepsilon,0}=0$.

We now want to exploit a similar argument as the one used in Theorem B.7.2 to get some estimates concerning the solution to the previous equation. Let $w:\mathbb{R}\to\mathbb{R}$ be an increasing smooth function with bounded derivatives and such that $w(0)=0$, $w(x)\to\pm 1$ as $x\to\pm\infty$, and define $W(x)=\int_0^x w(y)\,\mathrm{d}y$. Then, multiplying the equation by $\rho_\ell(\theta\cdot)w(\tilde{\mathscr{K}}_t^{\varepsilon,0})$, where $\theta,\ell>0$, and integrating, we have

$$\begin{aligned}
&\partial_t\|\rho_\ell(\theta\cdot)W(\nabla\mathscr{Y}_t^{\varepsilon,0})\|_{L^1}+m_\theta^2\|\rho_\ell(\theta\cdot)\,w(\nabla\mathscr{Y}_t^{\varepsilon,0})\tilde{\mathscr{K}}_t^{\varepsilon,0}\|_{L^1}+\|\rho_\ell(\theta\cdot)\,w'(\nabla\mathscr{Y}_t^{\varepsilon,0})(\nabla(\nabla\mathscr{Y}_t^{\varepsilon,0}))^2\|_{L^1}\\
&+\int\alpha^2\rho_\ell(\theta z)\,f_\varepsilon :e^{\alpha(g_\varepsilon*X_t)(z)}: e^{\alpha Y_t(z)}\,w(\nabla\mathscr{Y}_t^{\varepsilon,0})\nabla\mathscr{Y}_t^{\varepsilon,0}\,\mathrm{d}z\\
\lesssim&-\int\alpha^2\rho_{\ell/2}(\theta z)f_\varepsilon :e^{\alpha(g_\varepsilon*X_t)(z)}: e^{\alpha Y_t(z)}\,w(\nabla\mathscr{Y}_t^{\varepsilon,0})\,\rho_{\ell/2}(\theta z)(P_t(\mathscr{G}-\mathscr{G}_{\varepsilon,0}))\,\mathrm{d}z,
\end{aligned} \tag{B.3.23}$$

where $0<m_\theta\leqslant\sqrt{m^2-\left|\frac{\nabla\rho_\ell(\theta z)}{\rho_\ell(\theta z)}\right|}$ for every $z\in\mathbb{R}^2$, which holds for $\theta>0$ small enough.

If we choose the parameters as in Lemma B.3.13, we have

$$\begin{aligned}\rho_{\ell/2}(\theta\cdot)f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)}: e^{\alpha Y_t} &\in L^r_{\ell'''}(\mathbb{R}_+, B^{r,0}_{\exp}),\\ w(\nabla \mathscr{Y}^{\varepsilon,0}_t) &\in L^\infty(\mathbb{R}^2),\\ \rho_{\ell/2}(\theta\cdot)(P_t(\mathscr{G}-\mathscr{G}_{\varepsilon,0})) &\in L^\infty\big(|\cdot|^{-\kappa/2}\mathbb{R}_+, B^{-\gamma(q-1)-\delta+\kappa}_{q,q}(\mathbb{R}^2)\big).\end{aligned}$$

Therefore, by integrating with respect to time equation (B.3.23) and recalling that $\nabla\mathscr{Y}^{\varepsilon,0}_0=0$, we get

$$\begin{aligned}&\|\rho_\ell(\theta\cdot)W(\nabla\mathscr{Y}^{\varepsilon,0}_t)\|_{L^1}\lesssim\\ &\int_0^t \|\rho_{\ell/2}(\theta\cdot)f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)}: e^{\alpha Y_s}w(\nabla\mathscr{Y}^{\varepsilon,0}_s)\|_{B^{r,0}_{\exp}}\big\|\rho_{\ell/2}(\theta\cdot)\big(e^{-(-\Delta+m^2)s}(\mathscr{G}-\mathscr{G}_{\varepsilon,0})\big)\big\|_{B^{-\gamma(q-1)-\delta+\kappa}_{q,q}}\mathrm{d}s\end{aligned}$$

where we used the fact that $\frac{1}{r}+\frac{1}{q}<1$ and $-\gamma(r-1)-\gamma(q-1)-2\delta+\kappa>1$. By Proposition B.5.4, we have that

$$\|f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)}: e^{\alpha Y_s}w(\nabla\mathscr{Y}^{\varepsilon,0}_s)\|_{B^{r,\ell/2}_{\exp}}\lesssim\|f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)}:\|_{B^{r,\ell/2}_{\exp}}\|e^{\alpha Y_s}\|_{L^\infty}\|w(\nabla\mathscr{Y}^{\varepsilon,0}_s)\|_{L^\infty}.$$

Therefore, by Proposition B.5.6, we have

$$\|\rho_\ell(\theta\cdot)W(\nabla\mathscr{Y}^{\varepsilon,0}_t)\|_{L^1}\lesssim\int_0^t \|f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)}:\|_{B^{r,\ell/2}_{\exp}} s^{\kappa/2}\|\mathscr{G}-\mathscr{G}_{\varepsilon,0}\|_{B^{q,\ell/2}_{\exp}}\mathrm{d}s.$$

Multiply and divide then by $\rho_{\ell'''}(s)$ and apply Hölder inequality with respect to s to get

$$\begin{aligned}&\|\rho_\ell(\theta\cdot)W(\nabla\mathscr{Y}^{\varepsilon,0}_t)\|_{L^1}\\ \lesssim\;&\|\rho_{\ell/2}(\theta\cdot)f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)}:\|_{L^r_{\ell'''}(\mathbb{R}_+,B^{r,\ell/2}_{\exp})}\|\mathscr{G}-\mathscr{G}_{\varepsilon,0}\|_{B^{q,\ell/2}_{\exp}}\bigg(\int_0^t s^{q\kappa/2}\rho_{-\ell'''q}(s)\,\mathrm{d}s\bigg)^{1/q}.\end{aligned}$$

Since the expression $\big(\int_0^t s^{q\kappa/2}\rho_{-\ell'''q}(s)\,\mathrm{d}s\big)^{1/q}$ is bounded for $t\to0$ and grows polynomially in time as $t\to+\infty$, we deduce that the result holds. □

Proof of Proposition B.3.11. As in the proof of point *i.* of Proposition B.3.5 we can get

$$\nabla_{Y_0}G^\lambda_{\varepsilon,\bar\varepsilon}(X_0,Y_0)=\mathbb{E}_\xi\bigg[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y^{\varepsilon,\bar\varepsilon}_t)\,\nabla_{Y_0}Y^{\varepsilon,\bar\varepsilon}_t\,\mathrm{d}t\bigg].$$

Therefore, by Lemma B.3.12 and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{\bar\varepsilon\to0}\int\nabla_{Y_0}G^\lambda_{\varepsilon,\bar\varepsilon}(X_0,Y_0)(\mathscr{G}-\mathscr{G}_{\varepsilon,\bar\varepsilon})\,\mathrm{d}\mu &= \lim_{\bar\varepsilon\to0}\int\mathbb{E}_\xi\bigg[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y^{\varepsilon,\bar\varepsilon}_t)\,\nabla_{Y_0}Y^{\varepsilon,\bar\varepsilon}_t(\mathscr{G}-\mathscr{G}_{\varepsilon,\bar\varepsilon})\,\mathrm{d}t\bigg]\mathrm{d}\mu\\ &= \int\mathbb{E}_\xi\bigg[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y^{\varepsilon,0}_t)\,\nabla_{Y_0}Y^{\varepsilon,0}_t(\mathscr{G}-\mathscr{G}_{\varepsilon,0})\,\mathrm{d}t\bigg]\mathrm{d}\mu\\ &= \int\nabla_{Y_0}G^\lambda_{\varepsilon,0}(X_0,Y_0)(\mathscr{G}-\mathscr{G}_{\varepsilon,0})\,\mathrm{d}\mu.\end{aligned}$$

Now, recalling the representation (B.3.22) of $\nabla\mathscr{Y}^{\varepsilon,0}_t$, we have

$$\begin{aligned}&\int\mathbb{E}_\xi\bigg[\int_0^{+\infty}e^{-\lambda t}\mathrm{d}F(X_t,Y^{\varepsilon,0}_t)\,\nabla_{Y_0}Y^{\varepsilon,0}_t(\mathscr{G}-\mathscr{G}_{\varepsilon,0})\,\mathrm{d}t\bigg]\mathrm{d}\mu\\ \lesssim\;&\|\mathrm{d}F\|_{L^\infty}\int\mathbb{E}_\xi\bigg[\int_0^{+\infty}e^{-\lambda t}\|\nabla_{Y_0}Y^{\varepsilon,0}_t(\mathscr{G}-\mathscr{G}_{\varepsilon,0})\|_{B^{q,\ell/2}_{\exp}}\mathrm{d}t\bigg]\mathrm{d}\mu\\ \lesssim\;&\int\mathbb{E}_\xi\bigg[\int_0^{+\infty}e^{-\lambda t}\Big(\|P_t(\mathscr{G}-\mathscr{G}_{\varepsilon,0})\|_{B^{q,\ell/2}_{\exp}}+\|\nabla\mathscr{Y}^{\varepsilon,0}_t\|_{L^1_\ell(\mathbb{R}^2)}\Big)\mathrm{d}t\bigg]\mathrm{d}\mu.\end{aligned}$$

At this point, the proof follows the same steps as the one of point *iv.* in Proposition B.3.5. □

B.4 Existence of solutions via Lyapunov functions

In this section, we prove existence of solutions to Problem A" and B. Since existence for the Problem B implies existence for Problem A" (see point *iii.* in Theorem B.2.5), we only prove the first. We actually prove a stronger statement than the one given in Problem B, namely that equation (B.2.5) holds for every $\Phi \in \mathcal{F}$ and not only for $\Phi \in \mathrm{Cyl}_{B_X \times B_Y}$. We exploit a strategy based on Lyapunov functions.

Let us introduce a finite-dimensional approximation to the operator $\mathcal{L}$. Let $M, N \in \mathbb{N}$, $\varepsilon > 0$, and consider $\mathbb{T}_M^2$ the two-dimensional torus of size M which we identify hereafter with the subset $(-M\pi, M\pi]^2 \subset \mathbb{R}^2$. Consider the *Fejér operator* $Q_{N,M}$: $\mathcal{S}'(\mathbb{T}_M^2) \to C^\infty(\mathbb{T}_M^2)$ defined as, for $F \in \mathcal{S}'(\mathbb{T}_M^2)$,

$$Q_{N,M}(F) = \mathrm{Fej}_{N,M} * F, \qquad \mathrm{Fej}_{N,M}(x) = \sum_{|j_1|,|j_2| \leqslant N-1} \left(1 - \frac{|j_1|}{N}\right)\left(1 - \frac{|j_2|}{N}\right) e^{ijx/M}, \qquad x \in \mathbb{T}_M^2.$$

Let us stress that the operator $Q_{N,M}$ is both a positive operator, i.e. $\langle Q_{N,M}(F), F\rangle \geqslant 0$, and a positive preserving operator, namely, if F is a positive distribution, then $Q_{N,M}(F)$ is a positive function. The latter property following from the positivity of the kernel $\mathrm{Fej}_{N,M}$.

The new approximating operator $\mathcal{L}_{N,M,\varepsilon}$ is then given by, for $\Phi \in \mathcal{F}$, by

$$\begin{aligned} \mathcal{L}_{N,M,\varepsilon}(\Phi) &= \frac{1}{2} \mathrm{tr}_{L^2(\mathbb{T}_M^2)}(P_{\mathbb{T}_M^2}(\mathrm{Per}_{\mathbb{T}_M^2} \nabla_X^2 \Phi) P_{\mathbb{T}_M^2}) - \langle(-\Delta + m^2) X, \nabla_X \Phi\rangle \\ &\quad - \langle(-\Delta + m^2) Y + \mathcal{G}_{N,M,\varepsilon}(X, Y), \nabla_Y \Phi\rangle, \end{aligned}$$

where $\mathcal{G}_{N,M,\varepsilon}$ is the following approximation for the non-linearity in equation (B.3.4),

$$\mathcal{G}_{N,M,\varepsilon}(X, Y) = \alpha Q_{N,M}(g_\varepsilon * (:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: e^{\alpha Q_{N,M}(g_\varepsilon * Y)})). \tag{B.4.1}$$

Here, g_ε is defined as in Section B.2.2, $P_{\mathbb{T}_M^2}$ is the natural projection of $L^2(\mathbb{R}^2)$ on the space $L^2(\mathbb{T}_M^2)$, and $\mathrm{Per}_{\mathbb{T}_M^2}$ is the periodicization on the torus $\mathbb{T}_M^2$. Moreover, the term $:\exp(\alpha Q_{N,M}(g_\varepsilon * X)):$ is defined as follows

$$:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: = e^{\alpha Q_{N,M}(g_\varepsilon * X)} - \frac{\alpha^2}{2} c_{N,M,\varepsilon}, \tag{B.4.2}$$

where $c_{N,M,\varepsilon} = \int_{\mathbb{T}_M^2} Q_{N,M}(g_\varepsilon)(z)\,(-\Delta + m^2)^{-1}(Q_{N,M} g_\varepsilon)(z)\,\mathrm{d}z$, where the inverse $(-\Delta + m^2)^{-1}$ is taken with periodic boundary conditions on $\mathbb{T}_M^2$.

The system of equations for the flow is then given by

$$(\partial_t - \Delta + m^2) X_t^M = \xi_t^M, \tag{B.4.3}$$

$$(\partial_t - \Delta + m^2) Y_t^{N,M,\varepsilon} = -\mathcal{G}_{N,M,\varepsilon}(X_t^M, Y_t^{N,M,\varepsilon}), \tag{B.4.4}$$

where $Y^{N,M,\varepsilon}$ is negative, periodic, and it belongs to a subspace of $\mathfrak{H}_{N,M} := \mathrm{span}\{e^{in\cdot/M}; |n| \leqslant N\}$. As usual, we drop the dependence on the parameters N, M, ε when no ambiguity occurs.

B.4.1 Lyapunov functions

We introduce some Lyapunov functions V_1, V_2, and V_3 that will be crucial in the proof of the estimates for X and Y. In particular, we take them such that V_1 and V_2 depend on both X and Y, while V_3 depends on X only.

Lemma B.4.1. *Consider two Banach spaces B_1 and B_2. Let $\tilde{\mathscr{L}}$ be an operator taking values in some space of functions $\mathscr{F}$ and consider a measure $\tilde{\mu}$ such that $\int \tilde{\mathscr{L}}\Phi \,\mathrm{d}\tilde{\mu} = 0$ for every $\Phi \in \mathscr{F}$. Suppose that there exist some functions $V_1, V_2: B_1 \times B_2 \to \mathbb{R}$, and $V_3: B_1 \to \mathbb{R}$, such that we have V_1, $V_2, V_3 \in \mathscr{F}$ and*

i. V_2 and V_3 are positive,

ii. The inequality

$$\tilde{\mathscr{L}} V_1(X,Y) \leqslant -V_2(X,Y) + V_3(X)$$

holds true.

Then, we have

$$\int V_2(X,Y)\,\tilde{\mu}(\mathrm{d}X,\mathrm{d}Y) \leqslant \int V_3(X)\,\tilde{\mu}(\mathrm{d}X,\mathrm{d}Y). \tag{B.4.5}$$

Proof. The statement follows from the fact that $\int \tilde{\mathscr{L}} V_1(X,Y)\,\tilde{\mu}(\mathrm{d}X,\mathrm{d}Y) = 0$. □

We now work on the estimate for X. We will have to choose the aforementioned Lyapunov functions, we start with $V_1: B_X \to \mathbb{R}$, that we take to be, for $s' > 0$,

$$V_1(X) = \|X\|^p_{B^{-s'}_{p,p,\ell}}.$$

Note that, since X corresponds to the free field, the previous norm is finite. Notice also that the previous function is not cylindric. Moreover, if we consider the representation of Besov spaces given in Proposition B.5.9, then we have, for $-ps'/2 + 1 < 0$,

$$\|X\|^p_{B^{-s'}_{p,p,\ell}} \simeq \|\varphi_0(\mathrm{D})X\|^p_{L^p_\ell} + \int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\rho^k_\ell|^p}{t^{-ps'/2+1}} |P_t X(z)|^p \,\mathrm{d}z\mathrm{d}t =: V_{1,1}(X) + V_{1,2}(X),$$

where k is to be determined, similarly as in the proof of Proposition B.2.10.

Proposition B.4.2. *Let $s' > 0$, $p > 1$ and $\ell > 0$ such that $p\ell > 2$ and $-ps'/2 + 1 < 0$. Consider X to be the solution to equation (B.4.3) on the torus $\mathbb{T}^2_M$. For some $\delta \in (0,1)$, we have the inequality*

$$\mathscr{L}_{N,M,\varepsilon} V_1(X) = \mathscr{L}_{N,M,\varepsilon}(\|X\|^p_{B^{-s'}_{p,p,\ell}}) \lesssim -(1-\delta)\|X\|^p_{B^{-s'+2/p}_{p,p,\ell}} + \frac{1}{\delta} C. \tag{B.4.6}$$

Proof. We want to evaluate $\mathscr{L}_{N,M,\varepsilon} V_1(X)$. The gradient of V_1 is given by

$$\begin{aligned}
\nabla V_1(X)(h) &= \nabla \|\varphi_0(\mathrm{D})X\|^p_{L^p_\ell}(h) + \int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\rho^k_\ell|^p}{t^{-ps'/2+1}} \nabla(|P_t X(z)|^p)(h)\,\mathrm{d}z\mathrm{d}t \\
&= p\int_{\mathbb{R}^2} |\rho^k_\ell|^p |\varphi_0(\mathrm{D})X(z)|^{p-1} \mathrm{sign}(\varphi_0(\mathrm{D})X)\varphi_0(\mathrm{D})h\,\mathrm{d}z \\
&\quad + p\int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\rho^k_\ell|^p}{t^{-ps'/2+1}} |P_t X(z)|^{p-1} \mathrm{sign}(P_t X(z))\,P_t h\,\mathrm{d}z\mathrm{d}t,
\end{aligned}$$

while its second derivative is

$$\begin{aligned}
&\nabla^2 V_1(X)(h,h') \\
&= p\int_{\mathbb{R}^2} |\rho^k_\ell|^p\, \nabla(|\varphi_0(\mathrm{D})X(z)|^{p-1}\, \mathrm{sign}(\varphi_0(\mathrm{D})X)\,\varphi_0(\mathrm{D})h)(h')\,\mathrm{d}z \\
&\quad + p\int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\rho^k_\ell|^p}{t^{-ps'/2+1}}\, \nabla(|P_t X(z)|^{p-1}\, \mathrm{sign}(P_t X(z))\,P_t h)(h')\,\mathrm{d}z\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= p(p-1)\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^{p-2}(\varphi_0(\mathrm{D})h)(\varphi_0(\mathrm{D})h')\,\mathrm{d}z\\
&\quad +p(p-1)\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell^k|^p}{t^{-ps'/2+1}}|P_tX(z)|^{p-2}(P_th)(P_th')\,\mathrm{d}z\mathrm{d}t.
\end{aligned}
$$

The first term to deal with is $\langle(-\Delta+m^2)X,\nabla_XV_1(X)\rangle$. We have

$$
\begin{aligned}
-\langle(-\Delta+m^2)X,\nabla_XV_{1,1}(X)\rangle &= -p\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^{p-1}\mathrm{sign}(\varphi_0(\mathrm{D})X)(\varphi_0(\mathrm{D})(-\Delta+m^2)X)\,\mathrm{d}z\\
&= -p\int_{\mathbb{R}^2}\nabla(|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^{p-1}\mathrm{sign}(\varphi_0(\mathrm{D})X))\nabla(\varphi_0(\mathrm{D})X)\,\mathrm{d}z\\
&\quad -pm^2\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^p\,\mathrm{d}z\\
&= -p(p-1)\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^{p-2}|\nabla(\varphi_0(\mathrm{D})X)|^2\,\mathrm{d}z\\
&\quad -p^2\int_{\mathbb{R}^2}|\rho_\ell^k|^{p-1}|\varphi_0(\mathrm{D})X(z)|^{p-1}(\nabla\rho_\ell^k)\,\mathrm{sign}(\varphi_0(\mathrm{D})X)\nabla(\varphi_0(\mathrm{D})X)\,\mathrm{d}z\\
&\quad -pm^2\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^p\,\mathrm{d}z.
\end{aligned}
$$

Rearranging the second term on the right-hand side and exploiting Young's inequality, we have, for all $\sigma>0$,

$$
\begin{aligned}
-\langle(-\Delta+m^2)X,\nabla_XV_{1,1}(X)\rangle &\lesssim -p(p-1)\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^{p-2}|\nabla(\varphi_0(\mathrm{D})X)|^2\,\mathrm{d}z\\
&\quad +p^2\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^{p/2}\frac{|\nabla\rho_\ell^k|}{\rho_\ell^k}|\varphi_0(\mathrm{D})X|^{(p-2)/2}|\nabla(\varphi_0(\mathrm{D})X)|\,\mathrm{d}z\\
&\quad -pm^2\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^p\,\mathrm{d}z\\
&\lesssim -p(p-1)\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^{p-2}|\nabla(\varphi_0(\mathrm{D})X)|^2\,\mathrm{d}z\\
&\quad +\frac{p^2}{4\sigma}\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^p\frac{|\nabla\rho_\ell^k|^2}{\rho_{2\ell}^k}\,\mathrm{d}z\\
&\quad +p^2\sigma\int_{\mathbb{R}^2}|\rho_\ell^k|^p\,|\varphi_0(\mathrm{D})X|^{p-2}\,|\nabla(\varphi_0(\mathrm{D})X)|^2\,\mathrm{d}z\\
&\quad -pm^2\int_{\mathbb{R}^2}|\rho_\ell^k|^p|\varphi_0(\mathrm{D})X(z)|^p\,\mathrm{d}z,
\end{aligned}
$$

from which, choosing $\sigma>0$ small enough and $k>0$ large enough, and appropriately rearranging the terms, we get

$$
-\langle(-\Delta+m^2)X,\nabla_XV_{1,1}(X)\rangle\lesssim-\|\varphi_0(\mathrm{D})X\|_{L_\ell^p}^p.
$$

We also have

$$
\begin{aligned}
&-\langle(-\Delta+m^2)X,\nabla_XV_{1,2}(X)\rangle\\
&= -p\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell|^p}{t^{-ps'/2+1}}|P_tX(z)|^{p-1}\mathrm{sign}(P_tX(z))(P_t(-\Delta+m^2)X(z))\,\mathrm{d}z\mathrm{d}t\\
&= \int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell|^p}{t^{-ps'/2+1}}\partial_t|P_tX(z)|^p\,\mathrm{d}z\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^{+\infty}\int_{\mathbb{R}^2} \partial_t\left(\frac{|\rho_\ell|^p}{t^{-ps'/2+1}}\right)|P_tX(z)|^p\,\mathrm{d}z\mathrm{d}t \\
&= -\int_0^{+\infty}\int_{\mathbb{R}^2} \left(\frac{ps'}{2}-1\right)\frac{|\rho_\ell|^p}{t^{-ps'/2+2}}|P_tX(z)|^p\,\mathrm{d}z\mathrm{d}t \\
&= -\int_0^{+\infty}\int_{\mathbb{R}^2} \left(\frac{ps'}{2}-1\right)\frac{|\rho_\ell|^p}{t^{p(-s'+2/p)/2+1}}|P_tX(z)|^p\,\mathrm{d}z\mathrm{d}t \\
&\simeq -(\|X\|^p_{B^{-s'+2/p}_{p,p,\ell}} - \|\varphi_0(\mathrm{D})X\|^p_{L^p_\ell}).
\end{aligned}
$$

The second term to deal with is the trace of the second derivative, i.e. $\mathrm{tr}_{L^2(\mathbb{T}^2_M)}(\nabla^2 V_1(X))$. By the results presented in Section XI.V of [159], it suffices to consider the second derivative with $h'=h$, i.e. $\nabla^2 V_1(X)(h,h)$, and integrate with respect to h. Then, exploiting Young inequality for products and using the fact that $p\ell>2$, we have for any $\sigma>0$

$$
\begin{aligned}
\mathrm{tr}\,(\nabla^2 V_{1,1}(X)) &= p(p-1)\int_{\mathbb{T}^2_M}\int_{\mathbb{R}^2}|\rho^k_\ell|^p|\varphi_0(\mathrm{D})X(z)|^{p-2}|\hat{\varphi}_0(z-y)|^2\,\mathrm{d}z\mathrm{d}y \\
&= p(p-1)\int_{\mathbb{R}^2}|\rho^k_\ell|^p|\varphi_0(\mathrm{D})X(z)|^{p-2}\int_{\mathbb{T}^2_M}|\hat{\varphi}_0(z-y)|^2\,\mathrm{d}y\mathrm{d}z \\
&\lesssim p(p-1)\int_{\mathbb{R}^2}|\rho^k_\ell|^p|\varphi_0(\mathrm{D})X(z)|^{p-2}\,\mathrm{d}z \\
&\lesssim \frac{1}{\sigma}p(p-1)\int_{\mathbb{R}^2}|\rho^k_\ell|^p\,\mathrm{d}z + \sigma p(p-1)\int_{\mathbb{R}^2}|\rho^k_\ell|^p|\varphi_0(\mathrm{D})X(z)|^p\,\mathrm{d}z \\
&\lesssim \frac{1}{\sigma}C + \sigma\|\varphi_0(\mathrm{D})X\|^p_{L^p_\ell}.
\end{aligned}
$$

We get, if $\mathscr{P}^M_t$ is the heat kernel on the torus $\mathbb{T}^2_M$, that, for any $\sigma>0$,

$$
\begin{aligned}
\mathrm{tr}\,(\nabla^2 V_{1,2}(X)) &= p(p-1)\int_{\mathbb{T}^2_M}\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell|^p}{t^{-ps'/2+1}}|P_tX(z)|^{p-2}|\mathscr{P}^M_t(z-y)|^2\,\mathrm{d}z\mathrm{d}t\mathrm{d}y \\
&= p(p-1)\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell|^p}{t^{-ps'/2+1}}|P_tX(z)|^{p-2}\left(\int_{\mathbb{T}^2_M}|\mathscr{P}^M_t(z-y)|^2\,\mathrm{d}y\right)\mathrm{d}z\mathrm{d}t \\
&= p(p-1)\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell|^p}{t^{-ps'/2+1}}|P_tX(z)|^{p-2}\frac{Ce^{-m^2t}}{t}\,\mathrm{d}z\mathrm{d}t \\
&= p(p-1)\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{Ce^{-m^2t}|\rho_\ell|^p}{t^{-ps'/2+2}}|P_tX(z)|^{p-2}\,\mathrm{d}z\mathrm{d}t \\
&\lesssim \frac{1}{\sigma}C'\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell|^p e^{-q(p)m^2t}}{t^{-ps'/2+2}}\,\mathrm{d}z\mathrm{d}t + \sigma\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{|\rho_\ell|^p}{t^{-ps'/2+2}}|P_tX(z)|^p\,\mathrm{d}z\mathrm{d}t \\
&\lesssim \frac{1}{\sigma}C + \sigma\big(\|X\|^p_{B^{-s'+2/p}_{p,p,\ell}} - \|\varphi_0(\mathrm{D})X\|^p_{L^p_\ell}\big),
\end{aligned}
$$

where we used Young inequality as above.

Choosing $\sigma>0$ small enough, we get

$$
\mathscr{L}V_1(X) \lesssim -(1-\sigma)\|X\|^p_{B^{-s'+2/p}_{p,p,\ell}} + \frac{1}{\sigma}C,
$$

which gives the result. $\square$

Remark B.4.3. The result in Lemma B.4.1 with $B_2 = B_X$, no dependence on B_1 whatsoever, and with Lyapunov functions chosen as

$$\begin{aligned} V_1(X) &= \|X\|^p_{B^{-s'}_{p,p,\ell}}, \\ V_2(X) &= (1-\sigma)\|X\|^p_{B^{-s'+2/p}_{p,p,\ell}}, \\ V_3 &\equiv \frac{1}{\sigma}C, \end{aligned}$$

for some constant $C > 0$, gives the estimate

$$(1-\sigma)\int \|X\|^p_{B^{-s'+2/p}_{p,p,\ell}} \mu_{N,M,\varepsilon}(\mathrm{d}X, \mathrm{d}Y) \lesssim \frac{1}{\sigma}C.$$

Notice further that the bound (B.4.6) can be chosen to be uniform with respect to the size M of the torus $\mathbb{T}^2_M$ since the integrals $\int_{\mathbb{T}^2_M} |\hat{\varphi}_0(z-y)|^2 \mathrm{d}y$ and $\int_{\mathbb{T}^2_M} \|\mathscr{P}^M_t(z-y)|(z-y)|^2 \mathrm{d}y$ are uniformly bounded in M. It is worth to note also that by the Besov embedding (Proposition B.5.2) the space $B^{-s'+2/p}_{p,p,\ell}(\mathbb{R}^2)$ is embedded in $B_X = C^{-\delta}_\ell(\mathbb{R}^2)$.

Let us choose the Lyapunov functions in order to get an estimate for Y. We may need $X \mapsto V_1(X,\cdot)$ to be C^1 and $Y \mapsto V_1(\cdot, Y)$ to be C^2.

Let $s > 0$. Referring to the representation of Besov spaces given in Proposition B.5.9 and Remark B.5.10, take $(k - s/2)p > 1$, and consider

$$V_1(Y) = \|Y\|^p_{B^s_{p,p,\ell}} \simeq \|Y\|^p_{L^p_\ell} + \int_0^{+\infty}\int_{\mathbb{R}^2} \frac{t^{kp}|\rho_\ell|^p}{t^{ps/2+1}} |\partial_{t^k} P_t Y(z)|^p \mathrm{d}z\mathrm{d}t =: V_{1,1}(Y) + V_{1,2}(Y). \tag{B.4.7}$$

Theorem B.4.4. *Suppose that Y is a solution to equation (B.4.4). Let $\gamma = \alpha^2/(4\pi) < 2$, and $1 < p$, $q < +\infty$. Take s and r such that*

$$0 < s < \gamma + 2 - \sqrt{8\gamma}, \quad \frac{2+\gamma-s-\sqrt{(s-\gamma-2)^2-8\gamma}}{2\gamma} < r < \frac{2+\gamma-s+\sqrt{(s-\gamma-2)^2-8\gamma}}{2\gamma}, \quad \gamma r < 2. \tag{B.4.8}$$

If $:e^{\alpha X}: \in B^{r,\ell}_{\exp} = B^{-\gamma(r-1)-\delta}_{r,r,\ell}(\mathbb{R}^2)$, for ℓ large enough, then we have, for any $\sigma > 0$,

$$\mathscr{L}_{N,M,\varepsilon} V_1(Y) = \mathscr{L}_{N,M,\varepsilon}(\|Y\|^p_{B^s_{p,p,\ell}}) \lesssim -(1-\sigma)\|Y\|^p_{B^{s+2/p}_{p,p,\ell}} + \frac{1}{\sigma}\|\mathscr{G}_{N,M,\varepsilon}(X,Y)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}. \tag{B.4.9}$$

Proof. In the following proof, we neglect the term $V_{1,1}(Y)$ appearing in (B.4.7), which can be dealt with in a similar way as in the proof of Proposition B.4.2. The gradient of V_1 is given by

$$\begin{aligned} \nabla V_{1,2}(Y)(h) &= \int_0^{+\infty}\int_{\mathbb{R}^2} \frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+1}} \nabla(|\partial_{t^k} P_t Y(z)|^p)(h)\,\mathrm{d}z\mathrm{d}t \\ &= \int_0^{+\infty}\int_{\mathbb{R}^2} \frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+1}} p\,|\partial_{t^k} P_t Y(z)|^{p-1}\,\partial_{t^k} P_t h\,\mathrm{d}z\mathrm{d}t. \end{aligned}$$

Therefore, taking $h = -(-\Delta + m^2)Y - \mathscr{G}_{N,M,\varepsilon}(X,Y)$, we have

$$\begin{aligned} & -\langle(-\Delta+m^2)Y + \mathscr{G}_{N,M,\varepsilon}(X,Y), \nabla V_{1,2}(Y)\rangle \\ =\ & \int_0^{+\infty}\int_{\mathbb{R}^2} \frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+1}} p|\partial_{t^k} P_t Y(z)|^{p-1}\,\partial_{t^k} P_t(-(-\Delta+m^2)Y - \mathscr{G}_{N,M,\varepsilon}(X,Y))\,\mathrm{d}z\mathrm{d}t \\ =\ & \int_0^{+\infty}\int_{\mathbb{R}^2} \frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+1}} \partial_t(|\partial_{t^k} P_t Y|^p)\,\mathrm{d}z\mathrm{d}t \\ & + \int_0^{+\infty}\int_{\mathbb{R}^2} \frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+1}} p|\partial_{t^k} P_t Y(z)|^{p-1}\,\partial_{t^k} P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))\,\mathrm{d}z\mathrm{d}t \\ =:\ & I_1 + I_2. \end{aligned}$$

Let us focus on I_1. Integrating by parts, we get

$$\begin{aligned}
I_1 &= -\int_0^{+\infty}\int_{\mathbb{R}^2}\partial_t\left(\frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+1}}\right)|\partial_{t^k}P_tY|^p\,\mathrm{d}z\mathrm{d}t\\
&= -\int_0^{+\infty}\int_{\mathbb{R}^2}\left(pk-\frac{ps}{2}-1\right)\frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+2}}|\partial_{t^k}P_tY|^p\,\mathrm{d}z\mathrm{d}t\\
&= -\int_0^{+\infty}\int_{\mathbb{R}^2}\left(pk-\frac{ps}{2}-1\right)\frac{t^{pk}|\rho_\ell|^p}{t^{p(s+2/p)/2+1}}|\partial_{t^k}P_tY|^p\,\mathrm{d}z\mathrm{d}t\\
&\simeq -\left(\|Y\|^p_{B^{s+2/p}_{p,p,\ell}}-\|Y\|^p_{L^p_\ell}\right).
\end{aligned}$$

Consider now I_2. Exploiting Hölder inequality with q_1,p_1 and introducing $s'>0$ yields

$$\begin{aligned}
I_2 &= \int_0^{+\infty}\int_{\mathbb{R}^2}\frac{t^{pk}|\rho_\ell|^p}{t^{ps/2+1}}p|\partial_{t^k}P_tY(z)|^{p-1}\,\partial_{t^k}P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))\,\mathrm{d}z\mathrm{d}t\\
&= \int_0^{+\infty}\frac{1}{t}\int_{\mathbb{R}^2}\left[\frac{t^{(p-1)k}|\rho_\ell|^{p-1}}{t^{ps/2+s'/2}}p|\partial_{t^k}P_tY(z)|^{p-1}\right]\\
&\qquad\qquad\times[t^{k+s'/2}|\rho_\ell|\,\partial_{t^k}P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))]\,\mathrm{d}z\mathrm{d}t\\
&\leqslant \left(\int_0^{+\infty}\int_{\mathbb{R}^2}\left|\frac{t^{(p-1)k}|\rho_\ell|^{p-1}}{t^{ps/2+s'/2}}p|\partial_{t^k}P_tY(z)|^{p-1}\right|^{p_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{1/p_1}\\
&\qquad\qquad\times\left(\int_0^{+\infty}\int_{\mathbb{R}^2}|t^{s'/2}(t^k|\rho_\ell|)\,\partial_{t^k}P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))|^{q_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{1/q_1}.
\end{aligned}$$

If we apply Young inequality, we then have, for any $\sigma>0$,

$$\begin{aligned}
I_2 &\leqslant \sigma\left(\int_0^{+\infty}\int_{\mathbb{R}^2}\left|\frac{t^{(p-1)k}|\rho_\ell|^{p-1}}{t^{ps/2+s'/2}}p|\partial_{t^k}P_tY(z)|^{p-1}\right|^{p_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{p_2/p_1}\\
&\quad+\frac{1}{\sigma}\left(\int_0^{+\infty}\int_{\mathbb{R}^2}|t^{s'/2}(t^k|\rho_\ell|)\,\partial_{t^k}P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))|^{q_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{q''/q_1}\\
&= \sigma\left(\int_0^{+\infty}\int_{\mathbb{R}^2}\left(\frac{t^{(p-1)k}|\rho_\ell|^{p-1}}{t^{ps/2+s'/2}}\right)^{p_1}p^{p_1}|\partial_{t^k}P_tY(z)|^{p_1p-p_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{p_2/p_1}\\
&\quad+\frac{1}{\sigma}\left(\int_0^{+\infty}\int_{\mathbb{R}^2}|t^{s'/2}(t^k|\rho_\ell|)\,\partial_{t^k}P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))|^{q_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{q''/q_1}\\
&= \sigma\left(\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{t^{p_1(p-1)k}}{t^{psp_1/2+s'p_1/2}}\left(\frac{t^k}{t^{(ps+s')/(2(p-1))}}\right)^{p_1(p-1)}p^{p_1}\right.\\
&\qquad\qquad\left.\times|\rho_\ell|^{p_1(p-1)}|\partial_{t^k}P_tY(z)|^{p_1(p-1)}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{p_2/p_1}\\
&\quad+\frac{1}{\sigma}\left(\int_0^{+\infty}\int_{\mathbb{R}^2}|t^{k+s'/2}|\rho_\ell|\,\partial_{t^k}P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))|^{q_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{q_2/q_1}\\
&= \sigma\|Y\|^{p_2/p_1}_{B^{(ps+s')/(p-1)}_{p_1(p-1),p_1(p-1),\ell}}+\frac{1}{\sigma}\left(\int_0^{+\infty}\int_{\mathbb{R}^2}|t^{s'/2}(t^k|\rho_\ell|)\,\partial_{t^k}P_t(-\mathscr{G}_{N,M,\varepsilon}(X,Y))|^{q_1}\mathrm{d}z\frac{\mathrm{d}t}{t}\right)^{q_2/q_1}\\
&=: I_{2,1}+I_{2,2}.
\end{aligned}$$

We want to apply Besov embedding (see Proposition B.5.2) in order to reabsorb the term $I_{2,1}$ in I_1: considering the parameters of the involved norms, we get the condition

$$s>\frac{ps+s'}{p-1}-\frac{2}{p_1(p-1)}, \tag{B.4.10}$$

so that the reabsorbing procedure follows by a suitable choice of p_2.

On the other hand, we have by Proposition B.5.4

$$I_{2,2} \lesssim \frac{1}{\sigma}\|\mathscr{G}_{N,M,\varepsilon}(X,Y)\|^{q_2/q_1}_{B^{-s'}_{q_1,q_1,\ell}}.$$

All in all, we have the inequality

$$\mathscr{L}_{N,M,\varepsilon}V_1(Y)=\mathscr{L}_{N,M,\varepsilon}\|Y\|^p_{B^s_{p,p,\ell}}\lesssim-(1-\sigma)\|Y\|^p_{B^{s+2/p}_{p,p,\ell}}+\frac{1}{\sigma}\|\mathscr{G}_{N,M,\varepsilon}(X,Y)\|^{q_2/q_1}_{B^{-s'}_{q_1,q_1,\ell}},$$

provided condition (B.4.10) holds true and p_2 is chosen in an appropriate way.

Now, in order to obtain (B.4.9), we consider $q_1=r$ and $s'>\gamma(r-1)$, and have to check the condition

$$\gamma r^2+r(s-\gamma-2)+2<0,$$

which is satisfied with the choice of parameters given in (B.4.8). □

Remark B.4.5. If $Y\leqslant 0$, we have the bound

$$\|\mathscr{G}_{N,M,\varepsilon}(X,Y)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}\leqslant\|\mathscr{G}_{N,M,\varepsilon}(X,0)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}},$$

on the right-hand side of inequality (B.4.9). In that case, choosing the Lyapunov functions as follows

$$\begin{aligned} V_1(Y) &= \|Y\|^p_{B^s_{p,p,\ell}},\\ V_2(Y) &= C_\sigma\|Y\|^p_{B^{s+2/p}_{p,p,\ell}},\\ V_3(X) &= \frac{1}{\sigma}\|\mathscr{G}_{N,M,\varepsilon}(X,0)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}, \end{aligned}$$

we get that the conditions in Lemma B.4.1 are satisfied thanks to Theorem B.4.4. Therefore, we have

$$\int\|Y\|^p_{B^{s+2/p}_{p,p,\ell}}\mu_{N,M,\varepsilon}(\mathrm{d}X,\mathrm{d}Y)\leqslant\frac{1}{\sigma}\int\|\mathscr{G}_{N,M,\varepsilon}(X,0)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}\mu_{N,M,\varepsilon}(\mathrm{d}X,\mathrm{d}Y).$$

B.4.2 Measure of the approximating problem

Equations (B.4.3) and (B.4.4) induce a Markov process on $B_X\times B_Y^{\leqslant}$. This is due to the fact that, if we start from a negative initial condition for Y, then Y remains negative throughout all the evolution.

Proposition B.4.6. *The operator $(\mathscr{L}_{N,M,\varepsilon},\mathscr{F})$ is the restriction of the generator of the Markov process associated to equations (B.4.3) and (B.4.4) to the space of functions $\mathscr{F}$.*

Proof. The proof is based on the fact that we can apply Itô formula to the functions in $\mathscr{F}$. Exploiting Itô formula, the proof follows the argument of the proof of point *ii.* in Proposition B.3.5. □

By linearity, the invariant measure for equation (B.4.3) is given by the free field measure with mass m on the torus $\mathbb{T}^2_M$, that we can identify as usual with the periodic free field of mass m on the whole space $\mathbb{R}^2$.

We are interested in studying (infinitesimally) invariant measures for the aforementioned pair of equations, i.e. equations (B.4.3) and (B.4.4). Indeed, by Proposition B.4.6, an invariant measure for those two equations is a solution to a FPK equation associated with $\mathscr{L}_{N,M,\varepsilon}$ in the sense of Problem B. Let us start off by giving an argument for the existence of an infinitesimally invariant measure (for its definition and the relation with invariant measures see Chapter 5 in [37]) by means of the Lyapunov functions introduced in Section B.4.1.

Proposition B.4.7. *Equations (B.4.3) and (B.4.4) admit an infinitesimally invariant measure.*

Proof. The proof follows the same argument as Lemma 3.3 in [5]. We start equations (B.4.3) and (B.4.4) from deterministic initial conditions $(x_0, y_0) \in B_X \times B_Y^{\leqslant}$, and let $\mu_{(x_0,y_0),t}$ denote the measure of the solution of the equations starting at (x_0, y_0). Let

$$\tilde{\mu}_T = T^{-1} \int_0^T \mu_{(x_0,y_0),t} \, \mathrm{d}t.$$

If such a measure is tight with respect to T, and $\tilde{\mu}$ is its weak limit as $T \to +\infty$, then $\tilde{\mu}$ is an infinitesimal invariant measure of the equation. We consider the following Lyapunov functions, consisting of the sum of the one considered in Section B.4.1 where we dealt with X and Y separately (see Remarks B.4.3 and B.4.5), namely

$$V_1(X,Y) = \|X\|^p_{B^{-s'}_{p,p,\ell}} + \|Y\|^p_{B^{s}_{p,p,\ell}}, \tag{B.4.11}$$

$$V_2(X,Y) = (1-\sigma)\|X\|^p_{B^{-s'+2/p}_{p,p,\ell}} + C_\sigma \|Y\|^p_{B^{s+2/p}_{p,p,\ell}}, \tag{B.4.12}$$

$$V_3(X) = \frac{1}{\sigma}\big(C + \|\mathscr{G}_{N,M,\varepsilon}(X,0)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}\big). \tag{B.4.13}$$

Then, by Proposition B.4.2 and Theorem B.4.4, we have

$$\begin{aligned}
\mathbb{E}[V_1(X_t,Y_t) - V_1(x_0,y_0)] &= \mathbb{E}\int_0^t \mathscr{L}_{N,M,\varepsilon} V_1(X_\tau, Y_\tau) \, \mathrm{d}\tau \\
&\leqslant \int_0^t \mathbb{E}[-V_2(X_\tau, Y_\tau) + V_3(X_\tau)] \, \mathrm{d}\tau.
\end{aligned}$$

Recall that $X_\tau = e^{-(-\Delta+m^2)\tau} x_0 + \int_0^\tau e^{-(-\Delta+m^2)(\tau-s)} \xi_s \, \mathrm{d}s$, then $\mathbb{E}[\|\mathscr{G}_{N,M,\varepsilon}(X_\tau,0)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}]$ converges exponentially to a constant, and hence

$$K = \sup_{t \in [0,+\infty)} \frac{1}{t} \int_0^t \mathbb{E}\Big[\|\mathscr{G}_{N,M,\varepsilon}(X_\tau,0)\|^{(pr-r+1)/(pr^2)}_{B^{r,\ell}_{\exp}}\Big] \mathrm{d}\tau < +\infty.$$

Therefore, we have, for any $t > 0$,

$$\frac{1}{t}\mathbb{E}[V_1(X_t,Y_t) - V_1(x_0,y_0)] + \frac{1}{t}\int_0^t \mathbb{E}[V_2(X_\tau, Y_\tau)] \, \mathrm{d}\tau \lesssim K,$$

which yields

$$\frac{1}{t}\int_0^t \mathbb{E}[V_2(X_\tau,Y_\tau)] \, \mathrm{d}\tau \lesssim K + \frac{1}{t}\mathbb{E}[V_1(x_0,y_0)].$$

Taking the supremum over $t \in [0,+\infty)$, we have

$$\sup_{t\in[0,+\infty)} \frac{1}{t}\int_0^t \mathbb{E}[V_2(X_\tau,Y_\tau)] \, \mathrm{d}\tau < +\infty.$$

On the other hand, for any $t > 0$,

$$\int V_2(X,Y) \, \tilde{\mu}_t(\mathrm{d}X, \mathrm{d}Y) = \frac{1}{t}\int_0^t \mathbb{E}[V_2(X_\tau,Y_\tau)] \, \mathrm{d}s.$$

Since $V_2(X,Y)$ has compact sub-levels on $B_X \times B_Y^{\leqslant}$, tightness of $\tilde{\mu}_t$ is implied. □

We can actually prove a stronger result on the form of one invariant measure for equations (B.4.3) and (B.4.4). Indeed, if we start from $x_0 = 0$ and $y_0 = 0$ in the proof of Proposition B.4.7, then we have that the invariant measure that has been built there must be the measure of the process solving the equation

$$X_t = \int_{-\infty}^{t} P_{t-s}\xi_s \,\mathrm{d}s, \qquad Y_t = \int_{-\infty}^{t} P_{t-s}\mathscr{G}_{N,M,\varepsilon}(X_s, Y_s)\,\mathrm{d}s. \tag{B.4.14}$$

It is possible, in fact, to prove that the previous integral equations admit a unique solution.

Proposition B.4.8. *Equation (B.4.14) admits a unique solution in $B_X \times \mathfrak{H}_{N,M} \subset B_X \times B_Y$. Moreover, for every $t \in \mathbb{R}$, the law of (X_t, Y_t) is a solution to the FPK equation for $\mathscr{L}_{N,M,\varepsilon}$ in the sense of Problem B.*

Proof. Existence and uniqueness for equation (B.4.14) can be proved in the same way as in the proof of Theorem B.9.1.

In order to prove the second part of the result, we exploit the fact that

$$\mathrm{law}(X_t, Y_t) = \lim_{t \to +\infty} \mu_{((0,0),t)} = \lim_{t \to +\infty} \frac{1}{t}\int_0^t \mu_{((0,0),s)}\,\mathrm{d}s.$$

Indeed, if $X_t^{0,-T}$ and $Y_t^{0,-T}$ are the processes solution to equations (B.4.3) and (B.4.4) starting at time $-T$ with initial condition zero, then we have that

$$\mathrm{law}(X_t^{0,-T}, Y_t^{0,-T}) = \mu_{((0,0),t)},$$

and that $(X_t^{0,-T}, Y_t^{0,-T}) \to (X_t, Y_t)$ as $T \to +\infty$, almost surely. Therefore, by the proof of Proposition B.4.7, the law of (X_t, Y_t) is an invariant measure for equations (B.4.3) and (B.4.4), and thus a solution to the FPK equation for $\mathscr{L}_{N,M,\varepsilon}$. □

Hereafter, we denote by $\mu_{N,M,\varepsilon}$ the law at fixed time t of the process (X_t, Y_t) solution to equation (B.4.14), which is a solution to the FPK equation for $\mathscr{L}_{N,M,\varepsilon}$. Thanks to the representation given by Proposition B.4.8, we are able to establish some more precise properties of the measure $\mu_{N,M,\varepsilon}$.

Proposition B.4.9. *For every $M, N \in \mathbb{N}$, and every $\varepsilon > 0$, the measure $\mu_{N,M,\varepsilon}$ satisfies the following properties*

i. $\mathrm{supp}(\mu_{N,M,\varepsilon}) \subset B^2_{2,2,\ell}(\mathbb{T}^2_M) \times \mathfrak{H}_{N,M}$,

ii. For every $F \in \mathscr{F}$, we have

$$\int \mathscr{L}_{N,M,\varepsilon} F \,\mathrm{d}\mu_{N,M,\varepsilon} = 0,$$

iii. For every $F, G \in \mathfrak{F}$, we have

$$\int \mathscr{L}_{N,M,\varepsilon} F(X+Y)\, G(X+Y)\, \mu_{N,M,\varepsilon}(\mathrm{d}X, \mathrm{d}Y) = \int F(X+Y) \mathscr{L}_{N,M,\varepsilon} G(X+Y)\, \mu_{N,M,\varepsilon}(\mathrm{d}X, \mathrm{d}Y).$$

Proof. Property *i.* follows from the fact that the solution Y_t to the second equation in (B.4.14) is supported on the image of the projection of the operator $Q_{N,M}$, which is exactly $\mathfrak{H}_{N,M}$.

Property *ii.* is due to the fact that $\mu_{N,M,\varepsilon}$ solves the FPK equation for $\mathscr{L}_{N,M,\varepsilon}$. Indeed, the system (B.4.14) can be split in an infinite-dimensional linear equation and an independent finite-dimensional non-linear equation. The statement then follows from the results in Section 2.3 in [52] for the infinite-dimensional part and Theorem 5.2.2 in [37] for the non-linear finite-dimensional part.

As far as point *iii.* is concerned, letting $Z_t = X_t + Y_t$ and calling Z_t^1 the projection of Z_t on $\mathfrak{H}_{N,M}$ and Z_t^2 the projection on $\mathfrak{H}_{N,M}^{\perp}$, we have that Z_t^1 and Z_t^2 solve two independent equations: the equation for Z_t^1 is a non-linear finite-dimensional equation, since $\mathfrak{H}_{N,M}$ is a finite-dimensional Hilbert space a linear equation, while the equation for Z_t^2 is a linear equation in infinite dimensions. More precisely, we have

$$\begin{aligned}\partial_t Z_t^1 &= (\Delta - m^2) Z_t^1 + \frac{\delta V_{N,M,\varepsilon}}{\delta Z}(Z_t^1) + \xi_t^1,\\ \partial_t Z_t^2 &= (\Delta - m^2) Z_t^2 + \xi_t^2,\end{aligned}$$

where δ is the functional derivative of

$$V_{N,M,\varepsilon}(Z) = \int_{\mathbb{T}_M^2} \exp\Big(\alpha Q_{N,M} Z(z) - \frac{\alpha^2}{2} c_{N,M,\varepsilon}\Big) \mathrm{d}z,$$

with $c_{N,M,\varepsilon}$ is the constant appearing in the definition (B.4.2) of the Wick product appearing in the functional $\mathscr{G}_{N,M,\varepsilon}$, and ξ_t^1 and ξ_t^2 are the projection of ξ on the spaces $\mathfrak{H}_{N,M}$ and $\mathfrak{H}_{N,M}^{\perp}$, respectively.

Since the equation for Z_t^1 is a finite-dimensional equation with drift given by the gradient of a function and the equation for Z_t^2 is linear with self-adjoint drift, then the unique invariant measure of the process (Z^1, Z^2) satisfies property *iii.*, and hence also their sum $Z^1 + Z^2 = X + Y$ does. For further details on the relation between symmetric processes and the integration by parts formula see e.g. [19, 34, 76]. □

Remark B.4.10. From the proof of the previous result it is evident that the system consists of two independent equations: an infinite-dimensional linear one and a finite-dimensional non-linear one. This means that $\mu_{M,N,\varepsilon}$ is the unique invariant measure of the system.

B.4.3 Tightness of the measure

We prove tightness of the measures $\mu_{N,M,\varepsilon}$.

Theorem B.4.11. *The family of measures* $(\mu_{N,M,\varepsilon})_{N,M,\varepsilon}$ *is tight in* $B_X \times B_Y^{\leqslant}$.

Proof. Let V_1, V_2, and V_3 be the Lyapunov functions defined in equations (B.4.11), (B.4.12), and (B.4.13), respectively. Since the measures $\mu_{N,M,\varepsilon}$ are solutions to the FPK equation associated with $\mathscr{L}_{N,M,\varepsilon}$ in the sense of Problem B and $V_1 \in \mathscr{F}$, then

$$\int \mathscr{L}_{N,M,\varepsilon} V_1(X,Y)\, \mu_{N,M,\varepsilon}(\mathrm{d}X, \mathrm{d}Y) = 0.$$

Moreover, we have by Theorem B.4.2 together with Remark B.4.3 and by Theorem B.4.4 together with Remark B.4.5, that V_1 is a Lyapunov function for (X, Y), where X and Y solves equation (B.4.3) and (B.4.4), respectively. Thus, by Lemma B.4.1, we have

$$\int V_2(X,Y)\, \mu_{N,M,\varepsilon}(\mathrm{d}X,\mathrm{d}Y) \lesssim \int V_3(X)\, \mu_{N,M,\varepsilon}(\mathrm{d}X,\mathrm{d}Y) = \int V_3(X)\, \mu_M^{\mathrm{free}}(\mathrm{d}X).$$

Since $\sup_{M\in\mathbb{N}} \int V_3(X)\, \mu_M^{\mathrm{free}}(\mathrm{d}X) < +\infty$, and since V_2 has compact sub-levels, the thesis follows. □

We get the following consequence.

Corollary B.4.12. *There exist three sequences* $(N_k)_{k\in\mathbb{N}}\subset\mathbb{N}$, $(M_l)_{l\in\mathbb{N}}\subset\mathbb{N}$, *and* $(\varepsilon_j)_j\subset\mathbb{R}_+$ *such that* $N_k\to+\infty$, $M_l\to+\infty$, *and* $\varepsilon_j\to 0$ *as* $k,l,j\to+\infty$, *respectively, and some probability measures* μ_{M_l,ε_j}, μ_{ε_j}, *and* μ *such that, for any* M_l *and any* ε_j, *we have*

$$\lim_{k\to+\infty}\mu_{N_k,M_l,\varepsilon_j}=\mu_{M_l,\varepsilon_j},$$

for any ε_j *we have*

$$\lim_{l\to+\infty}\mu_{M_l,\varepsilon_j}=\mu_{\varepsilon_j},$$

and finally

$$\lim_{j\to+\infty}\mu_{\varepsilon_j}=\mu.$$

Proof. The result follows from Theorem B.4.11 and a diagonalization argument. □

B.4.4 Limit of the operator

We want now to prove that any limit measure μ built in Corollary B.4.12 solves the FPK equation in the sense of Problem B. We first prove that any measure appearing in Corollary B.4.12 solves the FPK equation associated with the corresponding operator. For simplicity, let us drop the dependence on the parameters k, l, and j introduced in Corollary B.4.12.

The aim of the section is to prove the following result.

Theorem B.4.13. *Let* $\Phi\in\mathscr{F}$. *We have the limit as* $N,M\to+\infty$ *and* $\varepsilon\to 0$,

$$\int\mathscr{L}_{N,M,\varepsilon}\Phi\,\mathrm{d}\mu_{N,M,\varepsilon}\to\int\mathscr{L}\Phi\,\mathrm{d}\mu. \tag{B.4.15}$$

In order to prove the limit (B.4.15), we proceed by first taking $N\to+\infty$, then $M\to+\infty$, and eventually $\varepsilon\to 0$, showing the convergence step by step.

B.4.4.1 Limit as $N\to+\infty$

As mentioned above, we start off by taking $N\to+\infty$, namely we want to show that

$$\int\mathscr{L}_{N,M,\varepsilon}(\Phi)\,\mathrm{d}\mu_{N,M,\varepsilon}\to\int\mathscr{L}_{M,\varepsilon}(\Phi)\,\mathrm{d}\mu_{M,\varepsilon},\qquad\text{as } N\to+\infty.$$

The term in the operator $\mathscr{L}_{M,N,\varepsilon}$ involving X only is independent of N, in particular we do not need to consider the trace. Instead, we need to work on the terms involving only the derivatives with respect to Y. Since, by Corollary B.4.12, $\mu_{N,M,\varepsilon}$ is tight and converges weakly up to subsequences to some limit $\mu_{M,\varepsilon}$, we have to show that

$$\int[\mathscr{L}_{N,M,\varepsilon}(\Phi(X,Y))-\mathscr{L}_{M,\varepsilon}(\Phi(X,Y))]\,\mu_{N,M,\varepsilon}(\mathrm{d}X,\mathrm{d}Y)\to 0,\qquad\text{as } N\to+\infty.$$

Let us compute the integrand and rearrange its terms. We have

$$\begin{aligned}&\mathscr{L}_{N,M,\varepsilon}(\Phi(X,Y))-\mathscr{L}_{M,\varepsilon}(\Phi(X,Y))\\ =\;&\alpha\langle Q_{N,M}(g_\varepsilon*(:e^{\alpha Q_{N,M}(g_\varepsilon*X)}\colon e^{\alpha Q_{N,M}(g_\varepsilon*Y)})),\nabla_Y\Phi\rangle-\alpha\langle g_\varepsilon*(:e^{\alpha(g_\varepsilon*X)}\colon e^{\alpha(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle\\ =\;&\mathbb{I}+\mathbb{II}+\mathbb{III}\end{aligned}$$

with

$$\begin{aligned}\mathbb{I}\;:=\;&\alpha\langle Q_{N,M}(g_\varepsilon*(:e^{\alpha Q_{N,M}(g_\varepsilon*X)}\colon e^{\alpha Q_{N,M}(g_\varepsilon*Y)})),\nabla_Y\Phi\rangle\\ &-\alpha\langle g_\varepsilon*(:e^{\alpha Q_{N,M}(g_\varepsilon*X)}\colon e^{\alpha Q_{N,M}(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle\\ =\;&\langle g_\varepsilon*(:e^{\alpha Q_{N,M}(g_\varepsilon*X)}\colon e^{\alpha Q_{N,M}(g_\varepsilon*Y)}),(Q_{N,M}-I)\,\mathrm{Per}_{\mathbb{T}^2_M}\nabla_Y\Phi\rangle\end{aligned}\tag{B.4.16}$$

$$\mathbb{II}:=\alpha\langle g_\varepsilon*(:e^{\alpha Q_{N,M}(g_\varepsilon*X)}\colon e^{\alpha Q_{N,M}(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle-\alpha\langle g_\varepsilon*(:e^{\alpha(g_\varepsilon*X)}\colon e^{\alpha Q_{N,M}(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle\tag{B.4.17}$$

and

$$\mathbb{III} := \alpha\langle g_\varepsilon * (:e^{\alpha(g_\varepsilon * X)}: e^{\alpha Q_{N,M}(g_\varepsilon * Y)}), \nabla_Y \Phi\rangle - \alpha\langle g_\varepsilon * (:e^{\alpha(g_\varepsilon * X)}: e^{\alpha(g_\varepsilon * Y)}), \nabla_Y \Phi\rangle \tag{B.4.18}$$

Let us deal with the term $\mathbb{I}$ (B.4.16). We need, for $p > 1$, the bound

$$\int \|:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: e^{\alpha Q_{N,M}(g_\varepsilon * Y)}\|_{L^\infty}^p \mathrm{d}\mu_{N,M,\varepsilon} < C_{M,\varepsilon}. \tag{B.4.19}$$

Indeed, if estimate (B.4.19) holds, then, by the regularization property of g_ε – namely the continuity for any $s > 0$ of the operator $g_\varepsilon * (\cdot): L^\infty(\mathbb{T}_M^2) \to B_{\infty,\infty}^s(\mathbb{T}_M^2)$ – and exploiting the fact that the norm $\|Q_{N,M} - I\|_{L(B_{\infty,\infty}^s, B_{\infty,\infty}^{s-\delta})}$ converges to zero as $N \to +\infty$ for any $s > 0$ and $\delta > 0$, we get the convergence of the term $\mathbb{I}$ to 0 as $N \to +\infty$. Let us prove the bound (B.4.19). The exponential involving Y disappears, since $Y \leqslant 0$. Moreover, since $\varepsilon > 0$, we have that the Wick exponential can be written as the exponential divided by some positive constant $C_{N,M,\varepsilon}$, i.e.

$$:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: = C_{N,M,\varepsilon}^{-1} e^{\alpha Q_{N,M}(g_\varepsilon * X)},$$

which converges to some finite number as $N \to +\infty$. Then, we have the inequality

$$\|:e^{\alpha Q_{N,M}(g_\varepsilon * X)}:\|_{L^\infty} \leqslant C_{N,M,\varepsilon}^{-1} e^{\|\alpha Q_{N,M}(g_\varepsilon * X)\|_{L^\infty}},$$

and hence, by positivity of $Q_{N,M}$, we obtain

$$\|:e^{\alpha Q_{N,M}(g_\varepsilon * X)}:\|_{L^\infty} \leqslant C_{N,M,\varepsilon}^{-1} e^{\|\alpha(g_\varepsilon * X)\|_{L^\infty}},$$

which is in $L^p(\mu_M^{\text{free}})$ by Fernique's theorem (see Theorem 2.8.5 in [33]).

We deal with the term $\mathbb{II}$ given by (B.4.17). We have

$$\mathbb{II} \leqslant \alpha\|g_\varepsilon\|_{L^1}\|:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: - :e^{\alpha(g_\varepsilon * X)}:\|_{L^p}\| e^{\alpha Q_{N,M}(g_\varepsilon * Y)}\|_{L^\infty}\|\mathrm{Per}_{\mathbb{T}_M^2} \nabla_Y \Phi\|_{L^q},$$

and, by stochastic estimates (see Proposition B.6.2), we have convergence of the mean of the term $\mathbb{II}$ to zero.

For the term $\mathbb{III}$ given by (B.4.18) we exploit the non-positivity of Y and replace the exponential by a bounded smooth function with bounded derivatives. Then we have that

$$e^{\alpha Q_{N,M}(g_\varepsilon * Y)} \to e^{\alpha(g_\varepsilon * Y)}, \qquad \text{in } L^\infty.$$

provided

$$Q_{N,M}(g_\varepsilon * Y) \to g_\varepsilon * Y, \qquad \text{in } L^\infty.$$

On the other hand, we have the bound

$$\|Q_{N,M}(g_\varepsilon * Y) - g_\varepsilon * Y\|_{L^\infty} \leqslant \|Q_{N,M} - I\|_{L(C^\delta, L^\infty)}\|g_\varepsilon * Y\|_{C^\delta}.$$

But $\|Q_{N,M} - I\|_{L(C^\delta, L^\infty)} \to 0$ depending only on M, while $\|g_\varepsilon * Y\|_{C^\delta} \in L^p$ uniformly since by the proof of Theorem B.4.11 we have that the integral $\int V_2(X, Y)\, \mu_{N,M,\varepsilon}(\mathrm{d}X, \mathrm{d}Y)$ is bounded uniformly with respect to M.

B.4.4.2 Limit as $M \to +\infty$

We now have to take $M \to +\infty$, namely we want to show that

$$\int \mathscr{L}_{M,\varepsilon}(\Phi)\, \mathrm{d}\mu_{M,\varepsilon} \to \int \mathscr{L}_\varepsilon(\Phi)\, \mathrm{d}\mu_\varepsilon, \qquad \text{as } M \to +\infty. \tag{B.4.20}$$

Notice that the only M-dependent term is now the trace-term $\mathrm{tr}_{L^2(\mathbb{T}_M^2)}(P_{\mathbb{T}_M^2} \nabla_X^2 \Phi P_{\mathbb{T}_M^2})$. By the same considerations as above, we have to show

$$\int [\mathscr{L}_{M,\varepsilon}(\Phi(X, Y)) - \mathscr{L}_\varepsilon(\Phi(X, Y))]\, \mu_{M,\varepsilon}(\mathrm{d}X, \mathrm{d}Y) \to 0, \qquad \text{as } M \to +\infty.$$

Since $\Phi\in\mathscr{F}$ (see Definition B.3.1 and Remark B.3.2), we have

$$\mathrm{tr}_{L^2}(|\nabla_X^2\Phi|)<f_\Phi(X),\qquad \text{and}\qquad \mathrm{tr}_{L^2}(|\rho_{-\ell}\nabla_X^2\Phi|)<f_\Phi(X),\quad \ell>1,$$

where $\int f_\Phi\,\mathrm{d}\mu^{\mathrm{free}}<+\infty$. First, note that

$$\begin{aligned}\mathrm{tr}_{L^2}(P_M\nabla_X^2\Phi P_M)-\mathrm{tr}_{L^2}(\nabla_X^2\Phi) &= \mathrm{tr}_{L^2}((P_M-I)\nabla_X^2\Phi P_M)+\mathrm{tr}_{L^2}(\nabla_X^2\Phi(P_M-I))\\ &= \mathrm{tr}_{L^2}((P_M-I)\nabla_X^2\Phi P_M)+\mathrm{tr}_{L^2}(\nabla_X^2\Phi\rho_\ell\rho_{-\ell}(P_M-I))\\ &= \mathrm{tr}_{L^2}(\nabla_X^2\Phi\rho_\ell\rho_{-\ell}(P_M-I)).\end{aligned}$$

Therefore, we have the bound

$$|\mathrm{tr}_{L^2}(P_M\nabla_X^2\Phi P_M)-\mathrm{tr}_{L^2}(\nabla_X^2\Phi)|\leqslant\mathrm{tr}_{L^2}(|\nabla_X^2\Phi\rho_{-\ell}|)\,\|\rho_\ell(P_M-I)\|_{L(L^2(\mathbb{R}^2),L^2(\mathbb{R}^2))}.$$

Now, let $h\in L^2(\mathbb{R}^2)$, then we have

$$(\rho_\ell(P_M-I))(h)=\rho_\ell\mathbb{1}_{\mathbb{R}^2\setminus\mathbb{T}_M^2}(\mathrm{Per}_{\mathbb{T}_M^2}h-h).$$

Take $\ell>\ell'>1$, then, considering the operator L^2 norm, we have

$$\|\rho_\ell(P_M-I)(h)\|_{L^2}^2\leqslant 2\sum_{y\in 2\pi\mathbb{Z}^2\setminus\{0\}}\int_{\mathbb{T}_M^2}\rho_\ell(z-y)^2h(z)^2\,\mathrm{d}z+2\int_{\mathbb{R}^2\setminus\mathbb{T}_M^2}\rho_\ell(z)^2h(z)^2\,\mathrm{d}z.$$

Since $\rho_\ell(z-y)=\rho_{\ell'}(z-y)\rho_{\ell-\ell'}(z-y)\leqslant\rho_{\ell'}(z-y)(1+M)^{-(\ell-\ell')}$, we get

$$\begin{aligned}&\|\rho_\ell(P_M-I)(h)\|_{L^2}^2\\ \lesssim\;& 2(1+M)^{-2(\ell-\ell')}\sum_{y\in 2\pi\mathbb{Z}^2\setminus\{0\}}\int_{\mathbb{T}_M^2}\rho_{\ell'}(z-y)^2h(z)^2\,\mathrm{d}z+2(1+M)^{-2\ell}\int_{\mathbb{R}^2\setminus\mathbb{T}_M^2}h(z)^2\,\mathrm{d}z.\end{aligned}$$

Now note

$$\sum_{y\in 2\pi\mathbb{Z}^2\setminus\{0\}}\int_{\mathbb{T}_M^2}\rho_{\ell'}(z-y)^2h(z)^2\,\mathrm{d}z\lesssim\int_{\mathbb{T}_M^2}(\mathrm{Per}_{\mathbb{T}_M^2}(\rho_{\ell'})(z))^2h(z)^2\,\mathrm{d}z,$$

and $\mathrm{Per}_{\mathbb{T}_M^2}(\rho_{\ell'})(z)\leqslant C_{\ell'}:=\sum_{y\in\mathbb{Z}^2}\rho_{\ell'}(y)$, to get

$$\begin{aligned}\|\rho_\ell(P_M-I)(h)\|_{L^2}^2 &\lesssim 2(1+M)^{-2(\ell-\ell')}C_{\ell'}\int_{\mathbb{T}_M^2}h(z)^2\,\mathrm{d}z+2(1+M)^{-2\ell}\int_{\mathbb{R}^2\setminus\mathbb{T}_M^2}h(z)^2\,\mathrm{d}z\\ &\lesssim 2(1+M)^{-2(\ell-\ell')}(C_{\ell'}+1)\int_{\mathbb{T}_M^2}h(z)^2\,\mathrm{d}z.\end{aligned}$$

By taking the supremum over all h with $\|h\|_{L^2(\mathbb{R}^2)}\leqslant 1$, we then obtain

$$\|\rho_\ell(P_M-I)\|_{L(L^2(\mathbb{R}^2),L^2(\mathbb{R}^2))}\lesssim(1+M)^{-2(\ell-\ell')}.$$

Letting $M\to+\infty$, we get the limit (B.4.20).

B.4.4.3 Limit as $\varepsilon\to 0$

Now, we are left to show the convergence

$$\int\mathscr{L}_\varepsilon(\Phi)\,\mathrm{d}\mu_\varepsilon\to\int\mathscr{L}(\Phi)\,\mathrm{d}\mu,\qquad \text{as }\varepsilon\to 0.$$

As above, we have to show that

$$\int[\mathscr{L}_\varepsilon(\Phi(X,Y))-\mathscr{L}(\Phi(X,Y))]\,\mu_\varepsilon(\mathrm{d}X,\mathrm{d}Y)\to 0,\qquad \text{as }\varepsilon\to 0.$$

Rewriting the integrand, we get

$$\begin{aligned}\mathscr{L}_\varepsilon(\Phi(X,Y))-\mathscr{L}(\Phi(X,Y)) &= \alpha\langle(g_\varepsilon*(:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)})),\nabla_Y\Phi\rangle-\alpha\langle:e^{\alpha X}:e^{\alpha Y},\nabla_Y\Phi\rangle\\ &= \mathbb{I}'+\mathbb{II}'+\mathbb{III}'\end{aligned}\tag{B.4.21}$$

with

$$\mathbb{I}' := \alpha\langle(g_\varepsilon*(:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)})),\nabla_Y\Phi\rangle-\alpha\langle(:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle\tag{B.4.22}$$

$$\mathbb{II}' := \alpha\langle(:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle-\alpha\langle(:e^{\alpha X}:e^{\alpha(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle\tag{B.4.23}$$

$$\mathbb{III}' := \alpha\langle:e^{\alpha X}:e^{\alpha(g_\varepsilon*Y)}),\nabla_Y\Phi\rangle-\alpha\langle(:e^{\alpha X}:e^{\alpha Y}),\nabla_Y\Phi\rangle\tag{B.4.24}$$

The stochastic estimates for $:\exp(\alpha(g_\varepsilon*X)):$ are done in Proposition B.6.1. We now deal with the term $\mathbb{I}'$ (B.4.22). We have the inequality

$$\begin{aligned}\mathbb{I}' &= \alpha\langle:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)},(g_\varepsilon-I)*\nabla_Y\Phi\rangle\\ &\leqslant \alpha\|:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)}\|_{B^s_{p,p,\ell}}\|(g_\varepsilon-I)*\nabla_Y\Phi\|_{B^{-s}_{q,q,-\ell}}\\ &\lesssim \alpha\|:e^{\alpha(g_\varepsilon*X)}:\|_{B^s_{p,p,\ell}}\|e^{\alpha(g_\varepsilon*Y)}\|_{L^\infty}\|g_\varepsilon-I\|_{L(B^{-s+\delta}_{q,q,-\ell},B^{-s}_{q,q,-\ell})}\|\nabla_Y\Phi\|_{B^{-s+\delta}_{q,q,-\ell}}.\end{aligned}$$

Taking the expectation and exploiting the negativity of Y, we have

$$\begin{aligned}&\mathbb{E}[\alpha\langle(g_\varepsilon*(:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)}))-:e^{\alpha(g_\varepsilon*X)}:e^{\alpha(g_\varepsilon*Y)},\nabla_Y\Phi\rangle]\\ \lesssim\;& \mathbb{E}\left[\|:e^{\alpha(g_\varepsilon*X)}:\|_{B^s_{p,p,\ell}}\right]\|g_\varepsilon-I\|_{L(B^{-s+\delta}_{q,q,-\ell},B^{-s}_{q,q,-\ell})}\|\nabla_Y\Phi\|_{L^\infty(B_X\times B_Y,B^{-s+\delta}_{q,q,-\ell})}\\ \lesssim\;& \|(g_\varepsilon-1)\|_{L(B^{-s+\delta}_{q,q,-\ell},B^{-s}_{q,q,-\ell})},\end{aligned}$$

and this last term converges to zero as $\varepsilon\to0$. The convergence to zero of the term $\mathbb{II}'$ (B.4.23) follows from Proposition B.6.1. Finally for the term $\mathbb{III}'$ (B.4.24), we proceed as follows. We have the bound

$$\mathbb{III}'\lesssim\|:e^{\alpha X}:\|_{B^s_{p,p,\ell}}\left(\left(\|Y\|_{B^{-s+\delta}_{q,q,\ell}}\|g_\varepsilon-I\|_{L(B^{-s+\delta}_{q,q,-\ell},B^{-s}_{q,q,-\ell})}\right)\wedge2\right)\|\nabla_Y\Phi\|_{B^{-s+\delta}_{q,q,-\ell}}.\tag{B.4.25}$$

Since μ_ε is tight and $\|:\exp(\alpha X):\|_{B^s_{p,p,\ell}}$ is uniformly integrable with respect to the measure μ_ε, then, for any $\tau>0$, there exists a Borel subset $\Omega_\tau\subset B_X\times B_Y$ such that

$$\int_{\Omega^c_\tau}\|:e^{\alpha X}:\|_{B^s_{p,p,\ell}}\mathrm{d}\mu_\varepsilon<\tau,$$

and $R_{\Omega_\tau}=\sup_{\Omega_\tau}\|Y\|_{B^{-s+\delta}_{q,q,\ell}}<+\infty$. Therefore, we have

$$\begin{aligned}\int|\mathbb{III}'|\mathrm{d}\mu_\varepsilon &\leqslant \int_{\Omega_\tau}|\alpha\langle:e^{\alpha X}:(e^{\alpha(g_\varepsilon*Y)}-e^{\alpha Y}),\nabla_Y\Phi\rangle|\mathrm{d}\mu_\varepsilon+\int_{\Omega^c_\tau}|\alpha\langle:e^{\alpha X}:(e^{\alpha(g_\varepsilon*Y)}-e^{\alpha Y}),\nabla_Y\Phi\rangle|\mathrm{d}\mu_\varepsilon\\ &\leqslant R_{\Omega_\tau}\|\nabla_Y\Phi\|_{B^{-s+\delta}_{q,q,-\ell}}\|g_\varepsilon-I\|_{L(B^{-s+\delta}_{q,q,-\ell},B^{-s}_{q,q,-\ell})}\int_\Omega\|:e^{\alpha X}:\|_{B^s_{p,p,\ell}}\mathrm{d}\mu+2\tau\|\nabla_Y\Phi\|_{B^{-s+\delta}_{q,q,-\ell}}.\end{aligned}$$

Thus,

$$\lim_{\varepsilon\to0}\int|\mathbb{III}'|\mathrm{d}\mu_\varepsilon\leqslant2\|\nabla_Y\Phi\|_{B^{-s+\delta}_{q,q,-\ell}}\tau,$$

which gives convergence to zero by arbitrary choice of δ. In order to show inequality (B.4.25), note that

$$\begin{aligned}|\mathbb{III}'| &\lesssim \alpha\,\|:\exp(\alpha X):(\exp(\alpha(g_\varepsilon*Y))-\exp(\alpha Y))\|_{B^{s-\delta}_{l',l',2\ell}}\|\nabla_Y\Phi\|_{B^{-s+\delta}_{l,l,-\ell}}\\ &\lesssim \alpha\,\|:\exp(\alpha X):(\exp(\alpha(g_\varepsilon*Y))-\exp(\alpha Y))\|_{B^s_{1,1,2\ell}}\|\nabla_Y\Phi\|_{B^{-s+\delta}_{l,l,-\ell}}\\ &\lesssim \alpha\left[(\|:\exp(\alpha X):\|_{B^s_{p,p,\ell}}\|(g_\varepsilon*Y)-Y\|_{B^{-s}_{q,q,\ell}})\wedge(2\|:\exp(\alpha X):\|_{B^s_{1,1,\ell}})\right]\|\nabla_Y\Phi\|_{B^{-s+\delta}_{l,l,-\ell}}\\ &\lesssim (\|:\exp(\alpha X):\|_{B^s_{p,p,\ell}}+\|:\exp(\alpha X):\|_{B^s_{1,1,\ell}})(\|(g_\varepsilon*Y)-Y\|_{B^{-s}_{q,q,\ell}}\wedge2)\|\nabla_Y\Phi\|_{B^{-s+\delta}_{l,l,-\ell}},\end{aligned}$$

where $l \in (1,+\infty)$ and $1/l' + 1/l = 1$ such that $\delta > 2 - 2/l'$. Now, if $|s|$ is small enough such that $B^{-s+\delta}_{q,q,\ell} \subset B_Y$, then we have the

$$(\|(g_\varepsilon * Y) - Y\|_{B^{-s}_{q,q,\ell}} \wedge 2) \leqslant \left(\|Y\|_{B^{-s+\delta}_{q,q,\ell}} \|g_\varepsilon - I\|_{L(B^{-s+\delta}_{q,q,-\ell}, B^{-s}_{q,q,-\ell})}\right) \wedge 2,$$

which is converging to zero for every $Y \in B^{-s+\delta}_{q,q,\ell} \subset B_Y$. We conclude the argument by Lebesgue dominated convergence theorem and with the reasonings similar as above.

B.5 Appendix: Besov spaces and heat semigroup

In this section, we collect some results about weighted Besov spaces. While we only focus on spaces defined on the whole space $\mathbb{R}^n$, the results hold also for Besov spaces on the n-dimensional torus $\mathbb{T}^n$.

Let us start by introducing Littlewood-Paley blocks. Let χ and φ be smooth non-negative functions from $\mathbb{R}^n$ into $\mathbb{R}$ satisfying the following properties:

- $\mathrm{supp}(\chi) \subset B_{4/3}(0)$ and $\mathrm{supp}(\varphi) \subset B_{8/3}(0) \setminus B_{3/4}(0)$,
- $\chi, \varphi \leqslant 1$ and $\chi(y) + \sum_{j \geqslant 0} \varphi(2^{-j}y) = 1$, for any $y \in \mathbb{R}^n$,
- $\mathrm{supp}(\chi) \cap \mathrm{supp}(\varphi(2^{-j}\cdot)) = \varnothing$, for $j \geqslant 1$,
- $\mathrm{supp}(\varphi(2^{-j}\cdot)) \cap \mathrm{supp}(\varphi(2^{-i}\cdot)) = \varnothing$, if $|i-j| > 1$,

where $B_r(x)$ denotes the ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$.

We introduce the following notations: $\varphi_{-1} = \chi$, $\varphi_j(\cdot) = \varphi(2^{-j}\cdot)$, $D_j = \hat{\varphi}_j$, and for any $f \in \mathcal{S}'(\mathbb{R}^n)$ we put $\Delta_j(f) = D_j * f$. Moreover, we write, for any $\ell > 0$, $\rho_\ell(y) = (1+|y|^2)^{-\ell/2}$, and let $L^p_\ell(\mathbb{R}^n)$ be the L^p-space with respect to the norm

$$\|f\|_{L^p_\ell} = \left(\int_{\mathbb{R}^n} (f(y)\rho_\ell(y))^p \,\mathrm{d}y\right)^{1/p},$$

where $p \in [1,+\infty]$.

Definition B.5.1. (Besov space $B^s_{p,q,\ell}$) *Let $s \in \mathbb{R}$, $p, q, \in [1,+\infty]$, and $\ell \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we define the norm*

$$\|f\|_{B^s_{p,q,\ell}} = \left(\sum_{j \geqslant -1} 2^{sqj} \|\Delta_j(f)\|^q_{L^p_\ell}\right)^{1/q}.$$

The space $B^s_{p,q,\ell}(\mathbb{R}^n)$ is the subset of $\mathcal{S}'(\mathbb{R}^n)$ such that the norm $\|\cdot\|_{B^s_{p,q,\ell}}$ is finite.

In the case where $p = q = +\infty$, the weighted Besov space $B^s_{\infty,\infty,\ell}(\mathbb{R}^n)$ is denoted by $C^s_\ell(\mathbb{R}^d)$ and it is called *weighted Besov-Hölder space* with regularity s. Moreover, if $s \in \mathbb{R}_+ \setminus \mathbb{Z}$, the space $C^s_\ell(\mathbb{R}^d)$ coincides with the Banach space of s-Hölder-continuous functions.

The relation between weighted Besov spaces is stated in the following result.

Proposition B.5.2. (Besov embedding) *Let $p_1, p_2, q_1, q_2 \in [1,+\infty]$, $\ell_1, \ell_2 \in \mathbb{R}$, and $s_1, s_2 \in \mathbb{R}$ be such that $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$ and $\ell_1 > \ell_2$. Then, we have the compact immersion*

$$B^{s_2}_{p_2,q_2,\ell_2} \subset B^{s_1}_{p_1,q_1,\ell_1}.$$

Proof. The proof can be found in Theorem 6.7 in [177]. □

If $p=q=2$, the Besov space $B^s_{2,2,\ell}$ coincides with the Sobolev space H^s_ℓ, i.e. the space of measurable tempered distributions f with bounded norm

$$\|f\|^2_{H^s_\ell}=\int_{\mathbb{R}^n}\rho^2_\ell(y)\,((-\Delta+1)^{s/2}f)^2(y)\,\mathrm{d}y.$$

The following theorem allows to extend products between a smooth function and a distribution to elements of Besov spaces.

Theorem B.5.3. (Paraproduct) *Let $p_1,p_2,p,q_1,q_2,q\in[1,+\infty]$, $\ell_1,\ell_2,\ell,s_1,s_2,s\in\mathbb{R}$, be such that*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\qquad \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2},\qquad \ell=\ell_1+\ell_2,\qquad s_1+s_2>0,\quad s=s_1\wedge s_2. \tag{B.5.1}$$

Consider the bilinear map $\Pi\colon\mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$, mapping $(f,g)\mapsto\Pi(f,g)=f\cdot g$. Then, there exists a unique continuous extension of Π as the map

$$\Pi\colon B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)\times B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)\to B^s_{p,q,\ell}(\mathbb{R}^n),$$

and we have, for any $f\in B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)$, $g\in B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$,

$$\|\Pi(f,g)\|_{B^s_{p,q,\ell}}\lesssim\|f\|_{B^{s_1}_{p_1,q_1,\ell_1}}\|g\|_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

Proof. See Section 3.3 in [141] for Besov spaces with exponential weights. The proof for polynomial weights follows in a similar way. □

In the case of products between a positive measure and an element of a Besov space, the previous result can be improved as follows.

Proposition B.5.4. *Consider the same parameters as in Theorem B.5.3 satisfying (B.5.1) and $s_1>0$, $s_2\leqslant 0$. Suppose that $f\in B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ and that $\mu\in B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^2)$ is a positive measure, then we have*

$$\|f\cdot\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}}\lesssim\|f\|_{L^\infty}\cdot\|\mu\|_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

Proof. See, e.g., Lemma 28 in [4]. □

The next result is an interpolation estimate for Besov spaces.

Proposition B.5.5. *Consider $p_1,p_2,q_1,q_2\in[1,+\infty]$, $\ell_1,\ell_2\in\mathbb{R}$ and $s_1,s_2\in\mathbb{R}$, and write, for any $\theta\in[0,1]$,*

$$\frac{1}{p_\theta}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2},\qquad \frac{1}{q_\theta}=\frac{\theta}{q_1}+\frac{1-\theta}{q_2},\qquad \ell_\theta=\theta\ell_1+(1-\theta)\ell_2,\qquad s_\theta=\theta s_1+(1-\theta)s_2.$$

If $f\in B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)\cap B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$, then $f\in B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}(\mathbb{R}^n)$, and furthermore

$$\|f\|_{B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}}\leqslant\|f\|^\theta_{B^{s_1}_{p_1,q_1,\ell_1}}\|f\|^{1-\theta}_{B^{s_2}_{p_2,q_2,\ell_2}}.$$

Proof. The proof is based on the fact that the complex interpolation of the two spaces $B^{s_1}_{p_1,q_1,\ell_1}(\mathbb{R}^n)$ and $B^{s_2}_{p_2,q_2,\ell_2}(\mathbb{R}^n)$ is given by $B^{s_\theta}_{p_\theta,q_\theta,\ell_\theta}(\mathbb{R}^n)$. Such an interpolation is shown in Theorem 6.4.5 in [32] for unweighted Besov spaces. The proof for weighted spaces follows from the fact that $f\in B^s_{p,q,\ell}$ if and only if $f\cdot\rho_\ell\in B^s_{p,q}$ (see Theorem 6.5 in [177]). □

We introduce now the heat kernel and present some of its properties. Let $P_t = e^{-(-\Delta+m^2)t}$, we consider $f \in L^r_{\ell_1}(\mathbb{R}, B^s_{p,q,\ell_2}(\mathbb{R}^2))$ and define the *heat kernel* on f as

$$e^{-(-\Delta+m^2)}f(t) = \int_{-\infty}^{t} P_{t-\tau}f(\tau)\,\mathrm{d}\tau.$$

Notice that $e^{-(-\Delta+m^2)}f(t)$ is a distribution.

More precisely, if $s>0$ and if $f(t,x)$, $(t,x) \in \mathbb{R}\times\mathbb{R}^2$, is a measurable function then

$$e^{-(-\Delta+m^2)}f(t,x) = \int_{-\infty}^{t}\int_{\mathbb{R}^2} \frac{1}{4\pi(t-\tau)} e^{-\frac{(x-y)^2}{4(t-\tau)}+m^2(t-\tau)} f(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau.$$

We have the following regularization property for $e^{-(-\Delta+m^2)}$. Let us remark that all the following results hold also in the case where $f \in L^r_{\ell_1}([t_1,t_2], B^s_{p,q,\ell_2}(\mathbb{R}^2))$, where $-\infty \leqslant t_1 < t_2 \leqslant +\infty$, and the operator $e^{-(-\Delta+m^2)}f(t)$ is defined as the integral from t_1 to $t \in [t_1,t_2]$.

Theorem B.5.6. *Consider $r \in [1,+\infty]$, $p,q \in [1,+\infty]$, $s \in \mathbb{R}$, and let $f \in L^r_{\ell_1}(\mathbb{R}, B^s_{p,q,\ell_2}(\mathbb{R}^2))$. Then, for any $\beta_1, \beta_2 > 0$ such that $\beta_1+\beta_2 < 1$, we have*

$$e^{-(-\Delta+m^2)}f \in B^{\beta_2}_{r,r,\ell_1}\big(\mathbb{R}, B^{s+2\beta_1}_{p,q,\ell_2}(\mathbb{R}^2)\big). \tag{B.5.2}$$

Notice that (B.5.2) states that we are gaining regularity β_2 in time and $2\beta_1$ in space.

In order to prove Theorem B.5.6, we need the following result saying that when we apply the heat kernel at time t we gain $2\beta_1$ in space-regularity, but we have to pay with a multiplicative factor of $t^{-\beta_1}$.

Lemma B.5.7. *Let $m>0$ and consider $g \in B^s_{p,q,\ell}(\mathbb{R}^2)$. We have, for every $t>0$,*

$$\|P_t g\|_{B^{s+2\beta_1}_{p,q,\ell}(\mathbb{R}^2)} \lesssim t^{-\beta_1} e^{-m^2 t} \|g\|_{B^s_{p,q,\ell}}.$$

Proof. See Proposition 5 in [141]. □

We will also need the following lemma saying that giving up some space-regularity we can gain a factor t^β on the right-hand side.

Lemma B.5.8. *Consider $0<\beta<1$ and $g \in B^s_{p,q,\ell}(\mathbb{R}^2)$. Then, for any $t>0$,*

$$\|(1-P_t)g\|_{B^{s-2\beta}_{p,q,\ell}} \lesssim t^\beta \|g\|_{B^s_{p,q,\ell}}.$$

Proof. See Proposition 6 in [141]. □

Proof of Theorem B.5.6. We give the proof for unweighted Besov spaces, the general case follows the same lines. We use the difference characterization of space-time Besov spaces (see e.g. Theorem 2.36 in [22] or Chapter 2.6.1 in [176]), which yields

$$\begin{aligned}\big\|e^{-(-\Delta-m^2)}f\big\|^r_{B^{\beta_2}_{r,r}(\mathbb{R}, B^{s+2\beta_1}_{p,q}(\mathbb{R}^2))} \sim{}& \big\|e^{-(-\Delta-m^2)}f\big\|^r_{L^r(\mathbb{R}, B^{s+2\beta_1}_{p,q}(\mathbb{R}^2))} \\ &+\int_{\mathbb{R}}\int_{|\Delta t|\leqslant 1} \frac{\big\|e^{-(-\Delta-m^2)}f(t+\Delta t) - e^{-(-\Delta-m^2)}f(t)\big\|^r_{B^s_{p,q}}}{|\Delta t|^{1+r\beta_2}}\,\mathrm{d}(\Delta t)\,\mathrm{d}t.\end{aligned}$$

First, we prove that the first term on the right-hand side is finite. Write $\tilde{f}=e^{-(-\Delta+m^2)}f$, we have

$$
\begin{aligned}
\|\tilde{f}\|^r_{L^r\left(\mathbb{R},B^{s+2\beta_1}_{p,q}(\mathbb{R}^2)\right)} &= \int_{\mathbb{R}} \|\tilde{f}(t)\|^r_{B^{s+2\beta_1}_{p,q}}\mathrm{d}t \\
&= \int_{\mathbb{R}} \left\| \int_{-\infty}^t P_{t-k}f(k)\,\mathrm{d}k \right\|^r_{B^{s+2\beta_1}_{p,q}}\mathrm{d}t \\
&\lesssim \int_{\mathbb{R}} \left(\int_{-\infty}^t e^{-m^2(t-k)}\|e^{\Delta(t-k)}f(k)\|_{B^{s+2\beta_1}_{p,q}}\mathrm{d}k \right)^r \mathrm{d}t
\end{aligned}
$$

Lemma B.5.7 and Young inequality yield

$$
\begin{aligned}
\|\tilde{f}\|^r_{L^r\left(\mathbb{R},B^{s+2\beta_1}_{p,q}(\mathbb{R}^2)\right)} &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\mathbb{I}_{[0,+\infty]}(t-k)}{(t-k)^{\beta_1}} e^{-m^2(t-k)}\|f(k)\|_{B^s_{p,q}}\mathrm{d}k \right)^r \mathrm{d}t \\
&\lesssim \left(\int_{\mathbb{R}} \frac{\mathbb{I}_{[0,+\infty]}(t-k)}{(t-k)^{\beta_1}} e^{-m^2(t-k)}\mathrm{d}k \right)^r + \|f\|^r_{L^r(\mathbb{R},B^s_{p,q}(\mathbb{R}^n))},
\end{aligned}
$$

where the first integral on the last step is finite if and only if $\beta_1<1$.

Consider now the difference term. We have

$$
\begin{aligned}
\tilde{f}(t+\Delta t)-\tilde{f}(t) &= \int_t^{t+\Delta t} P_{t+\Delta t-k}f(k)\,\mathrm{d}k + \left(1-e^{-(-\Delta+m^2)\Delta t}\right)\int_{-\infty}^t P_{t-k}f(k)\,\mathrm{d}k \\
&=: I_1+I_2.
\end{aligned}
$$

Now, by Lemma B.5.7 and Young inequality,

$$
\begin{aligned}
\left\|\left(\|I_1\|_{B^{s+2\beta_1}_{p,q}}\right)\right\|^r_{L^r} &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\mathbb{I}_{[-\Delta t,0]}(t-k)}{(t+\Delta t-k)^{\beta_1}} \|f(k)\|_{B^s_{p,q}} \right)^r \\
&\lesssim (\Delta t)^{1-\beta_1}\|f\|_{L^r(\mathbb{R},B^s_{p,q}(\mathbb{R}^n))}
\end{aligned}
$$

Consider $\beta_2<\tilde{\beta}<1-\beta_1$, then, by Lemma B.5.8,

$$
\begin{aligned}
\left\|\left(\|I_2\|_{B^{s+2\beta_1}_{p,q}}\right)\right\|^r_{L^r} &\lesssim (\Delta t)^{\tilde{\beta}}\left\|e^{-(-\Delta+m^2)}f\right\|_{L^r\left(\mathbb{R},B^{s+2\beta_1+2\tilde{\beta}}\right)} \\
&\lesssim (\Delta t)^{\tilde{\beta}}\|f\|_{L^r(\mathbb{R},B^s_{p,q})},
\end{aligned}
$$

where we used the first part of the proof in the last step and the fact that $\beta_1+\tilde{\beta}<1$. Putting everything together, we get

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{|\Delta t|\leqslant 1} \frac{\|\tilde{f}(t+\Delta t)-\tilde{f}(t)\|^r_{B^s_{p,q}}}{|\Delta t|^{1+r\beta_2}}\mathrm{d}(\Delta t)\,\mathrm{d}t &\lesssim \|f\|_{L^1(\mathbb{R},B^s_{p,q})}\int_{|\Delta t|\leqslant 1}\frac{(\Delta t)^{(1-\beta_1)r}+(\Delta t)^{\tilde{\beta}r}}{|\Delta t|^{1+r\beta_2}}\mathrm{d}(\Delta t) \\
&\lesssim \|f\|_{L^1(\mathbb{R},B^s_{p,q})}\int_{|\Delta t|\leqslant 1}\frac{1}{|\Delta t|^{1-(\tilde{\beta}-\beta_2)r}}\mathrm{d}(\Delta t) \\
&\lesssim \|f\|_{L^r(\mathbb{R},B^s_{p,q})},
\end{aligned}
$$

which gives the result. □

Thanks to the heat kernel, we have another representation for weighted Besov spaces.

Proposition B.5.9. *Let $s\in\mathbb{R}$, $p,q\in(0,+\infty]$, and $k\in\mathbb{N}_0$ be such that*

$$k>\frac{s}{2}.$$

Consider a smooth and compactly supported function φ_0. Then, for any $m>0$, we have the following equivalence between norms

$$\|f\|_{B^s_{p,q,\ell}}\simeq\|\mathscr{F}^{-1}(\varphi_0\mathscr{F}(f))\|_{L^p_\ell}+\left(\int_0^{+\infty} t^{\left(k-\frac{s}{2}\right)q}\|\partial_{t^k}(P_tf)\|^q_{L^p_\ell}\frac{\mathrm{d}t}{t}\right)^{1/q}. \tag{B.5.3}$$

Proof. See Section 2.6.4 in [176] for a version of the theorem with a mass-less heat kernel. In particular, such a result differs from the one presented above since the integral with respect to t in equation (B.5.3) goes from 0 to 1 instead of going from zero to $+\infty$. This extension is possible thanks to the regularization properties of the heat kernel and the exponential decay thereof, due to the presence of the positive mass $m>0$. □

Remark B.5.10. Under the same hypotheses as in Proposition B.5.9, if we assume further that $s>0$, then the norm $\|\mathcal{F}^{-1}(\varphi_0\mathcal{F}(f))\|_{L^p_\ell}$ appearing in equation (B.5.3) can be substituted by the weighted L^p-norm of f, since we have

$$\|\mathcal{F}^{-1}(\varphi_0\mathcal{F}(f))\|_{L^p_\ell}\lesssim\|f\|_{L^p_\ell}\lesssim\|f\|_{B^s_{p,q,\ell}}.$$

Namely, for $s>0$, equivalence (B.5.3) becomes

$$\|f\|_{B^s_{p,q,\ell}}\simeq\|f\|_{L^p_\ell}+\left(\int_0^1 t^{\left(k-\frac{s}{2}\right)q}\|\partial_{t^k}(P_tf)\|^q_{L^p_\ell}\frac{\mathrm{d}t}{t}\right)^{1/q}.$$

B.6 Appendix: Stochastic estimates for the Wick exponential

We prove here some stochastic estimates for the term $:e^{\alpha g_\varepsilon * X}:$. We need two different kind of estimates. One has to be at the initial time with respect to the Gaussian free field, while the second one needs to estimate the term in some L^p-space with respect to the variable t. Given g_ε as in Section B.2.2 and $m>0$, and μ^{free} being the Gaussian free field with mass m, we define the Wick exponential $:\exp(\alpha g_\varepsilon * X):$ as in equation (B.2.13).

The previous expression coincides with the standard Wick exponential in the case where we equip the space B_X with the free field measure μ^{free}.

Proposition B.6.1. *Let $\alpha^2<8\pi$, $\varepsilon>0$. Then, for every $r>1$ such that $\alpha^2 r/(4\pi)<2$, and for every $\ell>0$ such that $r\ell>2$, and for every $\delta>0$, we have that the sequence*

$$:\exp(\alpha g_\varepsilon * X):\to:\exp(\alpha X):,\qquad \text{in } L^r\big((B_X,\mu^{\mathrm{free}}),B^{-\gamma(r-1)-\delta}_{r,r,\ell}(\mathbb{R}^2)\big),$$

where $:\exp(\alpha X):$ is the unique (positive) limit random distribution and $\gamma=\alpha^2/(4\pi)$.

Proof. We report here the proof for the case $\alpha^2<4\pi$, the general case can be obtained mixing the method presented here and the techniques by [111]. We consider first the case $r=2$. We have, for $K_j=\mathcal{F}^{-1}(\varphi_j)$,

$$\mathbb{E}\left[\int_{\mathbb{R}^2}|[(:e^{\alpha(g_\varepsilon*X)}:-:e^{\alpha X}:)*K_j](z)|^2(\rho_\ell(z))^2\mathrm{d}z\right]=\int_{\mathbb{R}^2}\mathbb{E}[|[(:e^{\alpha(g_\varepsilon*X)}:-:e^{\alpha X}:)*K_j](z)|^2](\rho_\ell(z))^2\mathrm{d}z.$$

By translation invariance and orthogonality of Wick polynomials, we can consider

$$\begin{aligned}\mathbb{E}[|[(:e^{\alpha(g_\varepsilon*X)}:-:e^{\alpha X}:)*K_j](0)|^2] &= \mathbb{E}[|\langle(:e^{\alpha(g_\varepsilon*X)}:-:e^{\alpha X}:),K_j\rangle|^2]\\ &= \sum_{n=0}^{+\infty}\frac{\alpha^{2n}}{(n!)^2}\mathbb{E}[|\langle:(g_\varepsilon*X)^n:-:X^n:,K_j\rangle|^2].\end{aligned}$$

It then suffices to show that, for each $n\in\mathbb{N}$,

$$\mathbb{E}[|\langle:(g_\varepsilon*X)^n:-:X^n:,K_j\rangle|^2]\to 0. \qquad \text{(B.6.1)}$$

Indeed, for $|\alpha| < \sqrt{4\pi}$, we have

$$\sum_{n=0}^{+\infty} \frac{\alpha^{2n}}{(n!)^2} \mathbb{E}[|\langle :(g_\varepsilon * X)^n: - :X^n:, K_j\rangle|^2] \leqslant 2 \sum_{n=0}^{+\infty} \frac{\alpha^{2n}}{(n!)^2} \mathbb{E}[|\langle :(g_\varepsilon * X)^n:, K_j\rangle|^2 + |\langle :X^n:, K_j\rangle|^2].$$

We have then to show

$$\left| \frac{\alpha^{2n}}{(n!)^2} \mathbb{E}[|\langle :(g_\varepsilon * X)^n:, K_j\rangle|^2 + |\langle :(X)^n:, K_j\rangle|^2] \right| \leqslant c_n,$$

for some $\{c_n\} \in \ell^1(\mathbb{N})$.

We want an uniform bound on $\|\Delta_j :(g_\varepsilon * X)^n: \|_{L^2}$, which would then give, as $\varepsilon \to 0$, the convergence $\Delta_j :(g_\varepsilon * X)^n: \to \Delta_j :X^n:$ almost surely.

Notice that

$$\|\Delta_j : (g_\varepsilon * X)^n : \|_{L^2}^2 = n! \int_{\mathbb{T}_M^2} K_j(x) K_j(x') (\mathscr{X}_\varepsilon(x - x'))^n \, \mathrm{d}x \mathrm{d}x',$$

where

$$\mathscr{X}_\varepsilon = g_\varepsilon^{*2} * \mathscr{X} = \mathscr{F}^{-1}\left(\frac{|\hat{g}_\varepsilon|^2}{|\cdot|^2 + m^2} \right).$$

By Lemma 2.2 in [150], we have, for some constant $C > 0$,

$$\begin{aligned} \mathscr{X}_\varepsilon(x) &\lesssim -\frac{1}{2\pi} \log(|x| \wedge \varepsilon) + C, \\ \mathscr{X}(x) &\lesssim -\frac{1}{2\pi} \log|x| + C. \end{aligned}$$

Moreover, if $x \neq 0$, we have the point-wise limits

$$\mathscr{X}_\varepsilon(x) \xrightarrow{\varepsilon \to 0} \mathscr{X}(x).$$

Now,

$$\begin{aligned} \|\Delta_j :(g_\varepsilon * X)^n: \|_{L^2}^2 &= n! \int_{\mathbb{R}^2} K_j(x) K_j(x') (\mathscr{X}_\varepsilon(x - x'))^n \, \mathrm{d}x \mathrm{d}x' \\ &\lesssim n! \left(\frac{1}{2\pi} \log|2^j| \right)^n, \\ \|\Delta_j :X^n: \|_{L^2}^2 &= n! \int_{\mathbb{R}^2} K_j(x) K_j(x') (\mathscr{X}(x - x'))^n \, \mathrm{d}x \mathrm{d}x' \\ &\lesssim n! \left(\frac{1}{2\pi} \log|2^j| \right)^n, \end{aligned}$$

where the multiplicative constant absorbed in the symbol $\lesssim$ does not depend on j, ε, and n.

Summing up, we have

$$\left| \frac{\alpha^{2n}}{(n!)^2} \mathbb{E}[|\langle :(X^{N,M})^n:, K_j\rangle|^2 + |\langle :(X^M)^n:, K_j\rangle|^2] \right| \lesssim \left| \frac{\alpha^{2n}}{n!} \left(\frac{1}{2\pi} \log|2^j| \right)^n \right| =: c_n^j.$$

Then,

$$c^j = \sum_n c_n^j \lesssim \sum_n \left| \frac{\alpha^{2n}}{n!} \left(\frac{1}{2\pi} \log|2^j| \right)^n \right| \lesssim \mathrm{xp}\left(\frac{\alpha^2}{2\pi} \log|2^j| \right) \lesssim 2^{j\alpha^2/(2\pi)},$$

which yields

$$\sum_{j \geqslant -1} 2^{js} c^j \lesssim \sum_{j \geqslant -1} 2^{j(\alpha^2/(2\pi) + s)}.$$

Therefore, we need

$$s < -\frac{\alpha^2}{4\pi}.$$

Now, we have

$$\begin{aligned}\mathbb{E}\Big[\|{:}\exp(\alpha(g_\varepsilon * X)){:} - {:}\exp(\alpha X){:}\|^2_{B^s_{2,2,\ell}}\Big] &= \sum_{j\geqslant -1} 2^{2js}\mathbb{E}\|\Delta_j {:}\exp(\alpha(g_\varepsilon * X)){:} - \Delta_j {:}\exp(\alpha X){:}\|^2_{L^2}\\ &\lesssim \sum_{j\geqslant -1} 2^{2js}\mathbb{E}[\langle K_j, {:}\exp(\alpha(g_\varepsilon * X)){:} - {:}\exp(\alpha X){:}\rangle^2]\\ &\lesssim \sum_{j\geqslant -1}\sum_n 2^{2js}\frac{\alpha^{2n}}{(n!)^2}\mathbb{E}[\langle K_j, {:}(g_\varepsilon * X)^n{:} - {:}X^n{:}\rangle^2].\end{aligned}$$

Notice that each term in the sum converges to 0 in n and j. Moreover, we have an uniform bound in n and j, since

$$2^{2js}\frac{\alpha^{2n}}{(n!)^2}\mathbb{E}[\langle K_j, {:}(g_\varepsilon * X)^n{:} - {:}X^n{:}\rangle^2] \lesssim 2^{2js} c^j_n,$$

where the term on the right-hand side is summable, and therefore it is in $\ell^1(\mathbb{N}^2)$. By Lebesgue dominated convergence theorem, everything converges to zero.

We follow the proof of Theorem 3.8 in [25] to address the case $r > 2$. Let $r > 2$, take $\gamma > 1$, and recall hypercontractivity of Gaussian Wick monomials (see e.g. Chapter III in [179]):

$$\mathbb{E}[({:}X{:})^r] \leqslant (r-1)^{n/2}(\mathbb{E}[({:}X{:})^2])^{r/2}.$$

We have

$$\begin{aligned}&\mathbb{E}[|[({:}e^{\alpha(g_\varepsilon * X)}{:} - {:}e^{\alpha X}{:}) * K_j](0)|^r]^{1/r}\\ =\;& \mathbb{E}[|\langle({:}e^{\alpha(g_\varepsilon * X)}{:} - {:}e^{\alpha X}{:}), K_j\rangle|^r]^{1/r}\\ =\;& \sum_{n=0}^{+\infty}\frac{\alpha^n}{n!}\mathbb{E}[|\langle{:}(g_\varepsilon * X)^n{:} - {:}X^n{:}, K_j\rangle|^r]^{1/r}\\ \leqslant\;& \sum_{n=0}^{+\infty}\frac{(\alpha\sqrt{r-1})^n}{n!}\mathbb{E}[|\langle{:}(g_\varepsilon * X)^n{:} - {:}X^n{:}, K_j\rangle|^2]^{1/2}\\ \lesssim\;& \left(\frac{\kappa^2}{\kappa^2-1}\right)^{1/2}\left(\sum_{n=0}^{+\infty}\frac{\kappa^2\alpha^{2n}(r-1)^n}{(n!)^2}\mathbb{E}[|\langle{:}(g_\varepsilon * X)^n{:} - {:}X^n{:}, K_j\rangle|^2]\right)^{1/2}\\ \lesssim\;& \left(\frac{\kappa^2}{\kappa^2-1}\right)^{1/2}\mathbb{E}\Big[\Big|\langle{:}e^{\alpha\kappa\sqrt{r-1}(g_\varepsilon * X)}{:} - {:}e^{\alpha\kappa\sqrt{r-1}X}{:}, K_j\rangle\Big|^2\Big]^{1/2}.\end{aligned}$$

Taking κ such that

$$s < -\frac{\alpha^2\kappa^2(r-1)}{4\pi},$$

we conclude the proof in the same way as in the case $r = 2$. □

We need also a result for the periodic setting. In particular, we prove the convergence of the Wick exponential ${:}e^{\alpha Q_{N,M}(g_\varepsilon * X)}{:}$ introduced in equation (B.4.2), as $N \to +\infty$..

Proposition B.6.2. *Recall that $\gamma = \alpha^2/(4\pi)$. If $2 \leqslant p < 2/\gamma$, $\delta > 0$, $\ell > 0$, and $\ell' > \ell'_0(p)$, for a positive constant depending on p, then we have the convergence, as $N \to +\infty$,*

$$({:}e^{\alpha Q_{N,M}(g_\varepsilon * X)}{:} - {:}e^{\alpha(g_\varepsilon * X)}{:}) \to 0, \qquad \text{in } L^p((C^{-\delta}(\mathbb{T}^2_M), \mu^M_F), L^p(\mathbb{T}^2_M)).$$

Proof. Notice that, if Z is a Gaussian random variable, then

$${:}e^{\beta Z}{:} = e^{\beta Z - \frac{\beta^2}{2}\mathbb{E}[Z^2]}.$$

Therefore, applying the previous reasoning for $Z = Q_{N,M}(g_\varepsilon * X)$ and first $\beta = \alpha$, and then $\beta = p\alpha$, then we have

$$(:e^{\alpha Q_{N,M}(g_\varepsilon * X)}:)^p = e^{\alpha p Q_{N,M}(g_\varepsilon * X) - \frac{1}{2} p\alpha^2 \mathbb{E}[(Q_{N,M}(g_\varepsilon * X))^2]} =: e^{\alpha p Q_{N,M}(g_\varepsilon * X)} : e^{\frac{\alpha^2}{2}(p^2 - p)\mathbb{E}[(Q_{N,M}(g_\varepsilon * X))^2]}.$$

We observe that, if $X \in C^{-\delta}(\mathbb{T}_M^2)$, then, for every $x \in \mathbb{T}_M^2$,

$$Q_{N,M}(g_\varepsilon * X(x)) \to g_\varepsilon * X(x), \qquad \text{as } N \to +\infty.$$

Moreover, we have that

$$\mathbb{E}[(Q_{N,M}(g_\varepsilon * X))^2] \to \mathbb{E}[(g_\varepsilon * X)^2],$$

and also

$$:e^{\alpha Q_{N,M} g_\varepsilon * X}: \in L^p_{\mu_M^{\text{free}}}, \qquad \text{uniformly}.$$

These last properties imply that

$$:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: \to :e^{\alpha (g_\varepsilon * X)}:, \qquad \text{in } L^p_{\mu_M^{\text{free}}}. \tag{B.6.2}$$

Taking the norms, we have by translation invariance

$$\mathbb{E}\left[\int_{\mathbb{T}_M^2} |(:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: - :e^{\alpha(g_\varepsilon * X)}:)(z)|^p \mathrm{d}z\right] = (2\pi M)^2 \mathbb{E}[|(:e^{\alpha Q_{N,M}(g_\varepsilon * X)}: - :e^{\alpha (g_\varepsilon * X)}:)(0)|^p].$$

and the last term converges to zero by (B.6.2). □

We now prove convergence in space-time of the Wick exponential.

Proposition B.6.3. *Consider the same parameters α, p, ℓ, γ as in Proposition B.6.1. Let $(X_t)_{t \in \mathbb{R}_+}$ be the solution of equation (B.3.3) with X_0 distributed as a Gaussian free field of mass m. Then we have*

$$:\exp(\alpha g_\varepsilon * X_t): \to :\exp(\alpha X_t):, \qquad \textit{in } L^p\big(B_X \times \Omega, L^p_{\ell'}\big(\mathbb{R}_+, B^{-\gamma(p-1)-\delta}_{p,p,\ell}(\mathbb{R}^2)\big)\big),$$

where $\ell' > 0$ is such that $\ell' p > 1$.

Proof. The proof follows closely the one of Proposition B.6.1. See also Theorem 3.2 in [111] and Lemma 2.5 in [110]. □

B.7 Appendix: Estimates on linearized PDEs

Consider the partial differential equation

$$(\partial_t - \Delta + m^2)\psi(t,z) = -B(t,z,\psi(t,z))(A\psi(t,z) + C(t,z)), \qquad (t,z) \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{B.7.1}$$

where $B: \mathbb{R}_+ \times \mathbb{R}^2 \times H^1(\mathbb{R}^2) \to \mathbb{R}$ is a positive function with compact support with respect to $z \in \mathbb{R}^2$ independently of $t \in \mathbb{R}_+$, $A: B^s_{2,2,\ell}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ is a linear and bounded operator which is self-adjoint with respect to the $L^2(\mathbb{R}^2)$ Hilbert space structure and which commutes with the Laplacian $-\Delta$, while $C: \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a measurable function such that, for any $t \geqslant 0$, we have

$$\int_0^t \|B(s,\cdot,\psi(s,\cdot))C(s,\cdot)\|^2_{L^2} \mathrm{d}s < +\infty,$$

where $\psi \in L^2(\mathbb{R}_+, H^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^2))$.

In this section, we prove some a priori estimates for equation (B.7.1).

Theorem B.7.1. *Consider equation (B.7.1) with* $\psi(0,z)\equiv 0$. *We have the bound, for some constant* $K>0$ *and for every* $0<\sigma<1$,

$$\|A^{1/2}\psi(t,\cdot)\|_{L^2}^2+\int_0^t (\|A^{1/2}\psi(s,\cdot)\|_{H^1}^2+(m^2-K\sigma)\|A^{1/2}\psi(s,\cdot)\|_{L^2}^2)\,\mathrm{d}s\lesssim\int_0^t \|B(s,\cdot,\psi(s,\cdot))C(s,\cdot)\|_{L^2}^2\mathrm{d}s,$$

where the constant implied in the symbol $\lesssim$ *does not depend neither on B nor on C.*

Proof. Multiplying equation (B.7.1) by $A\psi(t,z)$ and integrating we have,

$$\begin{aligned}
&\|A^{1/2}\psi(t,\cdot)\|_{L^2}^2+\int_0^t (\|A^{1/2}\psi(s,\cdot)\|_{H^1}^2+m^2\|A^{1/2}\psi(s,\cdot)\|_{L^2}^2)\,\mathrm{d}s\\
= & -\int_0^t\int B(s,z,\psi(s,z))(A\psi(s,z)+C(s,z))A\psi(s,z)\,\mathrm{d}z\mathrm{d}s\\
= & -\int_0^t\int B(s,z,\psi(s,z))(A\psi(s,z))^2\,\mathrm{d}z\mathrm{d}s\\
& -\int_0^t\int B(s,z,\psi(s,z))C(s,z)A\psi(s,z)\,\mathrm{d}z\mathrm{d}s\\
\leqslant & -\int_0^t\int B(s,z,\psi(s,z))C(s,z)A\psi(s,z)\,\mathrm{d}z\mathrm{d}s.
\end{aligned}$$

Then, exploiting Young's inequality,

$$\begin{aligned}
&\|A^{1/2}\psi(t,\cdot)\|_{L^2}^2+\int_0^t (\|A^{1/2}\psi(s,\cdot)\|_{H^1}^2+m^2\|A^{1/2}\psi(s,\cdot)\|_{L^2}^2)\,\mathrm{d}s\\
\leqslant & -\int_0^t\int B(s,z,\psi(s,z))C(s,z)A\psi(s,z)\,\mathrm{d}z\mathrm{d}s\\
\lesssim & \ C_\sigma\int_0^t\int \big(B(s,z,\psi(s,z))C(s,z)\big)^2\,\mathrm{d}z\mathrm{d}s+C\sigma\int_0^t \|A\psi(s,\cdot)\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}$$

Since $\|A\psi(s,\cdot)\|_{L^2}=\|A^{1/2}A^{1/2}\psi(s,\cdot)\|_{L^2}\lesssim\|A^{1/2}\psi(s,\cdot)\|_{L^2}$, we can reabsorb the last term on the right-hand side and apply Young's convolution inequality to the remaining term to get

$$\begin{aligned}
&\|A^{1/2}\psi(t,\cdot)\|_{L^2}^2+\int_0^t (\|A^{1/2}\psi(s,\cdot)\|_{H^1}^2+(m^2-C\sigma)\|A^{1/2}\psi(s,\cdot)\|_{L^2}^2)\,\mathrm{d}s\\
\lesssim & \ C_\sigma\int_0^t\int \big(B(s,z,\psi(s,z))C(s,z)\big)^2\,\mathrm{d}z\mathrm{d}s\\
\lesssim & \int_0^t \|B(s,\cdot,\psi(s,\cdot))C(s,\cdot)\|_{L^2}^2\mathrm{d}s.
\end{aligned}$$

This concludes the proof. □

Let us modify the previous result in order to deal with weighted norms. In particular, we consider the case where $A\psi=g_\varepsilon*\psi$, where g_ε is defined as in Section B.3, with $A^{1/2}\psi=\tilde{g}_\varepsilon*\psi$. Let us also recall the definition $\rho_\ell^k(z)=(1+k|z|^2)^{-\ell/2}$, for $z\in\mathbb{R}^2$ and $k>0$.

Theorem B.7.2. *Consider equation (B.7.1) with* $\psi(0,z)\equiv 0$. *Assume further that*

$$\|B(\cdot,\cdot,\psi)\|_{L^\infty([0,t],L^\infty_{-\ell}(\mathbb{R}^2))}<+\infty.$$

We have the bound, for some constant $C_1, C_2 > 0$ and for every $0 < \sigma, \tilde{\sigma} < 1$,

$$\|A^{1/2}\psi(t,\cdot)\|^2_{L^2_{-\ell/2}} + \int_0^t \left((1-\sigma C_1)\|A^{1/2}\psi(s,\cdot)\|^2_{H^1_{-\ell/2}} + (m^2 - \tilde{\sigma} C_2)\|A^{1/2}\psi(s,\cdot)\|^2_{L^2_{-\ell/2}}\right) \mathrm{d}s$$
$$\lesssim \int_0^t \|B(s,\cdot,\psi(s,\cdot))\, C(s,\cdot)\|^2_{L^2_{-\ell/2}} \mathrm{d}s.$$

Proof. Instead of multiplying equation (B.7.1) by $A\psi(t,z)$, we multiply it by $A^{1/2}(\rho^k_{-\ell} A^{1/2}\psi(t,z))$. Proceeding as in the previous proof and noticing that

$$|A^{1/2}(\rho^k_{-\ell} A^{1/2}\psi(t,z))| \lesssim \rho^k_{-\ell}(z)|A^{1/2}\psi(t,z)|,$$

we have, integrating by parts,

$$\|\rho^k_{-\ell/2} A^{1/2}\psi(t,\cdot)\|^2_{L^2} + \int_0^t (\|\rho^k_{-\ell/2} A^{1/2}\psi(s,\cdot)\|^2_{H^1} + m^2\|\rho^k_{-\ell/2} A^{1/2}\psi(s,\cdot)\|^2_{L^2})\, \mathrm{d}s$$
$$+\int_0^t \int (\nabla A^{1/2}\psi(s,z))(A^{1/2}\psi(s,z))\nabla\rho^k_{-\ell}(z)\, \mathrm{d}z\mathrm{d}s$$
$$= -\int_0^t \int B(s,z,\psi(s,z))(A\psi(s,z) + C(s,z))A^{1/2}(\rho^k_{-\ell} A^{1/2}\psi(s,z))\, \mathrm{d}z\mathrm{d}s$$
$$= -\int_0^t \int B(s,z,\psi(s,z))(A\psi(s,z))(A^{1/2}(\rho^k_{-\ell}(z)\, A^{1/2}\psi(s,z)))\, \mathrm{d}z\mathrm{d}s$$
$$-\int_0^t \int B(s,z,\psi(s,z))C(s,z)(A^{1/2}(\rho^k_{-\ell}(z)\, A^{1/2}\psi(s,z)))\, \mathrm{d}z\mathrm{d}s.$$

Let us focus on the term

$$\int_0^t \int (\nabla A^{1/2}\psi(s,z))\, (A^{1/2}\psi(s,z))\, \nabla\rho^k_{-\ell}(z)\, \mathrm{d}z\mathrm{d}s.$$

Multiplying and dividing by $\rho^k_{-\ell}(z)$ inside the integrals gives

$$\int_0^t \int \rho^k_{-\ell}(z)\, (\nabla A^{1/2}\psi(s,z))\, (A^{1/2}\psi(s,z)) \frac{\nabla\rho^k_{-\ell}(z)}{\rho^k_{-\ell}(z)} \mathrm{d}z\mathrm{d}s.$$

By Young's inequality we have

$$\left|\int_0^t \int \rho^k_{-\ell}(z)\, (\nabla A^{1/2}\psi(s,z))\, (A^{1/2}\psi(s,z)) \frac{\nabla\rho^k_{-\ell}(z)}{\rho^k_{-\ell}(z)} \mathrm{d}z\mathrm{d}s\right|$$
$$\lesssim C_\sigma \int_0^t \int \rho^k_{-\ell}(z)\, (A^{1/2}\psi(s,z))^2 \left(\frac{\nabla\rho^k_{-\ell}(z)}{\rho^k_{-\ell}(z)}\right)^2 \mathrm{d}z\mathrm{d}s$$
$$+\sigma C_1 \int_0^t \int \rho^k_{-\ell}(z)(\nabla A^{1/2}\psi(s,z))^2\, \mathrm{d}z\mathrm{d}s,$$

and now we have

$$\frac{\nabla\rho^k_{-\ell}(z)}{\rho^k_{-\ell}(z)} \simeq \frac{\ell k|z_1|}{1+k|z|^2} \leqslant \ell\sqrt{k} \sup_{z\in\mathbb{R}^2} \frac{|z_1|}{1+|z|^2} \leqslant \sqrt{k}\, C_2.$$

This yields

$$\|\rho^k_{-\ell/2} A^{1/2}\psi(t,\cdot)\|^2_{L^2} + \int_0^t \left((1-\sigma C_1)\|\rho^k_{-\ell/2} A^{1/2}\psi(s,\cdot)\|^2_{H^1} + (m^2 - \sqrt{k}\, C_\sigma)\|\rho^k_{-\ell/2} A^{1/2}\psi(s,\cdot)\|^2_{L^2}\right) \mathrm{d}s$$
$$\simeq -\int_0^t \int B(s,z,\psi(s,z))(A\psi(s,z))(A^{1/2}(\rho^k_{-\ell}(z)\, A^{1/2}\psi(s,z)))\, \mathrm{d}z\mathrm{d}s$$
$$-\int_0^t \int B(s,z,\psi(s,z))C(s,z)(A^{1/2}(\rho^k_{-\ell}(z)\, A^{1/2}\psi(s,z)))\, \mathrm{d}z\mathrm{d}s.$$

Applying Young's inequality we get

$$\|\rho^k_{-\ell/2}A^{1/2}\psi(t,\cdot)\|^2_{L^2}+\int_0^t \big((1-\sigma C_1)\|\rho^k_{-\ell/2}A^{1/2}\psi(s,\cdot)\|^2_{H^1}+(m^2-\sqrt{k}C_\sigma)\|\rho^k_{-\ell/2}A^{1/2}\psi(s,\cdot)\|^2_{L^2}\big)\,\mathrm{d}s$$

$$\lesssim C_{\sigma_2}\int_0^t\int \rho^k_{-\ell}(z)\,|B(s,z,\psi(s,z))|^2\,|A\psi(s,z)|^2\,\mathrm{d}z\mathrm{d}s+\sigma_2\int_0^t\int \rho^k_{-\ell}(z)\,|A^{1/2}\psi(s,z)|^2\,\mathrm{d}z\mathrm{d}s$$

$$+C_{\sigma_3}\int_0^t\int (\rho^k_{-\ell/2}(z)B(s,z,\psi(s,z))C(s,z))^2\,\mathrm{d}z\mathrm{d}s+\sigma_3\int_0^t\int (\rho^k_{-\ell/2}(z)\,A^{1/2}\psi(s,z))^2\,\mathrm{d}z\mathrm{d}s$$

$$\lesssim C_{\sigma_2}\|B(\cdot,\cdot,\psi)\|_{L^\infty([0,t],L^\infty_{-\ell}(\mathbb{R}^2))}\int_0^t\int |B(s,z,\psi(s,z))|\,|A\psi(s,z)|^2\,\mathrm{d}z\mathrm{d}s$$

$$+\sigma_2\int_0^t \|\rho^k_{-\ell/2}A^{1/2}\psi(s,\cdot)\|^2_{L^2}\,\mathrm{d}s$$

$$+C_{\sigma_3}\int_0^t \|\rho^k_{-\ell/2}\,B(s,\cdot,\psi(s,\cdot))\,C(s,\cdot)\|^2_{L^2}\,\mathrm{d}s+\sigma_3\int_0^t \|\rho^k_{-\ell/2}\,A^{1/2}\psi(s,\cdot)\|^2_{L^2}\,\mathrm{d}s,$$

reabsorbing the terms multiplied by σ_2 and σ_3 respectively, to the left-hand side and noticing that from the proof of Theorem B.7.1 we also get

$$\int_0^t\int B(s,z,\psi(s,z))(A\psi(s,z))^2\,\mathrm{d}z\mathrm{d}s\lesssim\int_0^t \|B(s,\cdot,\psi(s,\cdot))C(s,\cdot)\|^2_{L^2}\,\mathrm{d}s,$$

we have, renaming the constants and introducing $\tilde{\sigma}$,

$$\|A^{1/2}\psi(t,\cdot)\|^2_{L^2_{-\ell/2}}+\int_0^t \big((1-\sigma C_1)\|A^{1/2}\psi(s,\cdot)\|^2_{H^1_{-\ell/2}}+(m^2-\tilde{\sigma}C_2)\|A^{1/2}\psi(s,\cdot)\|^2_{L^2_{-\ell/2}}\big)\,\mathrm{d}s$$

$$\lesssim \int_0^t \|B(s,\cdot,\psi(s,\cdot))\,C(s,\cdot)\|^2_{L^2_{-\ell/2}}\,\mathrm{d}s.$$

This concludes the proof. □

We now apply a bootstrap argument to the previous result to get the following statement.

Corollary B.7.3. *Under the same hypotheses of Theorem B.7.2, we have*

$$\|\psi\|_{B^\beta_{q,q,\ell'}(\mathbb{R}_+,B^{2-2\beta-\delta}_{p,p,-\ell}(\mathbb{R}^2))}\lesssim\mathfrak{P}_2(\|B\|_{L^\infty_{\ell'}(\mathbb{R}_+,L^\infty_{-\ell}(\mathbb{R}^2))},\|C\|_{L^\infty_{\ell'}(\mathbb{R}_+,L^\infty_{-\ell}(\mathbb{R}^2))}),$$

where $\mathfrak{P}_2$ is a second degree polynomial.

Proof. From Theorem B.7.2, we know that $A^{1/2}\psi$ lives in $L^2_{\ell'}(\mathbb{R}_+,H^1_{-\ell}(\mathbb{R}^2))\cap L^\infty_{\ell'}(\mathbb{R}_+,L^2_{-\ell}(\mathbb{R}^2))$. By interpolation and using the fact that $H^1_{-\ell}(\mathbb{R}^2)\subset B^0_{p,p,-\ell}(\mathbb{R}^2)$, for every $p<+\infty$, we have

$$\|A^{1/2}\psi\|^2_{L^q_{\ell'/2}(\mathbb{R}_+,B^0_{p,p,-\ell/2}(\mathbb{R}^2))}\lesssim\int_{\mathbb{R}_+}\rho_{\ell'}(s)\|B(s,\cdot,\psi(s,\cdot))\,C(s,\cdot)\|^2_{L^2_{-\ell/2}}\,\mathrm{d}s.$$

Applying the heat kernel to the equation (B.7.1), we have

$$\psi(t,z)=\int_0^t P_{t-s}\big(B(s,z,\psi(s,z))(A\psi(s,z)+C(s,z))\big)\mathrm{d}s,$$

and therefore, if $1/q+1/p=1$ and $\beta>0$, we have by Theorem B.5.6,

$$\|\psi\|_{B^\beta_{q,q,\ell'}(\mathbb{R}_+,B^{2-2\beta-\delta}_{p,p,-\ell}(\mathbb{R}^2))}\lesssim\|B(\cdot,\cdot,\psi(\cdot,\cdot))(A\psi(\cdot,\cdot)+C(\cdot,\cdot))\|_{L^q_{\ell'}(\mathbb{R}_+,B^0_{p,p,-\ell}(\mathbb{R}^2))}.$$

On the other hand, for some second degree polynomial $\mathfrak{P}_2$, we have

$$\|B(\cdot,\cdot,\psi(\cdot,\cdot))(A\psi(\cdot,\cdot)+C(\cdot,\cdot))\|_{L^q_{\ell'}(\mathbb{R}_+,B^0_{p,p,-\ell}(\mathbb{R}^2))}$$
$$\leqslant\mathfrak{P}_2(\|B\|_{L^\infty_{\ell'}(\mathbb{R}_+,L^\infty_{-\ell}(\mathbb{R}^2))},\|C\|_{L^\infty_{\ell'}(\mathbb{R}_+,L^\infty_{-\ell}(\mathbb{R}^2))}).$$

This concludes the proof. □

B.8 Appendix: Proof of Lemma B.3.4

We give here the proof of Lemma B.3.4. We only show the first part of the result, the second one following in a straightforward way. It is sufficient to prove the following property: for any $G\in\mathscr{F}$ there exists a sequence of cylinder functions $(G_k)_{k\in\mathbb{N}}$ (here we consider a sequence $(G_{N,M})_{N,M\in\mathbb{N}}$ depending on two parameters) such that $\mathscr{L}G_k\to\mathscr{L}G$ point-wise as $k\to+\infty$, and we have the uniform bound

$$|\mathscr{L}G_k|\leqslant F_G(X), \tag{B.8.1}$$

for some measurable $F_G\in L^1(\mu^{\text{free}})$, e.g. some polynomial of X. And secondly we have to show that $\mathscr{L}_\varepsilon G\to\mathscr{L}G$ point-wise as $\varepsilon\to 0$ with a bound $|\mathscr{L}_\varepsilon G|\leqslant F_G(X)$, for some $F_G\in L^1(\mu^{\text{free}})$. Since we take $\mu\in\mathscr{M}_{B_Y}$, the result follows from Lebesgue's dominated convergence theorem.

Take $G\in\mathscr{F}$ and $N,M,\in\mathbb{N}$, and define

$$G_{N,M}(X,Y)=G(f_MQ_{N,M}(f_MX),f_MQ_{N,M}(f_MY)),$$

where $f_M=f(\cdot/M)$, $f\colon\mathbb{R}^2\to[0,1]$ being a compactly supported smooth function in $\mathbb{T}_1^2$ which is identically 1 in a neighborhood of the origin, and $Q_{N,M}$ is the operator defined in Section B.4. We need to show that

$$\begin{aligned}
\nabla_X G_{N,M}(X,Y) &\to \nabla_X G(X,Y), && \text{point-wise in } B^{2-\delta}_{1,1,-\ell}(\mathbb{R}^2), && \text{(B.8.2)}\\
\nabla_Y G_{N,M}(X,Y) &\to \nabla_Y G(X,Y), && \text{point-wise in } B^{(2-s)\wedge(\gamma(r-1))+\delta}_{l,l,-\ell}(\mathbb{R}^2) \text{ for } l\in(1,\infty), && \text{(B.8.3)}\\
\operatorname{tr}(\nabla^2_X G_{N,M}(X,Y)) &\to \operatorname{tr}(\nabla^2_X G(X,Y)), && \text{point-wise.} && \text{(B.8.4)}
\end{aligned}$$

We focus only on the proof of (B.8.2) and (B.8.4) since the limit (B.8.3) follows with similar arguments. We have,

$$\begin{aligned}
& \nabla_X G_{N,M}(X,Y)[h]\\
= & \nabla_X G(f_MQ_{N,M}(f_MX),f_MQ_{N,M}(f_MY))[f_MQ_{N,M}(f_Mh)],
\end{aligned}$$

On the other hand, since the integrability parameters in the Besov spaces $\tilde{B}_X$, B_Y, $B^{2-\delta}_{1,1,-\ell}(\mathbb{R}^2)$, and $B^{(2-s)\wedge(\gamma(r-1))+\delta}_{l,l,-\ell}(\mathbb{R}^2)$ are finite, the linear operator $Z\mapsto f_MQ_{N,M}(f_MZ)$, where $Z\in\mathfrak{B}$, where $\mathfrak{B}$ is anyone of the Besov spaces listed before, strongly converges to the identity $\mathrm{id}_{\mathfrak{B}}$ on $\mathfrak{B}$. Therefore, by continuity of $\nabla_X G$, we get that the following convergence holds as $N,M\to+\infty$:

$$\nabla_X G(f_MQ_{N,M}(f_MX),f_MQ_{N,M}(f_MY))\to\nabla_X G(X,Y),\qquad \text{in } B^{2-\delta}_{1,1,-\ell}(\mathbb{R}^2).$$

By strong convergence of the operator $Z\mapsto f_MQ_{N,M}(f_MZ)$ in $B^{2-\delta}_{1,1,-\ell}(\mathbb{R}^2)$ and the fact that the composition of strongly convergent operators is strongly convergent, we get the limit in (B.8.2).

Let us now prove (B.8.4). We have, with the notation $p_{N,M}=(f_MQ_{N,M}(f_MX),f_MQ_{N,M}(f_MY))$, and $p=(X,Y)$,

$$\begin{aligned}
& \operatorname{tr}(f_MQ_{N,M}f_M\nabla^2G(p_{N,M})f_MQ_{N,M}f_M)-\operatorname{tr}(\nabla^2G(p))\\
= & \operatorname{tr}(f_MQ_{N,M}f_M\nabla^2G(p_{N,M})f_MQ_{N,M}f_M)-\operatorname{tr}(f_MQ_{N,M}f_M\nabla^2G(p)f_MQ_{N,M}f_M)\\
& +\operatorname{tr}(f_MQ_{N,M}f_M\nabla^2G(p)f_MQ_{N,M}f_M)-\operatorname{tr}(\nabla^2G(p)).
\end{aligned}\tag{B.8.5}$$

Let us deal first with the second term of equation (B.8.5), that is

$$\operatorname{tr}(f_MQ_{N,M}f_M\nabla^2G(p)f_MQ_{N,M}f_M)-\operatorname{tr}(\nabla^2G(p)). \tag{B.8.6}$$

We have

$$\begin{aligned}
\mathrm{tr}(f_M Q_{N,M} f_M \nabla^2 G(p) f_M Q_{N,M} f_M) &= \mathrm{tr}(\nabla^2 G(p)(f_M Q_{N,M} f_M)^2)\\
&= \sum_{n\in\mathbb{N}} (\nabla^2 G(p)(f_M Q_{N,M} f_M)^2 h_n, h_n)_{L^2}\\
&= \sum_{n\in\mathbb{N}} ((f_M Q_{N,M} f_M)^2 h_n, \nabla^2 G(p) h_n)_{L^2}\\
&= \sum_{n\in\mathbb{N}} ((f_M Q_{N,M} f_M)^2 h_n^G, \nabla^2 G(p) h_n^G)_{L^2}\\
&= \sum_{n\in\mathbb{N}} \lambda_n ((f_M Q_{N,M} f_M)^2 h_n^G, h_n^G)_{L^2},
\end{aligned}$$

where we use that f_M, $Q_{N,M}$, and $\nabla^2 G(p)$ are self-adjoint operators in $L^2(\mathbb{R}^2)$, and $(h_n^G)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$ of eigenvectors related to the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of the operator $\nabla^2 G(p)$ (which exists being $\nabla^2 G(p)$ a compact operator). Since $\nabla^2 G(p)$ is also a trace-class operator, we have that $(|\lambda_n|)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Furthermore, we get

$$|\lambda_n((f_M Q_{N,M} f_M)^2 h_n^Q, h_n^Q)| \leqslant |\lambda_n| \|(f_M Q_{N,M} f_M)^2\|_{L(L^2,L^2)} \leqslant |\lambda_n| \left(\sup_{N,M\in\mathbb{N}} \|(f_M Q_{N,M} f_M)^2\|_{L(L^2,L^2)} \right) \lesssim |\lambda_n|.$$

Since we have the strong convergences $f_M \to \mathrm{id}_{L^2(\mathbb{R}^2)}$ as $M\to+\infty$, and $Q_{N,M} \to \mathrm{id}_{L^2(\mathbb{T}_M^2)}$ as $N\to+\infty$, and so $\sup_M \|f_M\|_{L(L^2,L^2)}, \sup_{N,M} \|Q_{N,M}\|_{L(L^2,L^2)} < +\infty$, then we have

$$(f_M Q_{N,M} f_M)^2 \to \mathrm{id}_{L^2(\mathbb{R}^2)} \text{ strongly.}$$

Thus, we have $(f_M Q_{N,M} f_M)^2 h_n^G \to h_n^G$ in L^2 as $N, M \to +\infty$, and therefore

$$\{\lambda_n((f_M Q_{N,M} f_M)^2 h_n^G, h_n^G)\}_{n\in\mathbb{N}} \to \{\lambda_n\}_{n\in\mathbb{N}} \text{ point-wise.}$$

Then the term (B.8.6) converges to zero by Lebesgue's dominated convergence theorem.

We now show that the first term in equation (B.8.5), i.e.

$$\mathrm{tr}(f_M Q_{N,M} f_M \nabla^2 G(p_{N,M}) f_M Q_{N,M} f_M) - \mathrm{tr}(f_M Q_{N,M} f_M \nabla^2 G(p) f_M Q_{N,M} f_M). \tag{B.8.7}$$

converges to zero. Let $\mathfrak{M}\colon H^{\kappa}_{-\ell}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the natural isomorphism between the two spaces, where κ and ℓ are the same parameter as in point *iii.* of Definition B.3.1. The natural identification of $L^2(\mathbb{R}^2)$ with its dual, allows us to identify the dual map $\mathfrak{M}^*$ of $\mathfrak{M}$ with the natural isomorphism between $L^2(\mathbb{R}^2)$ and $H_{\ell}^{-\kappa}(\mathbb{R}^2)$. We can then write

$$\nabla^2 G(p) = \mathfrak{M}^{-1} \mathfrak{M} \nabla^2 G(p) \mathfrak{M}^* (\mathfrak{M}^*)^{-1},$$

and therefore, for two points $p, p_{N,M} \in \tilde{B}_X \times B_Y$,

$$\begin{aligned}
&\mathrm{tr}(|\nabla^2 G(p) - \nabla^2 G(p_{N,M})|)\\
\leqslant\ & \mathrm{tr}(\mathfrak{M}^{-1} |\mathfrak{M} \nabla^2 G(p) \mathfrak{M}^* - \mathfrak{M} \nabla^2 G(p_{N,M}) \mathfrak{M}^*| (\mathfrak{M}^*)^{-1})\\
=\ & \sum_{n\in\mathbb{N}} (\mathfrak{M}^{-1} |\mathfrak{M} \nabla^2 G(p) \mathfrak{M}^* - \mathfrak{M} \nabla^2 G(p_{N,M}) \mathfrak{M}^*| (\mathfrak{M}^*)^{-1} h_n, h_n)_{L^2}\\
=\ & \sum_{n\in\mathbb{N}} (|\mathfrak{M} \nabla^2 G(p) \mathfrak{M}^* - \mathfrak{M} \nabla^2 G(p_{N,M}) \mathfrak{M}^*| (\mathfrak{M}^*)^{-1} h_n, (\mathfrak{M}^*)^{-1} h_n)_{L^2}\\
\leqslant\ & \|\mathfrak{M} \nabla^2 G(p) \mathfrak{M}^* - \mathfrak{M} \nabla^2 G(p_{N,M}) \mathfrak{M}^*\|_{L(L^2,L^2)} \sum_{n\in\mathbb{N}} \|(\mathfrak{M}^*)^{-1} h_n\|_{L^2}^2\\
\leqslant\ & \|\mathfrak{M} \nabla^2 G(p) \mathfrak{M}^* - \mathfrak{M} \nabla^2 G(p_{N,M}) \mathfrak{M}^*\|_{L(L^2,L^2)} \mathrm{tr}(\iota_{H^{\kappa}_{-\ell} \hookrightarrow L^2} \mathfrak{M}^{-1} (\mathfrak{M}^*)^{-1} \iota_{L^2 \hookrightarrow H_{\ell}^{-\kappa}}),
\end{aligned}$$

which is finite since point *iii.* in Definition B.3.1 holds and the operator $\iota_{H^{\kappa}_{-\ell}\hookrightarrow L^2}\mathfrak{M}^{-1}(\mathfrak{M}^*)^{-1}\iota_{L^2\hookrightarrow H^{-\kappa}_{\ell}}$ is trace class because $\kappa>1$ and $\ell>1$ (see Remark B.3.2). By continuity of the map $\nabla^2 G: \tilde{B}_X \times B_Y \rightarrow L(H^{-\kappa}_{\ell}(\mathbb{R}^2), H^{\kappa}_{-\ell}(\mathbb{R}^2))$ and the fact that, by similar arguments as the one exploited to show (B.8.2), $p_{N,M}\rightarrow p$ as $N,M\rightarrow+\infty$, we have that $\operatorname{tr}(|\nabla^2 G(p)-\nabla^2 G(p_{N,M})|)\rightarrow 0$ as $N,M\rightarrow+\infty$.

The argument is then concluded by the following observation: if we take the absolute value of the difference in (B.8.7), then

$$\begin{aligned}
&|\operatorname{tr}(f_M Q_{N,M} f_M(\nabla^2 G(p_{N,M})-\nabla^2 G(p)) f_M Q_{N,M} f_M)|\\
\leqslant\ &\operatorname{tr}(f_M Q_{N,M} f_M|\nabla^2 G(p_{N,M})-\nabla^2 G(p)| f_M Q_{N,M} f_M)\\
\leqslant\ &\sup_{N,M\in\mathbb{N}}\|f_M Q_{N,M} f_M\|^2 \operatorname{tr}(|\nabla^2 G(p_{N,M})-\nabla^2 G(p)|),
\end{aligned}$$

which converges to zero since $\sup_{N,M\in\mathbb{N}}\|f_M Q_{N,M} f_M\|^2$ is finite.

In order to get inequality (B.8.1), we notice that

$$\|\nabla_X G_{N,M}(X,Y)\| \lesssim \|\nabla_X G(X,Y)\| \leqslant F_G(X), \tag{B.8.8}$$

where F_G plays the role of f_Φ in Definition B.3.1, as well as similar inequalities for $\nabla_Y G$ and $\operatorname{tr}(\nabla^2 G)$, which is due to the fact that the norm of operator $f_M Q_{N,M} f_M$ is uniformly bounded in N, $M\in\mathbb{N}$.

The convergence $\mathscr{L}_\varepsilon G\rightarrow\mathscr{L}G$ as $\varepsilon\rightarrow 0$ is proved in Section B.4.4.3.

B.9 Appendix: Technical results for the resolvent equation

We consider here the system of equations (B.3.3)–(B.3.4), and give a proof of Proposition B.3.8. For notation simplicity, we will write Y instead of Y^ε when no confusion occurs. Moreover, if not explicitly specified, all the appearing parameters are assumed to be taken as in Definition B.2.6.

B.9.1 Flow equations

We start with a result about existence and uniqueness of solutions to the system of equations (B.3.3)–(B.3.4), that is the first part of the statement in Proposition B.3.8.

Proposition B.9.1. *For any $\varepsilon>0$, if $(X_0,Y_0)\in\hat{B}_X\times\{B_Y\cup B^{r,\ell}_{\exp}\}$, then there exists a unique solution (X,Y) to equations (B.3.3)–(B.3.4) such that*

$$(X_t,Y_t)\in\hat{B}_X\times\{B_Y\cup B^{r,\ell}_{\exp}\},\qquad t\in\mathbb{R}_+.$$

Proof. The unique solution to equation (B.3.3) is given by the explicit formula

$$X_t = P_t X_0 + \int_0^t P_{t-s}\xi_s\,\mathrm{d}s \tag{B.9.1}$$

(see, e.g., Theorem 5.4 in [59]). Showing existence of a solution to equation (B.3.4) is equivalent to showing existence for

$$\tilde{Y}_t = Y_t - P_t Y_0.$$

We proceed by a Schaefer's fixed point argument (see Theorem 4 in Section 9.2.2 in [69]) to get the result up to a fixed time $T>0$. Consider the map $\mathscr{J}$ given by, for $t\in[0,T]$,

$$\mathscr{J}_t(\tilde{Y},Y_0,X) = -\int_0^t P_{t-s}\alpha f_\varepsilon :e^{\alpha(g_\varepsilon * X_s)}: e^{\alpha(g_\varepsilon*\tilde{Y}_s)} e^{\alpha P_s(g_\varepsilon * Y_0)}\,\mathrm{d}s.$$

We need to show that $\mathcal{J}_\cdot(\cdot, Y_0, X): A \to \bar{A}$ is a continuous and bounded map, where A and $\bar{A}$ are the convex closed subsets, for $\kappa > 0$ small enough, $\delta > 0$, $\theta \in (\kappa \vee \delta, 1)$, $\ell > 0$ large enough, consisting of non-negative functions, and such that

$$A \subset C^{\theta-\kappa}([0,T], C_{-\ell+\kappa}^{2-2\theta-\delta-\kappa}(\mathbb{R}^2)), \qquad \bar{A} \subset C^{\theta}([0,T], C_{-\ell}^{2-2\theta-\delta}(\mathbb{R}^2)).$$

Exploiting the compact embedding

$$C^{\theta-\kappa}([0,T], C_{-\ell+\kappa}^{2-2\theta-\delta-\kappa}(\mathbb{R}^2)) \hookrightarrow C^{\theta}([0,T], C_{-\ell}^{2-2\theta-\delta}(\mathbb{R}^2)),$$

given by Besov embedding and Corollary 3 in [172], we can then proceed applying Schaefer's fixed point theorem to get existence for every compact subset of $\mathbb{R}_+$ of the form $[0, \tau]$, for some $\tau > 0$. We have

$$\begin{aligned}
&\|\mathcal{J}_s(\tilde{Y}, Y_0, X)\|_{C^\theta([0,T], C_{-\ell}^{2-2\theta-\delta}(\mathbb{R}^2))}\\
\lesssim\; &\left\| \alpha f_\varepsilon : e^{\alpha(g_\varepsilon * X_s)} : e^{\alpha(g_\varepsilon * \tilde{Y}_s)} e^{\alpha P_s(g_\varepsilon * Y_0)} \right\|_{L^\infty([0,T], L^\infty(\mathrm{supp}(f_\varepsilon)))} \\
\lesssim\; &\|\alpha f_\varepsilon : e^{\alpha(g_\varepsilon * X_s)} : e^{\alpha P_s(g_\varepsilon * Y_0)}\|_{L^\infty([0,T], L^\infty(\mathrm{supp}(f_\varepsilon)))} < +\infty,
\end{aligned} \tag{B.9.2}$$

by the regularization property of the convolution with g_ε.

We have to prove continuity. For every $\tilde{Y}, \tilde{Y}' \in A$, we have

$$\begin{aligned}
&\|\mathcal{J}(\tilde{Y}, Y_0, X) - \mathcal{J}(\tilde{Y}', Y_0, X)\|_{C^\theta([0,T], C_{-\ell}^{2-2\theta-\delta}(\mathbb{R}^2))}\\
\lesssim\; &\left\| \alpha f_\varepsilon : e^{\alpha(g_\varepsilon * X_s)} : \left(e^{\alpha(g_\varepsilon * \tilde{Y}_s)} - e^{\alpha(g_\varepsilon * \tilde{Y}_s')}\right) e^{\alpha P_s(g_\varepsilon * Y_0)} \right\|_{L^\infty([0,T], L^\infty(\mathrm{supp}(f_\varepsilon)))}\\
\lesssim\; &\left\| e^{\alpha(g_\varepsilon * \tilde{Y}_s)} - e^{\alpha(g_\varepsilon * \tilde{Y}_s')} \right\|_{L^\infty([0,T], L^\infty(\mathrm{supp}(f_\varepsilon)))}\\
\lesssim\; &\|g_\varepsilon * \tilde{Y}_s - g_\varepsilon * \tilde{Y}_s'\|_{L^\infty([0,T], L^\infty(\mathrm{supp}(f_\varepsilon)))}\\
\lesssim\; &\|g_\varepsilon\|_{L^1(\mathbb{R}^2)} \|\tilde{Y}_s - \tilde{Y}_s'\|_{L^\infty([0,T], L^\infty(\mathrm{supp}(f_\varepsilon)))}\\
\lesssim\; &\|\tilde{Y}_s - \tilde{Y}_s'\|_{C^{\theta-\kappa}([0,T], C_{-\ell+\kappa}^{2-2\theta-\delta-\kappa}(\mathbb{R}^2))}.
\end{aligned}$$

We are left to show uniqueness. Take two solutions Y and Y' to equation (B.3.4). Notice that their difference is given by $Y - Y' = \tilde{Y} - \tilde{Y}'$ and it satisfies

$$(\partial_t - \Delta + m^2)(\tilde{Y}_t - \tilde{Y}_t') = -\alpha f_\varepsilon : e^{\alpha(g_\varepsilon * X)} : (e^{\alpha(g_\varepsilon * Y_t)} - e^{\alpha(g_\varepsilon * Y_t')}).$$

Introduce a positive function $h: \mathbb{R} \to \mathbb{R}$ such that $h(t) \to 0$ when $t \to -\infty$, and with $|h'(t)| \leqslant C h(t)$, for some constant $C > 0$. Multiplying the previous expression by $h(t)(\tilde{Y}_t - \tilde{Y}_t')$ and integrating with respect to time and space, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} h(t)(\tilde{Y}_t - \tilde{Y}_t')(\partial_t - \Delta + m^2)(\tilde{Y}_t - \tilde{Y}_t')\, \mathrm{d}z\, \mathrm{d}t\\
=\; &-\alpha \int_0^T \int_{\mathbb{R}^2} h(t)(\tilde{Y}_t - \tilde{Y}_t') f_\varepsilon : e^{\alpha(g_\varepsilon * X)} : (e^{\alpha(g_\varepsilon * Y_t)} - e^{\alpha(g_\varepsilon * Y_t')})\, \mathrm{d}z\, \mathrm{d}t.
\end{aligned}$$

For the left-hand side, we get by integration by parts

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} h(t)(\tilde{Y}_t - \tilde{Y}_t')(\partial_t - \Delta + m^2)(\tilde{Y}_t - \tilde{Y}_t')\, \mathrm{d}z\, \mathrm{d}t\\
=\; &h(T)\|\tilde{Y}_T - \tilde{Y}_T'\|_{L^2}^2 - \int_0^T h'(t)\|\tilde{Y}_t - \tilde{Y}_t'\|_{L^2}^2 \mathrm{d}t + \int_0^T h(t)(\|\tilde{Y}_t - \tilde{Y}_t'\|_{H^1}^2 + m^2\|\tilde{Y}_t - \tilde{Y}_t'\|_{L^2}^2)\, \mathrm{d}t\\
\geqslant\; &h(T)\|\tilde{Y}_T - \tilde{Y}_T'\|_{L^2}^2 - \int_0^T |h'(t)|\|\tilde{Y}_t - \tilde{Y}_t'\|_{L^2}^2 \mathrm{d}t + \int_0^T h(t)(\|\tilde{Y}_t - \tilde{Y}_t'\|_{H^1}^2 + m^2\|\tilde{Y}_t - \tilde{Y}_t'\|_{L^2}^2)\, \mathrm{d}t\\
\geqslant\; &h(T)\|\tilde{Y}_T - \tilde{Y}_T'\|_{L^2}^2 + \int_0^T h(t)(\|\tilde{Y}_t - \tilde{Y}_t'\|_{H^1}^2 + (m^2 - C)\|\tilde{Y}_t - \tilde{Y}_t'\|_{L^2}^2)\, \mathrm{d}t.
\end{aligned}$$

Notice that the last line is positive. Therefore,

$$h(T)\|\tilde{Y}_T-\tilde{Y}_T'\|_{L^2}^2+\int_0^T h(t)(\|\tilde{Y}_t-\tilde{Y}_t'\|_{H^1}^2+(m^2-C)\|\tilde{Y}_t-\tilde{Y}_t'\|_{L^2}^2)\,\mathrm{d}t$$
$$\leqslant -\alpha\int_0^T\int_{\mathbb{R}^2} h(t)(\tilde{Y}_t-\tilde{Y}_t'):e^{\alpha(g_\varepsilon*X)}:(e^{\alpha(g_\varepsilon*Y_t)}-e^{\alpha(g_\varepsilon*Y_t')})\,\mathrm{d}z\mathrm{d}t\leqslant 0,$$

which yields uniqueness. □

Remark B.9.2. The growth with respect to T of the norm of the solution Y to equation (B.3.4) is polynomial. Indeed, recalling that in the proof of Proposition B.9.1 we have

$$\tilde{Y}_t^\varepsilon=-\int_0^t P_{t-s}\,\alpha f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:e^{\alpha(g_\varepsilon*\tilde{Y}_s)}e^{\alpha P_s(g_\varepsilon*Y_0)}\,\mathrm{d}s,$$

we get that the time growth of $\tilde{Y}_t$ is determined by the growth with respect to s of the term

$$\alpha f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:e^{\alpha(g_\varepsilon*\tilde{Y}_s)}e^{\alpha P_s(g_\varepsilon*Y_0)}.$$

Now, $\exp(\alpha e^{-(-\Delta+m^2)s}(g_\varepsilon*Y_0))$ does not increase in time, while $\exp(\alpha(g_\varepsilon*\tilde{Y}_s))$ is bounded since $\tilde{Y}_s\leqslant 0$, and hence the growth with respect to s is determined only by the remaining term, that is the exponential $f_\varepsilon:\exp(\alpha(g_\varepsilon*X_s)):$.

In the next result, we prove continuity of the solutions to equations (B.3.3)–(B.3.4) with respect to the initial data.

Lemma B.9.3. *For every $\varepsilon>0$, the solution (X,Y) to equations (B.3.3)–(B.3.4) are continuous with respect to X_0 in $\hat{B}_X$ and with respect to Y_0 in $B_Y\cup B_{\exp}^{r,\ell}$, respectively.*

Proof. As far as X is concerned, continuity follows from the linearity of its representation (B.9.1). Let us focus on Y. Consider a sequence $(Y_0^n)_{n\in\mathbb{N}}$ converging to some limit Y_0 in $B_Y\cup B_{\exp}^{r,\ell}$. Then $(Y_0^n)_{n\in\mathbb{N}}$ is bounded in $B_Y\cup B_{\exp}^{r,\ell}$, and, by the regularization properties of g_ε, we have that $\exp(\alpha P_t Y_0^n)$ is uniformly bounded with respect to $z\in\mathbb{R}^2$, t and n on the support of g_ε. From inequality (B.9.2), it follows that the solution $\tilde{Y}_t(Y_0^{n_k})$ is bounded in the space $\bar{A}\subset C^\theta([0,\tau],C_{-\ell}^{2-2\theta-\delta}(\mathbb{R}^2))$ and pre-compact in $A\subset C^{\theta-\kappa}([0,\tau],C_{-\ell+\kappa}^{2-2\theta-\delta-\kappa}(\mathbb{R}^2))$. Thus, there exists a subsequence n_k such that $\tilde{Y}_t(Y_0^{n_k})$ converges to some $\tilde{Z}$ in A.

On the other hand, since $\tilde{Y}_t(Y_0^{n_k})$ solves the fixed-point equation

$$\tilde{Y}_t(Y_0^{n_k})=\mathcal{J}_t(\tilde{Y}_t(Y_0^{n_k}),Y_0^{n_k},X),$$

and since $\mathcal{J}$ is continuous with respect to all its variables (see the proof of Proposition B.9.1), we have

$$\tilde{Z}=\mathcal{J}_t(\tilde{Z},Y_0,X).$$

By uniqueness of solutions to the fixed point equation, we have that $\tilde{Y}_t(Y_0)=\tilde{Z}$. Finally, since the solution to equation (B.3.4) with initial data $Y_0^{n_k}$ is given by

$$Y_t(Y_0^{n_k})=\tilde{Y}_t(Y_0^{n_k})+P_tY_0^{n_k},$$

by the continuity of the heat kernel and of $\tilde{Y}_t$, we get the thesis. □

B.9.2 Derivatives of the flow

In this section, we denote by X and Y the solutions to the system of equations (B.3.3)–(B.3.4) and we study their derivatives with respect to the initial data X_0 and Y_0.

B.9.2.1 Existence and equations

We now show point *i.* in Proposition B.3.8.

We have that X is differentiable and that its derivatives $\nabla_{X_0} X_t$, $\nabla^2_{X_0} X_t$, and $\nabla_{Y_0} X_t$ solve equations (B.3.5), (B.3.6), and (B.3.7), respectively. This is because we have the following explicit representation (cf. equation (B.9.1)) of the solution

$$X_t = P_t X_0 + \int_0^t P_{t-s} \xi_s \, \mathrm{d}s,$$

which is linear with respect to the initial data. We immediately get that $\nabla^2_{X_0} X \equiv 0$ and $\nabla_{Y_0} X \equiv 0$.

We now focus on the derivatives $\nabla_{X_0} Y$, $\nabla^2_{X_0} Y$, and $\nabla_{Y_0} Y$ of Y, and show that they exist and satisfy equations (B.3.8), (B.3.9), and (B.3.10), respectively. Furthermore, they are all continuous functions with respect to X_0 and Y_0.

Proposition B.9.4. *For every $\varepsilon > 0$, we have that that the derivatives $\nabla_{Y_0} Y$, $\nabla_{X_0} Y$, and $\nabla^2_{X_0} Y$ of the solution Y to equation (B.3.4) exist and satisfy equations (B.3.8), (B.3.9), and (B.3.10), respectively.*

Proof. We only give the proof for $\nabla_{Y_0} Y$, the other cases follows in a similar way. Consider the approximating equation (B.3.4), that is

$$(\partial_t - \Delta + m^2) Y = -\mathscr{G}_\varepsilon(X, Y) = -\alpha f_\varepsilon :e^{\alpha(g_\varepsilon * X)}: e^{\alpha(g_\varepsilon * Y)}, \qquad Y(0) = Y_0, \tag{B.9.3}$$

with $Y_0 \leqslant 0$. We denote the solution to equation (B.9.3) as $Y_t(Z)$, in order to stress the initial condition $Z \in B_Y \cup B^{r,\ell}_{\exp}$. We have the integral representation of the solution Y to (B.9.3):

$$Y_t(Z) = P_t Z - \int_0^t P_{t-s} \mathscr{G}_\varepsilon(X_s, Y_s(Z)) \, \mathrm{d}s.$$

In order to compute the derivative with respect to the initial data, we need to perturb it. Therefore, taking $\lambda > 0$ and $h \in B_Y \cup B^{r,\ell}_{\exp}$, and considering the difference, we have

$$Y_t(Y_0 + \lambda h) - Y_t(Y_0) = P_t h \lambda - \int_0^t P_{t-s}(\mathscr{G}_\varepsilon(X_s, Y_s(Y_0 + \lambda h)) - \mathscr{G}_\varepsilon(X_s, Y_s(Y_0))) \, \mathrm{d}s.$$

But

$$\begin{aligned}
&\int_0^t P_{t-s}(\mathscr{G}_\varepsilon(X_s, Y_s(Y_0 + \lambda h)) - \mathscr{G}_\varepsilon(X_s, Y_s(Y_0))) \, \mathrm{d}s \\
&= \int_0^t \int_0^1 P_{t-s} \mathrm{D}_Y \mathscr{G}_\varepsilon\big(X_s, \varsigma Y_s(Y_0 + \lambda h) + (1 - \varsigma) Y_s(Y_0)\big) \big(g_\varepsilon * (Y_s(Y_0 + \lambda h) - Y_s(Y_0))\big) \, \mathrm{d}\varsigma \, \mathrm{d}s,
\end{aligned}$$

since $\mathscr{G}_\varepsilon$ is differentiable in the direction $g_\varepsilon * (Y_s(Y_0 + \lambda h) - Y_s(Y_0))$.

Define

$$H(s, \lambda, h) = \int_0^1 \mathrm{D}_Y \mathscr{G}_\varepsilon\big(X_s, \varsigma Y_s(Y_0 + \lambda h) + (1 - \varsigma) Y_s(Y_0)\big) \, \mathrm{d}\varsigma,$$

so that we can write

$$Y_t(Y_0 + \lambda h) - Y_t(Y_0) = P_t h \lambda - \int_0^t P_{t-s} H(s, \lambda, h) \cdot (g_\varepsilon * (Y_s(Y_0 + \lambda h) - Y_s(Y_0))) \, \mathrm{d}s.$$

Let us write down an equation for

$$D_{\lambda,h}Y_t(Y_0)=\frac{Y_t(Y_0+\lambda h)-Y_t(Y_0)}{\lambda}.$$

Notice that

$$D_{\lambda,h}Y_t(Y_0)=P_th-\int_0^t P_{t-s}H(s,\lambda,h)\cdot(g_\varepsilon*D_{\lambda,h}Y_t(Y_0))\,\mathrm{d}s,$$

and therefore

$$\partial_t(D_{\lambda,h}Y_t(Y_0))=-(-\Delta+m^2)D_{\lambda,h}Y_t(Y_0)-H(t,\lambda,h)(g_\varepsilon*D_{\lambda,h}Y_t(Y_0)),\qquad D_{\lambda,h}Y_0(Y_0)=h. \tag{B.9.4}$$

Now consider, as in the proof of Proposition B.9.1,

$$\tilde{Y}_t=Y_t-P_tY_0.$$

Then, we have

$$D_{\lambda,h}\tilde{Y}_t(Y_0)=\frac{\tilde{Y}_t(Y_0+\lambda h)-\tilde{Y}_t(Y_0)}{\lambda}=D_{\lambda,h}Y_t(Y_0)-P_th,$$

and such a difference satisfies the following equation

$$(\partial_t-\Delta+m^2)D_{\lambda,h}\tilde{Y}_t(Y_0)=-H(t,\lambda,h)(g_\varepsilon*D_{\lambda,h}\tilde{Y}_t(Y_0)+P_t(g_\varepsilon*h)),\quad D_{\lambda,h}Y_0(Y_0)=0. \tag{B.9.5}$$

By Theorem B.7.1, we have then the following bound, for some constant $K>0$ and every $0<\sigma<1$:

$$\begin{aligned}&\|\tilde{g}_\varepsilon*D_{\lambda,h}\tilde{Y}_t(Y_0)\|_{L^2}^2+\int_0^t(\|\tilde{g}_\varepsilon*D_{\lambda,h}\tilde{Y}_s(Y_0)\|_{H^1}^2+(m^2-K\sigma)\|\tilde{g}_\varepsilon*D_{\lambda,h}\tilde{Y}_s(Y_0)\|_{L^2}^2)\,\mathrm{d}s\\ \lesssim&\int_0^t\|H(t,\lambda,h)P_t(g_\varepsilon*h)\|_{L^2}^2\,\mathrm{d}s.\end{aligned} \tag{B.9.6}$$

Consider

$$\hat{Y}_t^\lambda(Y_0)=Y_t(Y_0+\lambda h)-P_th\lambda=P_tY_0-\int_0^t P_{t-s}\mathscr{G}_\varepsilon(X_s,Y_s(Y_0+\lambda h))\,\mathrm{d}s.$$

Notice that $\hat{Y}_t^\lambda(Y_0)$ is negative and moreover it solves

$$\hat{Y}_t^\lambda(Y_0)=P_tY_0-\alpha\int_0^t P_{t-s}f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:e^{\alpha(g_\varepsilon*\hat{Y}_s^\lambda(Y_0))}e^{\lambda\alpha\left(g_{\bar{\varepsilon}}*P_sh\right)}\,\mathrm{d}s.$$

Recall that g_ε is such that $g_\varepsilon*h\in L^\infty_{\mathrm{loc}}$ and f_ε is compactly supported. Therefore,

$$\begin{array}{ll}\exp(\alpha(1-\varsigma)(g_\varepsilon*Y_s)(Y_0)),\exp(\alpha g_\varepsilon*\hat{Y}_s^\lambda(Y_0))\in L^\infty, & \text{since the exponents are negative,}\\ \mathbb{I}_{\mathrm{supp}(f_\varepsilon)}\exp(\lambda\alpha(g_\varepsilon*P_sh))\in L^\infty, & \text{since the exponent is negative on supp}(f_\varepsilon).\end{array}$$

We then have the uniform estimate on H given by, for any $s>0$ small enough and $1<p<+\infty$,

$$\begin{aligned}\|H(s,\lambda,h)\|_{B^{-s}_{p,p,-\ell}}&\leqslant\|f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:\|_{B^{-s}_{p,p,-\ell}}\left\|e^{\alpha\varsigma(g_\varepsilon*\hat{Y}_s^\lambda(Y_0))}e^{\lambda\varsigma\alpha(g_\varepsilon*P_sh)}e^{\alpha(1-\varsigma)(g_\varepsilon*Y_s)(Y_0)}\right\|_{L^\infty(\mathrm{supp}(f_\varepsilon))}\\&\leqslant\|f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:\|_{B^{-s}_{p,p,-\ell}}\|e^{\lambda\varsigma\alpha(g_\varepsilon*P_sh)}\|_{L^\infty(\mathrm{supp}(f_\varepsilon))}\\&\leqslant\|f_\varepsilon:e^{\alpha(g_\varepsilon*X_s)}:\|_{B^{-s}_{p,p,-\ell}}\|e^{\alpha|g_\varepsilon*P_sh|}\|_{L^\infty(\mathrm{supp}(f_\varepsilon))},\end{aligned} \tag{B.9.7}$$

which is uniform both in λ and ξ.

This gives existence of a limit for $D_{\lambda,h}Y_t(Y_0)$ as $\lambda\to0$.

We are left to show that the limit satisfies equation (B.3.8). First, we have to prove that the following limit holds

$$\lim_{\lambda\to0}H(t,\lambda,h)=\alpha^2f_\varepsilon:e^{\alpha(g_\varepsilon*X_t)}:e^{\alpha(g_\varepsilon*Y_t(Y_0))},$$

in some suitable space.

By Lemma B.9.3, we have that $Y_t(Y_0+\lambda h)\to Y_t(Y_0)$ as $\lambda\to 0$ in $B_Y\cup B^{r,\ell}_{\exp}$, and hence, thanks to the regularization properties of g_ε, $g_\varepsilon * Y_t(Y_0+\lambda h)$ converges to $g_\varepsilon * Y_t(Y_0)$ uniformly on the support of f_ε. Thus, we have the weak convergence

$$H(t,\lambda,h)\to\alpha^2 f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)}: e^{\alpha(g_\varepsilon * Y_t(Y_0))}, \qquad \text{as } \lambda\to 0, \tag{B.9.8}$$

and, by the uniform estimates (B.9.7) and the compact embedding $B^{-s}_{p,p,-\ell}\hookrightarrow B^{-s'}_{p,p,0}$, with $s'>s$ (see Proposition B.5.2), we get the strong convergence in the space $B^{-s'}_{p,p,0}$.

We then have a weak limit for $D_{\lambda,h}Y_t(Y_0)$ as $\lambda\to 0$. Thus, proceeding as in the proof of Lemma B.9.3, thanks to the a priori estimate (B.9.6), to the uniform bound (B.9.7), and to the convergence (B.9.8), we get that the limit of $D_{\lambda,h}Y_t(Y_0)$ as $\lambda\to 0$ is a solution to equation (B.3.8). □

B.9.2.2 Properties of the flow derivatives

We prove here some bounds on $\nabla_{X_0}Y_t(Y_0)$, $\nabla_{Y_0}Y_t(Y_0)$, and on the trace of $\nabla^2_{X_0}Y_t(Y_0)$. Let us recall that the sets $\tilde{B}_Y$ and $\tilde{B}_X$ are defined as in (B.3.11).

Proposition B.9.5. *For every $\delta\in(0,1)$, $\theta\in(0,1-\delta)$, $\ell,\ell'\geqslant 1$ and $h\in\tilde{B}_Y$, we have the estimate*

$$\begin{aligned}&\|\nabla_{Y_0}Y_t(Y_0)(h)\|_{C^\theta_{\ell'}(\mathbb{R}_+,C^{2-2\theta-2\delta}_{-\ell}(\mathbb{R}^2))\oplus L^\infty(\mathbb{R}_+,\tilde{B}_Y)}\\ \lesssim_{g_\varepsilon}\ &\mathfrak{P}_2\big(\|f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)}: e^{\alpha P_t(g_\varepsilon * h)}\|_{L^\infty_{\ell'}(\mathbb{R}_+,L^\infty_{-\ell}(\mathbb{R}^2))},\|h\|_{\tilde{B}_Y}\big),\end{aligned}$$

where $\mathfrak{P}_2$ is a second degree polynomial.

Proof. We use the same notation as in the proof of Proposition B.9.4. Recall that

$$D_{\lambda,h}Y_t(Y_0)=D_{\lambda,h}\tilde{Y}_t(Y_0)+P_t h,$$

and then, since $P_t h$ is uniformly bounded in $\tilde{B}_Y$, in order to prove the result, it suffices to give an estimate on $D_{\lambda,h}\tilde{Y}_t(Y_0)$ in the space $C^\theta_{\ell'}(\mathbb{R}_+,C^{2-2\theta-2\delta}_{-\ell}(\mathbb{R}^2))$.

Let us consider a time-weight $\rho_{\ell'}(t)$ and a space-weight $\rho_{-\ell}(x)$ defines as

$$\rho^k_l(z)=(1+k|z|^2)^{-l/2},\qquad z\in\mathbb{R}_+,\mathbb{R}^2.$$

Then, by Theorem B.7.2 we have, for some $0<\sigma<1$,

$$\begin{aligned}&\rho_{2\ell'}(t)\|\rho_{-\ell}(\tilde{g}_\varepsilon * D_{\lambda,h}\tilde{Y}_t(Y_0))\|^2_{L^2}\\ &+\int_0^t\rho_{2\ell'}(s)(\|\rho_{-\ell}(\tilde{g}_\varepsilon * D_{\lambda,h}\tilde{Y}_s(Y_0))\|^2_{H^1}+(m^2-\sigma C)\|\rho_{-\ell}(\tilde{g}_\varepsilon * D_{\lambda,h}\tilde{Y}_s(Y_0))\|^2_{L^2})\,\mathrm{d}s\\ \lesssim\ &C_\delta\int_0^t\rho_{2\ell'}(s)\int\rho_{-2\ell}(x)\big(\tilde{g}_\varepsilon *(H(s,\lambda,h)e^{-(-\Delta+m^2)s}(g_\varepsilon * h))\big)^2\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Moreover, applying Corollary B.7.3 yields

$$\begin{aligned}&\|H(s,\lambda,h)(\tilde{g}_\varepsilon * D_{\lambda,h}\tilde{Y}_s(Y_0)+P_s(\tilde{g}_\varepsilon * h))\|_{L^q_{\ell'}(\mathbb{R}_+,B^0_{p,p,-\ell}(\mathbb{R}^2))}\\ \leqslant\ &\mathfrak{P}_2(\|H\|_{L^\infty_{\ell'}(\mathbb{R}_+,L^\infty_{-\ell}(\mathbb{R}^2))},\|\tilde{g}_\varepsilon * h\|_{L^\infty_{-\ell}(\mathbb{R}^2)}).\end{aligned}$$

Together with the previous estimate, this yields uniform bounds on the norm $\|D_{\lambda,h}\tilde{Y}_t(Y_0)\|_{B^\beta_{q,q,\ell'}(\mathbb{R}_+,B^{2-2\beta-\delta}_{p,p,-\ell}(\mathbb{R}^2))}$, and choosing β and p accordingly we then deduce that we have $D_{\lambda,h}\tilde{Y}_t(Y_0)\in C^\delta_{\ell'}(\mathbb{R}_+,C^{2-\delta'}_{-\ell}(\mathbb{R}^2))$, uniformly in λ. Now, letting $\lambda\to 0$ yields the result. □

Proposition B.9.6. *For every $\delta\in(0,1)$, $\theta\in(0,1-\delta)$, $\ell,\ell'\geqslant 1$ and $h\in\tilde{B}_X$, we have the estimate*

$$\|\nabla_{X_0}Y_t(Y_0)(h)\|_{C^\theta_{\ell'}(\mathbb{R}_+,C^{2-2\theta-2\delta}_{-\ell}(\mathbb{R}^2))}\lesssim_{g_\varepsilon}P_2(\|f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)}: e^{\alpha P_t(g_\varepsilon * h)}\|_{L^\infty_{\ell'}(\mathbb{R}_+,L^\infty_{-\ell}(\mathbb{R}^2))}),$$

where P_2 is a second degree polynomial.

Proof. The proof is similar to the one of Proposition B.9.5, the only difference is that here the initial data of $\nabla_{X_0} Y_t$ is zero and therefore we do not need to subtract it before doing the estimates in Theorem B.7.2 and in Corollary B.7.3. □

We now deal with the trace term appearing in the definition of the operator $\mathscr{L}$.

Proposition B.9.7. *For every $\ell, \kappa \geqslant 0$, there exist $\beta, \delta > 0$ such that*

$$\|\nabla_{X_0}^2 Y_t(Y_0)\|_{L(H_\ell^{-\kappa}, H_{-\ell}^\kappa)} \lesssim \left(\int_{\mathbb{R}^2} \alpha^2 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} e^{(\delta - m^2)t} (1 + |z'|^\beta) \mathrm{d}z' \right)^2.$$

It follows that, whenever $\ell > 1$ and $\kappa > 1$,

$$\mathrm{tr}(|\nabla_{X_0}^2 Y_t(Y_0) \rho_{-\ell}|) \lesssim \left(\int_{\mathbb{R}^2} \alpha^2 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} e^{(\delta - m^2)t} (1 + |z'|^\beta) \mathrm{d}z' \right)^2.$$

Proof. We suppose that all the computations involving equations (B.3.9) and (B.3.10) make sense since a rigorous proof of this fact can be given in a similar way as in the proof of Proposition B.9.5.

Let $\delta_\cdot$ be the Dirac delta distribution, and consider the map, for $X_0 \in B_X$ and $t \in \mathbb{R}_+$,

$$T_{X_0,t}: H_\ell^{-\kappa}(\mathbb{R}^2) \to H_{-\ell}^\kappa(\mathbb{R}^2), \quad h \mapsto \nabla_{X_0}^2 Y_t(Y_0)(h, \delta_\cdot).$$

Hereafter, we drop the dependence on Y_0 in the derivatives of Y. Let us then evaluate equations (B.3.9) and (B.3.10) at $\delta_z(\cdot)$, and $(h, \delta_z(\cdot))$, respectively, to get

$$\begin{aligned}
&(\partial_t - (\Delta - m^2)) \nabla_{X_0} Y_t(\delta_z)(z') \\
&= -\alpha^2 f_\varepsilon(z') :e^{\alpha g_\varepsilon * X_t(z')}: e^{\alpha g_\varepsilon * Y_t(z')} (P_t(g_\varepsilon * \delta_z)(z') + (g_\varepsilon * \nabla_{X_0} Y_t(\delta_z))(z')) \\
&= -\alpha^2 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (P_t g_\varepsilon(z - z') + (g_\varepsilon * \nabla_{X_0} Y_t(\delta_z))(z')),
\end{aligned} \tag{B.9.9}$$

and

$$\begin{aligned}
&(\partial_t - (\Delta - m^2)) \nabla_{X_0}^2 Y_t(h, \delta_z)(z') \\
&= -\alpha^2 f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (g_\varepsilon * \nabla_{X_0}^2 Y_t(h, \delta_z))(z') \\
&\quad -\alpha^3 f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (g_\varepsilon * P_t h)(z') ((g_\varepsilon * P_t \delta_z)(z') + (g_\varepsilon * \nabla_{X_0} Y_t(\delta_z))(z')) \\
&\quad -\alpha^3 f_\varepsilon :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (g_\varepsilon * \nabla_{X_0} Y_t(h))(z') ((g_\varepsilon * P_t \delta_z)(z') + (g_\varepsilon * \nabla_{X_0} Y_t(\delta_z))(z')).
\end{aligned} \tag{B.9.10}$$

Using similar methods as in the proofs of Proposition B.9.4 and Proposition B.9.5, it is possible to prove that $\nabla_{X_0} Y_t(\delta_z)$ and $\nabla_{X_0}^2 Y_t(h, \delta_z)$ are differentiable infinitely many times with respect to z and their derivatives with respect to z for any multi-index β satisfy

$$\partial_z^\beta (g_\varepsilon * \nabla_{X_0} Y_t(\delta_z))(z') = \int g_\varepsilon(z' - y) \, \partial_z^\beta \nabla_{X_0} Y_t(\delta_z)(y) \, \mathrm{d}y.$$

Furthermore, by equations (B.9.9) and (B.9.10), we get

$$\begin{aligned}
(\partial_t - (\Delta - m^2)) \partial_z^\beta \nabla_{X_0} Y_t(\delta_z)(z') &= -\alpha^2 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} P_t \partial_z^\beta g_\varepsilon(z - z') \\
&\quad -\alpha^2 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (g_\varepsilon * \partial_z^\beta \nabla_{X_0} Y_t(\delta_z))(z'),
\end{aligned} \tag{B.9.11}$$

and

$$\begin{aligned}
&(\partial_t - (\Delta - m^2)) \partial_z^\beta \nabla_{X_0}^2 Y_t(h, \delta_z)(z') \\
&= -\alpha^2 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (\partial_\cdot^\beta g_\varepsilon * \nabla_{X_0}^2 Y_t(h, \delta_z))(z') \\
&\quad -\alpha^3 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (g_\varepsilon * P_t h)(z') ((\partial_\cdot^\beta g_\varepsilon * P_t \delta_z)(z') + (g_\varepsilon * \partial_z^\beta \nabla_{X_0} Y_t(\delta_z))(z')) \\
&\quad -\alpha^3 f_\varepsilon(z') :e^{\alpha(g_\varepsilon * X_t)(z')}: e^{\alpha(g_\varepsilon * Y_t)(z')} (g_\varepsilon * \nabla_{X_0} Y_t(h))(z') ((\partial_\cdot^\beta g_\varepsilon * P_t \delta_z)(z') + (g_\varepsilon * \partial_z^\beta \nabla_{X_0} Y_t(\delta_z))(z')).
\end{aligned} \tag{B.9.12}$$

Exploiting Theorem B.7.2 applied to equation (B.9.11), we get

$$\begin{aligned}&\|\tilde{g}_\varepsilon * \partial_z^\beta \nabla_{X_0} Y_t(\delta_z)\|_{L^2}^2 + \int_0^t \|\tilde{g}_\varepsilon * \partial_z^\beta \nabla_{X_0} Y_\tau(\delta_z)\|_{H^1}^2 \,\mathrm{d}\tau \\ &\lesssim \int \alpha^2 f_\varepsilon(z') : e^{\alpha(g_\varepsilon * X_t)(z')} : e^{\alpha(g_\varepsilon * Y_t)(z')} |P_t \partial_z^\beta g_\varepsilon(z-z')|^2 \,\mathrm{d}z' =: F_\beta(z).\end{aligned}$$

By a bootstrap argument, i.e. applying Corollary B.7.3, we can conclude that

$$\|\partial_z^\beta \nabla_{X_0} Y_t(\delta_z)\|_{B_{2,2}^s}^2 \leqslant \rho_{-\ell'}(t) F_\beta(z) q_s(F_\beta(z)),$$

where q_s is an s-dependent polynomial, because of the relation of the heat kernel with the support of f_ε.

By a similar method as above, we get the following estimate concerning the second derivative

$$\|\partial_z^\beta \nabla_{X_0}^2 Y_t(h, \delta_z)\|_{B_{2,2}^s} \leqslant \|h\|_{H_\ell^{-\kappa}} F_\beta(z) \tilde{q}_s(F_\beta(z)),$$

where $\tilde{q}_s$ is another s-dependent polynomial. Taking s large enough, we have

$$\|\partial_z^\beta \nabla_{X_0}^2 Y_t(h, \delta_z)\|_{L^\infty} \leqslant \|h\|_{H_\ell^{-\kappa}} F_\beta(z)\, \tilde{q}_s(F_\beta(z)).$$

This proves that the map is linear with respect to h as a map from L^2, moreover, if $s \in \mathbb{N}$ and fixing z' and h, the norm with respect to z is given by

$$\begin{aligned}\|\nabla_{X_0}^2 Y_t(h, \delta_\cdot)(z')\|_{B_{2,2,-\ell}^s(\mathrm{d}z)}^2 &= \sum_{|\beta| \leqslant s} \int \rho_{-\ell}^2(z) \,(\partial_z^\beta \nabla_{X_0}^2 Y_t(h, \delta_z)(z'))^2 \,\mathrm{d}z \\ &\lesssim \sum_{|\beta| \leqslant s} \int \rho_{-\ell}^2(z) \,\|\partial_z^\beta \nabla_{X_0}^2 Y_t(h, \delta_z)(z')\|_{L^\infty(\mathrm{d}z')}^2 \,\mathrm{d}z \\ &\lesssim \|h\|_{H_\ell^{-\kappa}} \sum_{|\beta| \leqslant s} \int \rho_{-\ell}^2(z) \,|F_\beta(z) \tilde{q}_s(F_\beta(z))|^2 \,\mathrm{d}z. \qquad \text{(B.9.13)}\end{aligned}$$

We are left to show that the last integral is finite. Recall

$$F_\beta(z) = \int \alpha^2 f_\varepsilon(z') : e^{\alpha(g_\varepsilon * X_t)(z')} : e^{\alpha(g_\varepsilon * Y_t)(z')} |P_t \partial_z^\beta g_\varepsilon(z-z')|^2 \,\mathrm{d}z'.$$

Since we have to show that the integral is finite for some polynomial, we multiply (and divide) the heat kernel by some weight $\rho_{-2\ell}$, to get

$$F_\beta(z)\, \rho_{-2\ell}(z) = \int \alpha^2 f_\varepsilon(z') : e^{\alpha(g_\varepsilon * X_t)(z')} : e^{\alpha(g_\varepsilon * Y_t)(z')} |\rho_{-\ell}(z)\, P_t \partial_z^\beta g_\varepsilon(z-z')|^2 \,\mathrm{d}z'.$$

By inequality (6.2) in Section 6 of [177], we have, for some $\kappa > 0$,

$$\rho_{-\ell}(z) \leqslant \rho_{-\ell}(z - \tilde{z})(1 + |\tilde{z}|)^\kappa.$$

Therefore,

$$\begin{aligned}|\rho_{-\ell}(z)\, P_t \partial_z^\beta g_\varepsilon(z-z')| &= \left| \int_{\mathbb{R}^2} \frac{1}{2\pi t} e^{-\frac{|\tilde{z}|^2}{2t} - m^2 t} \,\partial_z^\beta g_\varepsilon(z - z' - \tilde{z}) \rho_{-\ell}(z) \,\mathrm{d}\tilde{z} \right| \\ &\leqslant \int_{\mathbb{R}^2} \frac{1}{2\pi t} e^{-\frac{|\tilde{z}|^2}{2t} - m^2 t} |\partial_z^\beta g_\varepsilon(z - z' - \tilde{z})|\, \rho_{-\ell}(z) \,\mathrm{d}\tilde{z} \\ &\leqslant \int_{\mathbb{R}^2} \frac{1}{2\pi t} e^{-\frac{|\tilde{z}|^2}{2t} - m^2 t} |\partial_z^\beta g_\varepsilon(z - z' - \tilde{z})|\, \rho_{-\ell}(z - z' - \tilde{z})\,(1 + |z' + \tilde{z}|)^\kappa \,\mathrm{d}\tilde{z}.\end{aligned}$$

By the compact support of g_ε, we have

$$\begin{aligned}|\rho_{-\ell}(z)\,P_t\partial_z^\beta g_\varepsilon(z-z')| &\leqslant C_{\ell,g,\varepsilon,\beta}\int_{\mathbb{R}^2}\frac{1}{2\pi t}e^{-\frac{|\tilde{z}|^2}{2t}-m^2t}(1+|z'+\tilde{z}|)^\kappa\,\mathrm{d}\tilde{z}\\ &= C_{\ell,g,\varepsilon,\beta}e^{-m^2t}\,\mathbb{E}_W[(1+|z'+W_t|)^\kappa],\end{aligned}$$

where W is some two-dimensional Brownian motion. Thus, if we take κ large enough, we get

$$\begin{aligned}|\rho_{-\ell}(z)\,P_t\partial_z^\beta g_\varepsilon(z-z')| &\leqslant C_{\kappa,\ell,g,\varepsilon,\beta}e^{-m^2t}(1+|z'|^\kappa+\mathbb{E}[|W_t|^\kappa])\\ &\leqslant C_{\kappa,\ell,g,\varepsilon,\beta,\delta}e^{(\delta-m^2)t}(1+|z'|^\beta).\end{aligned}$$

Then,

$$F_\beta(z)\rho_{-2\ell}(z)\leqslant C_{\kappa,\ell,g,\varepsilon,\beta,\delta}\int\alpha^2 f_\varepsilon(z'):e^{\alpha(g_\varepsilon*X_t)(z')}:e^{\alpha(g_\varepsilon*Y_t)(z')}e^{(\delta-m^2)t}(1+|z'|^\kappa)\,\mathrm{d}z',$$

which is finite, since f_ε is compactly supported. □

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